On the Number of Edges of a Fan-Crossing Free Graph

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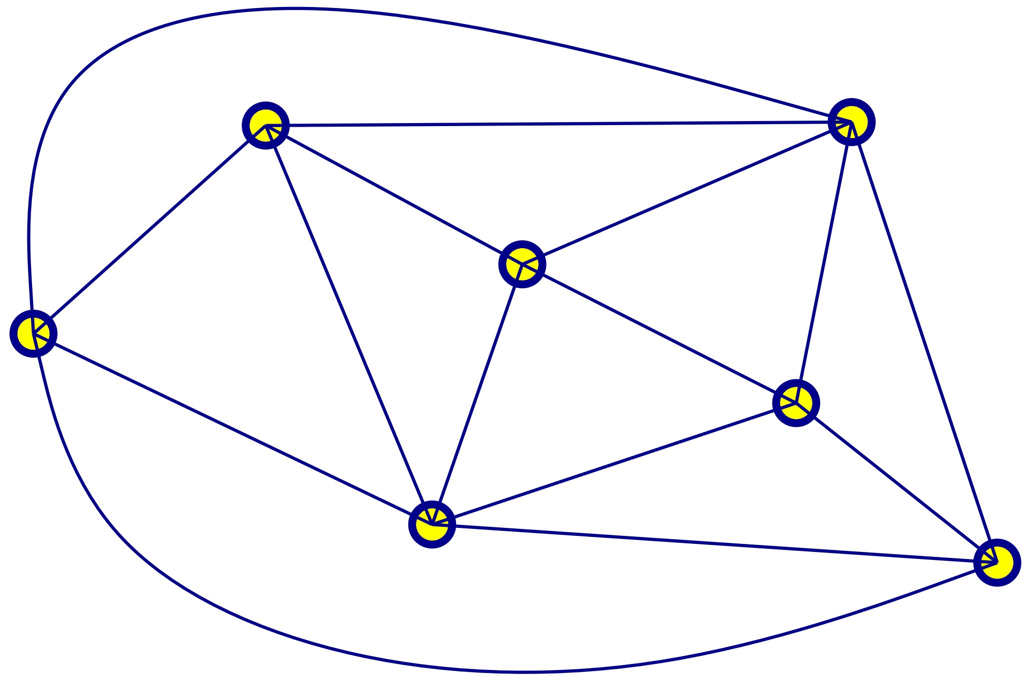
Motivation on Planar graphs

When do planar graphs achieve *the maximum number of edges*?

Motivation on Planar graphs

When do planar graphs achieve
the maximum number of edges?

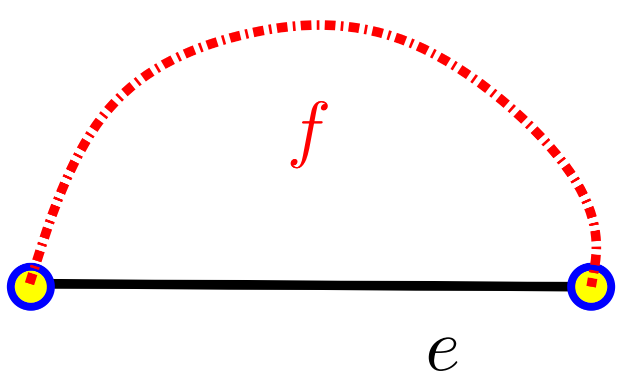


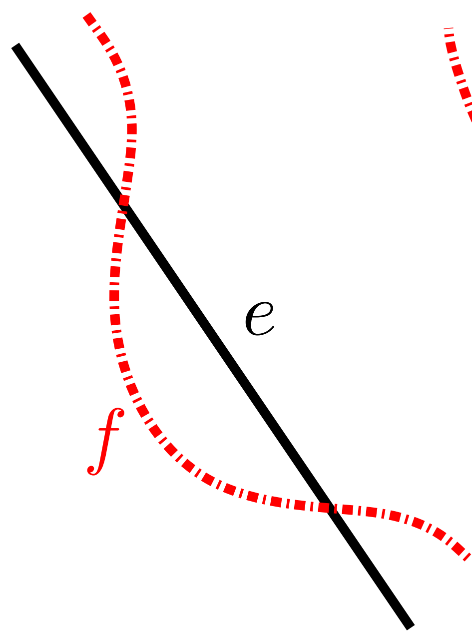
By Euler's formular, $e \leq 3n - 6$ where
e : #edges, n : #vertices

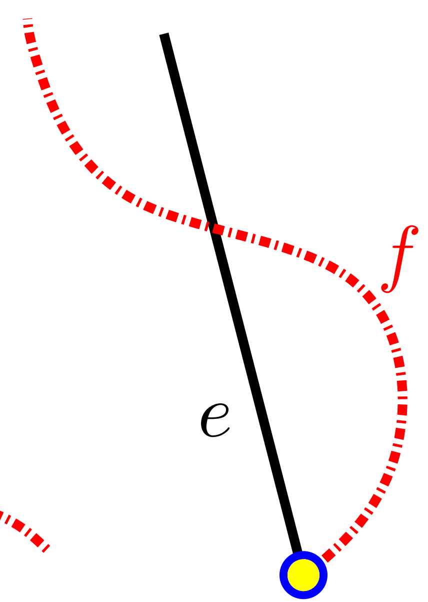
Problem Settings

- a *topological* graph
  - vertices = points on the plane
  - edges = Jordan curves connecting the vertices

- *simple*



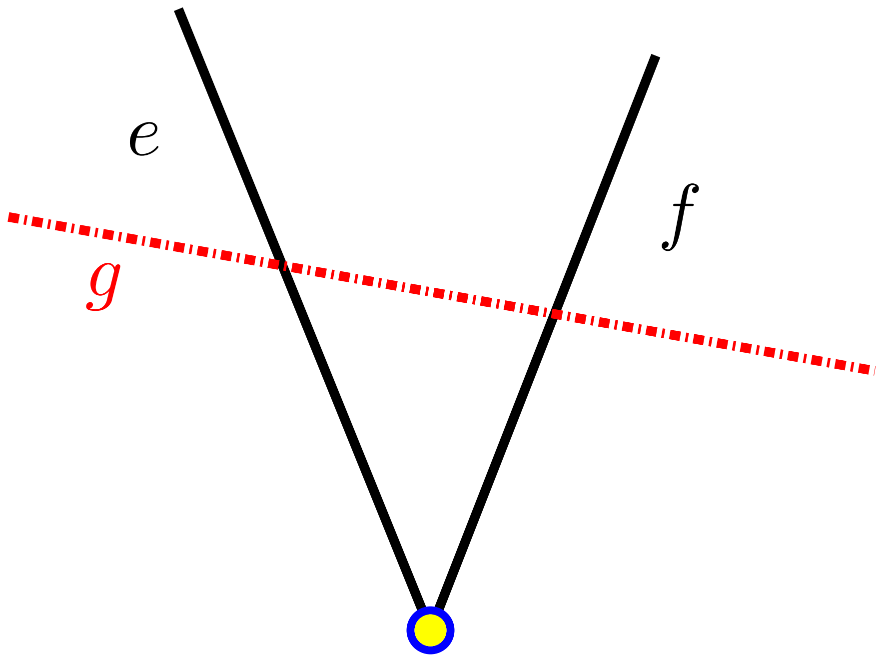




Definition

- A *radial (p, q)-grid*
  - a set of $p + q$ edges such that
  - p edges have a common endpoint, and
  - these p edges cross the remaining q edges.

- A k-**fan-crossing**
  - a radial (k,1)-grid
- A **fan-crossing**
  - a radial (2,1)-grid



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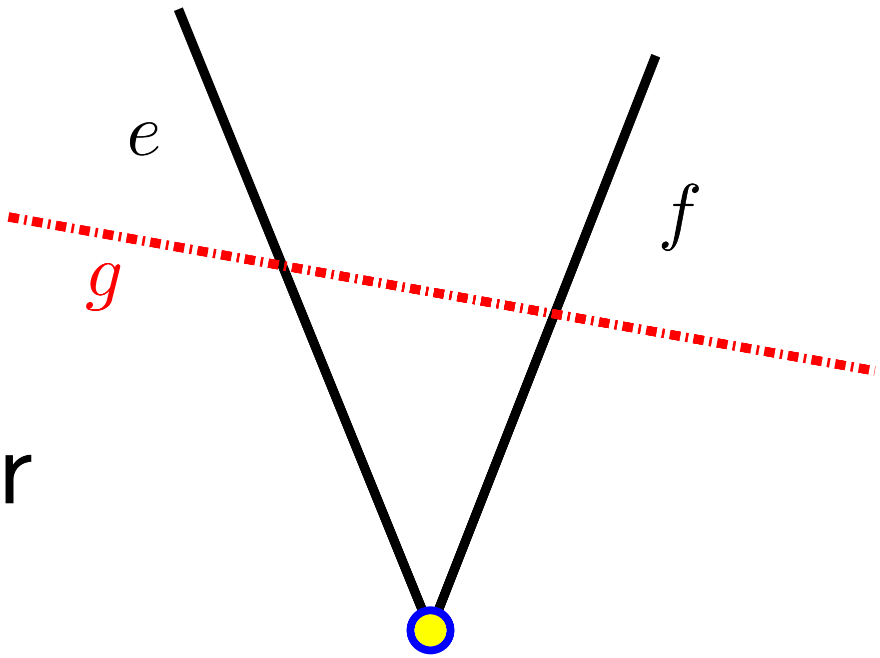
Question:

the maximum number of edges?



Results : the Maximum Number of Edges?

($n > 5$:#vertices, $k > 3$)

- Tight bounds
  - A *fan-crossing free* graph for *all* $n \neq 6, 7, 9$: $4n - 8$ edges
  - A *fan-crossing free* graph for $n = 6, 7, 9$: $4n - 9$ edges
  - A *straight fan-crossing free* graph for *all* $n \neq 6$: $4n - 9$ edges

- Upper bounds
  - A k*-fan-crossing free* graph : $\leq 3(k-1)(n-2)$ edges
  - A graph G s.t $|E(G_e)| = O(n^{1+\alpha})$ *for all* $e \in E(G)$: $O(n^{1+\alpha})$ edges if $\alpha > 0$, and $O(n \log^2 n)$ edges if $\alpha = 0$
    - G_e : the subgraph of G containing exactly those edges that cross e
    - G_e must have a monotone graph property.

Results : the Maximum Number of Edges?

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  - A *fan-crossing free* graph for *all* $n \neq 6, 7, 9$: $4n - 8$ edges
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Outline

- Lower bounds
- Upper bounds

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Lower Bounds: Fan-crossing Free Graphs

When do fan-crossing free graphs achieve *the maximum number of edges*?

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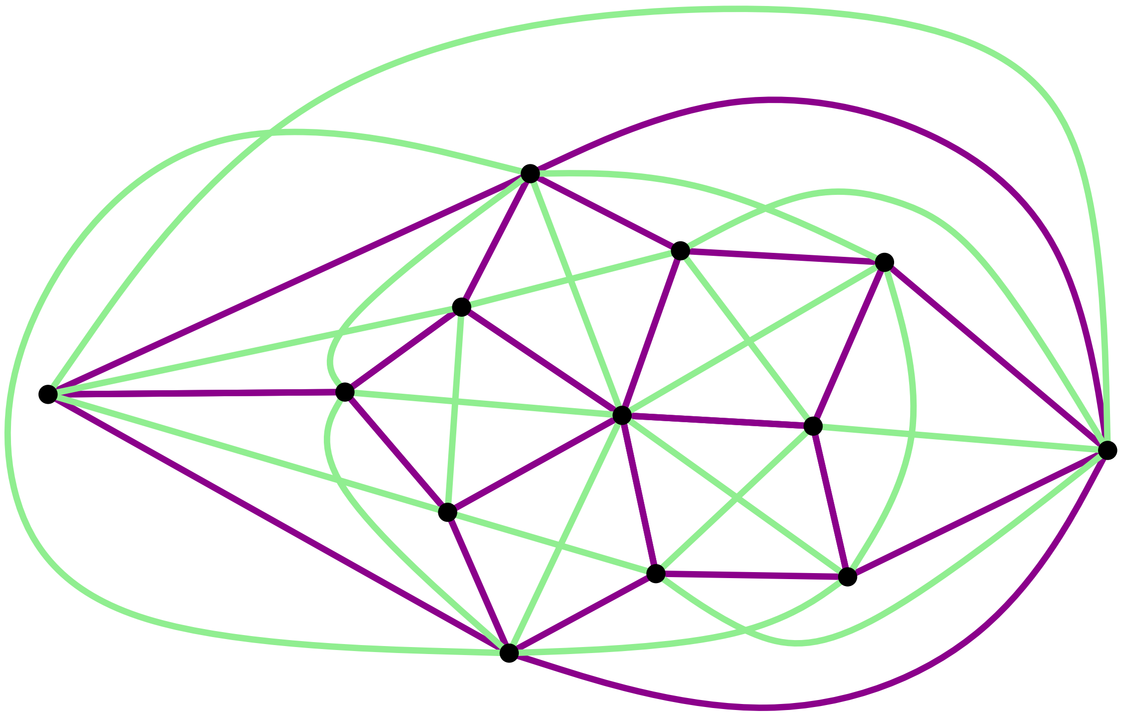
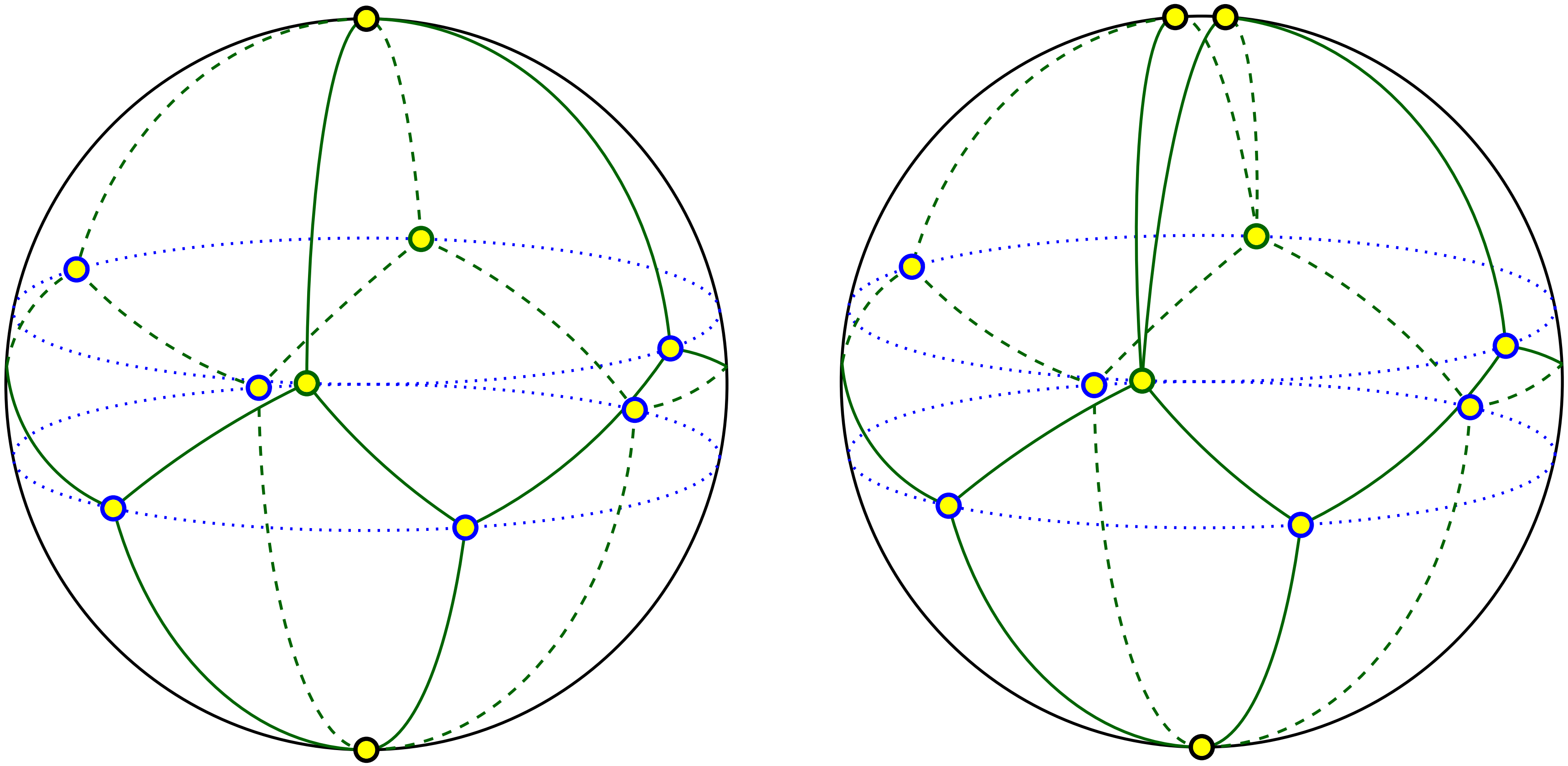


ex)
13 vertices
44 edges

$$3n - 6 + \frac{2n-4}{2} = 4n - 8 \text{ edges}$$

Lower Bounds: Fan-crossing Free Graphs

When do fan-crossing free graphs achieve *the maximum number of edges*?



Outline

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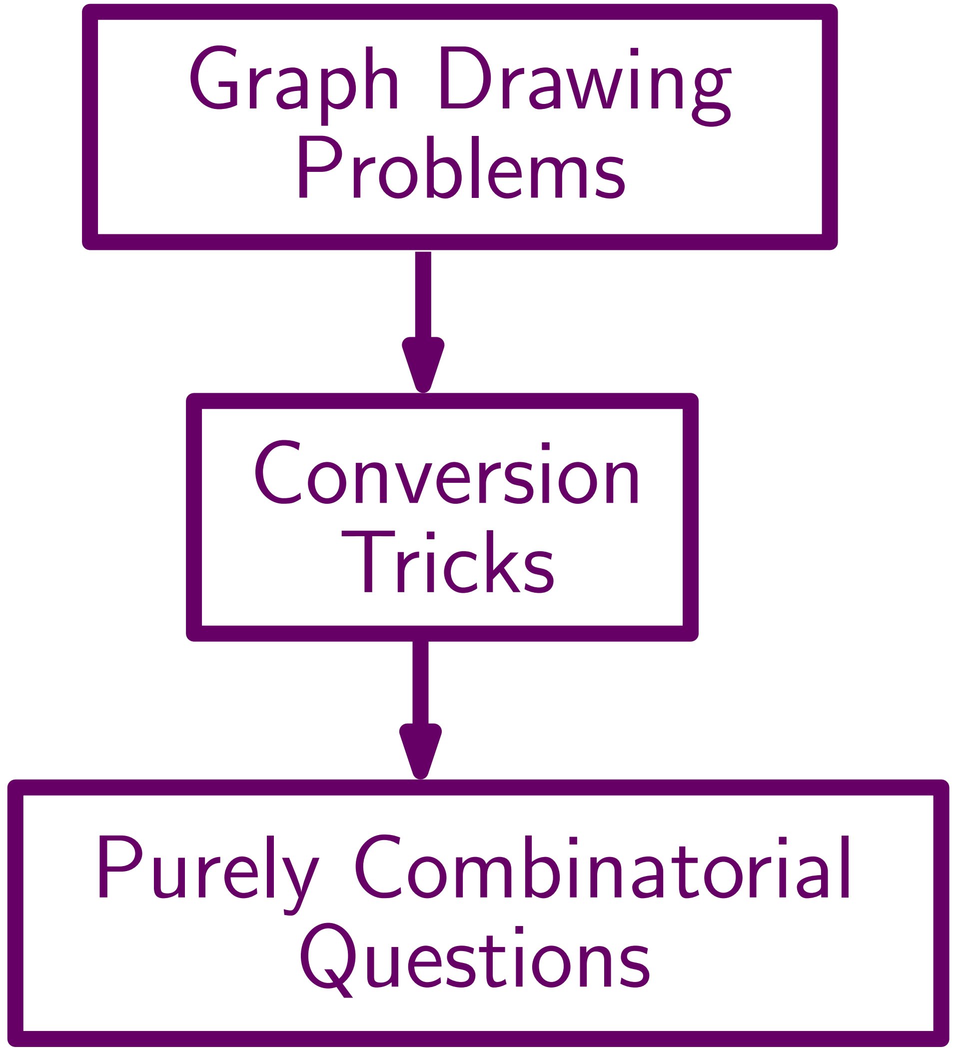
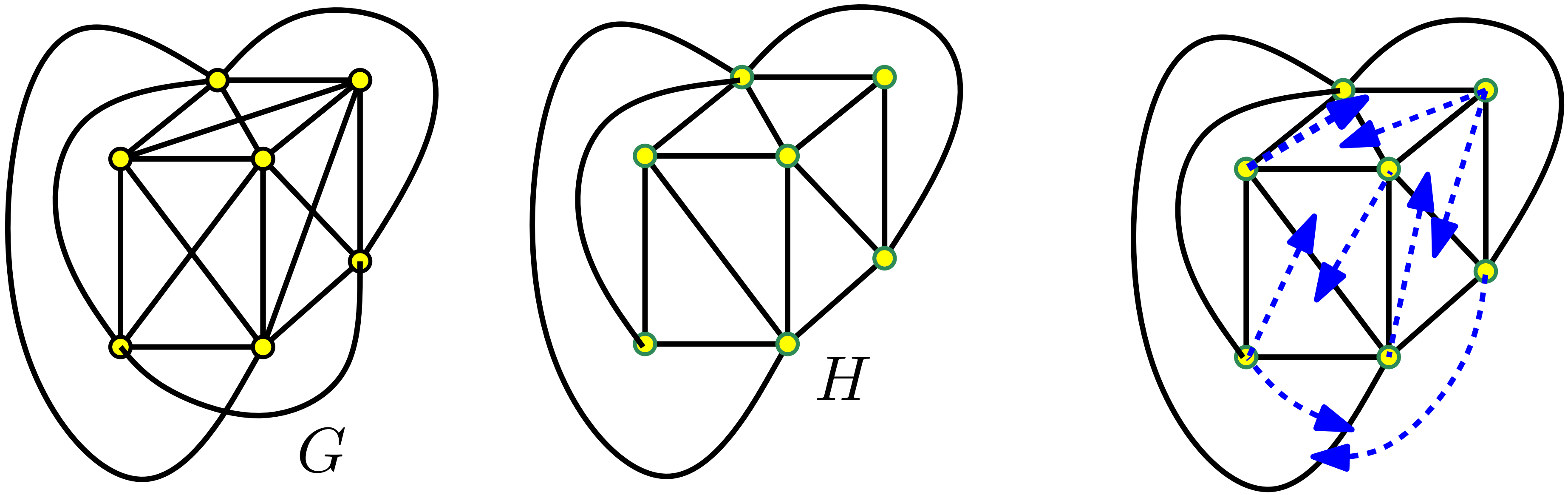


A Toy Example of Counting Edges of G



For a face ψ of a maximal plane subgraph H of G,

- $m(\psi)$: # edges on the boundary of ψ; the complexity of ψ
- $a(\psi)$: # arrows that ψ possesses

$$2|E(G)| = \sum_{\psi \in \mathcal{F}_H} (m(\psi) + a(\psi)) = 8 \cdot (3 + 1) + 2 \cdot 3$$

When are we able to bound $a(\psi)$ in terms of $m(\psi)$?

Outline

- Lower bounds
- Upper bounds

Graph Drawing Problems

Conversion Tricks

Purely Combinatorial Questions

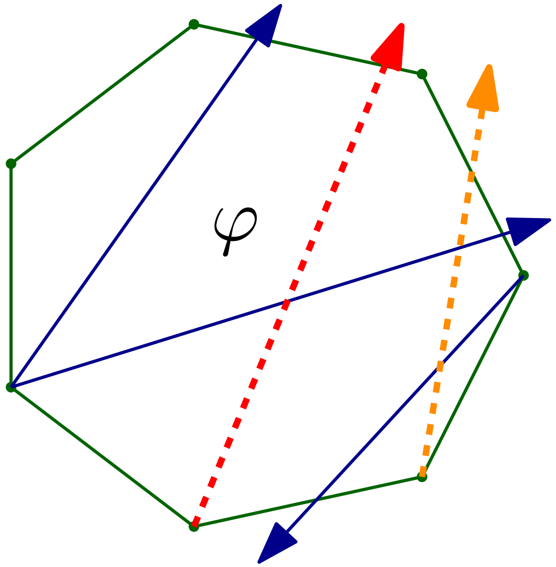
What is the maximum number of edges that can be drawn without fan crossings?

conversion to m-stars

An m-star

- What is an m-star?
- How many arrows in an m-star?
- How an m-star looks like

m-star



m-star :
A *purely combinatorial* m-gon φ
with a set of *arrows*.

- *short* arrows : of length 1
- *long* arrows : of length greater than 1

at most $f_l(m)$ *long* arrows,
at most $f_s(m)$ *short* arrows

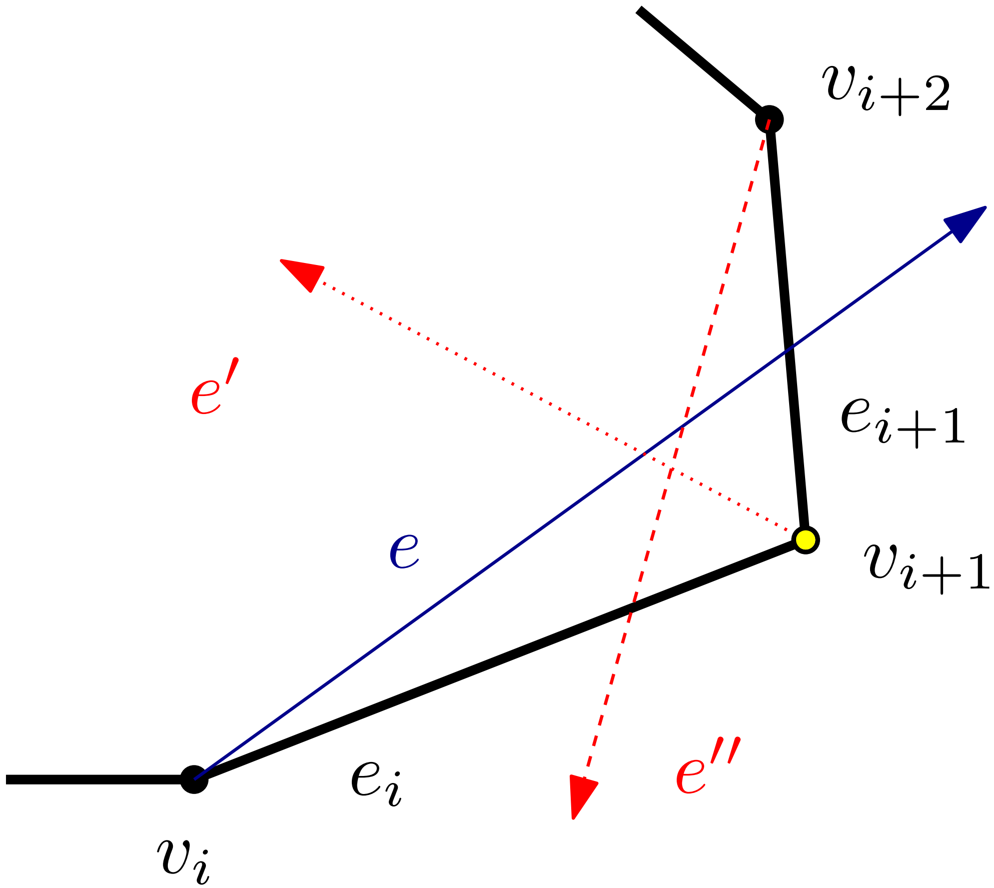
Bound the Number of Arrows

For an m-star in a fan-crossing free graph,

$$f_s(m) \leq \quad m - 1 \qquad \text{if } m \geq 3$$

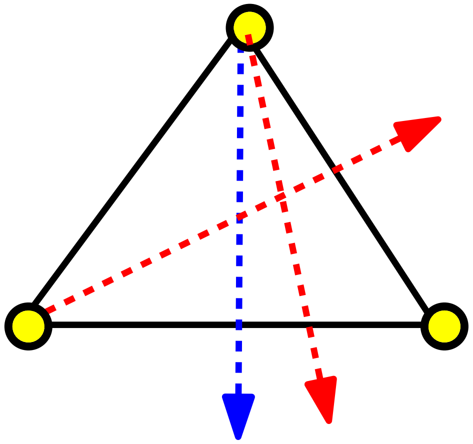
$$f_l(m) \leq \begin{cases} 0 & \text{if } m = 3, 4 \\ 2(m-4) & \text{if } m \geq 5 \end{cases}$$

Proof for Short Arrows

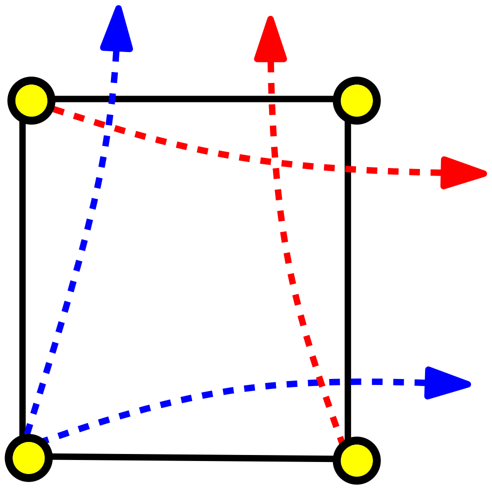


$$f_s(m) \leq \; m-1 \quad \text{if } m \geq 3$$

Base Cases : Long Arrows



- 3-star
  - no long arrows

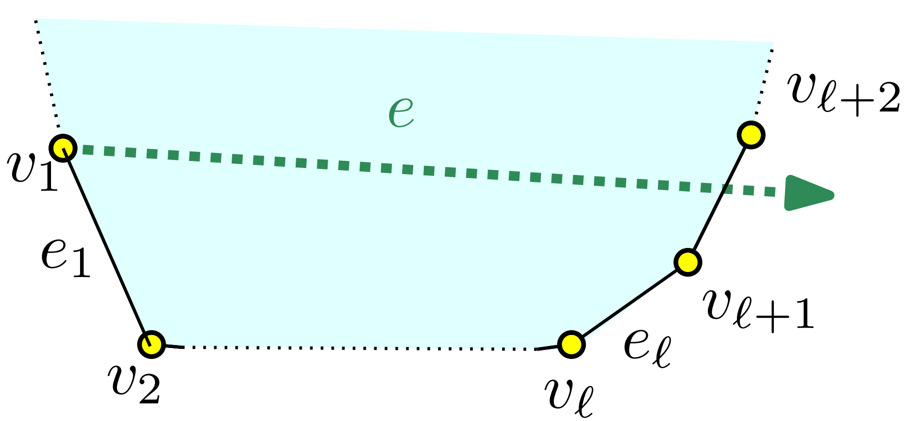


- 4-star
  - no long arrows

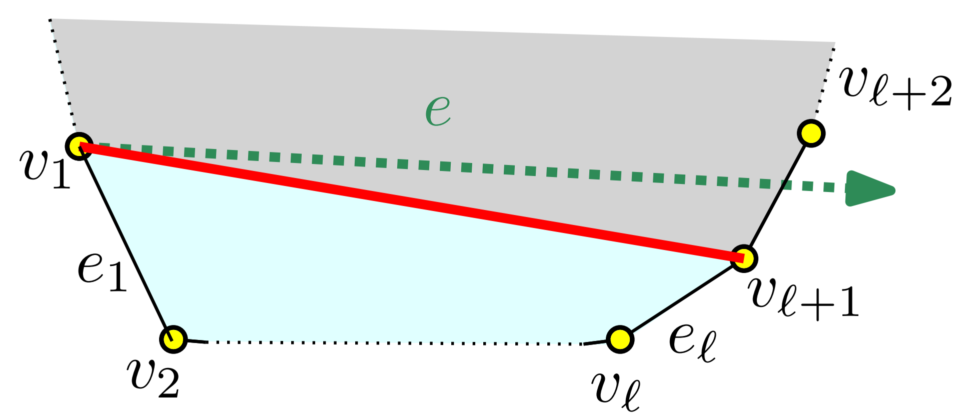
Induction : Long Arrows

- $m \geq 5$. We delete all short arrows.
- e : a shortest long arrow, $\ell \geq 2$: length of e

an m-star



an $(m - \ell + 1)$-star



$$f_l(m - \ell + 1) \geq f_l(m) - |\text{all arrows from } v_2 \sim v_\ell| - |\text{new short arrows}|$$

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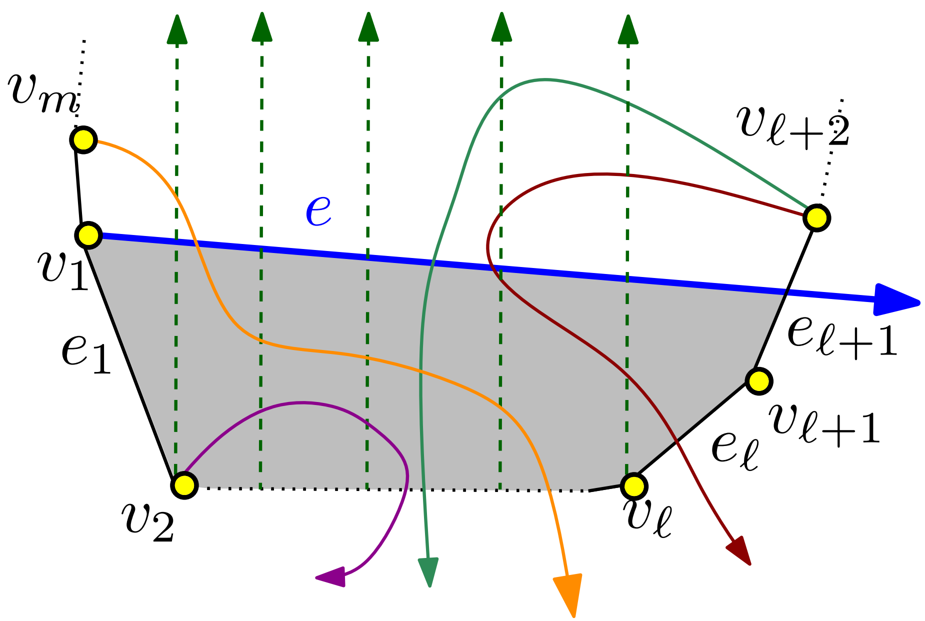
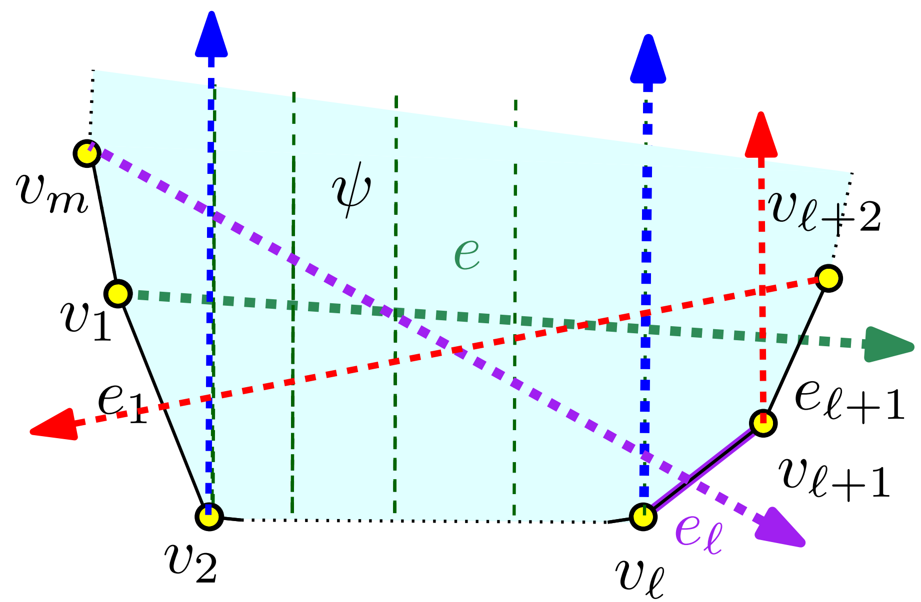
- e : a shortest long arrow, $\ell \geq 2$: length of e

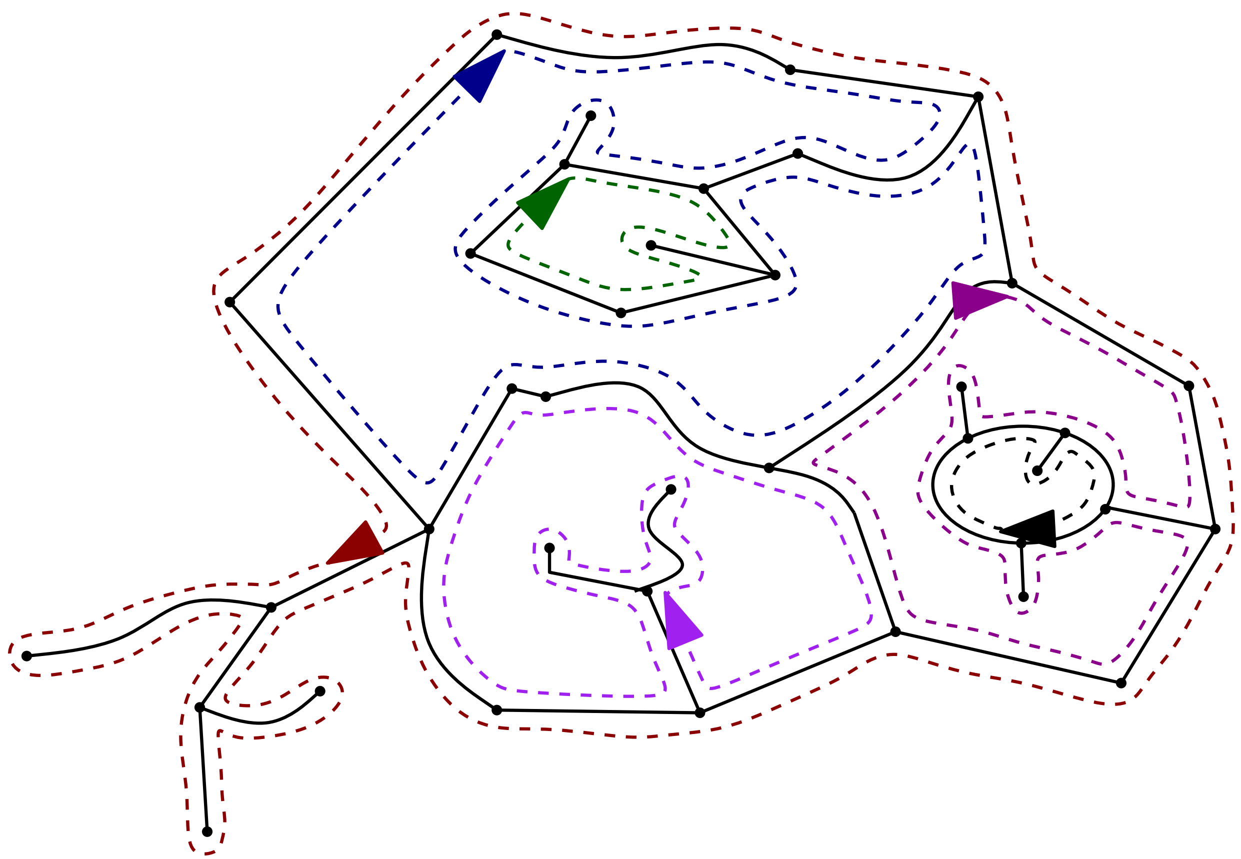




$$f_l(m - \ell + 1) \geq f_l(m) - |\text{all arrows from } v_2 \sim v_\ell| - |\text{new short arrows}|$$

$$f_l(m) \leq f_l(m - \ell + 1) + \ell$$

How does an m-star look like?



If a maximal plane subgraph H is connected,

each face ψ : *$m(\psi)$-star*

$$a(\psi) \leq f_l(m(\psi)) + f_s(m(\psi))$$

What if H is not connected...

Outline

- Lower bounds
- Upper bounds

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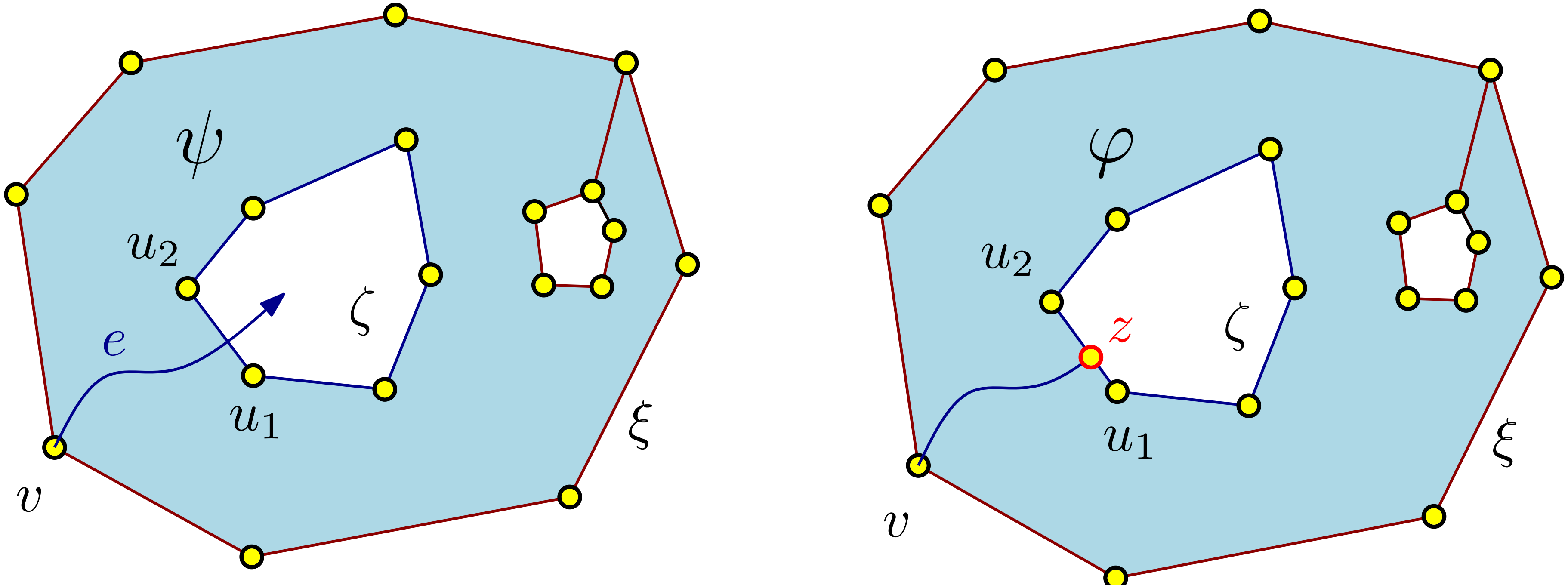
If a face ψ of H is not an m-star

Assume there is an arrow starting in a vertex of ξ and ending in an edge of ζ (otherwise, count them seperately).



- $p(\psi)$: #(boundary chains in ψ)

- $m(\varphi) = m(\psi) + 3(p(\psi) - 1)$
- $w(\varphi) = m(\psi) + p(\psi) - 1$

- $a(\psi) \leq f_l(m(\varphi)) + f_s(w(\varphi)) + (p(\psi) - 1)$

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Graph Drawing Problems

↓

Conversion Tricks

↓

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A bound of $a(\psi)$ for ψ in general

Remind that

$$\begin{aligned} a(\psi) &\leq f_l(m(\psi)) + f_s(m(\psi)) && \text{if } p(\psi) = 1 \\ a(\psi) &\leq f_l(m(\varphi)) + f_s(w(\varphi)) + (p(\psi) - 1) && \text{if } p(\psi) > 1 \end{aligned}$$

fan-crossing free :

$$a(\psi) \leq 3m(\psi) + 8p(\psi) - 17$$

k-fan-crossing free ($k \geq 3$) :

$$a(\psi) \leq 3(k-1)(m(\psi) + 2p(\psi) - 4) - (2m - 3)$$

Compute the sum of $m(\psi) + a(\psi)$

For a radial $(k, 1)$-grid free graph G on n vertices and e edges, H : a maximal plane subgraph, $\mathcal{F}$: the set of faces in H.

$$
\begin{aligned}
2e &\le \sum_{\psi \in \mathcal{F}} m(\psi) + \sum_{\psi \in \mathcal{F}} a(\psi) \\
&\le \begin{cases} \sum_{\psi \in \mathcal{F}} & (m(\psi) + 3m(\psi) + 8p(\psi) - 16) \\ \sum_{\psi \in \mathcal{F}} & \{3(k-1)(m(\psi) + 2p(\psi) - 4)\} \end{cases} \\
&\le \begin{cases} 8n - 16 & \text{if } k = 2 \\ 6(k-1)(n-2) & \text{if } k \ge 3 \end{cases}
\end{aligned}
$$

using Euler's formula($n - m + r = 1 + p$) on H with $\sum_{\psi \in \mathcal{F}} m(\psi) = 2m$ and $\sum_{\psi \in \mathcal{F}} (p(\psi) - 1) = p - 1$.

Questions

- What is the better bounds for k-fan-crossing free graphs?
- For a given adjacency matrix of a fan-crossing free graph, find an embedding (a drawing) of the graph without fan crossings.
- For a given adjacency matrix of a graph, test if the graph is fan-crossing free. (Testing if a graph is 1-planar is NP-hard.)
- When a graph is not a topological graph, for example, vertices are points on a torus, find the maximum number of edges of a fan-crossing free graph.

Thank you

Problem Settings

- a **monotone** *graph property* $\mathcal{P}$
  - preserved under edge-deletions

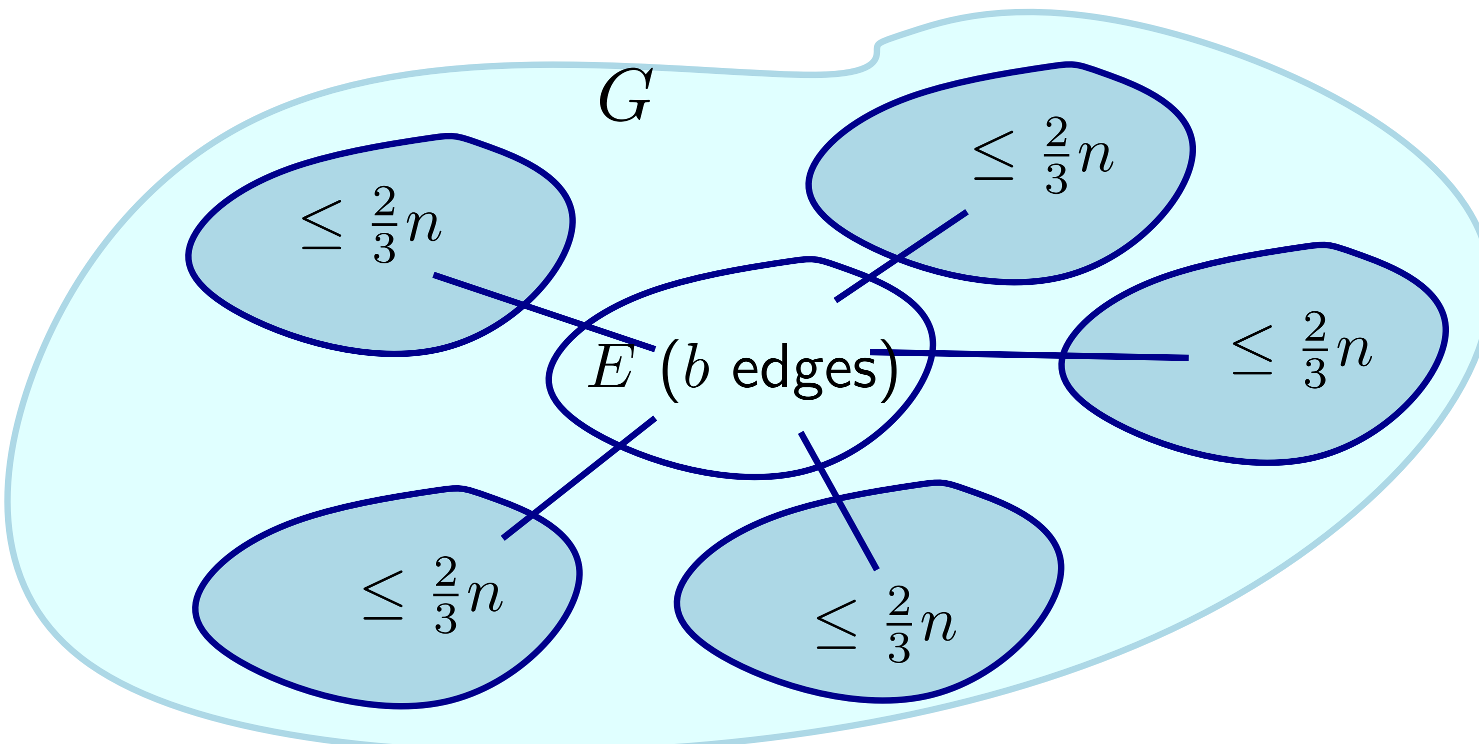
- A graph $G = (V, E)$ has a **derived** *graph property* $\mathcal{P}^*$ of a monotone graph property $\mathcal{P}$ *if G_e has $\mathcal{P}$ for all e in E.*
  - G_e : the subgraph of G containing exactly those edges that cross e

- A special case of $\mathcal{P}$ and $\mathcal{P}^*$
  - $\mathcal{P}$: G is $K_{1,k}$ free $\Rightarrow$ $\mathcal{P}^*$: a radial $(k, 1)$-grid free

General bounds

Lemma 1 (Theorem 2.1 , Pach et al.) *Let G be a graph with n vertices of degree $d_1, \ldots, d_n$ and crossing number χ. Then there is a subset E of b edges of G such that removing E from G creates components of size at most $2n/3$, and*

$$b^2 \leq (1.58)^2 \Big(16\chi + \sum_{i=1}^{n} d_i^2\Big).$$



General bounds

For a given graph G on n vertices and m edges,
G_e : the subgraph of G containing exactly those edges that cross e.
G_e has a monotone graph property.

If G_e has $O(n^{1+\alpha})$ for all e in G,
G has $O(n^{1+\alpha})$ *edges if* $\alpha > 0$, *and* $O(n \log^2 n)$ *edges if* $\alpha = 0$.

Proof idea. Recursively subdivide G by removing $b_0, b_1, \ldots, b_p$ edges.
$b_i = O(\sqrt{mn}\left(\left(\frac{2}{3}\right)^i n\right)^{\alpha/2})$ if $\alpha > 0$ and $b_i = O(\sqrt{mn})$ by Lemma 1 since $\chi = O(n^{1+\alpha})$. Then, $m \leq \sum_{i=1}^{p} b_i$ and $p = O(\log n)$