

# DISC Course: Nonlinear Control Systems

## Assignment 4

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### Exercise 1

**Solution:** (a) Let  $x = [x_1, x_2]^T = [q, \dot{q}]^T$ , then we have

$$\dot{x}_1 = x_2 \quad (1)$$

$$\dot{x}_2 = \frac{1}{m}(-dx_2 - g + u) \quad (2)$$

where  $m = m(x_1)$ ,  $d = d(x_1, x_2)$ ,  $g = g(x_1)$  are functions describing the mass, damping and spring. The above equation can be written in the following state-space form

$$\dot{x} = f(x) + k(x)u = \begin{bmatrix} x_2 \\ -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{m(x_1)} \end{bmatrix} u \quad (3)$$

$$y = h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} x \quad (4)$$

Lie derivatives

$$\mathcal{L}_f h(x) = x_2 \quad (5)$$

$$\mathcal{L}_k h(x) = 0 \quad (6)$$

$$\mathcal{L}_f^2 h(x) = -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) \quad (7)$$

$$\mathcal{L}_k \mathcal{L}_f h(x) = -\frac{1}{m(x_1)} \quad (8)$$

Therefore, the system has uniform relative degree 2. Let

$$\xi_1 = h(x) = x_1 \quad (9)$$

$$\xi_2 = \mathcal{L}_f h(x) = x_2 \quad (10)$$

Find a coordinate  $\eta_1 = \phi_1(x)$  such that  $\mathcal{L}_k \phi_1(x) = 0$ , i.e.,

$$\frac{\partial \phi_1}{\partial x}(x)k(x) = -\frac{\partial \phi_1}{\partial x_2} \frac{1}{m(x_1)} = 0 \quad (11)$$

A solution is  $\eta_1 = \phi_1(x) = x_1$ . Coordinate transformation

$$T(x) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \phi_1(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \quad (12)$$

Normal form

$$\dot{\xi}_1 = \xi_2 \quad (13)$$

$$\dot{\xi}_2 = -\frac{1}{m(\xi_1)}(d(\xi_1, \xi_2) + g(\xi_1)) + \frac{1}{m(\xi_1)}u \quad (14)$$

$$\dot{\eta}_1 = \xi_1 \quad (15)$$

(b) In equation (1), let

$$x_2 = -x_1 := k_1(x_1) \quad (16)$$

Take  $V_1(x_1) = \frac{1}{2}x_1^2$  as the Lyapunov function candidate, then  $\dot{V}_1(x_1) = x_1\dot{x}_1 = -x_1^2$ . Thus, the feedback control law stabilizes the system described by equation (1).

Let  $w_1 = x_2 - k_1(x_1)$ , then

$$\dot{x}_1 = x_2 = w_1 + k_1(x_1) = -x_1 + w_1 \quad (17)$$

$$\dot{w}_1 = \dot{x}_2 - \dot{k}_1(x_1) = -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2 \quad (18)$$

Take  $V(x) = V(x_1, w_1) = V_1(x_1) + \frac{1}{2}w_1^2 = \frac{1}{2}x_1^2 + \frac{1}{2}w_1^2$ , then

$$\begin{aligned} \dot{V}(x_1, w_1) &= x_1\dot{x}_1 + w_1\dot{w}_1 \\ &= x_1(-x_1 + w_1) + w_1\left(-\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2\right) \\ &= -x_1^2 + w_1\left(x_1 - \frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2\right) \end{aligned} \quad (19)$$

Let

$$\begin{aligned} u &= m(x_1)\left(-x_2 + \frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) - x_1 - c_1w_1\right) \\ &= -m(x_1)(c_1 + 1)(x_1 + x_2) + d(x_1, x_2)x_2 + g(x_1) \\ &:= \alpha(x) \end{aligned} \quad (20)$$

where  $c_1 > 0$ . Thus

$$\dot{V}(x_1, w_1) = -x_1^2 - c_1w_1^2 \leq 0 \quad (21)$$

Therefore, the feedback law (20) stabilizes the origin of the nonlinear system.

## Exercise 2

**Solution:** (a) The nonlinear system is described as

$$\dot{x}_1 = x_1^3 + x_2 \quad (22)$$

$$\dot{x}_2 = x_2 + x_1 + u \quad (23)$$

In equation (22), Let

$$x_2 = -2x_1^3 := k_1(x_1) \quad (24)$$

Take  $V_1(x_1) = \frac{1}{4}x_1^4$  as the Lyapunov function, then

$$\dot{V}_1(x_1) = x_1^3 \dot{x}_1 = x_1^3(x_1^3 - 2x_1^3) = -x_1^6 \leq 0 \quad (25)$$

Thus, the feedback law (24) stabilize the system (22).

Let  $w_1 = x_2 - k_1(x_1) = x_2 + 2x_1^3$ , then we have

$$\dot{x}_1 = x_1^3 + x_2 = -x_1^3 + w_1 \quad (26)$$

$$\dot{w}_1 = \dot{x}_2 + 6x_1^2 \dot{x}_1 = -6x_1^5 + 6x_1^2 w_1 + x_2 + x_1 + u \quad (27)$$

Take  $V(x) = V(x_1, w_1) = V_1(x_1) + \frac{1}{2}w_1^2 = \frac{1}{4}x_1^4 + \frac{1}{2}w_1^2$  as the Lyapunov function candidate, then

$$\begin{aligned} \dot{V}(x_1, w_1) &= x_1^3 \dot{x}_1 + \dot{w}_1 w_1 \\ &= x_1^3(-x_1^3 + w_1) + w_1(-6x_1^5 + 6x_1^2 w_1 + x_2 + x_1 + u) \\ &= -x_1^6 + w_1(x_1^3 - 6x_1^5 + 6x_1^2 w_1 + x_1 + x_2 + u) \end{aligned} \quad (28)$$

Let

$$\begin{aligned} u &= -x_1^3 + 6x_1^5 - 6x_1^2 w_1 - x_1 - x_2 - w_1 \\ &= -6x_1^5 - 3x_1^3 - 6x_1^2 x_2 - x_1 - 2x_2 \\ &:= \alpha(x) \end{aligned} \quad (29)$$

then

$$\dot{V}(x_1, w_1) = -x_1^6 - w_1^2 \leq 0 \quad (30)$$

which implies that feedback law (29) stabilizes the given nonlinear system.

(b) Let  $d = [d_1, d_2]^T$ ,  $x = [x_1, w_1]^T$ , considering the uncertainty in the system and base on the results in (a), we have

$$\begin{aligned} \dot{V}(x_1, w_1) &= x_1^3 \dot{x}_1 + \dot{w}_1 w_1 \\ &= -x_1^6 - w_1^2 + x_1^3 d_1 + w_1(d_2 + 6x_1^2 d_1) \\ &= -x_1^6 - w_1^2 + x_1^3 d_1 + w_1 d_2 + 6x_1^2 w_1 d_1 \\ &\leq -x_1^6 - w_1^2 + \left(\frac{x_1^6}{2} + \frac{d_1^2}{2}\right) + \left(\frac{w_1^2}{4} + d_2^2\right) + \left(\frac{w_1^2}{2} + \frac{x_1^6}{3} + 6^5 d_1^6\right) \\ &= -\frac{1}{6}x_1^6 - \frac{1}{4}w_1^2 + \left(\frac{1}{2} + 6^5\right)d_1^2 + d_2^2 \\ &\leq -\gamma_1(\|x\|) + \gamma_2(\|d\|) \end{aligned} \quad (31)$$

where  $\gamma_1(z) = \frac{1}{6}z^6$ ,  $\gamma_2(z) = (\frac{1}{2} + 6^5)z^2$  and  $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ .

Hence, with the feedback control law  $u = \alpha(x)$ , the closed-loop system is ISS with respect to  $d_1$  and  $d_2$ .