MATB44 - Week 12 Tutorial Teaching Notes

Week 12 Topics

Linear Systems, Straight-Line Solutions, and Operator Elimination

- Converting a linear system into dy/dx form
- Determining straight-line trajectories $y = cx$
- Solving the quadratic equation for the slope c
- Using the discriminant to classify linear solutions
- Connection with eigenvalues and eigenvectors
- Operator elimination method: reducing a 2×2 first-order system to a single higher-order equation

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1. Linear Systems and Straight-Line Solutions

1.1 Starting Linear System

Consider a linear system:

$$\begin{cases} \dot{x} = a_1 x + b_1 y, \\ \dot{y} = a_2 x + b_2 y. \end{cases}$$

As long as $\dot{x} \neq 0$, we can write:

$$\frac{dy}{dx} = \frac{a_2 x + b_2 y}{a_1 x + b_1 y}.$$

We want to determine whether there exists a straight-line solution of the form:

$$y = cx.$$

1.2 Substituting $y = cx$

If a trajectory is a straight line through the origin, then:

$$\frac{dy}{dx} = c.$$

Substitute $y = cx$ into the slope formula:

$$c = \frac{a_2 + b_2 c}{a_1 + b_1 c}.$$

Multiply both sides:

$$c(a_1 + b_1 c) = a_2 + b_2 c.$$

This simplifies to a quadratic equation for c:

$$b_1 c^2 + (a_1 - b_2)c - a_2 = 0.$$

Roots c are slopes of straight-line solutions.

1.3 Discriminant and Classification

Let the discriminant be:

$$\Delta = (a_1 - b_2)^2 + 4a_2 b_1.$$

Equivalent expression using system determinant:

$$D = a_1 b_2 - a_2 b_1,$$

$$\Delta = (a_2 + b_1)^2 + 4D.$$

Interpretation:

Case	Δ	Straight-line solutions?	Interpretation
1	$\Delta > 0$	Two distinct straight lines	Two real eigenvectors
2	$\Delta = 0$	One straight line (repeated root)	Defective matrix, one eigenvector
3	$\Delta < 0$	No straight-line solution	Complex eigenvalues (spiral/center)

1.4 Geometric Meaning

- Straight lines $y = cx$ correspond to **eigenvector directions**, i.e., invariant lines.
- If $\Delta > 0$: two real eigenvectors $\to$ node or saddle.
- If $\Delta = 0$: only one direction $\to$ repeated eigenvalue, defective.
- If $\Delta < 0$: no invariant straight lines $\to$ spirals/centers.

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2. Operator Elimination Method

A central technique:
Convert a 2×2 first-order system into a single higher-order ODE by eliminating one variable.

Given:

$$\begin{cases} P_1(D)x + Q_1(D)y = f_1(t) \quad (e_1) \\ P_2(D)x + Q_2(D)y = f_2(t) \quad (e_2) \end{cases}$$

Goal: eliminate y.

Key idea

Find operators $A(D)$, $B(D)$ such that:

$$A(D)Q_1(D) + B(D)Q_2(D) = 0,$$

so that:

$$A(D)e_1 + B(D)e_2$$

eliminates y, leaving an equation in x only.

This parallells row operations in linear algebra — but using differential operators.

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3. Example Set (from tutorial)

Example 1 — Operator Elimination (from your notes)

$$\begin{cases} (D+4)x - (D-1)y = 3t \quad (e_1) \\ (D^2 - D)x + y = e^t \quad (e_2) \end{cases}$$

Use:

$$e_1 \to e_1 + (D-1)e_2.$$

Then y is eliminated and we obtain:

$$(D^3 - 2D^2 + 2D + 4)x = 3t.$$

Example 2 — Operator Elimination

$$\begin{cases} (D^2 + 2D)x + (D-5)y = 0 \quad (e_1) \\ (D+1)x + y = \sin t \quad (e_2) \end{cases}$$

Use:

$$e_1 \to e_1 - (D-5)e_2.$$

Eliminate y:

$$(6D + 5)x = -\cos t + 5 \sin t.$$

Example 3 — Operator Elimination

$$\begin{cases} (D-2)x + (2D+1)y = t^3 \\ (D+3)x - (D+1)y = e^{2t} \end{cases}$$

Use:

$$e_1 \to (D+1)e_1 + (2D+1)e_2.$$

Result:

$$(3D^2 + 6D + 1)x = t^3 + 3t^2 + 5e^{2t}.$$

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4. Example — Straight-Line Solutions

Consider:

$$\begin{cases} \dot{x} = 3x + 2y, \\ \dot{y} = -4x + y. \end{cases}$$

Solve:

$$c = \frac{-4 + c}{3 + 2c}.$$

We get:

$$2c^2 + 3c + 4 = 0, \quad \Delta < 0.$$

No straight-line solutions (complex eigenvalues).

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5. Example — Straight-Line Solutions (Two Lines)

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = x + 3y. \end{cases}$$

Equation:

$$c = \frac{1 + 3c}{2 + c} \Rightarrow c^2 - c - 1 = 0.$$

Two real roots:

$$c = \frac{1 \pm \sqrt{5}}{2}.$$

$\rightarrow$ **two invariant lines**.

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6. Example — Direction Ratio y(t)/x(t)

Suppose:

$$x(t) = e^t + 2te^t, \quad y(t) = 3e^t + 5te^t.$$

Then:

$$\frac{y(t)}{x(t)} = \frac{3 + 5t}{1 + 2t} \rightarrow \frac{5}{2}.$$

Interpretation:
As $t \to \infty$, solution approaches direction:

$$y = \frac{5}{2}x.$$

This is exactly the eigenvector direction corresponding to the dominant eigenvalue.

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7. Practice Problems

1. Determine whether the system
   $\dot{x} = 4x + y,\ \dot{y} = -2x + y$
   has any straight-line solutions $y = cx$.

2. For the system
   $\dot{x} = x + 4y,\ \dot{y} = 2x + y,$
   compute the two invariant lines.

3. Use operator elimination to eliminate y:

$$\begin{cases} (D+2)x + Dy = t, \\ (D-3)x - (D+1)y = e^t. \end{cases}$$

4. For

$$x(t) = 3e^{2t} + te^{2t}, \quad y(t) = 5e^{2t} + 4te^{2t},$$

   compute $\lim_{t\to\infty} y(t)/x(t)$.

5. Show that

$$c = \frac{a_2 + b_2 c}{a_1 + b_1 c}$$

   is equivalent to the eigenvector equation for
   $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$.
