

MATB44 - Week 13 Tutorial Teaching Notes

Week 13 Topics

Laplace Transform Method

- Definition and basic properties of Laplace transforms
- Solving ODEs using Laplace transforms
- Initial value problems and convolution
- Inverse Laplace transforms and partial fractions

Frobenius Method

- Power series solutions for regular singular points
 - Indicial equation and roots
 - Finding series solutions near singular points
 - Cases: distinct roots, repeated roots, roots differing by an integer
-

1. Laplace Transform Method

1.1 Definition

The Laplace transform of a function $f(t)$ (defined for $t \geq 0$) is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt,$$

where s is a complex variable, and the integral converges for sufficiently large $\text{Re}(s)$.

Basic Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n ($n \in \mathbb{N}$)	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\delta(t-a)$ (Dirac delta)	e^{-as}

1.2 Key Properties

Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

First Derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Second Derivative

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

n-th Derivative

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Shifting Property (First Shifting Theorem)

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Second Shifting Theorem

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s),$$

where $u(t-a)$ is the unit step function (Heaviside function).

Convolution

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\},$$

where $(f * g)(t) = \int_0^t f(\tau)g(t-\tau) d\tau.$

1.3 Solving ODEs with Laplace Transform

General Procedure:

1. Take the Laplace transform of both sides of the ODE.
 2. Use initial conditions to simplify.
 3. Solve algebraically for $Y(s) = \mathcal{L}\{y(t)\}.$
 4. Find the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\{Y(s)\}.$
-

1.4 Inverse Laplace Transform

Partial Fractions

To find $\mathcal{L}^{-1}\{F(s)\}$, often we need to decompose $F(s)$ into partial fractions.

Common cases:

1. Distinct linear factors:

$$\frac{A}{s-a} + \frac{B}{s-b} \Rightarrow Ae^{at} + Be^{bt}$$

2. Repeated linear factors:

$$\frac{A}{(s-a)^n} \Rightarrow \frac{A}{(n-1)!} t^{n-1} e^{at}$$

3. Quadratic factors:

$$\frac{As + B}{s^2 + \omega^2} \Rightarrow A \cos(\omega t) + \frac{B}{\omega} \sin(\omega t)$$

1.5 Examples

Example 1: First-Order ODE

Solve:

$$y' + 2y = e^{-t}, \quad y(0) = 1$$

Solution:

Step 1. Take Laplace transform:

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

Step 2. Apply properties:

$$sY(s) - y(0) + 2Y(s) = \frac{1}{s+1}$$

Substitute $y(0) = 1$:

$$sY(s) - 1 + 2Y(s) = \frac{1}{s+1}$$

Step 3. Solve for $Y(s)$:

$$Y(s)(s+2) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}$$

$$Y(s) = \frac{1}{s+1}$$

Step 4. Inverse transform:

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} = e^{-t}$$

Example 2: Second-Order ODE

Solve:

$$y'' + 4y = \sin(2t), \quad y(0) = 0, \quad y'(0) = 1$$

Solution:

Step 1. Take Laplace transform:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

Step 2. Apply properties:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{2}{s^2 + 4}$$

Substitute initial conditions:

$$s^2Y(s) - 1 + 4Y(s) = \frac{2}{s^2 + 4}$$

Step 3. Solve for $Y(s)$:

$$Y(s)(s^2 + 4) = 1 + \frac{2}{s^2 + 4} = \frac{s^2 + 6}{s^2 + 4}$$

$$Y(s) = \frac{s^2 + 6}{(s^2 + 4)^2} = \frac{1}{s^2 + 4} + \frac{2}{(s^2 + 4)^2}$$

Step 4. Inverse transform:

$$y(t) = \frac{1}{2} \sin(2t) + \frac{1}{4} [\sin(2t) - 2t \cos(2t)] = \frac{3}{4} \sin(2t) - \frac{t}{2} \cos(2t)$$

Example 3: ODE with Step Function

Solve:

$$y'' + y = u(t - 1), \quad y(0) = 0, \quad y'(0) = 0$$

where $u(t - 1)$ is the unit step function.

Solution:

Step 1. Laplace transform:

$$s^2Y(s) + Y(s) = \frac{e^{-s}}{s}$$

Step 2. Solve:

$$Y(s) = \frac{e^{-s}}{s(s^2 + 1)} = e^{-s} \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right)$$

Step 3. Inverse transform (using second shifting theorem):

$$y(t) = u(t - 1) [1 - \cos(t - 1)]$$

Example 4: Using Convolution

Solve:

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

Solution:

Taking Laplace transform:

$$s^2Y(s) + Y(s) = F(s) \Rightarrow Y(s) = \frac{F(s)}{s^2 + 1}$$

Using convolution:

$$y(t) = \int_0^t \sin(t-\tau)f(\tau) d\tau$$

2. Frobenius Method

2.1 Regular Singular Points

Consider a second-order linear ODE:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

A point x_0 is a **regular singular point** if:

- $(x - x_0)\frac{Q(x)}{P(x)}$ is analytic at x_0
- $(x - x_0)^2 \frac{R(x)}{P(x)}$ is analytic at x_0

In standard form (dividing by $P(x)$):

$$y'' + p(x)y' + q(x)y = 0,$$

x_0 is a regular singular point if:

- $(x - x_0)p(x)$ is analytic at x_0
- $(x - x_0)^2q(x)$ is analytic at x_0

2.2 Frobenius Method Overview

Goal: Find power series solutions near a regular singular point x_0 (usually $x_0 = 0$).

Assumption: Solution of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

Steps:

1. Substitute the series into the ODE.
 2. Find the **indicial equation** (determines r).
 3. For each root r , find recurrence relation for a_n .
 4. Write the general solution.
-

2.3 Indicial Equation

After substitution, the lowest power of x gives the **indicial equation**:

$$r(r - 1) + p_0r + q_0 = 0,$$

where:

- $p_0 = \lim_{x \rightarrow 0} xp(x)$
- $q_0 = \lim_{x \rightarrow 0} x^2 q(x)$

This is a quadratic in r with roots r_1 and r_2 .

2.4 Three Cases

Case 1: Distinct Roots Not Differing by an Integer

If $r_1 \neq r_2$ and $r_1 - r_2 \notin \mathbb{Z}$, then two linearly independent solutions:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

General solution: $y(x) = C_1 y_1(x) + C_2 y_2(x)$.

Case 2: Repeated Root ($r_1 = r_2$)

If $r_1 = r_2 = r$, then:

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

Second solution:

$$y_2(x) = y_1(x) \ln x + x^r \sum_{n=1}^{\infty} b_n x^n$$

Case 3: Roots Differing by an Integer

If $r_1 - r_2 = N \in \mathbb{Z}^+$ (assume $r_1 > r_2$), then:

First solution:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

Second solution:

- If the recurrence relation gives a valid solution for r_2 , then:

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

- Otherwise (if a_N becomes undefined), the second solution contains a logarithm:

$$y_2(x) = C y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n$$

2.5 Examples

Example 4: Roots Differing by an Integer

Solve the ODE near the regular singular point $x = 0$:

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solution:

Step 1. Frobenius form and indicial equation

Assume:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

Then:

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

Substitute into the equation:

$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

We get:

$$\sum_{n=0}^{\infty} \left[(n+r)^2 - \frac{1}{4} \right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Setting coefficients to zero, we obtain the recurrence relation:

$$\left[(n+r)^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0, \quad n \geq 0$$

where we define $a_{-1} = a_{-2} = 0$.

For $n = 0$, we get the **indicial equation**:

$$r^2 - \frac{1}{4} = 0 \quad \Rightarrow \quad r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$$

The roots differ by 1 (an integer).

Step 2. Solution for $r = \frac{1}{2}$

With $r = \frac{1}{2}$, the recurrence becomes:

$$\left[(n + \frac{1}{2})^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0 \quad \Rightarrow \quad n(n+1)a_n + a_{n-2} = 0$$

That is:

$$a_n = -\frac{a_{n-2}}{n(n+1)}$$

Taking $a_0 = 1$:

- $n = 1: 1 \cdot 2 \cdot a_1 = 0 \Rightarrow a_1 = 0$ (all odd terms are 0)
- $n = 2: a_2 = -\frac{a_0}{2 \cdot 3} = -\frac{1}{6}$
- $n = 4: a_4 = -\frac{a_2}{4 \cdot 5} = \frac{1}{120}$
- $n = 6: a_6 = -\frac{a_4}{6 \cdot 7} = -\frac{1}{5040}$
- ...

Therefore:

$$\begin{aligned} y_1(x) &= x^{1/2} (a_0 + a_2 x^2 + a_4 x^4 + \dots) \\ &= x^{1/2} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right) \end{aligned}$$

The series in parentheses is $\frac{\sin x}{x}$, so:

$$y_1(x) = x^{1/2} \cdot \frac{\sin x}{x} = x^{-1/2} \sin x$$

Step 3. Solution for $r = -\frac{1}{2}$

With $r = -\frac{1}{2}$, the recurrence becomes:

$$\left[(n - \frac{1}{2})^2 - \frac{1}{4} \right] a_n + a_{n-2} = 0 \quad \Rightarrow \quad n(n-1)a_n + a_{n-2} = 0$$

That is:

$$a_n = -\frac{a_{n-2}}{n(n-1)}$$

For $n = 0, 1$, the coefficient is 0, so a_0 and a_1 are free constants.

To get a solution linearly independent from y_1 , we take $a_0 = 1, a_1 = 0$.

- $n = 2$: $a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{1}{2}$
- $n = 4$: $a_4 = -\frac{a_2}{4 \cdot 3} = \frac{1}{24}$
- $n = 6$: $a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{1}{720}$
- ...

Therefore:

$$\begin{aligned}y_2(x) &= x^{-1/2} (a_0 + a_2 x^2 + a_4 x^4 + \dots) \\&= x^{-1/2} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)\end{aligned}$$

The series in parentheses is $\cos x$, so:

$$y_2(x) = x^{-1/2} \cos x$$

Note: Although the indicial roots differ by an integer, the second solution is a regular Frobenius series solution and does not contain $\ln x$.

Step 4. General solution

The general solution near $x = 0$ is:

$$y(x) = C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x$$

where C_1, C_2 are arbitrary constants.
