The goal of this part is to classify representations of $\mathbf{SU}(3)$.

Note that

- The finite-dimensional representations of $\mathbf{SU}(3)$ are in 1-1 correspondance with the finite-dimensional representations of $\mathbf{su}(3)$,since $\mathbf{SU}(3)$ is simply connected, by the theorem about Lie group and Lie Algebra Homomorphism.

- A representation of $\mathbf{SU}(3)$ is irreducible iff the associated representation of $\mathbf{su}(3)$ is irreducible ,since $\mathbf{SU}(3)$ is connected, by the theorem about irreducible representations.

- The complex representations of $\mathbf{su}(3)$ are in 1-1 correspondance with the complex-linear representation of $\mathbf{su}_{\mathbb{C}}(3) = \mathbf{sl}(3;\mathbb{C})$. Also, a complex representation of $\mathbf{su}(3)$ is irreducible iff it is irreducible as a representation of $\mathbf{su}_{\mathbb{C}}$. These are by the theorem about complexification of Lie algebra.

Proposition 0.1. *There is a one to one correspondence between the finite-dimensional complex representation* Π *of* $\mathbf{SU}(3)$ *and the finite-dimensional complex-linear representation* π *of* $\mathbf{sl}(3;\mathbb{C})$.

Also, since $\mathbf{SU}(3)$ is compact, so $\mathbf{SU}(3)$is completely reducible by the theorem about complete reducibility. Thus, it is enough to consider only (irreducible representations of) $\mathbf{sl}(3;\mathbb{C})$, which has a nicer basis, instead of $\mathbf{SU}(3)$.

We will use the following basis for $\mathbf{sl}(3;\mathbb{C})$;

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

0.1 Definitions

Definition 1. *Let* $\pi : \mathbf{sl}(3;\mathbb{C}) \to \mathbf{GL}(V)$ *be a representation.*

- *Weight vector : a simultaneous eigenvector of* $\pi(H_1)$ *and* $\pi(H_2)$*,*
- *Weight : an ordered pair of two corresponding eigenvalues,*
- *Weight space : the space of all weight vectors corresponding to a weight,*
- *Multiplicity of a weight : the dimension of the corresponding weight space.*

Every representation of $\mathbf{sl}(3;\mathbb{C})$ has at least one weight and elements of a weight of it are always integers.

Definition 2.

- *Root vector : a simultaneous eigenvector of ad_{H_1} and ad_{H_2},*
- *Root : an ordered pair of two corresponding eigenvalues.*

There are six roots for $\mathbf{sl}(3;\mathbb{C})$:

roots	linear combination of the positive simple roots	root vectors							
$(2,-1)$	α_1	X_1		$[H_1,X_1]$	$=$	$2X_1$	$[H_2,X_1]$	$=$	$-X_1$
$(-1,2)$	α_2	X_2		$[H_1,X_2]$	$=$	$-X_2$	$[H_2,X_2]$	$=$	$2X_2$
$(1,1)$	$\alpha_1+\alpha_2$	X_3	since	$[H_1,X_3]$	$=$	X_3	$[H_2,X_2]$	$=$	X_3
$(-2,1)$	$-\alpha_1$	Y_1		$[H_1,Y_1]$	$=$	$-2Y_1$	$[H_2,Y_1]$	$=$	Y_1
$(1,-2)$	$-\alpha_2$	Y_2		$[H_1,Y_2]$	$=$	Y_2	$[H_2,Y_2]$	$=$	-2 Y_2
$(-1,-1)$	$-\alpha_1-\alpha_2$	Y_3		$[H_1,Y_3]$	$=$	$-Y_3$	$[H_2,Y_3]$	$=$	$-Y_3$

and the first two one α_1,α_2 are called the positive simple roots.

Definition 3. *A weight μ_1 is higher than μ_2 (or μ_2 is lower than μ_1), $\mu_1 \succeq \mu_2$, if $\mu_1-\mu_2$ is a conical combination of the positive simple roots. A weight μ_0 is a highest weight if there is no weight higher than μ_0.*

Definition 4. *A representation $\pi : \mathbf{sl}(3;\mathbb{C}) \to \mathbf{GL}(V)$ is a highest weight cyclic representation with weight μ_0 if there exists a cyclic vector v in V. A cyclic vector v is a weight vector of μ_0 in V that satisfies*

1. *$\pi(X_1)v$ and $\pi(X_2)v$ vanish.*
2. *V itself is the smallest invariant subspace of V containing v.*

0.2 The Theorem of the Highest Weight

The theorem below can be informally stated as

"Irreducible representations of $\mathbf{sl}(3;\mathbb{C})$ can be classified by their higest weight, which is a pair of two non-negative integers."

Theorem 0.2. *Let $\pi : \mathbf{sl}(3;\mathbb{C}) \to \mathbf{GL}(V)$ be an irreducible representation*

(1) V is the direct sum of its weight spaces; $\pi(H_1)$ and $\pi(H_2)$ are simultaneously diagonalizable.

(2-1) π has a unique highest weight.

(2-2) Equivalent irreducible representations of $\mathbf{sl}(3;\mathbb{C})$ have the same highest weight.

(2-3) Irreducible representations of $\mathbf{sl}(3;\mathbb{C})$ with the same highest weight are equivalent.

(3-1) The highest weight of π is a pair of two non-negative integers.

(3-2) For a given $\mu_0 = (m_1, m_2)$ where m_1, m_2 are non-negative integers, there exists an irreducible representation of $\mathbf{sl}(3;\mathbb{C})$ with having μ_0 as the highest weight.

Proof. (1) Let W be the direct sum of the weight spaces in V, the linear space spanned by simultaneous eigenvectors of $\pi(H_1)$ and $\pi(H_2)$.

- $W \neq \{0\}$.
  Claim π has at least one weight. Let U be the eigenspace of $\pi(H_1)$ and $v \in U$. Since $[H_1, H_2] = 0$, $\pi(H_1)(\pi(H_2)v) = \pi(H_2)(\pi(H_1)v) = m_1(\pi(H_2)v)$ so $\pi(H_2)(U) \subseteq U$ and $\pi(H_2)$ has at least one eigenvector w and eigenvalue in U.
- W is invariant under $\mathbf{sl}(3;\mathbb{C})$.
  Let $\mu = (m_1, m_2)$ and v be a weight and the corresponding weight vector in W and let $\alpha = (a_1, a_2)$ and Z_α be a root and the corresponding root vector. Then from $[H_1, Z_\alpha] = a_1 Z_\alpha$, $\pi(H_1)\pi(Z_\alpha)v = (\pi(Z_\alpha)\pi(H_1) + a_1\pi(Z_\alpha))v = (m_1 + a_1)\pi(Z_\alpha)v$, and similarly $\mu + \alpha$ is also weight. Therefore, $\pi(Z_\alpha)(W) \subseteq W$; that is, W is an invariant subspace of V under all bases of $\mathbf{sl}(3;\mathbb{C})$, consequently under $\mathbf{sl}(3;\mathbb{C})$.
- Since π is irreducible, $W = V$.

(2) We will show (2-1) $\sim$ (2-3) by using the fact that a highest weight cyclic representation of $\mathbf{sl}(3;\mathbb{C})$is an irreducible representation and vice versa.

- Every irreducible representation of $\mathbf{sl}(3;\mathbb{C})$is a highest weight cyclic representation.
  Since the representation is finite-dimensional, there are finitely many weight; thus, $\exists \mu_0$ with the weight vector v such that there is no weight higher than μ_0. Then, $\pi(X_1)v = \pi(X_2)v = 0$. (Otherwise, $\pi(H_i)\pi(X_j)v = (\mu_0 + \alpha_j)_i \pi(X_j)v$, which implies $\pi(X_j)v \succeq v$.)
- A highest weight cyclic representation of $\mathbf{sl}(3;\mathbb{C})$has a unique highest weight.
  (Existence) Claim a weight mu_0 corresponding to the cyclic vector v is the highest weight. The set $W = \{w | w = \pi(Y_{i_1}) \ldots \pi(Y_{i_n})v, i_1, \ldots, i_n = 1 or 2 or 3, n \geqslant 1\}$ is invariant and $v \in W$, so $W = V$. Since $\pi(H_1)w = (\mu_0 - n_1\alpha_1 - n_2\alpha)w$, for $n_1, n_2 \geqslant 0$, μ_0 is the highest weight.
  (Uniqueness) By definition of the highest weight.

- Every highest weight cyclic representation of $\mathbf{sl}(3;\mathbb{C})$ is irreducible.

  According to (1) and complete irreducibility, $V = \bigoplus_i V_i$ where each of V_i's is the direct sum of its weight spaces. The cyclic vector must be (up to constant) the only vector contained in a weight space in some V_i. Therefore, V_i is invariant so $V_i = V$; V is irreducible.

- Irreducible representations of $\mathbf{sl}(3;\mathbb{C})$ with the same highest weight are equivalent.

  From previous steps, π is irreducible $\Leftrightarrow$ π is highest weight cyclic.

$\square$