

# MATB44 - Week 9 Tutorial Teaching Notes

## Week 9 Topics

### Non-Homogeneous Linear ODEs with Constant Coefficients

- Methods: Undetermined Coefficients, Variation of Parameters, (optional) Annihilators
- Resonance and multiplicity adjustments
- RHS types: polynomials, exponentials, sines/cosines, and their products

## Definition and Theory

### General Form and Structure

- Linear ODE (constant coefficients):

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x), \quad a_n \neq 0.$$

- Homogeneous part:

$$a_n \frac{d^n y}{dx^n} + \cdots + a_0 y = 0 \quad \Rightarrow \quad y_h \text{ from characteristic roots.}$$

- Particular solution  $y_p$  satisfies the full equation. The general solution is  $y = y_h + y_p$ .
- Superposition for RHS: if  $g(x) = g_1(x) + g_2(x)$ , then  $y_p = y_{p,1} + y_{p,2}$  where each solves its part.

### Method of Undetermined Coefficients

- Choose a trial form similar to the RHS:
  - $g(x) = P_m(x)$  (polynomial of degree  $m$ ): try  $y_p = Q_m(x)$  (degree  $m$ ).
  - $g(x) = e^{ax} P_m(x)$ : try  $y_p = e^{ax} Q_m(x)$ .
  - $g(x) = \cos(bx)$  or  $\sin(bx)$ : try  $y_p = A \cos(bx) + B \sin(bx)$ .
  - $g(x) = e^{ax} \cos(bx)$  or  $e^{ax} \sin(bx)$ : try  $y_p = e^{ax} (A \cos bx + B \sin bx)$ .
- Resonance rule: if the trial form overlaps  $y_h$ , multiply by  $x^s$ , where  $s$  = multiplicity of the overlapping root (or conjugate pair) in the characteristic equation.
- Determine coefficients by substitution.

### Variation of Parameters (2nd order, standard form)

For  $y'' + p(x)y' + q(x)y = g(x)$ , with a fundamental pair  $y_1, y_2$  and Wronskian  $W = y_1 y_2' - y_1' y_2$ , one particular solution is

$$y_p = -y_1 \int \frac{y_2 g}{W} dx + y_2 \int \frac{y_1 g}{W} dx.$$

Works for general RHS, even when undetermined coefficients is not applicable.

**Proof (sketch)** Take a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

with  $u_1, u_2$  differentiable. Impose the auxiliary condition

$$u'_1(x)y_1(x) + u'_2(x)y_2(x) = 0$$

to avoid second derivatives of  $u_1, u_2$ . Then

$$y'_p = u_1y'_1 + u_2y'_2, \quad y''_p = u'_1y'_1 + u'_2y'_2 + u_1y''_1 + u_2y''_2.$$

Substitute into  $y'' + py' + qy = g$  and use that  $y_j$  ( $j = 1, 2$ ) solve the homogeneous equation  $y''_j + py'_j + qy_j = 0$ . This yields

$$u'_1y'_1 + u'_2y'_2 = g.$$

Together with the auxiliary condition we have the linear system for  $u'_1, u'_2$ :

$$\begin{cases} u'_1y_1 + u'_2y_2 = 0, \\ u'_1y'_1 + u'_2y'_2 = g. \end{cases}$$

Let the Wronskian be  $W = y_1y'_2 - y'_1y_2 \neq 0$ . By Cramer's rule,

$$u'_1 = -\frac{y_2g}{W}, \quad u'_2 = \frac{y_1g}{W}.$$

Integrating gives

$$u_1 = -\int \frac{y_2g}{W} dx, \quad u_2 = \int \frac{y_1g}{W} dx,$$

and hence

$$y_p = -y_1 \int \frac{y_2g}{W} dx + y_2 \int \frac{y_1g}{W} dx.$$

### (Optional) Annihilator Method

- Idea: find a differential operator  $L$  that annihilates  $g(x)$  (i.e.,  $L[g] = 0$ ).
- Apply  $L$  to both sides to obtain a higher-order homogeneous ODE, solve it, and then select the part corresponding to  $y_p$ .

### Key Examples

#### Example 1: First-Order Equation

Solve:

$$y' + 2y = e^{-2x}$$

Solution: 1)  $y_h : y' + 2y = 0 \Rightarrow y_h = Ce^{-2x}$  2) Particular (details). Try  $y_p = Axe^{-2x}$  (resonance):

$$y'_p = Ae^{-2x} - 2Axe^{-2x}.$$

Substitute into LHS:

$$(y'_p + 2y_p) = (Ae^{-2x} - 2Axe^{-2x}) + 2(Axe^{-2x}) = Ae^{-2x}.$$

Match RHS  $e^{-2x}$   $A = 1$ . Hence  $y_p = xe^{-2x}$ . 3)  $y = Ce^{-2x} + xe^{-2x}$

### Example 2: First-Order with Exponential RHS (non-resonant)

Solve:

$$y' + 2y = e^{-x}$$

Solution: 1) Homogeneous:  $y' + 2y = 0 \Rightarrow y_h = Ce^{-2x}$ . 2) Particular (details). Try  $y_p = Ae^{-x}$  (non-resonant with  $e^{-2x}$ ):

$$y'_p = -Ae^{-x}.$$

LHS:

$$y'_p + 2y_p = (-A + 2A)e^{-x} = Ae^{-x}.$$

Match RHS  $e^{-x}$   $A = 1$ . Hence  $y_p = e^{-x}$ . 3)  $y = Ce^{-2x} + e^{-x}$

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### Example 3: Trigonometric RHS via Variation of Parameters

Solve:

$$y'' + y = \sin x$$

Solution: 1) Homogeneous:

$$y'' + y = 0 \Rightarrow y_h = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x, \quad W = y_1 y'_2 - y'_1 y_2 = 1$$

2) Particular via VOP:

$$y_p = -y_1 \int \frac{y_2 g}{W} dx + y_2 \int \frac{y_1 g}{W} dx$$

Substitute  $g = \sin x, W = 1$ :

$$y_p = -\cos x \int \sin^2 x dx + \sin x \int \sin x \cos x dx$$

3) Integrate and simplify:

$$\begin{aligned} \int \sin^2 x dx &= \frac{x}{2} - \frac{\sin(2x)}{4}, \quad \int \sin x \cos x dx = \frac{\sin^2 x}{2} \\ \Rightarrow y_p &= -\frac{x}{2} \cos x \quad (\text{terms in } \sin x \text{ can be absorbed by } y_h) \end{aligned}$$

4) Final solution:

$$y = C_1 \cos x + C_2 \sin x - \frac{1}{2}x \cos x$$


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