University of Toronto Scarborough

Department of Mathematics

MATB44 Differential Equations I

October 2025 Midterm Examination

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IC130, 3PM-5PM

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Instructions:

- Do not turn this page and start the exam until you are told to do so.
- Do not make use of personal notes, computers or books. **No aids are allowed.**
- Read the problems carefully and answer only what is being asked.
- **Answer all five** problems only on the designated space for each question.
- You may write your draft work on pages 15-16. However, **the pages 15-16 will not be graded**.
- Please write your solutions clearly and legibly.
- Maximum points 100.
- **Do not rip any pages from this booklet.**

Good Luck!

Good Luck!

Problem 1. (20 points)

Characterize each of the following statements as True or False. Explain your answer.

A. If the characteristic equation of a second-order linear homogeneous ODE with constant coefficients has a repeated real root r, the general solution is of the form $y(x) = c_1 e^{rx} + c_2 e^{rx}$.

Answer: **False.** The two terms $c_1 e^{rx}$ and $c_2 e^{rx}$ are not linearly independent. For a repeated root r, the two linearly independent solutions are e^{rx} and xe^{rx}. The correct general solution is $y(x) = c_1 e^{rx} + c_2 x e^{rx}$.

B. The differential equation $y'' + \sin(x) y' + y = e^x$ is a linear ODE.

Answer: **True.** The equation is linear because the dependent variable y and its derivatives (y' and y'') appear only to the first power, and their coefficients (1, $\sin(x)$, and 1) are functions of the independent variable x only. The equation fits the general form of a linear ODE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$.

C. There are two solutions to the differential equation $y' - y = -y^2$ which are linearly independent.

Answer: **True.** This is a Bernoulli equation. The general solution is

$$y(x) = \frac{1}{1 + Ce^{-x}}.$$

Take now $C = 0$, and $C = 1$ and this gives two functions which are linearly independent.

Good Luck!

Problem 1 cont'd.

D. The differential equation

$$x \cos\left(\frac{y}{x}\right)(y\,dx + x\,dy) = y \sin\left(\frac{y}{x}\right)(x\,dy - y\,dx)$$

is homogeneous.

Answer: **True.** We can rearrange the equation into the form $M(x,y)dx + N(x,y)dy = 0$.

$$\left[xy\cos\left(\frac{y}{x}\right) + y^2 \sin\left(\frac{y}{x}\right)\right] dx + \left[x^2 \cos\left(\frac{y}{x}\right) - xy \sin\left(\frac{y}{x}\right)\right] dy = 0$$

A differential equation is homogeneous if $M(tx,ty) = t^k M(x,y)$ and $N(tx,ty) = t^k N(x,y)$ for some constant k. Here, both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree 2, so the equation is homogeneous.

E. The differential equation

$$(y')^2 + 2y' = -1$$

has infinitely many solutions.

Answer: **True.** The equation can be rewritten as $(y')^2 + 2y' + 1 = 0$, which factors into $(y'+1)^2 = 0$. This implies that $y' + 1 = 0$, so $\frac{dy}{dx} = -1$. Integrating with respect to x gives the general solution $y(x) = -x + C$, where C is an arbitrary constant. Since C can be any real number, there are infinitely many solutions.

Good Luck!

Problem 2. (20 points)

Solve the ordinary differential equation (ODE):

$$y^{(4)} + 2y'' + y = 0.$$

with the following initial conditions at $x = 0$:

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \\ y''(0) &= -1 \\ y'''(0) &= 0 \end{aligned}$$

Solution of Problem 2.

This is a fourth-order linear homogeneous differential equation with constant coefficients. The characteristic equation is:

$$r^4 + 2r^2 + 1 = 0$$

This can be factored as:

$$(r^2 + 1)^2 = 0$$

The roots of $r^2 + 1 = 0$ are $r = \pm i$. Since the factor is squared, these roots each have a multiplicity of 2. The roots are $r_1 = i$ (multiplicity 2) and $r_2 = -i$ (multiplicity 2).

For repeated complex roots $\pm\beta i$ with multiplicity k, the linearly independent solutions are $\cos(\beta x), x\cos(\beta x), \ldots, x^{k-1}\cos(\beta x)$ and $\sin(\beta x), x\sin(\beta x), \ldots, x^{k-1}\sin(\beta x)$. Here, $\beta = 1$ and $k = 2$. The four linearly independent solutions are $\cos(x)$, $x\cos(x)$, $\sin(x)$, and $x\sin(x)$. The general solution is:

$$y(x) = c_1\cos(x) + c_2 x\cos(x) + c_3\sin(x) + c_4 x\sin(x)$$

Now we apply the initial conditions by finding the derivatives of $y(x)$:

$$y'(x) = -c_1\sin(x) + c_2(\cos(x) - x\sin(x)) + c_3\cos(x) + c_4(\sin(x) + x\cos(x))$$

$$y''(x) = -c_1\cos(x) + c_2(-2\sin(x) - x\cos(x)) - c_3\sin(x) + c_4(2\cos(x) - x\sin(x))$$

$$y'''(x) = c_1\sin(x) + c_2(-3\cos(x) + x\sin(x)) - c_3\cos(x) + c_4(-3\sin(x) - x\cos(x))$$

Evaluate at $x = 0$:

- $y(0) = 1 \implies c_1\cos(0) + 0 + c_3\sin(0) + 0 = 1 \implies \mathbf{c_1 = 1}$.
- $y'(0) = 0 \implies -c_1\sin(0) + c_2(\cos(0) - 0) + c_3\cos(0) + c_4(0 + 0) = 0 \implies c_2 + c_3 = 0$.

- $y''(0) = -1 \implies -c_1\cos(0) + c_2(0-0) - c_3\sin(0) + c_4(2\cos(0) - 0) = -1 \implies -c_1 + 2c_4 = -1$.
- $y'''(0) = 0 \implies c_1\sin(0) + c_2(-3\cos(0)+0) - c_3\cos(0) + c_4(0-0) = 0 \implies -3c_2 - c_3 = 0$.

We have a system of linear equations for the coefficients:

1. $c_1 = 1$
2. $c_2 + c_3 = 0$
3. $-c_1 + 2c_4 = -1$
4. $-3c_2 - c_3 = 0$

From (1), we know $c_1 = 1$. Substitute $c_1 = 1$ into (3): $-1 + 2c_4 = -1 \implies 2c_4 = 0 \implies \mathbf{c_4 = 0}$. From (2), $c_3 = -c_2$. Substitute this into (4): $-3c_2 - (-c_2) = 0 \implies -2c_2 = 0 \implies \mathbf{c_2 = 0}$. Since $c_3 = -c_2$, we have $\mathbf{c_3 = 0}$.

The coefficients are $c_1 = 1, c_2 = 0, c_3 = 0, c_4 = 0$. Substituting these values back into the general solution gives the particular solution:

$$y(x) = (1)\cos(x) + (0)x\cos(x) + (0)\sin(x) + (0)x\sin(x)$$

$$\mathbf{y(x)} = \cos(\mathbf{x})$$

Good Luck!

Solution of Problem 2 cont'd.

Good Luck!

Solution of Problem 2 cont'd.

Good Luck!

Problem 3. (20 points)

Solve the following ODE:

$$(x^2 + y^2)dx - xydy = 0.$$

Solution of Problem 3.

We have the equation $M(x, y)dx + N(x, y)dy = 0$, where $M(x, y) = x^2 + y^2$ and $N(x, y) = -xy$. Let's check if the equation is homogeneous.

$$M(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 M(x, y)$$

$$N(tx, ty) = -(tx)(ty) = -t^2 xy = t^2 N(x, y)$$

Since both M and N are homogeneous functions of the same degree (2), the differential equation is homogeneous.

We use the substitution $y = vx$. This implies $dy = vdx + xdv$. Substitute y and dy into the ODE:

$$(x^2 + (vx)^2)dx - x(vx)(vdx + xdv) = 0$$

Factor out x^2:

$$x^2(1 + v^2)dx - vx^2(vdx + xdv) = 0$$

$$x^2(1 + v^2)dx - v^2x^2dx - vx^3dv = 0$$

Assuming $x \neq 0$, we can divide by x^2:

$$(1 + v^2)dx - v^2dx - vxdv = 0$$

$$(1 + v^2 - v^2)dx - vxdv = 0$$

$$dx - vxdv = 0$$

This is a separable equation. We can rearrange it to separate the variables x and v:

$$dx = vxdv$$

Divide by x (assuming $x \neq 0$):

$$\frac{1}{x}dx = vdv$$

Now, integrate both sides:

$$\int \frac{1}{x}dx = \int vdv$$

$$\ln|x| = \frac{v^2}{2} + C_1$$

where C_1 is the constant of integration. Now, substitute back $v = \frac{y}{x}$:

$$\ln|x| = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C_1$$

$$\ln|x| = \frac{y^2}{2x^2} + C_1$$

We can rearrange this to express the solution implicitly:

$$y^2 = 2x^2(\ln|x| - C_1)$$

Let $C = -C_1$ be an arbitrary constant. The general solution is:

$$\mathbf{y^2 = 2x^2(\ln|x| + C)}$$

Good Luck!

Solution of Problem 3 cont'd.

Good Luck!

Solution of Problem 3 cont'd.

Good Luck!

Problem 4. (20 points)

Find all functions $y = y(x)$ for which:

$$(2x + 3y)dx + (6x + 9y)dy = 0$$

and $y(3) = -2$.

Solution of Problem 4.

The given differential equation is:

$$(2x + 3y)dx + (6x + 9y)dy = 0$$

We can factor the coefficient of dy:

$$(2x + 3y)dx + 3(2x + 3y)dy = 0$$

Now, factor out the common term $(2x + 3y)$:

$$(2x + 3y)(dx + 3dy) = 0$$

This equation is satisfied if either of the factors is zero. This leads to two possibilities for the solution.

Case 1: $2x + 3y = 0$ If $2x + 3y = 0$, then $y = -\frac{2}{3}x$. This is an algebraic relation. Let's verify if it is a solution to the ODE. If $y = -\frac{2}{3}x$, then $dy = -\frac{2}{3}dx$. Substituting y into the term $2x + 3y$ gives $2x + 3(-\frac{2}{3}x) = 2x - 2x = 0$. The ODE becomes $0 \cdot dx + (6x + 9y)dy = 0$, which is $0 = 0$. So, $y = -\frac{2}{3}x$ is a solution. Now we check the initial condition $y(3) = -2$:

$$-2 = -\frac{2}{3}(3) \implies -2 = -2$$

The initial condition is satisfied. Thus, $\mathbf{y(x) = -\frac{2}{3}x}$ is a solution to the initial value problem.

Case 2: $dx + 3dy = 0$ If $dx + 3dy = 0$, we can integrate this equation:

$$\int dx + \int 3dy = C_1$$

$$x + 3y = C_1$$

This is a family of solutions. To find the specific solution for the given initial condition, we substitute $x = 3$ and $y = -2$:

$$3 + 3(-2) = C_1$$

$$3 - 6 = C_1 \implies C_1 = -3$$

So, the particular solution from this case is $x + 3y = -3$. We can write this as an explicit function $y(x)$:

$$3y = -x - 3 \implies \mathbf{y(x) = -\frac{1}{3}x - 1}$$

The problem asks for all functions. Both functions satisfy the initial value problem. The reason for the non-uniqueness is that the equation can be written as $\frac{dy}{dx} = -\frac{2x+3y}{6x+9y} = -\frac{2x+3y}{3(2x+3y)}$. The existence and uniqueness theorem does not apply at the initial point $(3, -2)$ because the expression for $\frac{dy}{dx}$ is undefined, as $2x + 3y = 2(3) + 3(-2) = 0$ at this point.

The two solutions are $y(x) = -\frac{2}{3}x$ and $y(x) = -\frac{1}{3}x - 1$.

Good Luck!

Solution of Problem 4 cont'd.

Good Luck!

Problem 5. (20 points)

Solve the following ODE:

$$(3xy + y^2)dx + (x^2 + xy)dy = 0.$$

Solution of Problem 5.

We have the equation in the form $M(x,y)dx + N(x,y)dy = 0$, with $M(x,y) = 3xy + y^2$ and $N(x,y) = x^2 + xy$. First, let's check for exactness by computing the partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3xy + y^2) = 3x + 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy) = 2x + y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

We can look for an integrating factor μ. Let's check if there is an integrating factor that depends only on x. The test expression is $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$:

$$\frac{1}{x^2 + xy}\left((3x + 2y) - (2x + y)\right) = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

Since this expression depends only on x, there is an integrating factor $\mu(x)$. The integrating factor is given by:

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$$

We can choose $\mu(x) = x$ (the sign does not affect the final solution). Multiply the original ODE by $\mu(x) = x$:

$$x(3xy + y^2)dx + x(x^2 + xy)dy = 0$$

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

This new equation should be exact. Let's verify. Let $M^*(x,y) = 3x^2y + xy^2$ and $N^*(x,y) = x^3 + x^2y$.

$$\frac{\partial M^*}{\partial y} = \frac{\partial}{\partial y}(3x^2y + xy^2) = 3x^2 + 2xy$$

$$\frac{\partial N^*}{\partial x} = \frac{\partial}{\partial x}(x^3 + x^2y) = 3x^2 + 2xy$$

Indeed, $\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$, so the new equation is exact.

Now we find a potential function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$. Integrate M^* with respect to x:

$$F(x,y) = \int (3x^2y + xy^2)dx = y\int 3x^2dx + y^2\int xdx = y(x^3) + y^2\left(\frac{x^2}{2}\right) + g(y)$$

$$F(x, y) = x^3 y + \frac{1}{2}x^2 y^2 + g(y)$$

To find $g(y)$, we differentiate F with respect to y and set it equal to N^*:

$$\frac{\partial F}{\partial y} = x^3 + x^2 y + g'(y)$$

Setting this equal to N^*:

$$x^3 + x^2 y + g'(y) = x^3 + x^2 y$$

$$g'(y) = 0$$

This implies that $g(y)$ is a constant. We can choose $g(y) = 0$. So the potential function is $F(x, y) = x^3 y + \frac{1}{2}x^2 y^2$. The general solution to the ODE is given by $F(x, y) = C$, where C is an arbitrary constant.

$$x^3 y + \frac{1}{2}x^2 y^2 = C$$

To eliminate the fraction, we can multiply by 2 and let $C_1 = 2C$:

$$2x^3 y + x^2 y^2 = C_1$$

We can also write this solution by factoring out common terms:

$$\mathbf{x^2 y(2x + y) = C_1}$$

Good Luck!

Solution of Problem 5 cont'd.

Good Luck!

Solution of Problem 5 cont'd.

Good Luck!

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