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1.5 UNION-FIND

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why not.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

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1.5 Union-Find

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
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Dynamic connectivity problem

Given a set of N objects, support two operation:

- Connect two objects.
- Is there a path connecting the two objects?

connect 4 and 3

connect 3 and 8

connect 6 and 5

connect 9 and 4

connect 2 and 1

are 0 and 7 connected? ✗

are 8 and 9 connected? ✔

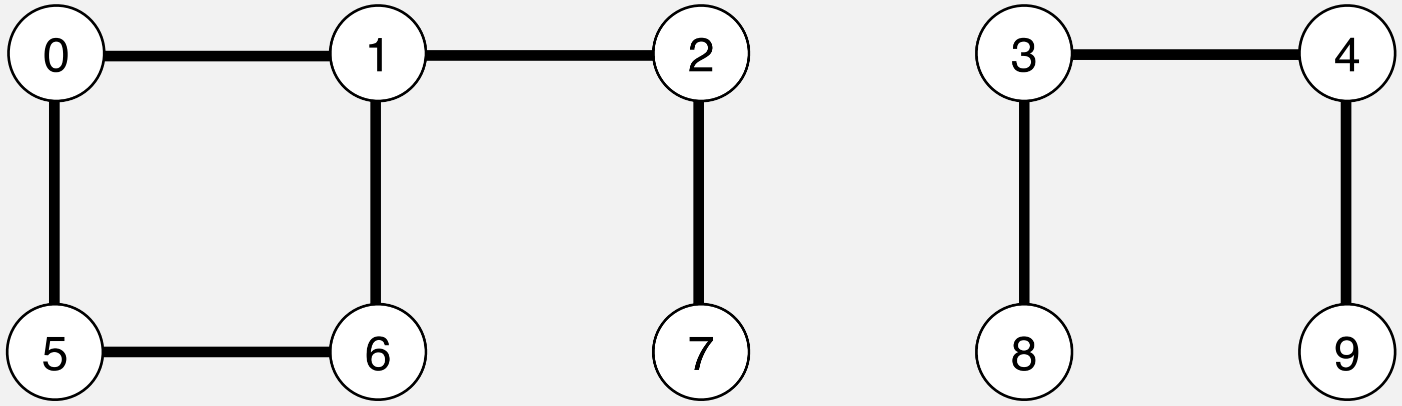
connect 5 and 0

connect 7 and 2

connect 6 and 1

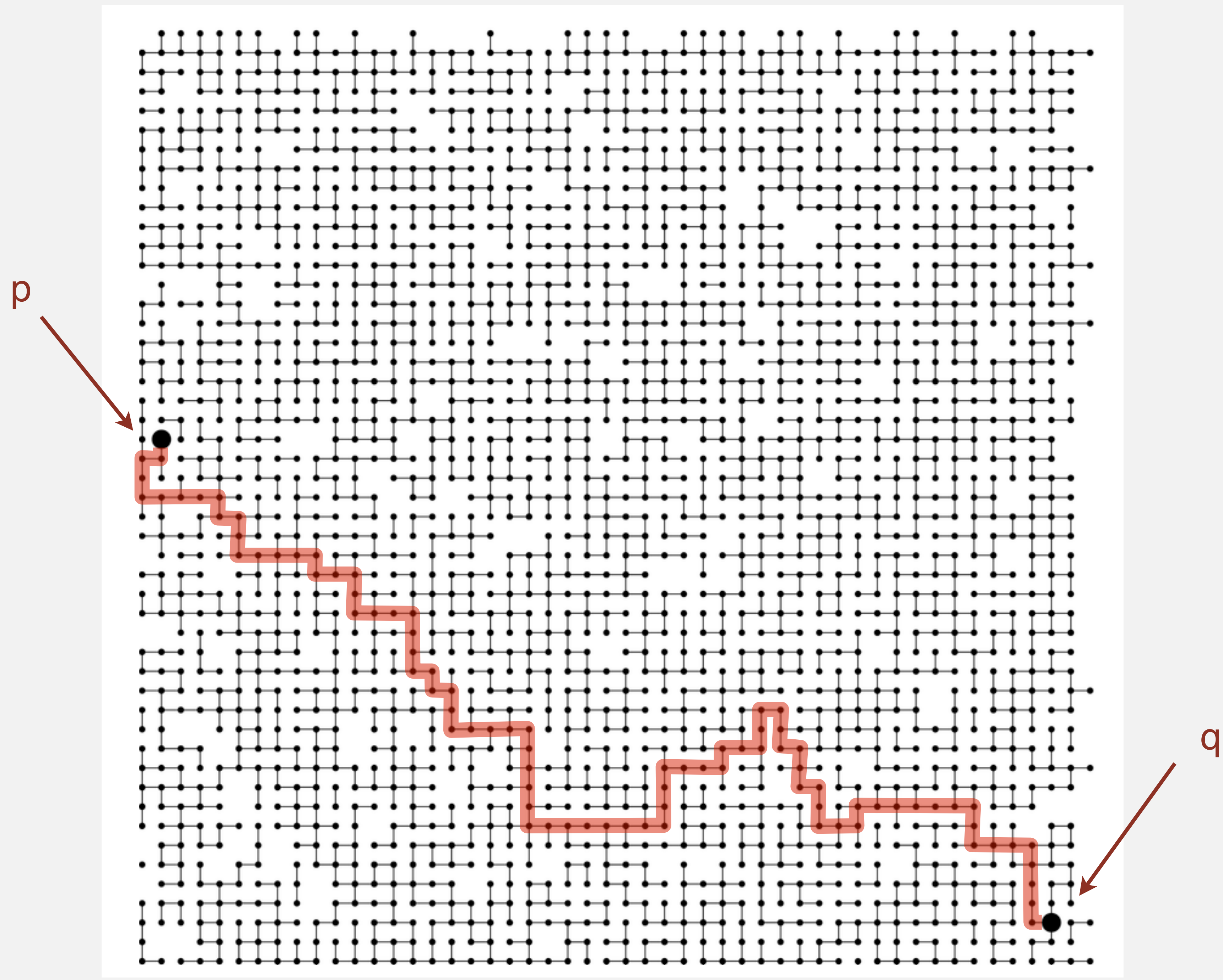
connect 1 and 0

are 0 and 7 connected? ✔



A larger connectivity example

Q. Is there a path connecting p and q ?



A. Yes.

Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to N – 1.

- Use integers as array index.
- Suppress details not relevant to union-find.

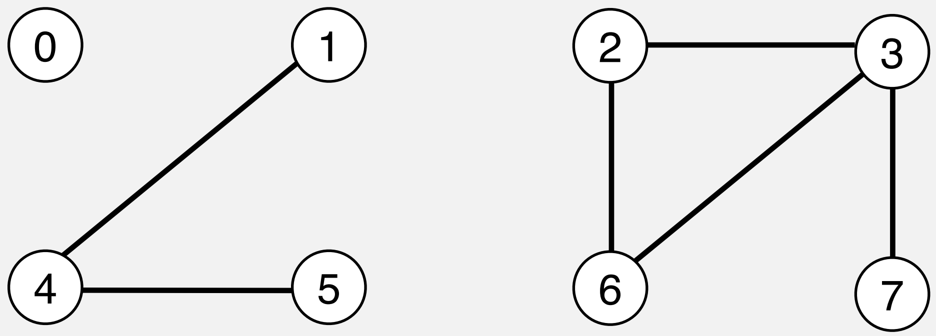
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.

Connected component. Maximal set of objects that are mutually connected.



{ 0 } { 1 4 5 } { 2 3 6 7 }

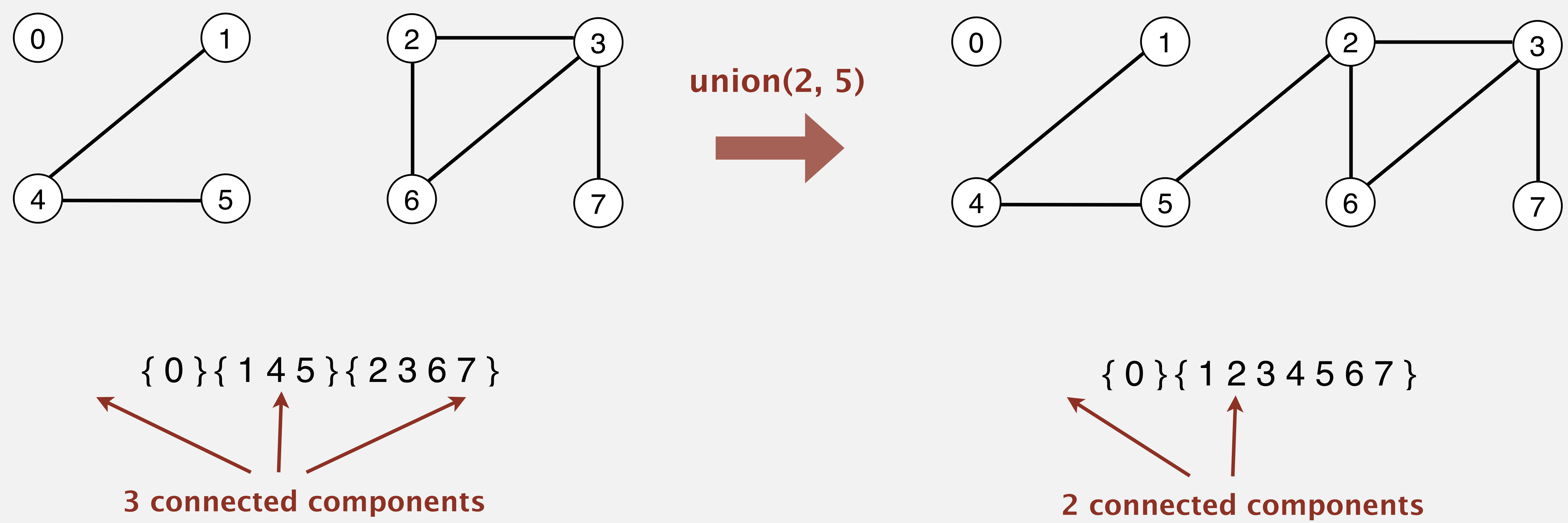
3 connected components

Implementing the operations

Find. In which component is object p ?

Connected. Are objects p and q in the same component?

Union. Replace components containing objects p and q with their union.



Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Union and find operations may be intermixed.

public class UF		
	UF(int N)	*initialize union-find data structure with N singleton objects (0 to N – 1)*
void	union(int p, int q)	*add connection between p and q*
int	find(int p)	*component identifier for p (0 to N – 1)*
boolean	connected(int p, int q)	*are p and q in the same component?*

```
public boolean connected(int p, int q)
{  return find(p) == find(q);  }
```

1-line implementation of connected()

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
  int N = StdIn.readInt();
  UF uf = new UF(N);
  while (!StdIn.isEmpty())
  {
    int p = StdIn.readInt();
    int q = StdIn.readInt();
    if (!uf.connected(p, q))
    {
      uf.union(p, q);
      StdOut.println(p + " " + q);
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

already connected

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1.5 Union-Find

- *dynamic connectivity*
- ***quick find***
- *quick union*
- *improvements*
- *applications*

Quick-find [eager approach]

Data structure.

if and only if

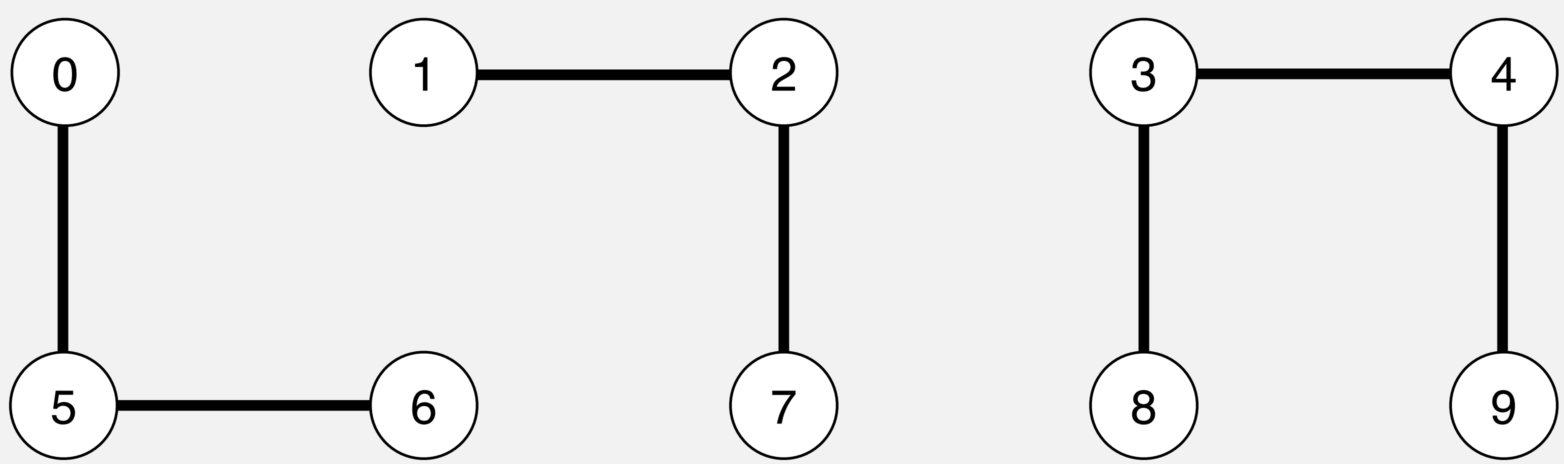
- Integer array id[] of length N.
- Interpretation: id[p] is the id of the component containing p.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected

1, 2, and 7 are connected

3, 4, 8, and 9 are connected



Quick-find [eager approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[p] is the id of the component containing p.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Find. What is the id of p?

Connected. Do p and q have the same id?

id[6] = 0; id[1] = 1

6 and 1 are not connected

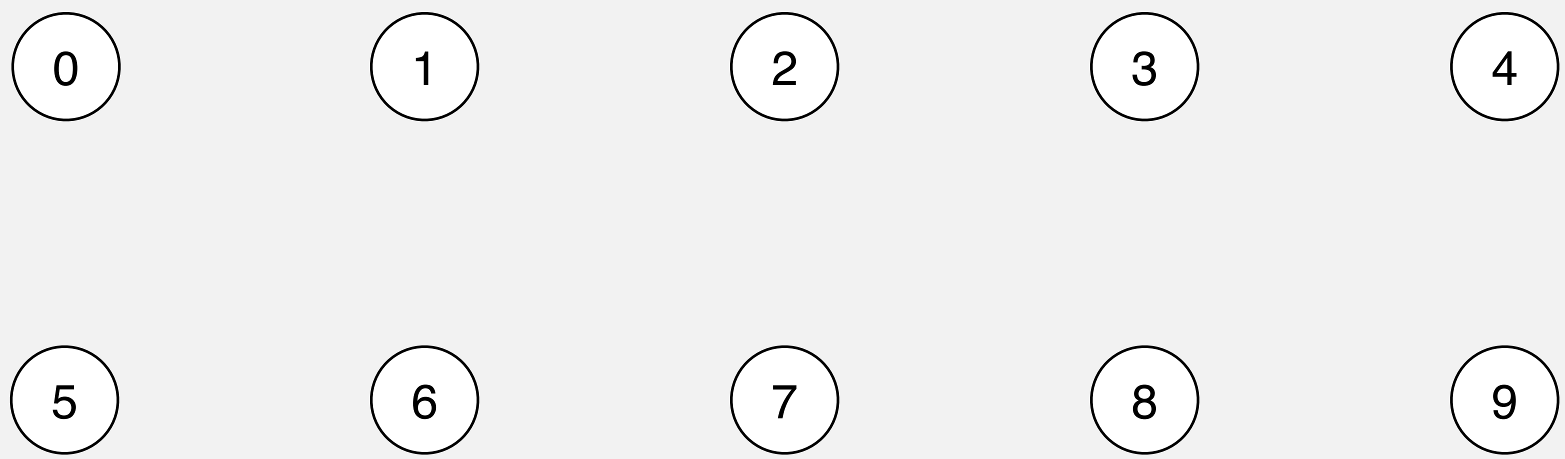
Union. To merge components containing p and q, change all entries whose id equals id[p] to id[q].

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

problem: many values can change

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-find demo

union(4, 3)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

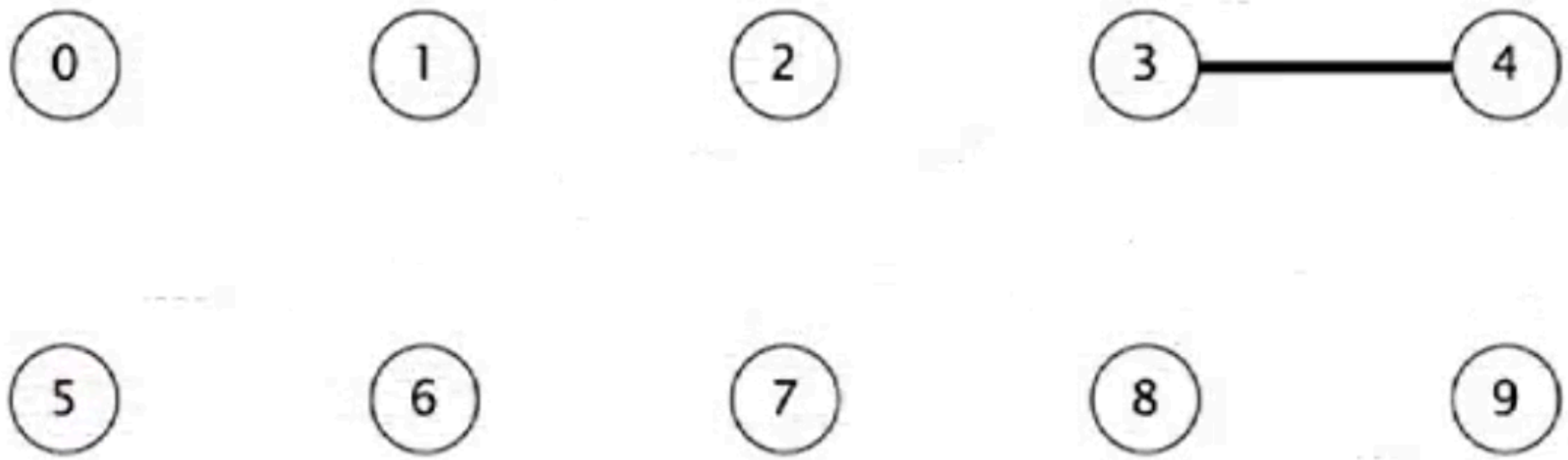
Quick-find demo

union(4, 3)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

Quick-find demo

union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

Quick-find demo

union(3, 8)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	6	7	8	9

Quick-find demo

union(3, 8)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	6	7	8	9

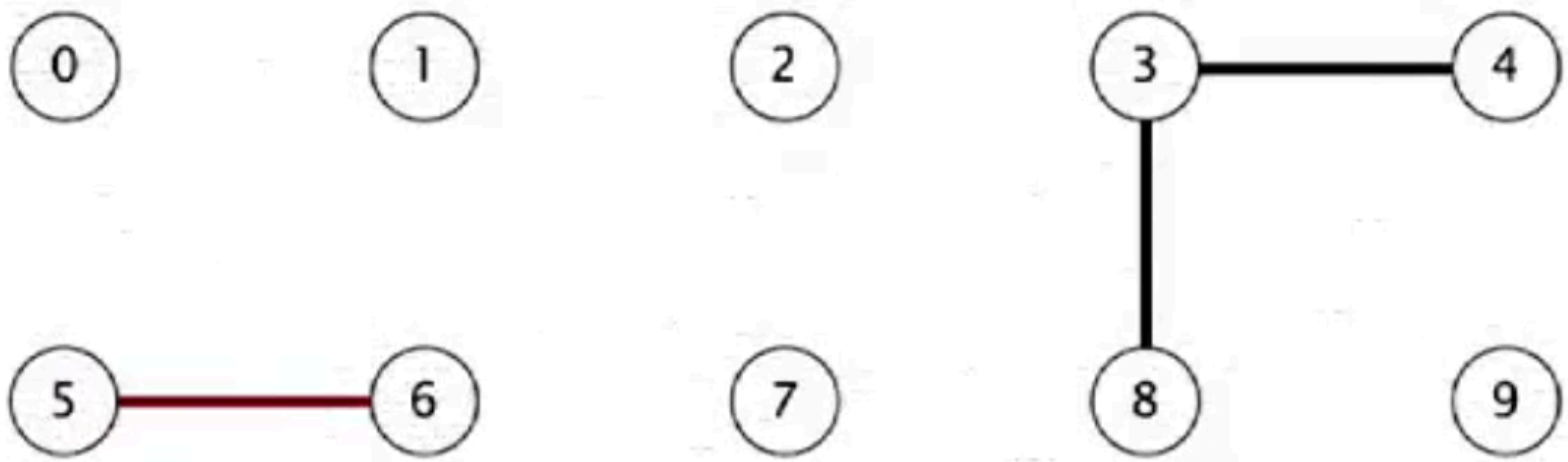
Quick-find demo

union(6, 5)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	6	7	8	9

Quick-find demo

union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	9

Quick-find demo

union(9, 4)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	9

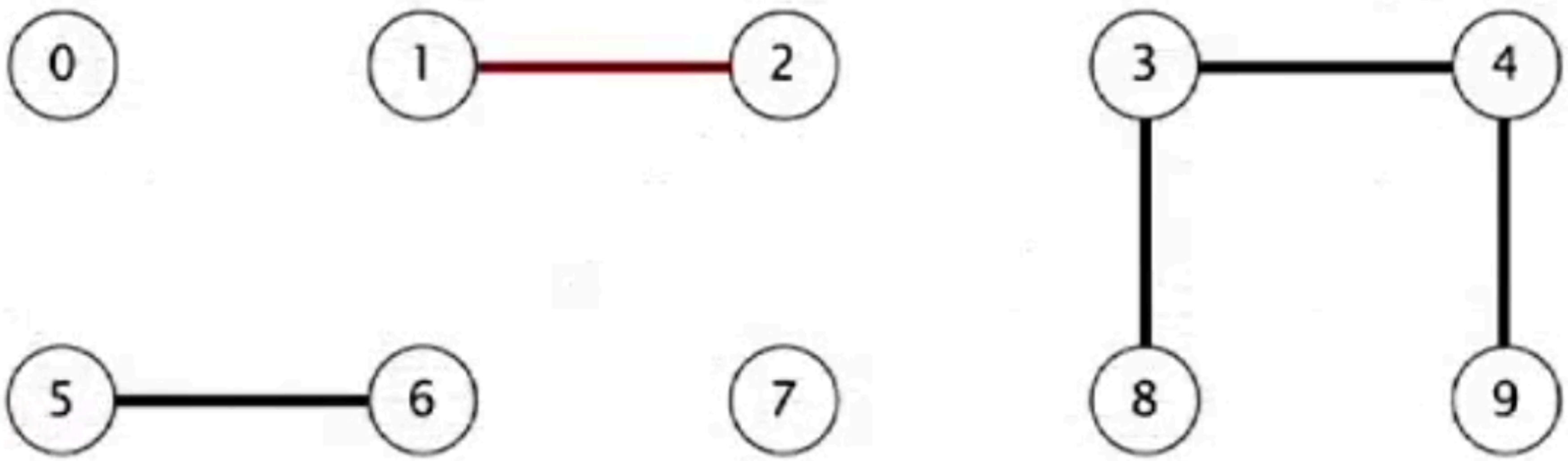
Quick-find demo

union(9, 4)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

Quick-find demo

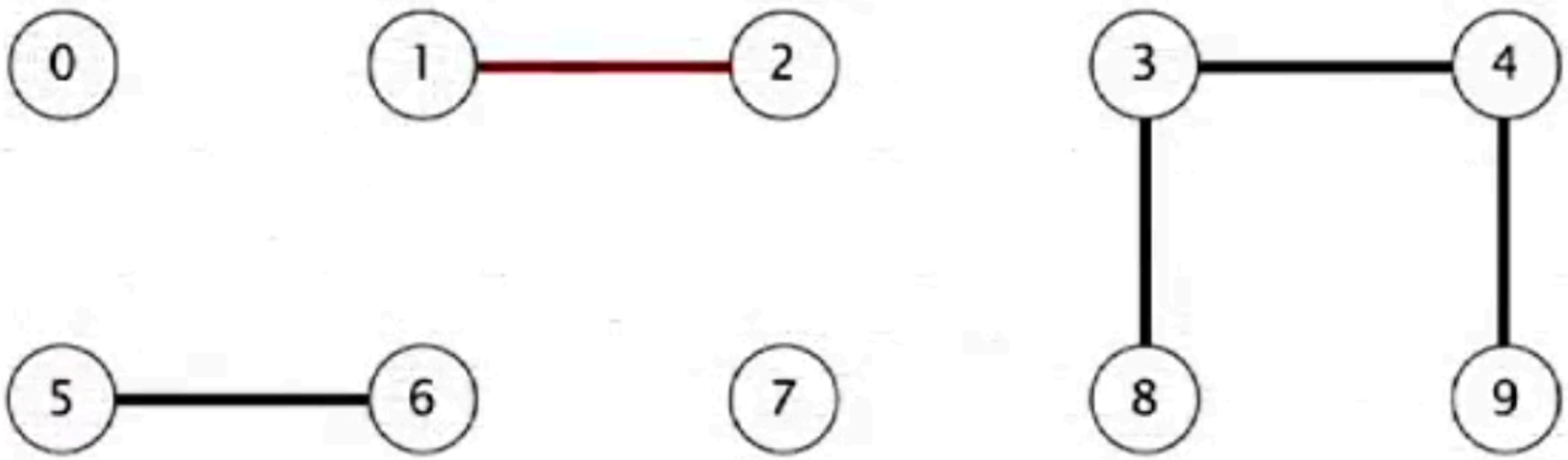
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	8	5	5	7	8	8

Quick-find demo

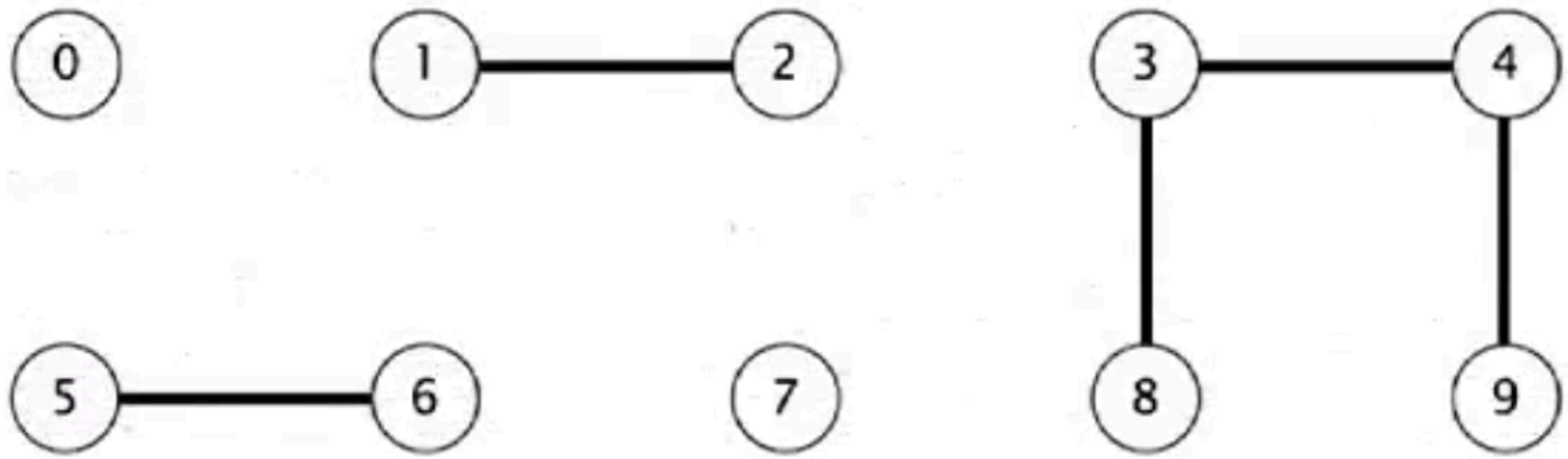
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

connected(8, 9)

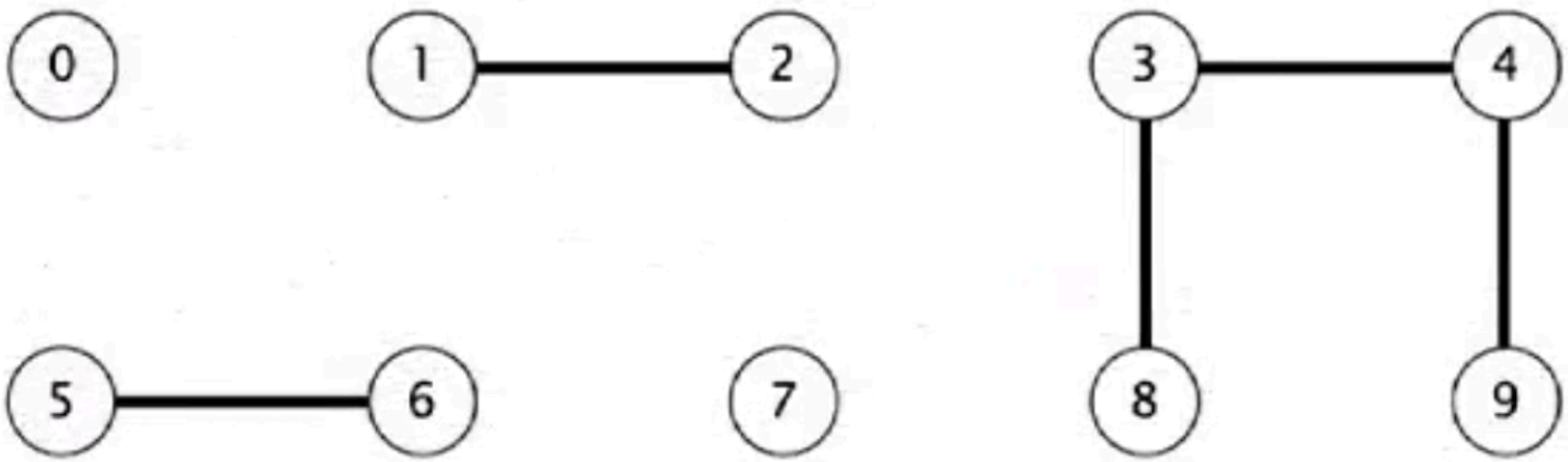


	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

already connected

Quick-find demo

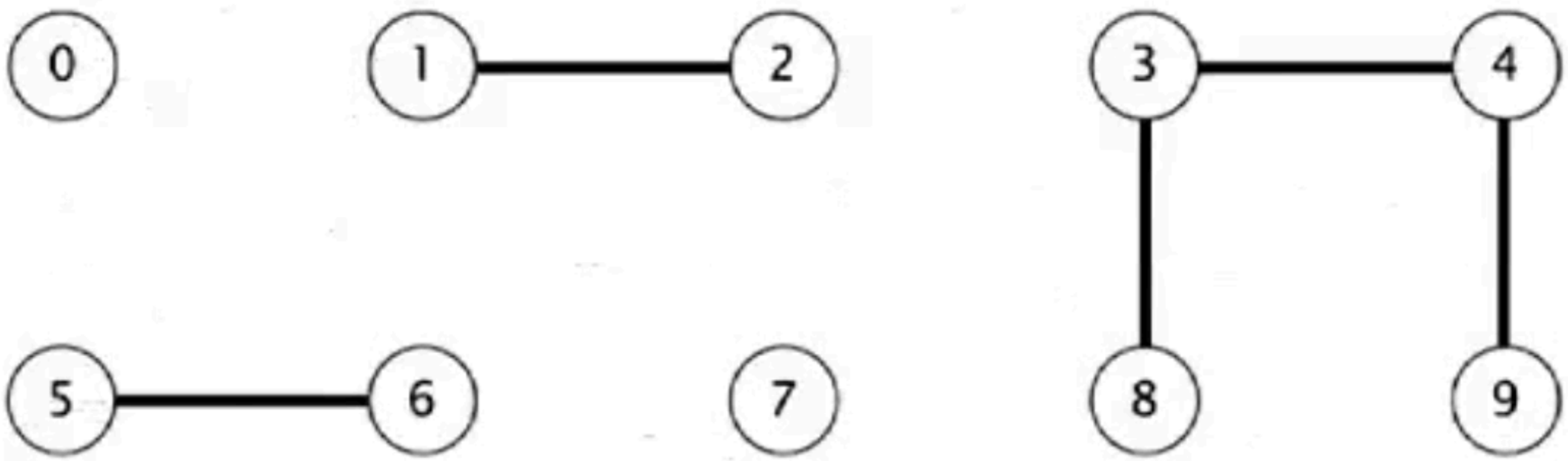
connected(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

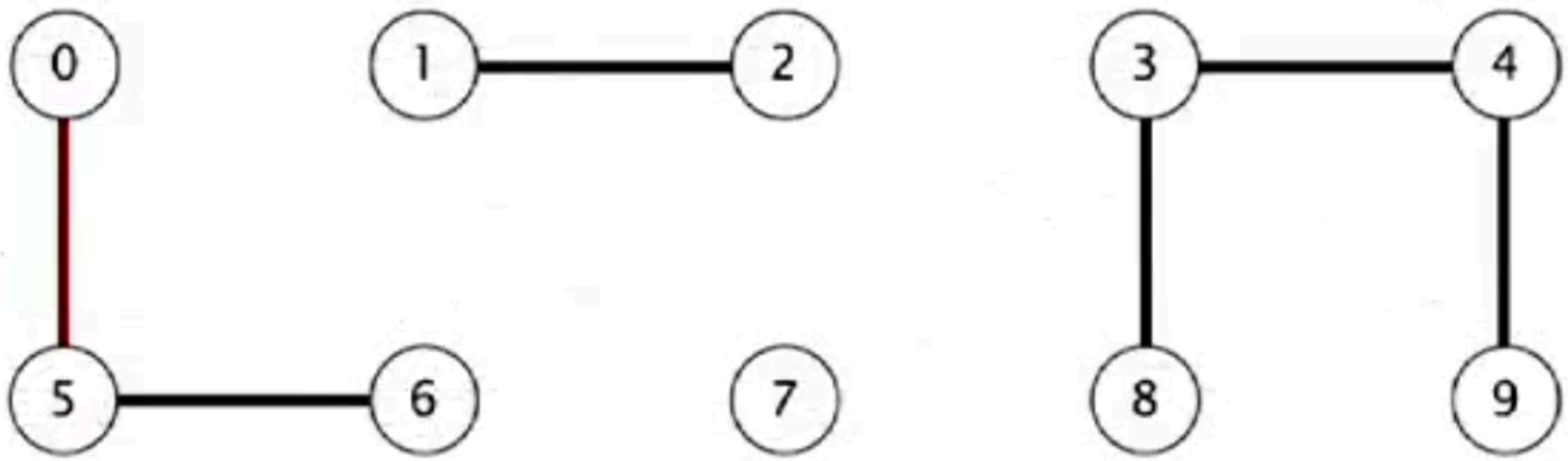
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	5	5	7	8	8

Quick-find demo

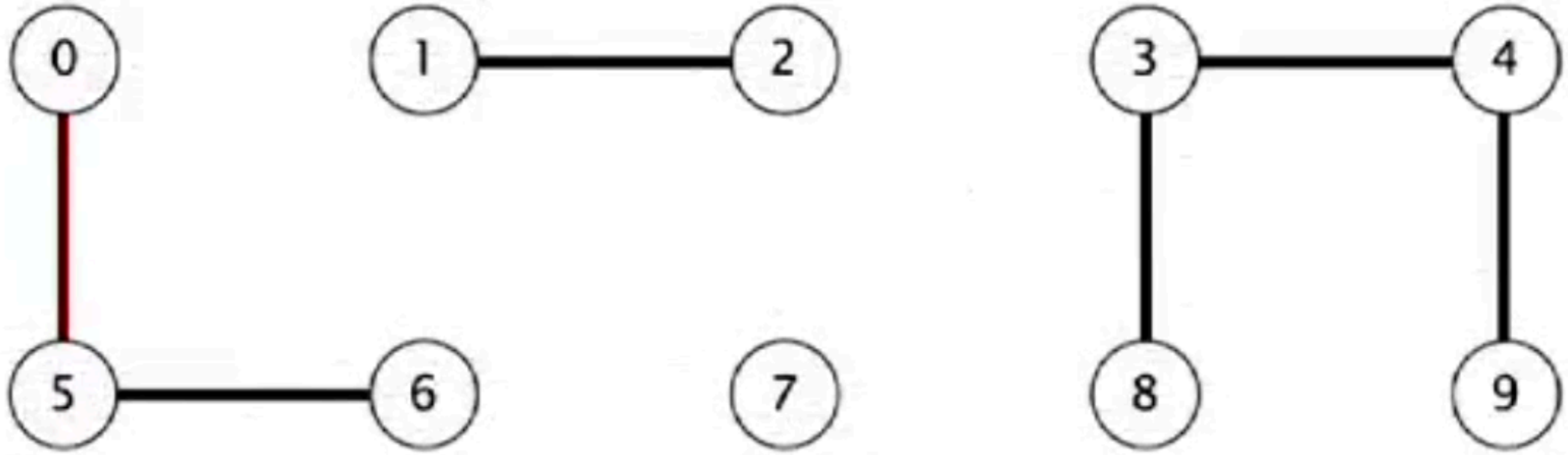
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	5	7	8	8

Quick-find demo

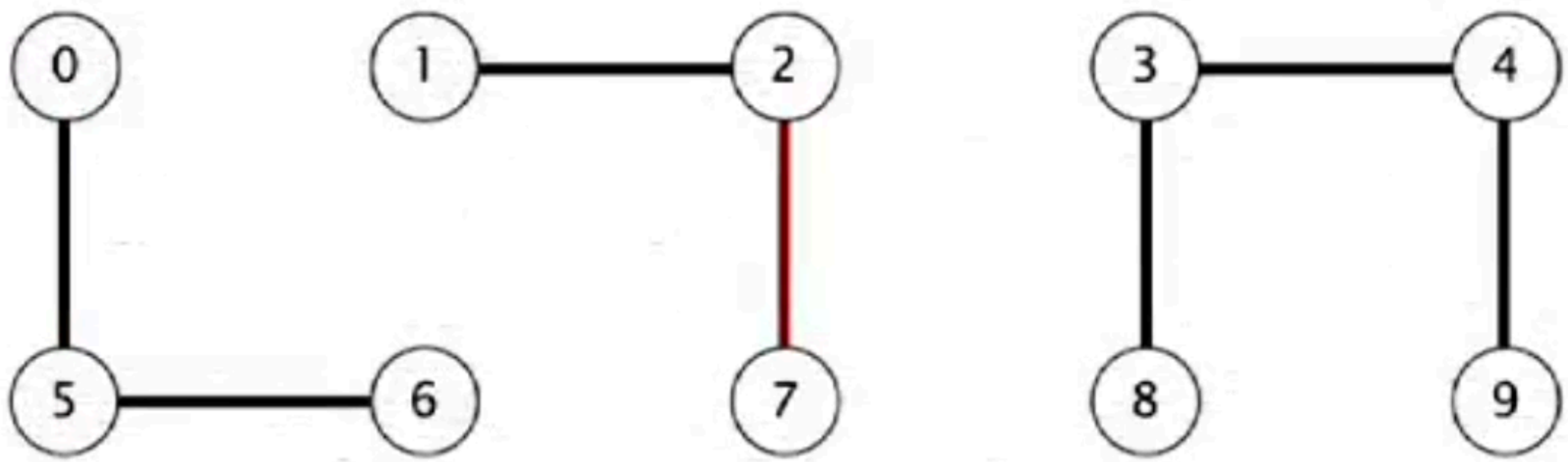
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

Quick-find demo

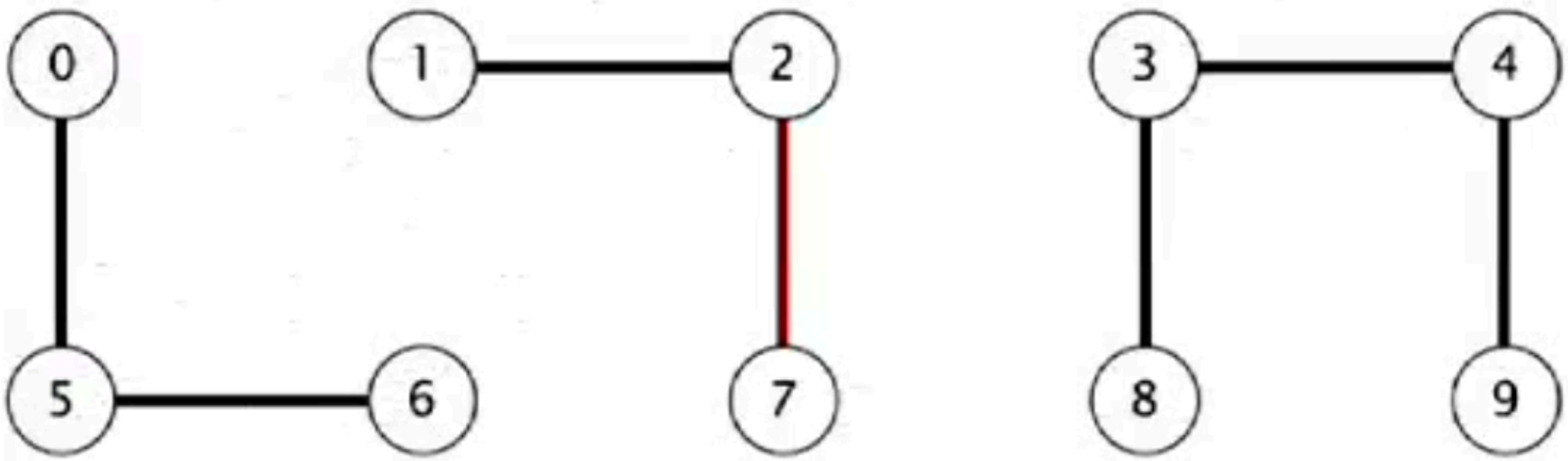
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	7	8	8

Quick-find demo

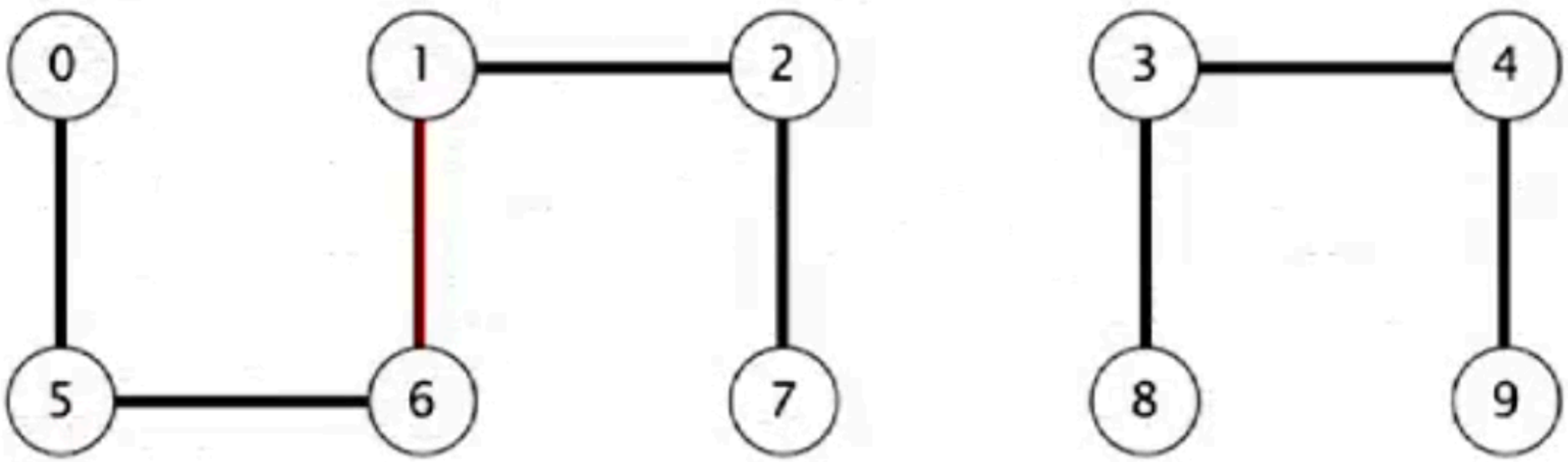
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

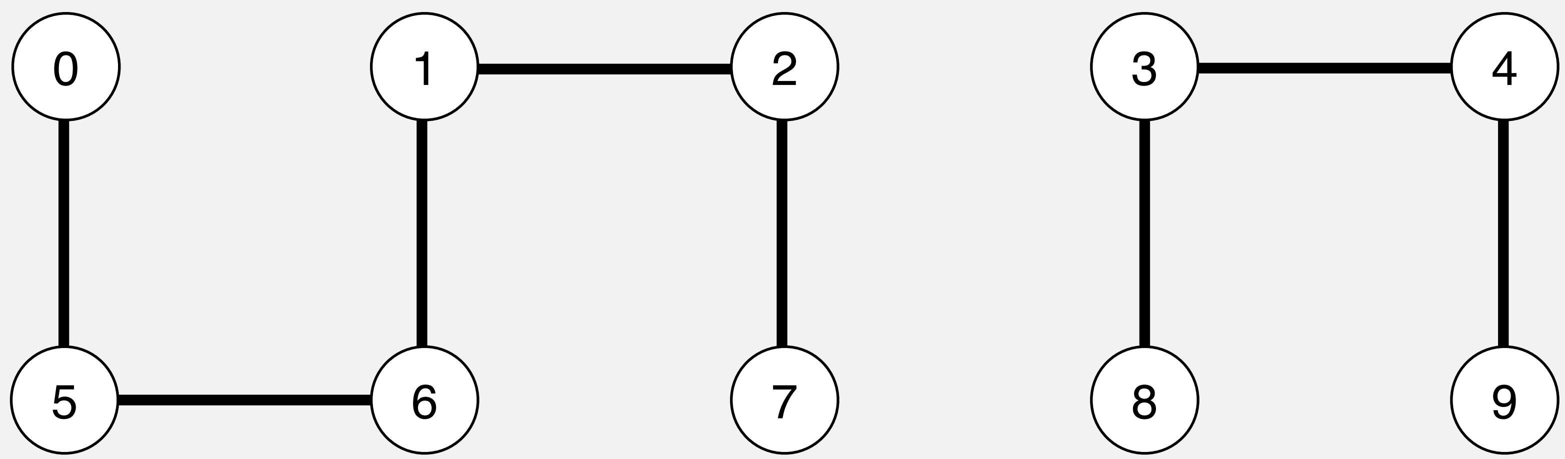
Quick-find demo

union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }

   public boolean find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
```

set id of each object to itself
(N array accesses)

return the id of p
(1 array access)

change all entries with id[p] to id[q]
(at most 2N + 2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

quadratic

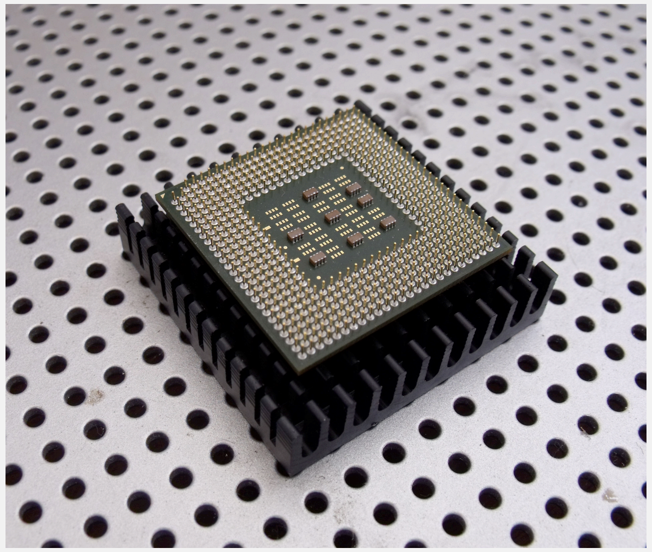
Union is too expensive. It takes N^2 array accesses to process a sequence of N union operations on N objects.

Quadratic algorithms do not scale

Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!

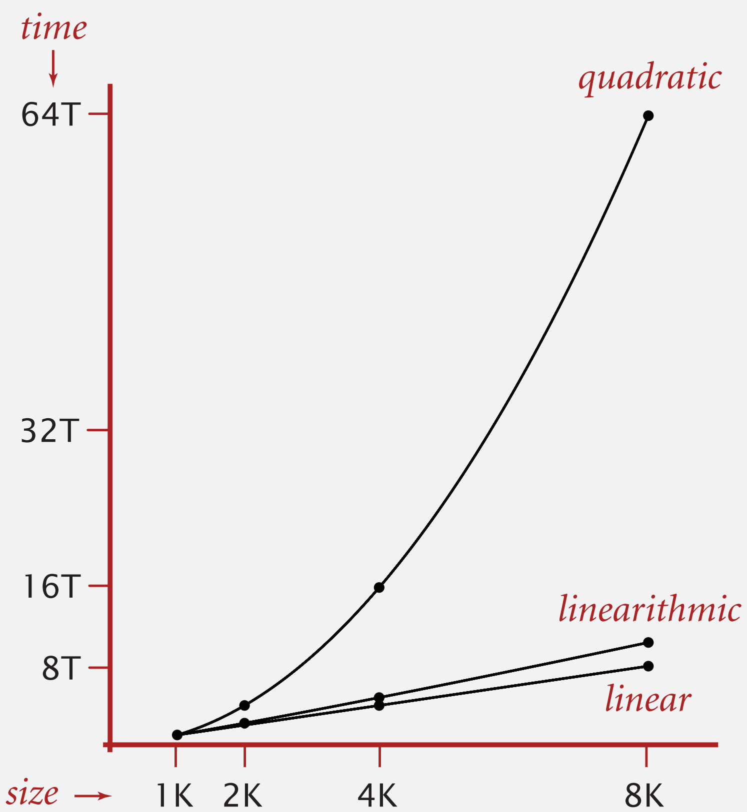


Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory $\Rightarrow$ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!



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1.5 Union-Find

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Quick-union [lazy approach]

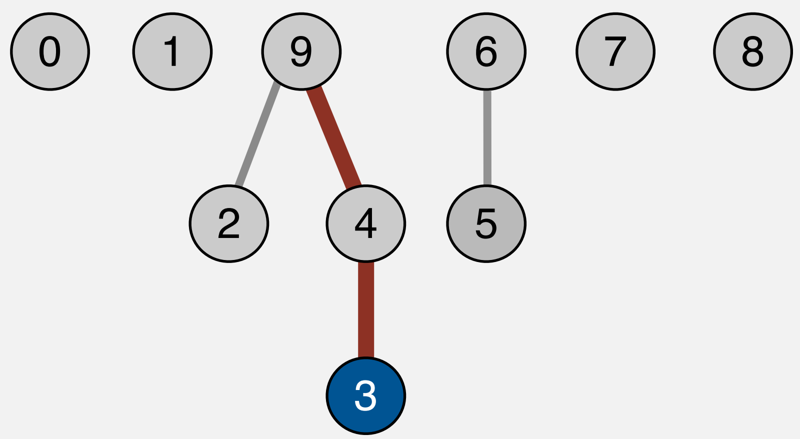
Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

keep going until it doesn't change

(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

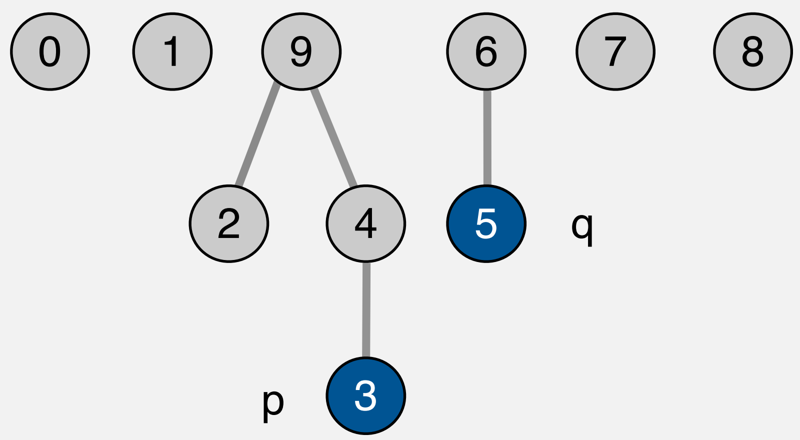
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array id[] of length N.
- Interpretation: id[i] is parent of i.
- Root of i is id[id[id[...id[i]...]]].

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



Find. What is the root of p?

Connected. Do p and q have the same root?

root of 3 is 9

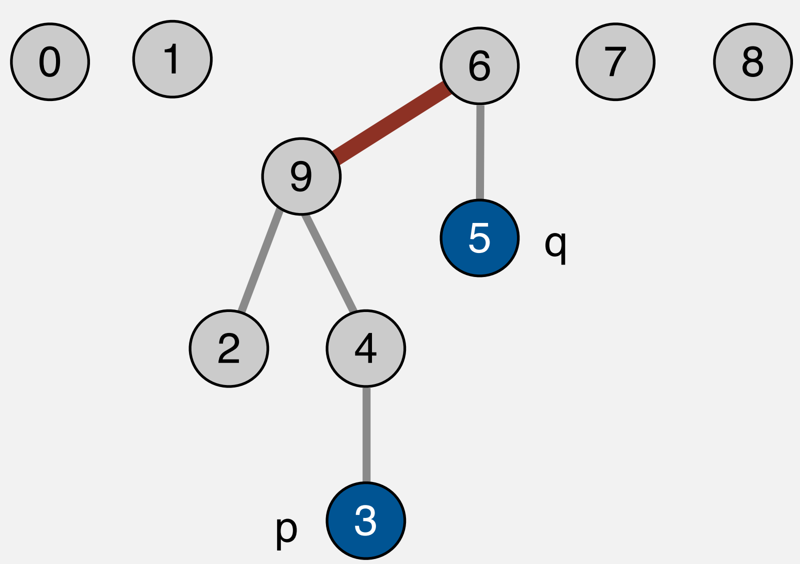
root of 5 is 6

3 and 5 are not connected

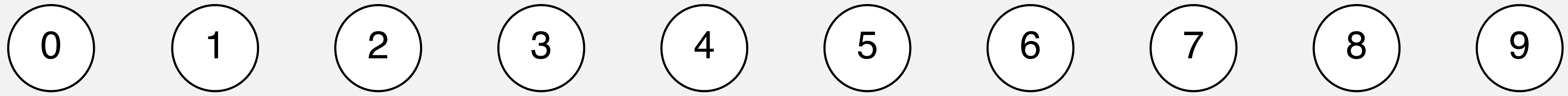
Union. To merge components containing p and q, set the id of p's root to the id of q's root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-union demo

union(4, 3)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

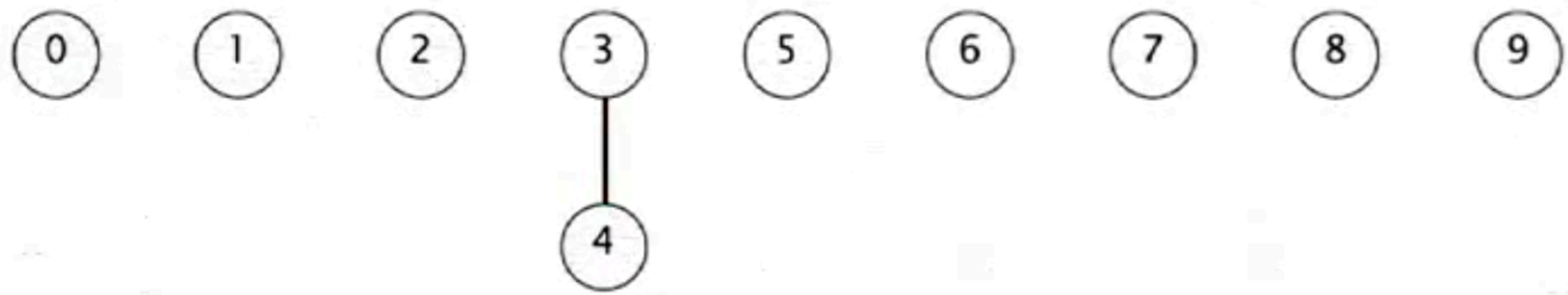
Quick-union demo

union(4, 3)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

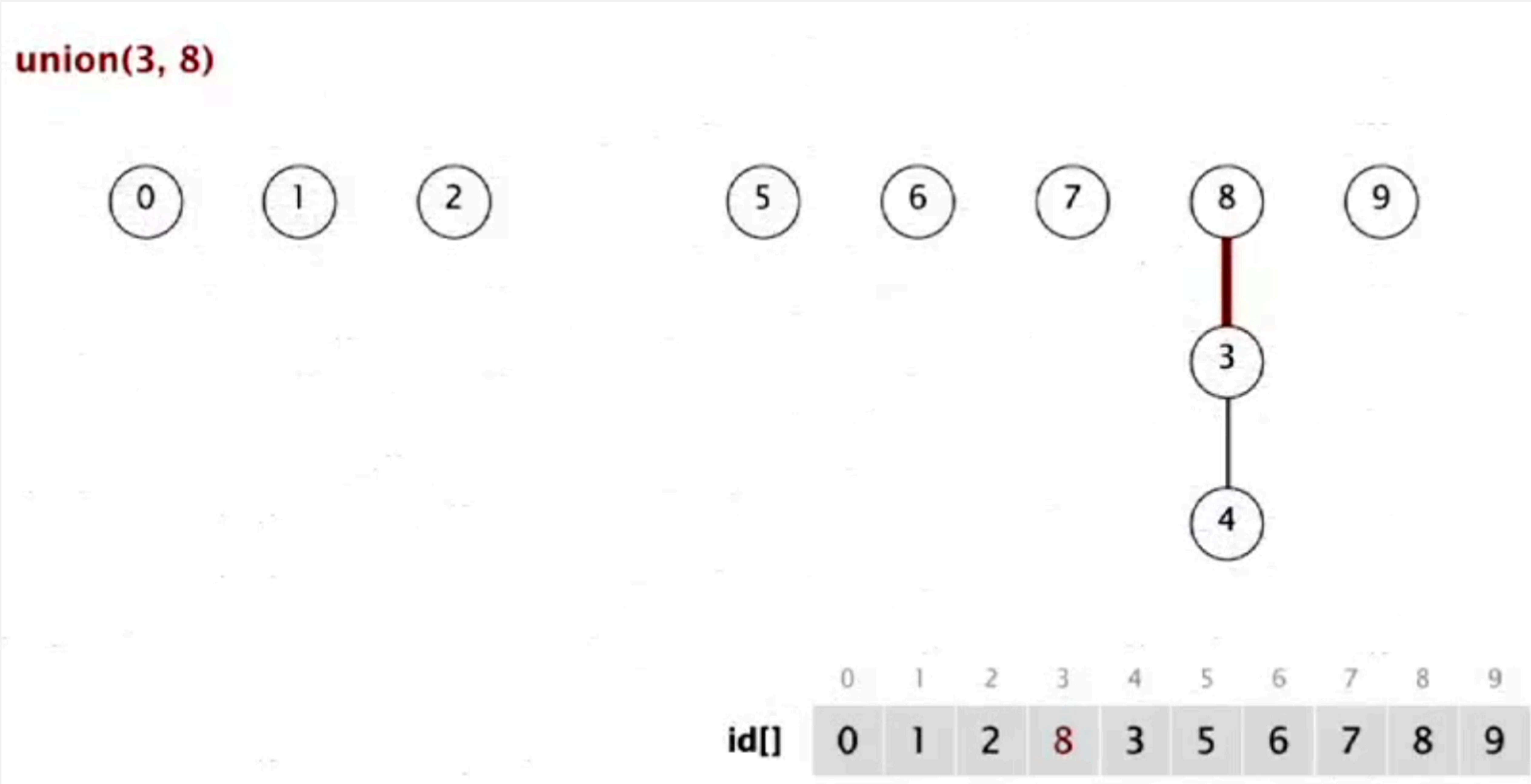
Quick-union demo

union(3, 8)



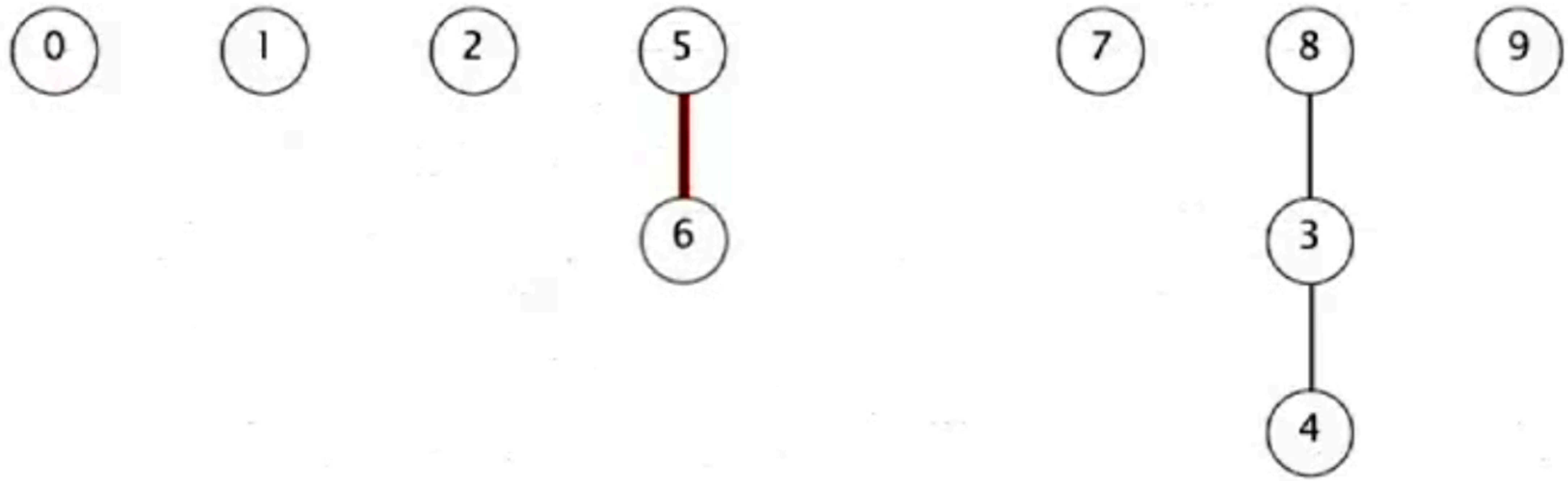
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	3	5	6	7	8	9

Quick-union demo



Quick-union demo

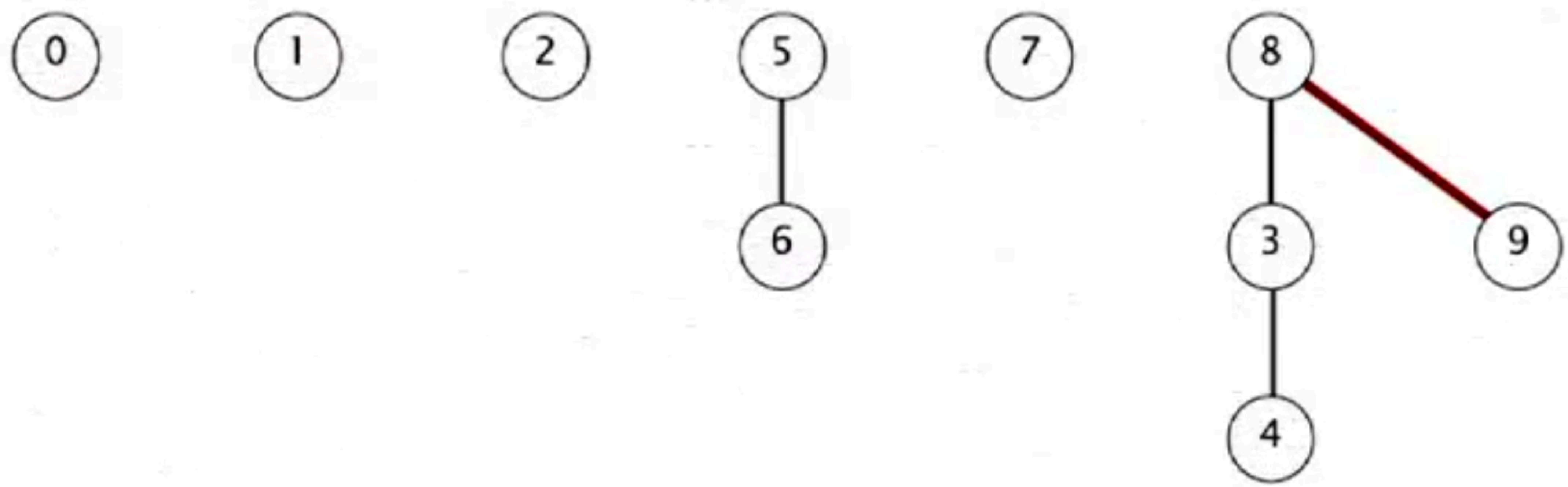
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	9

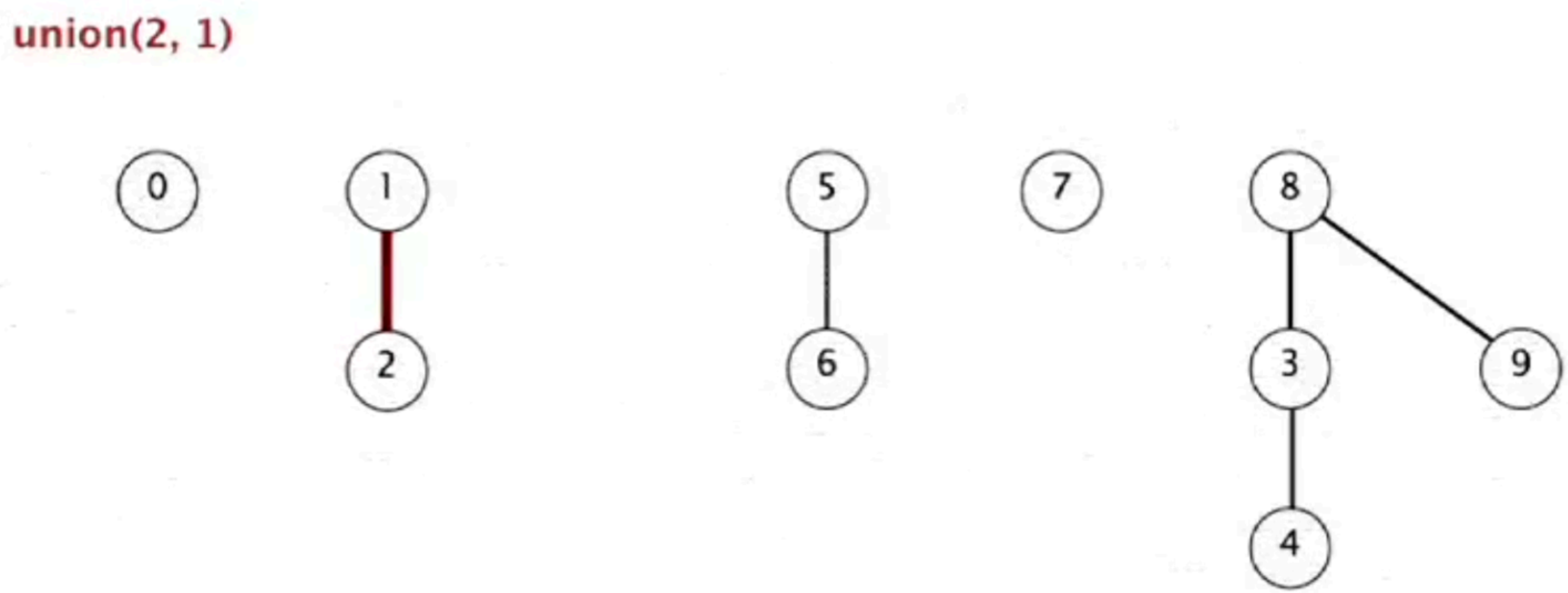
Quick-union demo

union(9, 4)



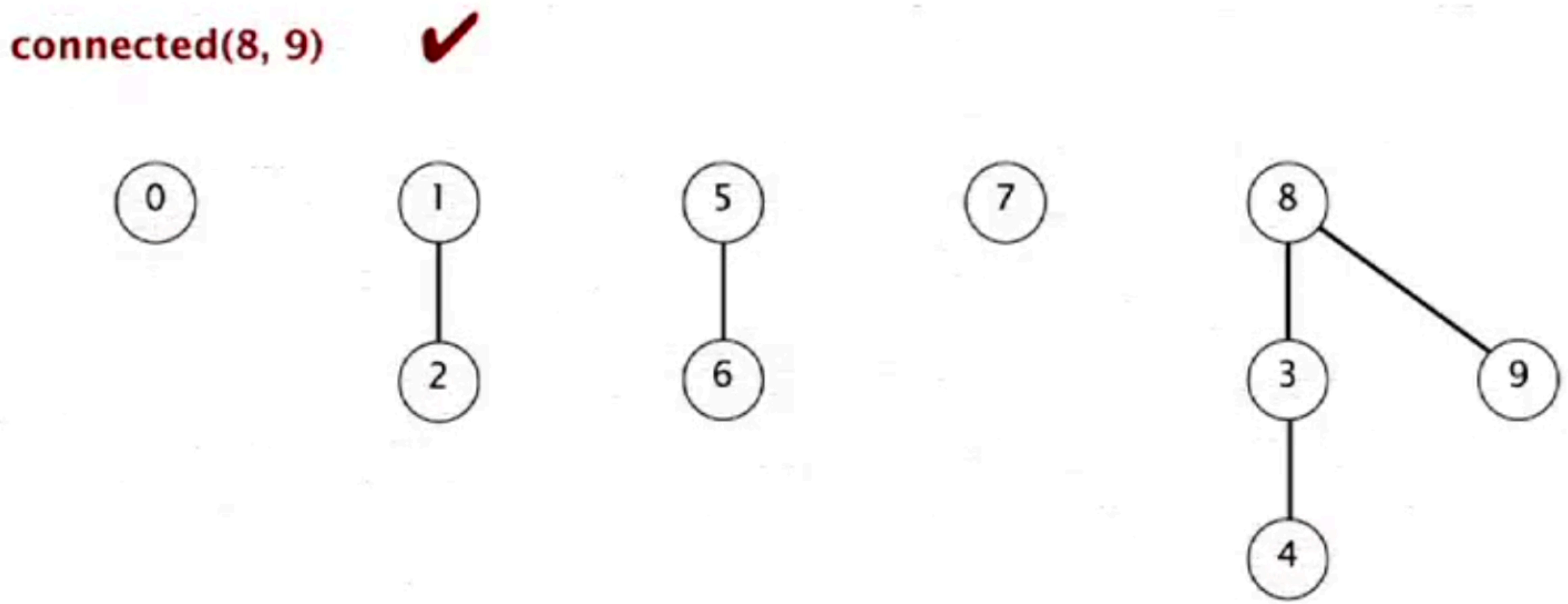
	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	8	3	5	5	7	8	8

Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

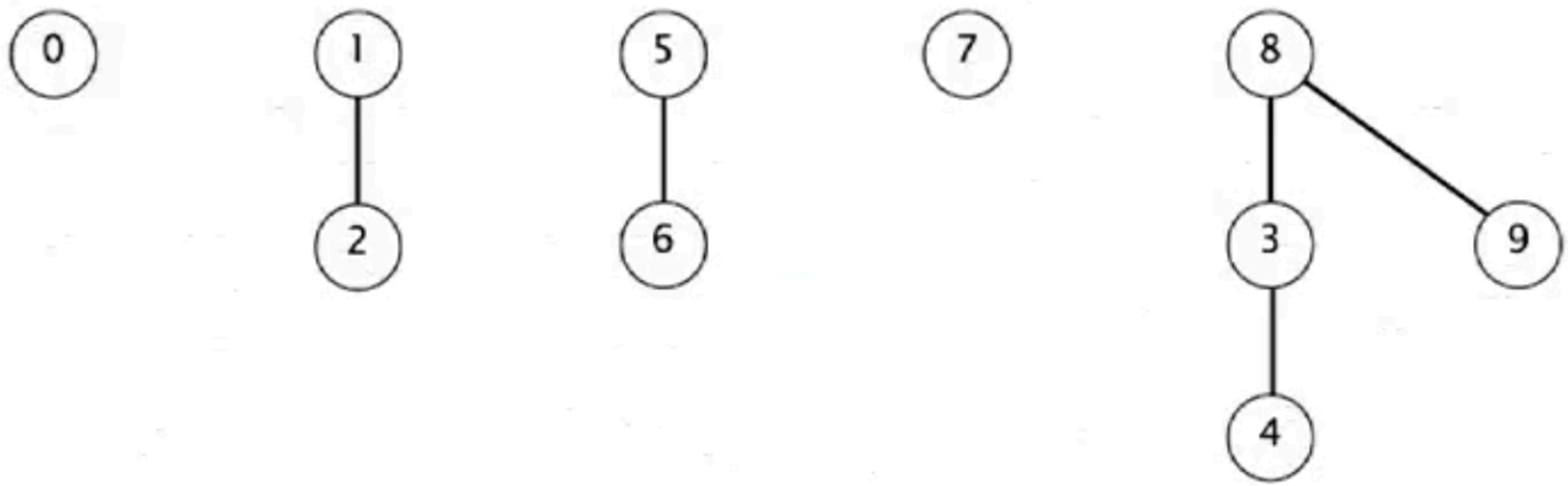
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

Quick-union demo

connected(5, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	5	5	7	8	8

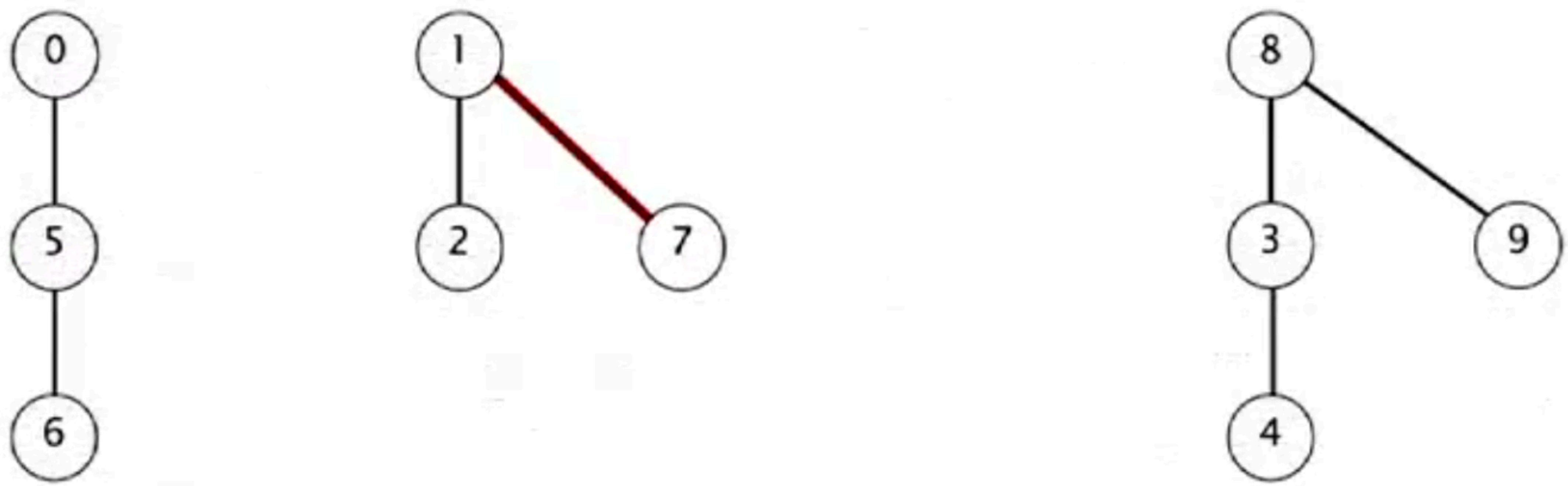
Quick-union demo

union(5, 0)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	7	8	8

Quick-union demo

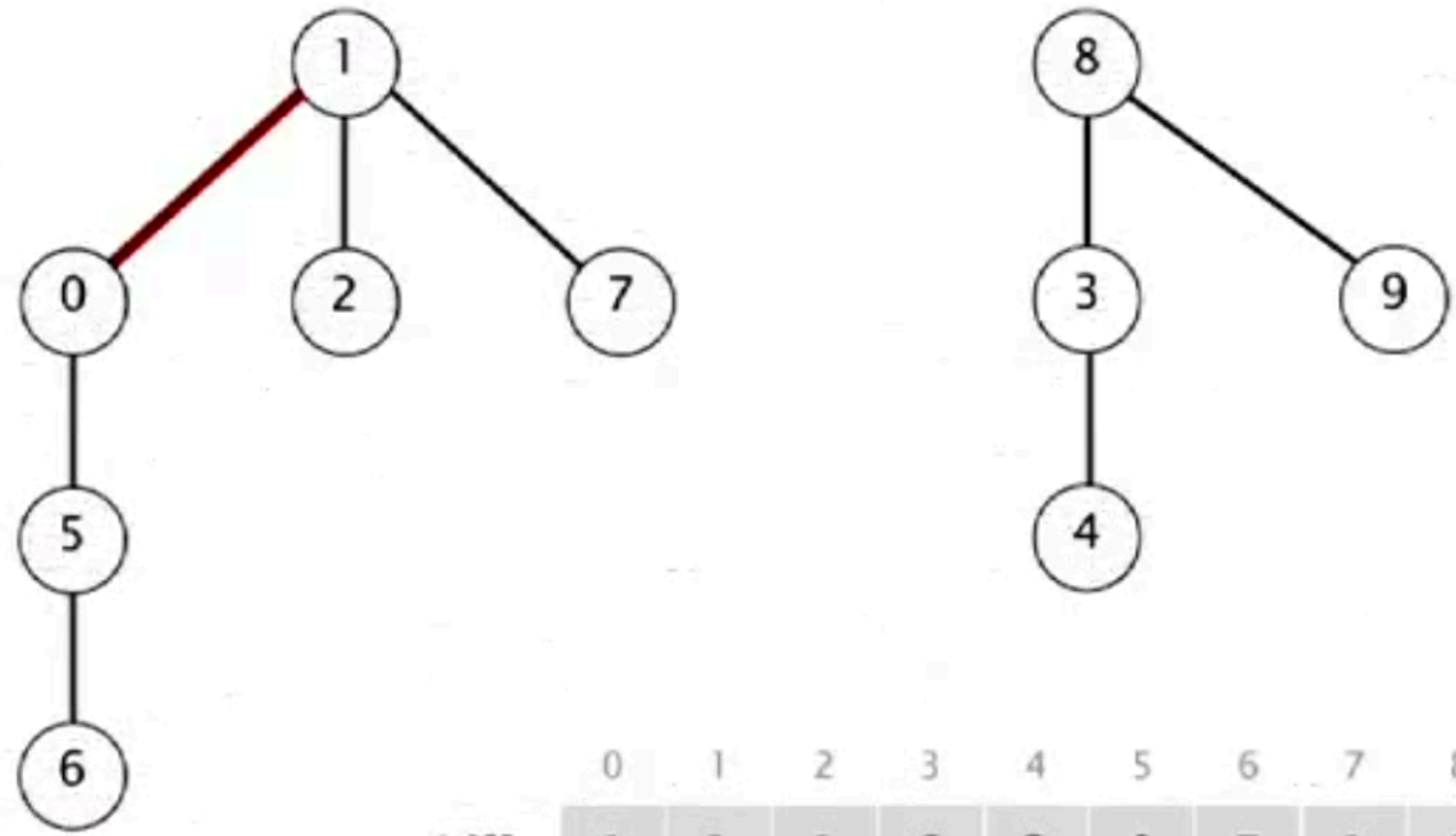
union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	3	0	5	1	8	8

Quick-union demo

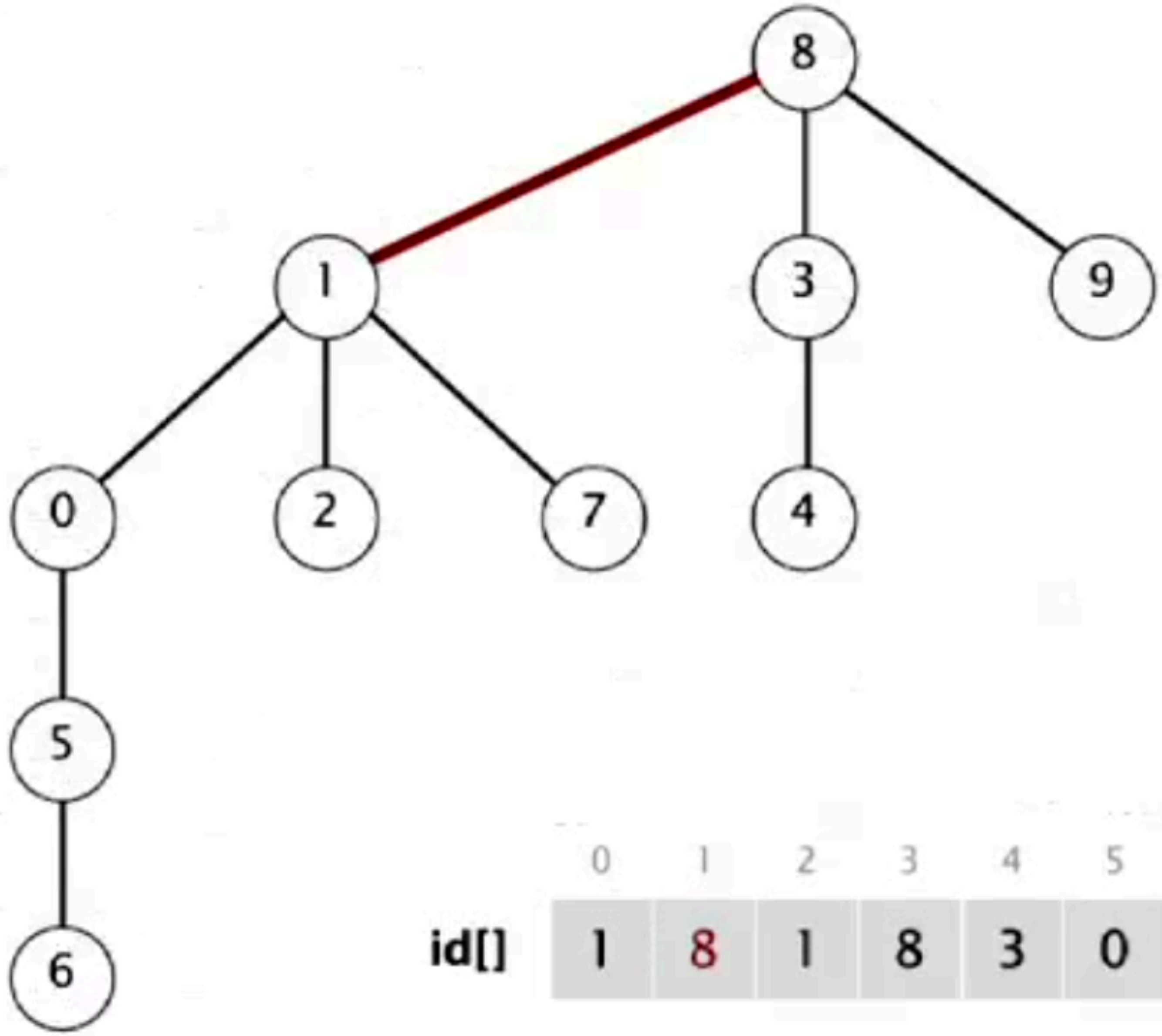
union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	3	0	5	1	8	8

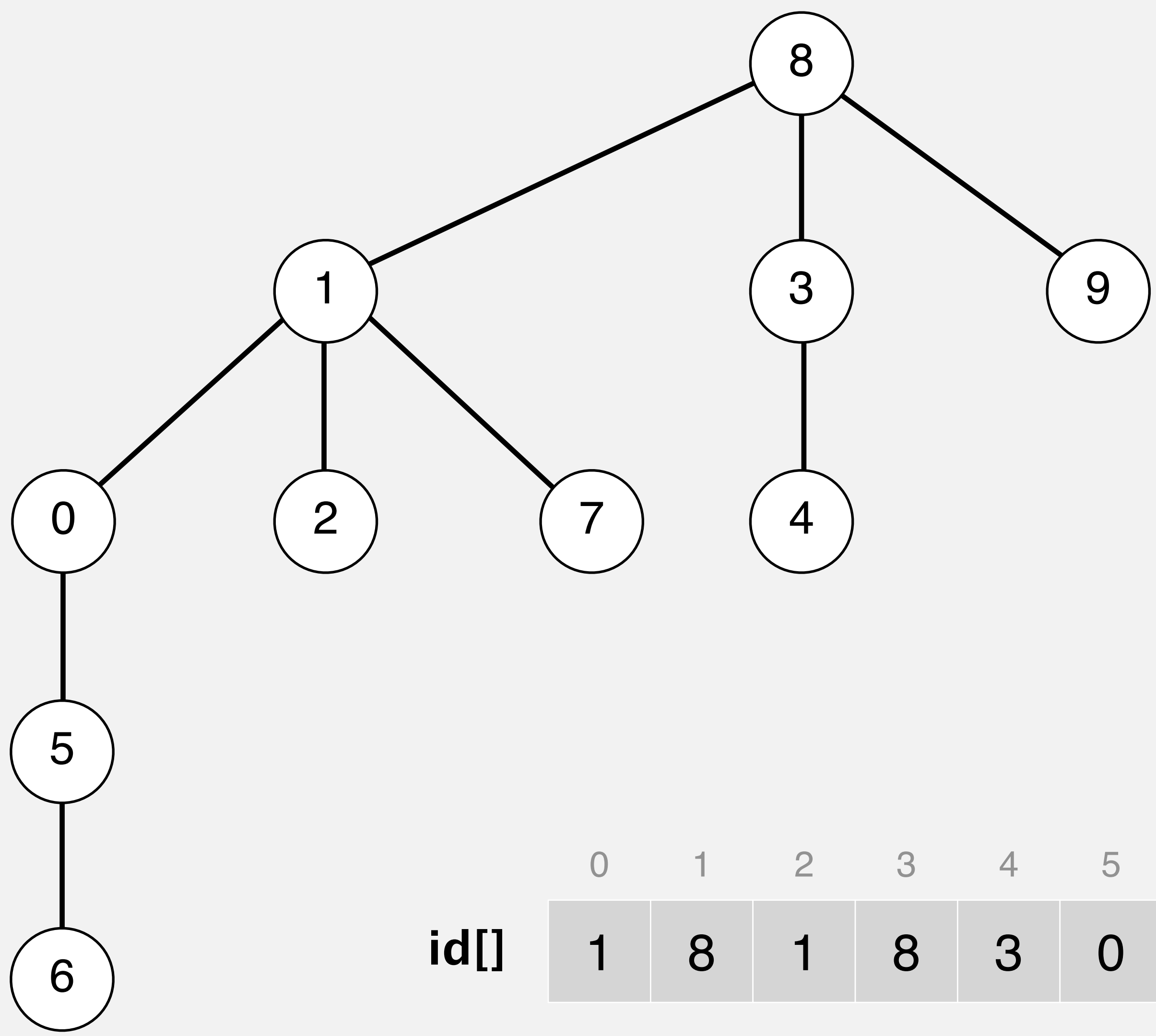
Quick-union demo

union(7, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   public int find(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }


   public void union(int p, int q)
   {

```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be N array accesses).

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1.5 Union-Find

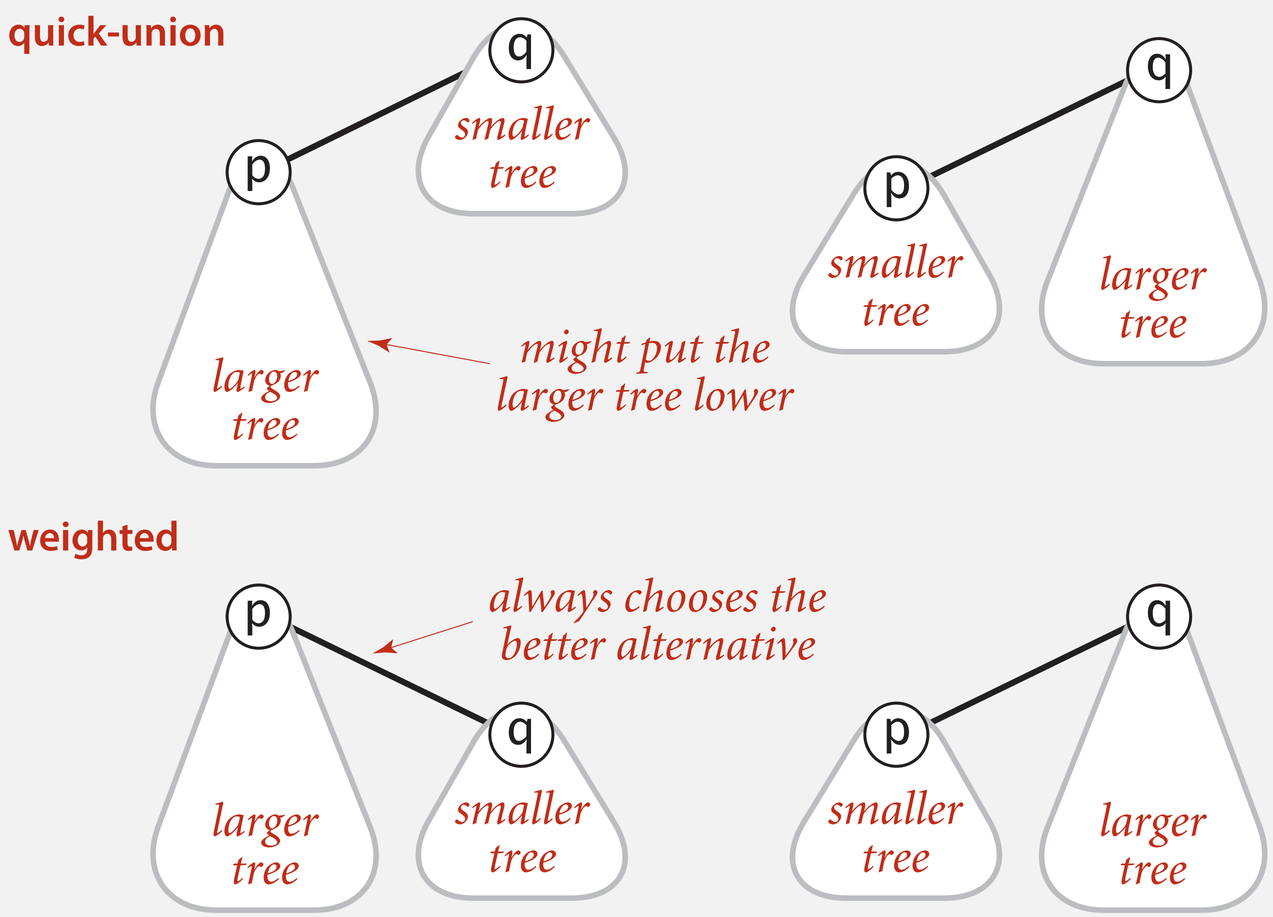
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Improvement 1: weighting

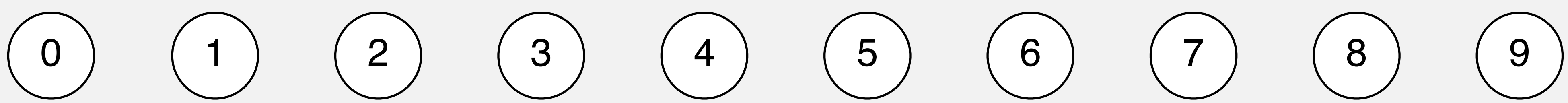
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"

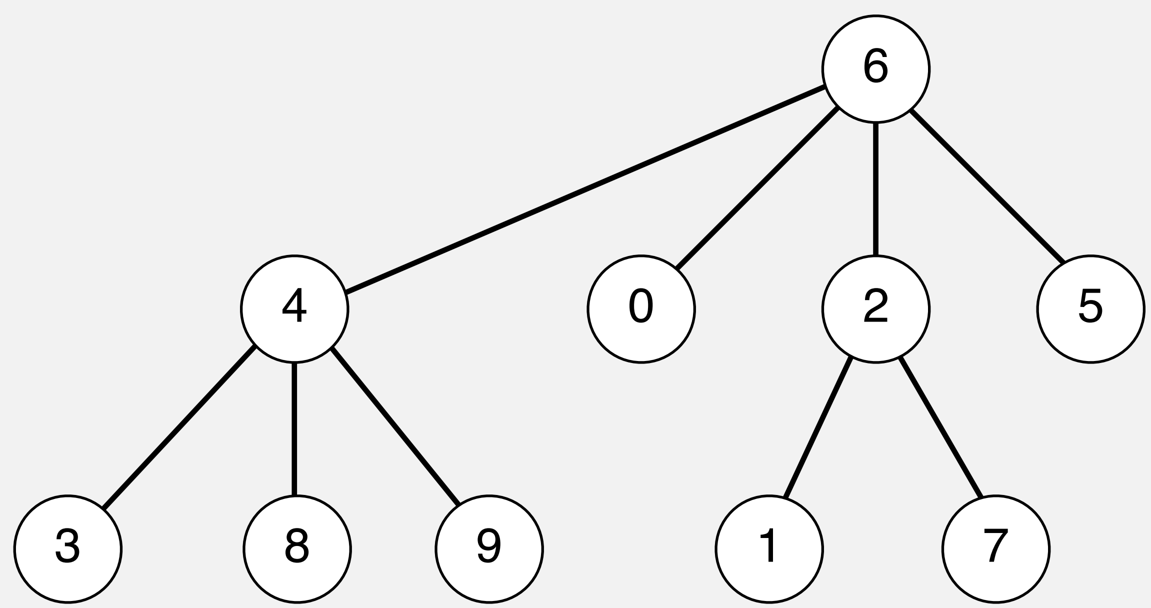


Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

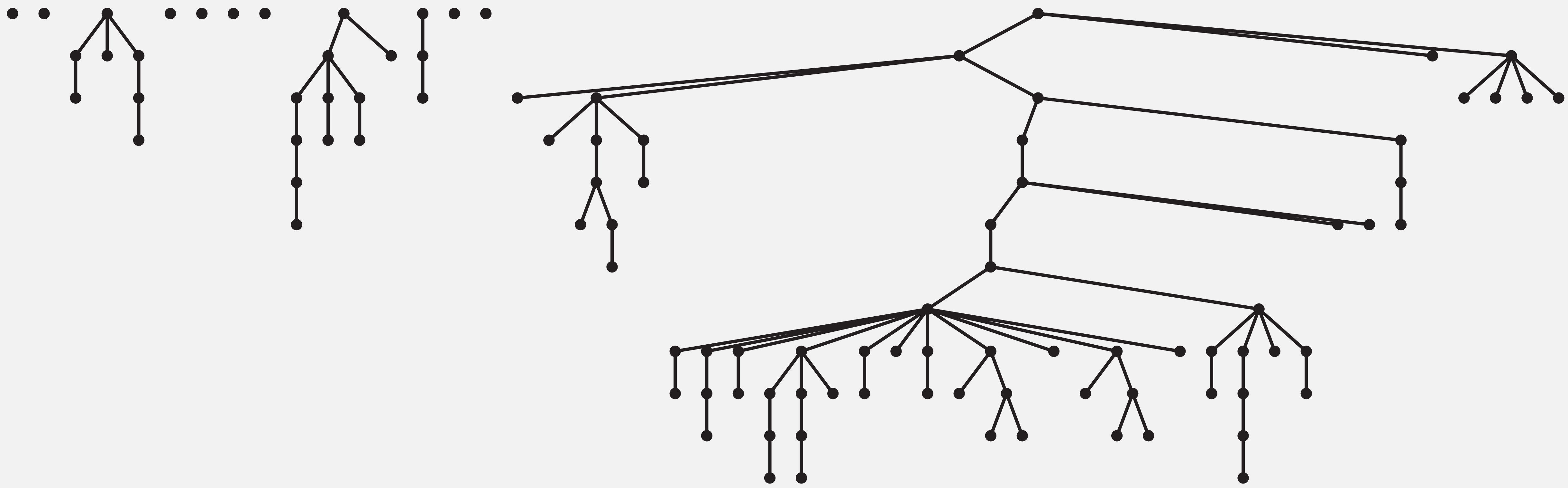
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

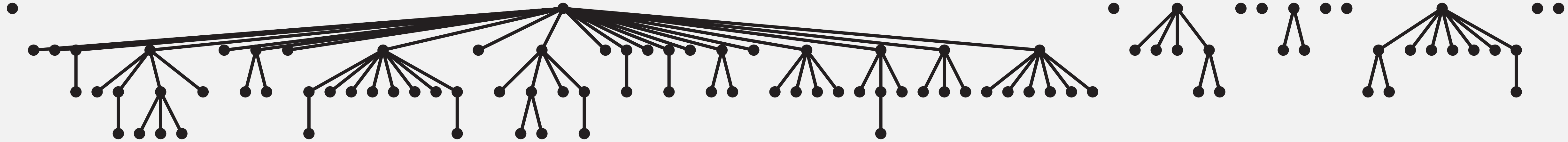
Quick-union and weighted quick-union example

quick-union



average distance to root: 5.11

weighted



average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array sz[i] to count number of objects in the tree rooted at i.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the sz[] array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

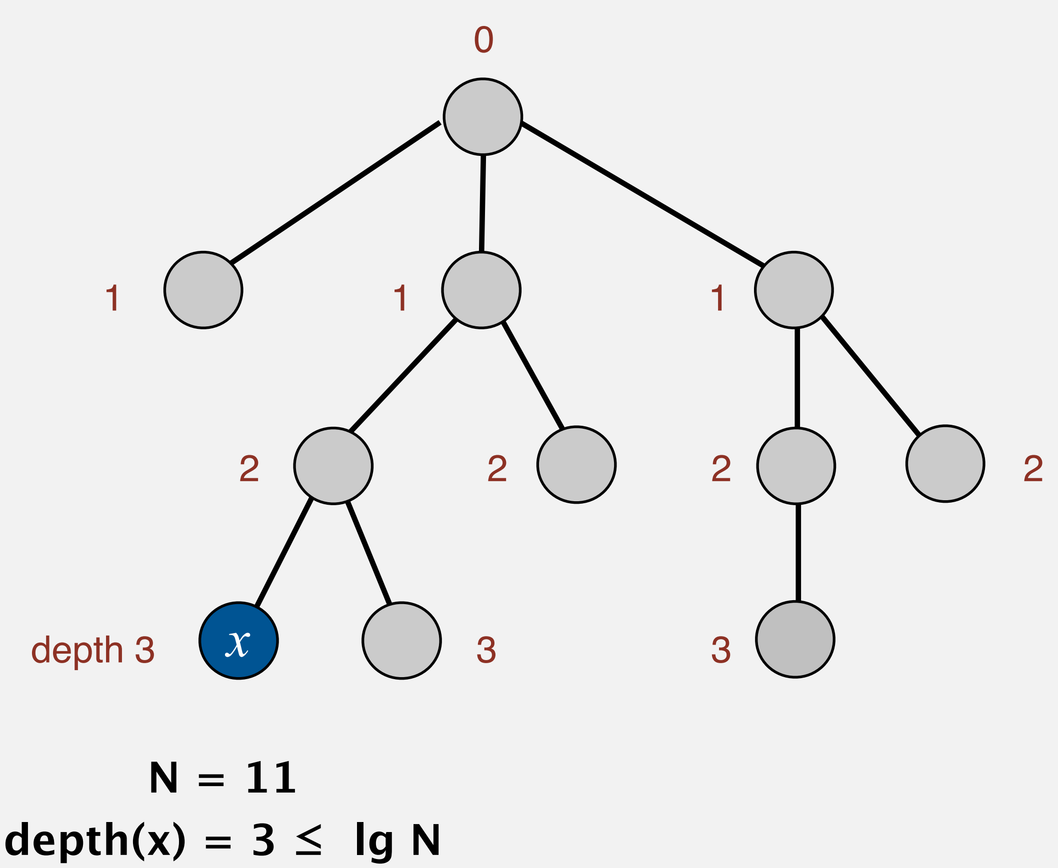
Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

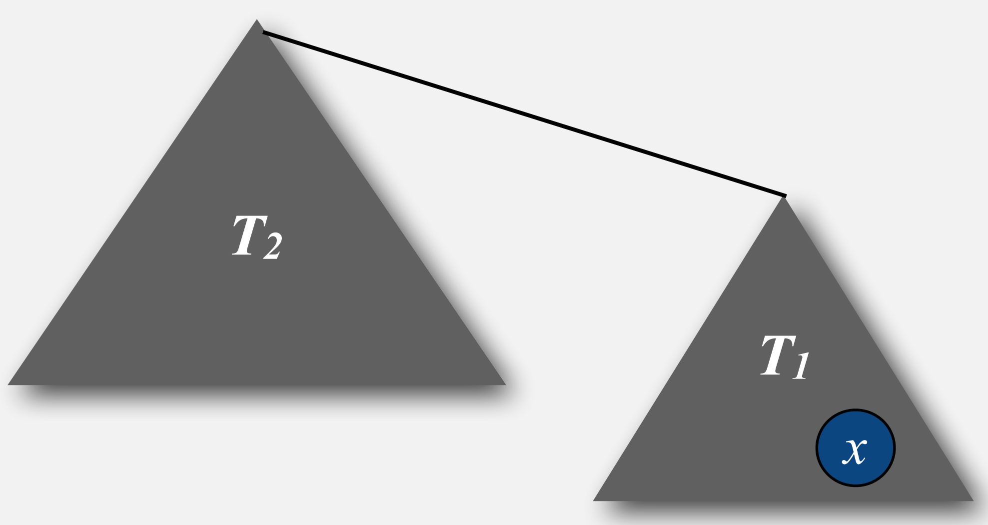
lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.

Pf. What causes the depth of object x to increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $| T_2 | \geq | T_1 |$.
- Size of tree containing x can double at most $\lg N$ times. Why?



$$\left.\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ \vdots \\ N \end{array}\right\} \lg N$$

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

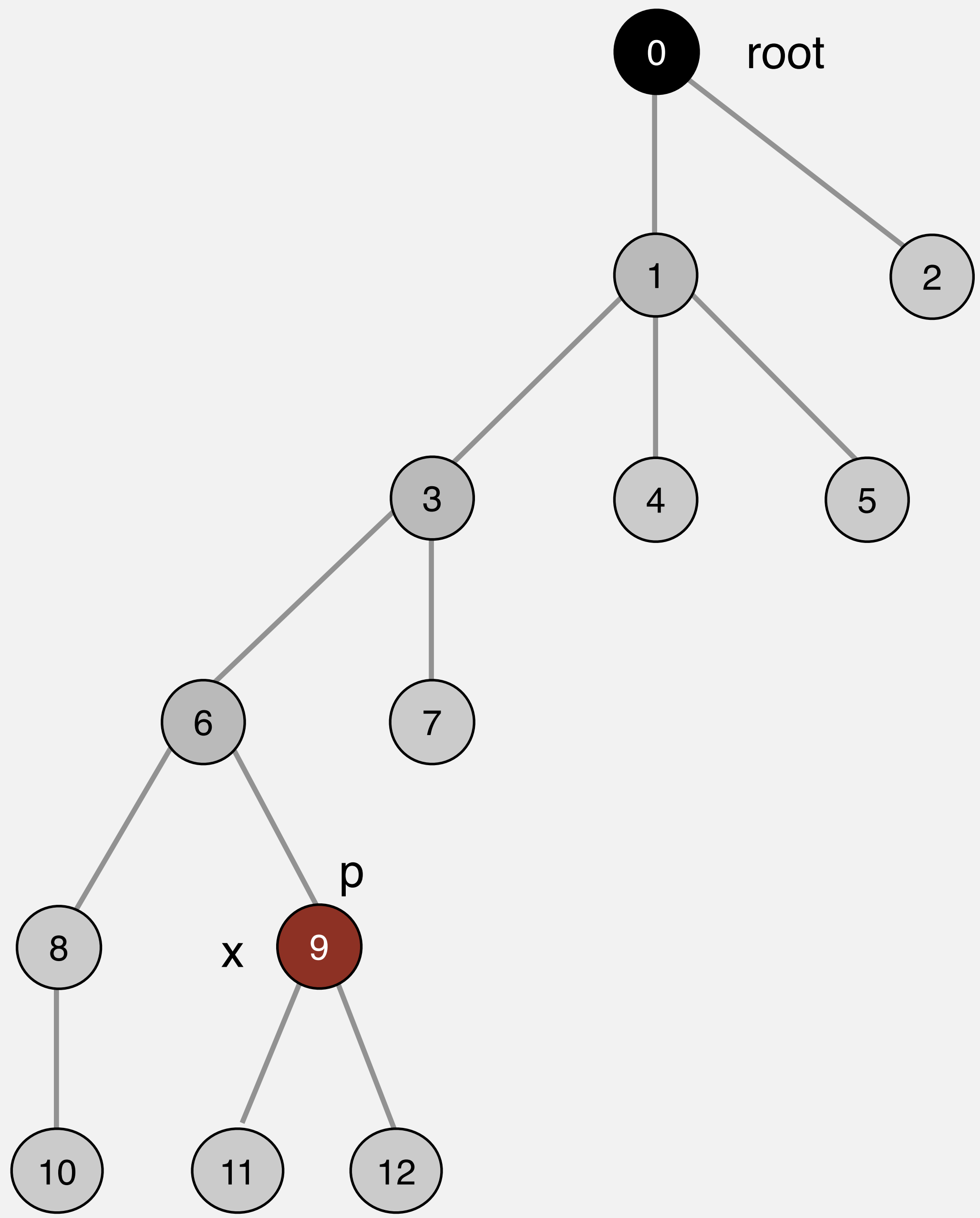
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id[] of each examined node to point to that root.



Improvement 2: path compression

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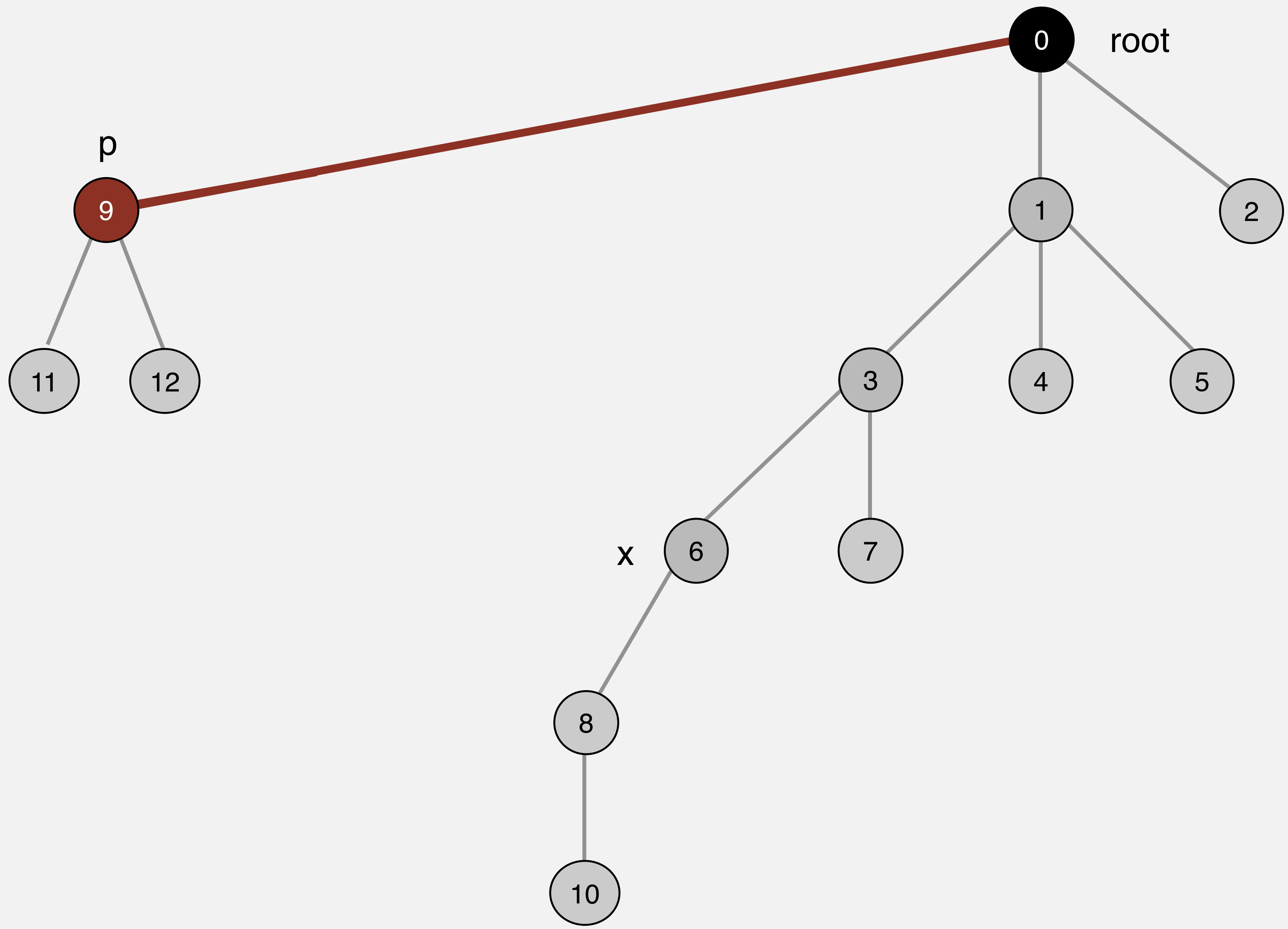

Improvement 2: path compression

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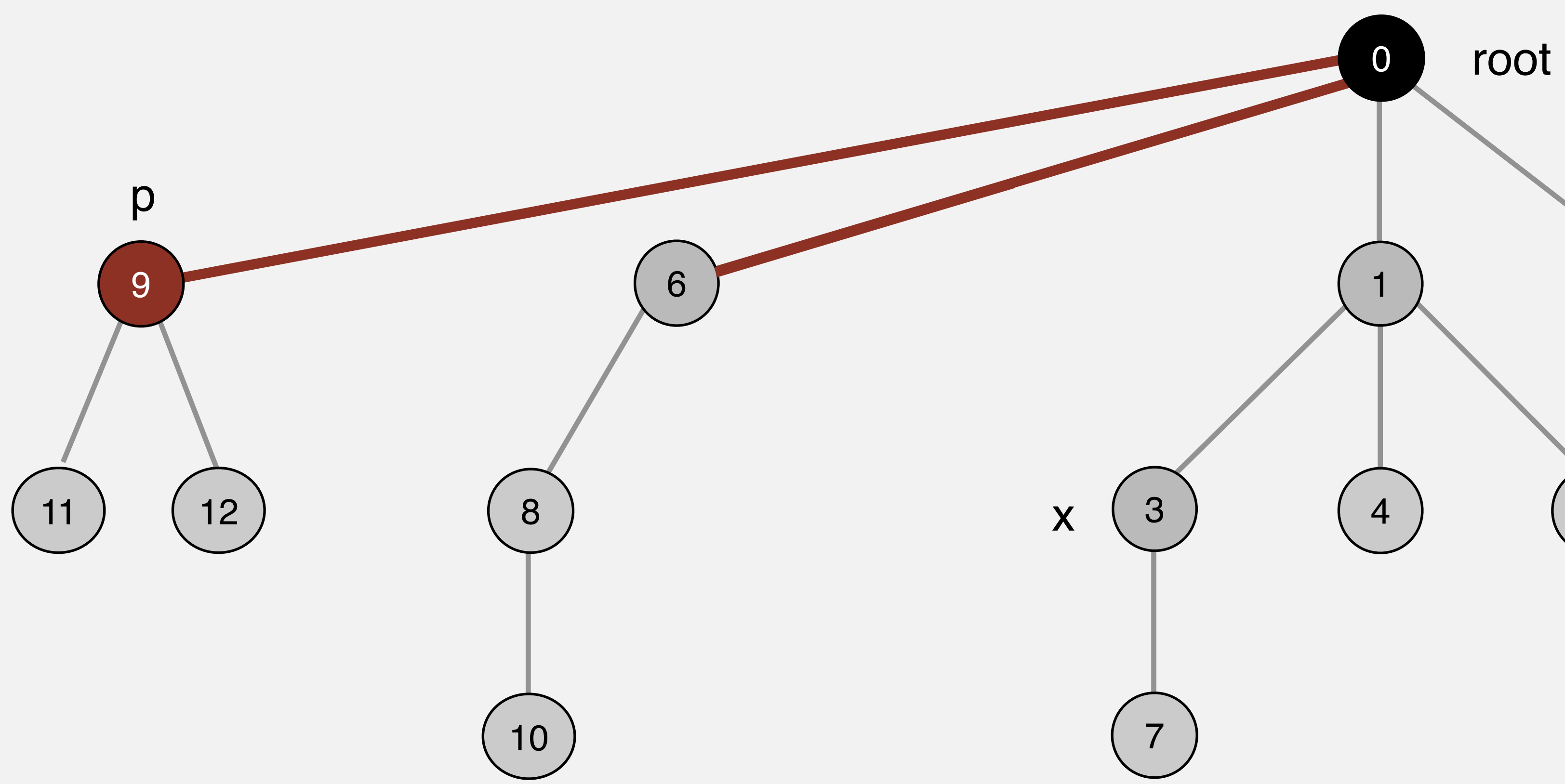
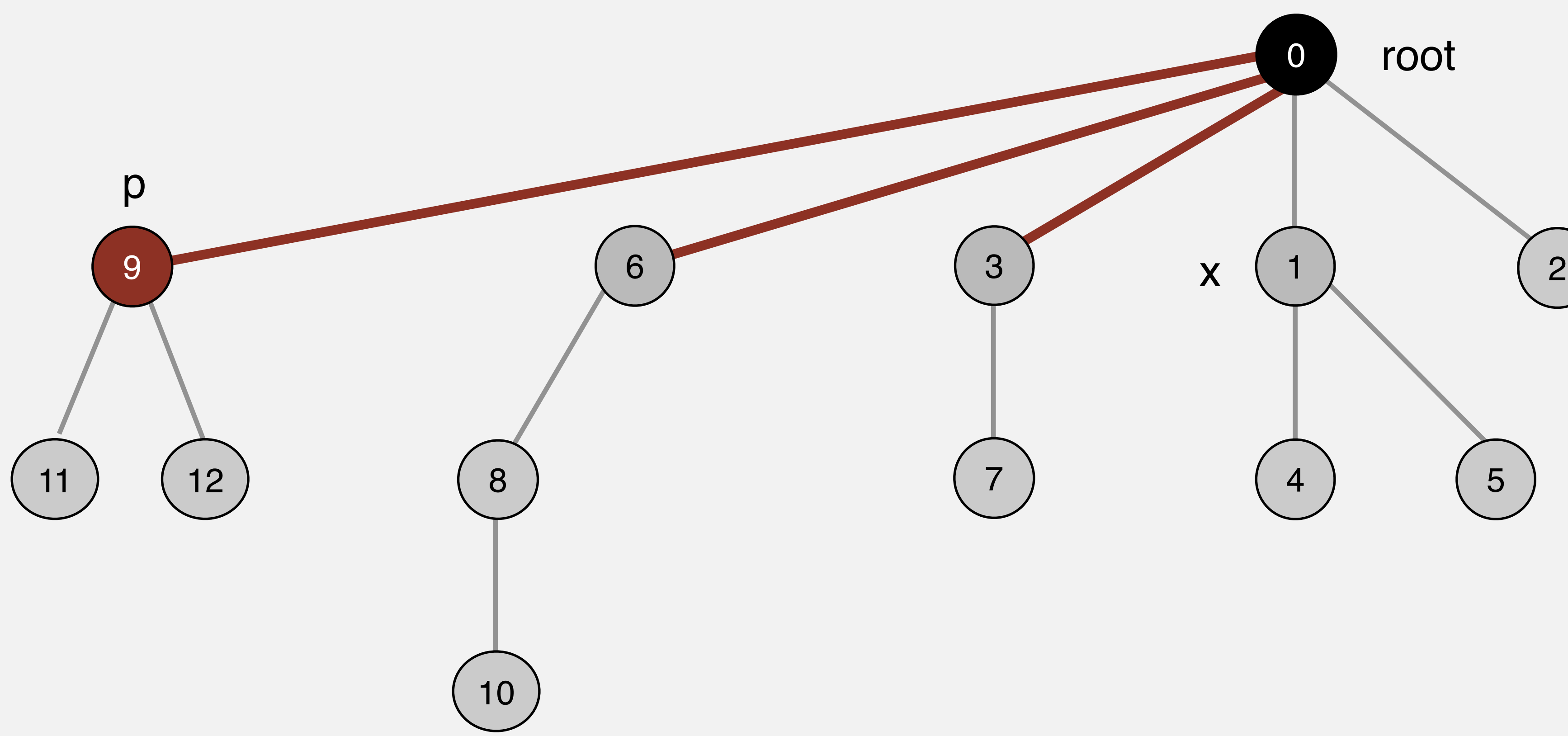


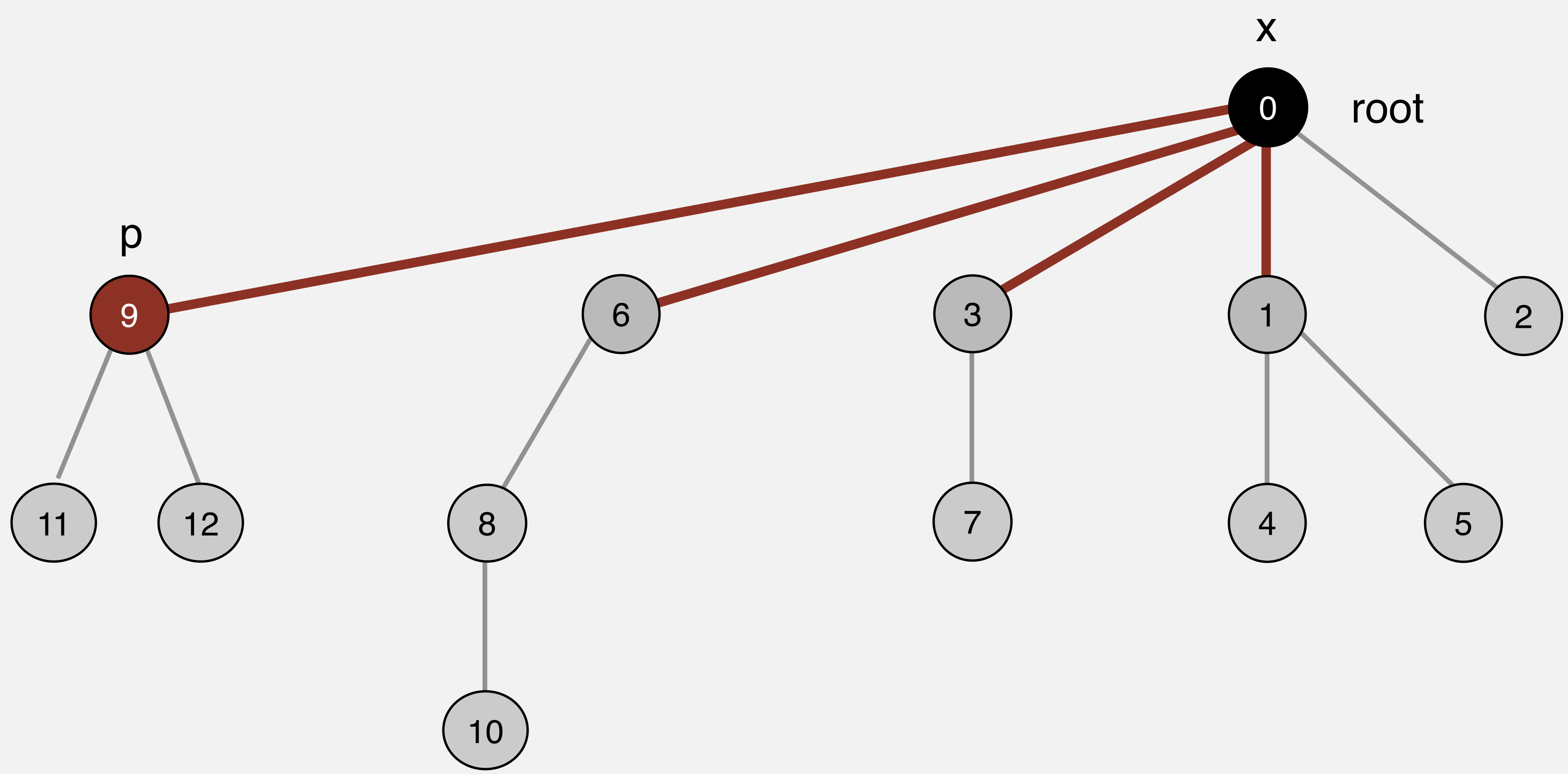
Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id[] of each examined node to point to that root.



Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id[] of each examined node to point to that root.



Bottom line. Now, find() has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to find() to set the id[] of each examined node to the root.

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Weighted quick-union with path compression: amortized analysis

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of M union–find ops on N objects makes ≤ $c\,(\,N + M \lg^* N\,)$ array accesses.

- Analysis can be improved to $N + M\,\alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterated lg function

Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N

order of growth for M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

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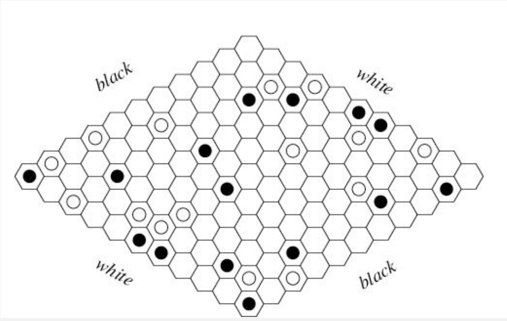
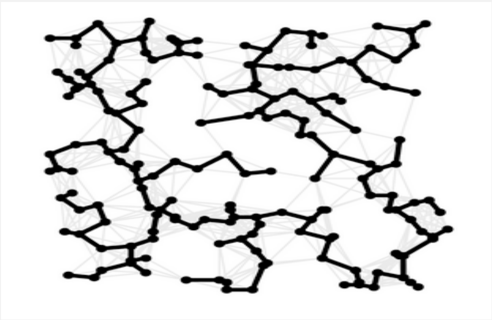
1.5 Union-Find

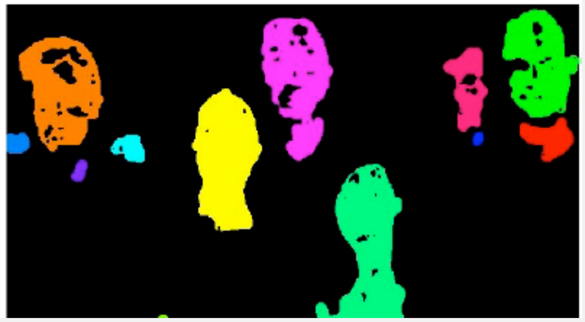
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Union-find applications

- Percolation.
- Games (Go, Hex).
- ✓ Dynamic connectivity.
- Least common ancestor.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hinley-Milner polymorphic type inference.
- Kruskal's minimum spanning tree algorithm.
- Compiling equivalence statements in Fortran.
- Morphological attribute openings and closings.
- Matlab's `bwlabel()` function in image processing.



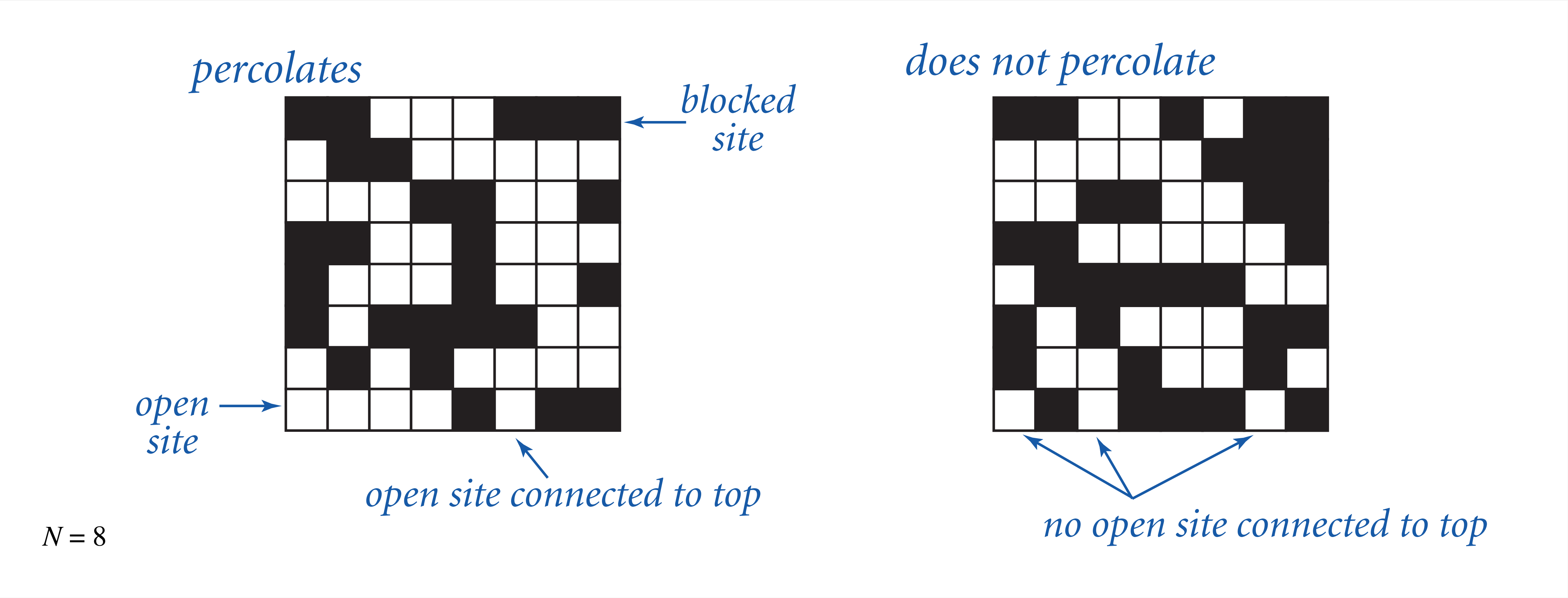




Percolation

An abstract model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.



Percolation

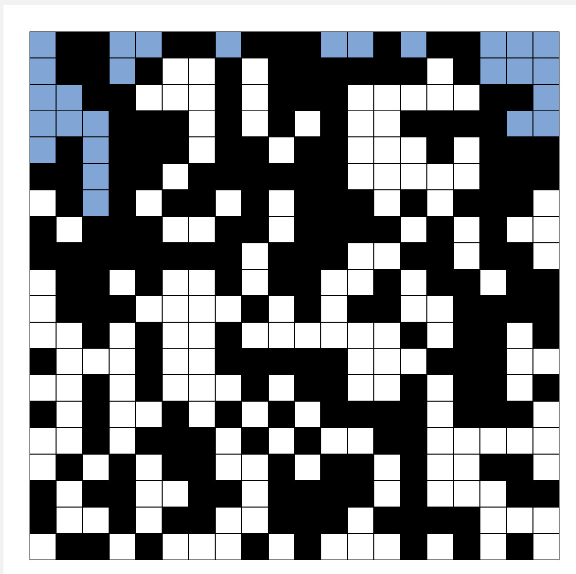
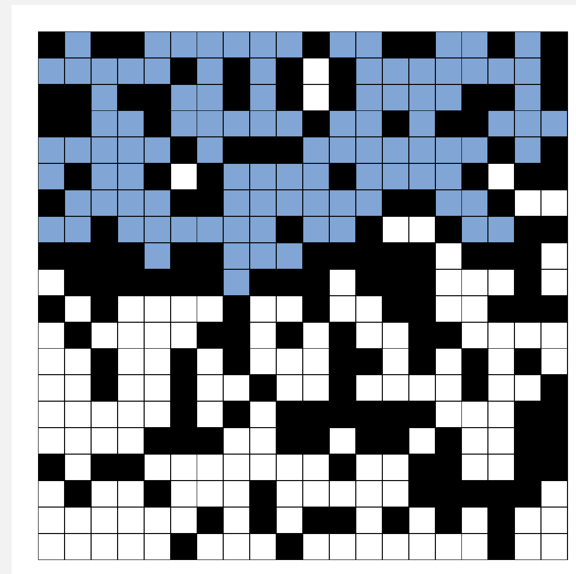
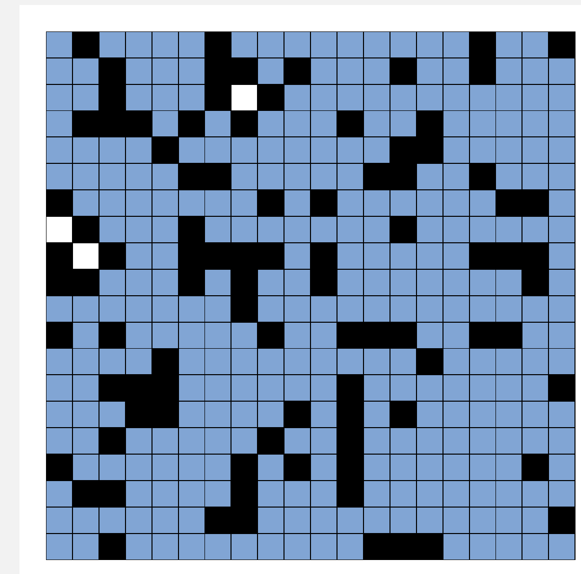
An abstract model for many physical systems:

- N-by-N grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

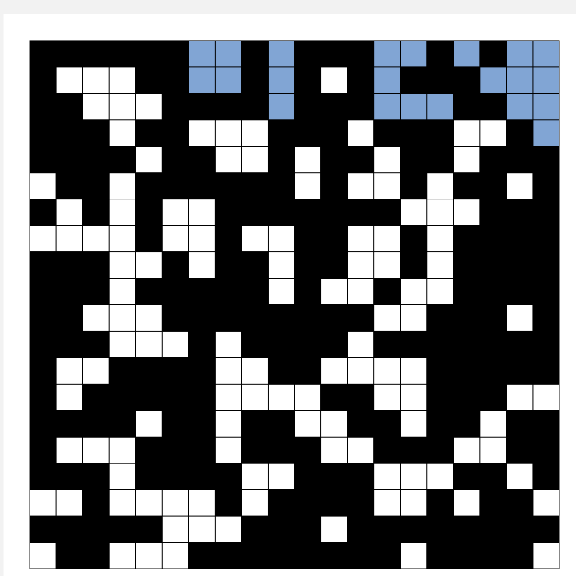
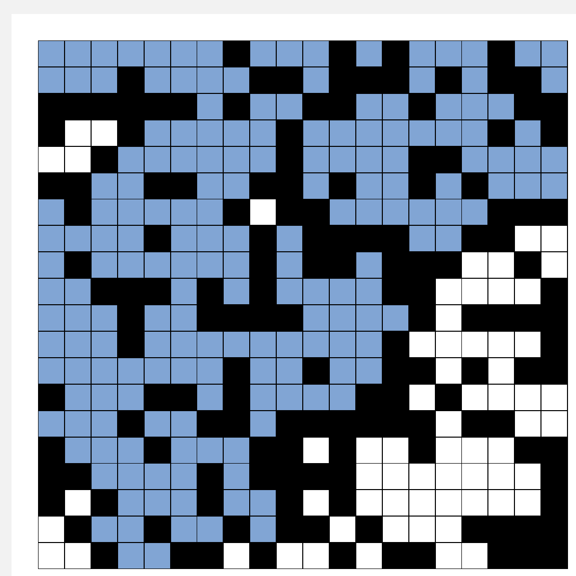
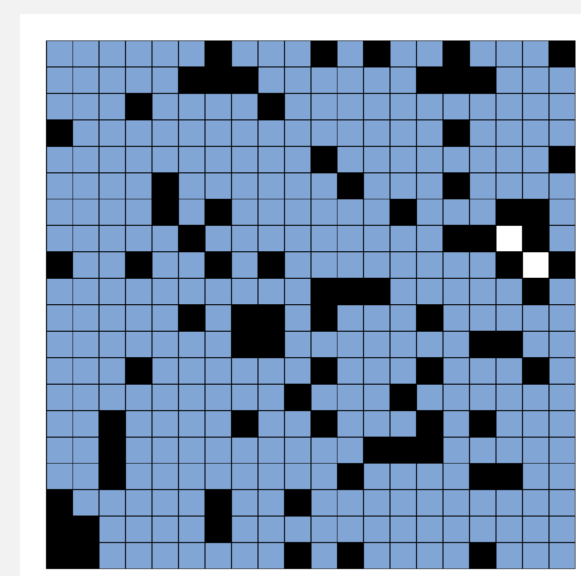
Depends on grid size N and site vacancy probability p.

p low (0.4)
does not percolate

p medium (0.6)
percolates?

p high (0.8)
percolates

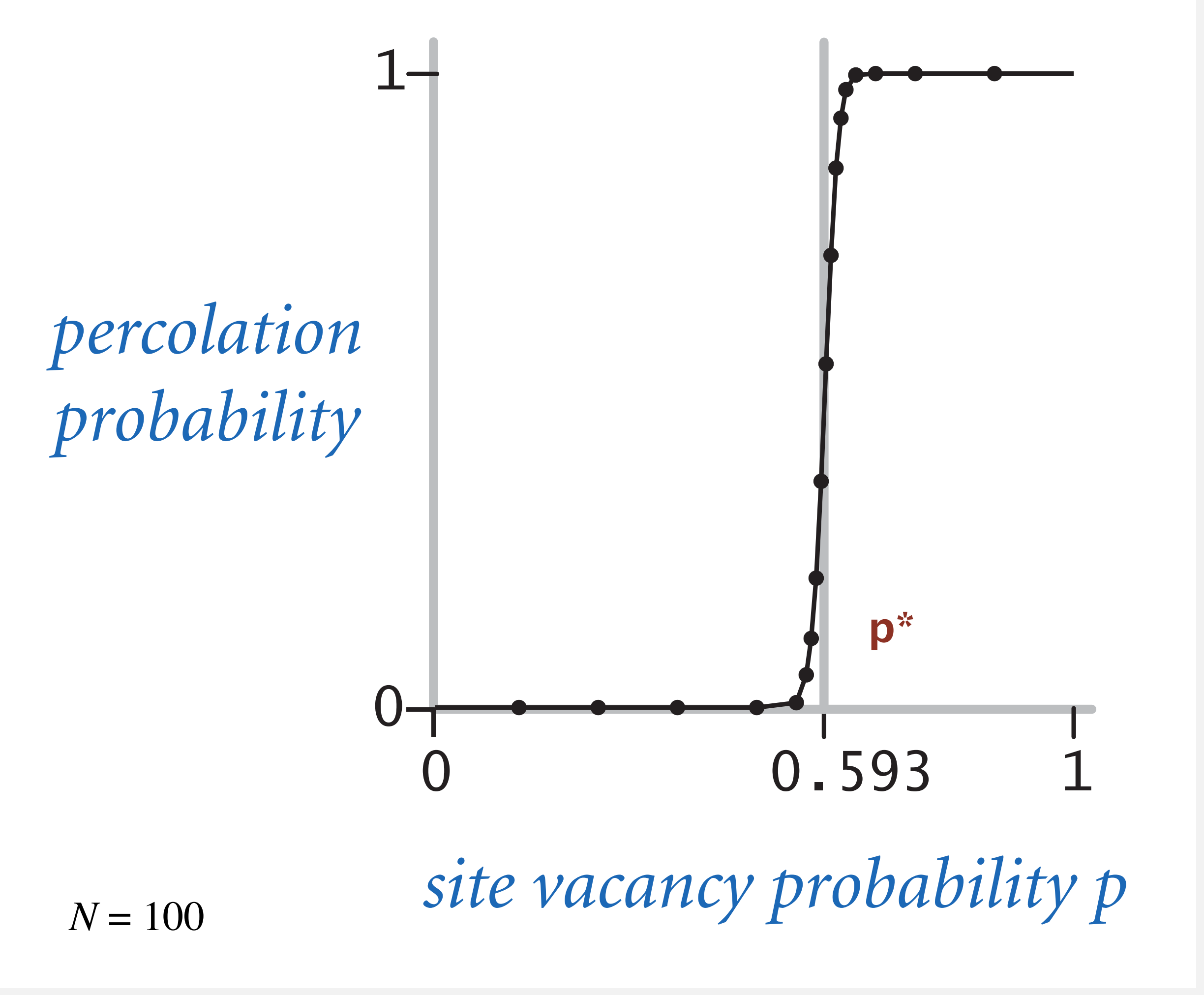




Percolation phase transition

When N is large, theory guarantees a sharp threshold p^*.

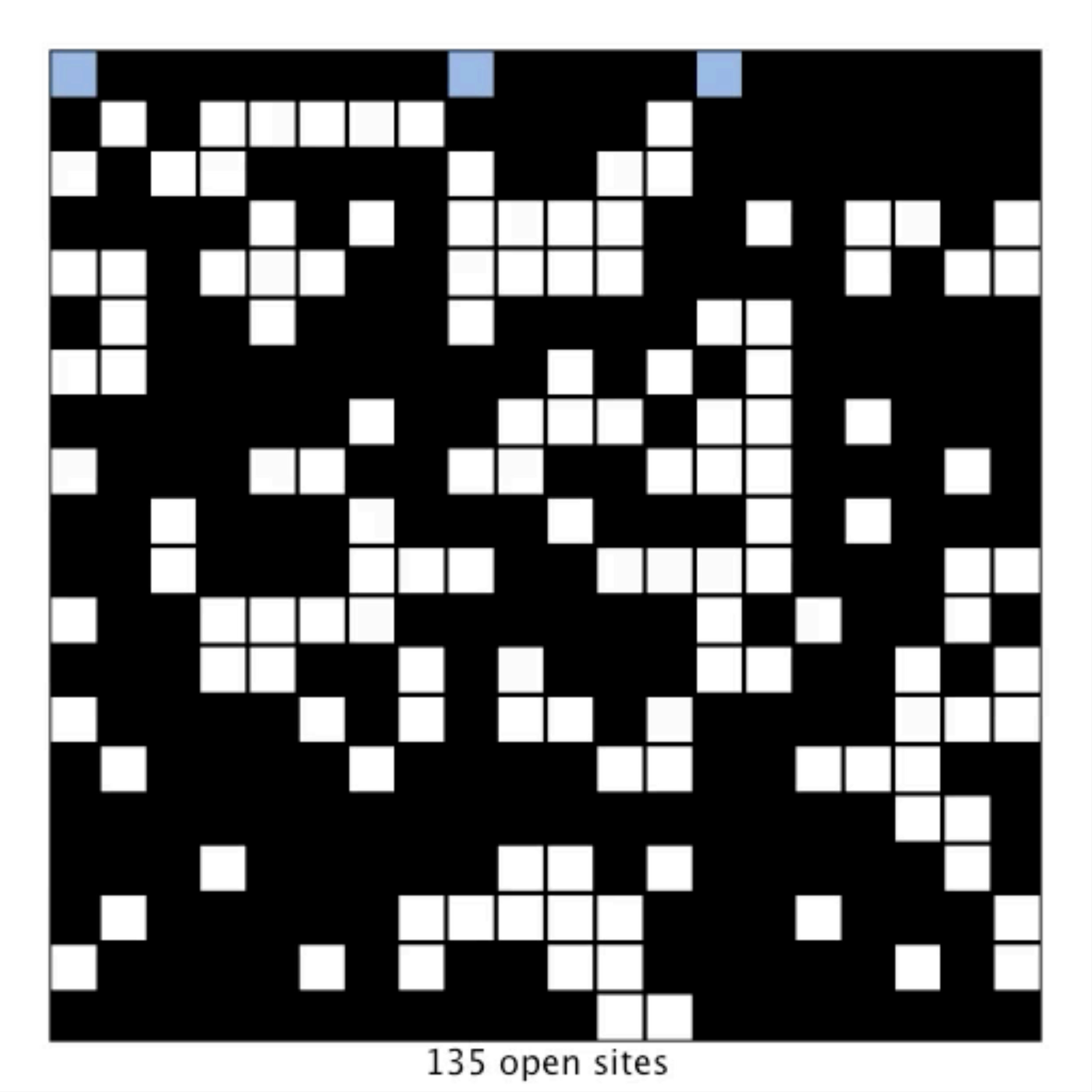
- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

Q. What is the value of p^* ?

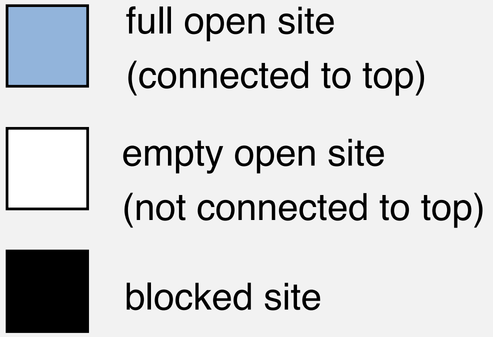


Monte Carlo simulation

- Initialize all sites in an N-by-N grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.



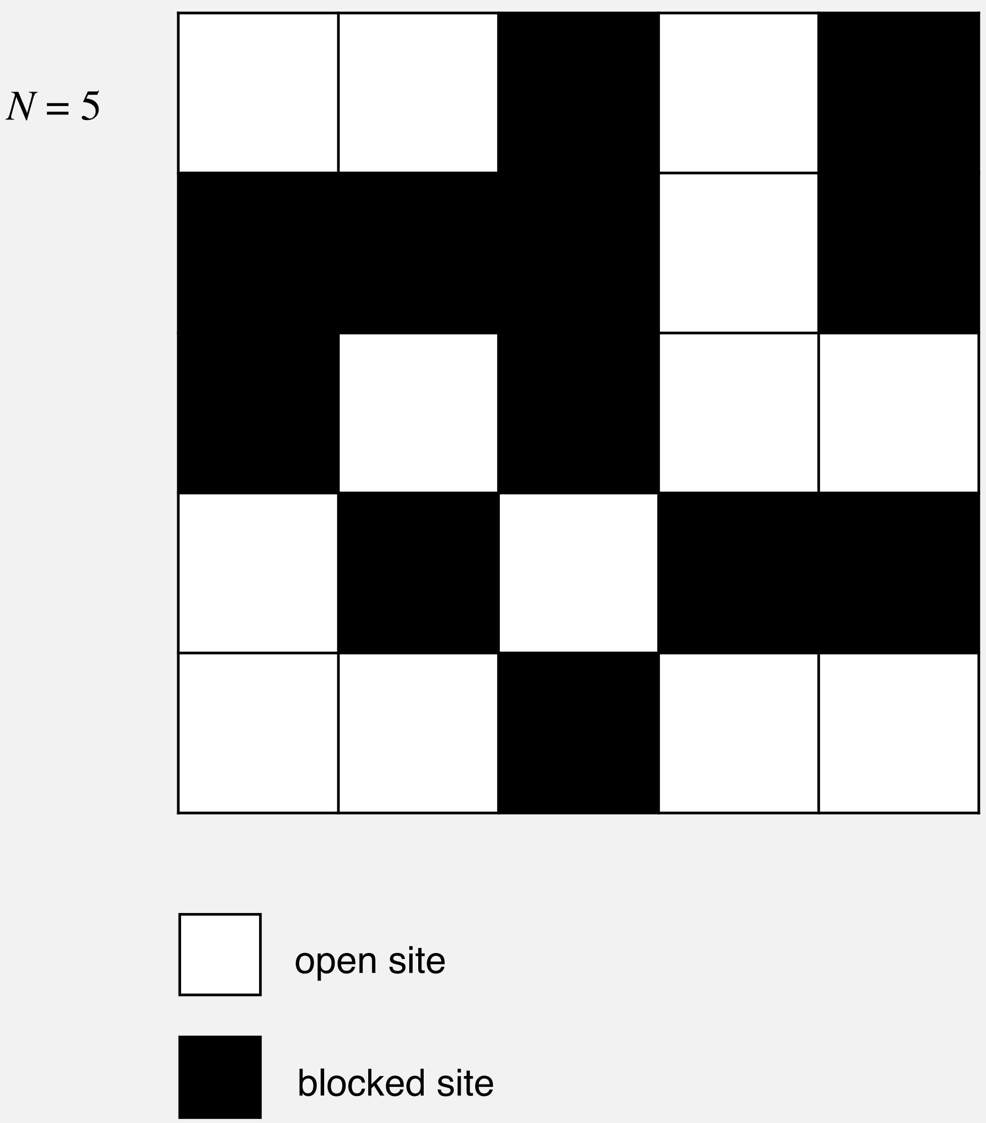
$N = 20$



Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

A. Model as a dynamic connectivity problem and use union-find.

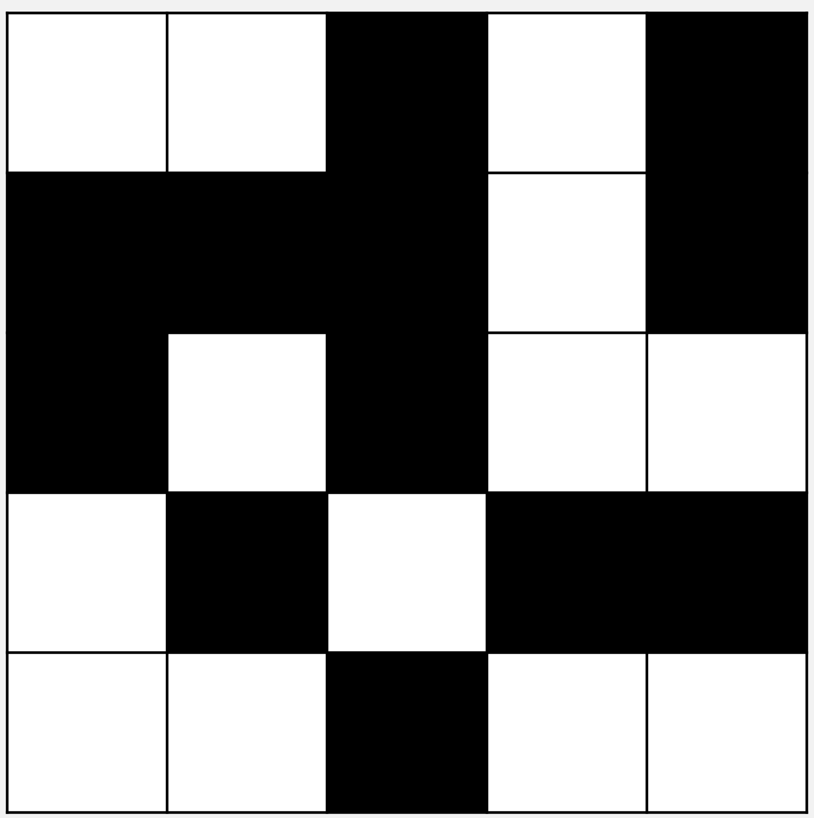


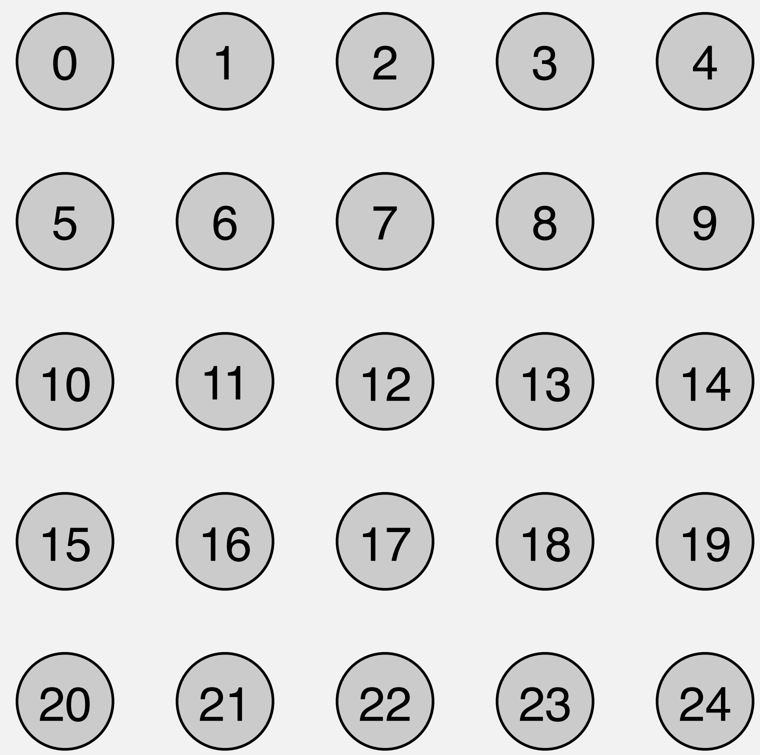
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.

$N = 5$





open site

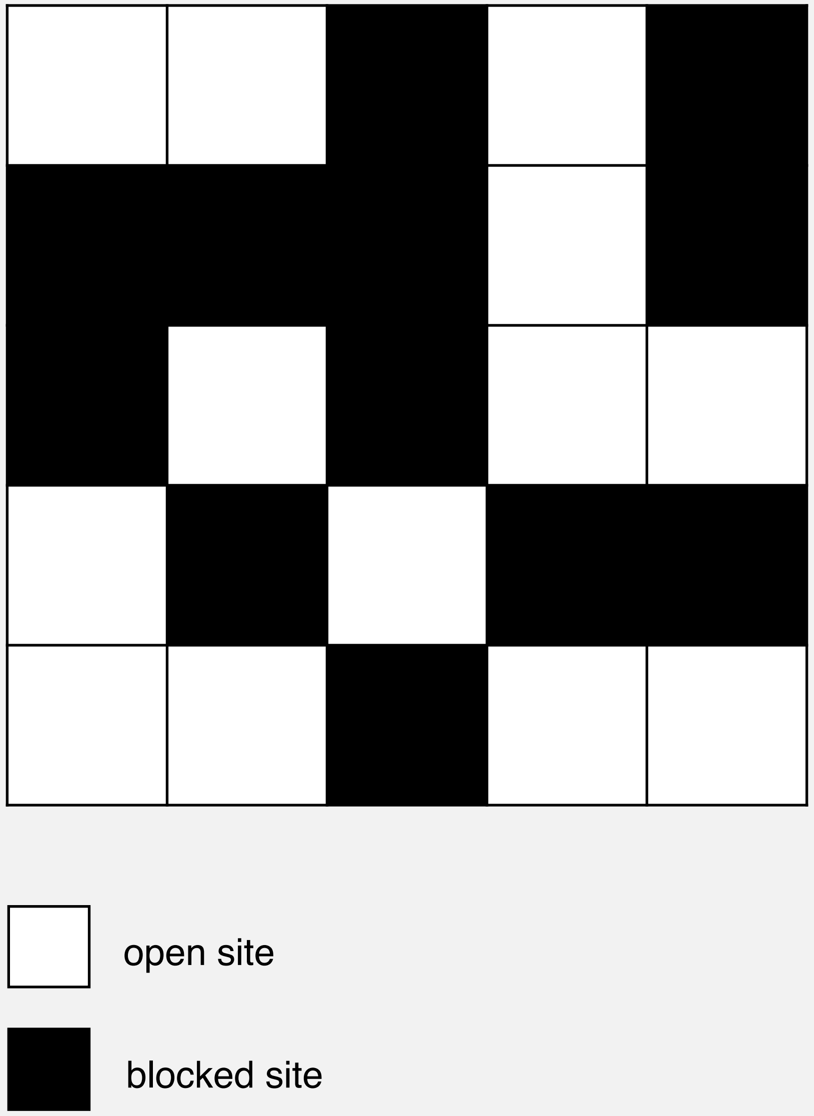
blocked site

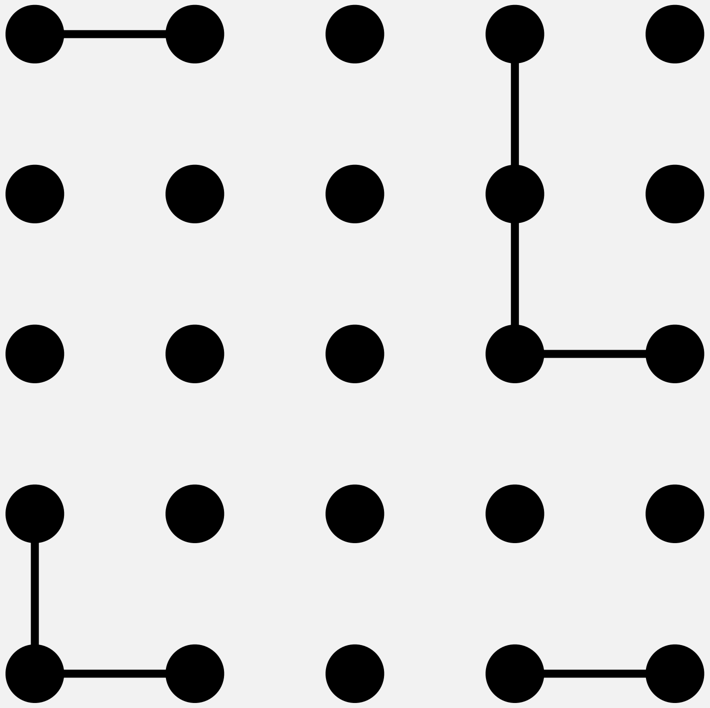
Dynamic connectivity solution to estimate percolation threshold

Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.

$N = 5$





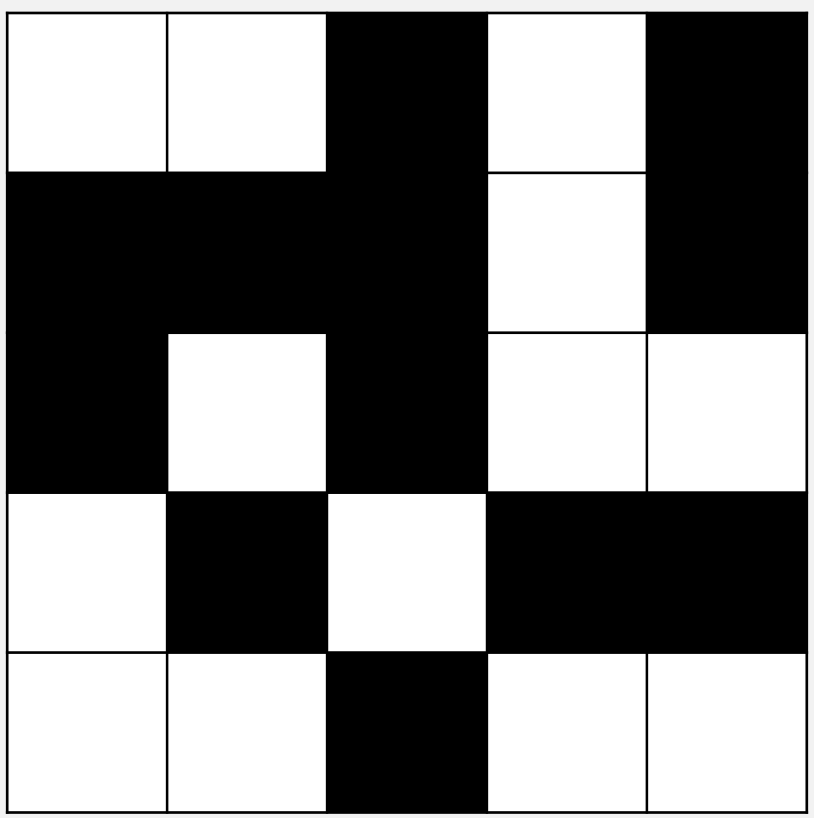
Dynamic connectivity solution to estimate percolation threshold

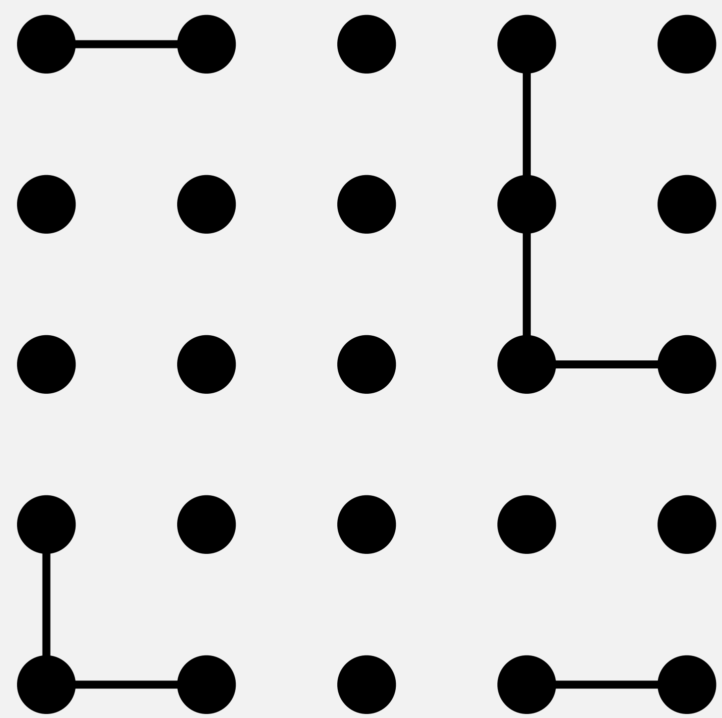
Q. How to check whether an N-by-N system percolates?

- Create an object for each site and name them 0 to $N^2 - 1$.
- Sites are in same component iff connected by open sites.
- Percolates iff any site on bottom row is connected to any site on top row.

brute-force algorithm: N^2 calls to connected()

$N = 5$





open site

blocked site

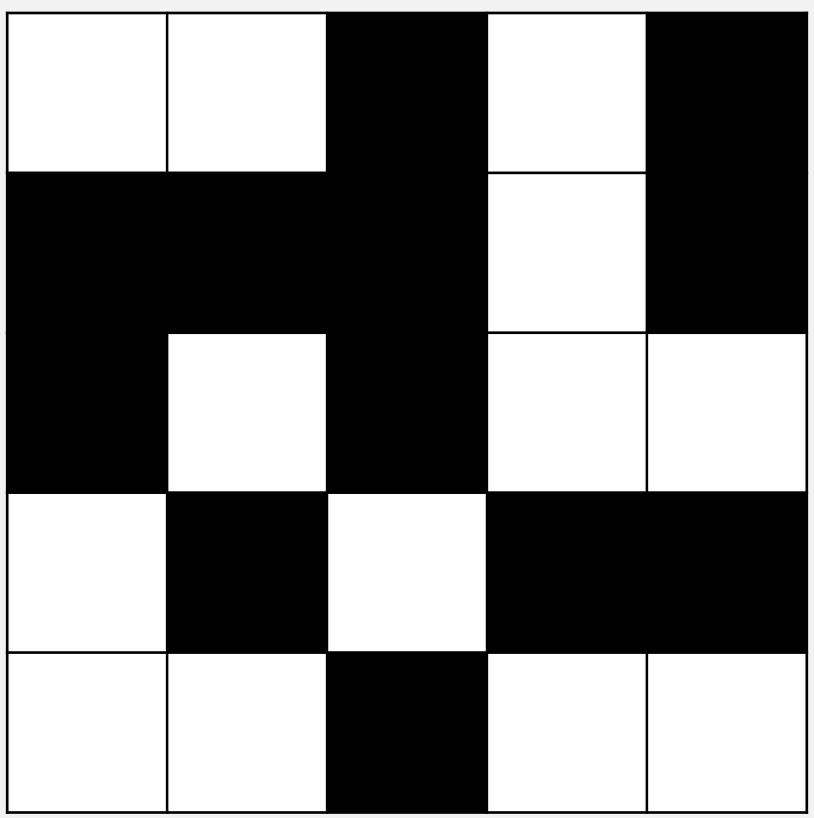
Dynamic connectivity solution to estimate percolation threshold

Clever trick. Introduce 2 virtual sites (and connections to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

more efficient algorithm: only 1 call to connected()

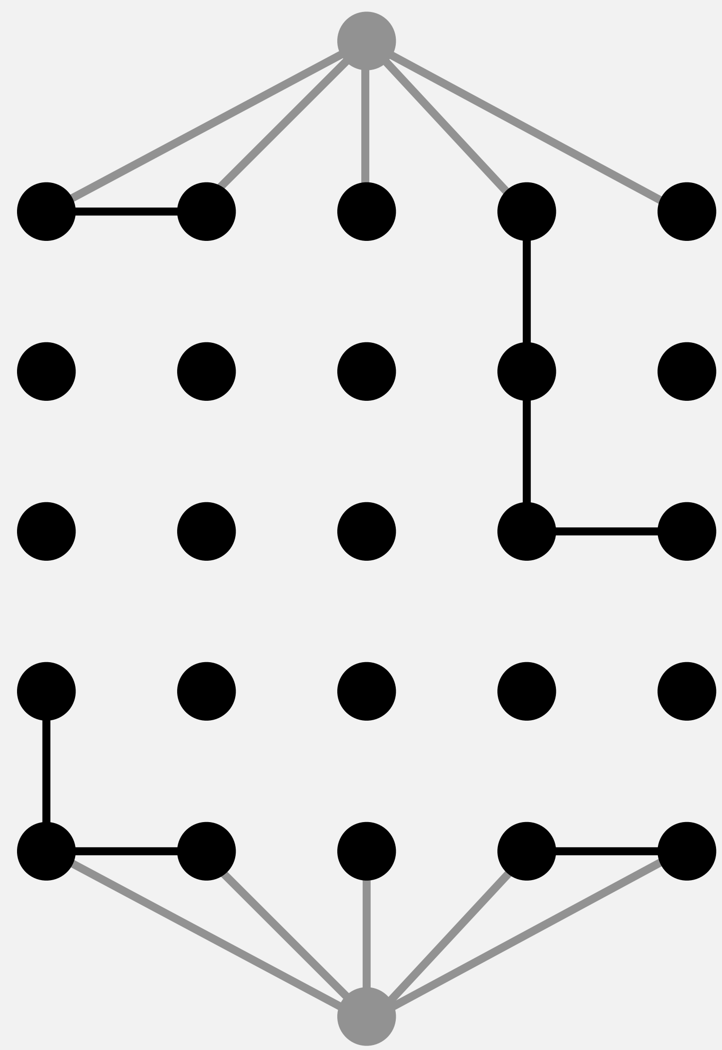
$N = 5$



open site

blocked site

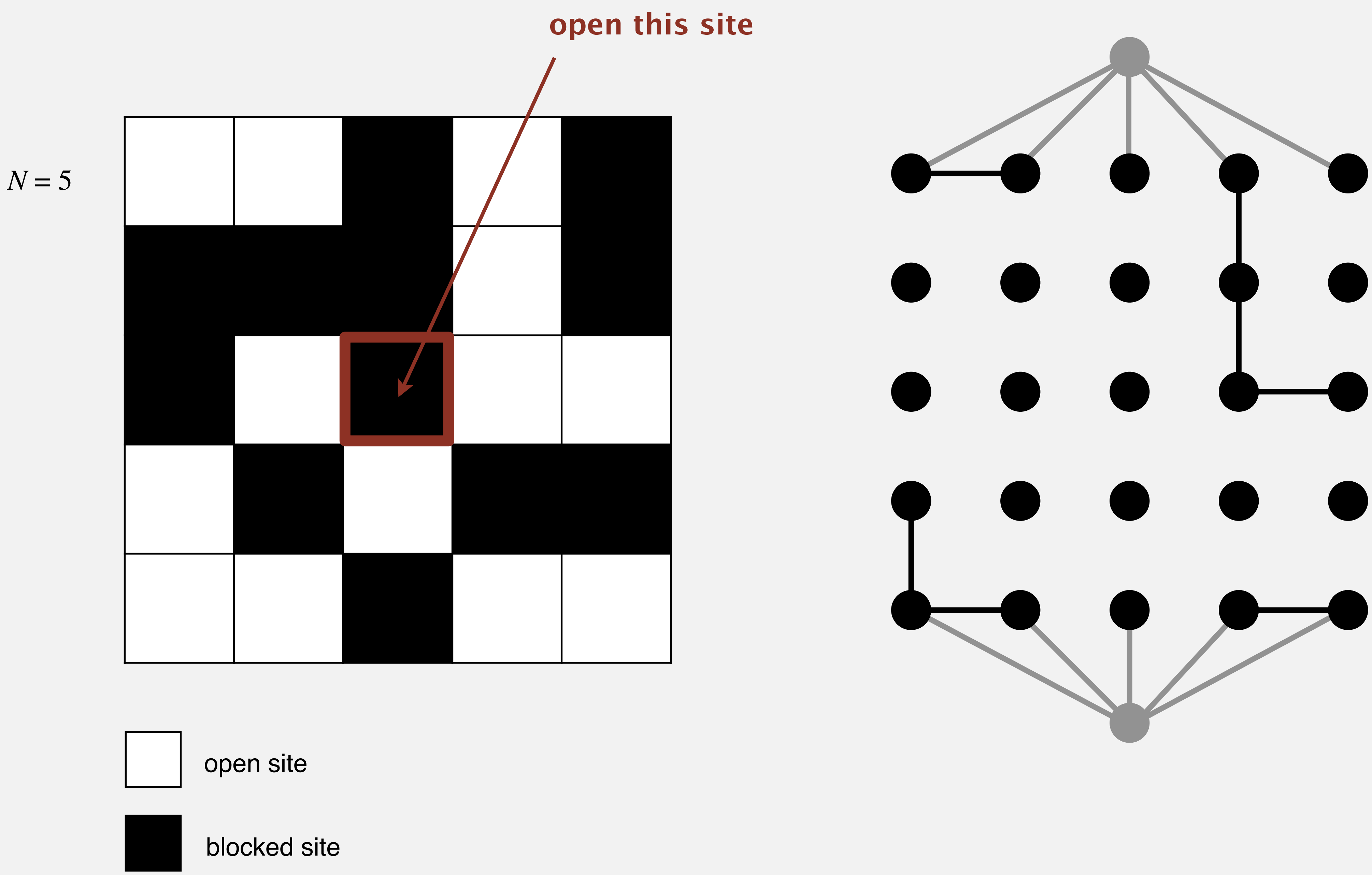
virtual top site



virtual bottom site

Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

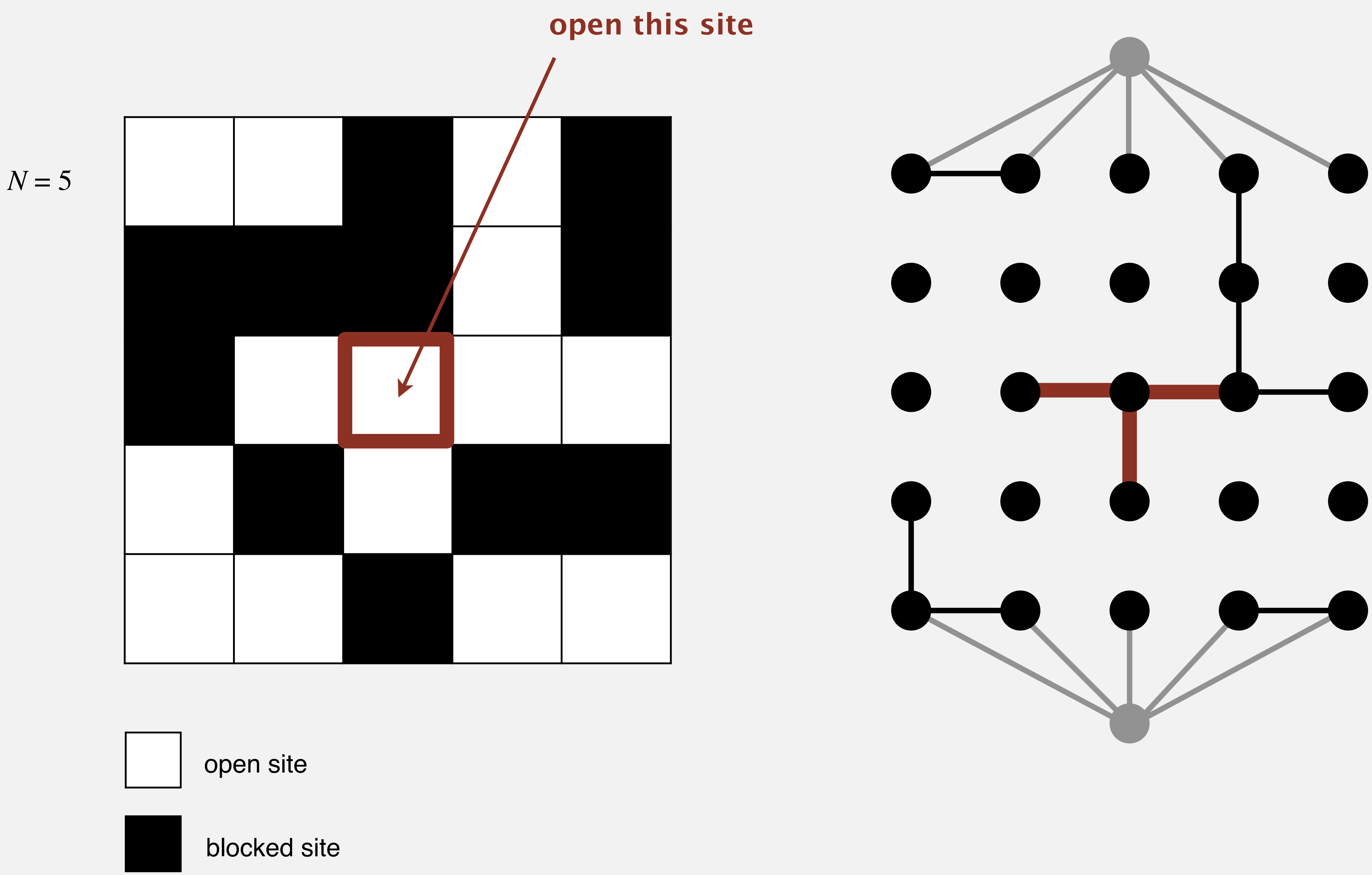


Dynamic connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

A. Mark new site as open; connect it to all of its adjacent open sites.

up to 4 calls to union()

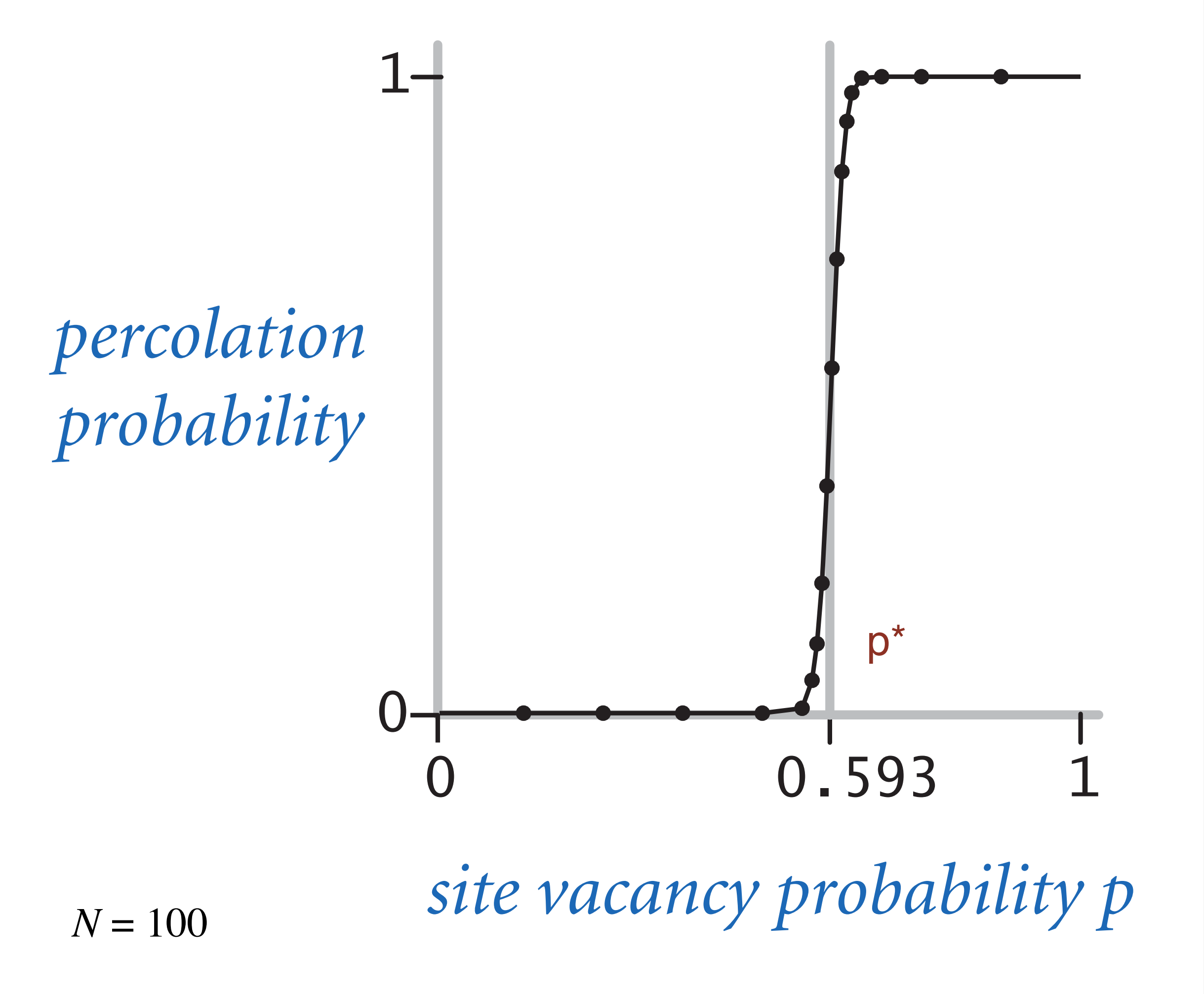


Percolation threshold

Q. What is percolation threshold p^* ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.