MATB44 - Week 4 Tutorial Teaching Notes

Week 4 Topics

Exact Differential Equations

Definition and Recognition

- General form: $M(x,y)dx + N(x,y)dy = 0$
- Exactness condition: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- Solution method: finding potential function $\phi(x,y)$
- Integration techniques and path independence

Method of Finding Integrating Factors

- When equations are not exact
- Types of integrating factors: $\mu(x)$, $\mu(y)$, $\mu(xy)$, $\mu(x^2 + y^2)$
- Systematic approach to finding appropriate integrating factors
- Reduction to exact form and solution

Definition and Theory

What are Exact Differential Equations?

An **exact differential equation** has the form: $$M(x,y)dx + N(x,y)dy = 0$$

where there exists a function $\phi(x,y)$ such that: $$\frac{\partial \phi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x,y)$$

Exactness Condition

The equation $M(x,y)dx + N(x,y)dy = 0$ is **exact** if and only if: $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution Method for Exact Equations

1. **Verify exactness**: Check that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. **Find potential function**: Integrate M with respect to x or N with respect to y
3. **Determine arbitrary function**: Use the other partial derivative condition
4. **Write general solution**: $\phi(x,y) = C$

Integrating Factors

When an equation is **not exact**, we can sometimes find an **integrating factor** $\mu(x,y)$ such that: $$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact.

Methods for Finding Integrating Factors

Method 1: $\mu = \mu(x)$ (depends only on x)

- Condition: $\frac{P_y - Q_x}{Q} = F(x)$
- Formula: $I(x) = e^{\int F(x)dx}$

Method 2: $\mu = \mu(y)$ (depends only on y)

- Condition: $\frac{P_y - Q_x}{P} = F(y)$
- Formula: $I(y) = e^{-\int F(y)dy}$

Method 3: $\mu = \mu(x/y)$ (depends on ratio x/y)

- Condition: $y^2 \cdot \frac{P_y - Q_x}{xP + yQ} = F(x/y)$
- Formula: $I(x/y) = e^{\int F(u)du}$ where $u = x/y$

Method 4: $\mu = \mu(xy)$ (depends on product xy)

- Condition: $\frac{P_y - Q_x}{yQ - xP} = F(xy)$
- Formula: $I(xy) = I(u) = e^{\int F(u)du}$ where $u = xy$

Method 5: Monomial Integrating Factor

- For equations with monomial terms $x^a y^b$
- General form: $P(x,y)dx + Q(x,y)dy = 0$
- Where $P = (x^{a_1} y^{a_2} + x^{b_1} y^{b_2})$ and $Q = (x^{c_0} y^{c_2} + x^{d_0} y^{d_2})$
- Integrating factor: $I = x^a y^b$ where a, b are to be determined

Key Examples

example 1 exact

Solve: $(2xy + 3x^2)dx + (x^2 + 2y)dy = 0$

Solution: **Step 1**: Check exactness

- $P(x,y) = 2xy + 3x^2$, $Q(x,y) = x^2 + 2y$
- $P_y = 2x$, $Q_x = 2x$

Since $P_y = Q_x = 2x$, the equation is **exact**.

Step 2: Find potential function From $\frac{\partial \phi}{\partial x} = P = 2xy + 3x^2$: $$\phi(x,y) = \int (2xy + 3x^2)dx = x^2y + x^3 + h(y)$$

Step 3: Determine $h(y)$ From $\frac{\partial \phi}{\partial y} = Q = x^2 + 2y$: $$\frac{\partial}{\partial y}(x^2y + x^3 + h(y)) = x^2 + h'(y) = x^2 + 2y$$

Therefore: $h'(y) = 2y$, so $h(y) = y^2 + C$

Final solution: $x^2y + x^3 + y^2 = C$

example 2 applying method 1

Solve: $((x^2+1)y + x),dx + x(x^2+1),dy = 0$

Solution: **Step 1**: Check exactness

- $P = (x^2+1)y + x$, $Q = x(x^2+1)$
- $P_y = x^2 + 1$, $Q_x = 3x^2 + 1$

Since $P_y \ne Q_x$, the equation is not exact.

Step 2: Try $\mu = \mu(x)$ Compute $$\frac{P_y - Q_x}{Q} = \frac{(x^2+1) - (3x^2+1)}{x(x^2+1)} = \frac{-2x^2}{x(x^2+1)} = -\frac{2x}{x^2+1} = F(x)$$ This depends only on x, so Method 1 applies.

Find the integrating factor: $$\mu(x) = e^{\int F(x),dx} = e^{\int -\frac{2x}{x^2+1}dx} = e^{-\ln(x^2+1)} = \frac{1}{x^2+1}$$

Step 3: Multiply by $\mu(x)$ and solve as exact Multiply the equation by $\mu(x) = \frac{1}{x^2+1}$: $$\Big(y + \frac{x}{x^2+1}\Big)dx + x,dy = 0$$ Now $\tilde P = y + \frac{x}{x^2+1}$, $\tilde Q = x$ with $\tilde P_y = 1$, $\tilde Q_x = 1$ so it is exact.

Integrate $\tilde P$ with respect to x: $$\phi(x,y) = \int \Big(y + \frac{x}{x^2+1}\Big)dx = xy + \tfrac{1}{2}\ln(x^2+1) + h(y)$$ Differentiate with respect to y and match $\tilde Q$: $$\phi_y = x + h'(y) = \tilde Q = x ;\Rightarrow; h'(y)=0 \Rightarrow h(y)=C$$

Final solution: $$xy + \tfrac{1}{2}\ln(x^2+1) = C$$

example 3 applying method 2

Solve: $((y^2+1)),dx + (xy),dy = 0$ with $P(y)=y^2+1$, $Q(x,y)=xy$

Solution: **Step 1**: Check condition for $\mu(y)$

- $P_y = 2y$, $Q_x = y$
- Compute $\dfrac{P_y - Q_x}{P} = \dfrac{2y - y}{y^2 + 1} = \dfrac{y}{y^2 + 1} = F(y)$, which depends only on y.

Step 2: Find integrating factor $\mu(y)$ (Method 2) $$\mu(y) = e^{-\int F(y),dy} = e^{-\int \frac{y}{y^2+1}dy} = e^{-\tfrac{1}{2}\ln(y^2+1)} = \frac{1}{\sqrt{y^2+1}}$$

Step 3: Multiply and check exactness After multiplying by $\mu(y)$:

- $\tilde P = \mu P = \dfrac{y^2+1}{\sqrt{y^2+1}} = \sqrt{y^2+1}$
- $\tilde Q = \mu Q = \dfrac{xy}{\sqrt{y^2+1}}$ Then $\tilde P_y = \dfrac{y}{\sqrt{y^2+1}}$ and $\tilde Q_x = \dfrac{y}{\sqrt{y^2+1}}$, so the equation is exact.

Step 4: Solve the exact equation Integrate $\tilde P$ with respect to x: $$\phi(x,y) = \int \sqrt{y^2+1},dx = x\sqrt{y^2+1} + h(y)$$ Differentiate w.r.t. y and match $\tilde Q$: $$\phi_y = x\frac{y}{\sqrt{y^2+1}} + h'(y) = \tilde Q = \frac{xy}{\sqrt{y^2+1}} \Rightarrow h'(y)=0$$ Thus $h(y)=C$.

Final solution: $$x\sqrt{y^2+1} = C$$

example 4 applying method 3

Solve: $\displaystyle \frac{x^2}{y},dx + x,dy = 0$ with $P(x,y)=\frac{x^2}{y}$, $Q(x,y)=x$

Solution (Method 3: $\mu=\mu(x/y)$): **Step 1**: Verify the condition depends only on $u=x/y$

- $P_y = -\dfrac{x^2}{y^2}$, $Q_x = 1$ (non-exact)
- Compute $$y^2,\frac{P_y - Q_x}{xP + yQ} = y^2,\frac{-\dfrac{x^2}{y^2} - 1}{x\cdot \dfrac{x^2}{y} + y\cdot x} = \frac{-(x^2+y^2)}{x^3/ y + xy} = -\frac{x^2+y^2}

{x^2(x/y)+y^2(x/y)} = -\frac{u^2+1}{u(u^2+1)} = -\frac{1}{u} = F(u)$$ This depends only on $u=x/y$, so Method 3 applies.

Step 2: Find integrating factor $\mu(u)$ $$\mu(u) = e^{\int F(u),du} = e^{\int -\frac{1}{u} du} = e^{-\ln|u|} = \frac{1}{|u|}$$ We take $\mu(x,y) = \dfrac{1}{x/y} = \dfrac{y}{x}$ (up to a constant factor).

Step 3: Multiply and check exactness Multiply by $\mu=\dfrac{y}{x}$:

- $\tilde P = \dfrac{y}{x}\cdot \dfrac{x^2}{y} = x$
- $\tilde Q = \dfrac{y}{x}\cdot x = y$ Then $\tilde P_y = 0$, $\tilde Q_x = 0$, so it is exact.

Step 4: Solve Integrate $\tilde P$ with respect to x: $$\phi(x,y) = \int x,dx = \tfrac{1}{2}x^2 + h(y)$$ Differentiate and match $\tilde Q$: $$\phi_y = h'(y) = y \Rightarrow h(y) = \tfrac{1}{2}y^2 + C$$

Final solution: $$\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = C $$

example 5 applying method 4

Solve: $x\big(1+(xy)^2\big),dx + y\big(1+(xy)^2\big),dy = 0$ with $P=x(1+(xy)^2)$, $Q=y(1+(xy)^2)$

Solution (Method 4: $\mu=\mu(xy)$): **Step 1**: Verify the condition depends only on $u=xy$

- $P_y = \partial_y\big[x(1+u^2)\big] = x\cdot 2u,u_y = x\cdot 2u\cdot x = 2x^2u = 2x^3y$
- $Q_x = \partial_x\big[y(1+u^2)\big] = y\cdot 2u,u_x = y\cdot 2u\cdot y = 2y^2u = 2xy^3$
- Compute $$\frac{P_y - Q_x}{yQ - xP} = \frac{2xy(x^2 - y^2)}{,(y^2 - x^2)(1+u^2),} = -\frac{2u}{1+u^2} = F(u)$$ This depends only on $u=xy$, so Method 4 applies.

Step 2: Find integrating factor $\mu(u)$ $$\mu(u) = e^{\int F(u),du} = e^{\int -\frac{2u}{1+u^2}du} = e^{-\ln(1+u^2)} = \frac{1}{1+u^2}$$

Step 3: Multiply and solve as exact Multiply by $\mu=\dfrac{1}{1+(xy)^2}$:

- $\tilde P = \dfrac{x(1+(xy)^2)}{1+(xy)^2} = x$
- $\tilde Q = \dfrac{y(1+(xy)^2)}{1+(xy)^2} = y$ Then the equation becomes $x,dx + y,dy = 0$, which is exact with potential $$\phi(x,y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = C$$

Final solution: $$\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 = C$$

example 6 applying method 5

Solve: $(y^2 + 3xy),dx + (x^2 + xy),dy = 0$ with $P=y^2+3xy$, $Q=x^2+xy$.

Solution (Method 5: Monomial Integrating Factor):

Step 1: Check exactness $$P_y = 2y + 3x, \qquad Q_x = 2x + y ;(\neq)$$

Step 2: Assume a monomial integrating factor $$\mu(x,y) = x^a y^b$$ and require exactness of $\mu P,dx + \mu Q,dy=0$.

Step 3: Expand μP and μQ $$x^a y^b P = x^a y^{b+2} + 3x^{a+1}y^{b+1}, \qquad x^a y^b Q = x^{a+2}y^{b} + x^{a+1}y^{b+1}.$$

Step 4: Differentiate $$\partial_y(x^a y^b P) = (b+2)x^a y^{b+1} + 3(b+1)x^{a+1}y^{b},$$ $$\partial_x(x^a y^b Q) = (a+2)x^{a+1}y^{b} + (a+1)x^{a}y^{b+1}.$$

Step 5: Match coefficients Match like monomials $x^a y^{b+1}$ and $x^{a+1}y^{b}$: $$b+2 = a+1, \qquad 3(b+1) = a+2 ;\Rightarrow; a=1,; b=0.$$

Step 6: Integrating factor $$\mu(x,y) = x.$$

Step 7: Multiply and check exactness After multiplying by x: $$\tilde P = x y^2 + 3x^2 y, \qquad \tilde Q = x^3 + x^2 y,$$ $$\tilde P_y = 2xy + 3x^2, \qquad \tilde Q_x = 3x^2 + 2xy ;\Rightarrow; \text{exact}.$$

Step 8: Potential function From $\phi_x = \tilde P$: $$\phi(x,y) = \int (x y^2 + 3x^2 y),dx = \tfrac{1}{2}x^2 y^2 + x^3 y + g(y).$$

Step 9: Determine $g(y)$ $$\phi_y = x^2 y + x^3 + g'(y) = \tilde Q = x^3 + x^2 y ;\Rightarrow; g'(y)=0.$$

Step 10: Implicit solution $$\tfrac{1}{2}x^2 y^2 + x^3 y = C.$$