MATB44 - Week 3 Tutorial Teaching Notes

Week 3 Topics

Linear Coefficient ODEs

Definition and Recognition

- General form: $(ax + by + c)dx + (dx + ey + f)dy = 0$
- When $\begin{vmatrix} a & b \\ d & e \end{vmatrix} \neq 0$
- Solution strategies based on coefficient relationships

Solution Methods

- **Case 1**: When $c = f = 0$ (homogeneous coefficients)
- **Case 2**: When $c \neq 0$ or $f \neq 0$ (non-homogeneous coefficients)
- Substitution techniques: $u = x - h$, $v = y - k$
- Finding critical points and transformations

Special Cases

- Exact equations
- Integrating factors
- Reduction to separable or homogeneous forms

Definition and Theory

What are Linear Coefficient ODEs?

A **linear coefficient ODE** has the form:

$$(ax + by + c)dx + (dx + ey + f)dy = 0$$

where a, b, c, d, e, f are constants and the coefficient matrix is non-singular:

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd \neq 0$$

Solution Strategy

The solution method depends on whether the constant terms are zero:

Case 1: Homogeneous Coefficients ($c = f = 0$) When $c = f = 0$, the equation becomes:

$$(ax + by)dx + (dx + ey)dy = 0$$

This is **homogeneous** and can be solved using the substitution $v = \frac{y}{x}$.

Case 2: Non-Homogeneous Coefficients ($c \neq 0$ or $f \neq 0$) When constant terms are present, we find the **critical point** (h, k) by solving:

$$\begin{cases} ah + bk + c = 0 \\ dh + ek + f = 0 \end{cases}$$

Then use the substitution: $u = x - h$, $v = y - k$

This transforms the equation to homogeneous form in (u, v).

Critical Point Method

Step 1: Check the coefficient matrix determinant:

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$$

- If $ae - bd \neq 0$: Unique critical point exists, proceed to Step 2
- If $ae - bd = 0$: No unique critical point, use different methods (homogeneous or special substitutions)

Step 2: Find critical point (h, k) from:

$$\begin{cases} ah + bk + c = 0 \\ dh + ek + f = 0 \end{cases}$$

Step 2: Substitute $u = x - h$, $v = y - k$ (so $du = dx$, $dv = dy$)

Step 3: The equation becomes:

$$(au + bv)du + (du + ev)dv = 0$$

Step 4: Solve this homogeneous equation using $w = \frac{v}{u}$

Step 5: Back-substitute to get solution in terms of x and y

Key Examples

Example 1: Non-Homogeneous Coefficients

Solve: $(2x + y - 1)dx + (x + 2y - 3)dy = 0$

Solution: **Step 1**: Make the substitution directly: Let $u = 2x - y + 1$ and $v = x + y$

From these equations: - $u = 2x - y + 1$ … (1) - $v = x + 2y - 3$ … (2)

Solving for x and y: From the system: - $u = 2x - y + 1$ … (1) - $v = x + 2y - 3$ … (2)

From (1): $y = 2x - u + 1$ Substitute into (2): $v = x + 2(2x - u + 1) - 3 = x + 4x - 2u + 2 - 3 = 5x - 2u - 1$ So: $x = \frac{v+2u+1}{5}$

And: $y = 2x - u + 1 = 2 \cdot \frac{v+2u+1}{5} - u + 1 = \frac{2v+4u+2}{5} - u + 1 = \frac{2v+4u+2-5u+5}{5} = \frac{2v-u+7}{5}$

Step 2: Find the differentials: $dx = \frac{2}{5}du + \frac{1}{5}dv$ and $dy = -\frac{1}{5}du + \frac{2}{5}dv$

Step 3: Substitute into the original equation: $(2x+y-1)dx+(x+2y-3)dy = 0$

Notice that: - $2x + y - 1 = u$ (by our substitution) - $x + 2y - 3 = v$ (by our substitution)

Step 4: The equation becomes: $u \cdot dx + v \cdot dy = 0$

Substituting the differentials: $u \cdot (\frac{2}{5}du + \frac{1}{5}dv) + v \cdot (-\frac{1}{5}du + \frac{2}{5}dv) = 0$

Expanding: $\frac{2u}{5}du + \frac{u}{5}dv - \frac{v}{5}du + \frac{2v}{5}dv = 0$

Collecting terms: $(\frac{2u}{5} - \frac{v}{5})du + (\frac{u}{5} + \frac{2v}{5})dv = 0$

Factoring out $\frac{1}{5}$: $(2u - v)du + (u + 2v)dv = 0$

Step 5: This is now a homogeneous equation in (u, v). We can solve using $w = \frac{v}{u}$:

Let $v = wu$, then $dv = wdu + udw$

$(2u - wu)du + (u + 2wu)(wdu + udw) = 0$

$u(2 - w)du + u(1 + 2w)(wdu + udw) = 0$

Dividing by u: $(2 - w)du + (1 + 2w)(wdu + udw) = 0$

$(2 - w + w + 2w^2)du + (1 + 2w)udw = 0$

$(2 + 2w^2)du + (1 + 2w)udw = 0$

Separating: $\frac{du}{u} = -\frac{(1+2w)dw}{2(1+w^2)}$

Final solution: After integration and back-substitution, we get the relationship between $u = 2x - y + 1$ and $v = x + 2y - 3$.

Example 2: Special Case - Determinant = 0

Solve: $(x + 2y - 3)dx + (2x + 4y + 1)dy = 0$

Solution: **Step 1**: Check the coefficient matrix determinant:

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 4 - 4 = 0$$

Since the determinant is 0, we cannot find a unique critical point. We need a different approach.

Step 2: Make substitution: Let $y = y$ and $u = x + 2y - 3$

From this: $x = u - 2y + 3$

Step 3: Find the differentials: $du = dx + 2dy$, so $dx = du - 2dy$

Step 4: Substitute into the original equation: $(x+2y-3)dx+(2x+4y+1)dy = 0$

Notice that: - $x+2y-3 = u$ - $2x+4y+1 = 2(x+2y)+1 = 2(u+3)+1 = 2u+7$

The equation becomes: $u \cdot dx + (2u + 7)dy = 0$

Substituting $dx = du - 2dy$: $u \cdot (du - 2dy) + (2u + 7)dy = 0$

$u\,du - 2u\,dy + (2u + 7)dy = 0$

$u\,du + (-2u + 2u + 7)dy = 0$

$u\,du + 7dy = 0$

Step 5: This is **separable**: $u\,du = -7dy$

Integrating both sides: $\int u\,du = -7 \int dy$

$\frac{u^2}{2} = -7y + C$

Final solution: $\frac{(x+2y-3)^2}{2} = -7y + C$, or $(x + 2y - 3)^2 + 14y = K$