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Shadows of a Closed Curve and Spheres

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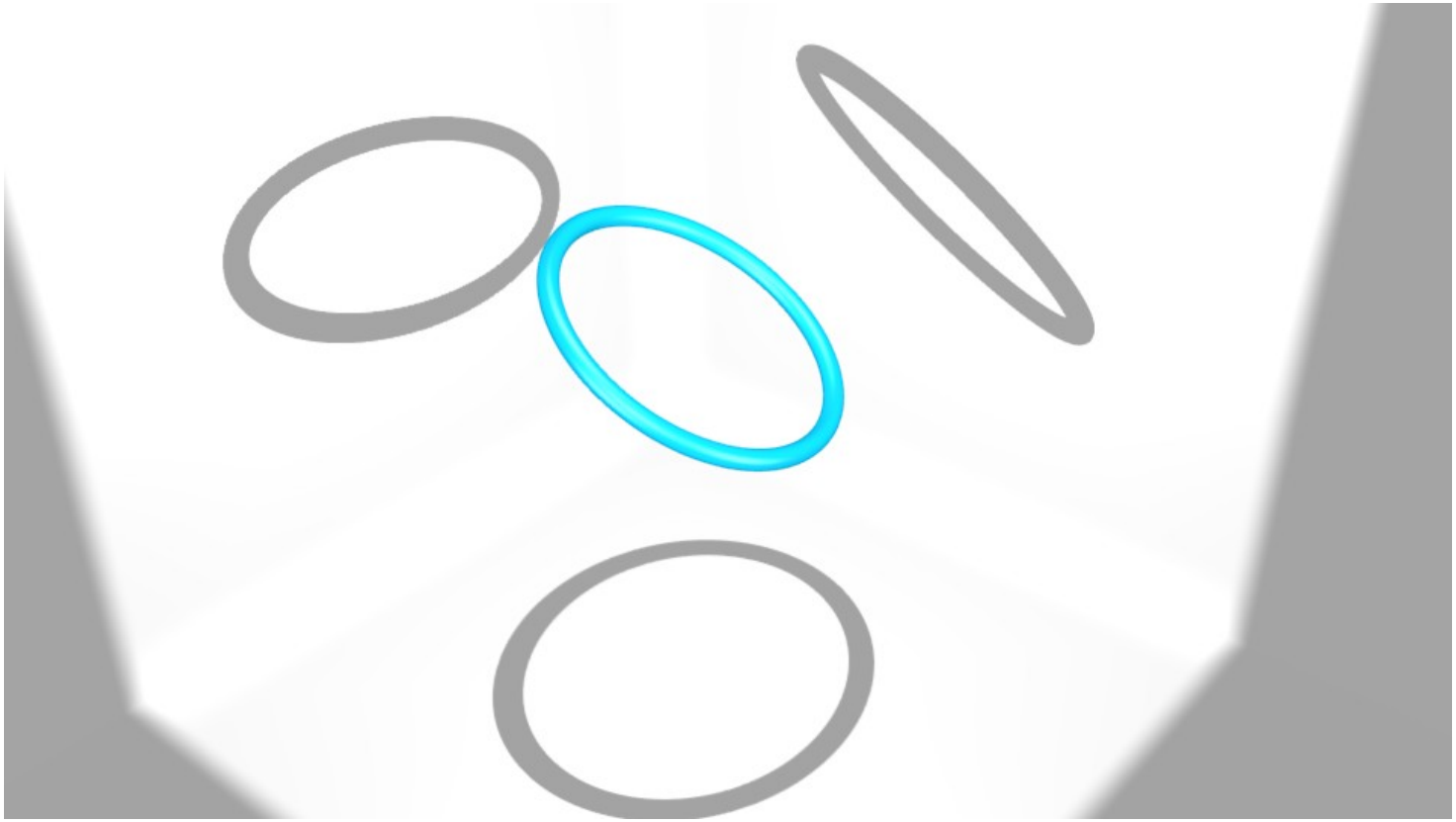
Joint work with P. K. Bose, J. D. Carufel, M. G. Dobbins, L. Montejano
E. Roldan-Pensado, G. Viglietta

Shadow Puzzles



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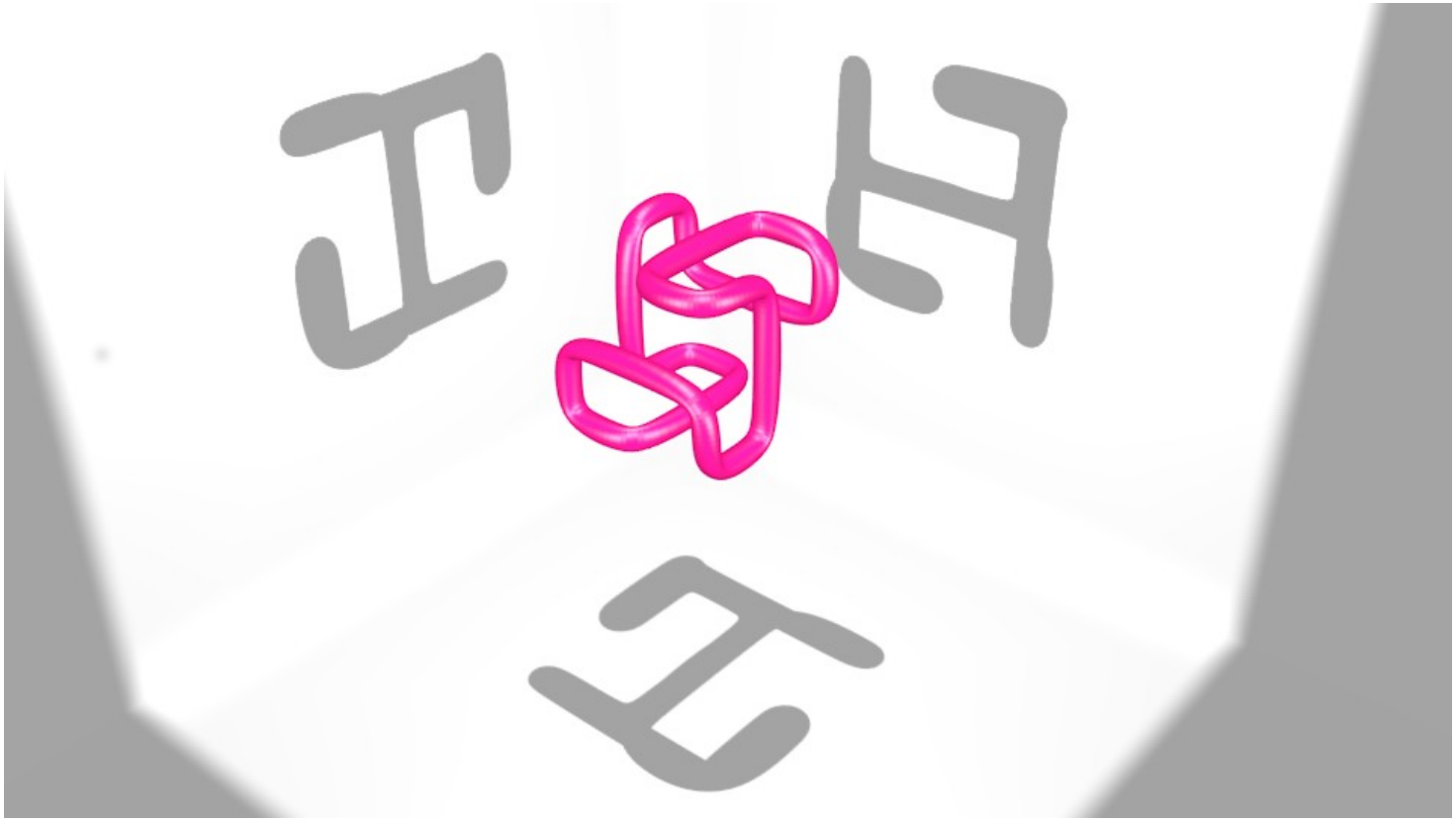
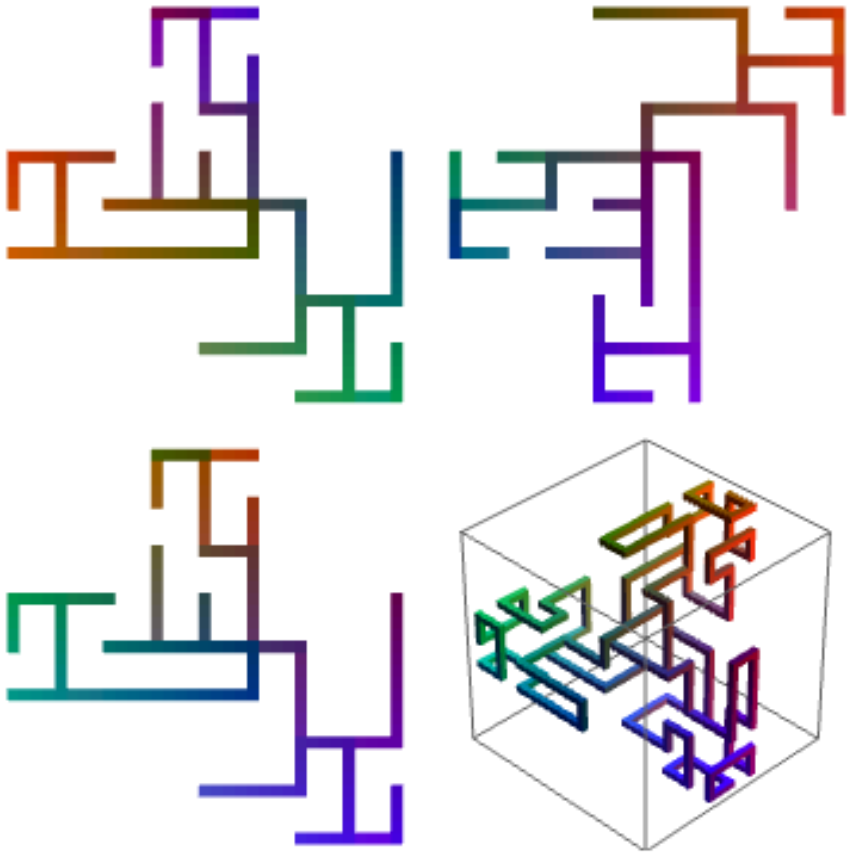


Rickard's Curve (1995)

Treefoil Knots for Trefoil Knots



Constructed and drawn by Adam P. Goucher (2012)

Generalizations

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Objects	Ambient Space	Shadows	Exist?
d-Spheres, $d > 0$	$(d+2)$-Space	Contractible	YES
Simple Closed Curves	3-Space	Paths	NO
Simple Closed Polygonal Curves	d-Space, $d \geq 3$	More than two of them are paths	NO
2-Spheres	4-Space	Disks	NO if there exists a transverse level set of one component
Polygonal Paths	3-Space	Non-degenerating Closed Curves	YES
d-Spheres, $d > 0$	$(d+2)$-Space	d-Balls	**?**

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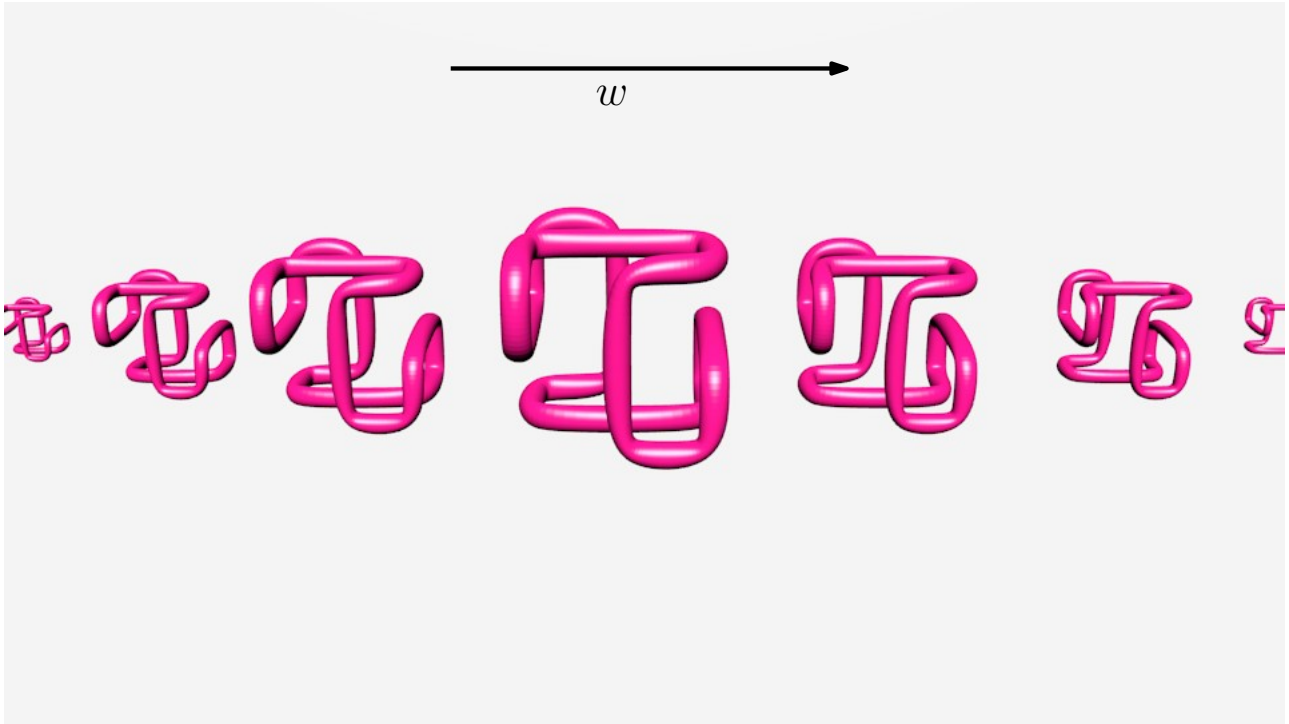
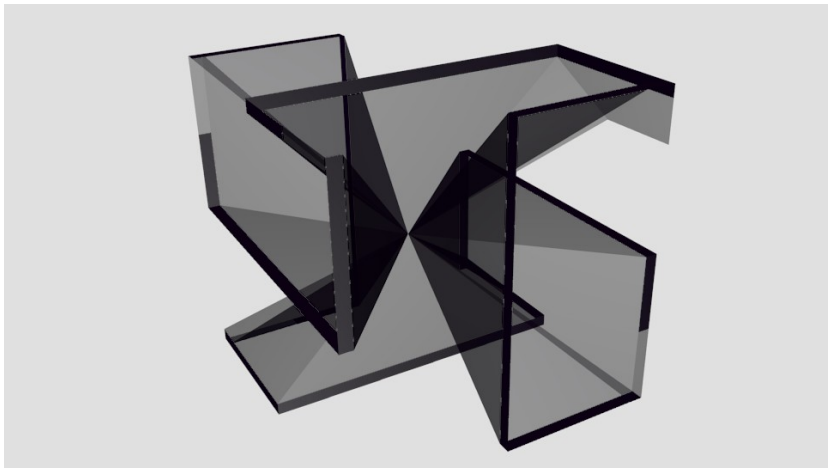
A Contractible Construction



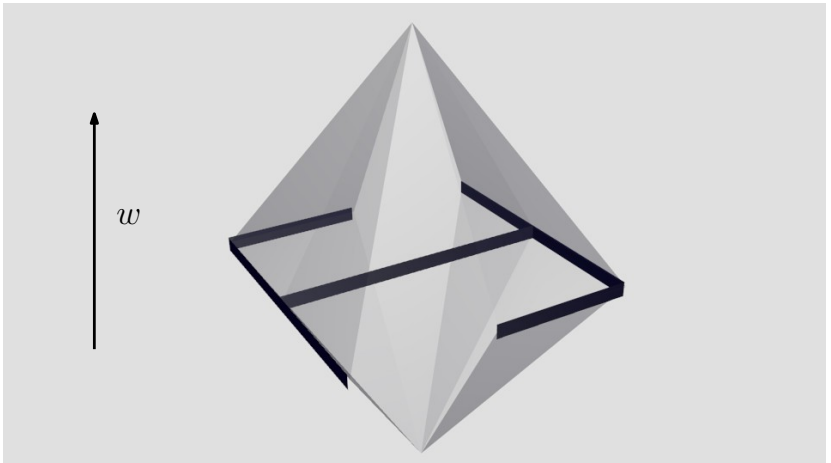
Inductive Construction by Suspension

$$S_{d+1} : \bigcup_{\lambda \in [-1,1]} (1 - |\lambda|) \cdot S_d \times \{\lambda\}$$

A Contractible Construction



in the $xyz-$plane



in other planes

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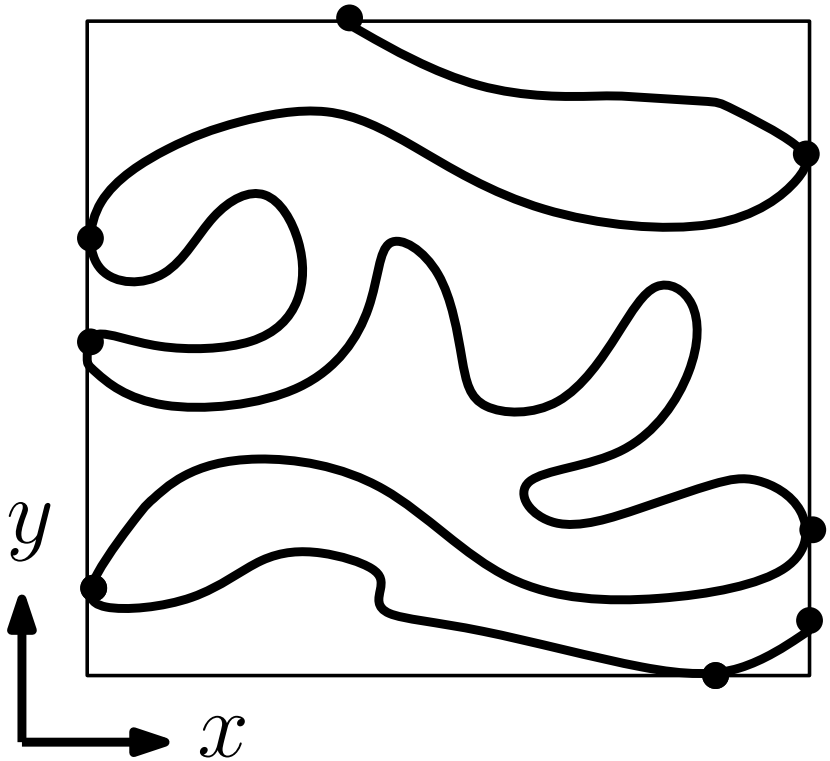
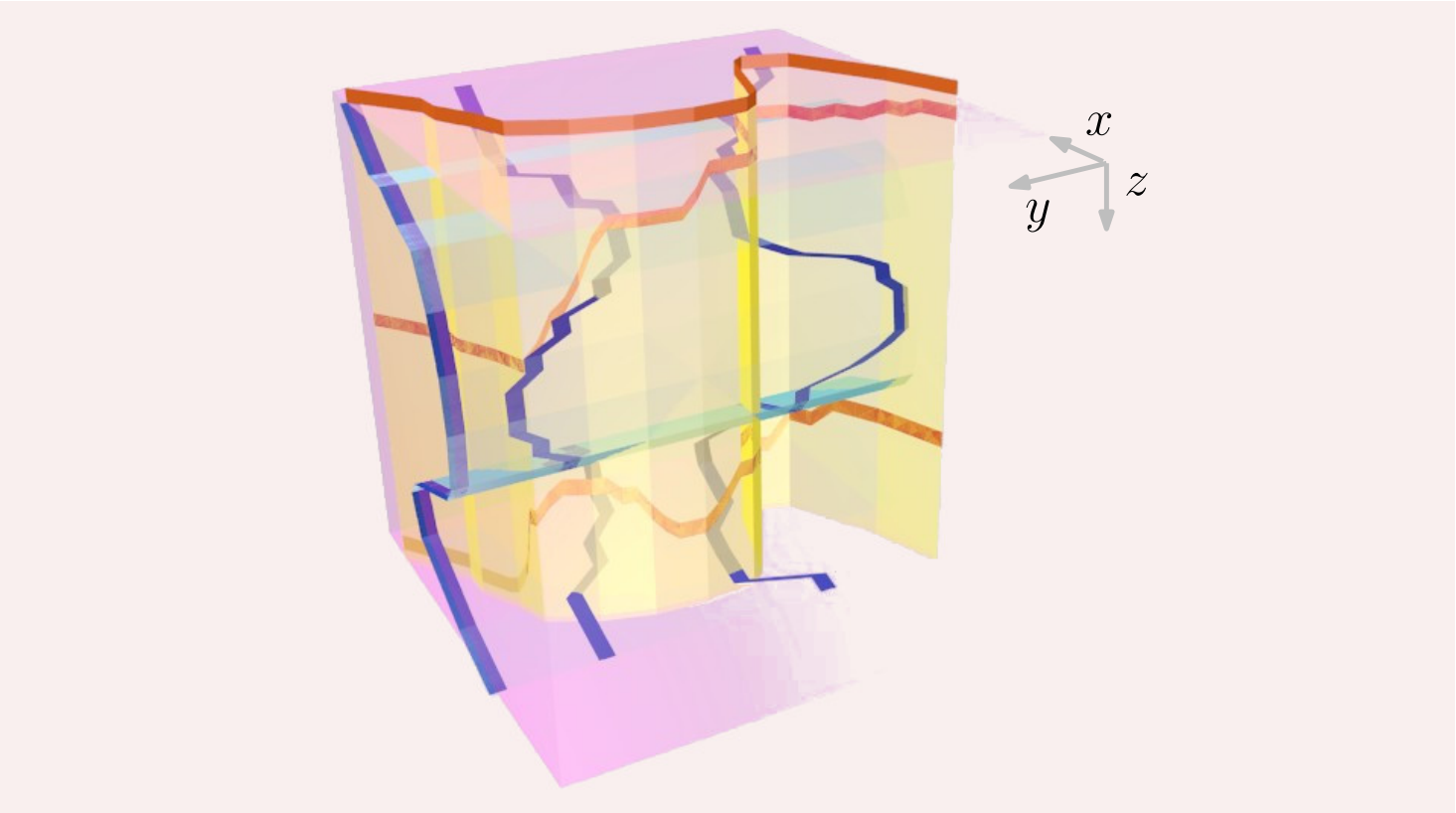
A Proof of Nonexistence: 3-Space



There exists a direction of the unique extreme path

A Proof of Nonexistence: 3-Space



The shadow in the yz-plane cannot be a path.

Generalizations

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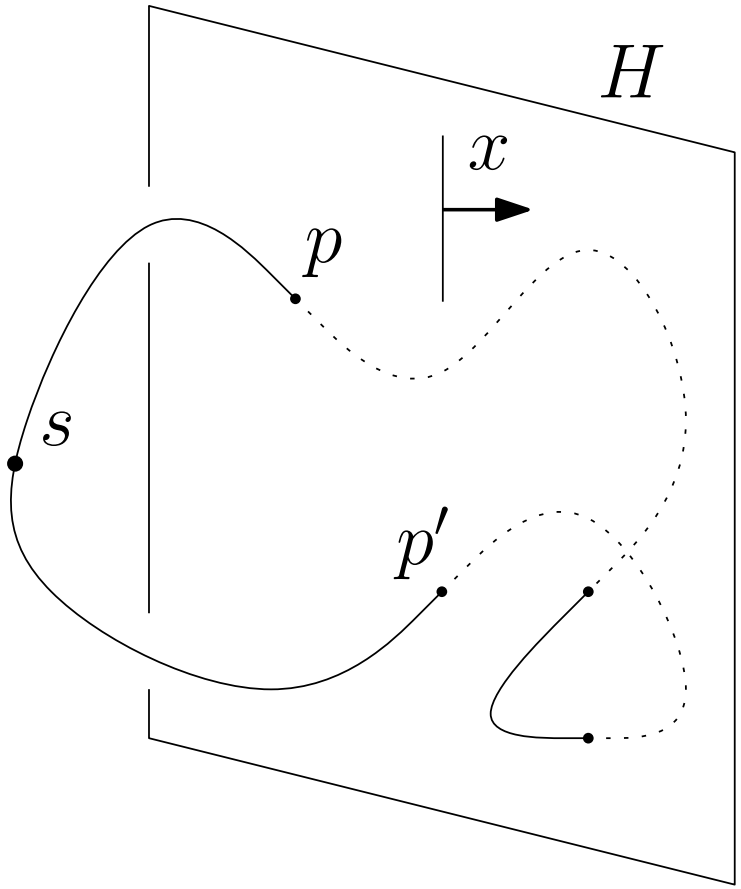


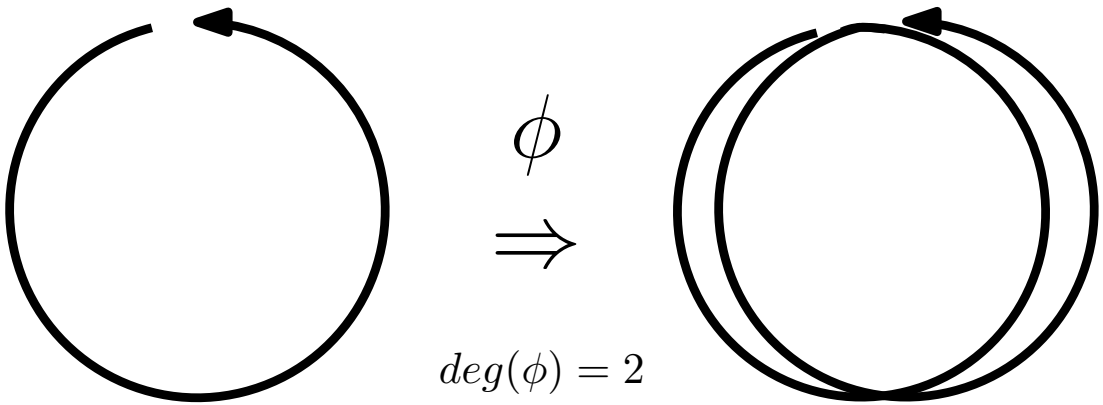
No common endpoints

(No points projected to endpoints in all x, y, z directions)

Degree of a function $\phi : \mathbb{S}^1 \to \mathbb{S}^1$

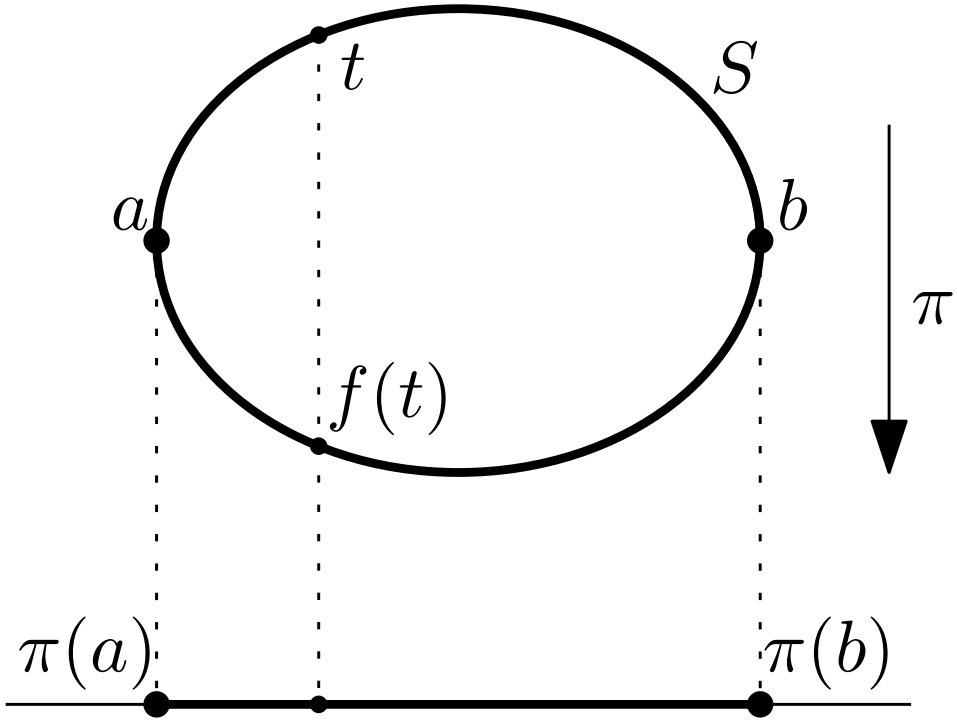
- Winding number

  e.g.)



- If $deg(\phi) \neq 1$, there is a fixed point $s \in \mathbb{S}^1$; $\phi(s) = s$.

A Proof of Nonexistence: d-Space

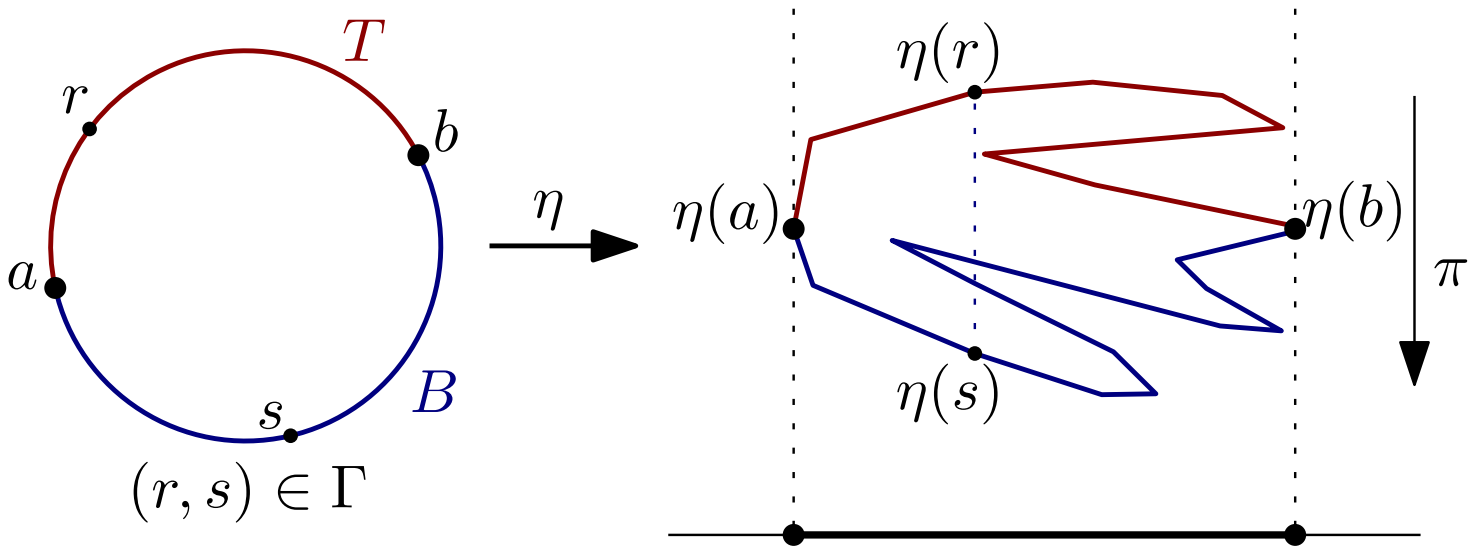


Special Case: π is a 2-to-1 map

$$deg(f_z \circ f_y \circ f_x) = -1$$

$$f_z \circ f_y \circ f_x(q_0) = q_0 : \qquad f_x(q_0) = q_1, \qquad f_y(q_1) = q_2, \qquad f_z(q_2) = q_0$$

A Proof of Nonexistence: d-Space



Relations Γ_α for $\alpha = x, y, z$

$$(r, s) \in \Gamma_\alpha \quad \text{if} \quad \pi_\alpha(\eta(r)) = \pi_\alpha(\eta(s))$$

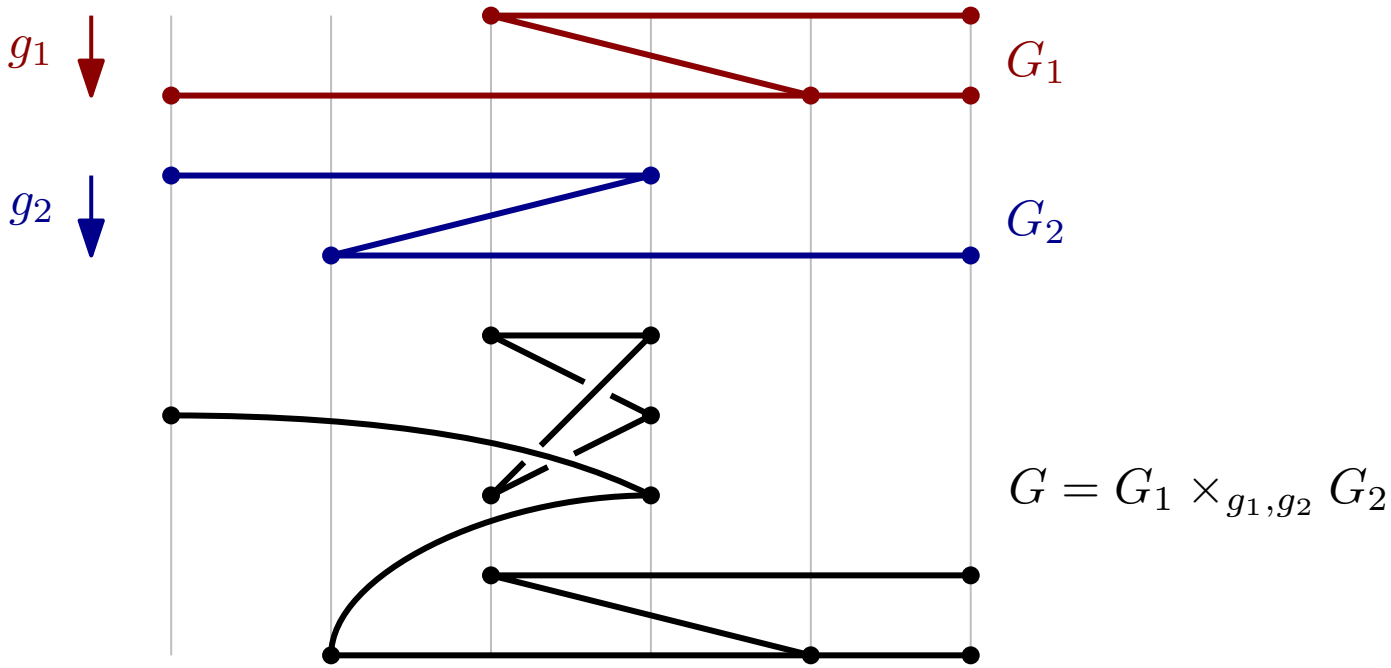
Degree of $\phi = (\phi_1, \phi_2) : \mathbb{S}^1 \to \mathbb{T}^2$

- $deg(\phi) = (deg(\phi_1), deg(\phi_2)) = (n, m)$
- $n \neq m \Rightarrow$

  ϕ crosses a diagonal of $\mathbb{T}^2$ (an analogue of fixed points)

- Is there such a ϕ in $\Gamma_z \circ \Gamma_y \circ \Gamma_x \subset \mathbb{T}^2$?

Fiber Product



$$G_1 \times_{g_1,g_2} G_2 = \{(p_1, p_2) \in G_1 \times G_2 | g_1(p_1) = g_2(p_2)\}$$

A Proof of Nonexistence: d-Space

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- Step 1: by examining $T_\alpha \times_{\pi_\alpha} B_\alpha$ for $\alpha = x, y, z$

  we find $\phi_\alpha = (\phi_{\alpha\beta}, \phi_{\alpha\beta'}) : \mathbb{S}^1 \to \Gamma_\alpha$ such that $deg(\phi_\alpha) = (1, -1)$

- Step 2: by using $(S_x \times_{\phi_{xz}, \phi_{yz}} S_y) \times_{\phi'_x, \phi_{zx}} S_z$

  where S_α: domain of ϕ_α

  there is $\phi : \mathbb{S}^1 \to \Gamma_z \circ \Gamma_y \circ \Gamma_x$ with $deg(\phi) = (n, -n)$

  - Conclusion: ϕ crosses a diagonal of $\mathbb{T}^2$
    ⇒ There exists a common endpoint.

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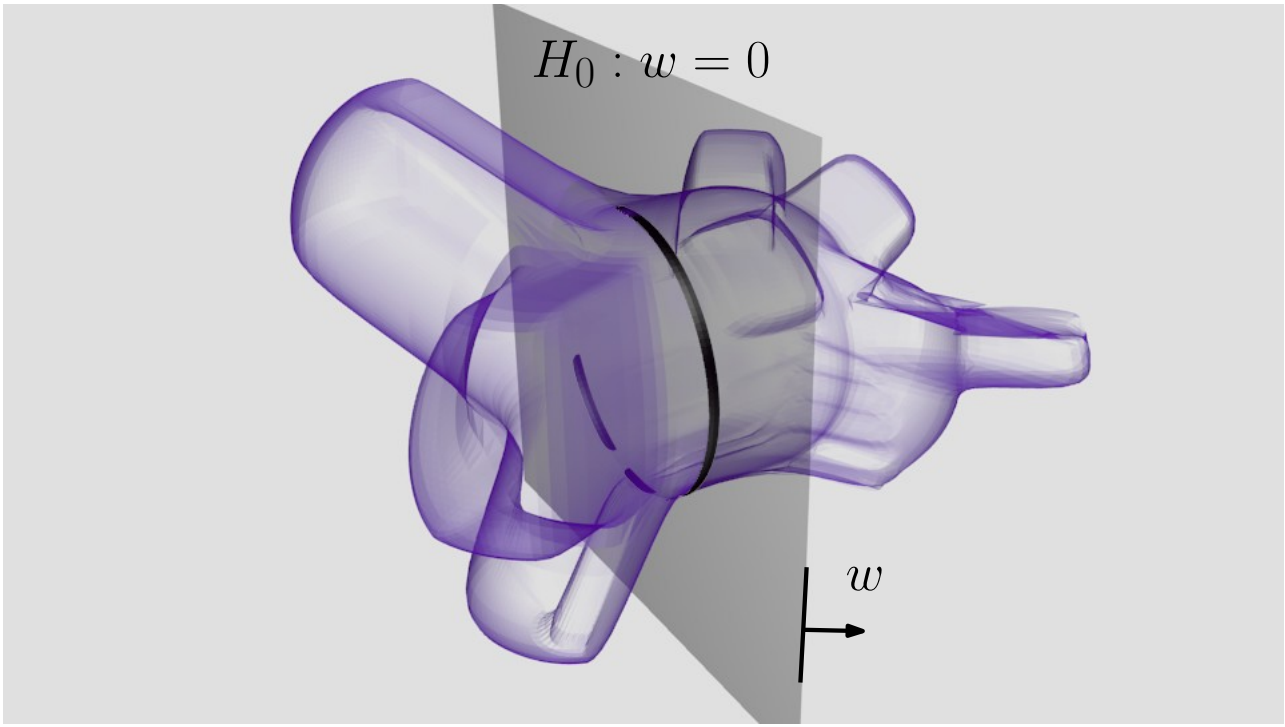
A Proof of Nonexistence: 4-Space



Mayer–Vietoris sequence

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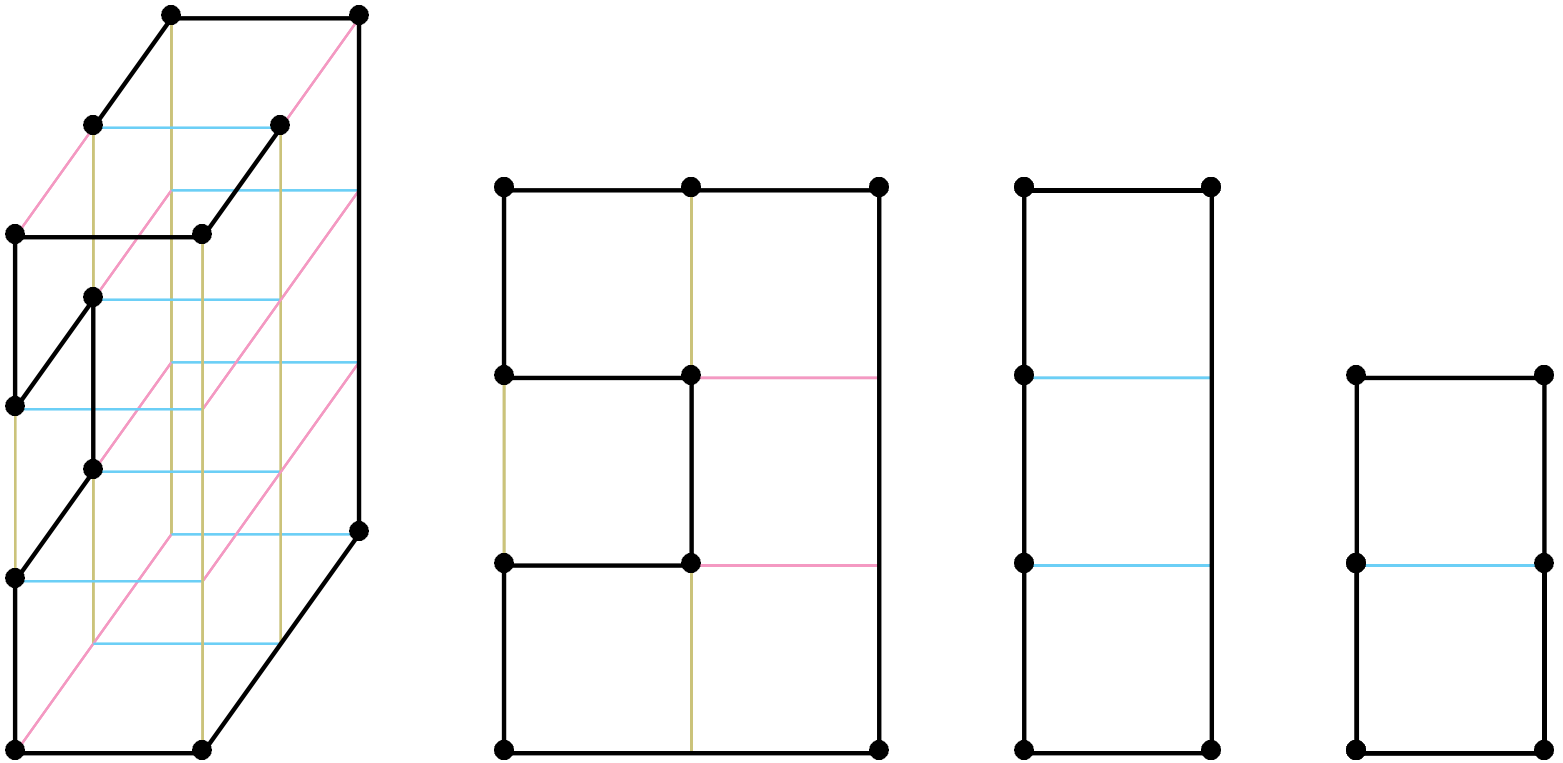
An Axis-Parallel Construction



Impossible for all the shadows to be convex

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