MATB44 - Week 9 Tutorial Teaching Notes

Week 9 Topics

Non-Homogeneous Linear ODEs with Constant Coefficients

- Methods: Undetermined Coefficients, Variation of Parameters, (optional) Annihilators
- Resonance and multiplicity adjustments
- RHS types: polynomials, exponentials, sines/cosines, and their products

Definition and Theory

General Form and Structure

- Linear ODE (constant coefficients):

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x), \quad a_n \neq 0.$$

- Homogeneous part:

$$a_n \frac{d^n y}{dx^n} + \cdots + a_0 y = 0 \quad \Rightarrow \quad y_h \text{ from characteristic roots.}$$

- Particular solution y_p satisfies the full equation. The general solution is $y = y_h + y_p$.
- Superposition for RHS: if $g(x) = g_1(x) + g_2(x)$, then $y_p = y_{p,1} + y_{p,2}$ where each solves its part.

Method of Undetermined Coefficients

- Choose a trial form similar to the RHS:
  - $g(x) = P_m(x)$ (polynomial of degree m): try $y_p = Q_m(x)$ (degree m).
  - $g(x) = e^{ax} P_m(x)$: try $y_p = e^{ax} Q_m(x)$.
  - $g(x) = \cos(bx)$ or $\sin(bx)$: try $y_p = A\cos(bx) + B\sin(bx)$.
  - $g(x) = e^{ax}\cos(bx)$ or $e^{ax}\sin(bx)$: try $y_p = e^{ax}(A\cos bx + B\sin bx)$.
- Resonance rule: if the trial form overlaps y_h, multiply by x^s, where s = multiplicity of the overlapping root (or conjugate pair) in the characteristic equation.
- Determine coefficients by substitution.

Variation of Parameters (2nd order, standard form)

For $y'' + p(x)y' + q(x)y = g(x)$, with a fundamental pair y_1, y_2 and Wronskian $W = y_1 y_2' - y_1' y_2$, one particular solution is

$$y_p = -y_1 \int \frac{y_2\, g}{W}\, dx + y_2 \int \frac{y_1\, g}{W}\, dx.$$

Works for general RHS, even when undetermined coefficients is not applicable.

Proof (sketch) Take a particular solution of the form

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

with u_1, u_2 differentiable. Impose the auxiliary condition

$$u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$$

to avoid second derivatives of u_1, u_2. Then

$$y_p' = u_1 y_1' + u_2 y_2', \qquad y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Substitute into $y'' + py' + qy = g$ and use that y_j $(j = 1, 2)$ solve the homogeneous equation $y_j'' + py_j' + qy_j = 0$. This yields

$$u_1' y_1' + u_2' y_2' = g.$$

Together with the auxiliary condition we have the linear system for u_1', u_2':

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = g. \end{cases}$$

Let the Wronskian be $W = y_1 y_2' - y_1' y_2 \neq 0$. By Cramer's rule,

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}.$$

Integrating gives

$$u_1 = -\int \frac{y_2 g}{W}\, dx, \qquad u_2 = \int \frac{y_1 g}{W}\, dx,$$

and hence

$$y_p = -y_1 \int \frac{y_2 g}{W}\, dx + y_2 \int \frac{y_1 g}{W}\, dx.$$

(Optional) Annihilator Method

- Idea: find a differential operator L that annihilates $g(x)$ (i.e., $L[g] = 0$).
- Apply L to both sides to obtain a higher-order homogeneous ODE, solve it, and then select the part corresponding to y_p.

Key Examples

Example 1: First-Order Equation

Solve:

$$y' + 2y = e^{-2x}$$

Solution: 1) $y_h : y' + 2y = 0 \Rightarrow y_h = Ce^{-2x}$ 2) Particular (details). Try $y_p = Axe^{-2x}$ (resonance):

$$y_p' = Ae^{-2x} - 2Axe^{-2x}.$$

Substitute into LHS:

$$(y_p' + 2y_p) = (Ae^{-2x} - 2Axe^{-2x}) + 2(Axe^{-2x}) = Ae^{-2x}.$$

Match RHS e^{-2x} $\quad A = 1$. Hence $y_p = xe^{-2x}$. 3) $y = Ce^{-2x} + xe^{-2x}$

Example 2: First-Order with Exponential RHS (non-resonant)

Solve:

$$y' + 2y = e^{-x}$$

Solution: 1) Homogeneous: $y' + 2y = 0 \Rightarrow y_h = Ce^{-2x}$. 2) Particular (details). Try $y_p = Ae^{-x}$ (non-resonant with e^{-2x}):

$$y_p' = -Ae^{-x}.$$

LHS:

$$y_p' + 2y_p = (-A + 2A)e^{-x} = Ae^{-x}.$$

Match RHS e^{-x} $A = 1$. Hence $y_p = e^{-x}$. 3) $y = Ce^{-2x} + e^{-x}$

Example 3: Trigonometric RHS via Variation of Parameters

Solve:

$$y'' + y = \sin x$$

Solution: 1) Homogeneous:

$$y'' + y = 0 \Rightarrow y_h = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x, \quad W = y_1 y_2' - y_1' y_2 = 1$$

2) Particular via VOP:

$$y_p = -y_1 \int \frac{y_2 g}{W}\, dx + y_2 \int \frac{y_1 g}{W}\, dx$$

Substitute $g = \sin x, W = 1$:

$$y_p = -\cos x \int \sin^2 x\, dx + \sin x \int \sin x \cos x\, dx$$

3) Integrate and simplify:

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin(2x)}{4}, \quad \int \sin x \cos x\, dx = \frac{\sin^2 x}{2}$$

$$\Rightarrow\ y_p = -\frac{x}{2}\cos x \quad \text{(terms in } \sin x \text{ can be absorbed by } y_h)$$

4) Final solution:

$$y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x$$

Example 4: Resonance with Mixed Polynomial–Trig–Exponential Forcing (Harder)

Solve:

$$y'' - 2y' + 2y = e^x(1+x)\cos x$$

Solution: 1) Homogeneous: characteristic $r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$.

$$y_h = e^x(C_1 \cos x + C_2 \sin x)$$

2) Substitute $y = e^x u$:

$$y' = e^x(u' + u), \quad y'' = e^x(u'' + 2u' + u),$$

so the ODE reduces to

$$u'' + u = (1+x)\cos x.$$

3) Undetermined coefficients with resonance. Try

$$u_p = x\big[(ax+b)\sin x + (cx+d)\cos x\big].$$

Matching coefficients in $u_p'' + u_p = (1+x)\cos x$ gives

$$a = \tfrac{1}{4}, \quad b = \tfrac{1}{2}, \quad c = 0, \quad d = \tfrac{1}{4},$$

hence

$$u_p = \frac{x}{4}\cos x + \left(\frac{x^2}{4} + \frac{x}{2}\right)\sin x.$$

4) Transform back:

$$y_p = e^x u_p = e^x\left[\frac{x}{4}\cos x + \left(\frac{x^2}{4} + \frac{x}{2}\right)\sin x\right].$$

Final solution:

$$y = e^x(C_1 \cos x + C_2 \sin x) + y_p.$$