Congruence Testing for Point Sets in 4-Space

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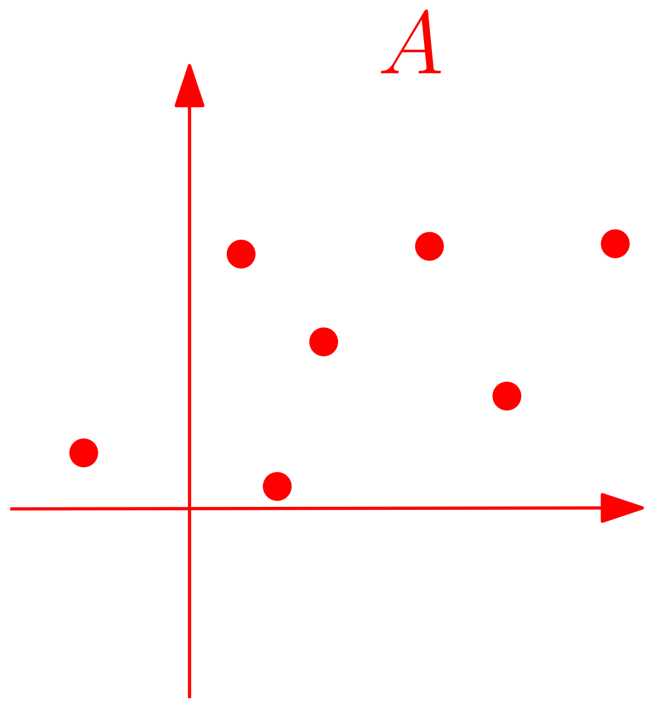
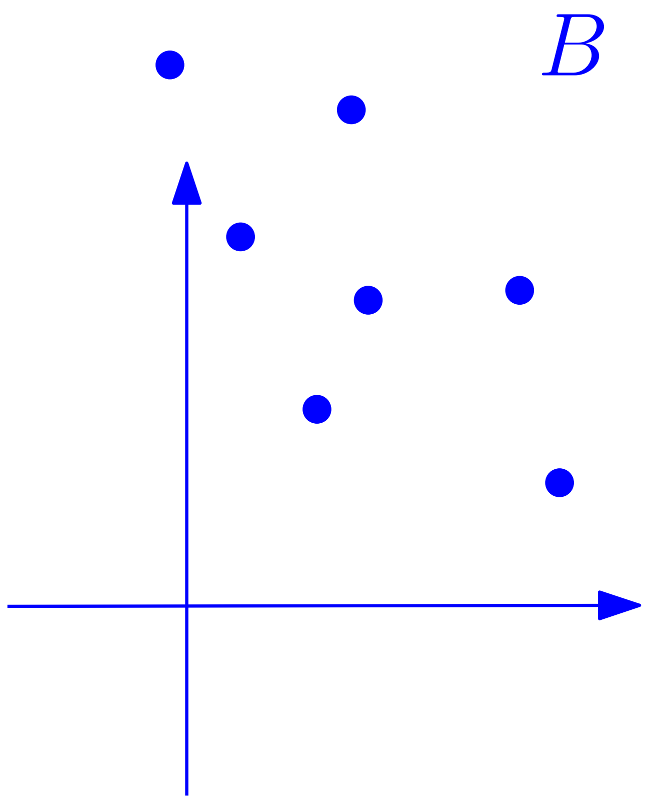
Congruence Testing Problem

Congruence $\Leftarrow$ Translations + Rotations

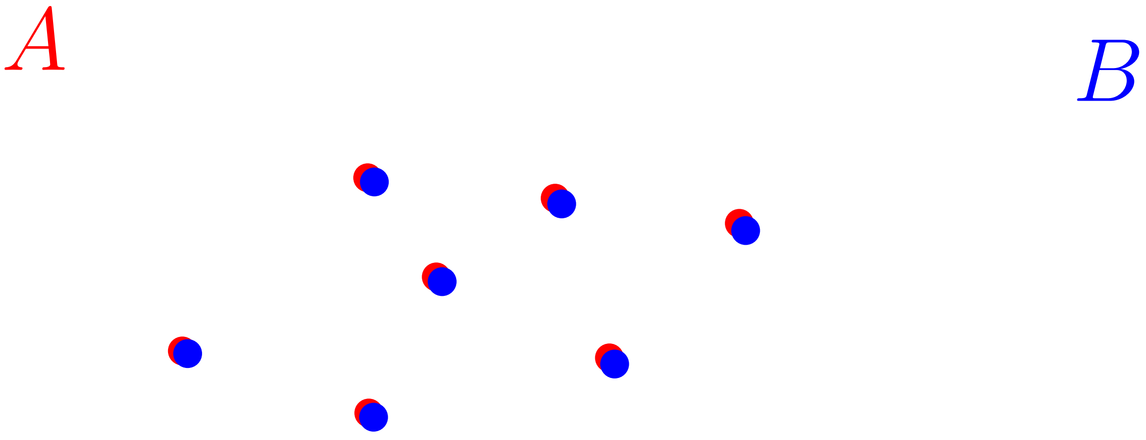




A, B : point sets

Congruence Testing Problem

Congruence $\Leftarrow$ Translations $+$ Rotations



A, B : point sets

Previous Algorithms

n :#points

- *Plane* (2-Space): $O(n \log n)$-*time* algorithms

  by Manacher (1976), Atallah (1985), Highnam (1986)

- *3-Space* : $O(n \log n)$-*time* algorithms

  by Sugihara (1984), Atkinson (1987),
  Alt, Mehlhorn, Wagener and Welzl (1988)

Previous Algorithms

– *d-Space* :

by Alt, Mehlhorn, Wagener and Welzl (1988)

$O(n^{d-2}\log n)$-time algorithm for $d \geq 3$

by Akutsu (1998) with an idea of Matoušek:

a randomized $O(n^{\lfloor d/2 \rfloor/2}\log n)$-time algorithm for $d \geq 6$

$O(n^{3/2}\log n)$-time algorithm for $d = 4, 5$

by Brass and Knauer (2003)

$O(n^{\lceil d/3 \rceil}\log n)$-time algorithm for $d \geq 3$

$\Rightarrow$ *Conjecture* : $O(n \log n)$ for any fixed d

4-Space

– Deterministic: $O(n^2 \log n)$-time by Brass and Knauer

– Randomized : $O(n^{3/2} \log n)$-time by Akutsu

$\Rightarrow$ *New Algorithm* :

$O(n \log n)$-time in 4-Space

Outline

– Problem definition

– Basic principles

Apply *condensing* principles
until *dimension reduction* is affordable
on a *closest-pair* graph

– New algorithm

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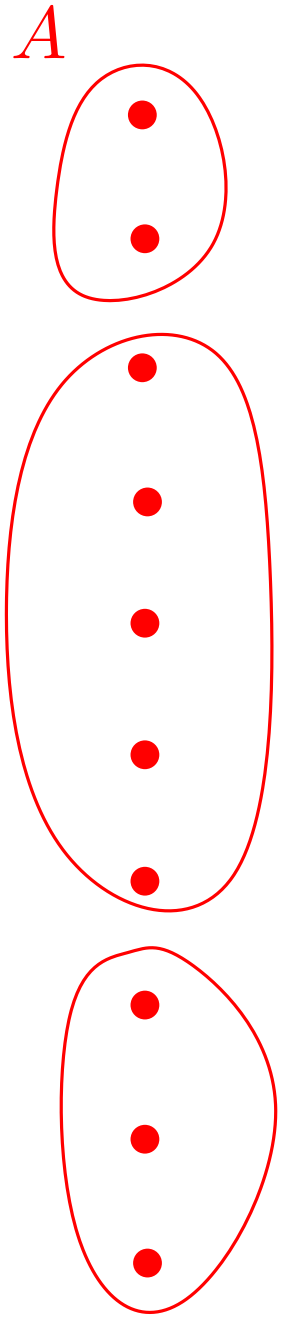
- Problem definition
- Basic principles
  - Condensing
  - Dimension Reduction
  - Closest-pair graphs
- New algorithm

Pruning

A

Pruning

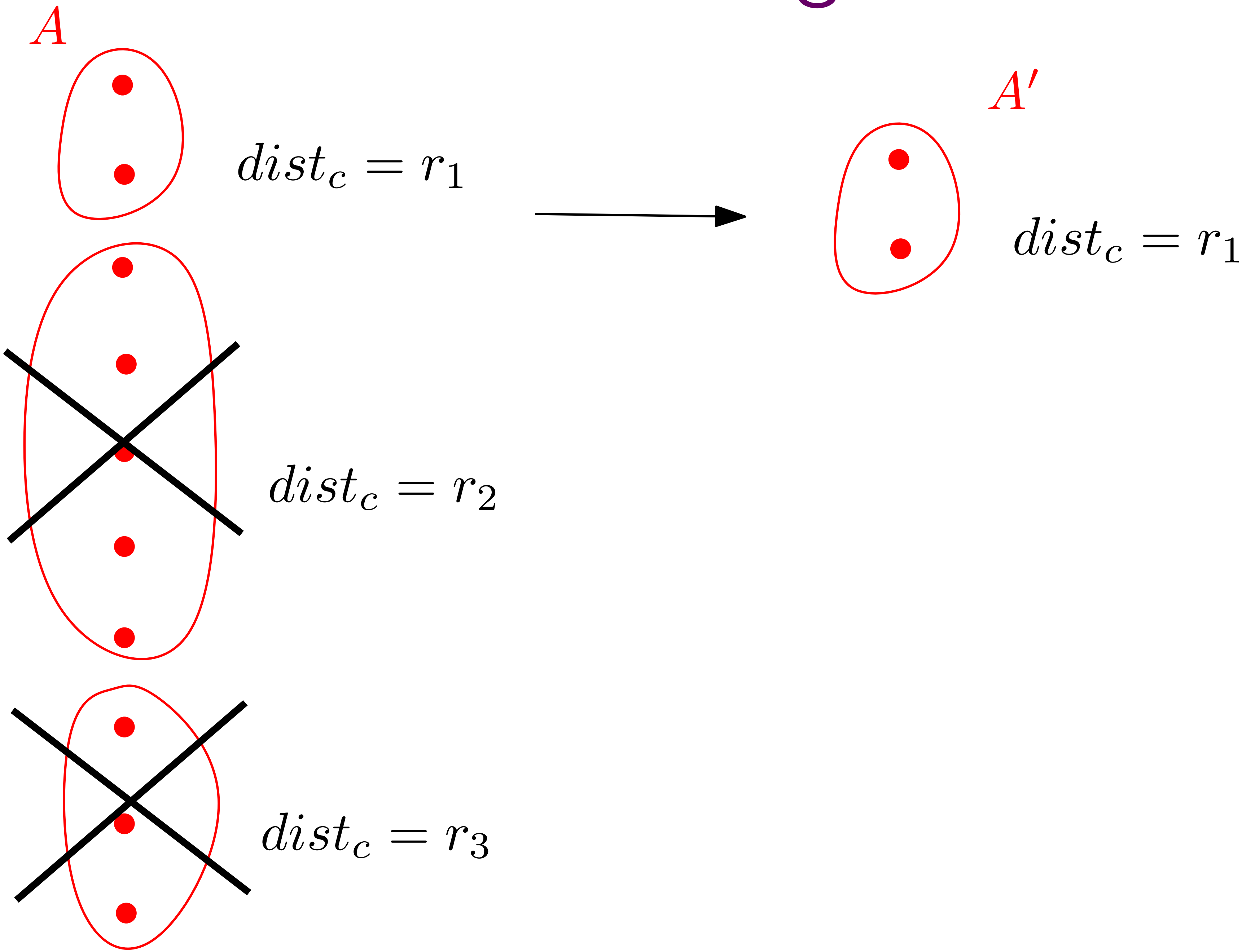


$dist_c = r_1$

$dist_c = r_2$

$dist_c = r_3$

Pruning



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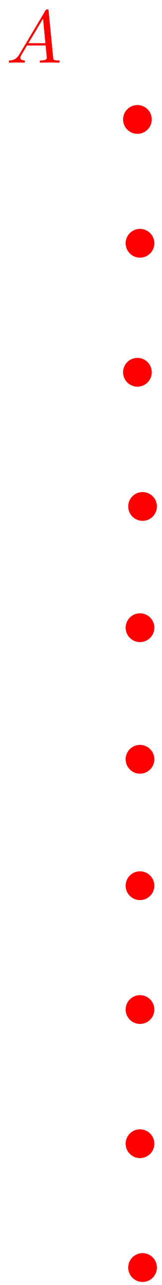
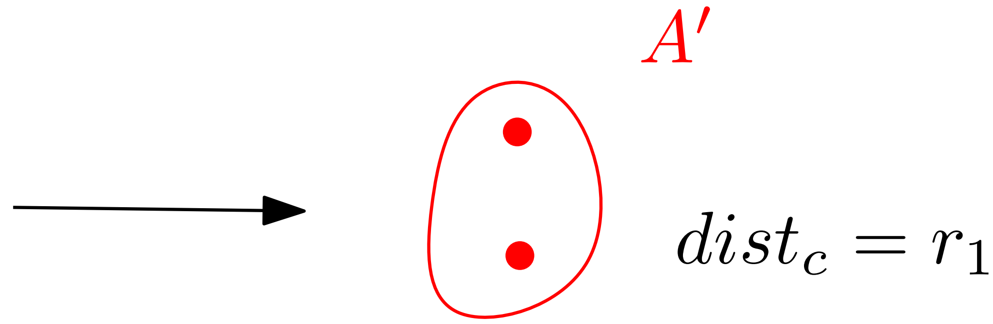




- *More structure*

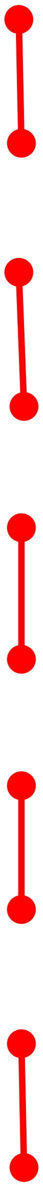
  e.g.) all points are on a sphere

- *Reduction of # points*

  e.g.) $|A'| = \frac{2}{3}|A|$ for each pruning,
  $O(|A| \log |A|) + O(\frac{2}{3}|A| \log |A|) +$
  $O(\frac{4}{9}|A| \log |A|) + ... = O(|A| \log |A|)$

Condensing

A



Condensing

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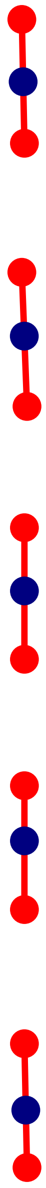
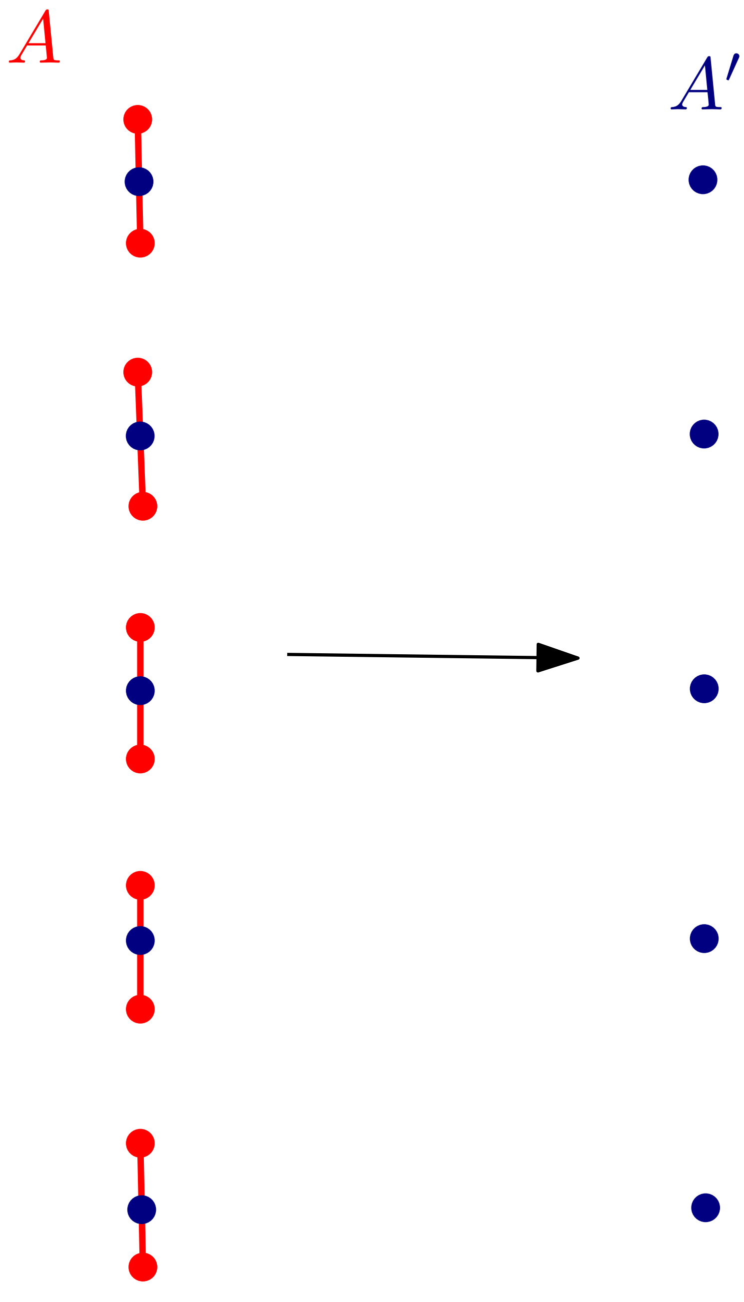
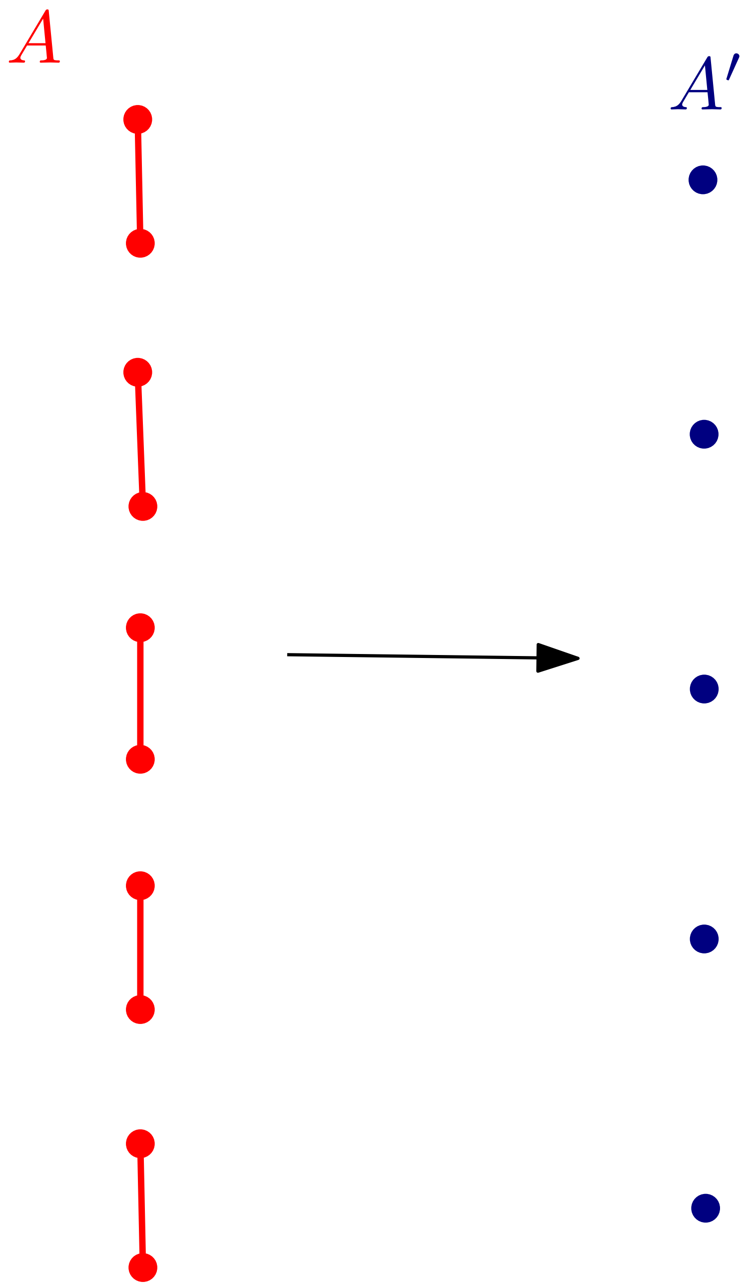


Midpoints!

Condensing



Condensing



Extensions of Pruning

any equivariant mappings f;

that is, $fR = Rf$ for any rotation R

Only Temporarily!!!

restore all points once

we can afford dimension reduction

Outline

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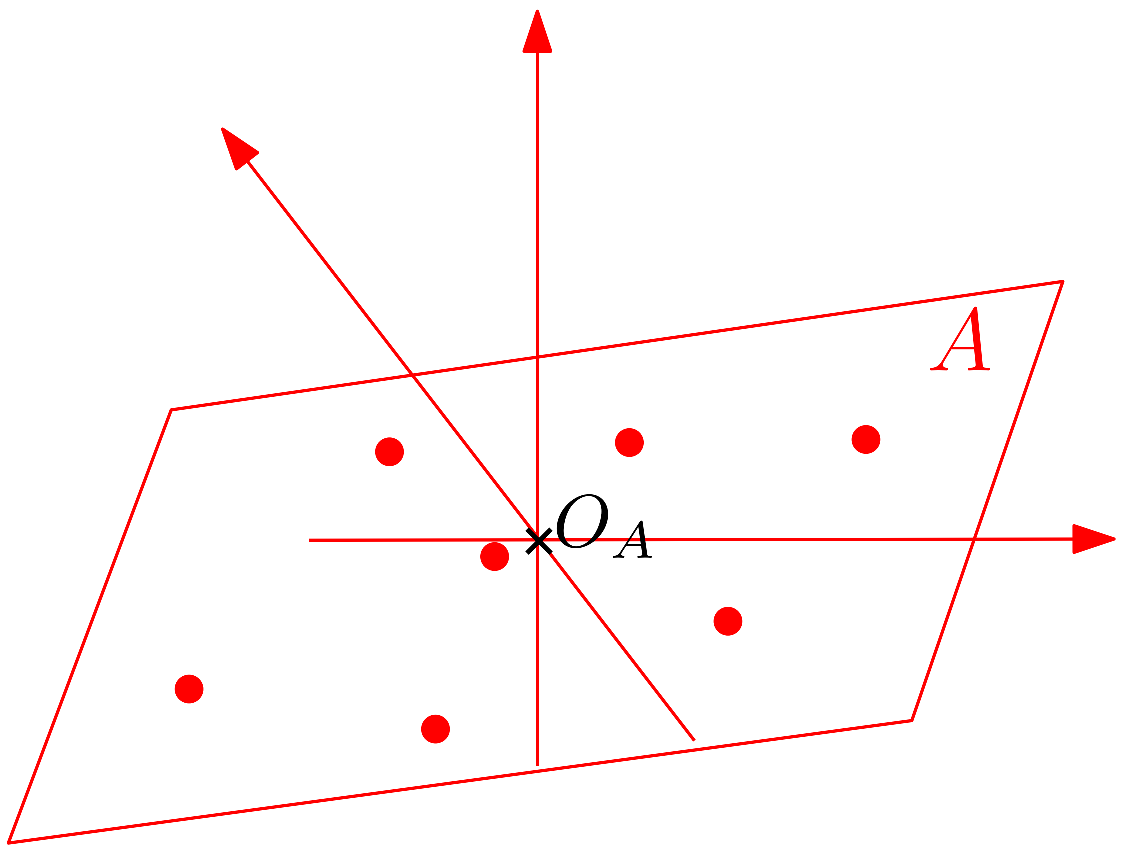
  - Dimension Reduction
  - Closest-pair graphs

- New algorithm

Condensing + Dimension Reduction



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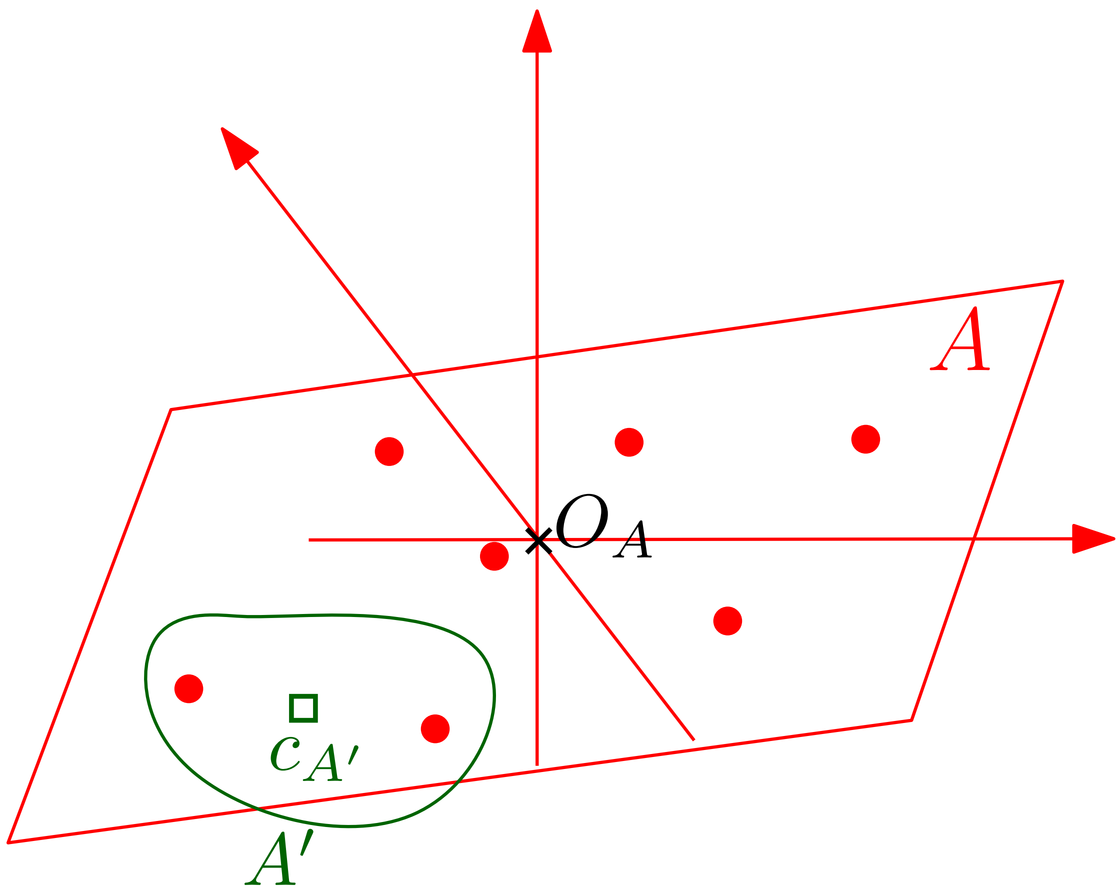

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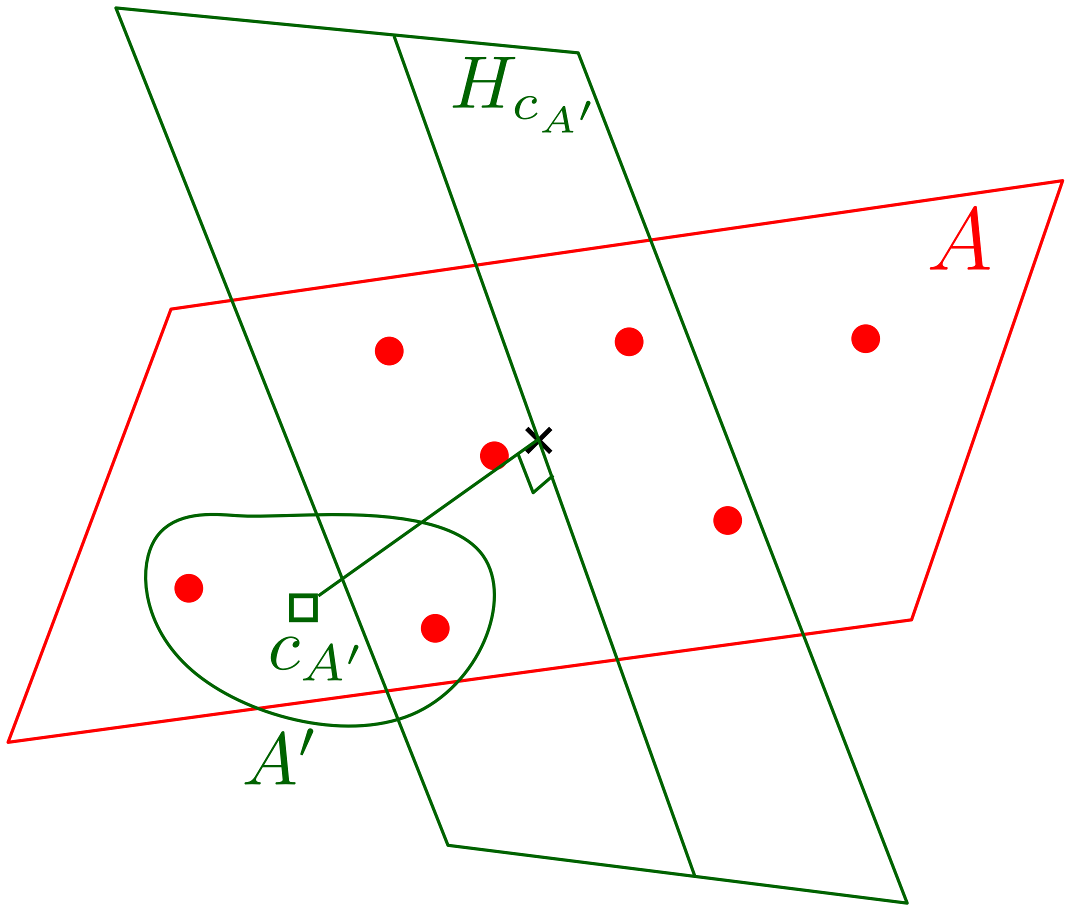

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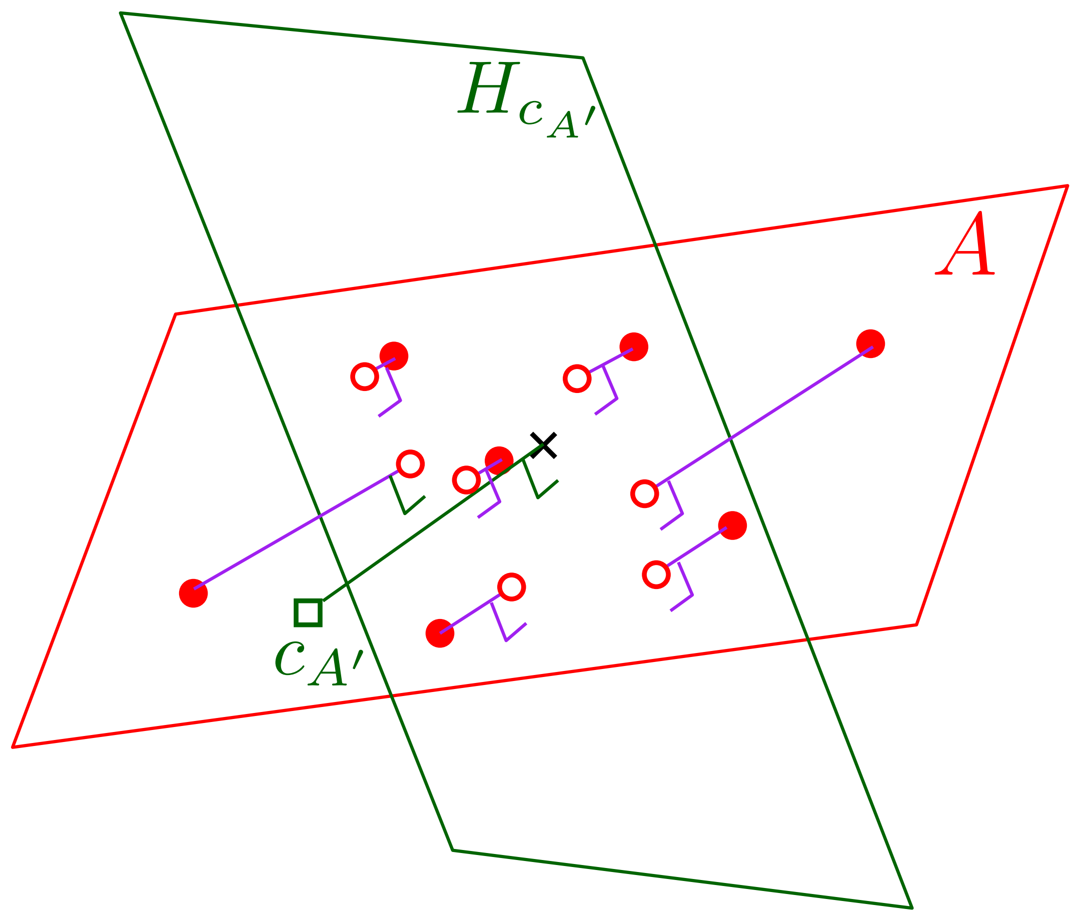

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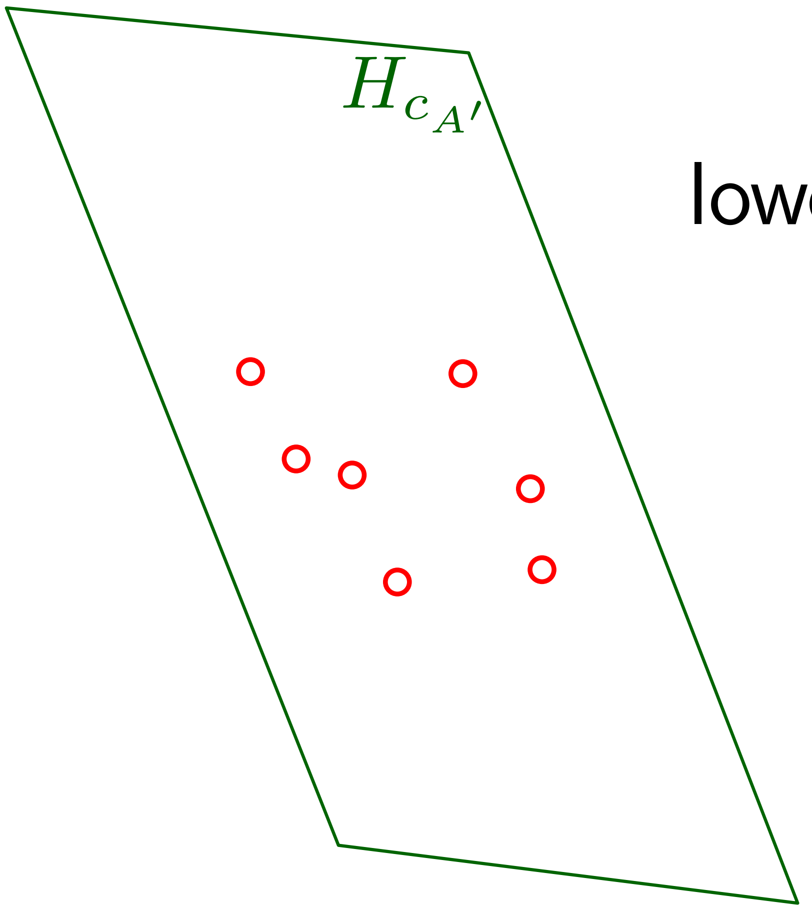

$\Rightarrow$ One lower-dimensional space

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Closest-Pair Graphs

$G = (V, E)$: $V = A$,
$E = \{(u, v)|\ dist(u, v)$: minimum $\}$

Construction : $O(n \log n)$ time in any fixed dimension d

The maximum degree : $C^d = O(1)$

$\Rightarrow \#$ edges $= O(n)$.

Outline

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- New algorithm

Outline

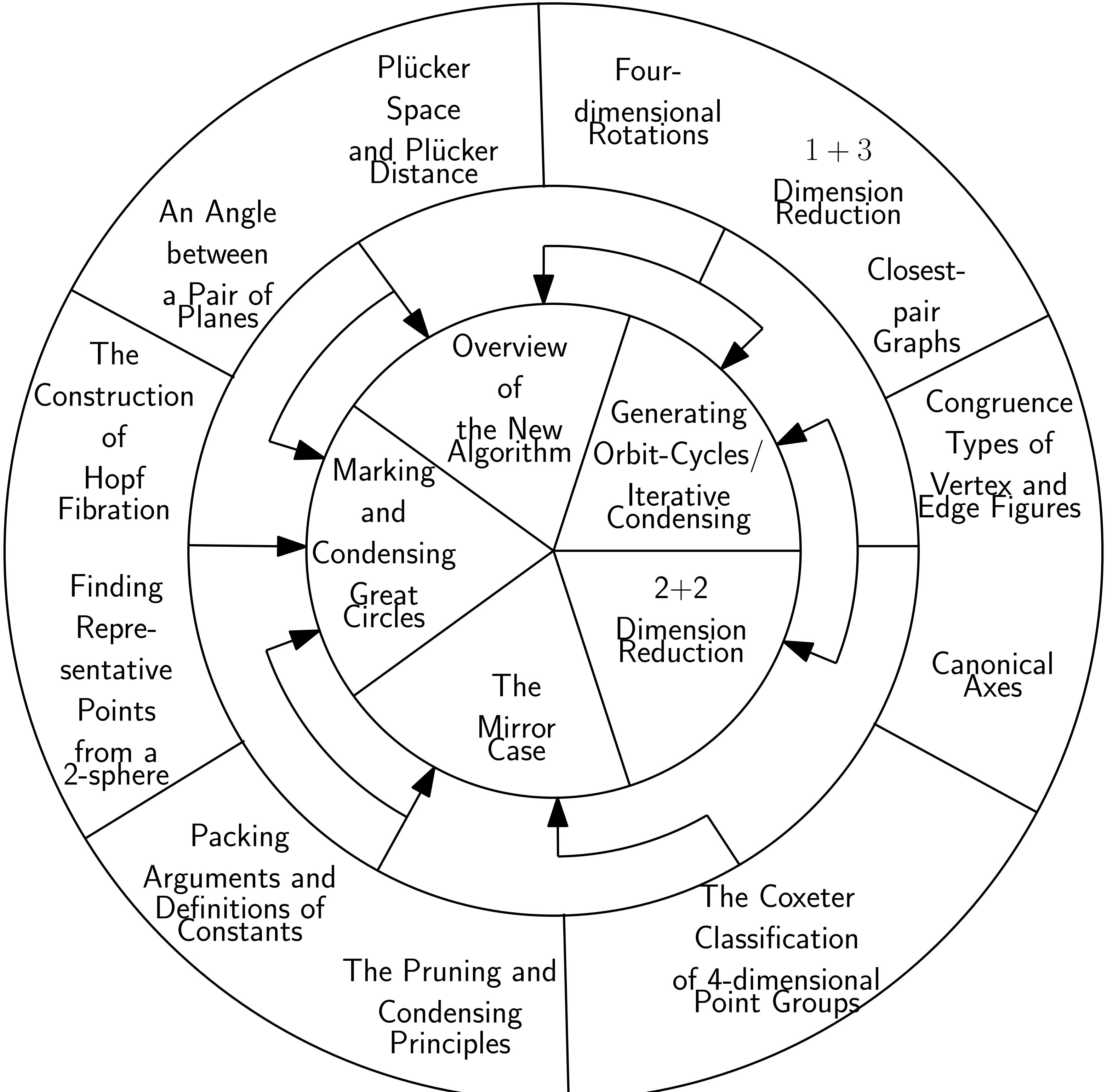
– Problem definition

– Basic principles

– New algorithm

  Overview

  An Example of Modules


Overview of the New Algorithm
Generating Orbit-Cycles/ Iterative Condensing
2+2 Dimension Reduction
The Mirror Case
Marking and Condensing Great Circles
Plücker Space and Plücker Distance
Four-dimensional Rotations
$1+3$ Dimension Reduction
Closest-pair Graphs
Congruence Types of Vertex and Edge Figures
Canonical Axes
The Coxeter Classification of 4-dimensional Point Groups
The Pruning and Condensing Principles
Packing Arguments and Definitions of Constants
Finding Repre-sentative Points from a 2-sphere
The Construction of Hopf Fibration
An Angle between a Pair of Planes

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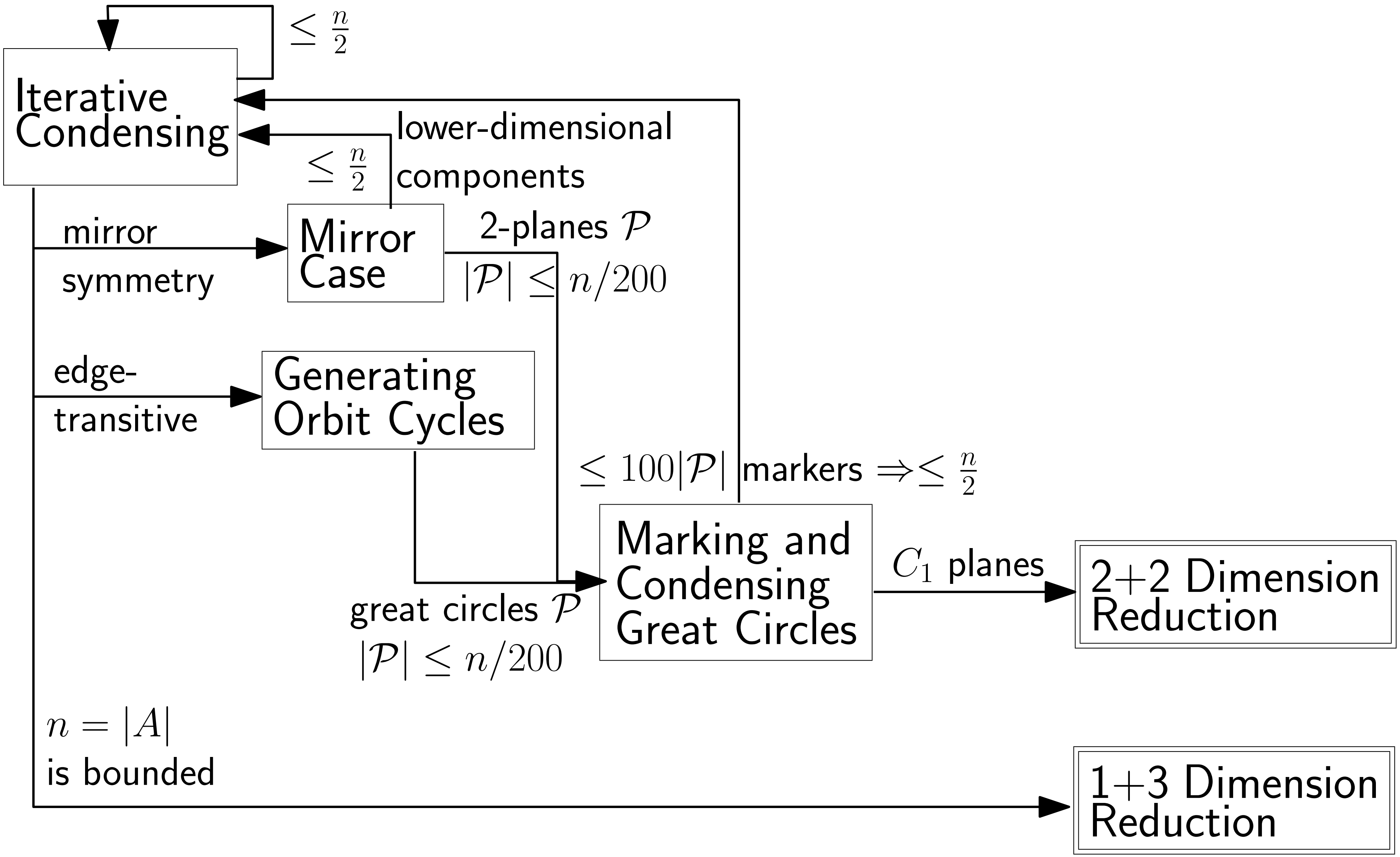

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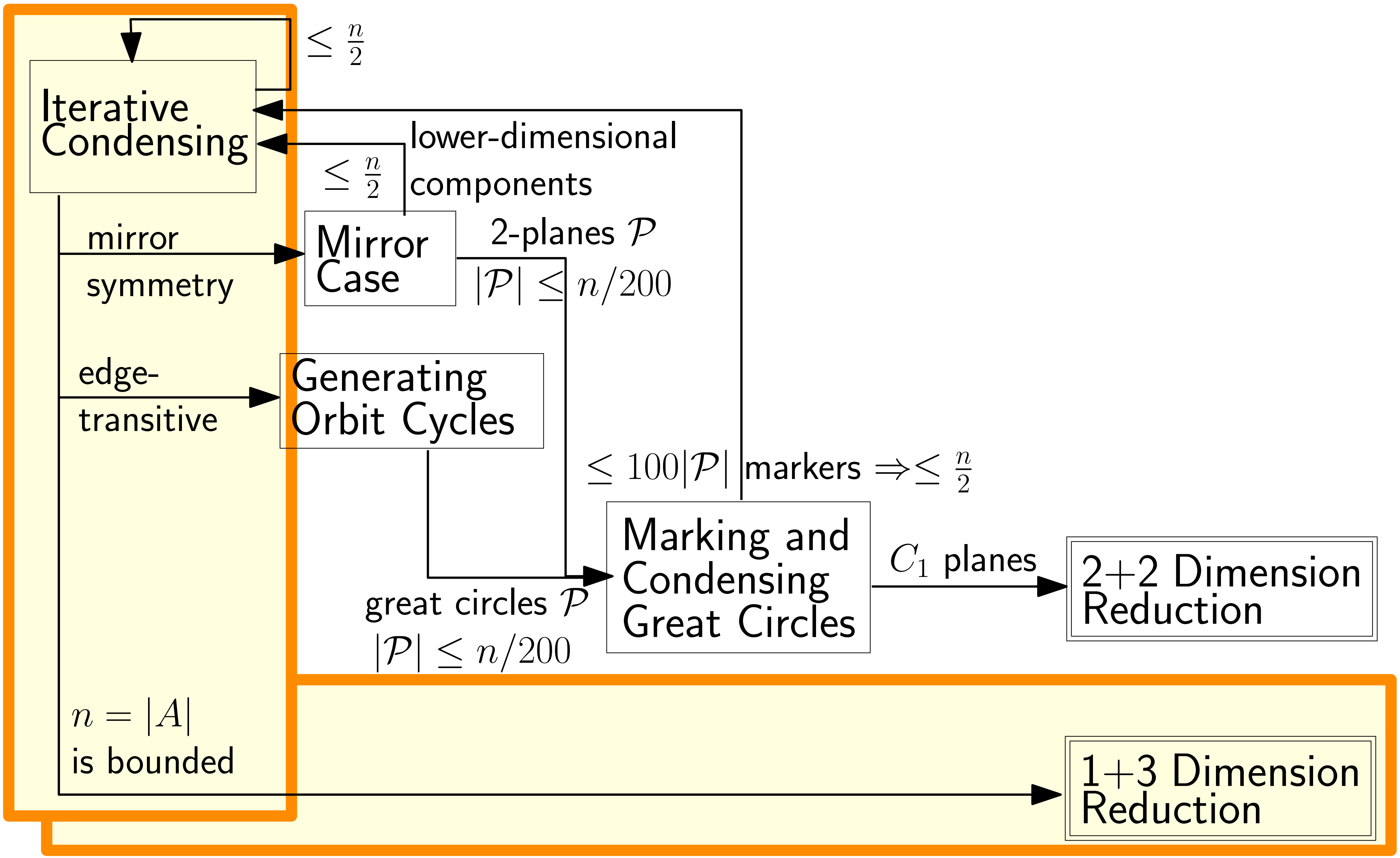

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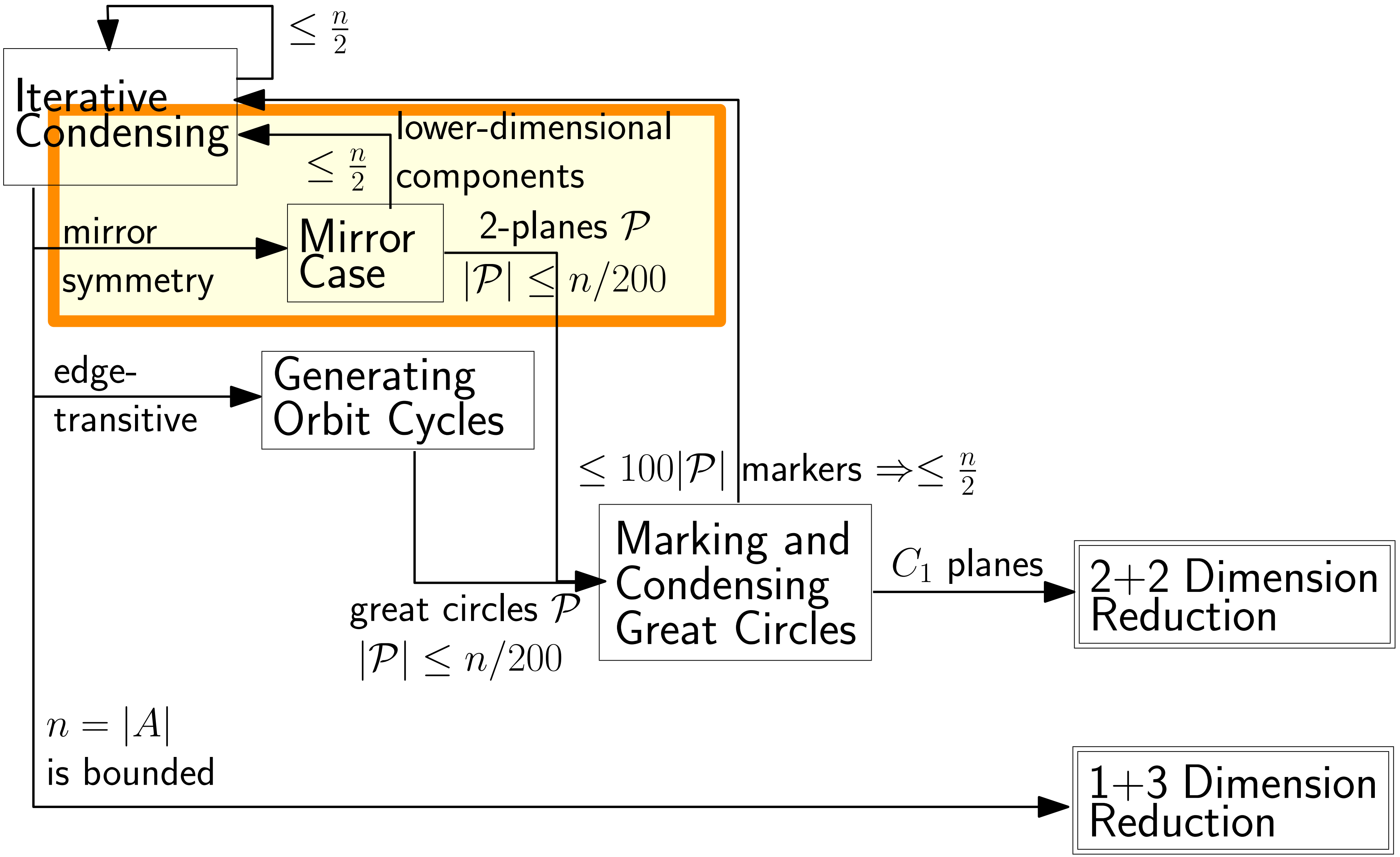

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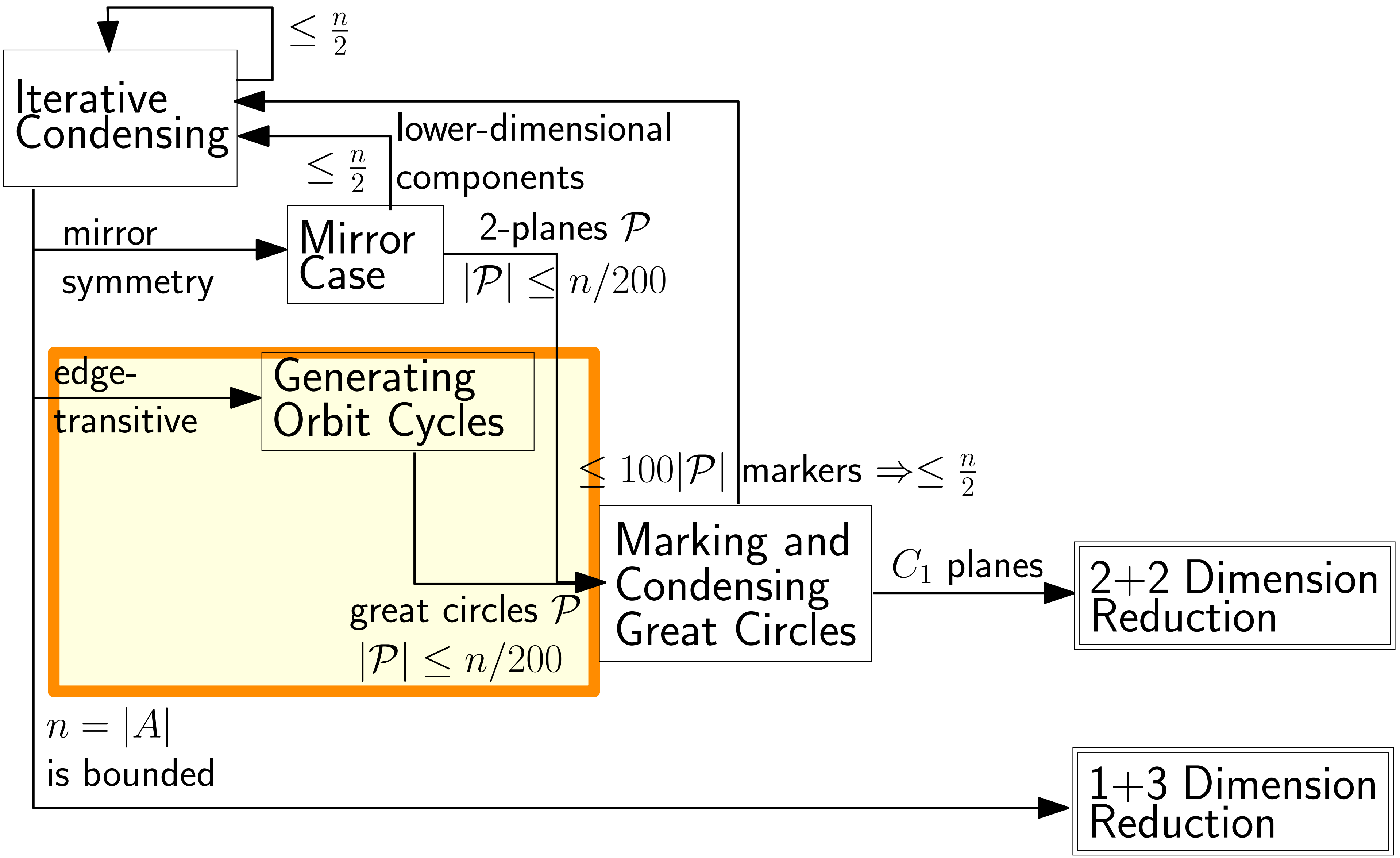
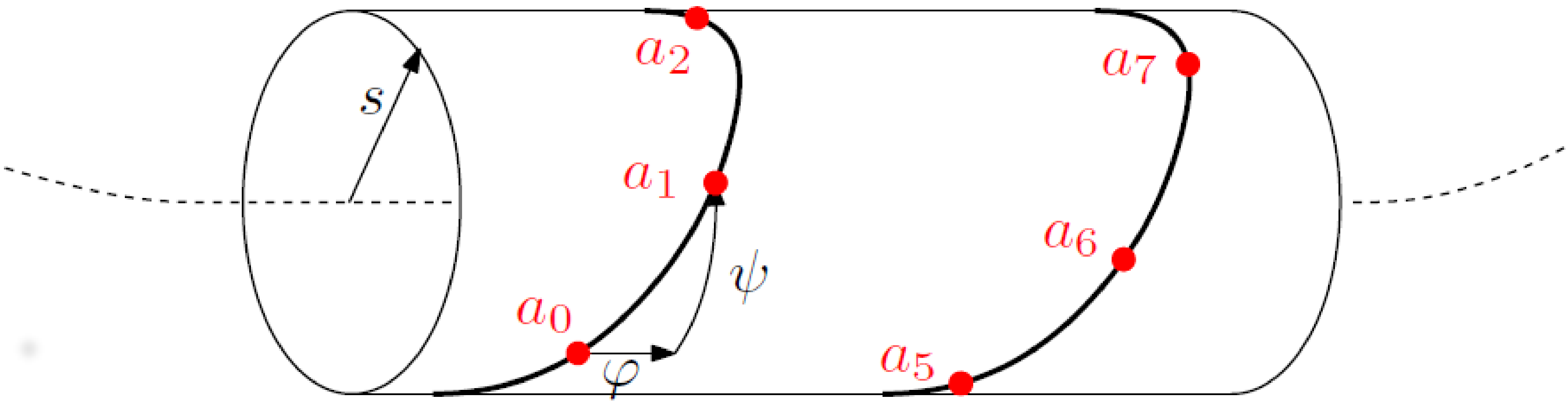
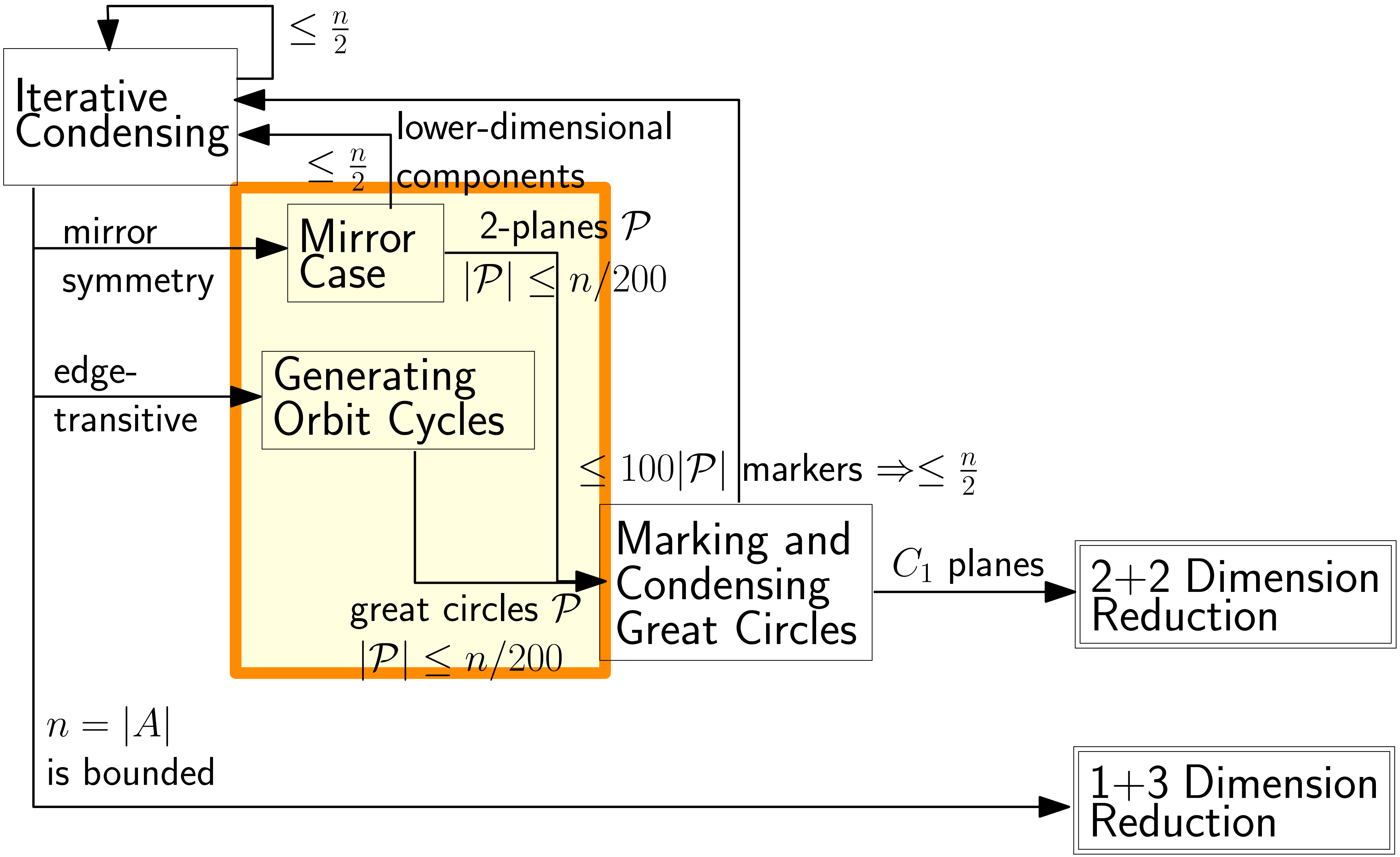




An orbit cycles:
a helix around a great circle

Overview of the New Algorithm



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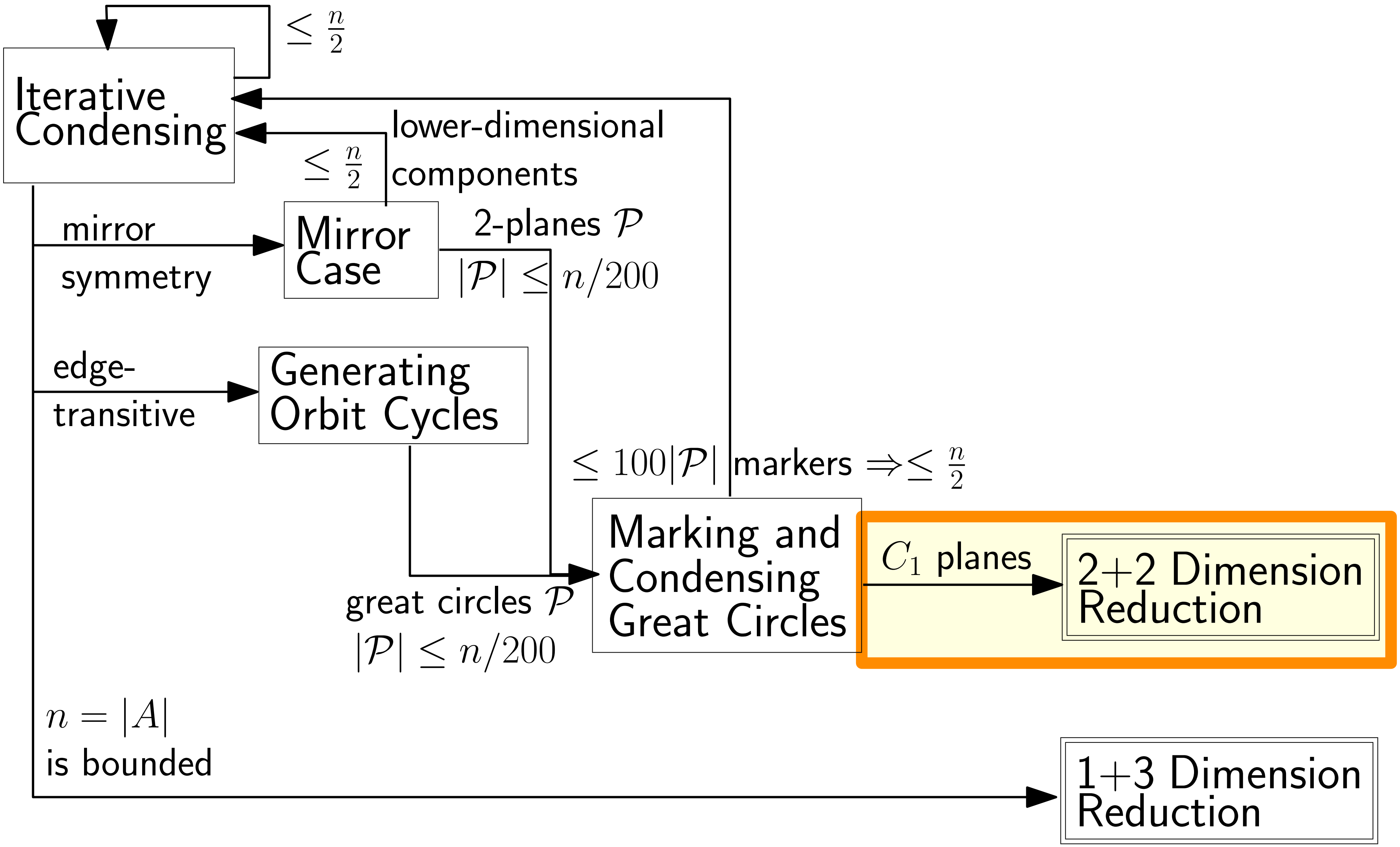

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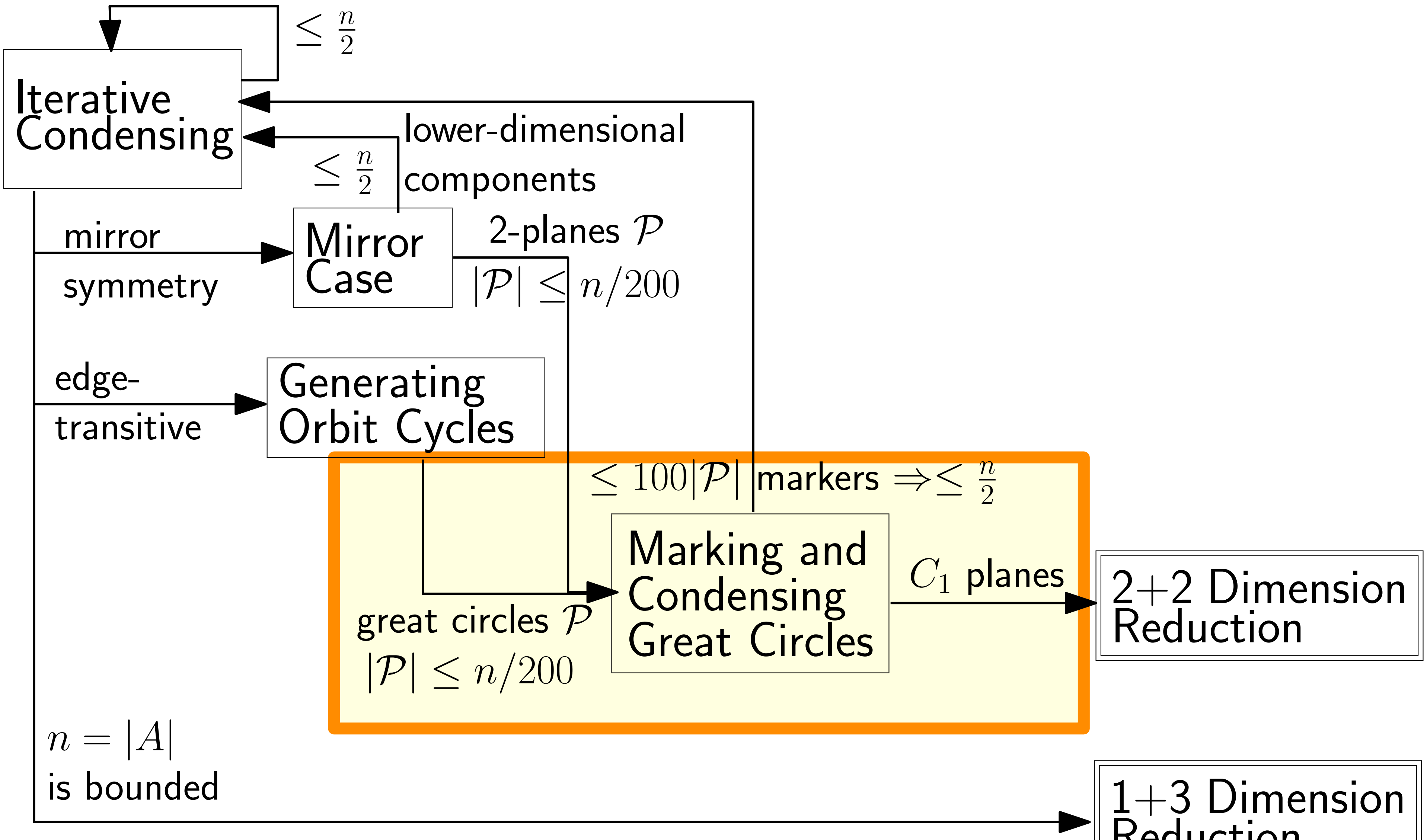
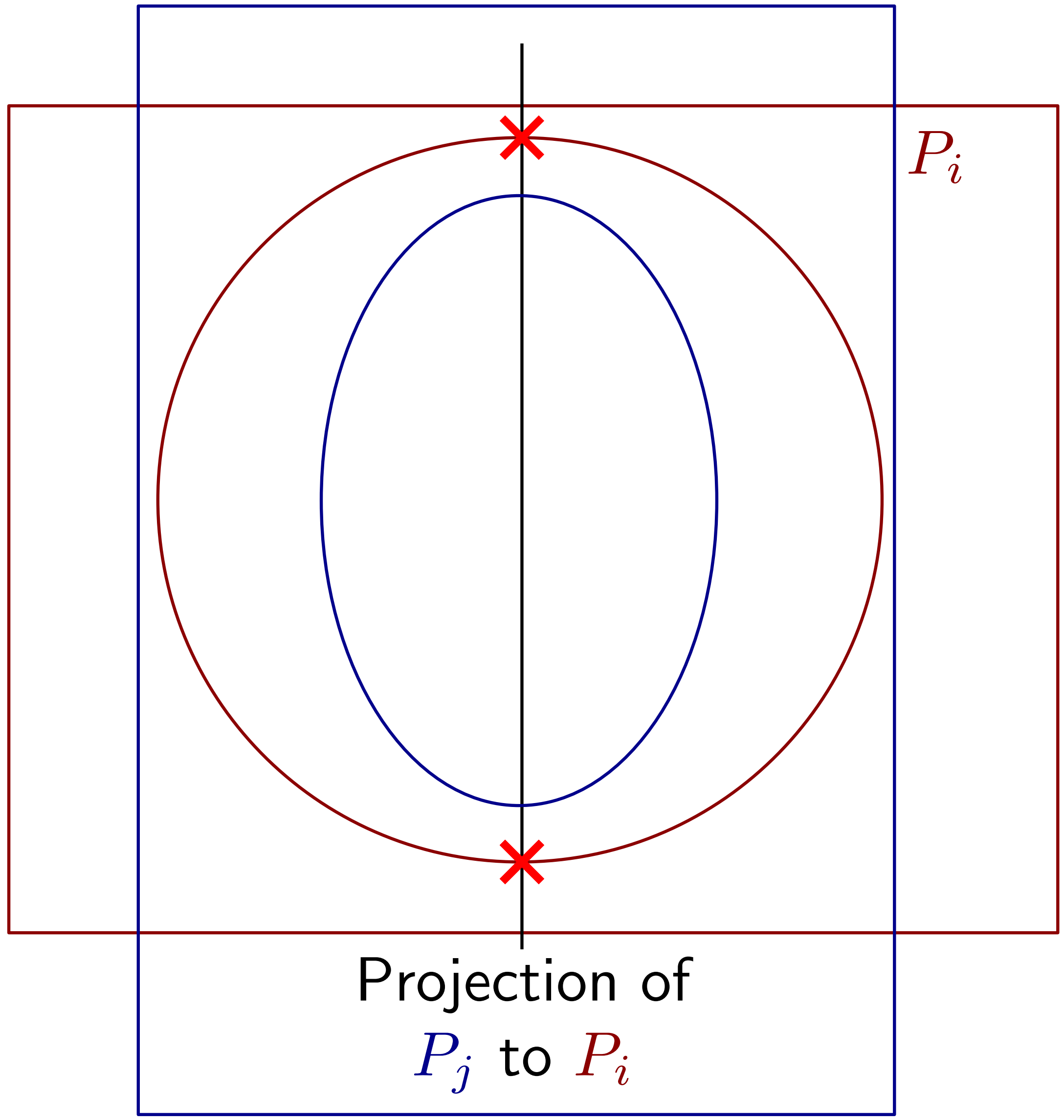


If Projections are Ellipses

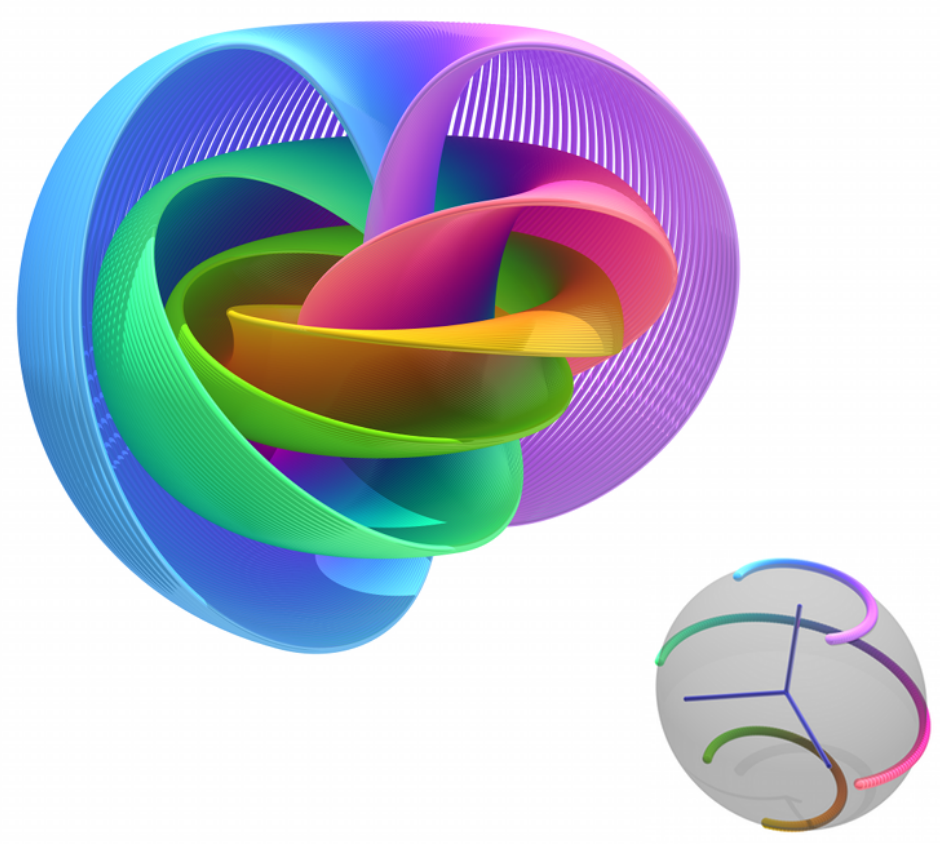


If Projections are Circles

– All the great circles are in the same *Hopf bundle*

$\Rightarrow$ Great circles = Points on a 2-sphere

Condense into at most 12 points on a 2-sphere



Picture by Niles Johnson, The Ohio State University, from Wikipedia

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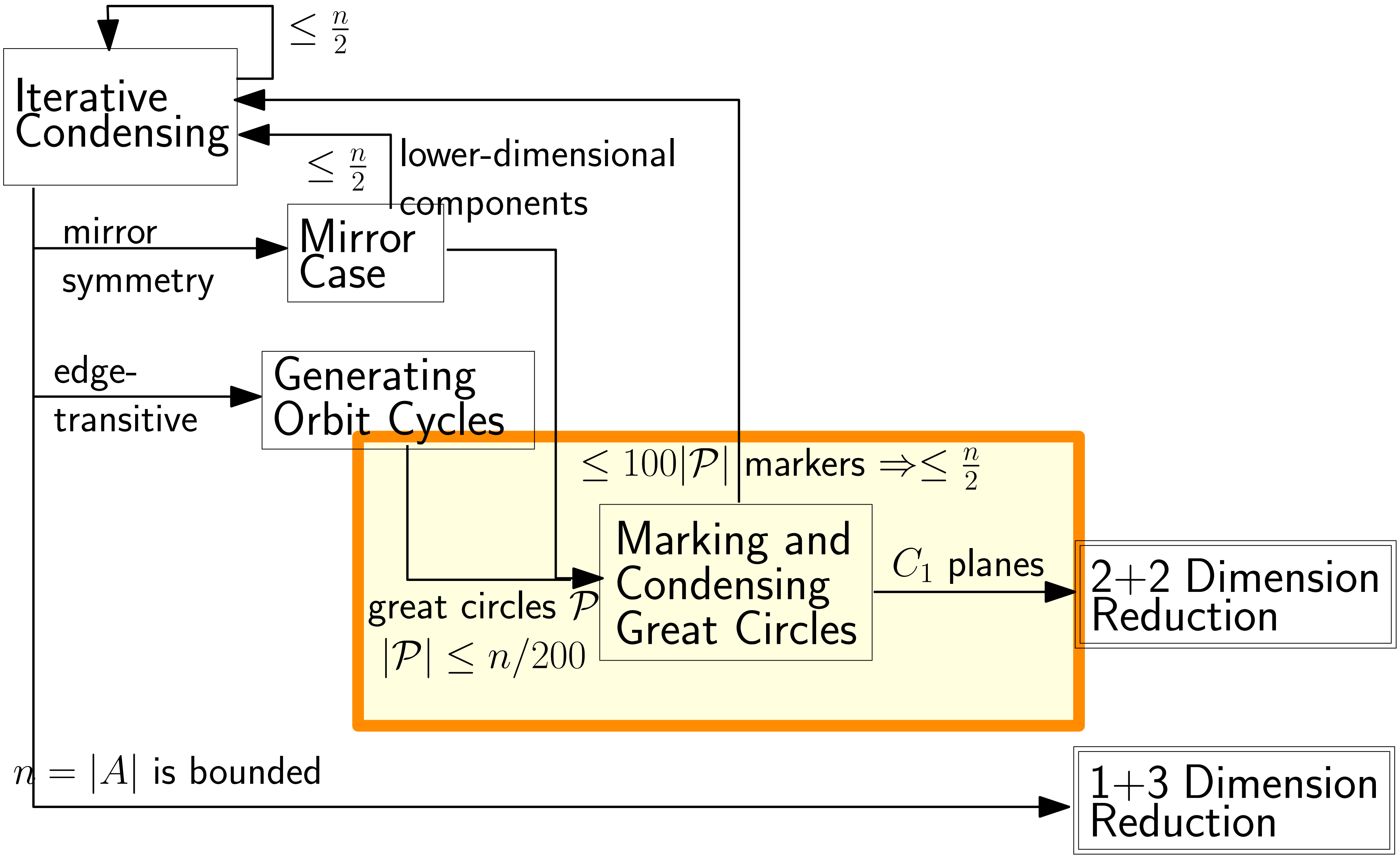
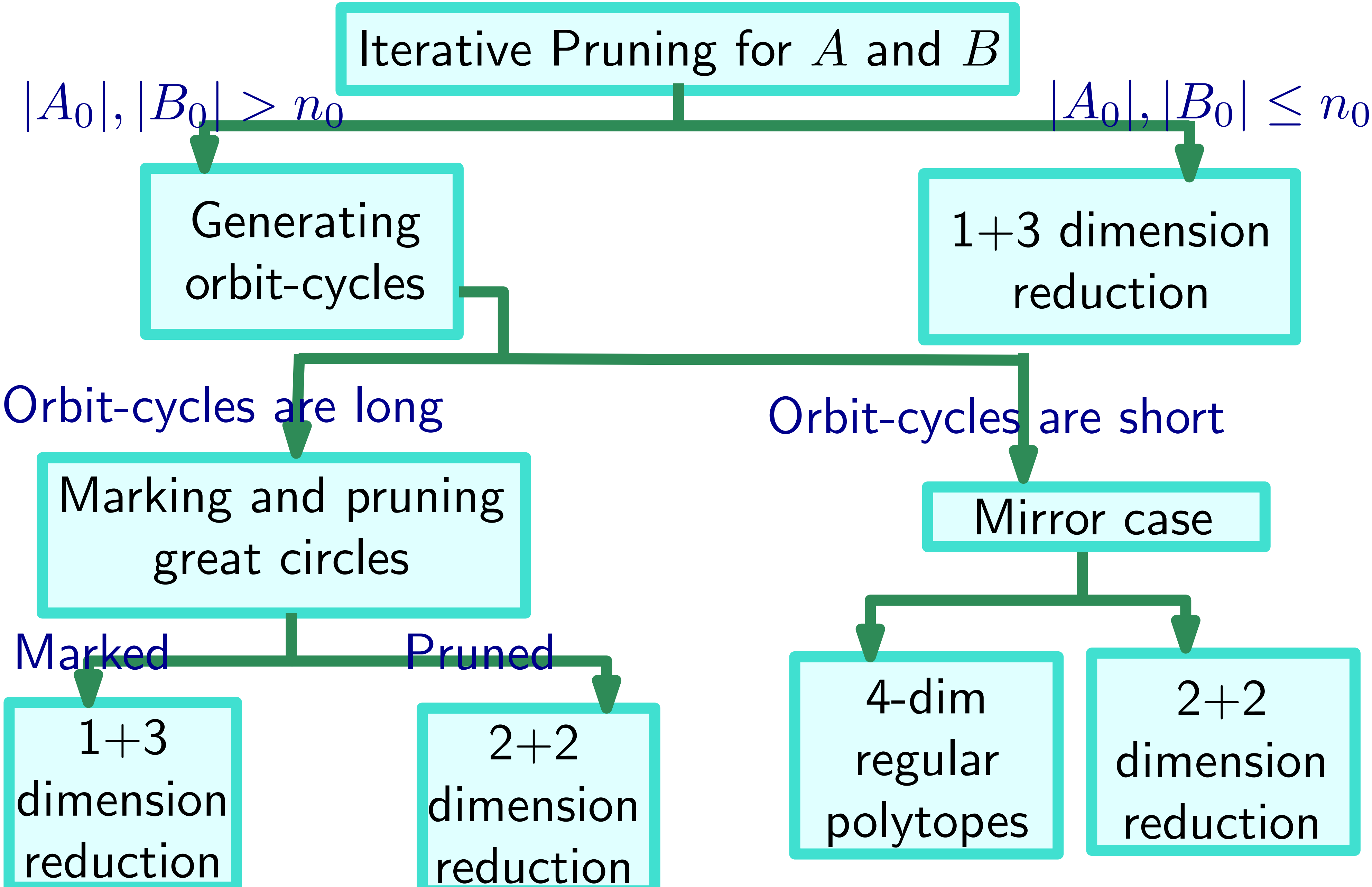


Closer to $O(n \log n)$ for fixed d?

Thank you

New Algorithm

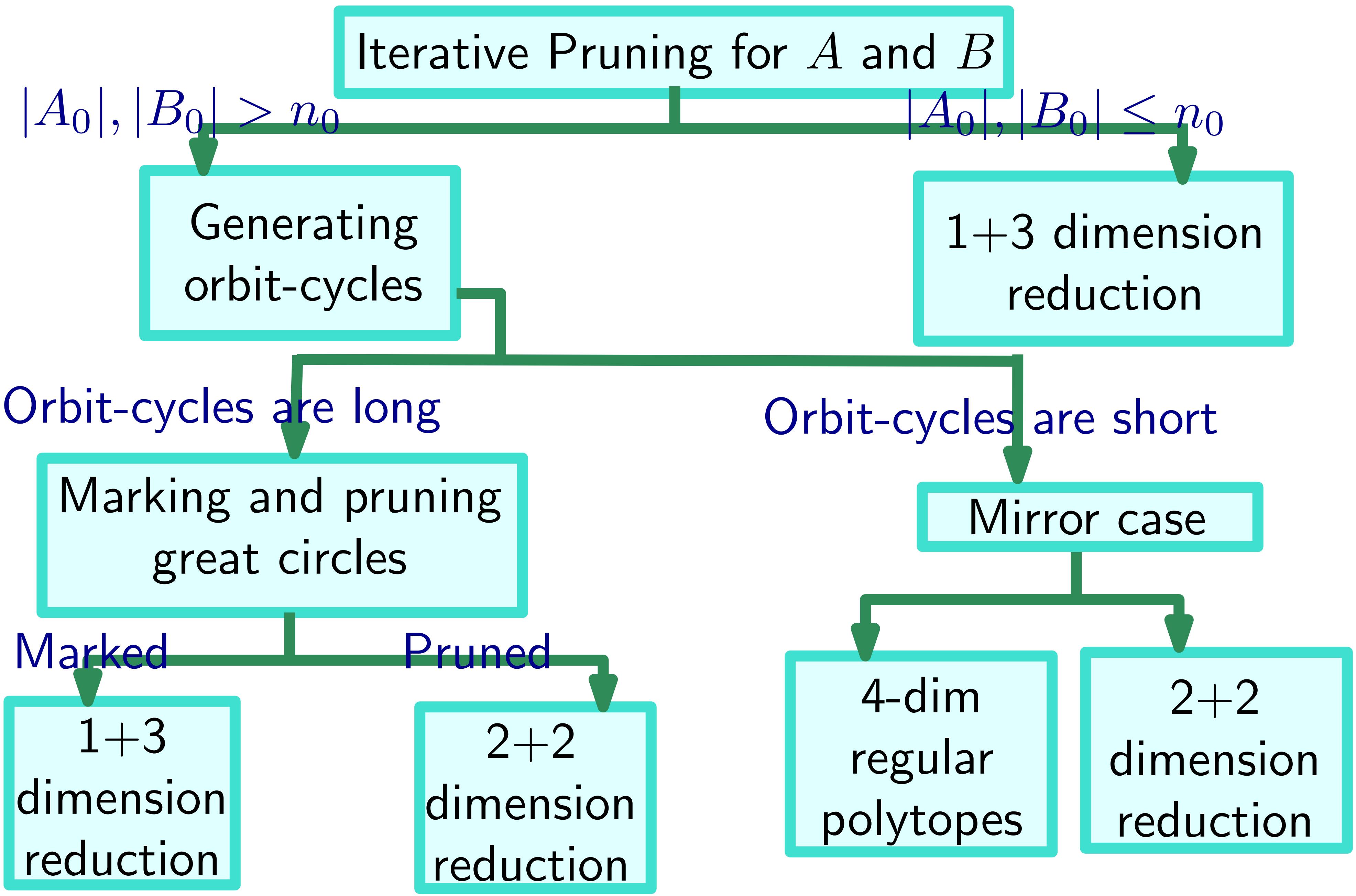


New Algorithm

Iterative Pruning for A and B

- Prune by distance from the origin
- Construct the closest-pair graph
- Prune by congruence type of edge figures
  - Edge figures
    two adjacent vertices and their neighbors

New Algorithm



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  - Dimension Reduction
  - Closest-pair graphs
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    - Generating orbit-cycles
    - Marking and pruning great circles
    - 2+2 dimension reduction
    - Mirror case

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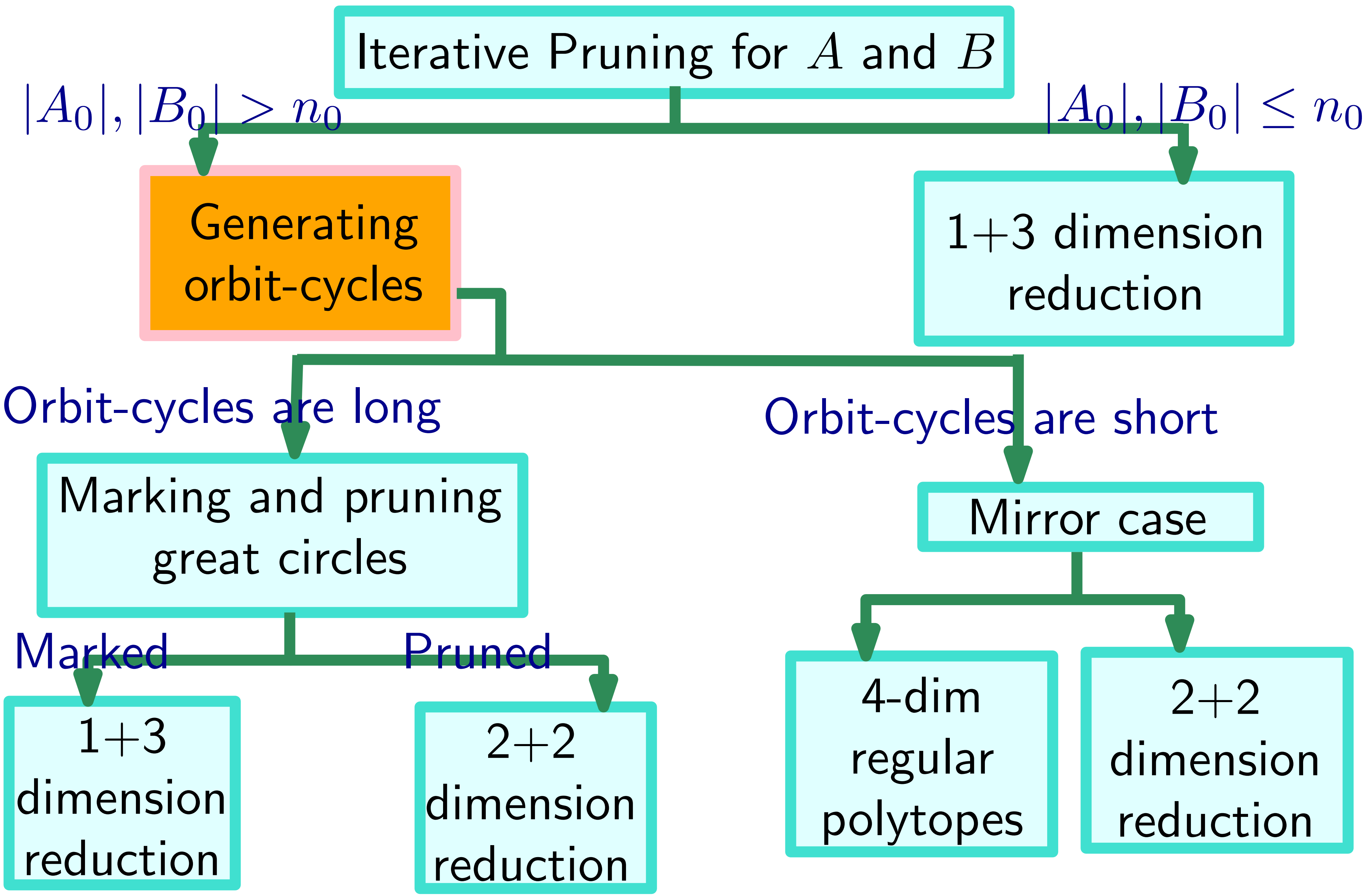


Generating Orbit-Cycles

A_0 is on the 3-sphere
G: the closest-pair graph of A_0
D: the directed version of G.

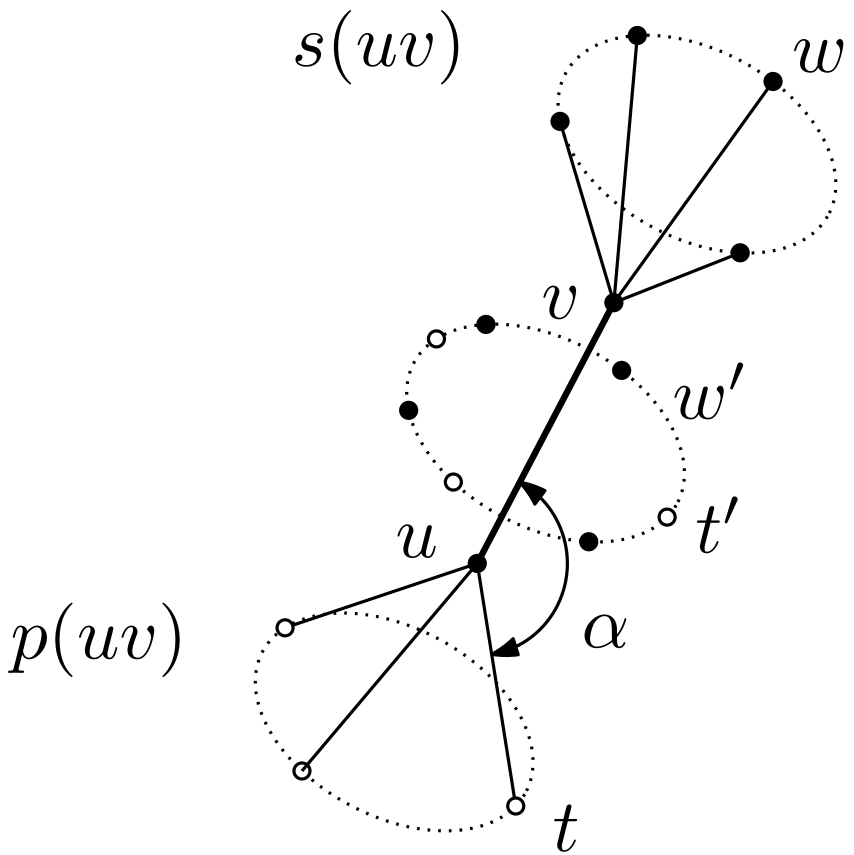
prune arcs in D
by congruence type of
edge figures and
predecessor-successor figures

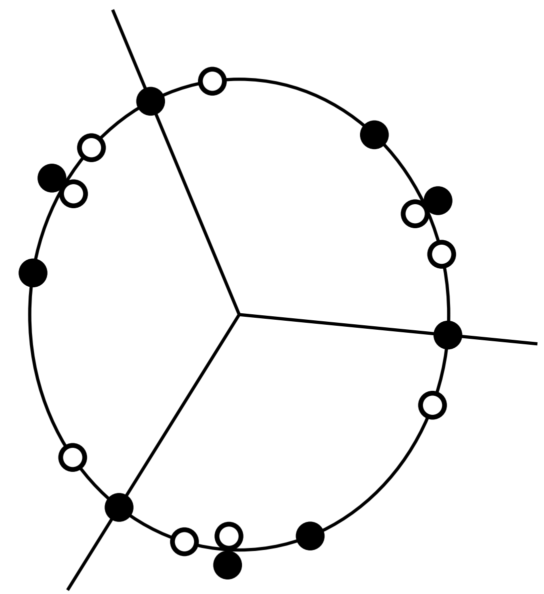
prune points in A_0
by indegrees and
outdegrees

Mirror case :
all the edge figures are
mirror-symmetric

Edge-transitive case :
successors and predecessors
are regular k-gons
$\Rightarrow$ *canonical orbit-cycles*

Predecessor-Successor Figures





$$s(uv) = \{vw : vw \in E, \angle uvw = \alpha\}$$

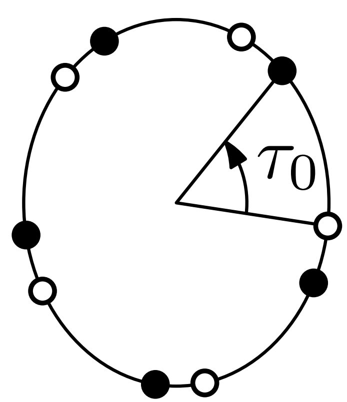
Canonical Orbit-Cycles

Fix $t_0u_0v_0w_0$.

For edge-transitive cases :

for every $a_3 \in s(a_1a_2)$,
there exists a unique $a_4 \in s(a_2a_3)$
such that $a_1a_2a_3a_4$ and $t_0u_0v_0w_0$ are congruent.

$R[a_1a_2a_3] = a_2a_3a_4$ is uniquely determined



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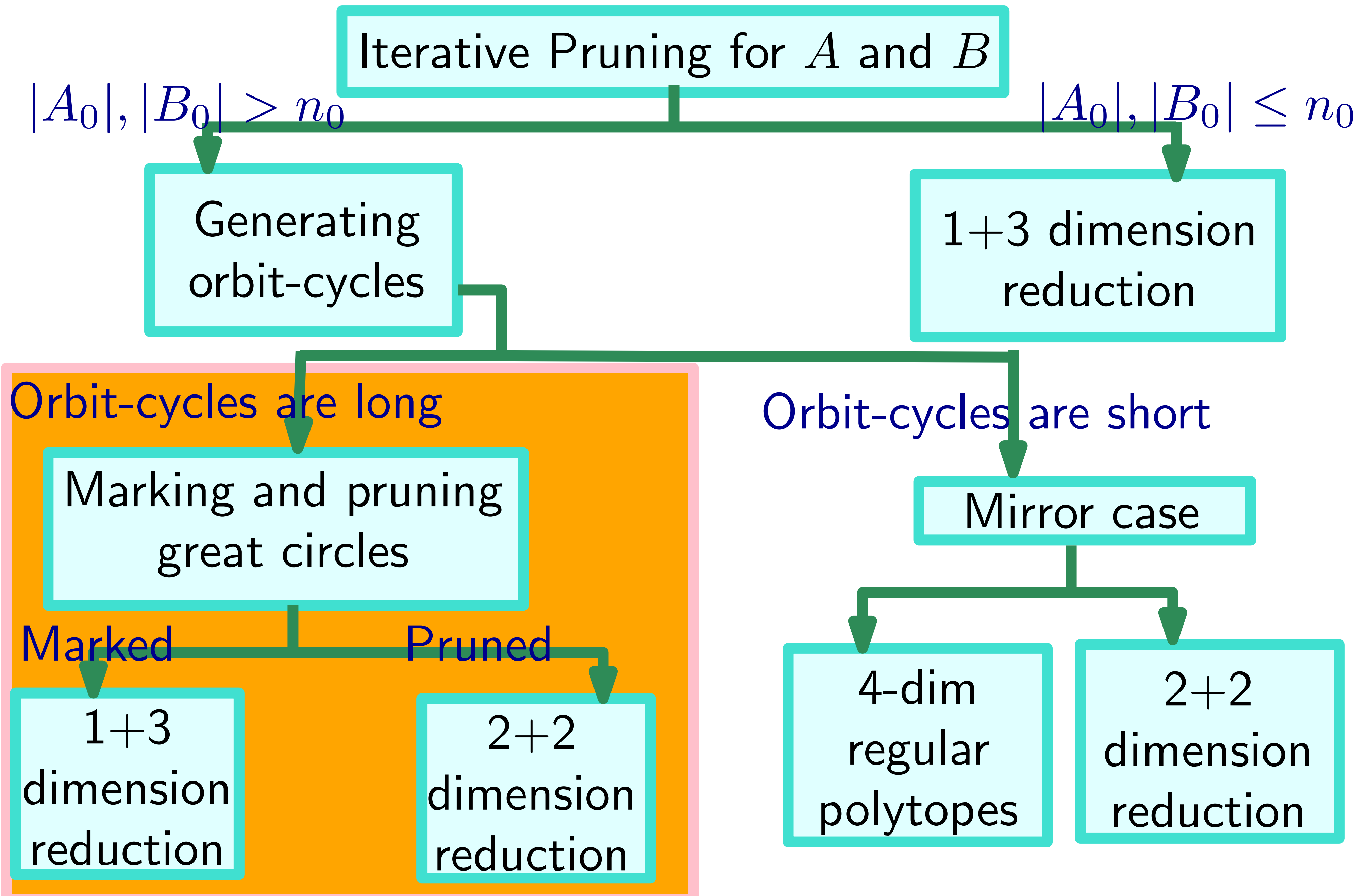
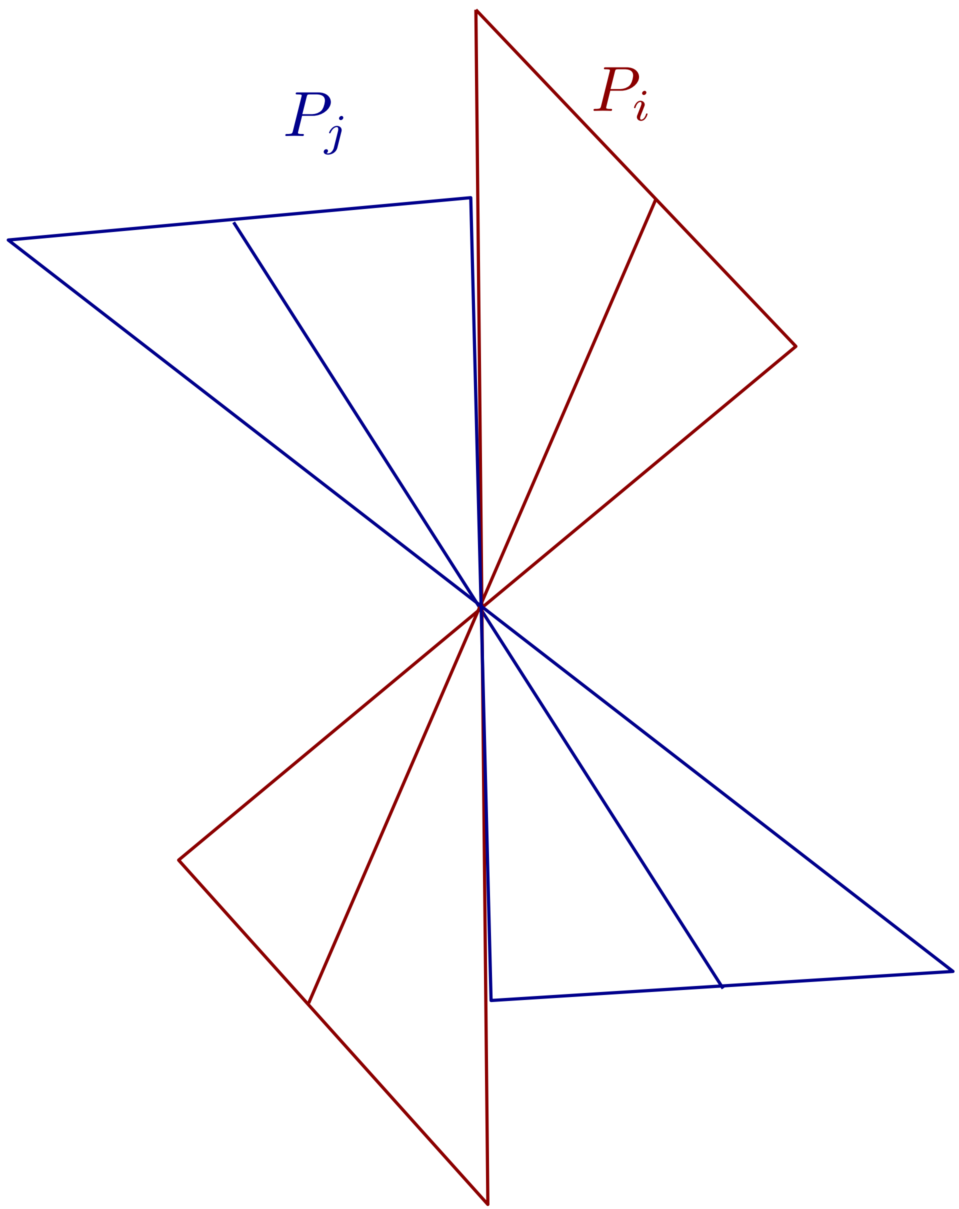
New Algorithm



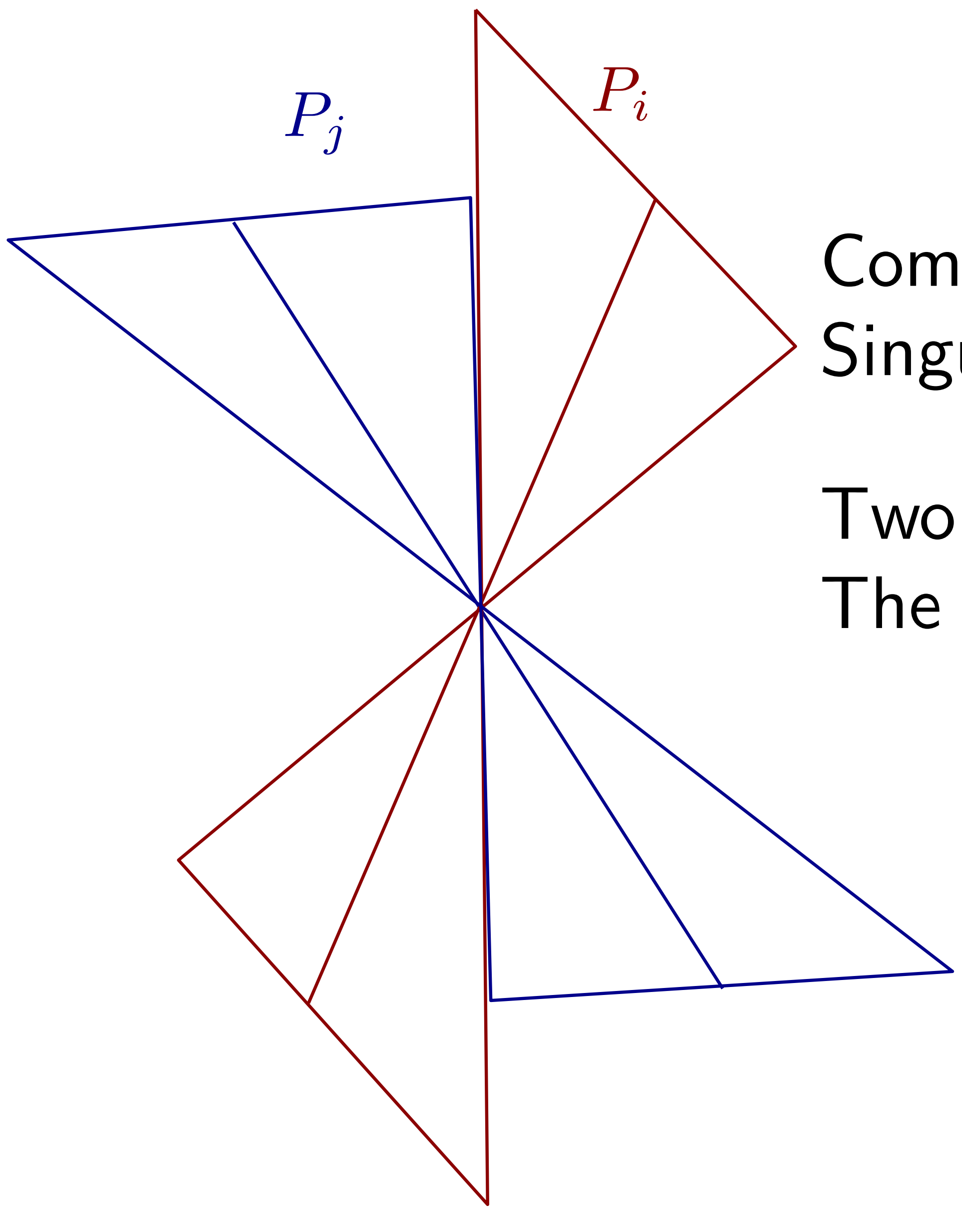
Angle between Two Planes



$\cos\alpha$:
the minor axis of
the projection
of a unit circle in P_i to P_j.

$\cos\beta$: the major axis of
the same ellipse

Angle between Two Planes

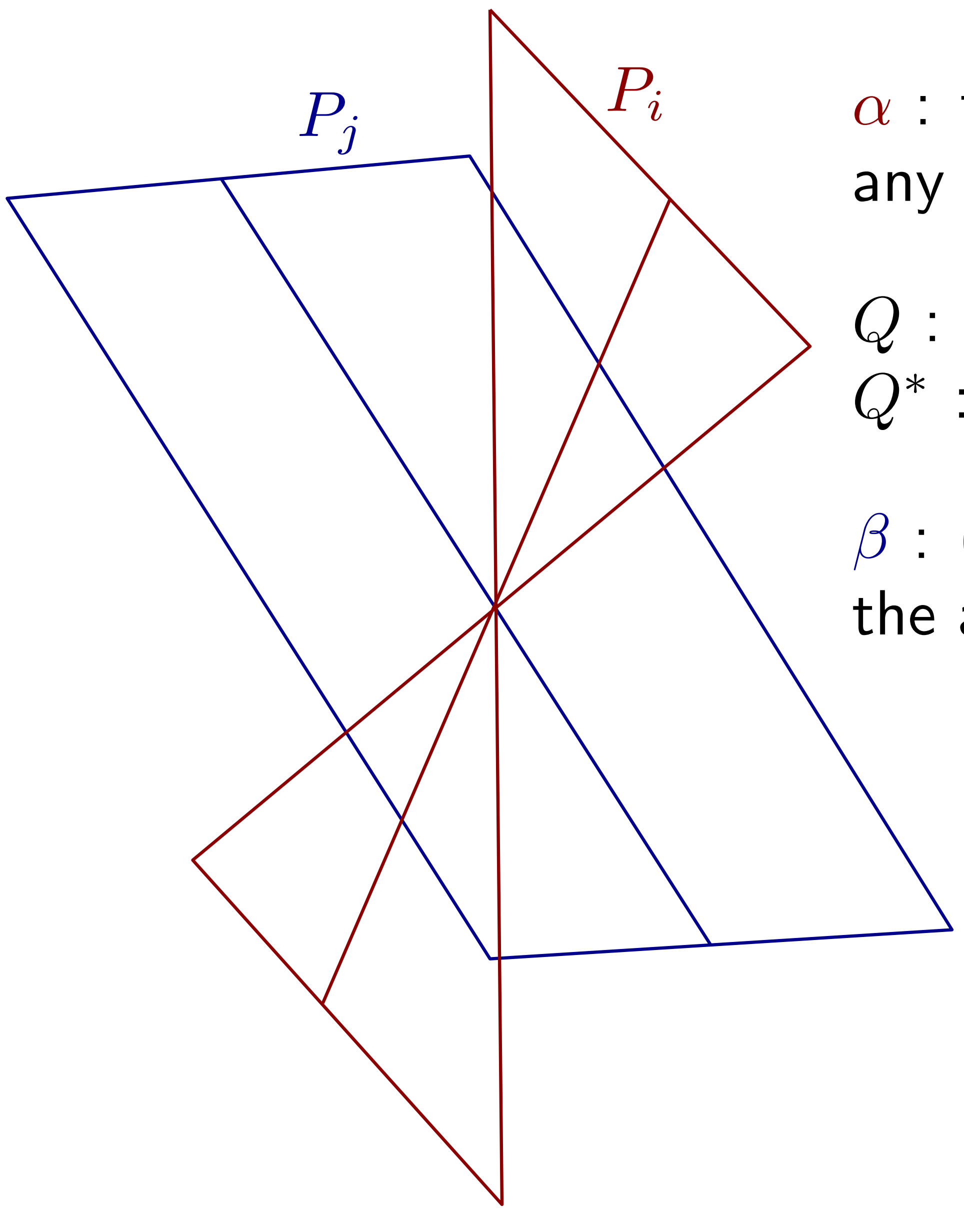


Computable by using
Singular Value Decomposition(SVD)

Two orthogonal planes : $(\frac{\pi}{2}, \frac{\pi}{2})$
The same plane : (0,0)

$\Rightarrow$ Prune by angles
(α, β)

Angle between Two Planes

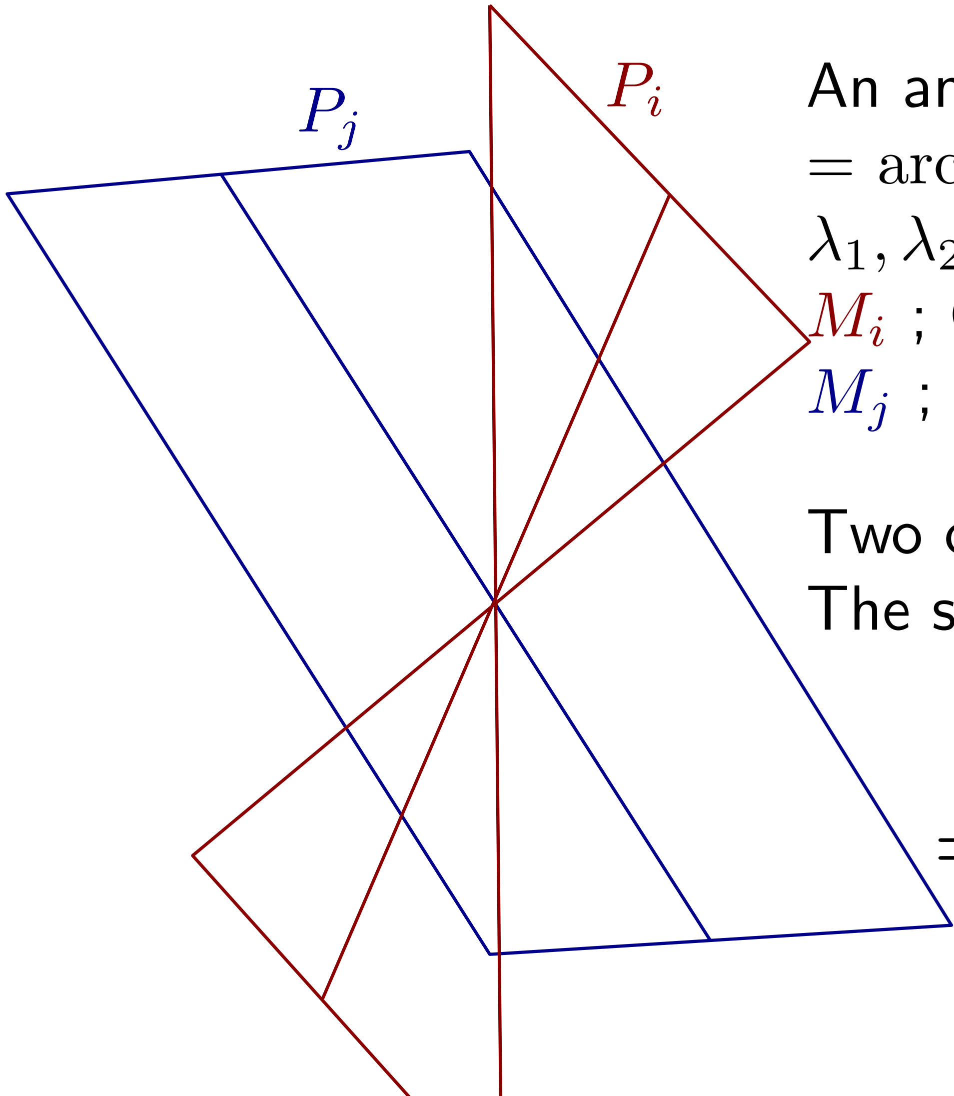


α : the minimum angle between any two half lines ℓ_i and ℓ_j in P_i and P_j.

Q : the plane containing ℓ_i and ℓ_j.
Q^* : the plane orthogonal to Q.

β : (minimum orthogonal angle to α) the angle of P_i and P_j defined by Q^*

Angle between Two Planes

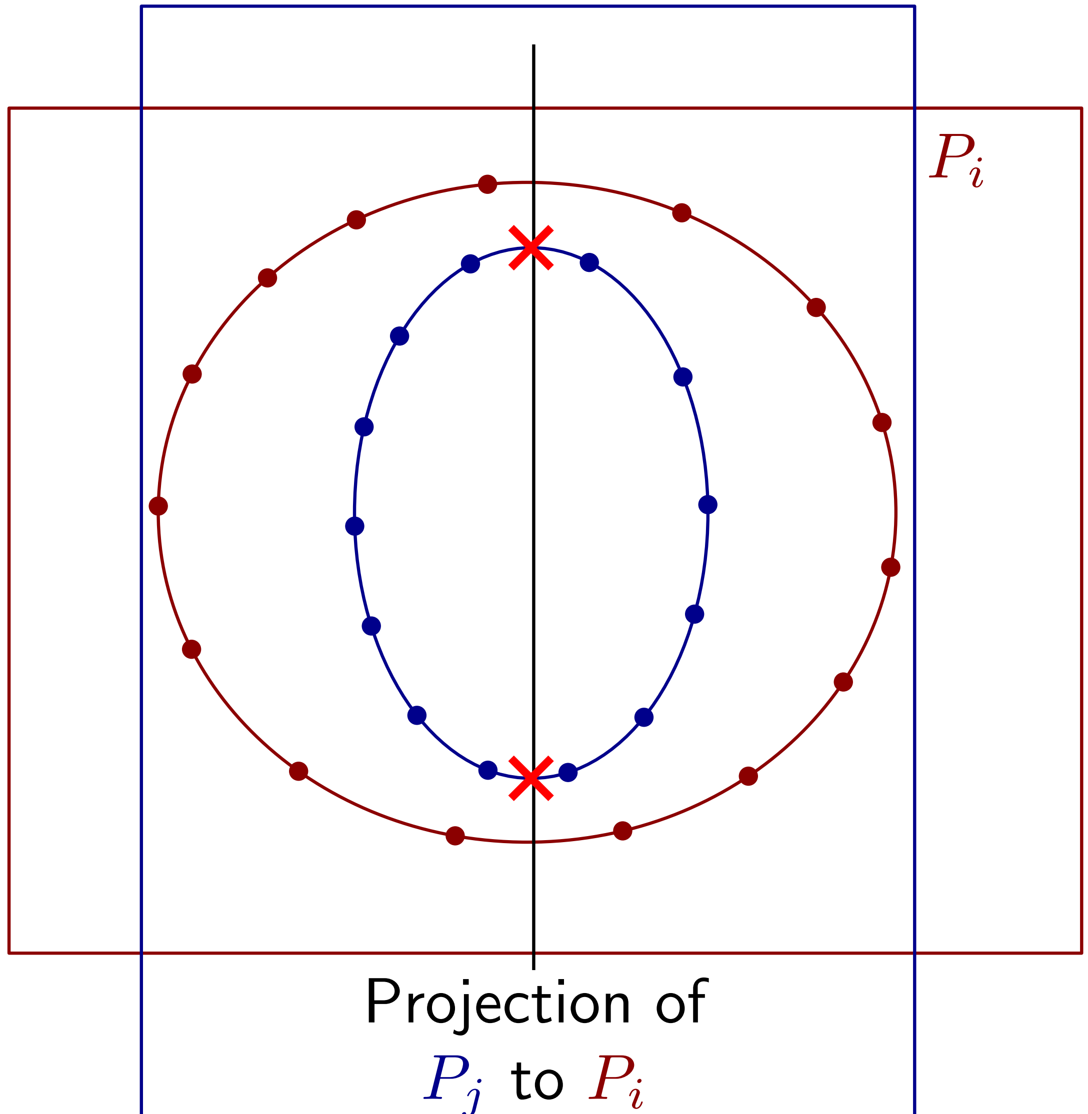


An angle (α, β) between P_i and P_j
$= \arccos \lambda_1, \arccos \lambda_2$
λ_1, λ_2: two largest singular values of $M_i M_j$
M_i ; Orthonormal basis of P_i,
M_j ; Orthonormal basis of P_j,

Two orthogonal planes : $(\frac{\pi}{2}, \frac{\pi}{2})$
The same plane : (0,0)

$\Rightarrow$ Prune by angles
(α, β)

$$\alpha \neq \beta$$

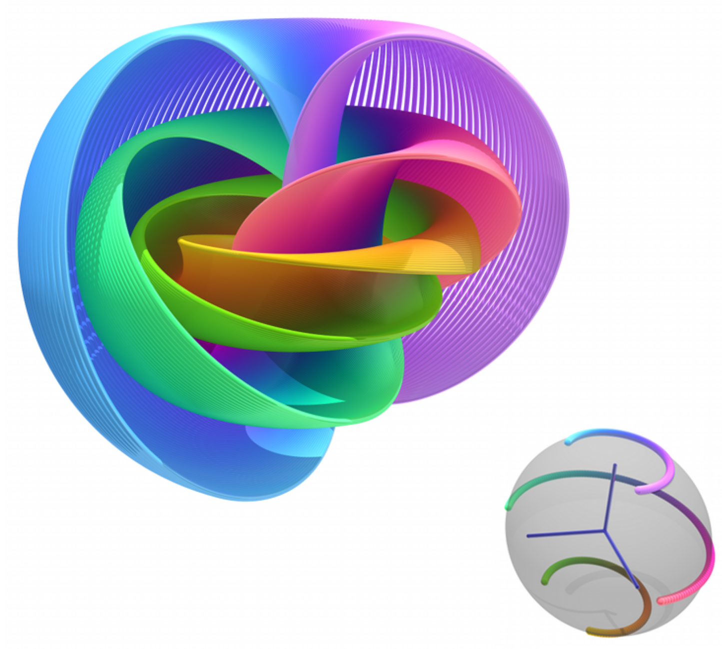


$$\alpha = \beta$$

- All the great circles are in the same *Hopf bundle*

- A set of great circles decides one *Hopf fibration*

$\Rightarrow$ Great circles = Points in a 2-sphere
Prune on the 2-sphere



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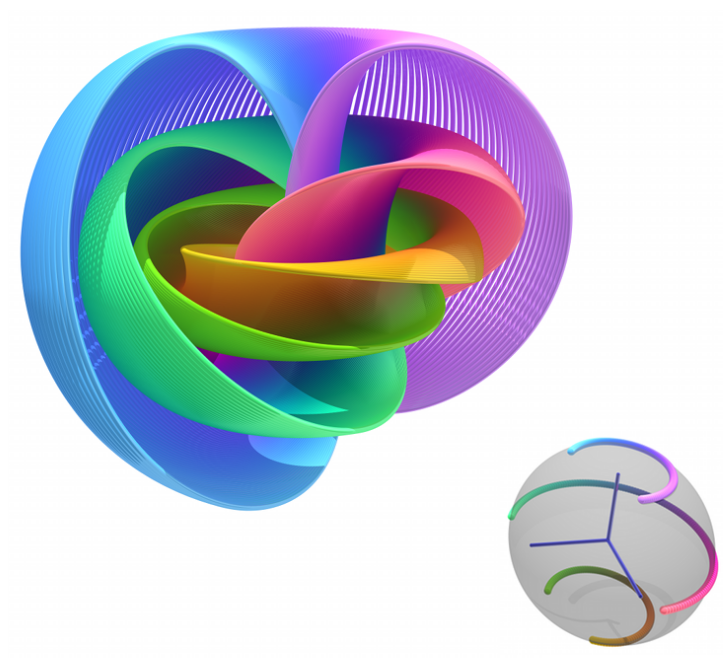

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Marking and Condensing Great Circles

If the pruned pairs of great circles have angles (α, β) such that

- $\alpha \neq \beta$
- $\alpha = \beta$ with the different chirality from the previous pairs
- $\alpha = \beta$ with the same chirality from the previous pairs

Marking and Condensing Great Circles

If the pruned pairs of great circles have angles (α, β) such that

- $\alpha \neq \beta$
- $\alpha = \beta$ with the different chirality from the previous pairs

Marking on Great Circles

- $\alpha = \beta$ with the same chirality from the previous pairs

Condensing by Using 2-Sphere

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  - Closest-pair graphs
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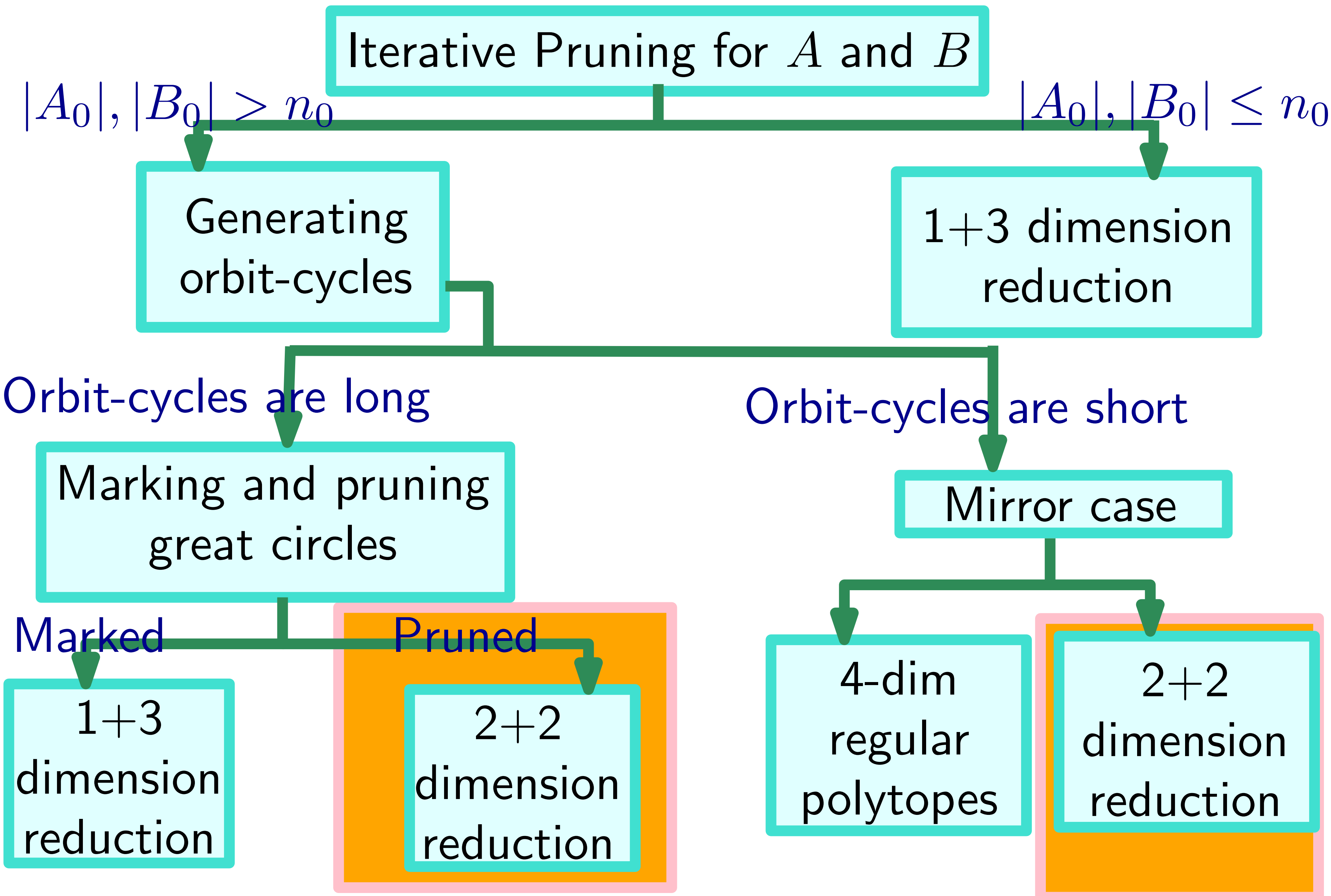
New Algorithm



2+2 dimension reduction

Fix a plane P and its orthogonal plane Q
as invariant subspaces.

Project to either
– a unit circle C_P in P or
– a unit circle C_Q in Q or
– a flat torus $C_P \times C_Q$

$\Rightarrow$ *translation* of labeled points in a *flat torus* $\mathbb{S}^1 \times \mathbb{S}^1$

Canonical Set Procedures

A map from A to A' for a subgroup Θ of symmetries in a space

1. Symmetries are preserved : $\mathrm{Sym}_\Theta(A') = \mathrm{Sym}_\Theta(A)$.
2. $\mathrm{Sym}_\Theta(A')$ acts transitively on A' :
for every $p, q \in A'$, there is $R \in \mathrm{Sym}_\Theta(A')$ s.t $R : p \mapsto q$.
3. A' is canonical :
if $RA = B$, $B' = RA'$ where $R \in \Theta$.

If $R \in \Theta : p \mapsto q, p \in A', q \in B'$, $RA = B$

for translation in $\mathbb{S}^1$: lexicographically smallest axis
for translation in $\mathbb{S}^1 \times \mathbb{S}^1$: prune by Delaunay triangulations and Voronoi regions in the plane

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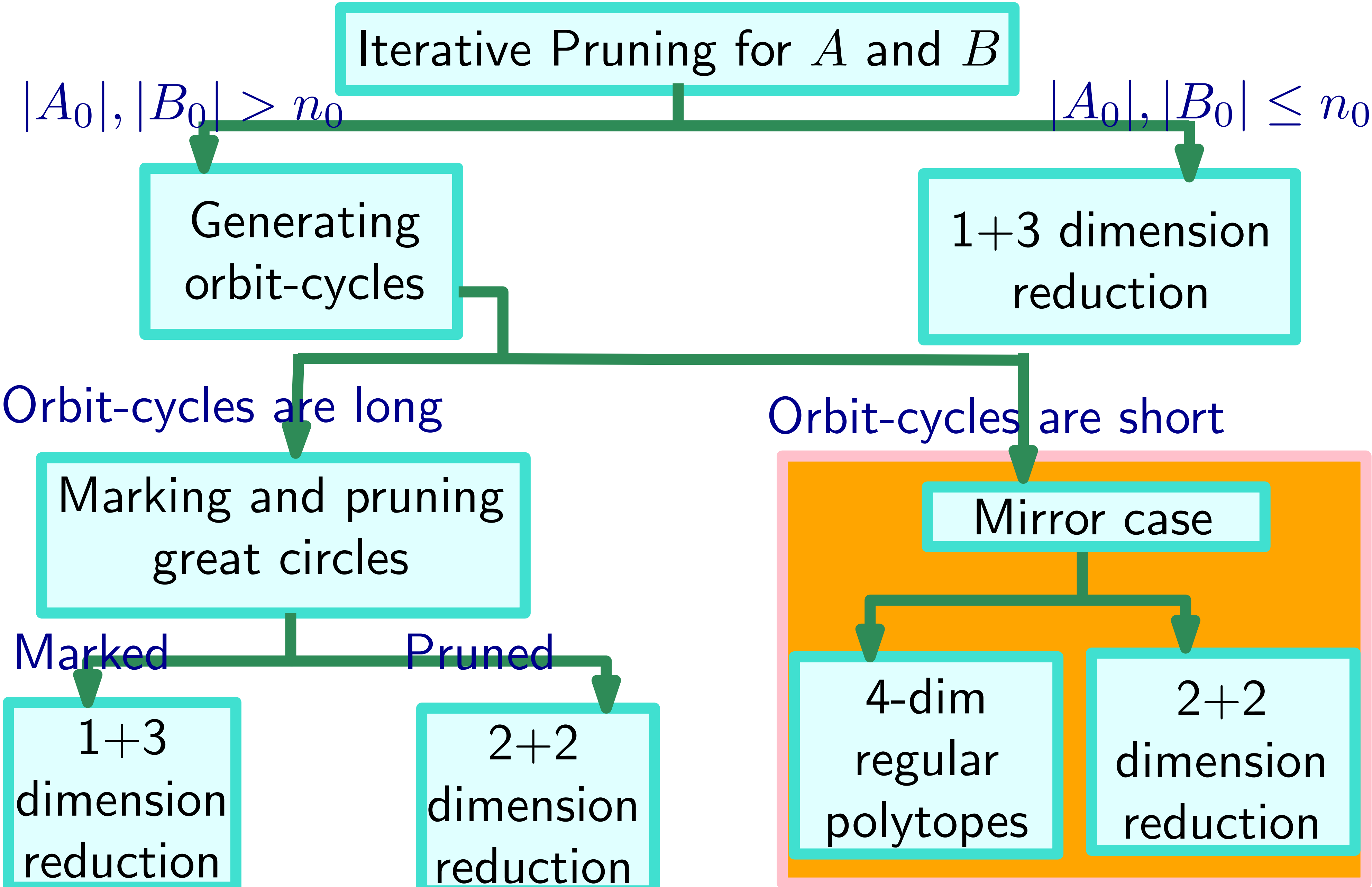


Mirror Case

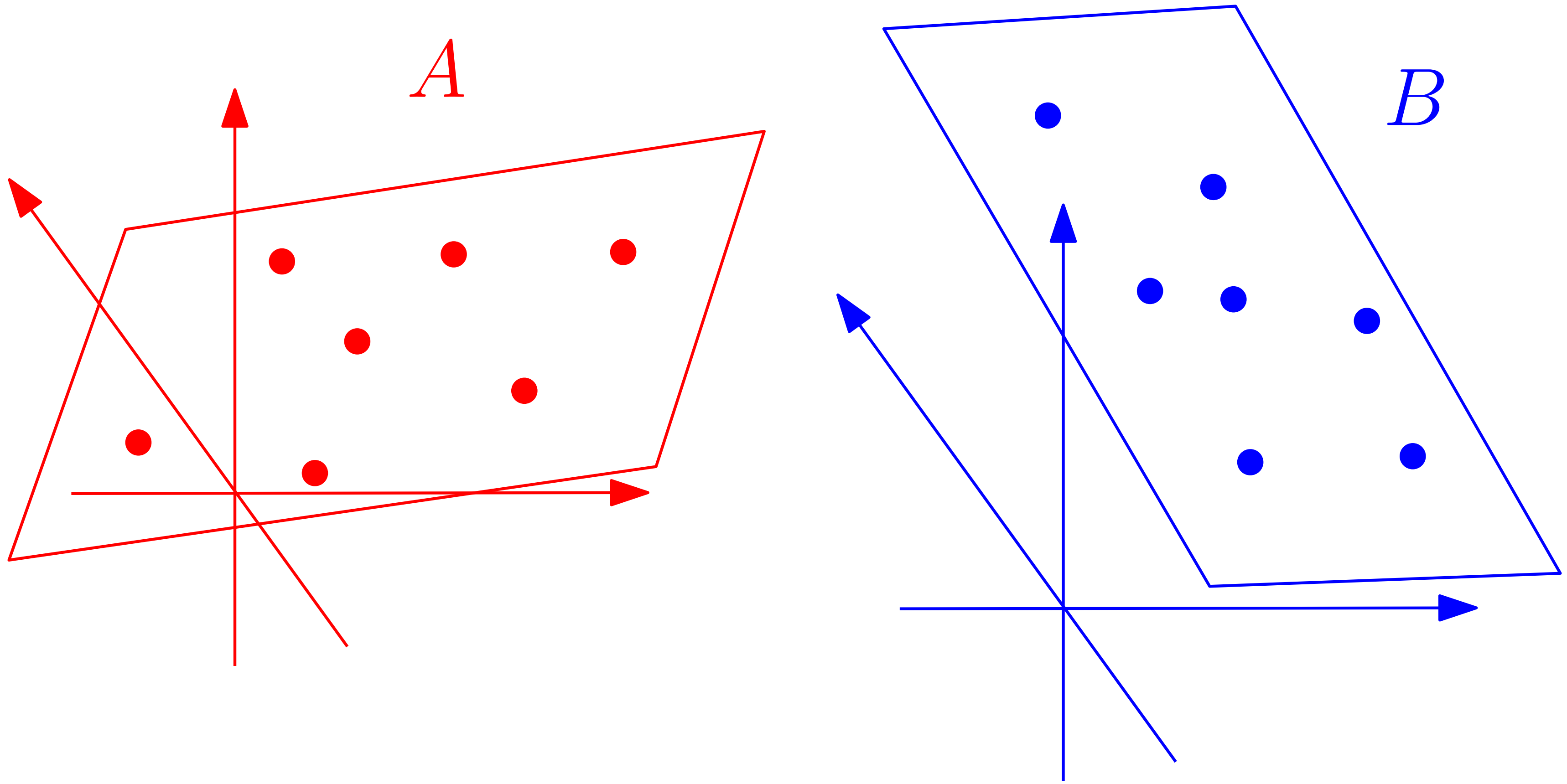
two main cases :

4-dimensional regular polytopes
$\rightarrow$ the number of vertices is bounded
$\Rightarrow$ 1+3 dimension reduction

the Cartesian product of
regular polygons in orthogonal planes
$\Rightarrow$ 2+2 dimension reduction

Exact Congruence Testing Problem

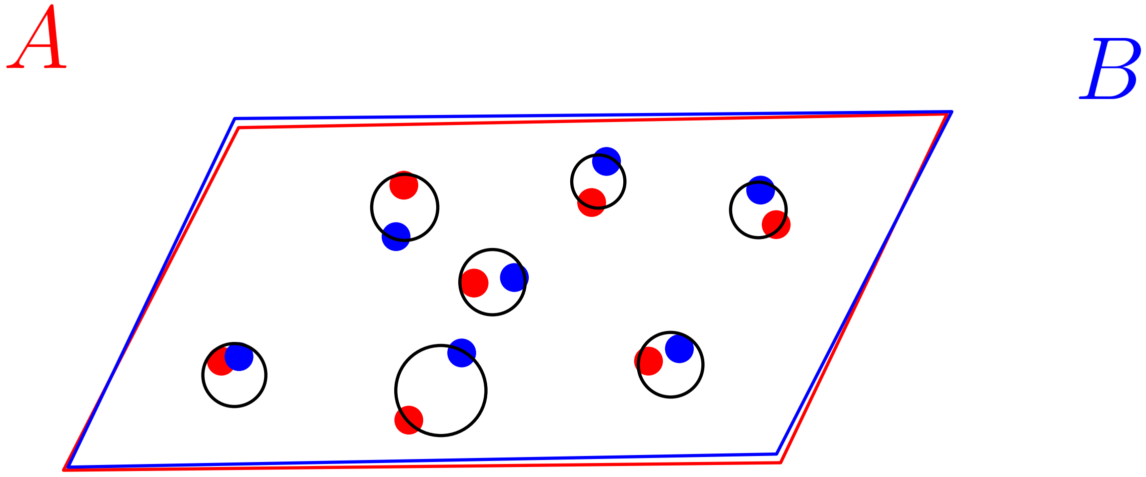
Approximate?



Exact Congruence Testing Problem

The *approximate* problem : *NP-hard*
[Iwanowski 1991, Dieckmann 2012]

$\Rightarrow$ Real-RAM(Random Access Machine) model



The Real-RAM Model

– *exact arithmetic* with *real* numbers

– examples of $O(1)$-time operations

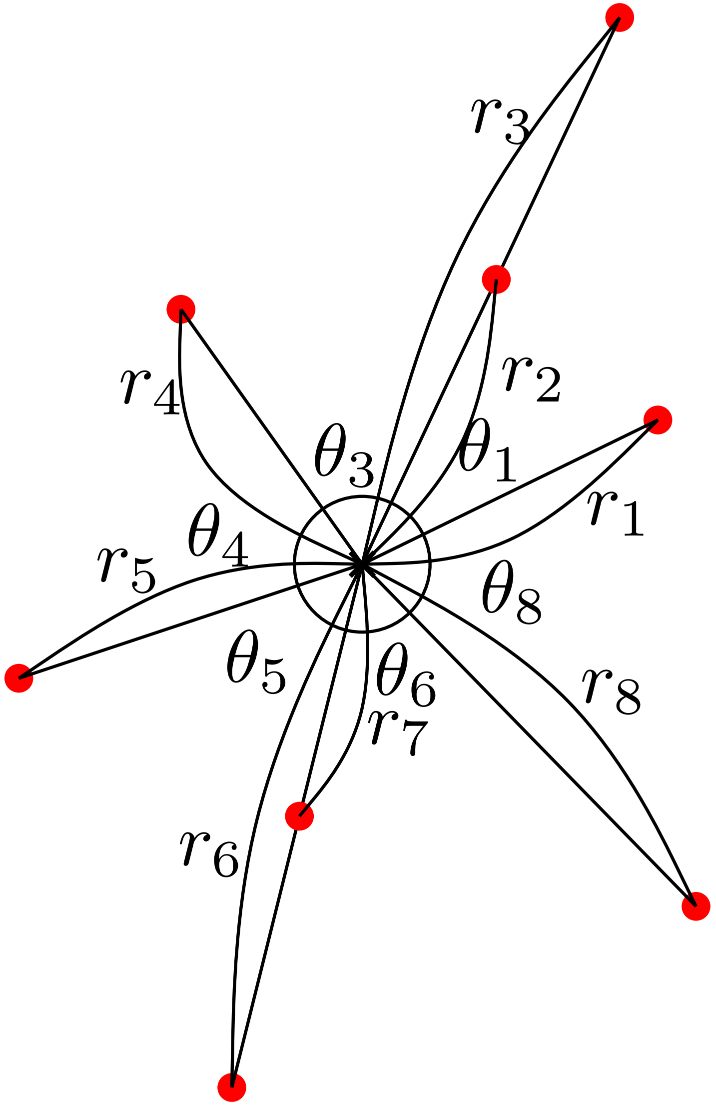
square roots,
sines and cosines,
eigenvalues of 4×4 matrices, ...

– Why?

only with rational coordinates,
a fivefold symmetry is impossible in any dimension.
not interesting...

Previous Algorithms

In 2-space, by string matching [Manacher 1976].



$$r_1\theta_1r_2r_3\theta_3r_4\theta_4r_5\theta_5r_7r_6\theta_6r_8\theta_8$$

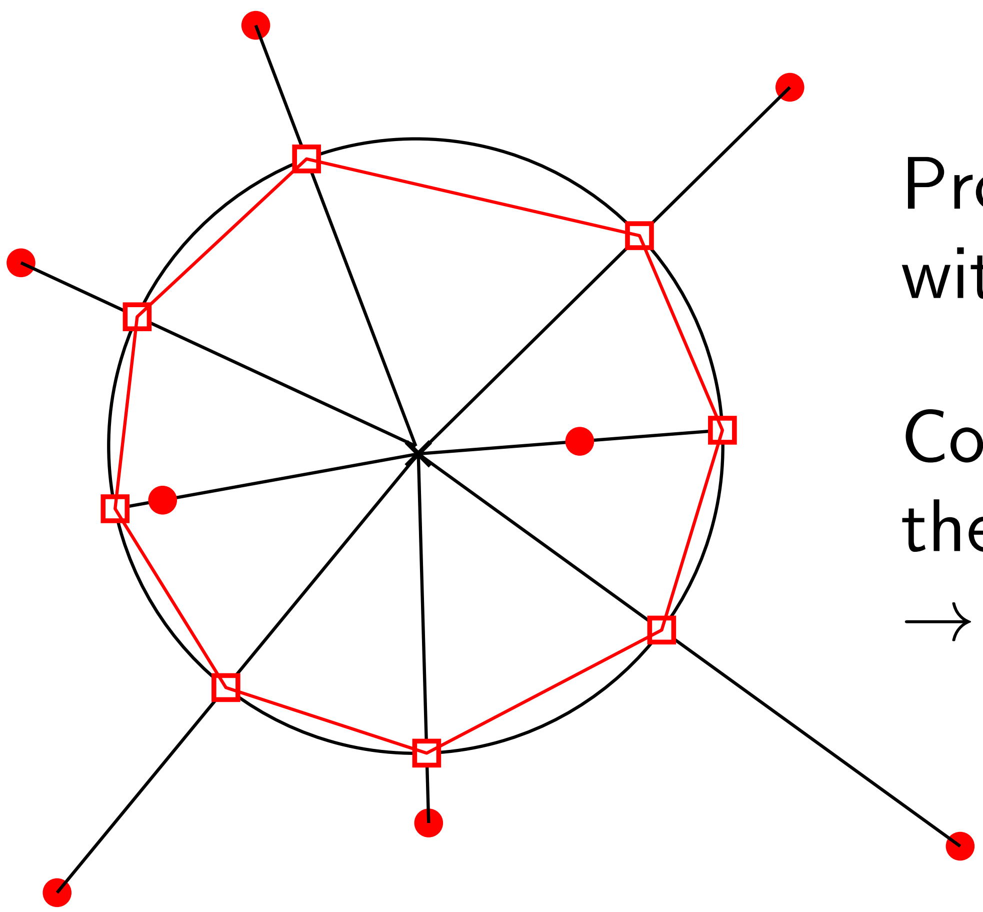
Alternate distances r_i and angle θ_j in a cyclic order.

If points and the origin are colinear, sort from the closest to the farthest

Cyclic Shifts?

Previous Algorithms

In 3-space, by labeled polyhedral (3-connected and planar) graph isomorphism [Alt et. al. 1988]



Project to a unit 2-sphere with labeling distances

Compute the convex hull of the projected points
$\rightarrow$ a polyhedral graph

Previous Algorithms

In d-space,

$O(n^{d-2}\log n)$ by matching one $a \in A$ and all $b \in B$ [Alt et. al. 1988]

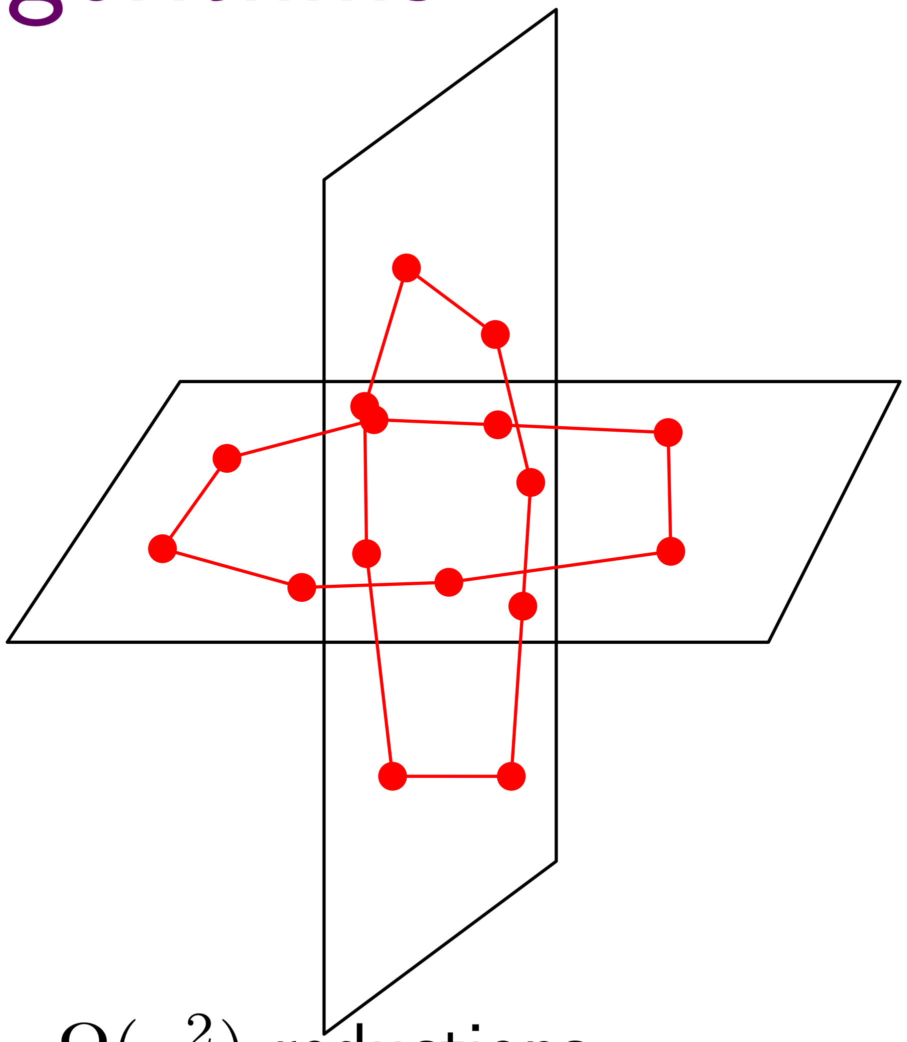
$O(n^{\lfloor \frac{d}{2} \rfloor}\log n)$ by matching a pair of points and all the pairs of points in the closest pair graph [Matoušek]

Previous Algorithms

In d-space,

$O(n^{\lceil \frac{d}{3} \rceil} \log n)$ by matching triples of points but still avoiding the inner isometry of the triples

[Brass and Knauer 2002]



$\Omega(n^2)$ reductions for any full-dim triples

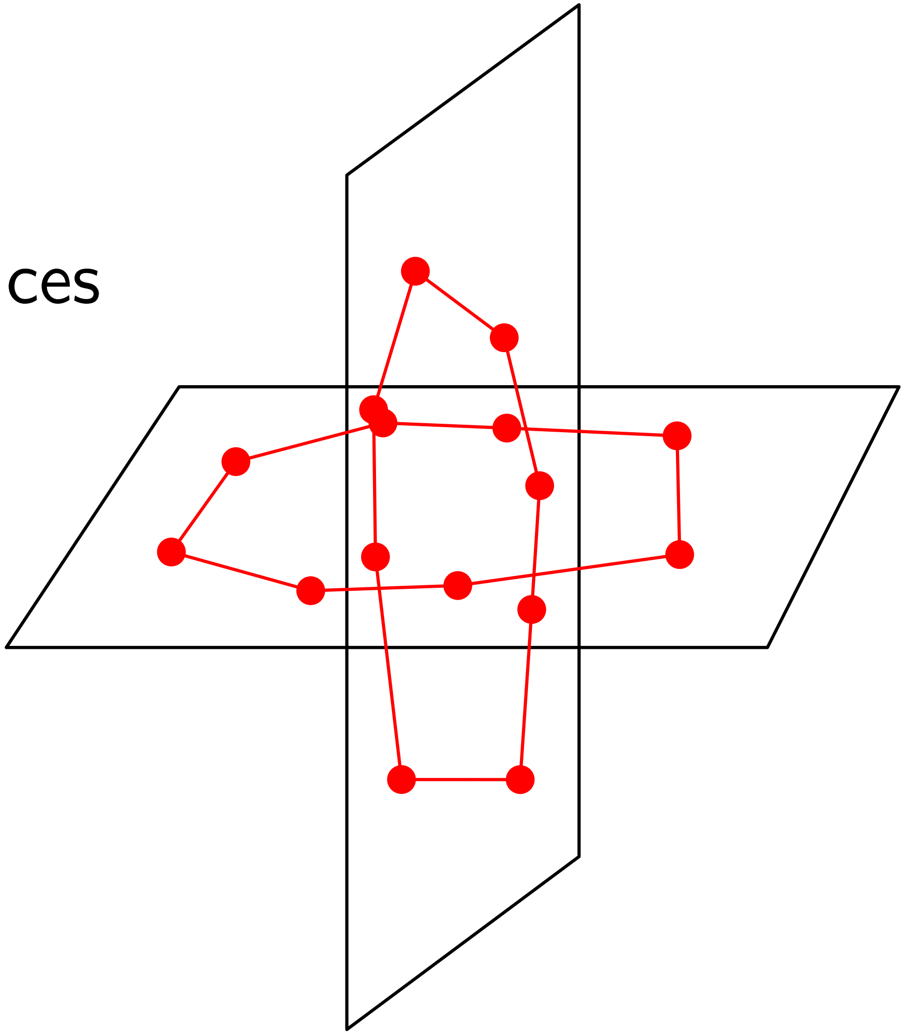
Previous Algorithms

In d-space,

Only problematic case :
point sets in orthogonal subspaces

Add anti-podal points.
Then q, q' are
in orthogonal subspaces
if and only if
$dist(\{q, -q\}, \{q', -q'\}) = \sqrt{2}$.

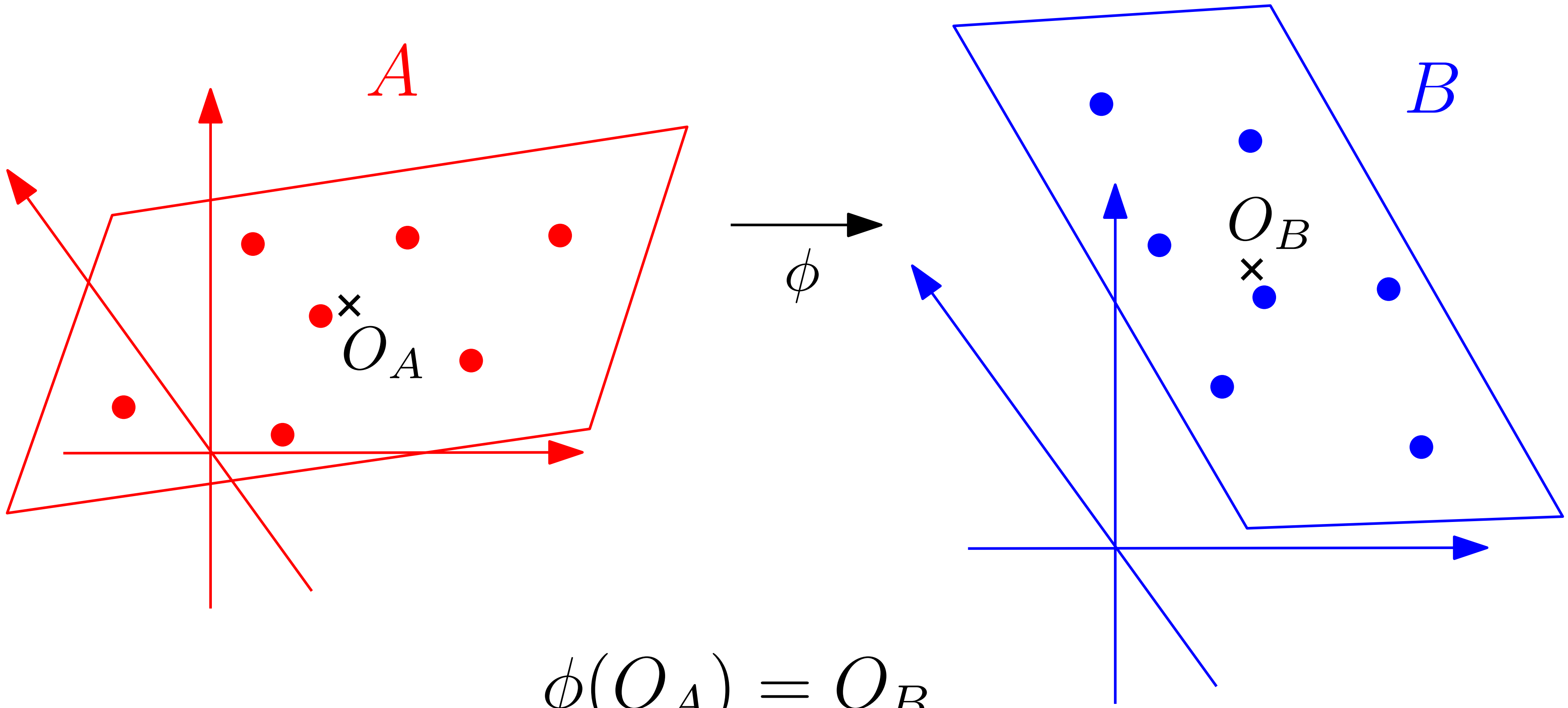
[Brass and Knauer 2002]



$\Omega(n^2)$ reductions

Translation

Congruence $=$ Translations $+$ Rotations

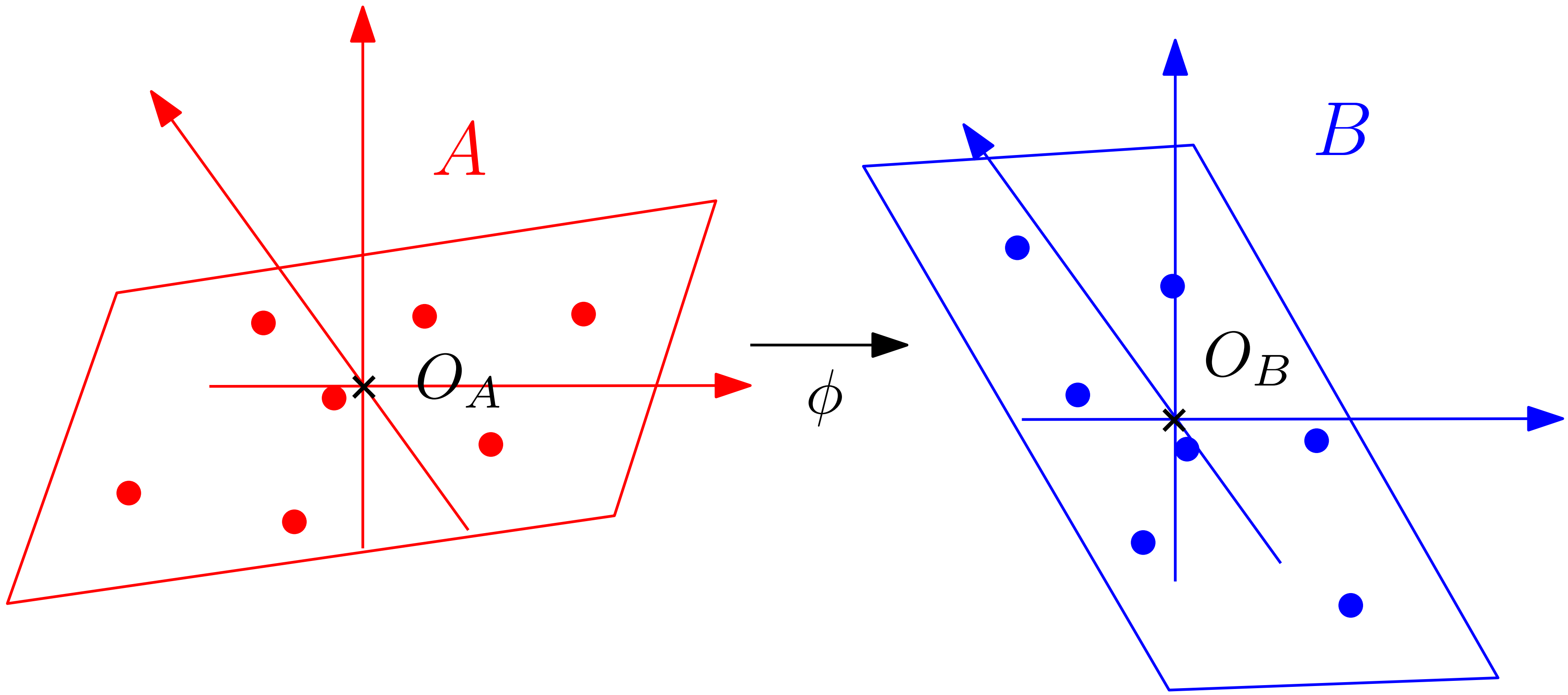


$$\phi(O_A) = O_B$$

Any congruence mapping maps
the centroid to the centroid

Translation

Congruence $=$ Translations $+$ Rotations



The centroids $\Rightarrow$ The origins

Dimension Reduction

Fix subspaces as invariant spaces and match them.

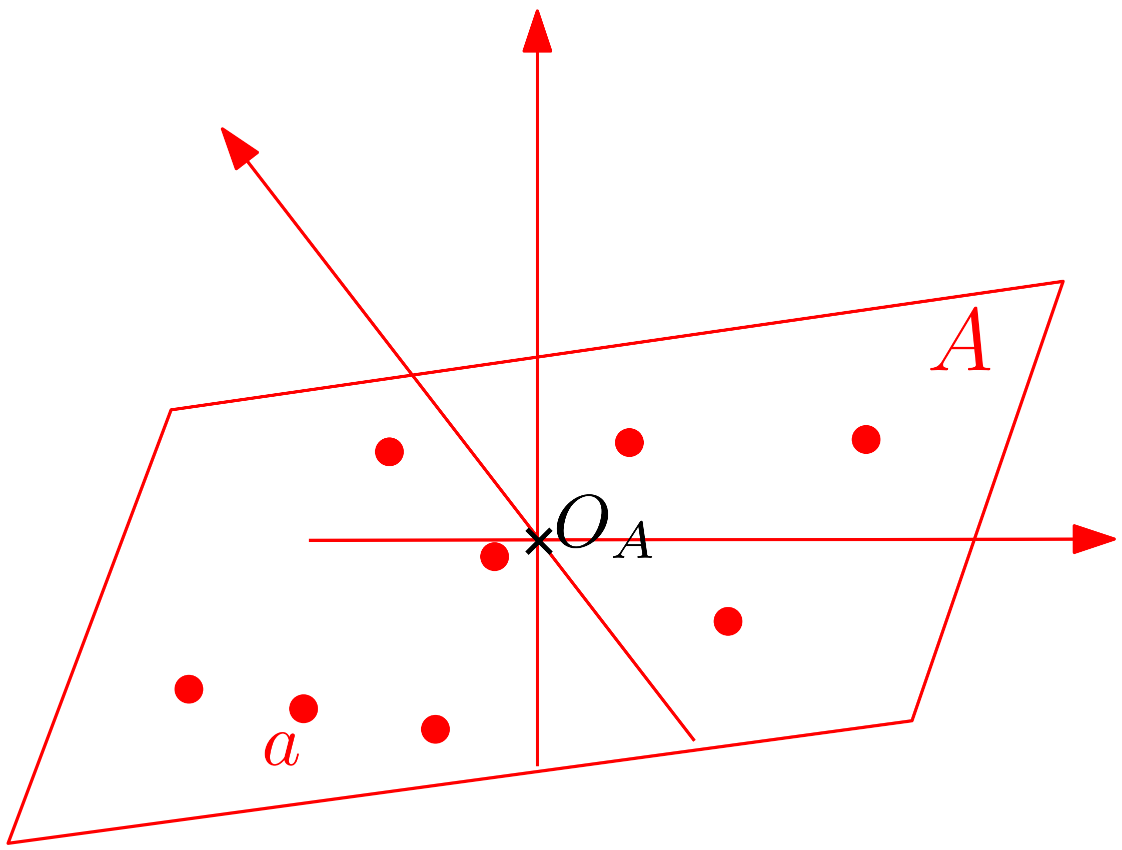
Example : 1+3 dimension reduction
If $|A| = |B| = O(1)$,

try to match *ONE* $a \in A$ to *ALL* $b \in B$.

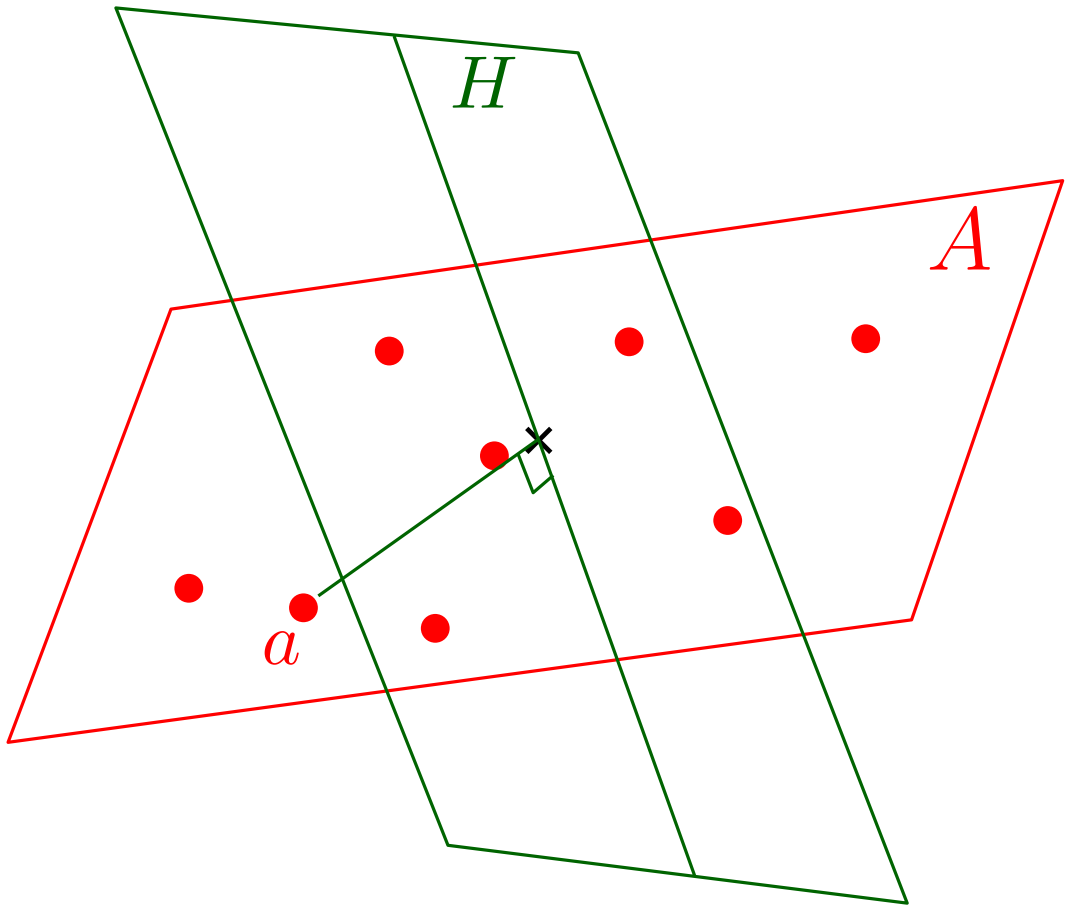
do congruence testing in H_a and H_b.

H_v : the hyperplane orthogonal to v.

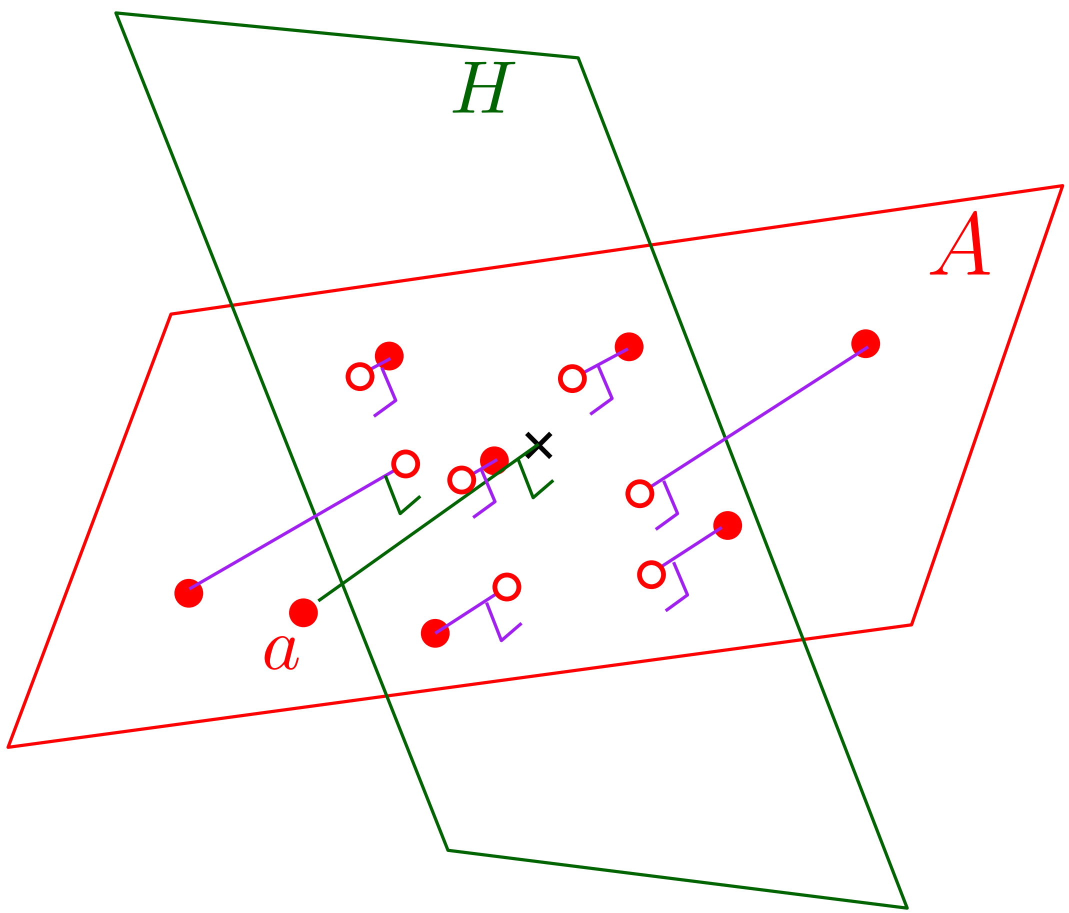
1+3 dimension reduction



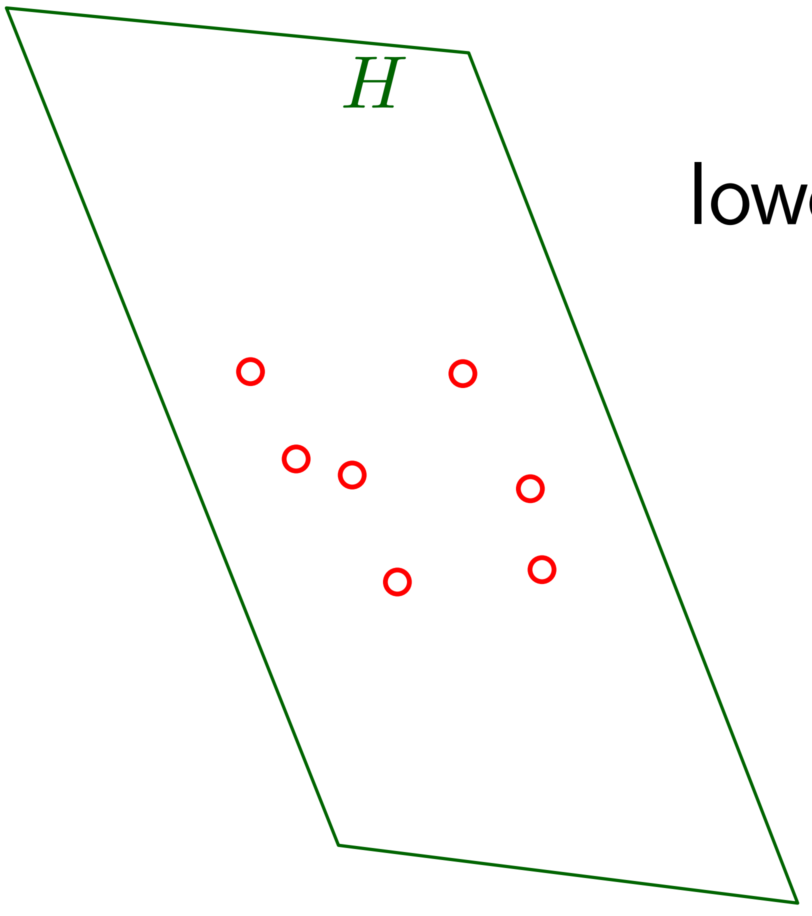
1+3 dimension reduction



1+3 dimension reduction

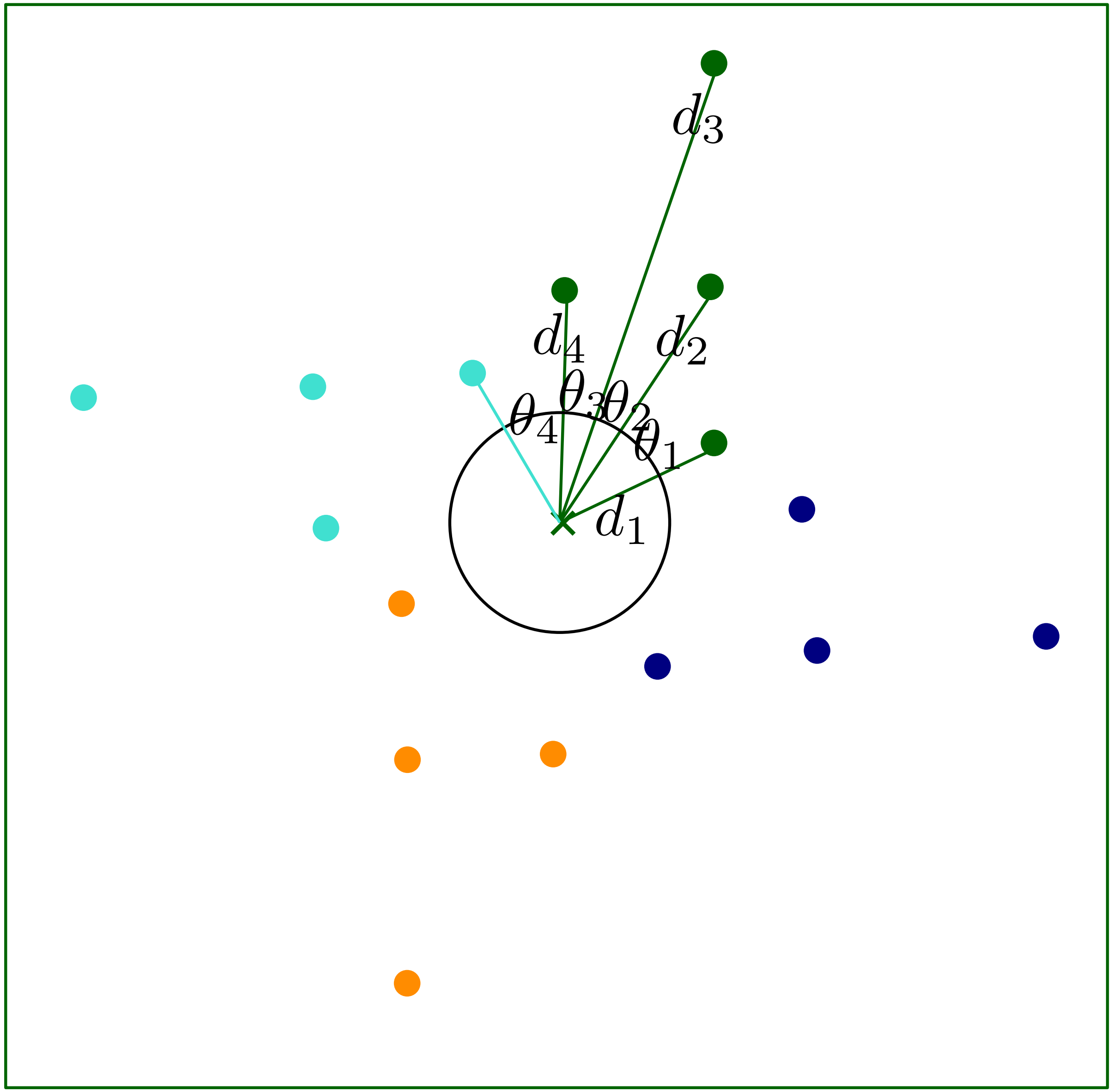


1+3 dimension reduction



$\Rightarrow$ One lower-dimensional space

Lexicographically Smallest Axes



$$d_1\theta_1 d_2\theta_2 d_3\theta_3 d_4\theta_4 \times 4$$

vs.

$$d_2\theta_2 d_3\theta_3 d_4\theta_4 d_1\theta_1 \times 4$$

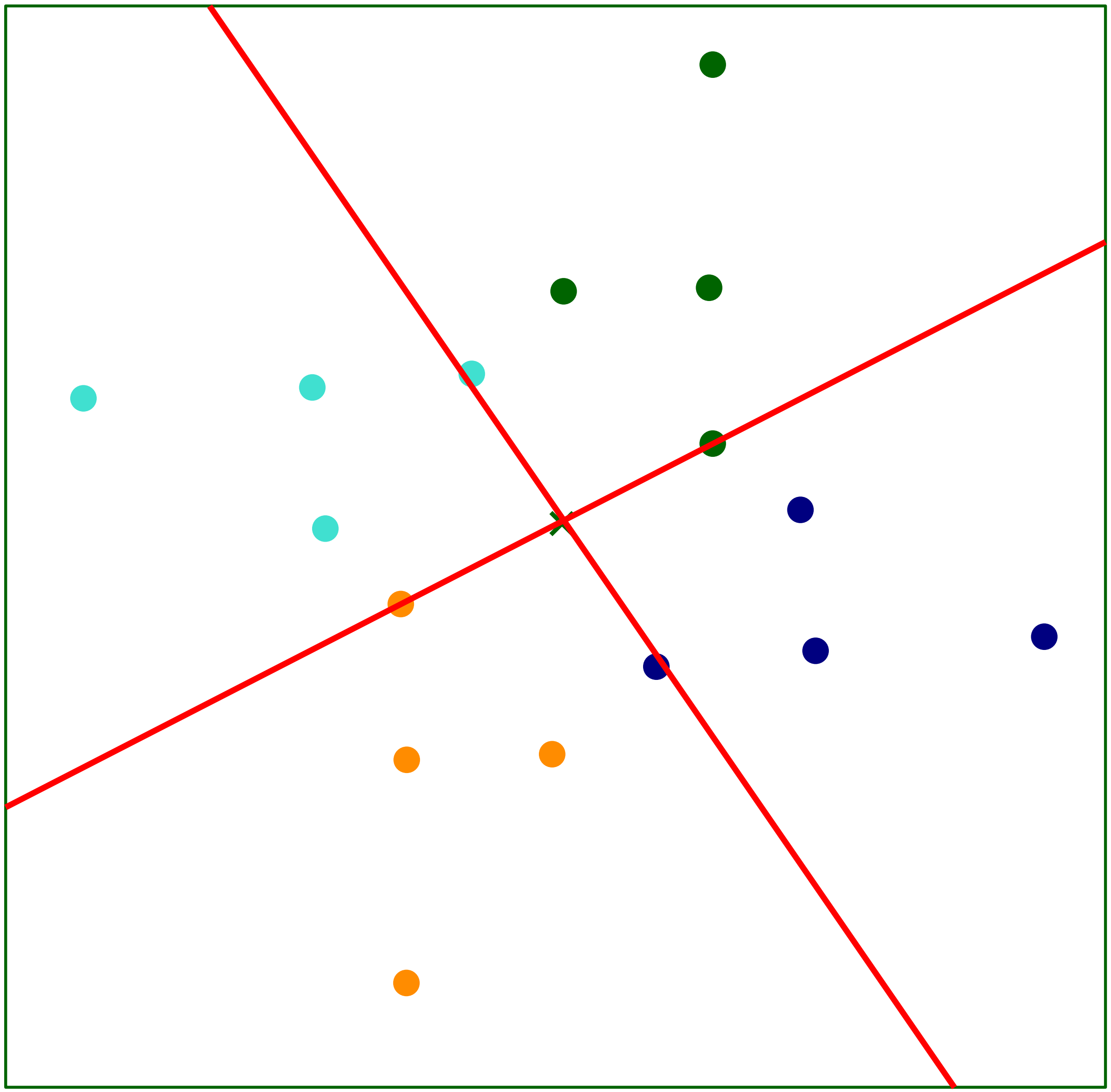
vs.

$$d_3\theta_3 d_4\theta_4 d_1\theta_1 d_2\theta_2 \times 4$$

vs.

$$d_4\theta_4 d_1\theta_1 d_2\theta_2 d_3\theta_3 \times 4$$

Lexicographically Smallest Axes



The lexicographically smallest string :
$d_1\theta_1 d_2\theta_2 d_3\theta_3 d_4\theta_4 \times 4$

Outline

- Basic principles
  - Pruning
  - Dimension Reduction
  - Closest-pair graphs
- New algorithm
  - Overview
  - Modules
    - Generating orbit-cycles
    - Marking and pruning great circles
    - 2+2 dimension reduction
    - Mirror case

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