

# MATB44 - Week 13 Tutorial Teaching Notes

## Week 13 Topics

### Laplace Transform Method

- Definition and basic properties of Laplace transforms
- Solving ODEs using Laplace transforms
- Initial value problems and convolution
- Inverse Laplace transforms and partial fractions

### Frobenius Method

- Power series solutions for regular singular points
  - Indicial equation and roots
  - Finding series solutions near singular points
  - Cases: distinct roots, repeated roots, roots differing by an integer
- 

## 1. Laplace Transform Method

### 1.1 Definition

The Laplace transform of a function  $f(t)$  (defined for  $t \geq 0$ ) is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

where  $s$  is a complex variable, and the integral converges for sufficiently large  $\text{Re}(s)$ .

### Basic Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
$t^n$ ( $n \in \mathbb{N}$ )	$\frac{n!}{s^{n+1}}$
$e^{at}$	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\delta(t-a)$ (Dirac delta)	$e^{-as}$

---

## 1.2 Key Properties

### Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

### First Derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

### Second Derivative

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

### $n$ -th Derivative

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

### Shifting Property (First Shifting Theorem)

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

### Second Shifting Theorem

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s),$$

where  $u(t-a)$  is the unit step function (Heaviside function).

### Convolution

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\},$$

where  $(f * g)(t) = \int_0^t f(\tau)g(t-\tau) d\tau$ .

---

## 1.3 Solving ODEs with Laplace Transform

### General Procedure:

1. Take the Laplace transform of both sides of the ODE.
  2. Use initial conditions to simplify.
  3. Solve algebraically for  $Y(s) = \mathcal{L}\{y(t)\}$ .
  4. Find the inverse Laplace transform  $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ .
- 

## 1.4 Inverse Laplace Transform

### Partial Fractions

To find  $\mathcal{L}^{-1}\{F(s)\}$ , often we need to decompose  $F(s)$  into partial fractions.

### Common cases:

1. **Distinct linear factors:**

$$\frac{A}{s-a} + \frac{B}{s-b} \Rightarrow Ae^{at} + Be^{bt}$$

2. **Repeated linear factors:**

$$\frac{A}{(s-a)^n} \Rightarrow \frac{A}{(n-1)!} t^{n-1} e^{at}$$

### 3. Quadratic factors:

$$\frac{As + B}{s^2 + \omega^2} \Rightarrow A \cos(\omega t) + \frac{B}{\omega} \sin(\omega t)$$

---

## 1.5 Examples

### Example 1: First-Order ODE

Solve:

$$y' + 2y = e^{-t}, \quad y(0) = 1$$

**Solution:**

**Step 1.** Take Laplace transform:

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

**Step 2.** Apply properties:

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+1}$$

Substitute  $y(0) = 1$ :

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+1}$$

**Step 3.** Solve for  $Y(s)$ :

$$Y(s)(s+2) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}$$

$$Y(s) = \frac{1}{s+1}$$

**Step 4.** Inverse transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

---

### Example 2: Second-Order ODE

Solve:

$$y'' + 4y = \sin(2t), \quad y(0) = 0, \quad y'(0) = 1$$

**Solution:**

**Step 1.** Take Laplace transform:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

**Step 2.** Apply properties:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{2}{s^2 + 4}$$

Substitute initial conditions:

$$s^2Y(s) - 1 + 4Y(s) = \frac{2}{s^2 + 4}$$

**Step 3.** Solve for  $Y(s)$ :

$$Y(s)(s^2 + 4) = 1 + \frac{2}{s^2 + 4} = \frac{s^2 + 6}{s^2 + 4}$$

$$Y(s) = \frac{s^2 + 6}{(s^2 + 4)^2} = \frac{1}{s^2 + 4} + \frac{2}{(s^2 + 4)^2}$$

**Step 4.** Inverse transform:

$$y(t) = \frac{1}{2} \sin(2t) + \frac{1}{4} [\sin(2t) - 2t \cos(2t)] = \frac{3}{4} \sin(2t) - \frac{t}{2} \cos(2t)$$


---

### Example 3: ODE with Step Function

Solve:

$$y'' + y = u(t - 1), \quad y(0) = 0, \quad y'(0) = 0$$

where  $u(t - 1)$  is the unit step function.

**Solution:**

**Step 1.** Laplace transform:

$$s^2Y(s) + Y(s) = \frac{e^{-s}}{s}$$

**Step 2.** Solve:

$$Y(s) = \frac{e^{-s}}{s(s^2 + 1)} = e^{-s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

**Step 3.** Inverse transform (using second shifting theorem):

$$y(t) = u(t - 1) [1 - \cos(t - 1)]$$


---

### Example 4: Using Convolution

Solve:

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

**Solution:**

Taking Laplace transform:

$$s^2 Y(s) + Y(s) = F(s) \Rightarrow Y(s) = \frac{F(s)}{s^2 + 1}$$

Using convolution:

$$y(t) = \int_0^t \sin(t - \tau) f(\tau) d\tau$$

---

## 2. Frobenius Method

### 2.1 Regular Singular Points

Consider a second-order linear ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

A point  $x_0$  is a **regular singular point** if:

- $(x - x_0) \frac{Q(x)}{P(x)}$  is analytic at  $x_0$
- $(x - x_0)^2 \frac{R(x)}{P(x)}$  is analytic at  $x_0$

In standard form (dividing by  $P(x)$ ):

$$y'' + p(x)y' + q(x)y = 0,$$

$x_0$  is a regular singular point if:

- $(x - x_0)p(x)$  is analytic at  $x_0$
  - $(x - x_0)^2 q(x)$  is analytic at  $x_0$
- 

### 2.2 Frobenius Method Overview

**Goal:** Find power series solutions near a regular singular point  $x_0$  (usually  $x_0 = 0$ ).

**Assumption:** Solution of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

**Steps:**

1. Substitute the series into the ODE.
  2. Find the **indicial equation** (determines  $r$ ).
  3. For each root  $r$ , find recurrence relation for  $a_n$ .
  4. Write the general solution.
- 

## 2.3 Indicial Equation

After substitution, the lowest power of  $x$  gives the **indicial equation**:

$$r(r-1) + p_0r + q_0 = 0,$$

where:

- $p_0 = \lim_{x \rightarrow 0} xp(x)$
- $q_0 = \lim_{x \rightarrow 0} x^2q(x)$

This is a quadratic in  $r$  with roots  $r_1$  and  $r_2$ .

---

## 2.4 Three Cases

### Case 1: Distinct Roots Not Differing by an Integer

If  $r_1 \neq r_2$  and  $r_1 - r_2 \notin \mathbb{Z}$ , then two linearly independent solutions:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

General solution:  $y(x) = C_1 y_1(x) + C_2 y_2(x)$ .

---

### Case 2: Repeated Root ( $r_1 = r_2$ )

If  $r_1 = r_2 = r$ , then:

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Second solution:

$$y_2(x) = y_1(x) \ln x + x^r \sum_{n=1}^{\infty} b_n x^n$$


---

### Case 3: Roots Differing by an Integer

If  $r_1 - r_2 = N \in \mathbb{Z}^+$  (assume  $r_1 > r_2$ ), then:

**First solution:**

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

**Second solution:**

- If the recurrence relation gives a valid solution for  $r_2$ , then:

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

- Otherwise (if  $a_N$  becomes undefined), the second solution contains a logarithm:

$$y_2(x) = C y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$


---

## 2.5 Examples

### Example 4: Roots Differing by an Integer

Solve the ODE near the regular singular point  $x = 0$ :

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$$

**Solution:**

**Step 1.** Frobenius form and indicial equation

Assume:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

Then:

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substitute into the equation:

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$$

We get:

$$\sum_{n=0}^{\infty} \left[ (n+r)^2 - \frac{1}{4} \right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Setting coefficients to zero, we obtain the recurrence relation:

$$\left[ (n+r)^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0, \quad n \geq 0$$

where we define  $a_{-1} = a_{-2} = 0$ .

For  $n = 0$ , we get the **indicial equation**:

$$r^2 - \frac{1}{4} = 0 \quad \Rightarrow \quad r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$$

The roots differ by 1 (an integer).

---

**Step 2.** Solution for  $r = \frac{1}{2}$

With  $r = \frac{1}{2}$ , the recurrence becomes:

$$\left[ \left( n + \frac{1}{2} \right)^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0 \quad \Rightarrow \quad n(n+1)a_n + a_{n-2} = 0$$

That is:

$$a_n = -\frac{a_{n-2}}{n(n+1)}$$

Taking  $a_0 = 1$ :

- $n = 1$ :  $1 \cdot 2 \cdot a_1 = 0 \Rightarrow a_1 = 0$  (all odd terms are 0)
- $n = 2$ :  $a_2 = -\frac{a_0}{2 \cdot 3} = -\frac{1}{6}$
- $n = 4$ :  $a_4 = -\frac{a_2}{4 \cdot 5} = \frac{1}{120}$
- $n = 6$ :  $a_6 = -\frac{a_4}{6 \cdot 7} = -\frac{1}{5040}$
- ...

Therefore:

$$\begin{aligned} y_1(x) &= x^{1/2} (a_0 + a_2 x^2 + a_4 x^4 + \dots) \\ &= x^{1/2} \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right) \end{aligned}$$

The series in parentheses is  $\frac{\sin x}{x}$ , so:

$$y_1(x) = x^{1/2} \cdot \frac{\sin x}{x} = x^{-1/2} \sin x$$


---

**Step 3.** Solution for  $r = -\frac{1}{2}$

With  $r = -\frac{1}{2}$ , the recurrence becomes:

$$\left[ \left( n - \frac{1}{2} \right)^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0 \quad \Rightarrow \quad n(n-1)a_n + a_{n-2} = 0$$

That is:

$$a_n = -\frac{a_{n-2}}{n(n-1)}$$

For  $n = 0, 1$ , the coefficient is 0, so  $a_0$  and  $a_1$  are free constants.

To get a solution linearly independent from  $y_1$ , we take  $a_0 = 1, a_1 = 0$ .

- $n = 2$ :  $a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{1}{2}$
- $n = 4$ :  $a_4 = -\frac{a_2}{4 \cdot 3} = \frac{1}{24}$
- $n = 6$ :  $a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{1}{720}$
- ...

Therefore:

$$\begin{aligned} y_2(x) &= x^{-1/2} (a_0 + a_2 x^2 + a_4 x^4 + \dots) \\ &= x^{-1/2} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \end{aligned}$$

The series in parentheses is  $\cos x$ , so:

$$y_2(x) = x^{-1/2} \cos x$$

**Note:** Although the indicial roots differ by an integer, the second solution is a regular Frobenius series solution and does not contain  $\ln x$ .

---

#### Step 4. General solution

The general solution near  $x = 0$  is:

$$y(x) = C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x$$

where  $C_1, C_2$  are arbitrary constants.

---