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## 1.5 UNION-FIND

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- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

# Subtext of today's lecture (and this course)

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## Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why not.
- Find a way to address the problem.
- Iterate until satisfied.

## The scientific method.

## Mathematical analysis.



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## 1.5 UNION-FIND

---

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# Dynamic connectivity problem

---

Given a set of N objects, support two operation:

- Connect two objects.
- Is there a path connecting the two objects?

*connect 4 and 3*

*connect 3 and 8*

*connect 6 and 5*

*connect 9 and 4*

*connect 2 and 1*

*are 0 and 7 connected?* ✗

*are 8 and 9 connected?* ✓

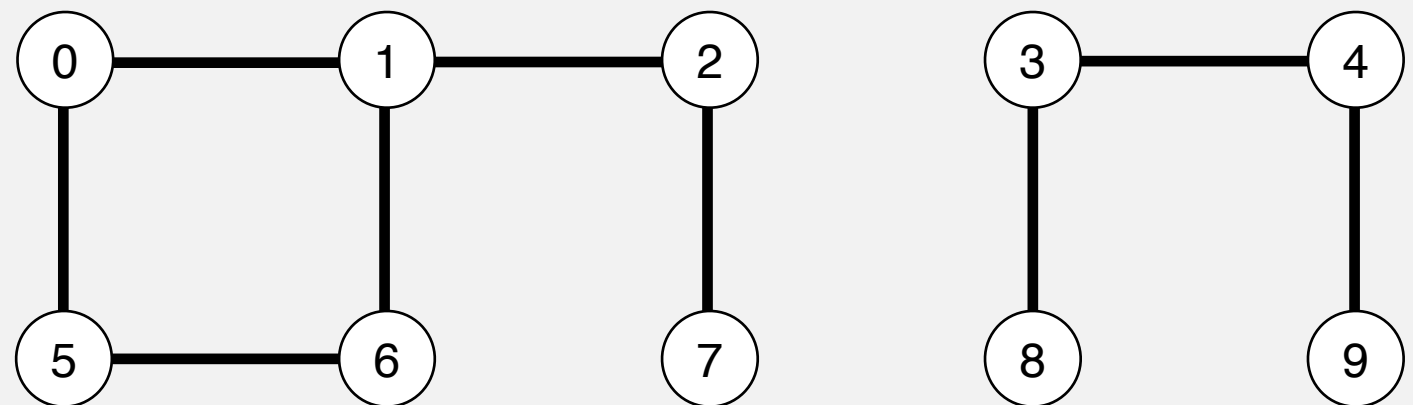
*connect 5 and 0*

*connect 7 and 2*

*connect 6 and 1*

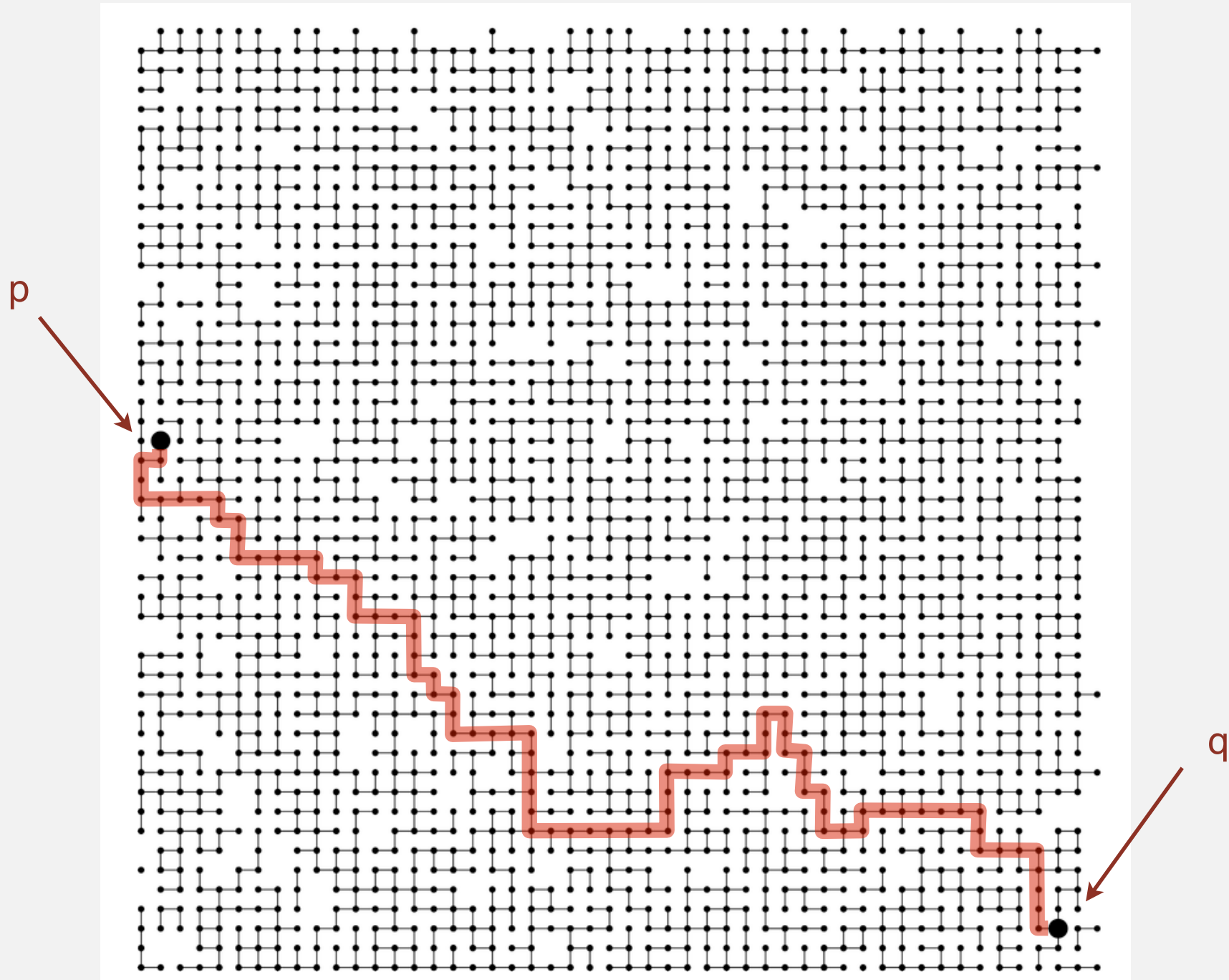
*connect 1 and 0*

*are 0 and 7 connected?* ✓



## A larger connectivity example

Q. Is there a path connecting  $p$  and  $q$  ?



A. Yes.

# Modeling the objects


---

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to  $N - 1$ .

- Use integers as array index.
- Suppress details not relevant to union-find.



can use symbol table to translate from site names  
to integers: stay tuned (Chapter 3)

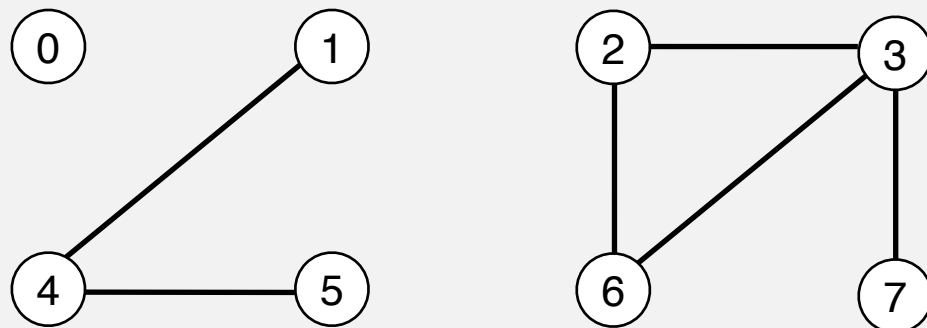
# Modeling the connections

---

We assume "is connected to" is an equivalence relation:

- Reflexive:  $p$  is connected to  $p$ .
- Symmetric: if  $p$  is connected to  $q$ , then  $q$  is connected to  $p$ .
- Transitive: if  $p$  is connected to  $q$  and  $q$  is connected to  $r$ , then  $p$  is connected to  $r$ .

**Connected component.** Maximal **set** of objects that are mutually connected.



$\{0\} \{1\ 4\ 5\} \{2\ 3\ 6\ 7\}$

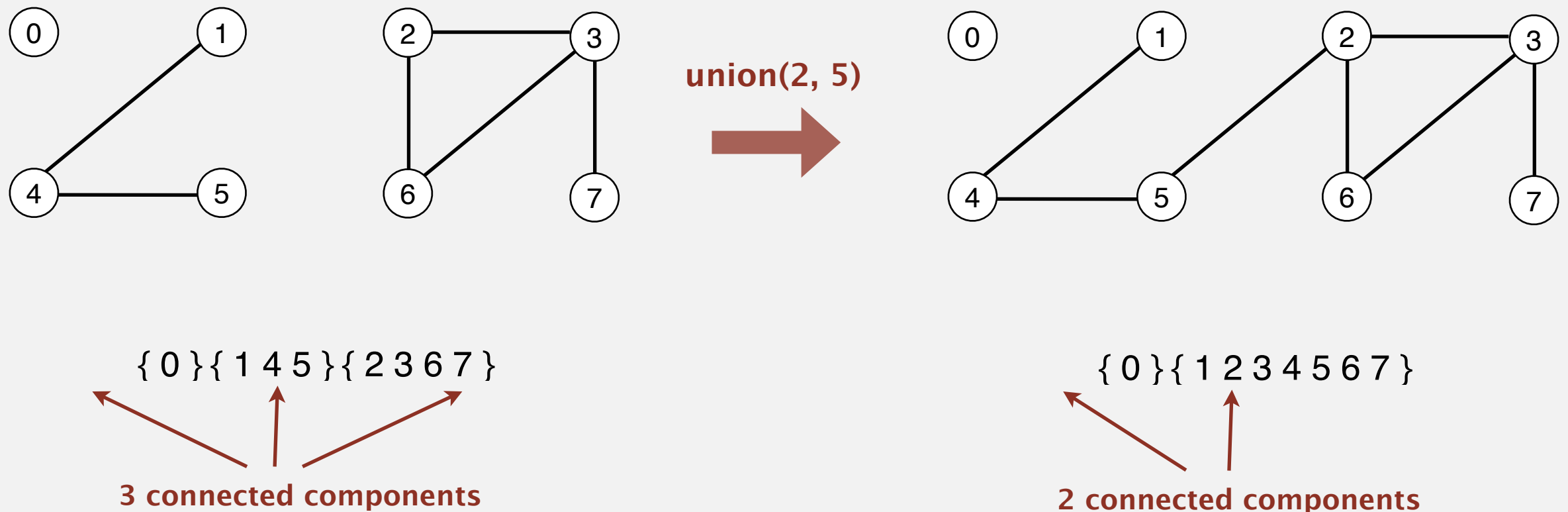
3 connected components

# Implementing the operations

**Find.** In which component is object  $p$  ?

**Connected.** Are objects  $p$  and  $q$  in the same component?

**Union.** Replace components containing objects  $p$  and  $q$  with their union.





# Union-find data type (API)

---

**Goal.** Design efficient data structure for union-find.

- Number of objects  $N$  can be huge.
- Number of operations  $M$  can be huge.
- Union and find operations may be intermixed.

```
public class UF
```

```
    UF(int N)
```

*initialize union-find data structure  
with  $N$  singleton objects (0 to  $N - 1$ )*

```
    void union(int p, int q)
```

*add connection between  $p$  and  $q$*

```
    int find(int p)
```

*component identifier for  $p$  (0 to  $N - 1$ )*

```
    boolean connected(int p, int q)
```

*are  $p$  and  $q$  in the same component?*

```
    public boolean connected(int p, int q)
    { return find(p) == find(q); }
```

**1-line implementation of connected()**

# Dynamic-connectivity client

---

- Read in number of objects  $N$  from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
    int N = StdIn.readInt();
    UF uf = new UF(N);
    while (!StdIn.isEmpty())
    {
        int p = StdIn.readInt();
        int q = StdIn.readInt();
        if (!uf.connected(p, q))
        {
            uf.union(p, q);
            StdOut.println(p + " " + q);
        }
    }
}
```

% more tinyUF.txt

10

4 3

3 8

6 5

9 4

2 1

8 9

5 0

7 2

6 1

1 0

6 7

already connected





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## 1.5 UNION-FIND

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- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

# Quick-find [eager approach]

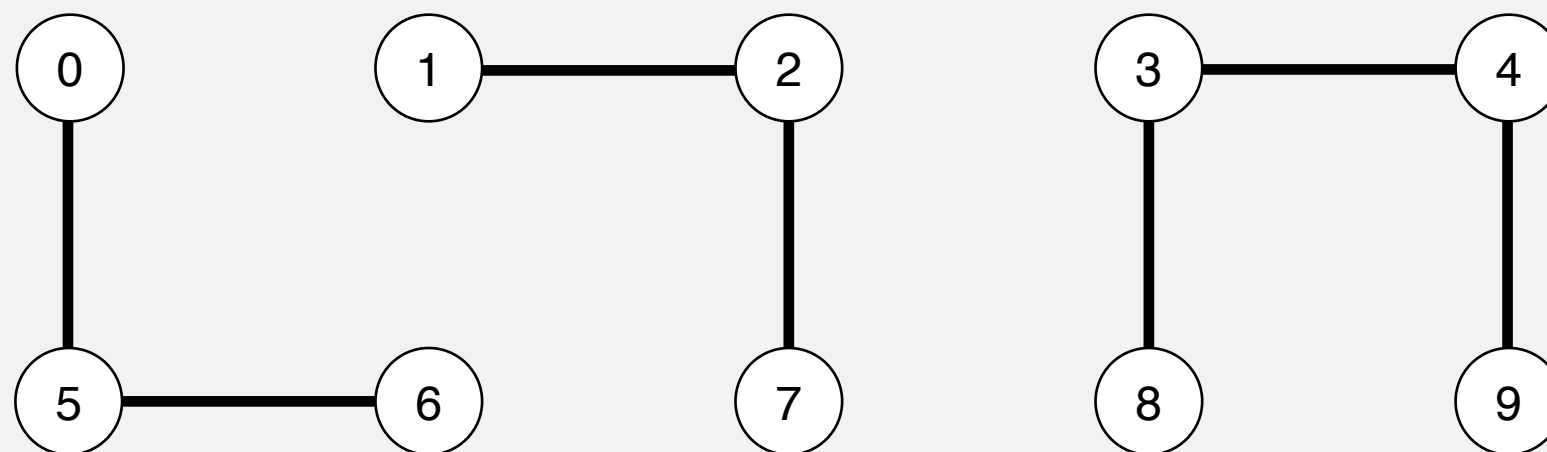
## Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[p]` is the id of the component containing `p`.

if and only if  
↙

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

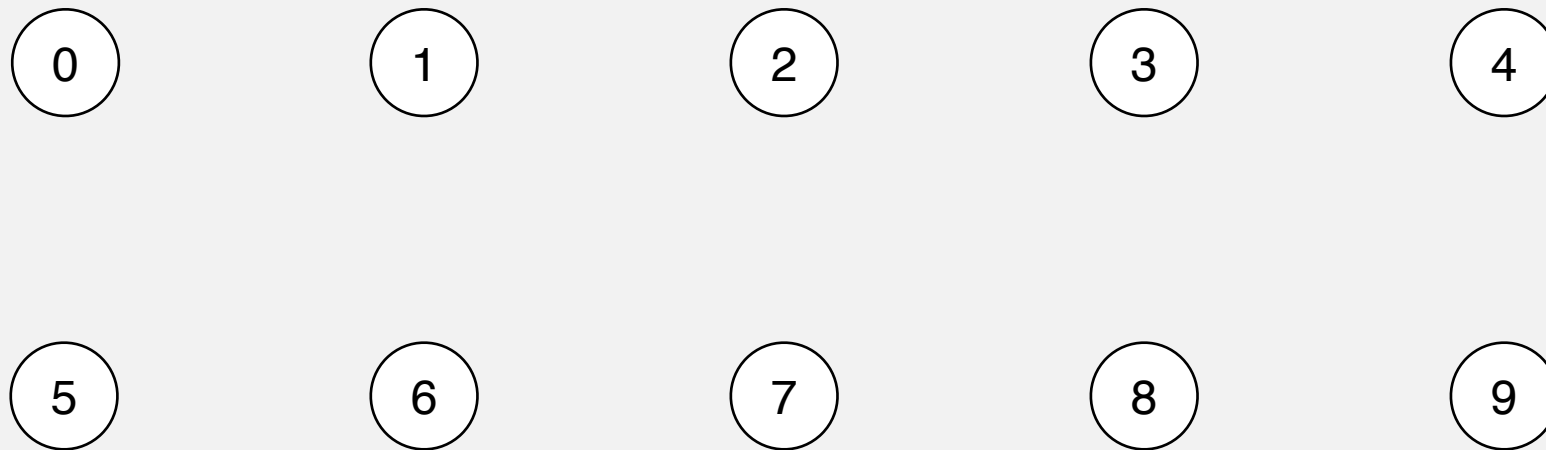
0, 5 and 6 are connected  
1, 2, and 7 are connected  
3, 4, 8, and 9 are connected





# Quick-find demo

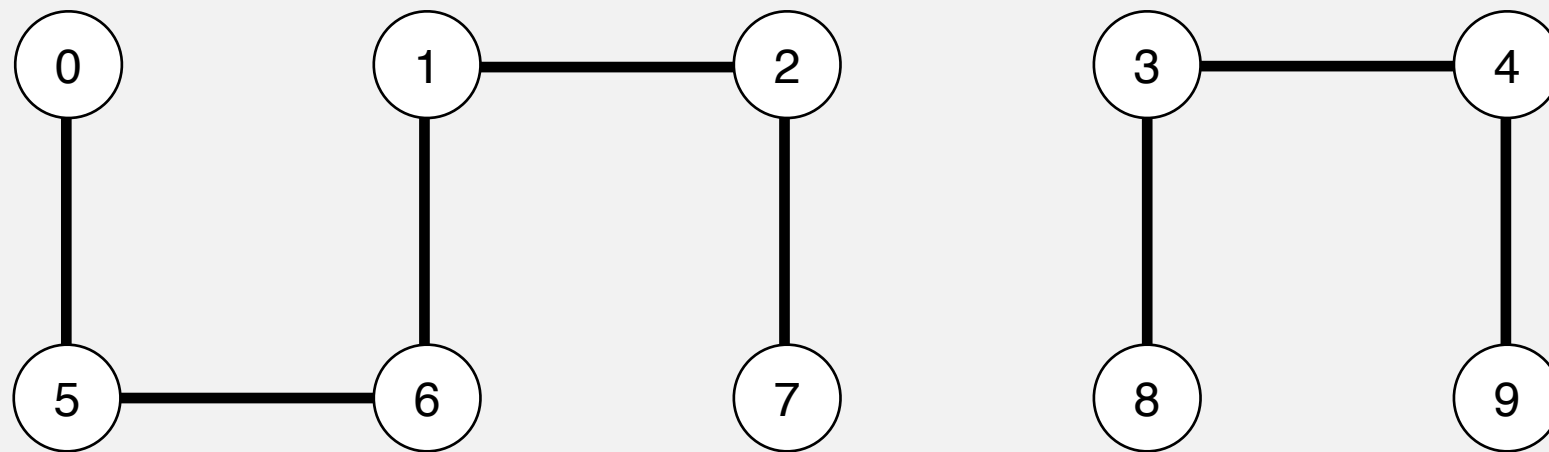
---



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

# Quick-find demo

---



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

# Quick-find: Java implementation

---

```
public class QuickFindUF
```

```
{
```

```
    private int[] id;
```

```
    public QuickFindUF(int N)
```

```
    {
```

```
        id = new int[N];
```

```
        for (int i = 0; i < N; i++)
```

```
            id[i] = i;
```

```
    }
```

← set id of each object to itself  
(N array accesses)

← return the id of p  
(1 array access)

```
    public boolean find(int p)
```

```
    { return id[p]; }
```

```
    public void union(int p, int q)
```

```
    {
```

```
        int pid = id[p];
```

```
        int qid = id[q];
```

```
        for (int i = 0; i < id.length; i++)
```

```
            if (id[i] == pid) id[i] = qid;
```

← change all entries with id[p] to id[q]  
(at most  $2N + 2$  array accesses)



# Quick-find is too slow


---

**Cost model.** Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
<b>quick-find</b>	N	N	1	1

order of growth of number of array accesses

**Union is too expensive.** It takes  $N^2$  array accesses to process a sequence of  $N$  union operations on  $N$  objects.

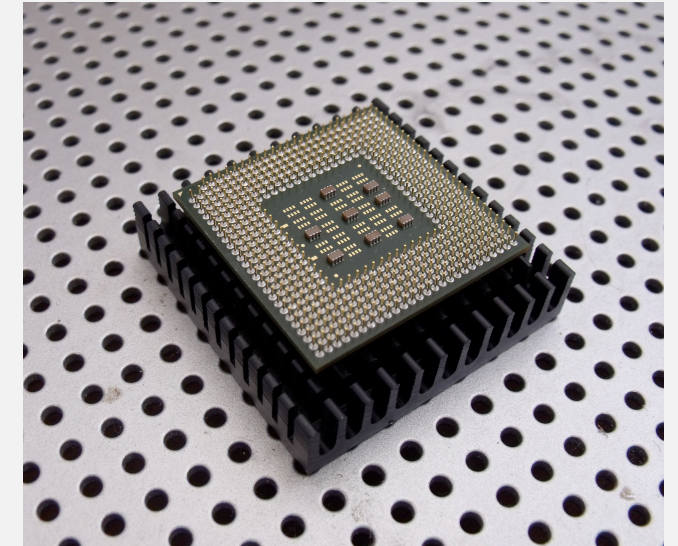
quadratic  


# Quadratic algorithms do not scale

## Rough standard (for now).

- $10^9$  operations per second.
- $10^9$  words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly)  
since 1950!

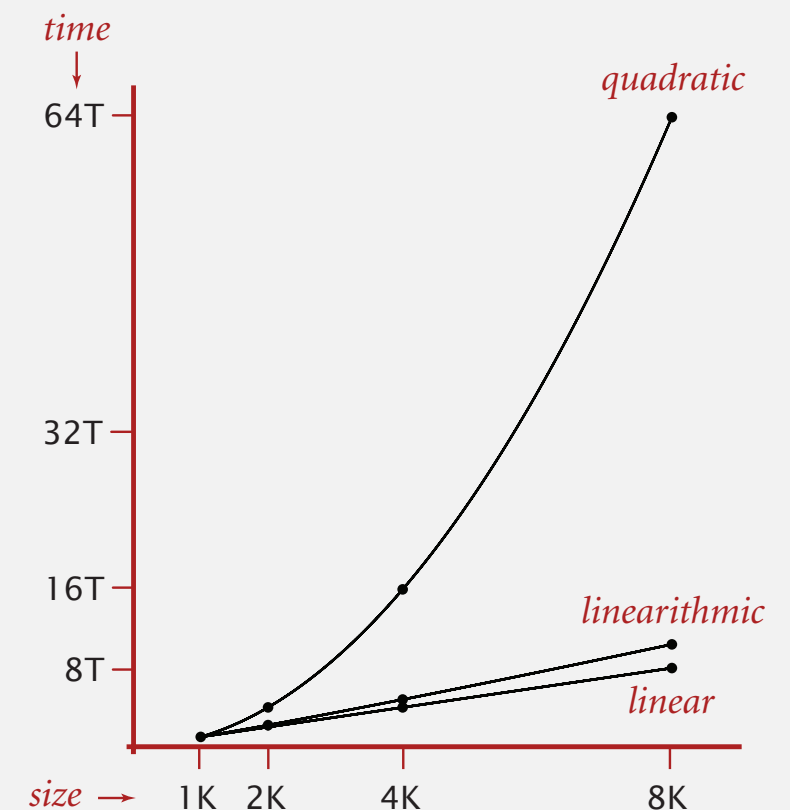


## Ex. Huge problem for quick-find.

- $10^9$  union commands on  $10^9$  objects.
- Quick-find takes more than  $10^{18}$  operations.
- 30+ years of computer time!

## Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory  $\Rightarrow$  want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!





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## 1.5 UNION-FIND

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- *dynamic connectivity*
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- *quick union*
- *improvements*
- *applications*

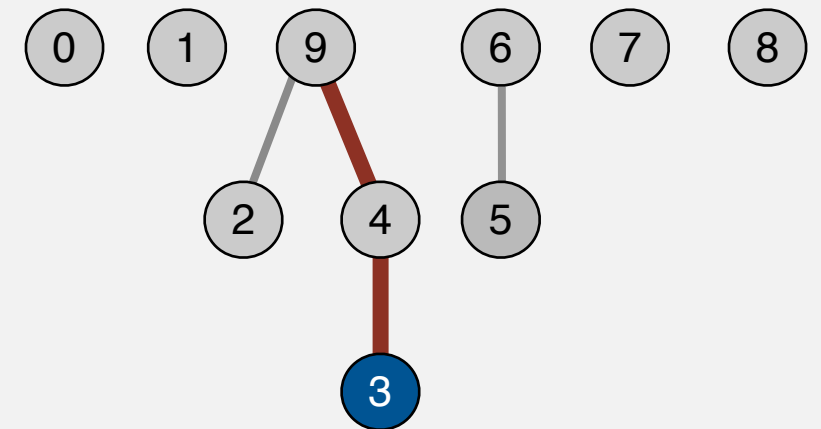
# Quick-union [lazy approach]

## Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- **Root** of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
<code>id[]</code>	0	1	9	4	9	6	6	7	8	9

keep going until it doesn't change  
(algorithm ensures no cycles)



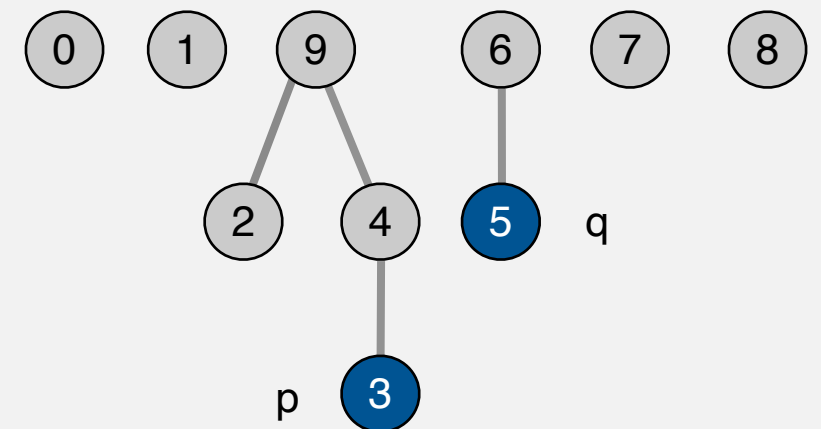
parent of 3 is 4  
root of 3 is 9

# Quick-union [lazy approach]

## Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[...id[i]...]]`.

	0	1	2	3	4	5	6	7	8	9
<code>id[]</code>	0	1	9	4	9	6	6	7	8	9



root of 3 is 9

root of 5 is 6

3 and 5 are not connected

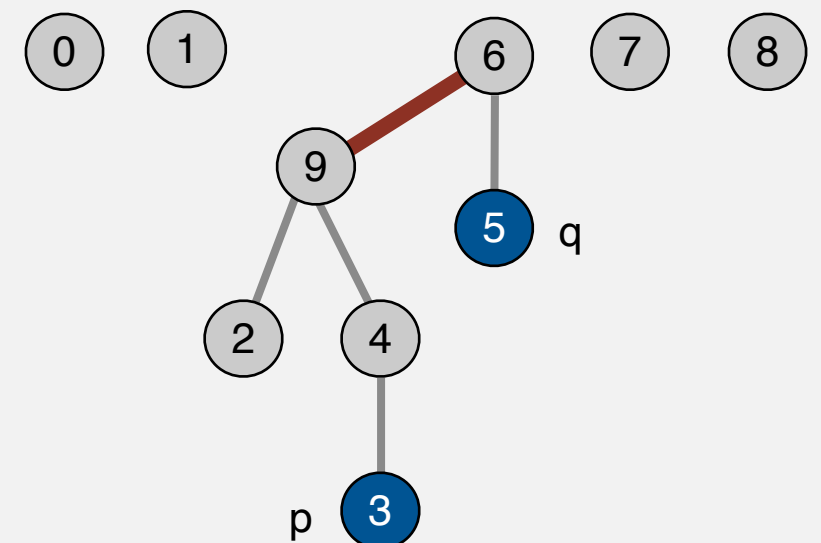
**Find.** What is the root of `p`?

**Connected.** Do `p` and `q` have the same root?

**Union.** To merge components containing `p` and `q`, set the `id` of `p`'s root to the `id` of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
<code>id[]</code>	0	1	9	4	9	6	6	7	8	6

↑  
only one value changes



# Quick-union demo

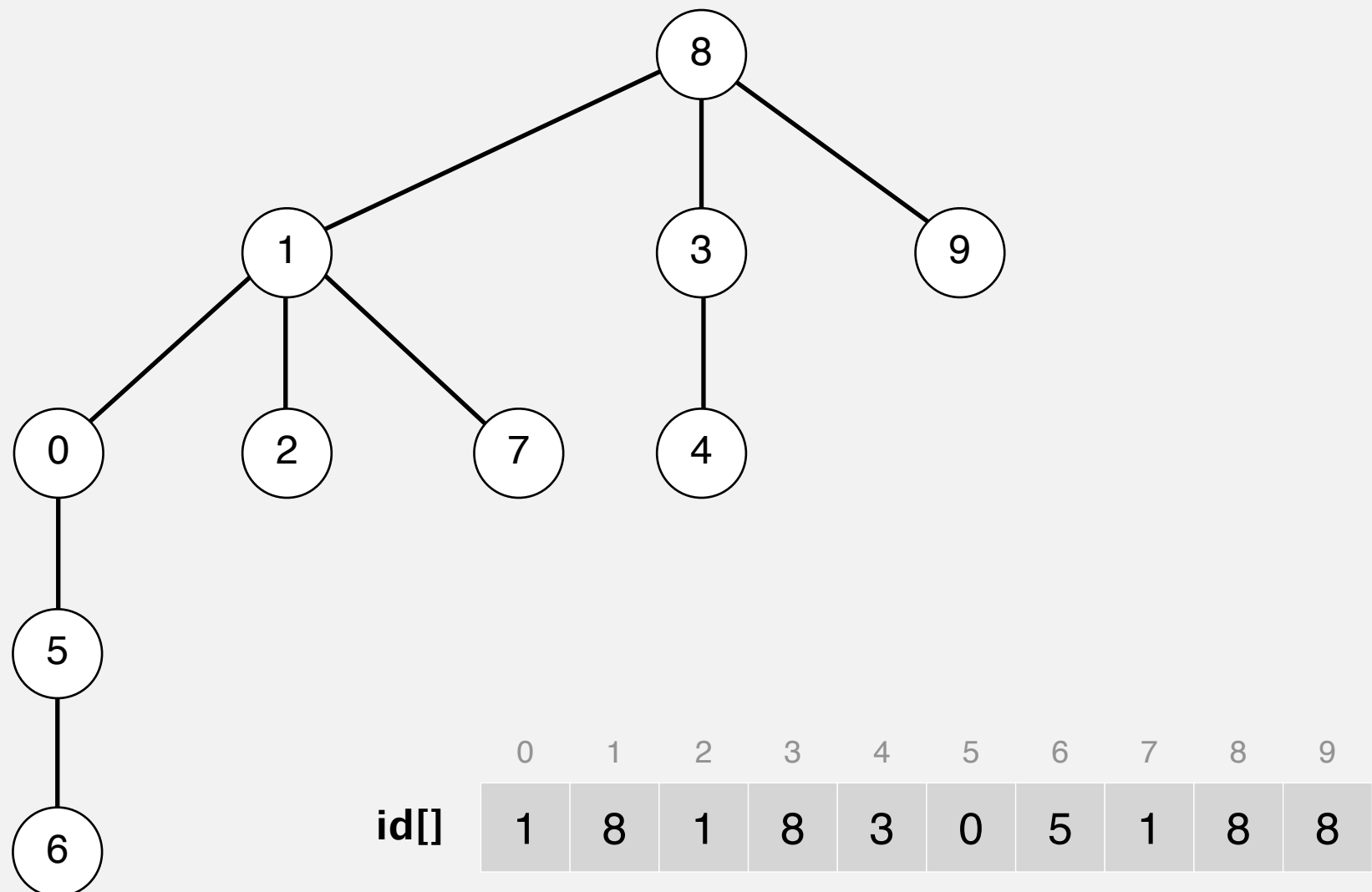
---



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

# Quick-union demo

---



# Quick-union: Java implementation

---

```
public class QuickUnionUF
{
    private int[] id;

    public QuickUnionUF(int N)
    {
        id = new int[N];
        for (int i = 0; i < N; i++) id[i] = i;
    }

    public int find(int i)
    {
        while (i != id[i]) i = id[i];
        return i;
    }

    public void union(int p, int q)
    {
```

← set id of each object to itself  
(N array accesses)

← chase parent pointers until reach root  
(depth of i array accesses)

← change root of p to point to root of q  
(depth of p and q array accesses)



# Quick-union is also too slow

---

**Cost model.** Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	$N$	$N$	1	1
quick-union	$N$	$N^\dagger$	$N$	$N$

← worst case

$^\dagger$  includes cost of finding roots

## Quick-find defect.

- Union too expensive ( $N$  array accesses).
- Trees are flat, but too expensive to keep them flat.

## Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be  $N$  array accesses).



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## 1.5 UNION-FIND

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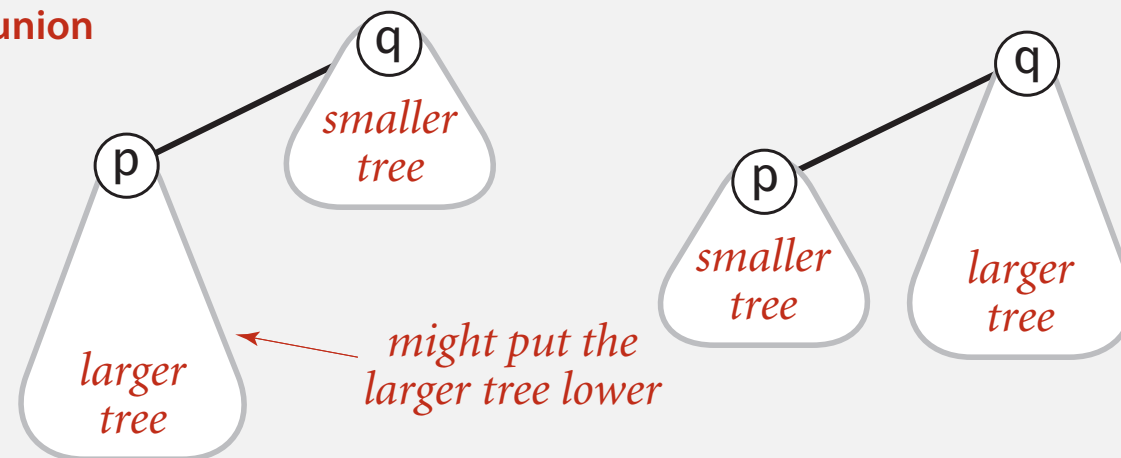
- *dynamic connectivity*
- *quick find*
- *quick union*
- ***improvements***
- *applications*

# Improvement 1: weighting

## Weighted quick-union.

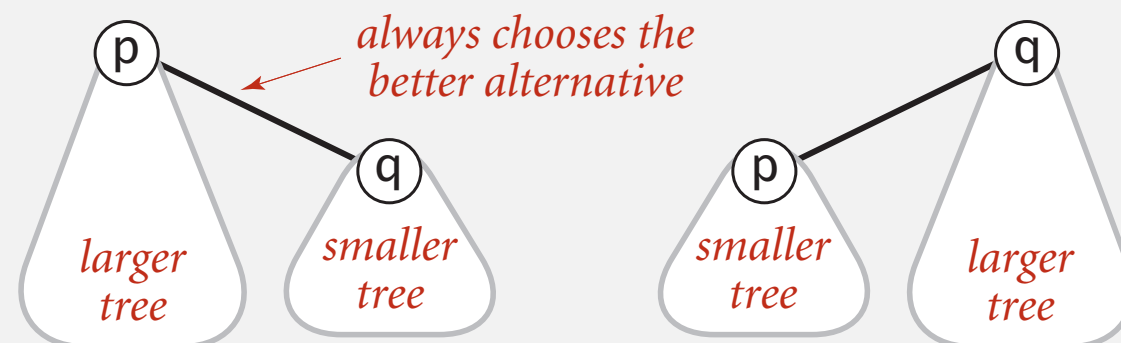
- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

quick-union



reasonable alternatives:  
union by height or "rank"

weighted



# Weighted quick-union demo

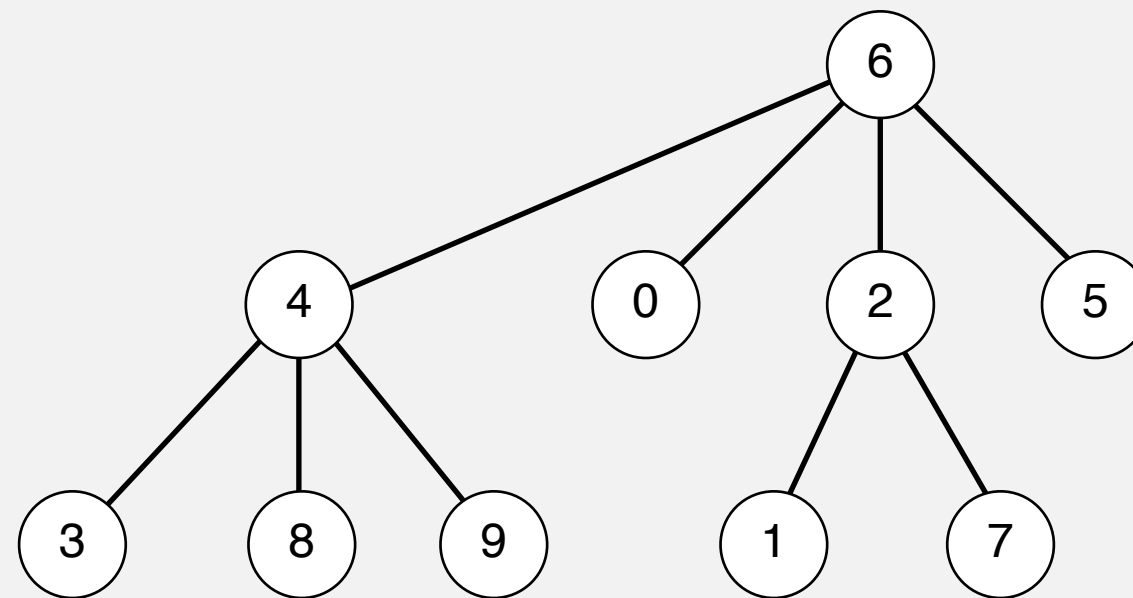
---



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

# Weighted quick-union demo

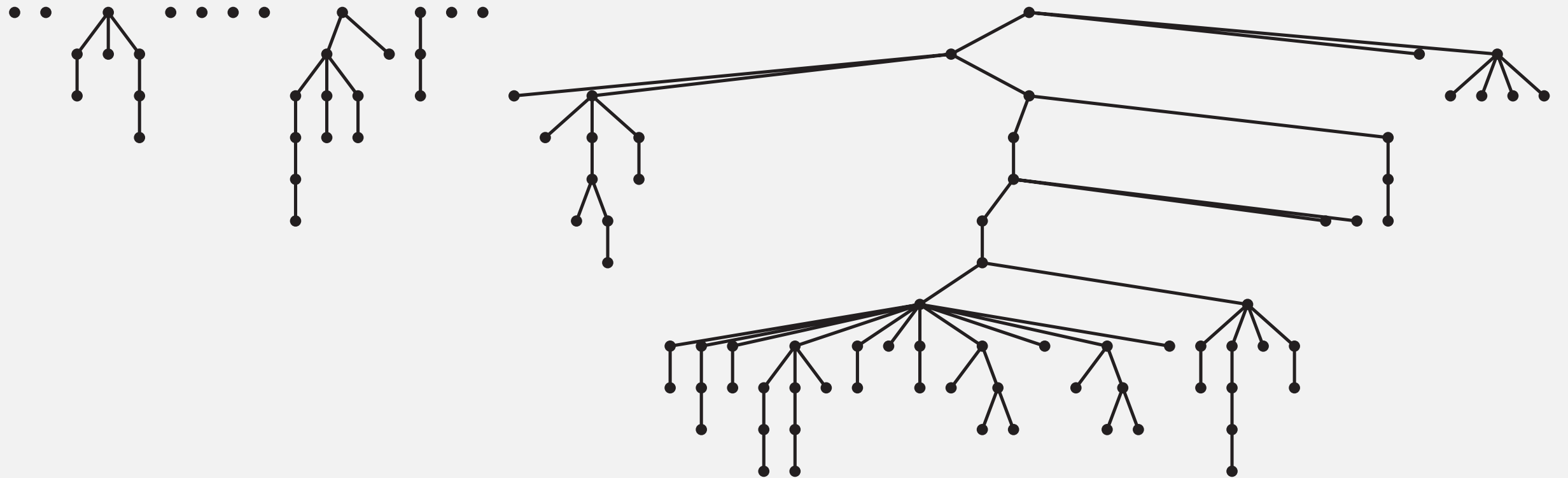
---



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

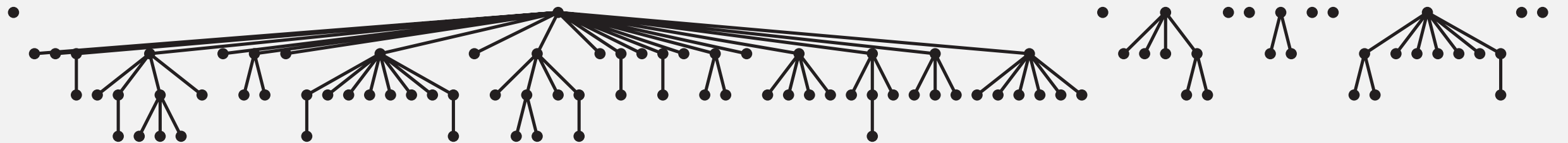
# Quick-union and weighted quick-union example

quick-union



*average distance to root: 5.11*

weighted



*average distance to root: 1.52*

Quick-union and weighted quick-union (100 sites, 88 union() operations)

# Weighted quick-union: Java implementation

---

**Data structure.** Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

**Find/connected.** Identical to quick-union.

**Union.** Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

```
int i = find(p);
int j = find(q);
if (i == j) return;

if (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

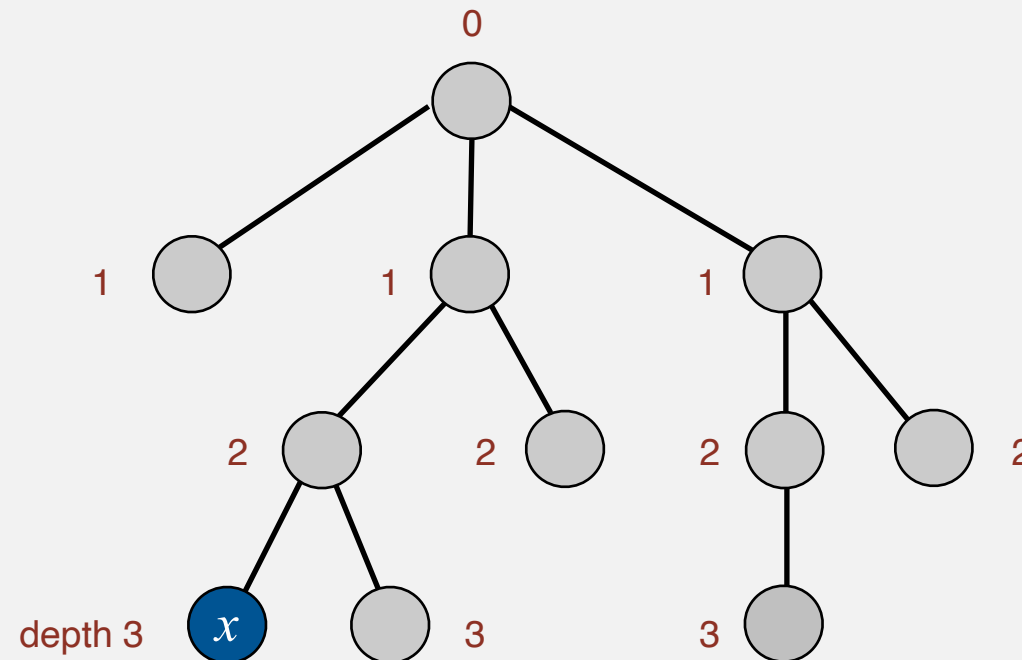
# Weighted quick-union analysis

## Running time.

- Find: takes time proportional to depth of  $p$ .
- Union: takes constant time, given roots.

**Proposition.** Depth of any node  $x$  is at most  $\lg N$ .

$\lg$  = base-2 logarithm



$$N = 11$$
$$\text{depth}(x) = 3 \leq \lg N$$



# Weighted quick-union analysis

## Running time.

- Find: takes time proportional to depth of  $p$ .
- Union: takes constant time, given roots.

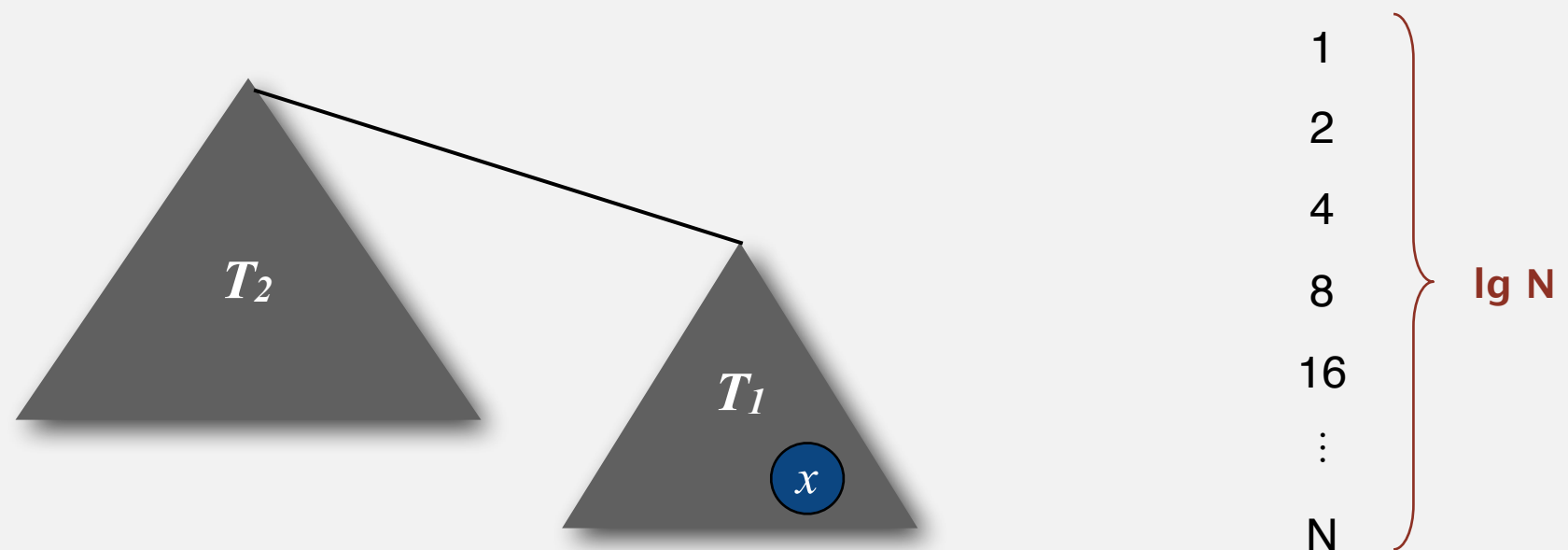
$\lg$  = base-2 logarithm

**Proposition.** Depth of any node  $x$  is at most  $\lg N$ .

**Pf.** What causes the depth of object  $x$  to increase?

Increases by 1 when tree  $T_1$  containing  $x$  is merged into another tree  $T_2$ .

- The size of the tree containing  $x$  at least doubles since  $|T_2| \geq |T_1|$ .
- Size of tree containing  $x$  can double at most  $\lg N$  times. Why?



# Weighted quick-union analysis

---

## Running time.

- Find: takes time proportional to depth of  $p$ .
- Union: takes constant time, given roots.

**Proposition.** Depth of any node  $x$  is at most  $\lg N$ .

algorithm	initialize	union	find	connected
<b>quick-find</b>	$N$	$N$	1	1
<b>quick-union</b>	$N$	$N^\dagger$	$N$	$N$
<b>weighted QU</b>	$N$	$\lg N^\dagger$	$\lg N$	$\lg N$

$^\dagger$  includes cost of finding roots

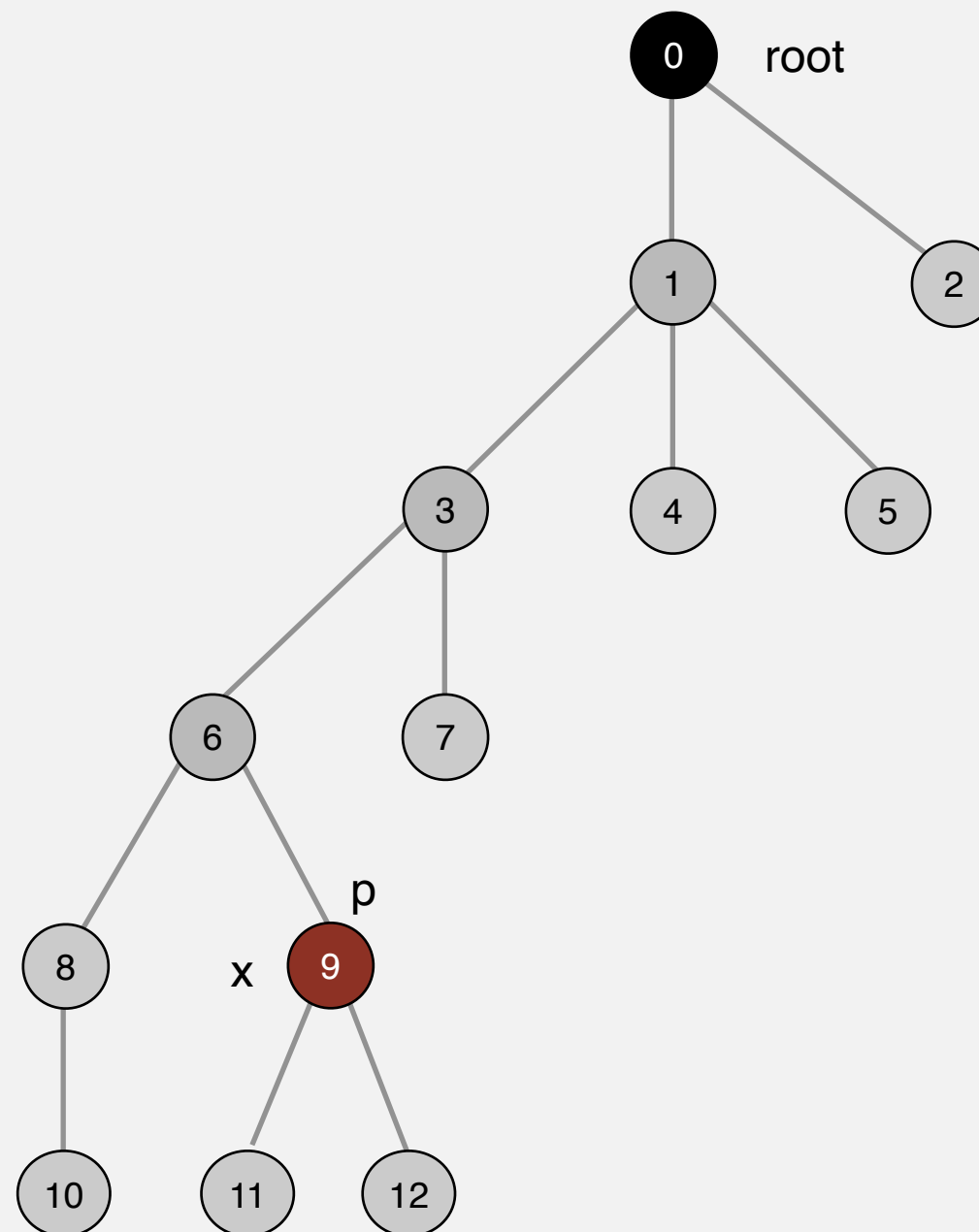
**Q.** Stop at guaranteed acceptable performance?

**A.** No, easy to improve further.

## Improvement 2: path compression

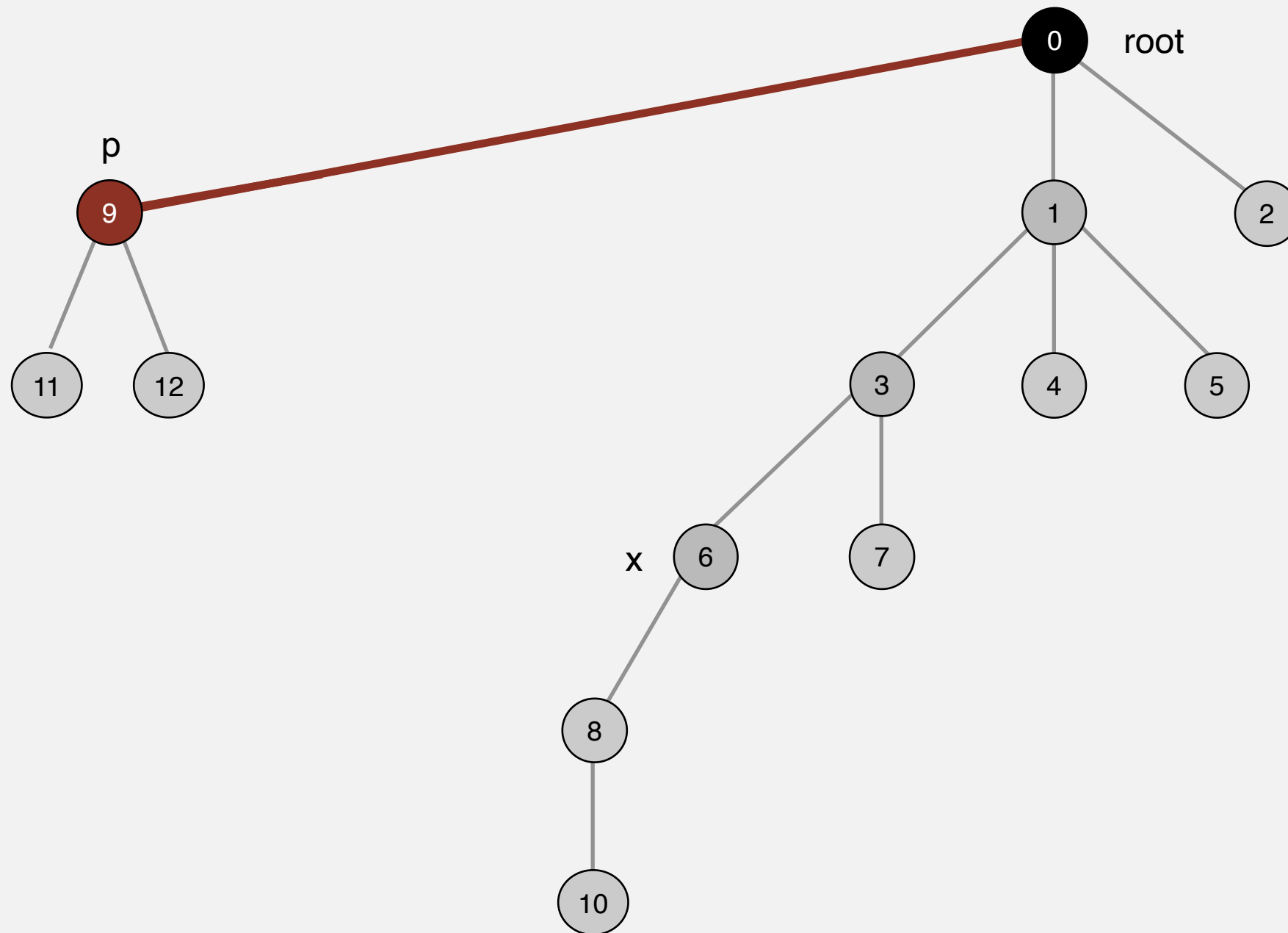
---

**Quick union with path compression.** Just after computing the root of  $p$ , set the `id[]` of each examined node to point to that root.



## Improvement 2: path compression

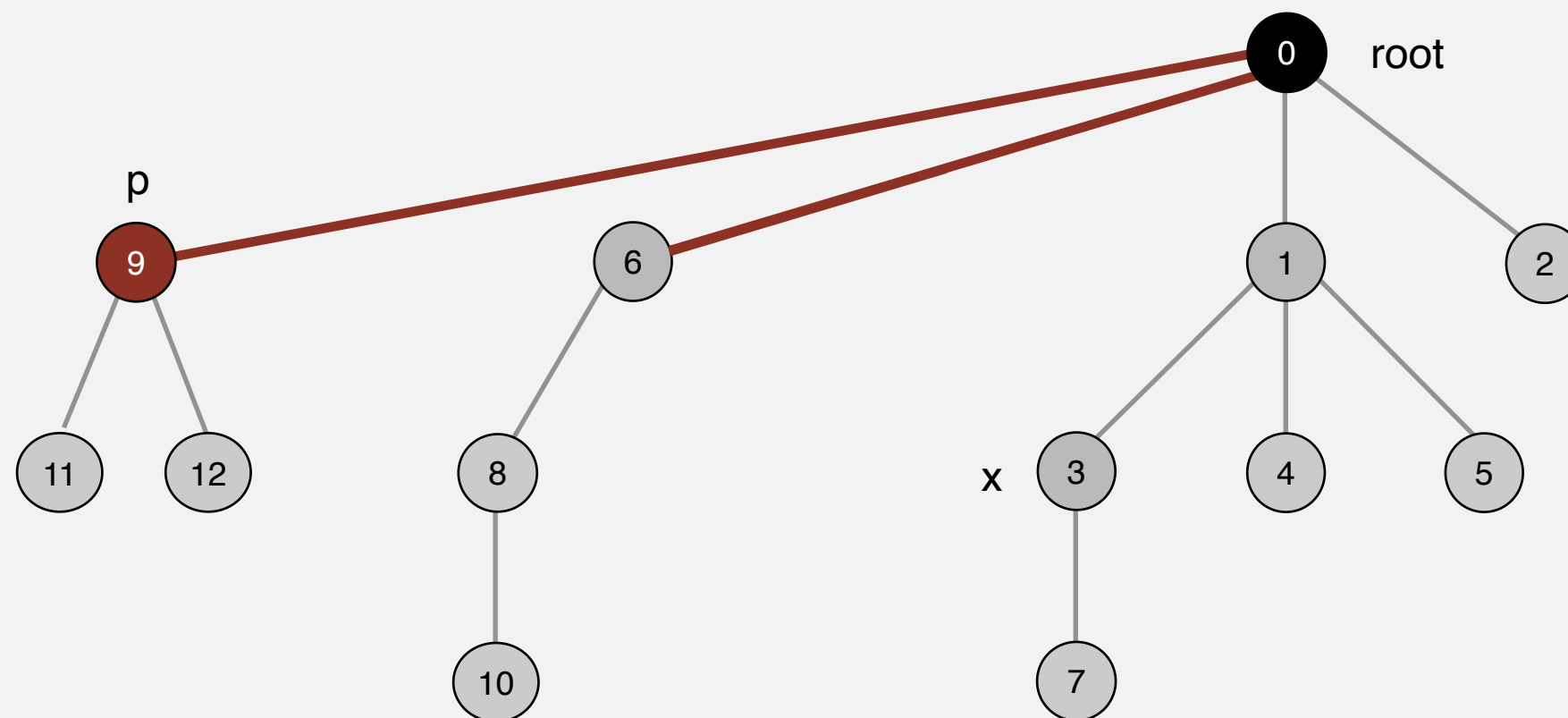
**Quick union with path compression.** Just after computing the root of  $p$ , set the `id[]` of each examined node to point to that root.



## Improvement 2: path compression

---

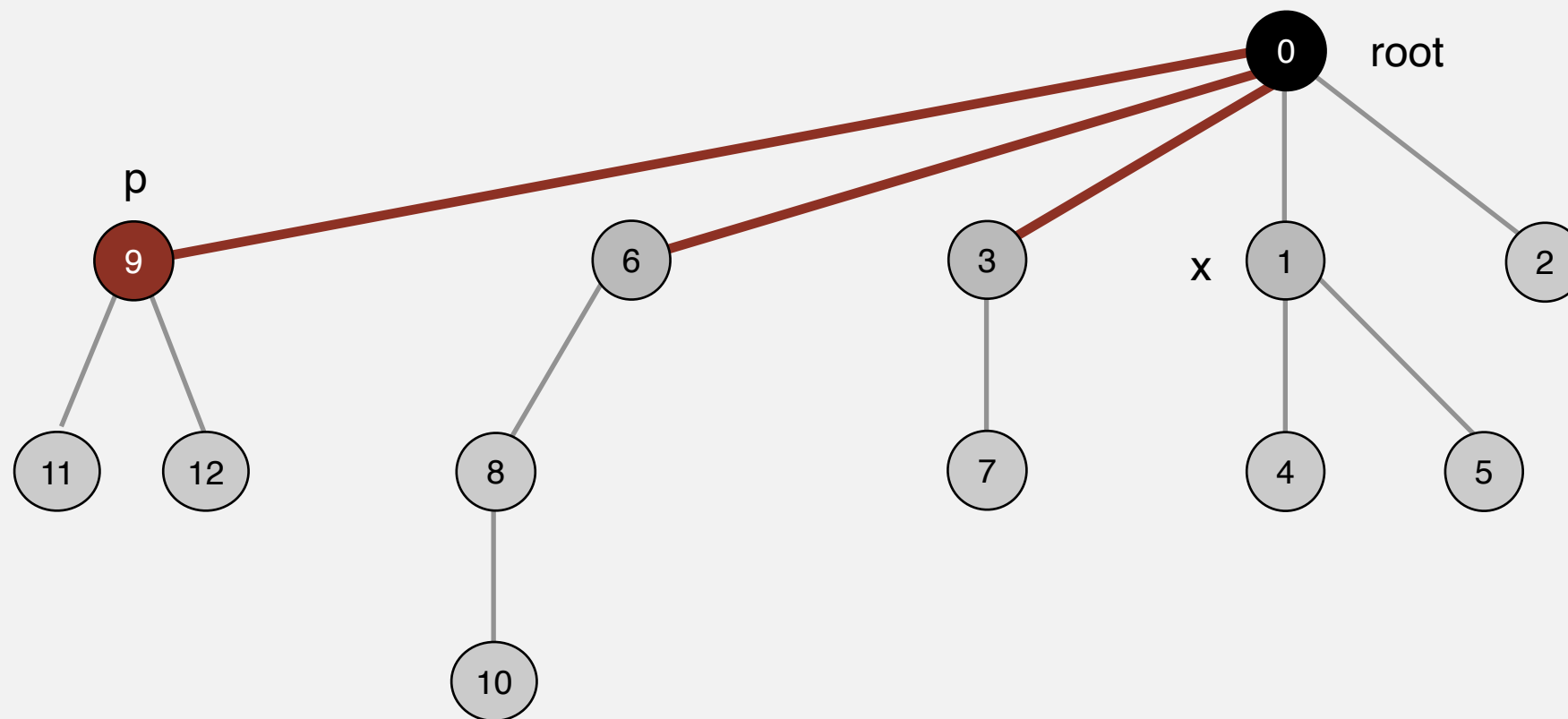
**Quick union with path compression.** Just after computing the root of  $p$ , set the `id[]` of each examined node to point to that root.



## Improvement 2: path compression

---

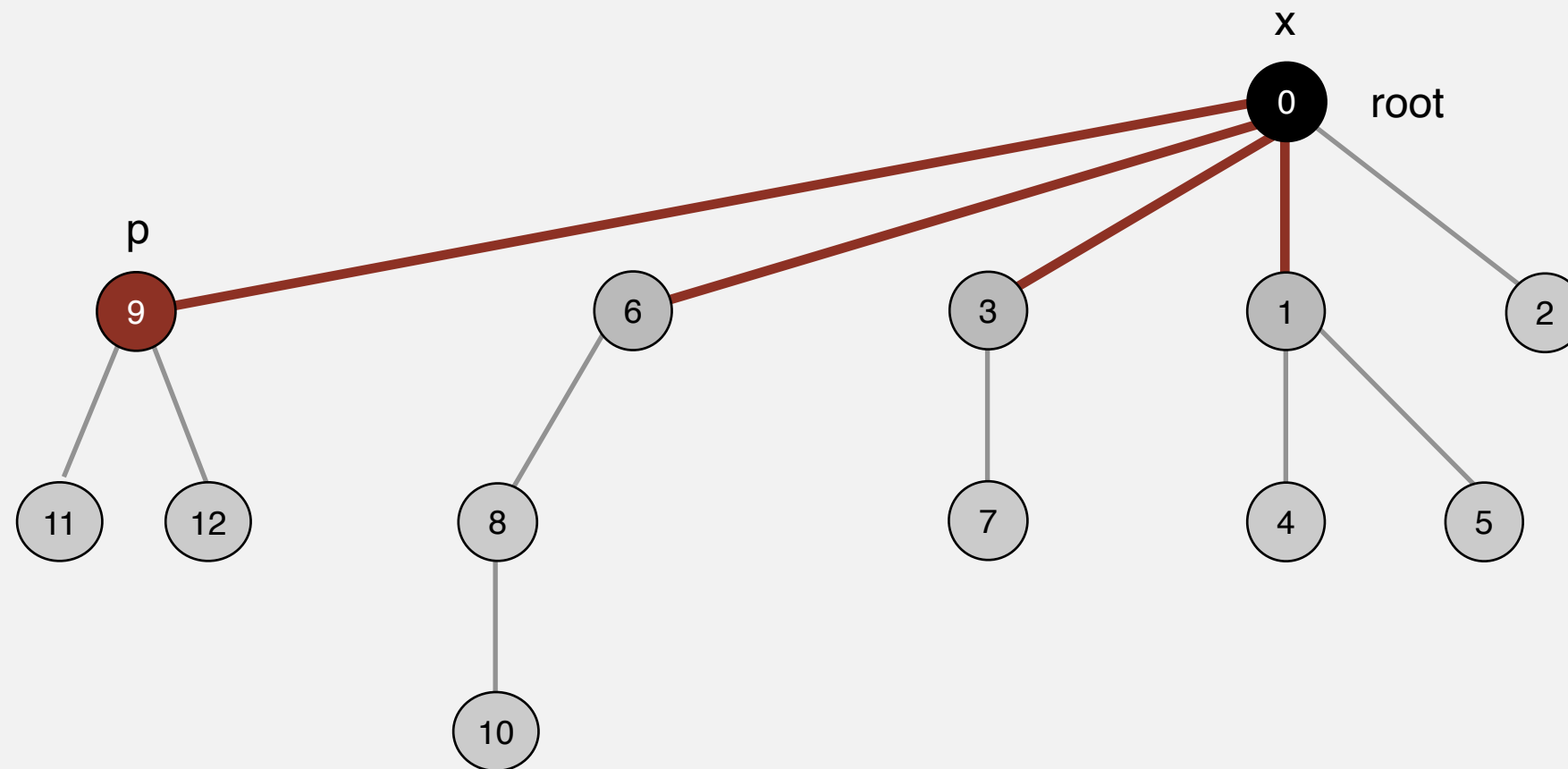
**Quick union with path compression.** Just after computing the root of  $p$ , set the `id[]` of each examined node to point to that root.



## Improvement 2: path compression

---

**Quick union with path compression.** Just after computing the root of  $p$ , set the `id[]` of each examined node to point to that root.



**Bottom line.** Now, `find()` has the side effect of compressing the tree.

# Path compression: Java implementation

---

**Two-pass implementation:** add second loop to find() to set the id[] of each examined node to the root.

**Simpler one-pass variant (path halving):** Make every other node in path point to its grandparent.

```
public int find(int i)
{
    while (i != id[i])
    {
        id[i] = id[id[i]];
        i = id[i];
    }
    return i;
}
```

← only one extra line of code !

**In practice.** No reason not to! Keeps tree almost completely flat.



# Weighted quick-union with path compression: amortized analysis

**Proposition.** [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of  $M$  union–find ops on  $N$  objects makes  $\leq c (N + M \lg^* N)$  array accesses.

- Analysis can be improved to  $N + M \alpha(M, N)$ .
- Simple algorithm with fascinating mathematics.


N	$\lg^* N$
1	0
2	1
4	2
16	3
65536	4
$2^{65536}$	5

iterated lg function

Linear-time algorithm for  $M$  union-find ops on  $N$  objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

**Amazing fact.** [Fredman-Saks] No linear-time algorithm exists.

  
in "cell-probe" model of computation

# Summary

---

**Key point.** Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
<b>quick-find</b>	$M N$
<b>quick-union</b>	$M N$
<b>weighted QU</b>	$N + M \log N$
<b>QU + path compression</b>	$N + M \log N$
<b>weighted QU + path compression</b>	$N + M \lg^* N$

order of growth for  $M$  union-find operations on a set of  $N$  objects

**Ex.** [ $10^9$  unions and finds with  $10^9$  objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.



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## 1.5 UNION-FIND

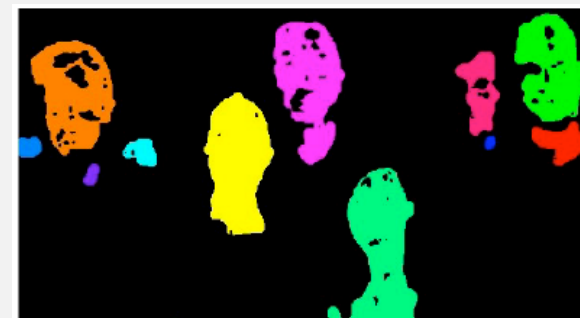
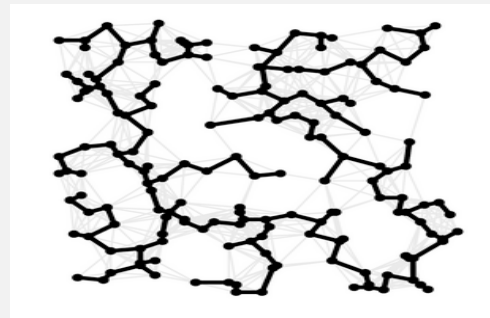
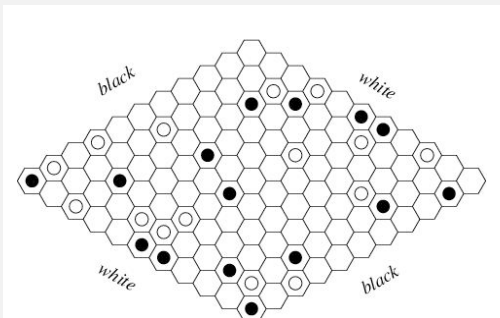
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- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

# Union-find applications

---

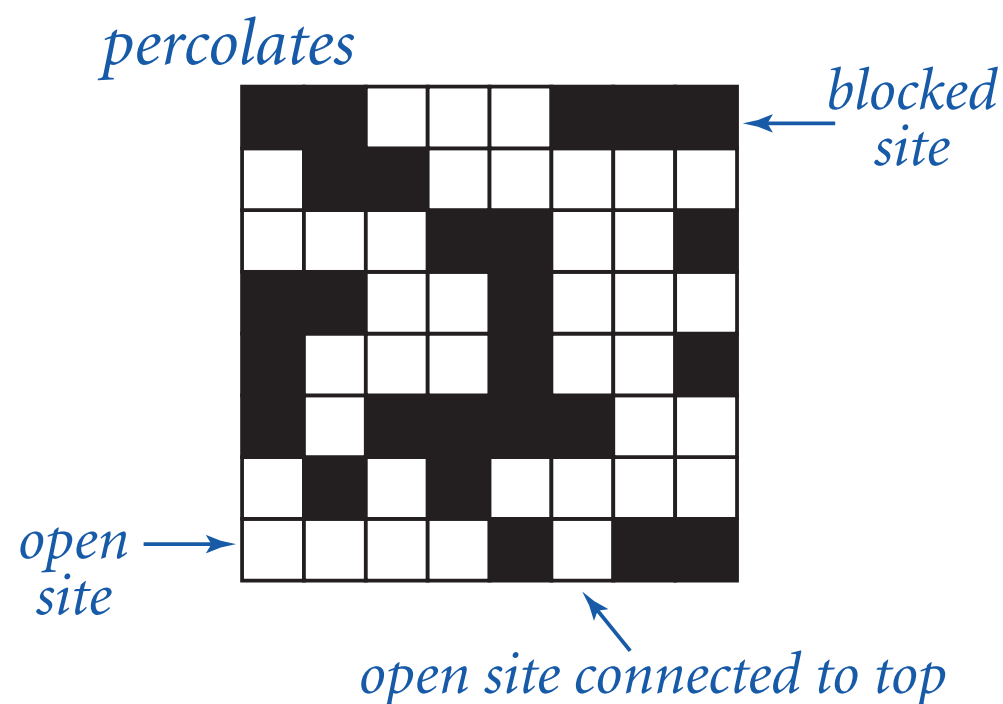
- Percolation.
- Games (Go, Hex).
- ✓ Dynamic connectivity.
  - Least common ancestor.
  - Equivalence of finite state automata.
  - Hoshen-Kopelman algorithm in physics.
  - Hinley-Milner polymorphic type inference.
  - Kruskal's minimum spanning tree algorithm.
  - Compiling equivalence statements in Fortran.
  - Morphological attribute openings and closings.
  - Matlab's `bwlabel()` function in image processing.



# Percolation

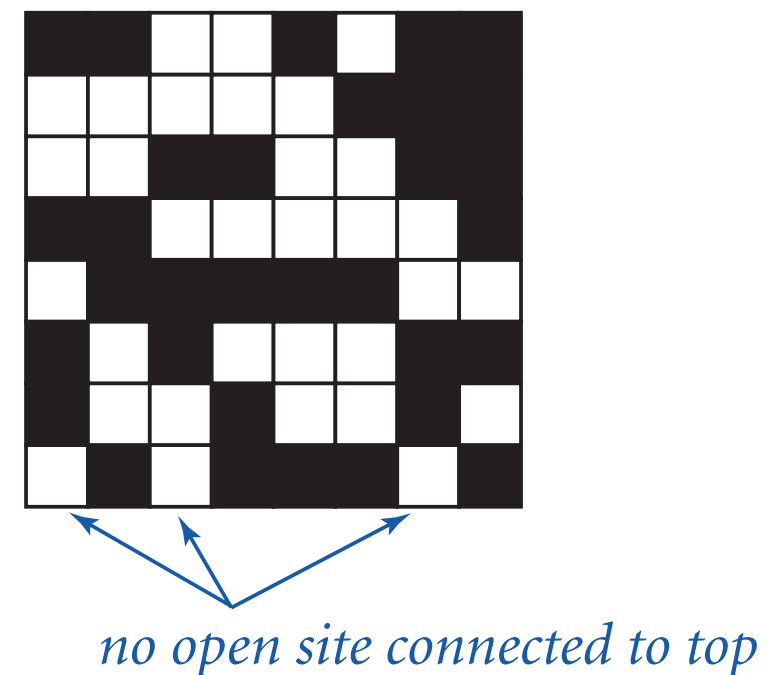
An abstract model for many physical systems:

- $N$ -by- $N$  grid of sites.
- Each site is open with probability  $p$  (and blocked with probability  $1 - p$ ).
- System **percolates** iff top and bottom are connected by open sites.



$N = 8$

*does not percolate*



# Percolation

---

An abstract model for many physical systems:

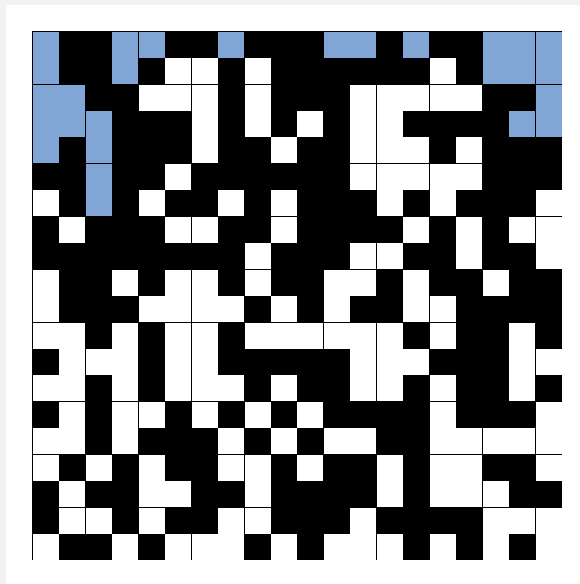
- $N$ -by- $N$  grid of sites.
- Each site is open with probability  $p$  (and blocked with probability  $1 - p$ ).
- System **percolates** iff top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

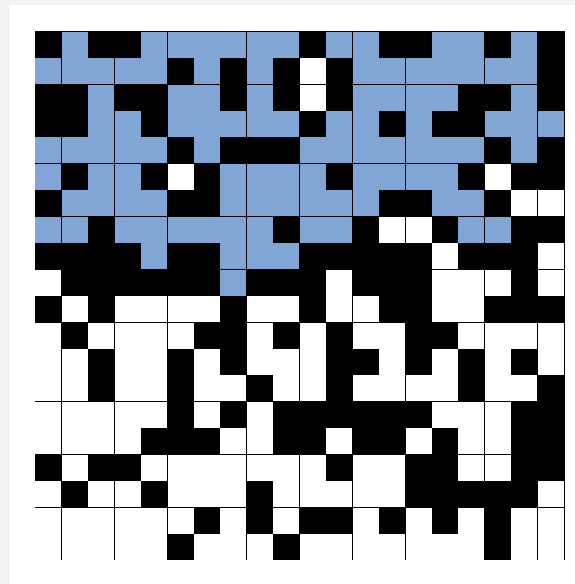
# Likelihood of percolation

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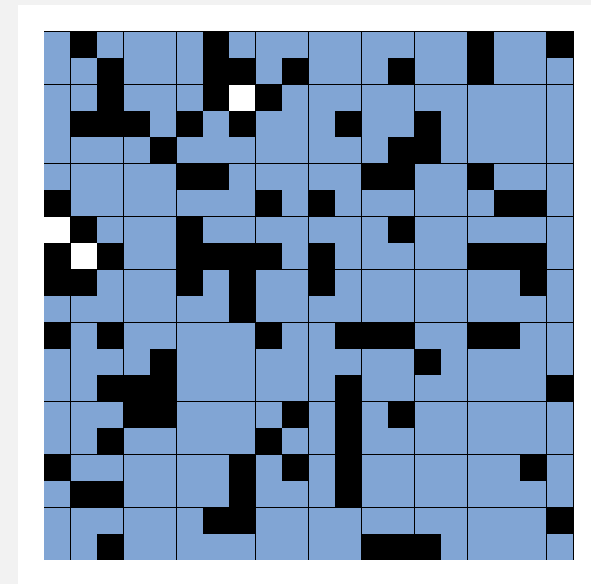
Depends on grid size  $N$  and site vacancy probability  $p$ .



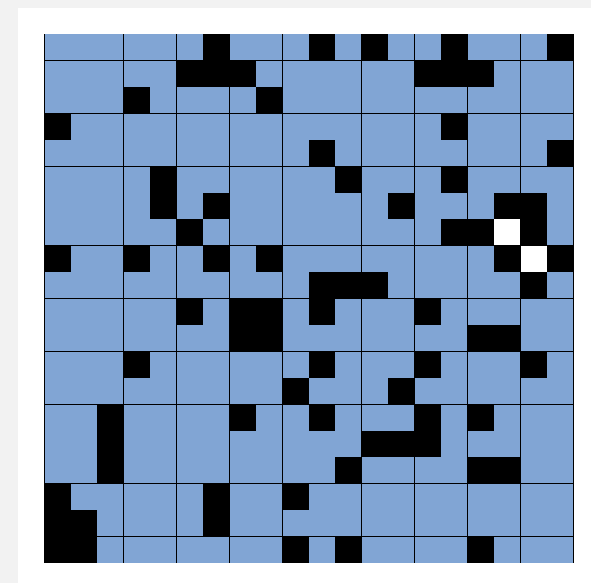
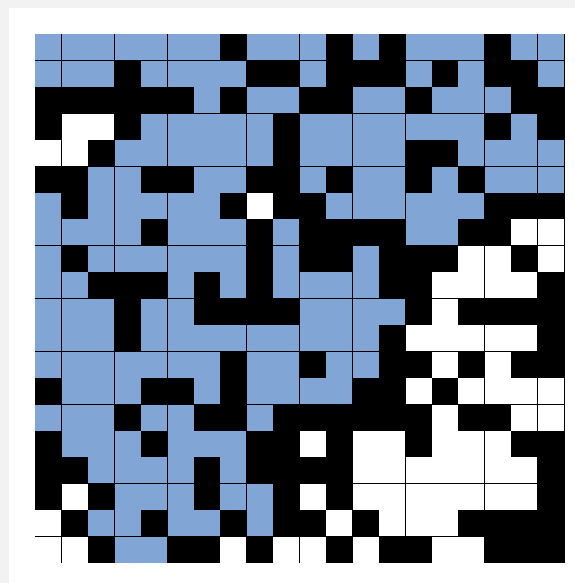
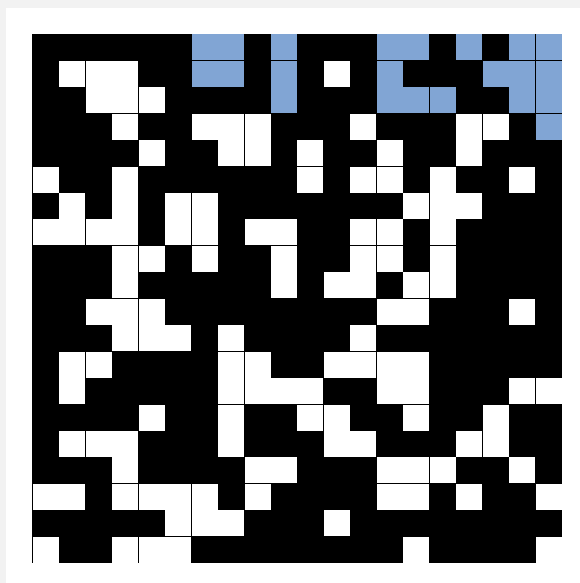
**p low (0.4)**  
**does not percolate**



**p medium (0.6)**  
**percolates?**



**p high (0.8)**  
**percolates**

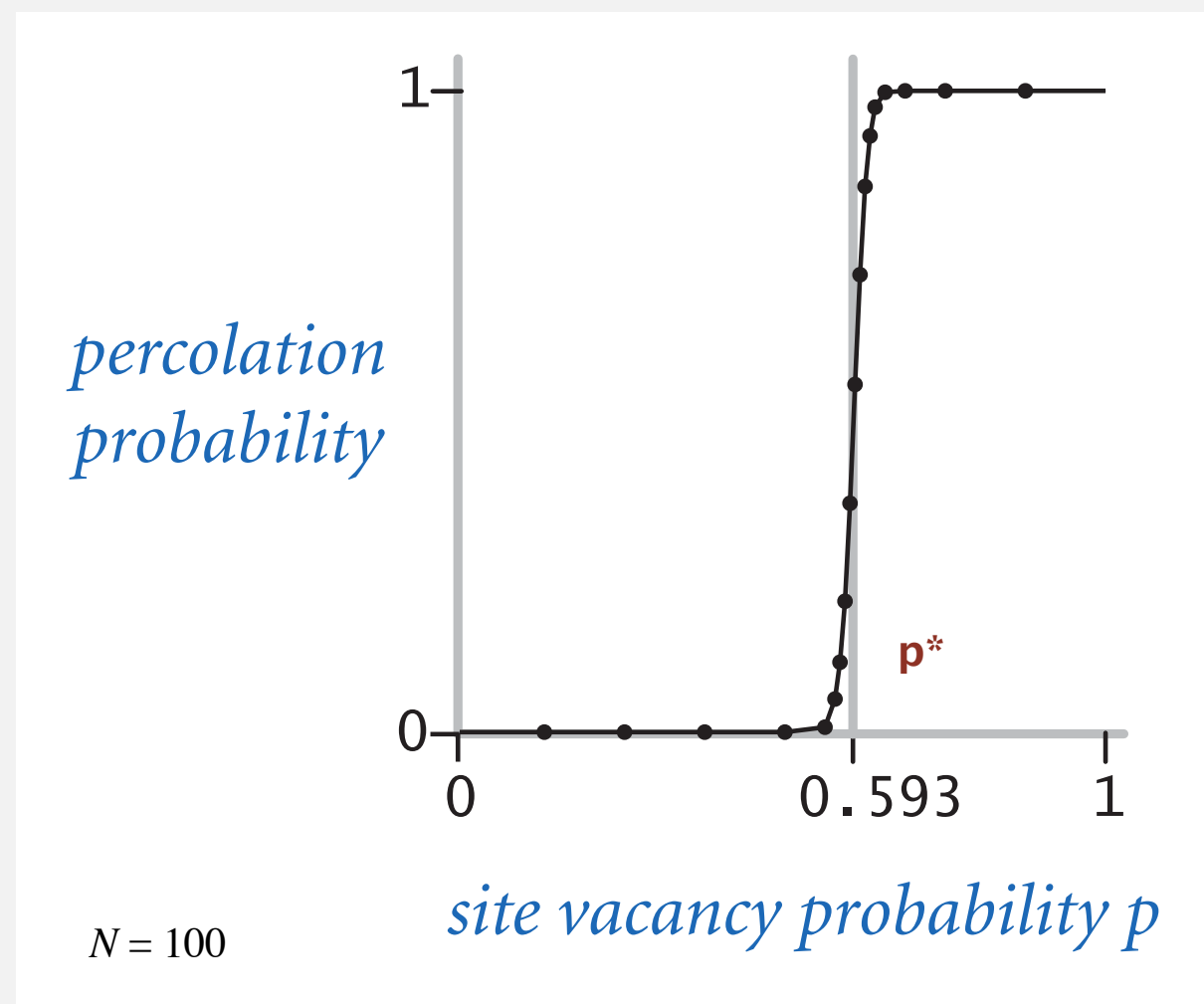


# Percolation phase transition

When  $N$  is large, theory guarantees a sharp threshold  $p^*$ .

- $p > p^*$ : almost certainly percolates.
- $p < p^*$ : almost certainly does not percolate.

Q. What is the value of  $p^*$  ?

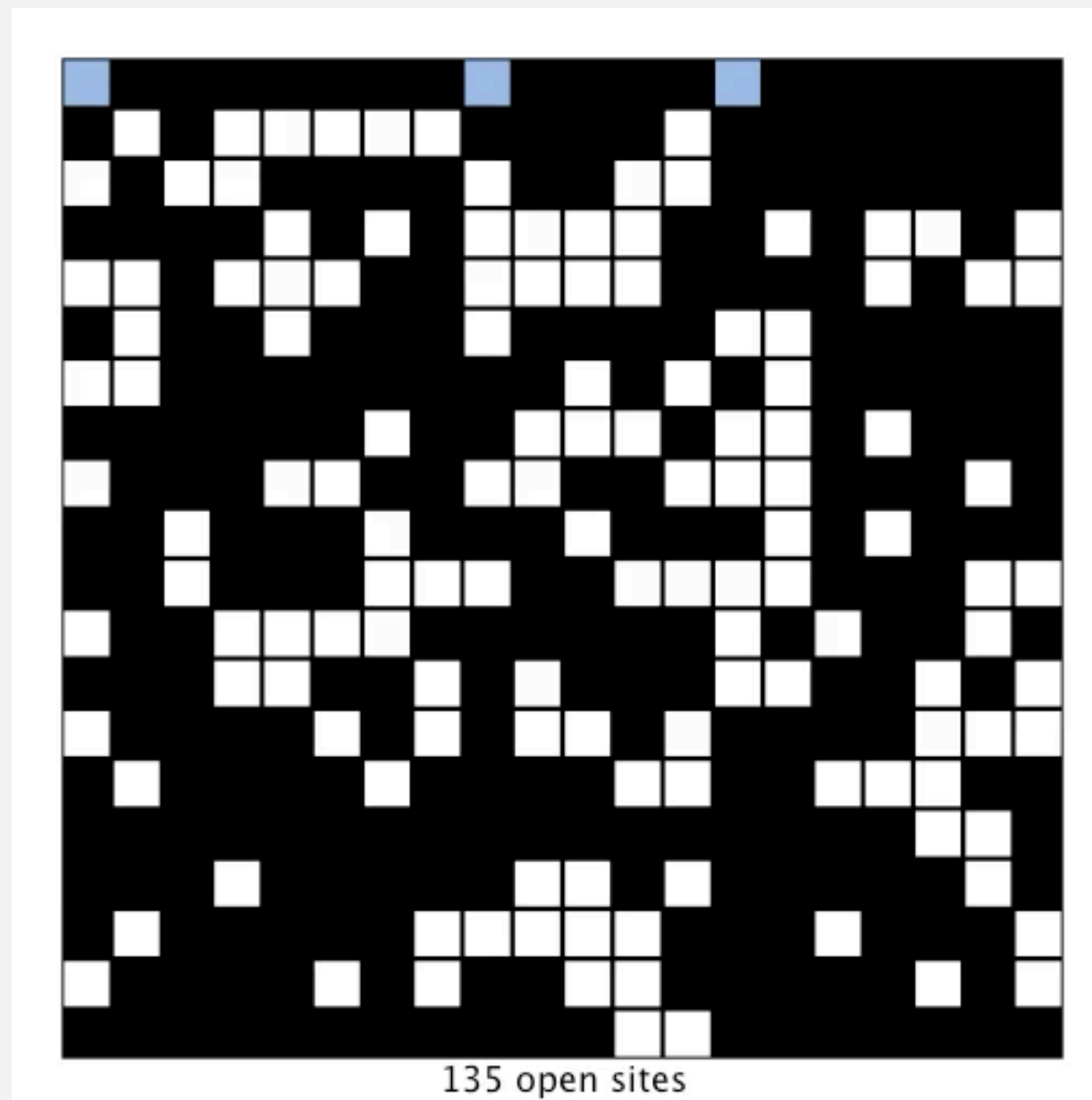




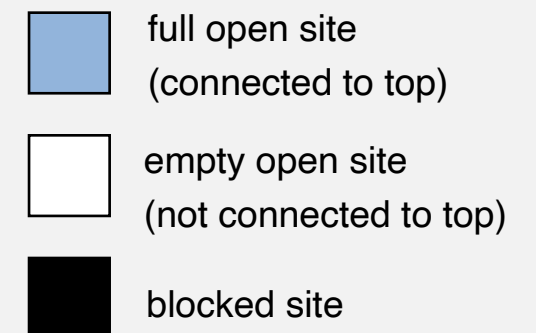
# Monte Carlo simulation

---

- Initialize all sites in an  $N$ -by- $N$  grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates  $p^*$ .



$N = 20$



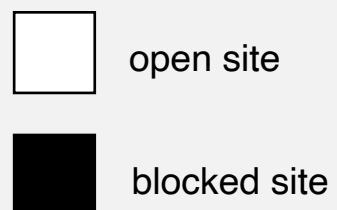
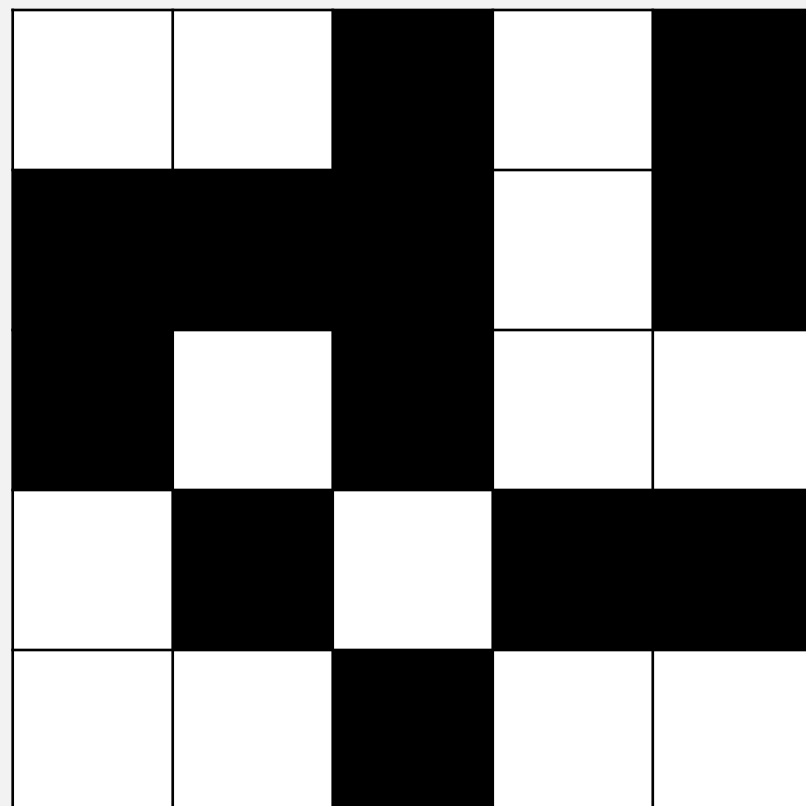
# Dynamic connectivity solution to estimate percolation threshold

---

Q. How to check whether an  $N$ -by- $N$  system percolates?

A. Model as a **dynamic connectivity** problem and use **union-find**.

$N = 5$

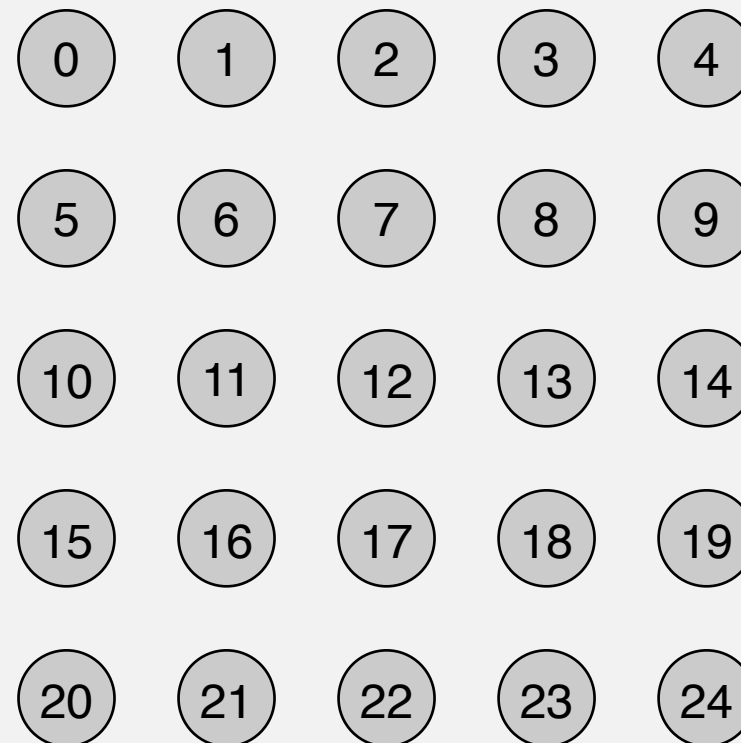
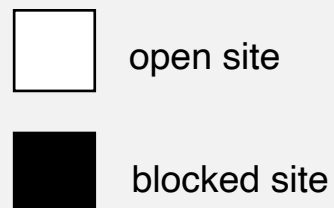
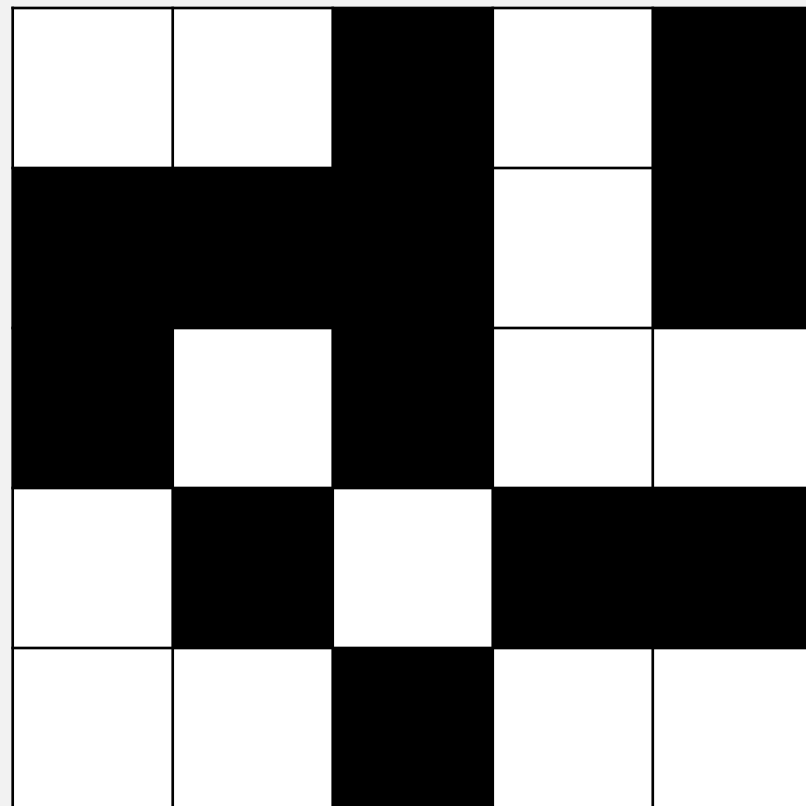


# Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an  $N$ -by- $N$  system percolates?

- Create an object for each site and name them 0 to  $N^2 - 1$ .

$N = 5$

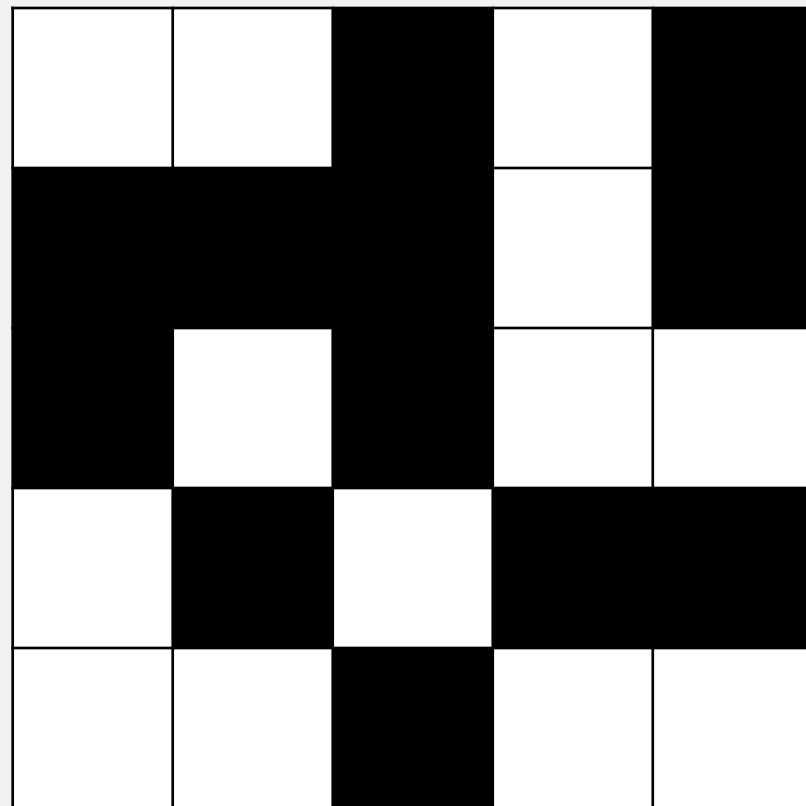


# Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an  $N$ -by- $N$  system percolates?

- Create an object for each site and name them  $0$  to  $N^2 - 1$ .
- Sites are in same component iff connected by open sites.

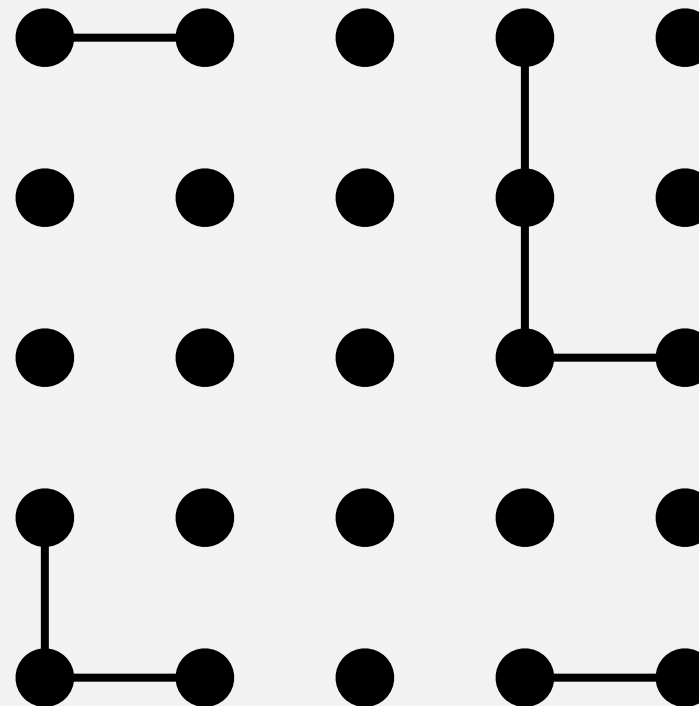
$N = 5$



open site



blocked site



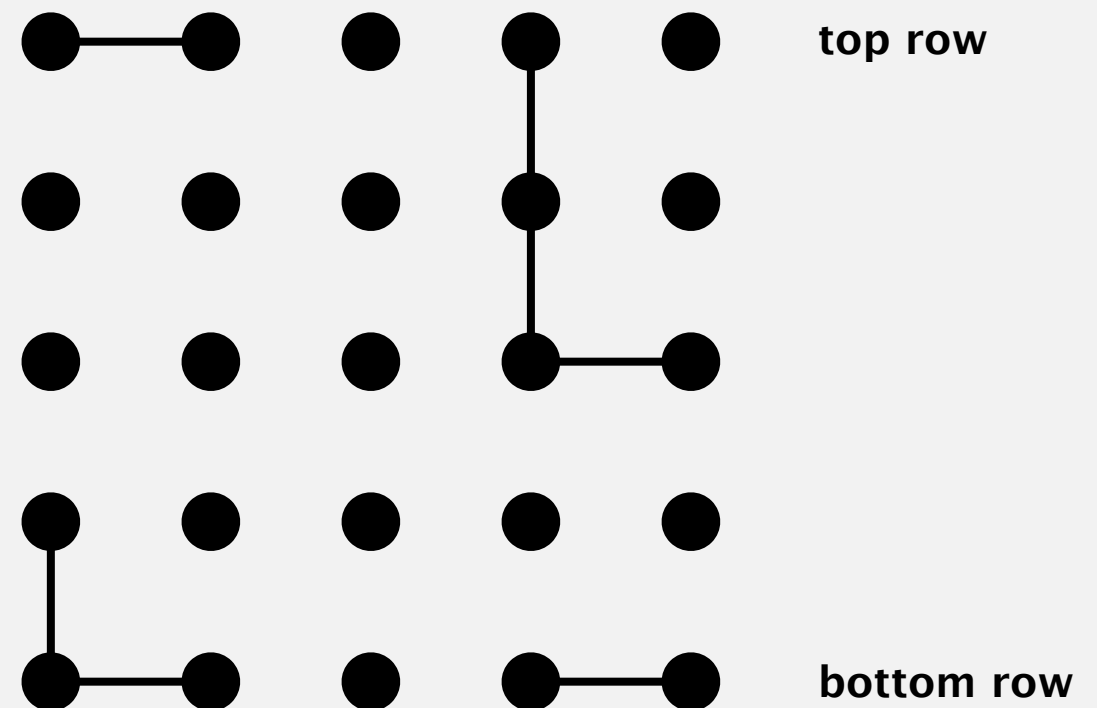
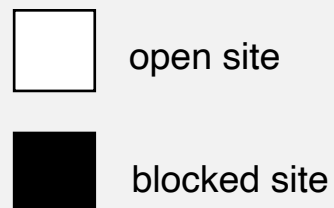
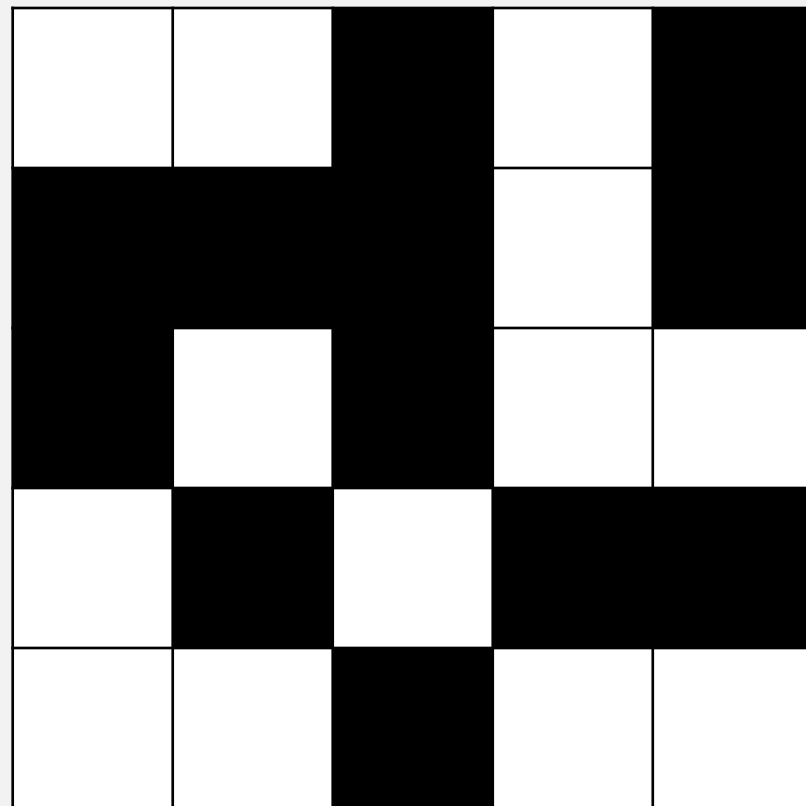
# Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an  $N$ -by- $N$  system percolates?

- Create an object for each site and name them  $0$  to  $N^2 - 1$ .
- Sites are in same component iff connected by open sites.
- Percolates iff any site on bottom row is connected to any site on top row.

brute-force algorithm:  $N^2$  calls to `connected()`

$N = 5$



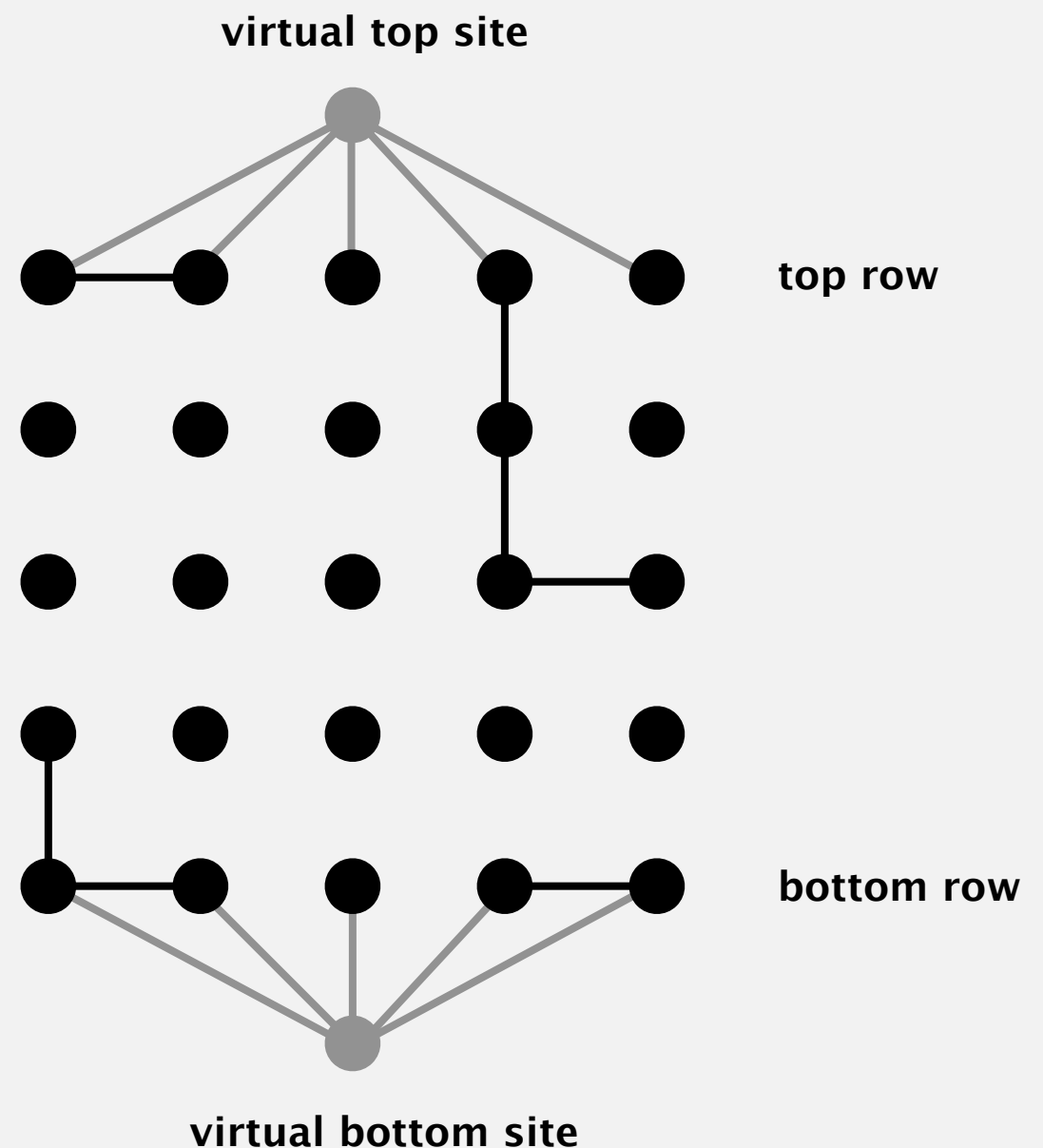
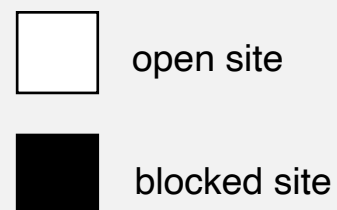
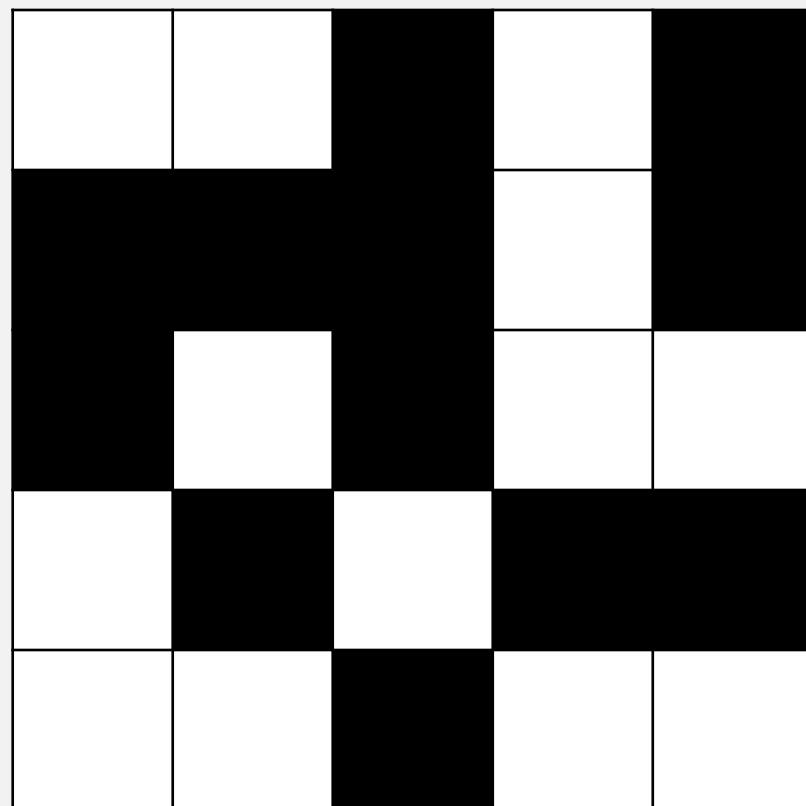
# Dynamic connectivity solution to estimate percolation threshold

**Clever trick.** Introduce 2 virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

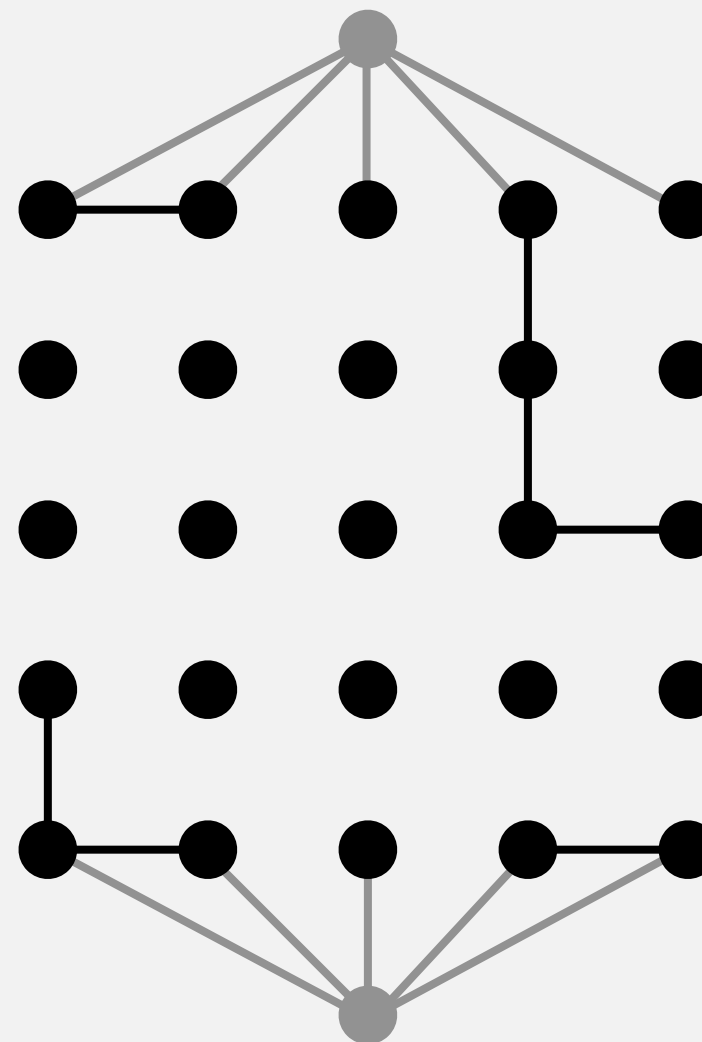
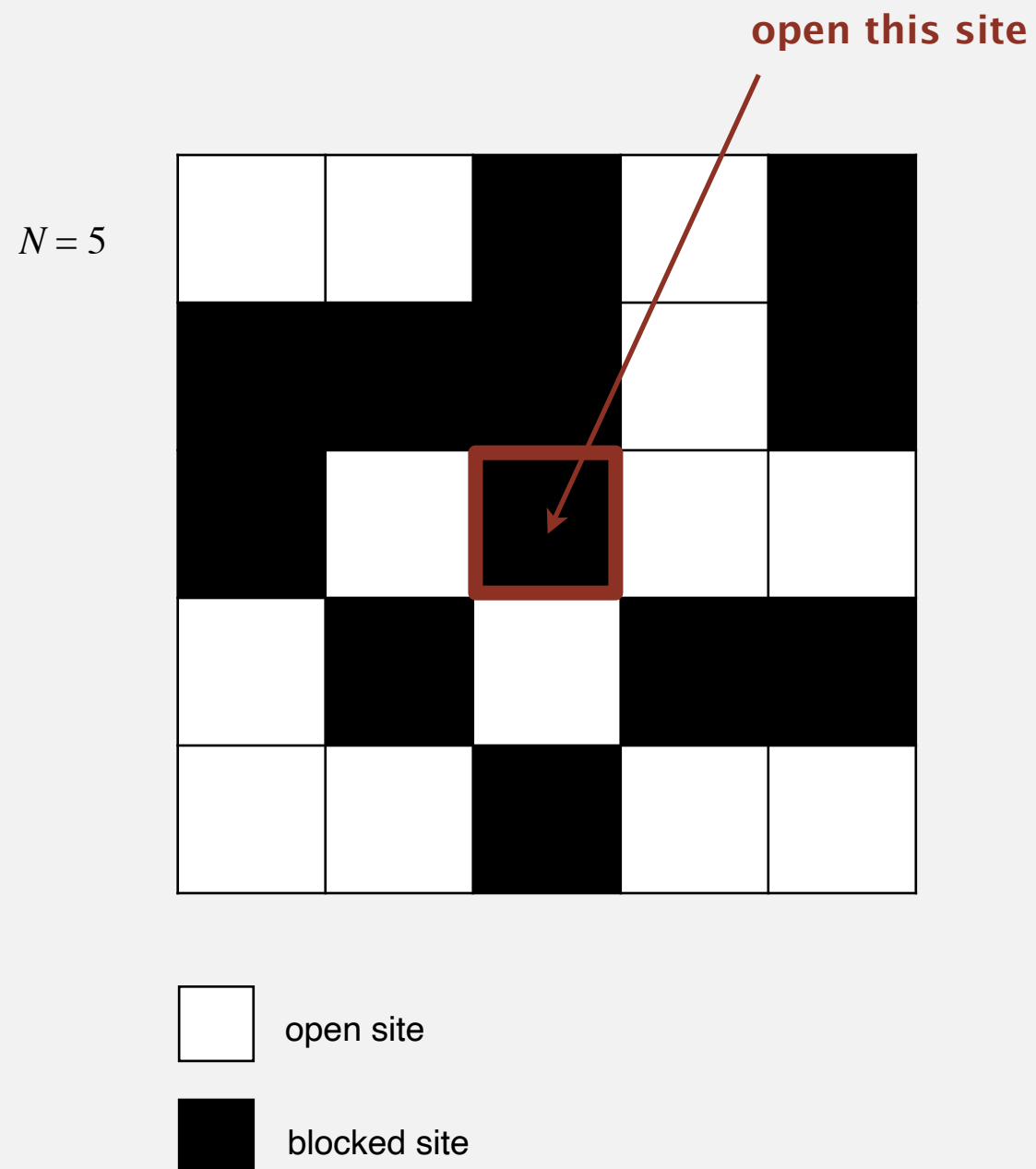
more efficient algorithm: only 1 call to connected()

$N = 5$



# Dynamic connectivity solution to estimate percolation threshold

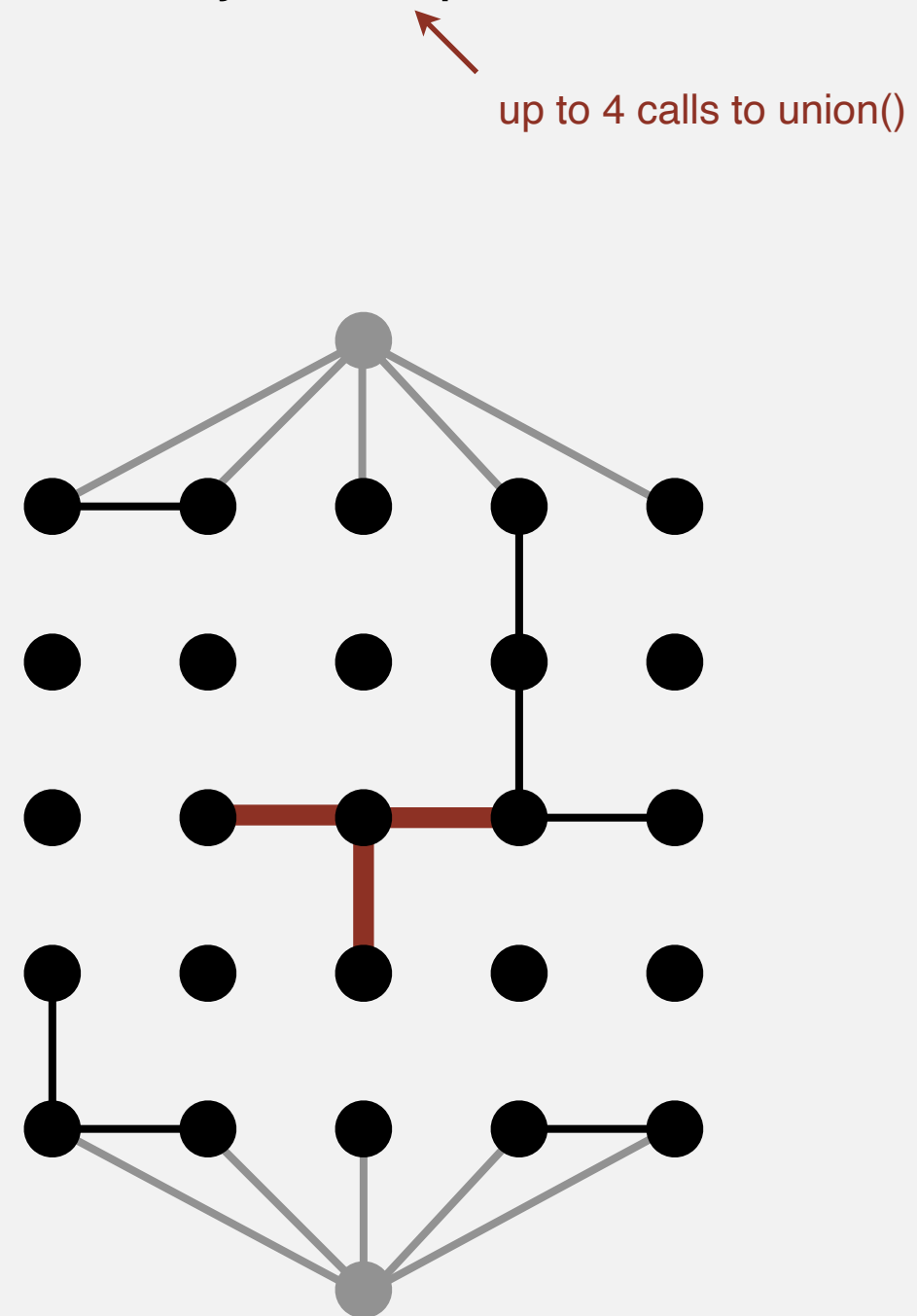
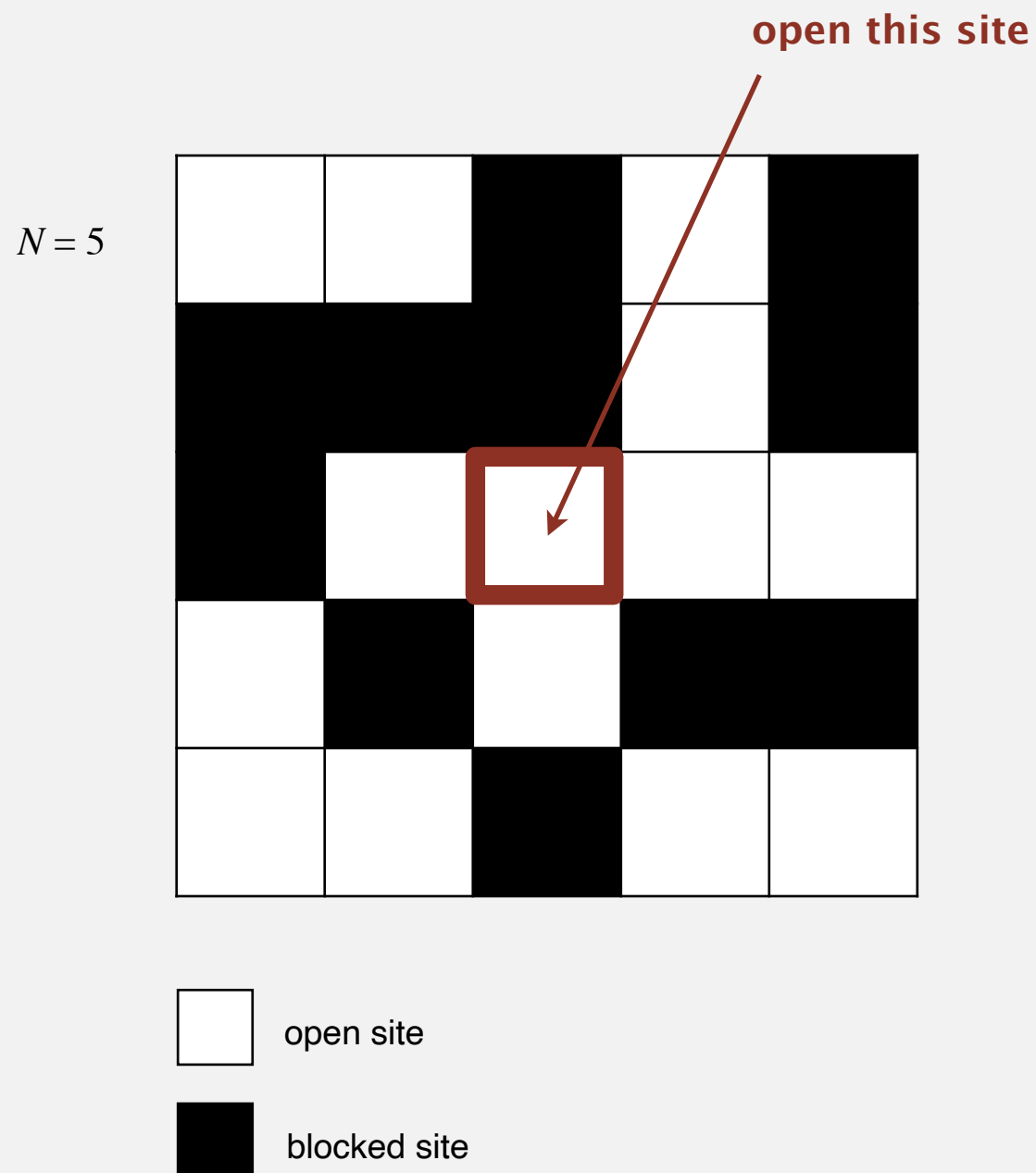
Q. How to model opening a new site?



# Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

A. Mark new site as open; connect it to all of its adjacent open sites.



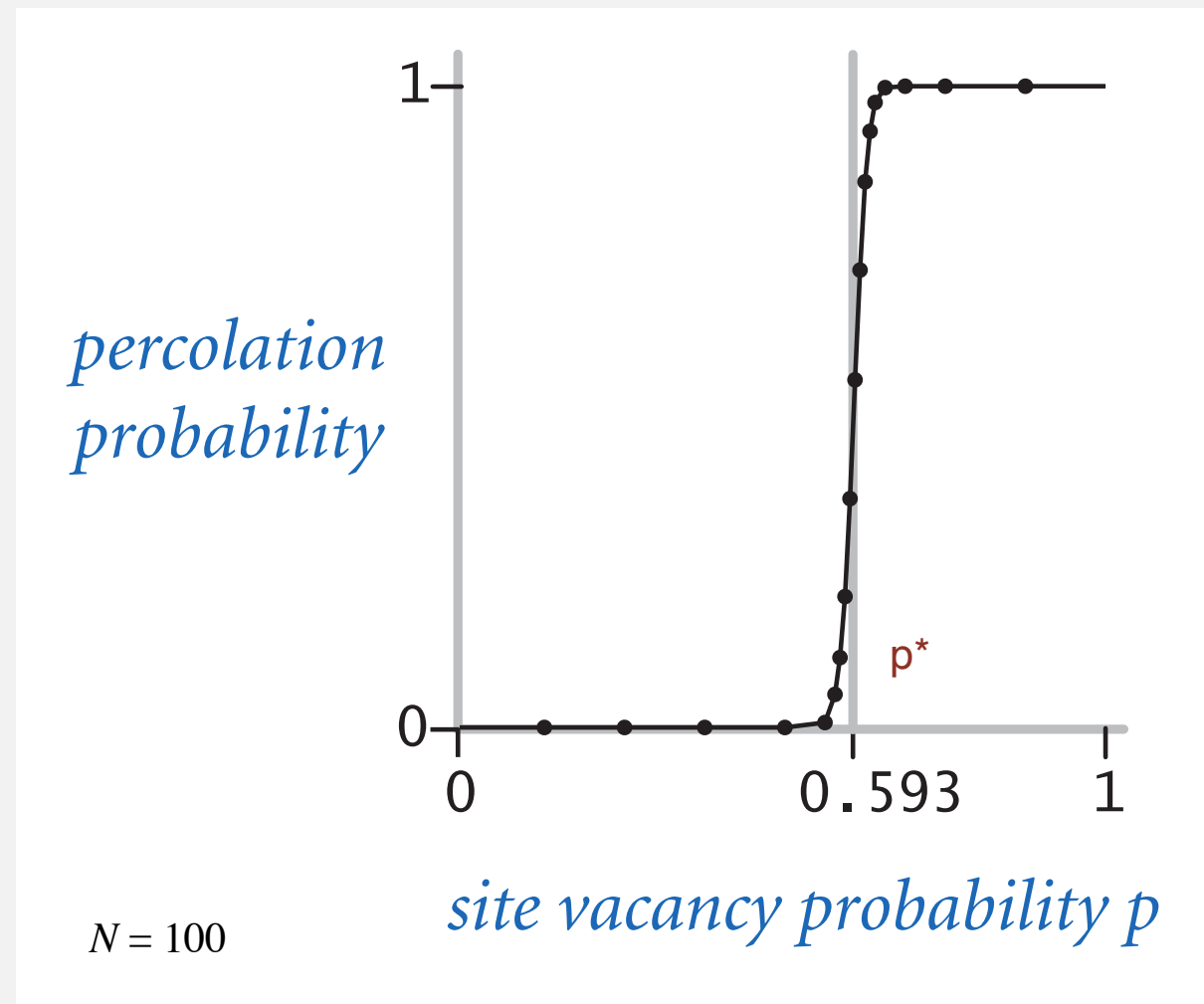


# Percolation threshold

Q. What is percolation threshold  $p^*$  ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm **enables** accurate answer to scientific question.

# Subtext of today's lecture (and this course)

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## Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

## The scientific method.

## Mathematical analysis.