DISC Course: Mathematical Models of Systems

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March 5, 2018

1 Homework 2

1.1 Chapter 4

1.1.1 Exercise 4.2

Solution: The system dynamics can be written

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} y \\ \frac{dy}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1.1}$$

Let $x = \begin{bmatrix} y \\ \frac{dy}{dt} \end{bmatrix}$, we have

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \ \ 1]x \end{aligned} \tag{1.2}$$

which is an i/s/o representation for the original system.

1.1.2 Exercise 4.6

Solution: Choose the following state

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix} = \begin{bmatrix} u(k-5) \\ u(k-4) \\ u(k-3) \\ u(k-2) \\ u(k-1) \end{bmatrix} = \begin{bmatrix} y(k) \\ y(k+1) \\ y(k+2) \\ y(k+3) \\ y(k+4) \end{bmatrix} \tag{1.3}$$

Thus, we have

$$x(k+1)=\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\x_3(k+1)\\x_4(k+1)\\x_5(k+1)\end{bmatrix}=\begin{bmatrix}y(k+1)\\y(k+2)\\y(k+3)\\y(k+4)\\y(k+5)\end{bmatrix}=\begin{bmatrix}x_2(k)\\x_3(k)\\x_4(k)\\x_5(k)\\u(k)\end{bmatrix} \tag{1.4}$$

Hence, an i/s/o representation of the discrete-time system can be written as

$$\begin{aligned} x(k+1) &= \begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix} x(k) + \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix}1&0&0&0&0\end{bmatrix} x(k) \end{aligned} \tag{1.5}$$

1.1.3 Exercise 4.8

Solution: Take x an infinite-dimensional state which is defined by

$$x=[x_0,\ldots,x_k,\ldots,x_1]^T \tag{1.6}$$

where $k\in(0,1)$. Then we can determine a state space model for the system, which is written as follows

$$\begin{aligned} x_0(t) &= u(t) \\ x_k(t) &= u(t-k), \quad k\in(0,1) \\ y(t) &= u(t-1)=x_1(t) \end{aligned} \tag{1.7}$$

1.1.4 Exercise 4.22

Solution: (a) First we prove that these three i/s/o representations define the same behavior. For case (a), let $x=[x_1,x_2]^T$, then we have

$$\begin{bmatrix}\frac{dx_1}{dt}\\\frac{dx_2}{dt}\end{bmatrix}=\begin{bmatrix}-1&0\\0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u \tag{1.8}$$

and

$$y=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1 \tag{1.9}$$

Thus, the behavior of the system is decided by $\frac{d}{dt}y(t)=-y(t)+u(t)$.

Similarly, we can easily determine the behavior of case (b) and (c), whose behavior are both decided by $\frac{d}{dt}y(t)=-y(t)+u(t)$.

Hence, these three i/s/o representations define the same behavior.

(b) Consider the following nonsingular matrix

$$\Sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.10}$$

Then we have

$$\Sigma A_1 \Sigma^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = A_2 \tag{1.11}$$

and

$$\Sigma B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B_2 \tag{1.12}$$

and

$$C_1 \Sigma^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} = C_2 \tag{1.13}$$

Hence, the first two systems are similar.

(c) The first and the third system are not similar. That is, there does not exist a nonsingular matrix such that

$$\Sigma A_1 \Sigma^{-1} = A_3, \quad \Sigma B_1 = B_3, \quad C_1 \Sigma^{-1} = C_3 \tag{1.14}$$

Assume that there exist such a matrix $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ which satisfies the above equations. First, from $\Sigma B_1 = B_3$, we have

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.15}$$

which results in $\sigma_{11} = 1, \sigma_{21} = 0$. Since Σ is nonsingular, we have $\sigma_{22} \neq 0$.

Next, from $C_1 \Sigma^{-1} = C_3$, we have

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{\sigma_{12}}{\sigma_{22}} \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{1.16}$$

which results in $\sigma_{12} = 0$. Thus, we have

$$\Sigma A_1 \Sigma^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_{22} \end{bmatrix} \begin{bmatrix} -11 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \neq A_3 \tag{1.17}$$

Hence, the first and the third system are not similar.

1.2 Chapter 5

1.2.1 Exercise 5.8

Solution: (a) The controllability matrix of the system is given by

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 6 & -12 \end{bmatrix} \tag{1.18}$$

It is obvious that $\text{rank}(\mathcal{C}) = 2$. Hence, the system is controllable.

(b) The input function can be calculated based on the following equations

$$u(t) = B^T e^{-A^T t} z(t) \tag{1.19}$$
$$z(t) = K^{-1}(-x_0 + e^{-At} x_1) \tag{1.20}$$
$$K = \int_0^{t_1} e^{-A\tau} B B^T e^{-A^T \tau} d\tau \tag{1.21}$$

Let $t_1 = \log 2, x_0 = [0, 0]^T, x_1 = [1, 0]^T$. Since $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, we have

$$e^{-At} = e^{-A^T t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \tag{1.22}$$

$$B^T e^{-A^T \tau} = \begin{bmatrix} 2e^{\tau} & 6e^{2\tau} \end{bmatrix} \tag{1.23}$$

Thus, we can obtain

$$K = \begin{bmatrix} 6 & 28 \\ 28 & 135 \end{bmatrix} \tag{1.24}$$

Hence, we have

$$\begin{aligned} u(t) &= B^T e^{-A^T t} z(t) \\ &= \begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 6 & 28 \\ 28 & 135 \end{bmatrix}^{-1} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad = -\frac{84}{13} e^{3t} + \frac{135}{13} e^{2t} \end{aligned} \tag{1.25}$$

1.2.2 Exercise 5.10

Solution: (a) Denote $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. Thus,

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2} \end{bmatrix} \tag{1.26}$$

where

$$e^{A_1 t} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{1.27}$$

$$e^{A_2 t} = \begin{bmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{bmatrix} \tag{1.28}$$

Hence, we have

$$e^{At} = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos 2t & \frac{1}{2}\sin 2t \\ 0 & 0 & -2\sin 2t & \cos 2t \end{bmatrix} \tag{1.29}$$

(b) Similar with Exercise 5.8, let $x_0 = [0\ 1\ 0\ -1]^T, x_1 = [0\ 0\ 0\ 0]^T$ and $t_1 = 2\pi$, then we can calculate the input function which is given below

$$u(t) = \frac{\cos 2t}{2\pi} - \frac{2\cos t}{\pi} \tag{1.30}$$

(c) The controllability matrix of the system is

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0.5 & 0 & -0.5 \\ 0.5 & 0 & -0.5 & 0 \\ 0 & 2 & 0 & -8 \\ 2 & 0 & -8 & 0 \end{bmatrix} \end{aligned} \tag{1.31}$$

whose rank is rank$(C) = 4$. Therefore, the system is controllable. Hence, there exists an output function u that derives the system from equilibrium at t = 0 to state $[1, 0, -1, 0]^T$ of at $t = 1$.

(d) Suppose that there exists such a control input which makes the state keep at $[1, 0, -1, 0]^T$. Then we have

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 2 \end{bmatrix} u(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{1.32}$$

which results in

$$\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 2 \end{bmatrix} u(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -4 \end{bmatrix} \tag{1.33}$$

It can be verified that there does not exist a function $u(t)$ which can satisfy the above equation. Hence, there does not exist an input function u that drives the system from equilibrium at $t = 0$ to state $[1, 0, -1, 0]^T$ of at $t = 1$.

(e) Denote $x = [x_1, x_2, x_3, x_4]^T$ and let

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 2 \end{bmatrix} u(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{1.34}$$

we can obtain

$$
\begin{aligned}
x_1 &= -\frac{1}{2}u(t) \\
x_2 &= 0 \\
x_3 &= -\frac{1}{2}u(t) \\
x_4 &= 0
\end{aligned}
\tag{1.35}
$$

which indicates that the two masses should be at the same position and their velocities should should be zero at time.

1.2.3 Exercise 5.12

Solution: It is obvious that in this exercise, question (a) is only a special case of question (b). Therefore, we solve question (b) directly and state why question (a) is also solved after proving the statement in (b).

Define

$$M_1(\lambda) = \lambda I_{n_1} - A_1 \tag{1.36}$$
$$M_2(\lambda) = \lambda I_{n_2} - A_2 \tag{1.37}$$
$$\tilde{M}(\lambda) = \lambda I_{n_1+n_2} - \tilde{A} \tag{1.38}$$

Then we have

$$\tilde{M}(\lambda) = \begin{bmatrix} M_1(\lambda) & 0 \\ 0 & M_2(\lambda) \end{bmatrix} \tag{1.39}$$

Note that A_1 and A_2 have a common eigenvalue, which is denoted as λ_k. Then we have

$$\text{rank} M_1(\lambda_k) \leq n_1 - 1 \tag{1.40}$$
$$\text{rank} M_2(\lambda_k) \leq n_2 - 1 \tag{1.41}$$

Thus,

$$\text{rank}\tilde{M}(\lambda_k) \leq \text{rank}M_1(\lambda_k) + \text{rank}M_2(\lambda_k) \leq n_1 + n_2 - 2 \tag{1.42}$$

Furthermore,

$$\text{rank}\begin{bmatrix} \lambda_k I_{n_1+n_2} - \tilde{A} & \tilde{b} \end{bmatrix} = \text{rank}\begin{bmatrix} \tilde{M}(\lambda_k) & \tilde{b} \end{bmatrix} \leq n_1 + n_2 - 1 \tag{1.43}$$

That is, the matrix $\begin{bmatrix} \lambda I_{n_1+n_2} - \tilde{A} & \tilde{b} \end{bmatrix}$ is not full of rank at least at $\lambda = \lambda_k$. Hence, $(\tilde{A}, \tilde{b})$ is not controllable for any $\tilde{b} \in \mathbb{R}^{2n \times 1}$. This completes the proof of question (b).

For question (a), it is a special case of question (b) when $A_1 = A_2 = A$. Obviously their eigenvalues are the same. Hence, the statement in question (a) is also true.

1.2.4 Exercise 5.13

Solution: (a) The matrix $R(\xi) \in \mathbb{R}^{3\times 4}[\xi]$ is

$$R(\xi) = \begin{bmatrix} k_1 + k_3 + d_1\xi + M_1\xi^2 & 0 & -k_3 & 0 \\ 0 & k_2 + k_4 + d_2\xi + M_2\xi^2 & -k_4 & 0 \\ -k_3 & -k_4 & k_3 + k_4 + M_3\xi^2 & -1 \end{bmatrix} \tag{1.44}$$

(b) Denote

$$r_1(\xi) = k_1 + k_3 + d_1\xi + M_1\xi^2 \tag{1.45}$$
$$r_2(\xi) = k_2 + k_4 + d_2\xi + M_2\xi^2 \tag{1.46}$$
$$r_3(\xi) = k_3 + k_4 + M_3\xi^2 \tag{1.47}$$

We can write $R(\xi)$ in the following form

$$R(\xi) = \begin{bmatrix} r_1(\xi) & 0 & -k_3 & 0 \\ 0 & r_2(\xi) & -k_4 & 0 \\ -k_3 & -k_4 & r_3(\xi) & -1 \end{bmatrix} = \begin{bmatrix} R_1(\xi) \\ R_2(\xi) \\ R_3(\xi) \end{bmatrix} \tag{1.48}$$

The system is controllable if and only if $\text{rank}(R(\lambda))$ is the same for all $\lambda \in \mathbb{C}$. Since $\text{rank}(R(0)) = 3$, we have $\text{rank}(R(\lambda)) = 3, \forall\lambda \in \mathbb{C}$, which is true if and only if $R_1(\xi)$ and $R_2(\xi)$ are coprime. Now, we prove that it is equivalent to that $r_1(\xi)$ and $r_2(\xi)$ are coprime.

First, if $R_1(\xi)$ and $R_2(\xi)$ are coprime, we prove that $r_1(\xi)$ and $r_2(\xi)$ are also coprime by contradiction. Suppose that $r_1(\xi)$ and $r_2(\xi)$ are not coprime and thus can be expressed by

$$r_1(\xi) = g(\xi)\bar{r}_1(\xi) \tag{1.49}$$
$$r_2(\xi) = g(\xi)\bar{r}_2(\xi) \tag{1.50}$$

where $\bar{r}_1(\xi)$ and $\bar{r}_2(\xi)$ are coprime and $g(\xi) \not\equiv 0$ is the common factor. Note that

$$\alpha R_1(\xi) + \beta R_2(\xi) = \begin{bmatrix} \alpha g(\xi)\bar{r}_1(\xi) & \beta g(\xi)\bar{r}_2(\xi) & -\alpha k_3 - \beta k_4 & 0 \end{bmatrix} \tag{1.51}$$

If we take $\alpha = k_4 \neq 0, \beta = -k_3 \neq 0$ and $\xi = \{\xi \in \mathbb{C} | g(\xi) = 0\}$, then we have $\alpha R_1(\xi) + \beta R_2(\xi) = 0$, which indicates that $R_1(\xi)$ and $R_2(\xi)$ are not coprime. This is contradictive with the fact that they are coprime. Hence, $r_1(\xi)$ and $r_2(\xi)$ are coprime.

Second, if $r_1(\xi)$ and $r_2(\xi)$ are coprime, we prove that $R_1(\xi)$ and $R_2(\xi)$ are also coprime. Note that

$$\alpha R_1(\xi) + \beta R_2(\xi) = \begin{bmatrix} \alpha r_1(\xi) & \beta r_2(\xi) & -\alpha k_3 - \beta k_4 & 0 \end{bmatrix} \tag{1.52}$$

Since $r_1(\xi)$ and $r_2(\xi)$ are coprime, we have

$$\alpha R_1(\xi) + \beta R_2(\xi) = 0 \implies \alpha r_1(\xi) = 0, \beta r_2(\xi) = 0 \tag{1.53}$$
$$\implies \alpha = 0, \beta = 0 \tag{1.54}$$

which implies that $R_1(\xi)$ and $R_2(\xi)$ are coprime.

In sum, the system is controllable if and only if $r_1(\xi)$ and $r_2(\xi)$ are coprime.

(c) Substituting $r_1(\xi)$, $r_2(\xi)$ and $a(\xi) = a_1\xi + a_0$, $b(\xi) = b_1\xi + b_0$ into the equation

$$a(\xi)r_1(\xi) + b(\xi)r_2(\xi) = 1 \tag{1.55}$$

we can obtain

$$\begin{aligned} a_1M_1 + b_1M_2 &= 0 \\ a_0M_1 + a_1d_1 + b_0M_2 + b_1d_2 &= 0 \\ a_0d_1 + a_1(k_1+k_3) + b_0d_2 + b_1(k_2+k_4) &= 0 \\ a_0(k_1+k_3) + b_0(k_2+k_4) - 1 &= 0 \end{aligned} \tag{1.56}$$

which results in

$$\begin{bmatrix} 0 & M_1 & 0 & M_2 \\ M_1 & d_1 & M_2 & d_2 \\ d_1 & k_1+k_3 & d_2 & k_2+k_4 \\ k_1+k_3 & 0 & k_2+k_4 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.57}$$

(d) In equation (1.57), let

$$A = \begin{bmatrix} 0 & M_1 & 0 & M_2 \\ M_1 & d_1 & M_2 & d_2 \\ d_1 & k_1+k_3 & d_2 & k_2+k_4 \\ k_1+k_3 & 0 & k_2+k_4 & 0 \end{bmatrix} \tag{1.58}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.59}$$

Then it has a solution if and only if

$$\text{rank}([A \;\; B]) = \text{rank}(A) \tag{1.60}$$

Now we prove that the above equation is equivalent with that A is nonsingular.

First, we prove that "rank$([A \;\; B]) = \text{rank}(A) \implies A$ is nonsingular" by contradiction. Suppose that A is singular, then we have rank$(A) \leq 3$. Furthermore, rank$([A \;\; B]) = \text{rank}(A) + 1 \neq \text{rank}(A)$, which is contradictive with the fact. Hence, A is singular.

Second, we prove that if A is nonsingular, then rank$([A \;\; B]) = \text{rank}(A)$. It is obvious that if A is nonsingular, rank$([A \;\; B]) = 4 = \text{rank}(A)$. This completes the proof.

In sum, the equation (1.57) has a solution if and only the coefficient matrix A is nonsingular.

(e) Based on previous results, we know that the system is not controllable if and only if A is singular, which implies

$$\begin{aligned}\det(A) =&(k_1+k_3)M_1d_2^2+(k_2+k_4)(d_1^2-2(k_1+k_3)M_1)M_2+(k_2+k_4)^2M_1^2 \quad (1.61)\\ &+(k_1+k_3)^2M_2^2-((k_2+k_4)M_1+(k_1+k_3)M_2)d_1d_2=0 \quad (1.62)\end{aligned}$$

That is, the values of the parameters satisfy an algebraic equation.

(f) We rewrite the system into the following form

$$P(\frac{d}{dt})\begin{bmatrix}w_1\\w_2\end{bmatrix}=Q(\frac{d}{dt})\begin{bmatrix}w_3\\w_4\end{bmatrix} \tag{1.63}$$

where

$$P(\xi)=\begin{bmatrix}2+\xi+\xi^2 & 0\\ 0 & 2+\xi+\xi^2\\ -1 & -1\end{bmatrix} \tag{1.64}$$

$$Q(\xi)=\begin{bmatrix}1 & 0\\ 1 & 0\\ -2-\xi^2 & 1\end{bmatrix} \tag{1.65}$$

Let $\xi^\star=\{\xi\in\mathbb{C}|2+\xi+\xi^2=0\}$, then we have rank$(R(\xi^\star))=1$. That is, $R(\xi^\star)$ is not full column rank. Hence, (w_1,w_2) is not observable from (w_3,w_4).

(g) Let

$$x=[w_1,\dot{w}_1,w_2,\dot{w}_2,w_3,\dot{w}_3]^T \tag{1.66}$$

then we have

$$\dot{x}=Ax+Bw_4 \tag{1.67}$$

$$y=Cx \tag{1.68}$$

where

$$A=\begin{bmatrix}0&1&0&0&0&0\\ -\frac{k_1+k_3}{M_1}&\frac{d_1}{M_1}&0&0&\frac{k_3}{M_1}&0\\ 0&0&0&1&0&0\\ 0&0&-\frac{k_2+k_4}{M_2}&-\frac{d_2}{M_2}&-\frac{k_4}{M_2}&0\\ 0&0&0&0&0&1\\ \frac{k_3}{M_3}&0&\frac{k_4}{M_3}&0&-\frac{k_3+k_4}{M_3}&0\end{bmatrix} \tag{1.69}$$

$$B=\begin{bmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{M_3}\end{bmatrix} \tag{1.70}$$

$$C=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\end{bmatrix} \tag{1.71}$$

1.3 Chapter 6

1.3.1 Exercise 6.2

Solution: (a) Since $M_1(\xi)$ and $M_2(\xi)$ have no common factor, there exists a unimodular matrix $U(\xi)$ such that the left-most column of it is $M(\xi) = [M_1(\xi), M_2(\xi)]^T$. Thus, we can denote

$$U(\xi) = \begin{bmatrix} M_1(\xi) & M_3(\xi) \\ M_2(\xi) & M_4(\xi) \end{bmatrix} \tag{1.72}$$

Thus,

$$U^{-1}(\xi) = \frac{1}{C} \begin{bmatrix} M_4(\xi)- & M_3(\xi) \\ -M_2(\xi) & M_1(\xi) \end{bmatrix} \tag{1.73}$$

$$U(\xi) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = M(\xi) \tag{1.74}$$

$$U^{-1}(\xi)M(\xi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.75}$$

where $C = \det(U(\xi))$. For the system differential equation,

$$R(\frac{d}{dt})w = M(\frac{d}{dt})l \tag{1.76}$$

Pre-multiple it by $U^{-1}(\frac{d}{dt})$, then we obtain

$$U^{-1}(\frac{d}{dt})R(\frac{d}{dt})w = U^{-1}(\frac{d}{dt})M(\frac{d}{dt})l \tag{1.77}$$

Substituting (1.72) and (1.73), we have

$$\frac{1}{C} \begin{bmatrix} M_4(\frac{d}{dt})- & M_3(\frac{d}{dt}) \\ -M_2(\frac{d}{dt}) & M_1(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} R_1(\frac{d}{dt}) \\ R_2(\frac{d}{dt}) \end{bmatrix} w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} l \tag{1.78}$$

whose second row results in

$$(M_2(\frac{d}{dt})R_1(\frac{d}{dt}) - M_1(\frac{d}{dt})R_2(\frac{d}{dt}))w = 0 \tag{1.79}$$

This completes the proof.

(b) Suppose the common factor of $M_1(\xi)$ and $M_2(\xi)$ is $g(\xi)$. That is,

$$M_1(\xi) = g(\xi)\tilde{M}_1(\xi) \tag{1.80}$$

$$M_2(\xi) = g(\xi)\tilde{M}_2(\xi) \tag{1.81}$$

where $\tilde{M}_1(\xi)$ and $\tilde{M}_2(\xi)$ have no common factor. Thus we have

$$M(\xi) = \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix} = g(\xi) \begin{bmatrix} \tilde{M}_1(\xi) \\ \tilde{M}_2(\xi) \end{bmatrix} \tag{1.82}$$

Similarly, take $\tilde{U}(\xi)$ a unimodular matrix such that

$$\tilde{U}(\xi) = \begin{bmatrix} \tilde{M}_1(\xi) & \tilde{M}_3(\xi) \\ \tilde{M}_2(\xi) & \tilde{M}_4(\xi) \end{bmatrix} \tag{1.83}$$

$$\tilde{U}^{-1}(\xi) = \frac{1}{\tilde{C}} \begin{bmatrix} \tilde{M}_4(\xi) - & \tilde{M}_3(\xi) \\ -\tilde{M}_2(\xi) & \tilde{M}_1(\xi) \end{bmatrix} \tag{1.84}$$

where $\tilde{C} = \det(\tilde{U}(\xi))$. Then, for the system differential equation, we have

$$\tilde{U}^{-1}(\frac{d}{dt})R(\frac{d}{dt})w = \tilde{U}^{-1}(\frac{d}{dt})M(\frac{d}{dt})l \tag{1.85}$$

$$\frac{1}{\tilde{C}} \begin{bmatrix} \tilde{M}_4(\frac{d}{dt}) - & \tilde{M}_3(\frac{d}{dt}) \\ -\tilde{M}_2(\frac{d}{dt}) & \tilde{M}_1(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} R_1(\frac{d}{dt}) \\ R_2(\frac{d}{dt}) \end{bmatrix} w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} g(\frac{d}{dt})l \tag{1.86}$$

whose second row results in

$$(\tilde{M}_2(\frac{d}{dt})R_1(\frac{d}{dt}) - \tilde{M}_1(\frac{d}{dt})R_2(\frac{d}{dt}))w = 0 \tag{1.87}$$

which is the differential equation for the manifest behavior.

1.3.2 Exercise 6.3

Solution: (a) Consider the SISO systems

$$\Sigma_1 : \; p_1(\frac{d}{dt})y_1 = q_1(\frac{d}{dt})u_1 \tag{1.88}$$

$$\Sigma_2 : \; p_2(\frac{d}{dt})y_2 = q_2(\frac{d}{dt})u_2 \tag{1.89}$$

and

$$u_1 = u + y_2 \tag{1.90}$$

$$u_2 = y_1 = y \tag{1.91}$$

We can rearrange the equations into

$$p_1(\frac{d}{dt})y - q_1(\frac{d}{dt})u = q_1(\frac{d}{dt})y_2 \tag{1.92}$$

$$q_2(\frac{d}{dt})y = p_2(\frac{d}{dt})y_2 \tag{1.93}$$

which can be further written in the form

$$R(\frac{d}{dt})w = M(\frac{d}{dt})l \tag{1.94}$$

where

$$w = \begin{bmatrix} u \\ y \end{bmatrix}, \; l = \begin{bmatrix} u_1 \\ y_2 \end{bmatrix}, \; R(\xi) = \begin{bmatrix} -q_1(\xi) & p_1(\xi) \\ 0 & q_2(\xi) \end{bmatrix}, \; M(\xi) = \begin{bmatrix} 0 & q_1(\xi) \\ 0 & p_2(\xi) \end{bmatrix} \tag{1.95}$$

(b) This is similar with Exercise 6.2. Since $\bar{p}_2(\xi)$ and $\bar{q}_1(\xi)$ have no common factor, there exists a unimodular matrix $U(\xi)$ such that the left-most column of it is $[\bar{q}_1(\xi), \bar{p}_2(\xi)]^T$. Thus, we can denote

$$U(\xi) = \begin{bmatrix} \bar{q}_1(\xi) & a(\xi) \\ \bar{p}_2(\xi) & b(\xi) \end{bmatrix} \tag{1.96}$$

Thus,

$$U^{-1}(\xi) = \frac{1}{C} \begin{bmatrix} b(\xi) & -a(\xi) \\ -\bar{p}_2(\xi) & \bar{q}_1(\xi) \end{bmatrix} \tag{1.97}$$

$$U(\xi) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{q}_1(\xi) \\ \bar{p}_2(\xi) \end{bmatrix} \tag{1.98}$$

$$U^{-1}(\xi) \begin{bmatrix} \bar{q}_1(\xi) \\ \bar{p}_2(\xi) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.99}$$

where $C = \det(U(\xi))$. For the system differential equation,

$$R(\frac{d}{dt})w = M(\frac{d}{dt})l \tag{1.100}$$

we have

$$U^{-1}(\frac{d}{dt})R(\frac{d}{dt})w = U^{-1}(\frac{d}{dt})M(\frac{d}{dt})l \tag{1.101}$$

Substituting (1.94) and (1.95), we have

$$\frac{1}{C} \begin{bmatrix} M_4(\frac{d}{dt}) - & M_3(\frac{d}{dt}) \\ -M_2(\frac{d}{dt}) & M_1(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} -q_1(\frac{d}{dt}) & p_1(\frac{d}{dt}) \\ 0 & q_2(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = c(\frac{d}{dt}) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} \tag{1.102}$$

whose second row results in

$$(p_1(\frac{d}{dt})\bar{p}_2(\frac{d}{dt}) - \bar{q}_1(\frac{d}{dt})q_2(\frac{d}{dt}))y = \bar{p}_2(\frac{d}{dt})\bar{q}_1(\frac{d}{dt})u \tag{1.103}$$

This completes the proof.

1.4 Additional Exercise

Proof: For the matrix $R(\xi)$ given by

$$R(\xi) = \begin{bmatrix} 3 + 3\xi & 2 + 5\xi + \xi^2 \\ -5 + 3\xi^2 & -5 - 4\xi + 4\xi^2 + \xi^3 \end{bmatrix} \tag{1.104}$$

Consider the following unimodular matrix

$$U(\xi) = \begin{bmatrix} 1 + \xi - \xi^2 & \xi \\ 1 - \xi & 1 \end{bmatrix} \tag{1.105}$$

where $\det U(\xi) = 1$. Let

$$\tilde{R}(\xi) = U(\xi)R(\xi) = \begin{bmatrix} 3+\xi & 2+2\xi \\ -2 & -3-\xi \end{bmatrix} \tag{1.106}$$

$$= \begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \xi \tag{1.107}$$

$$= B + A\xi \tag{1.108}$$

Let $\tilde{R}(\frac{d}{dt})x = 0$, then we have

$$\tilde{R}(\frac{d}{dt})x = Bx + A\frac{dx}{dt} = 0 \tag{1.109}$$

and

$$\frac{dx}{dt} = -A^{-1}Bx = \begin{bmatrix} -1 & -4 \\ 2 & 3 \end{bmatrix} x \tag{1.110}$$

which is a state space representation with $x = [x_1, x_2]^T$ as the state. Since the behavior described by $R(\xi)$ and $\tilde{R}(\xi)$ are equivalent (due to the face that $\tilde{R}(\xi) = U(\xi)R(\xi)$), the system described by $R(\frac{d}{dt})x$ is a state space representation with x as the state.