MATB44 - Week 2 Tutorial Teaching Notes

Week 2 Topics

Separable Ordinary Differential Equations

- Definition and identification
- Solution method
- Initial value problems
- Key examples

Definition and Method

What is a Separable ODE?

A **separable ODE** has the form:

$$\frac{dy}{dx} = f(x)g(y)$$

Homogeneous First Order ODEs

A **homogeneous first order ODE** has the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

where F is a function of the ratio $\frac{y}{x}$ only.

Solution Method for Homogeneous ODEs: 1. **Substitution**: Let $v = \frac{y}{x}$, so $y = vx$ 2. **Differentiate**: $\frac{dy}{dx} = v + x\frac{dv}{dx}$ 3. **Substitute**: $v + x\frac{dv}{dx} = F(v)$ 4. **Separate**: $\frac{dv}{F(v)-v} = \frac{dx}{x}$ 5. **Integrate and solve**

Solution Method for Separable ODEs: 1. **Separate**: $\frac{dy}{g(y)} = f(x)dx$ 2. **Integrate**: $\int \frac{dy}{g(y)} = \int f(x)dx$ 3. **Solve for** y

Summary of ODE Types (i made huge mistake in the last tutorial because that there is difference between the first oder ode and linear ode)

Type	General Form	Standard Form
Autonomous	$y' = F(x, y)$	$F(x, y) = F(y)$
Separable	$F(x, y) = P(x) \cdot Q(y)$	$P(x, y) = P(x)$; $Q(x, y) = Q(y)$
Homogeneous	$F = F(x, y)$: homogeneous of degree 0	$P(x, y), Q(x, y)$: homogeneous of degree n
Linear Coefficient		$P(x, y) = ax + by + c$; $Q(x, y) = a'x + b'y + c'$ where $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \neq 0$

Key Examples

Example 1: Separable ODE with Rational Functions

Solve:

$$\begin{cases} y' = \frac{2x^2+3x+1}{y^2+2y+2}, \\ y(0) = 0 \end{cases}$$

Solution: This is separable: $(y^2 + 2y + 2)dy = (2x^2 + 3x + 1)dx$

Integrating both sides:

$$\int (y^2 + 2y + 2)dy = \int (2x^2 + 3x + 1)dx$$

$$\frac{y^3}{3} + y^2 + 2y = \frac{2x^3}{3} + \frac{3x^2}{2} + x + C$$

Using initial condition $y(0) = 0$: $C = 0$

Final solution: $\frac{y^3}{3} + y^2 + 2y = \frac{2x^3}{3} + \frac{3x^2}{2} + x$

Example 2: Separable ODE with Trigonometric Functions

Solve:

$$\begin{cases} y' = \frac{xy^3}{\sqrt{1+x^2}}, \\ y(0) = -2 \end{cases}$$

Solution: This is separable: $\frac{dy}{y^3} = \frac{xdx}{\sqrt{1+x^2}}$

Integrating both sides:

$$\int y^{-3}dy = \int \frac{xdx}{\sqrt{1+x^2}}$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + C$$

Using initial condition $y(0) = -2$: $-\frac{1}{8} = 1 + C \Rightarrow C = -\frac{9}{8}$

Final solution: $-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{9}{8}$

Solving for y: $y^2 = \frac{-4}{\sqrt{1+x^2}-\frac{9}{8}}$

Domain: Solution exists where $\sqrt{1+x^2} < \frac{9}{8}$, which gives $|x| < \frac{5}{8}$

Example 3: Homogeneous ODE Method

Solve: $(x+4y)dx - (x+y)dy = 0$

Method for Homogeneous ODEs: 1. **Check if homogeneous**: Verify that $P(x,y)$ and $Q(x,y)$ are homogeneous of the same degree 2. **Use substitution**: Let $v = \frac{y}{x}$, so $y = vx$ 3. **Find derivative**: $dy = xdv + vdx$ 4. **Substitute and simplify**: Replace y and dy in the original equation 5. **Separate variables**: Get equation in terms of v and x only 6. **Integrate**: Solve the separated equation 7. **Back-substitute**: Replace v with $\frac{y}{x}$ to get final solution

Solution: Here $P(x,y) = x+4y$ and $Q(x,y) = x+y$ are both homogeneous of degree 1.

Let $v = \frac{y}{x}$, so $y = vx$ and $dy = xdv + vdx$

Substituting: $(x+4vx)dx - (x+vx)(xdv+vdx) = 0$

Simplifying: $(x+4vx)dx - (x+vx)xdv - (x+vx)vdx = 0$

Factor out x: $x[(1+4v)dx - (1+v)xdv - (1+v)vdx] = 0$

Rearranging: $x[(1+4v-v-v^2)dx - (1+v)xdv] = 0$

Simplifying: $(1+3v-v^2)dx = (1+v)xdv$

Separating: $\frac{dx}{x} = \frac{(1+v)dv}{1+3v-v^2}$

Final form: $\ln|x| = \int \frac{(1+v)dv}{1+3v-v^2} + C$

Example 4: Similar Homogeneous ODE

Solve: $(2x+3y)dx - (x+2y)dy = 0$

Solution: $P(x,y) = 2x+3y$ and $Q(x,y) = x+2y$ are homogeneous of degree 1.

Let $v = \frac{y}{x}$, so $y = vx$ and $dy = xdv + vdx$

Substituting: $(2x+3vx)dx - (x+2vx)(xdv+vdx) = 0$

Expanding: $(2x + 3vx)dx - (x + 2vx)xdv - (x + 2vx)vdx = 0$

Factor out x: $x[(2 + 3v)dx - (1 + 2v)xdv - (1 + 2v)vdx] = 0$

Simplifying: $(2 + 3v - v - 2v^2)dx = (1 + 2v)xdv$

Rearranging: $(2 + 2v - 2v^2)dx = (1 + 2v)xdv$

Separating: $\frac{dx}{x} = \frac{(1+2v)dv}{2+2v-2v^2} = \frac{(1+2v)dv}{2(1+v-v^2)}$

Integrating: $\ln|x| = \frac{1}{2}\int \frac{(1+2v)dv}{1+v-v^2} + C$

Final solution: $\ln|x| = \frac{1}{2}\ln|1 + v - v^2| + C$, where $v = \frac{y}{x}$