

# MATB44 - Week 1 tut1 Teaching Notes

## Ta Information

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## Week 1 Topics

### 1. Introduction to Ordinary Differential Equations (ODEs)

- Basic concepts of differential equations
- Definition and classification of differential equations
- Historical context and applications

### 2. Types of Differential Equations

- Ordinary vs Partial Differential Equations
- Linear vs Nonlinear differential equations
- Classification by order (first-order, second-order, etc.)
- Autonomous vs Non-autonomous equations
- Homogeneous vs Non-homogeneous equations

## Detailed Content

### 1. Introduction to Ordinary Differential Equations (ODEs)

**What is a Differential Equation?** A **differential equation** is an equation that relates a function with its derivatives. It describes how a quantity changes over time or space.

**General form:**  $F(x, y, y', y'', \dots, y^{(n)}) = 0$

**Example:**  $\frac{dy}{dx} = 2x + 3$

### Why Study Differential Equations?

- **Modeling:** Describe natural phenomena (population growth, heat transfer, motion)
- **Prediction:** Forecast future behavior
- **Control:** Design systems to achieve desired outcomes

## 2. Classification of Differential Equations

### A. Ordinary vs Partial Differential Equations    Ordinary Differential Equation (ODE):

- Contains only ordinary derivatives
- One independent variable
- Example:  $\frac{dy}{dx} = x^2 + y$

### Partial Differential Equation (PDE):

- Contains partial derivatives
- Multiple independent variables
- Example:  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$  (heat equation)

### B. Linear vs Nonlinear    Linear ODE:

- Can be written as:  $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$
- Example:  $y'' + 3y' + 2y = e^x$

### Nonlinear ODE:

- Contains products or powers of  $y$  and its derivatives
- Example:  $y'' + (y')^2 + y^3 = 0$

### C. Classification by Order    Order: The highest derivative present in the equation

- **First-order:**  $\frac{dy}{dx} = f(x, y)$
- **Second-order:**  $\frac{d^2y}{dx^2} = f(x, y, y')$
- **Higher-order:** Contains derivatives of order 3 or higher

### D. Autonomous vs Non-autonomous    Autonomous: Right-hand side doesn't explicitly depend on independent variable

- Example:  $\frac{dy}{dt} = y(1 - y)$

### Non-autonomous: Right-hand side explicitly depends on independent variable

- Example:  $\frac{dy}{dt} = ty + \sin(t)$

**E. Homogeneous vs Non-homogeneous Homogeneous:** For linear ODEs, when the right-hand side is zero

- General form:  $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$
- Example:  $y'' + 3y' + 2y = 0$

**Non-homogeneous:** For linear ODEs, when the right-hand side is non-zero

- General form:  $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$  where  $g(x) \neq 0$
- Example:  $y'' + 3y' + 2y = e^x$

**Note:** The concept of homogeneous/non-homogeneous applies only to linear differential equations.

## Classification Practice Problems

**Exercise:** For each differential equation, determine:

**Order:** The highest derivative present

**Linear:** Yes/No

**Autonomous:** Yes/No

**Homogeneous:** Yes/No (only applies to linear equations)

1.  $\frac{dy}{dx} = 3x^2 + 2y$
2.  $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$
3.  $y'' + (y')^2 + y^3 = x$
4.  $\frac{dy}{dt} = y^2 - 4y$
5.  $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$
6.  $x^2y'' + xy' + y = 0$
7.  $\frac{dy}{dx} = \sqrt{x+y}$
8.  $y''' + 2y'' - y' + y = \sin(x)$
9.  $\frac{d^2y}{dx^2} = y^2$
10.  $\frac{dy}{dt} = ty + \sin(t)$

**Solutions:**

1. **Order:** 1, **Linear:** Yes, **Autonomous:** No, **Homogeneous:** No
2. **Order:** 2, **Linear:** Yes, **Autonomous:** Yes, **Homogeneous:** Yes
3. **Order:** 2, **Linear:** No, **Autonomous:** No, **Homogeneous:** N/A
4. **Order:** 1, **Linear:** No, **Autonomous:** Yes, **Homogeneous:** N/A

5. **Order:** 3, **Linear:** Yes, **Autonomous:** No, **Homogeneous:** No
6. **Order:** 2, **Linear:** Yes, **Autonomous:** No, **Homogeneous:** Yes
7. **Order:** 1, **Linear:** No, **Autonomous:** No, **Homogeneous:** N/A
8. **Order:** 3, **Linear:** Yes, **Autonomous:** No, **Homogeneous:** No
9. **Order:** 2, **Linear:** No, **Autonomous:** Yes, **Homogeneous:** N/A
10. **Order:** 1, **Linear:** Yes, **Autonomous:** No, **Homogeneous:** No

### Advanced Exercise: Autonomous Equations Property

**Problem:** Let  $x^{(k)} = f(x, x^{(1)}, \dots, x^{(k-1)})$  be an autonomous equation (or system).

Show that if  $\phi(t)$  is a solution, so is  $\phi(t - t_0)$ .

**Solution:**

Since the equation is autonomous, the right-hand side  $f$  does not explicitly depend on the independent variable  $t$ .

Given that  $\phi(t)$  is a solution, we have:

$$\phi^{(k)}(t) = f(\phi(t), \phi^{(1)}(t), \dots, \phi^{(k-1)}(t))$$

Now consider  $\psi(t) = \phi(t - t_0)$ . We need to show that  $\psi(t)$  is also a solution.

Taking derivatives of  $\psi(t)$ : -  $\psi^{(1)}(t) = \phi^{(1)}(t - t_0)$  -  $\psi^{(2)}(t) = \phi^{(2)}(t - t_0)$  - ...  
 -  $\psi^{(k)}(t) = \phi^{(k)}(t - t_0)$

Substituting into the original equation:

$$\psi^{(k)}(t) = \phi^{(k)}(t - t_0) = f(\phi(t - t_0), \phi^{(1)}(t - t_0), \dots, \phi^{(k-1)}(t - t_0))$$

Since  $\phi(t - t_0) = \psi(t)$ ,  $\phi^{(1)}(t - t_0) = \psi^{(1)}(t)$ , etc., we have:

$$\psi^{(k)}(t) = f(\psi(t), \psi^{(1)}(t), \dots, \psi^{(k-1)}(t))$$

Therefore,  $\psi(t) = \phi(t - t_0)$  is indeed a solution.

**Interpretation:** This property means that autonomous equations are time-invariant - if we shift a solution in time, we get another valid solution. This is a fundamental characteristic of autonomous systems.