DISC Course: Nonlinear Control Systems

Assignment 1

Hai Zhu
Delft University of Technology
h.zhu@tudelft.nl

Monday 5 February 2018

Exercise 1 (4.15 from [1]): Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \quad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions *(we need h_2 to be at least Lipschitz at 0)* that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.
(b) Show that $V(x) = \int_0^{x_1} h_1(y)dy + x_2^2/2 + \int_0^{x_3} h_2(y)dy$ is positive definite for all $x \in \mathbb{R}^3$.
(c) Show that the origin is asymptotically stable.
(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable.

Solution: (a) Assume $\bar{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]^T$ is an equilibrium point of the system. According to the definition of equilibrium points, we have

$$\bar{x}_2 = 0 \tag{1a}$$
$$-h_1(\bar{x}_1) - \bar{x}_2 - h_2(\bar{x}_3) = 0 \tag{1b}$$
$$\bar{x}_2 - \bar{x}_3 = 0 \tag{1c}$$

This results $\bar{x}_2 = 0$, $\bar{x}_3 = 0$ and $h_1(\bar{x}_1) = -h_2(0) = 0$. Since $yh_i(y) > 0$ for all $y \neq 0$, we have $h_i(y) \neq 0$ for all $y \neq 0$. Therefore, $h_1(\bar{x}_1) = 0 \Rightarrow \bar{x}_1 = 0$. Hence, the system has a unique equilibrium point at the origin, i.e., $[\bar{x}_1, \bar{x}_2, \bar{x}_3]^T = [0, 0, 0]^T$.

(b) To show that $V(x)$ is positive definite for all $x \in \mathbb{R}^3$, we need to show

$$V(0) = 0 \tag{2a}$$
$$V(x) > 0, \quad \forall x \in \mathbb{R}^3, x \neq 0 \tag{2b}$$

For the first condition, it is trivial that $V(0) = 0$. For the second condition, notice that $V(x)$ is a sum of three non-negative terms, thus we have $V(x) \geq 0$. Furthermore, since $yh_i(y) > 0$ for all $y \neq 0$, the two integral terms $\int_0^{x_1} h_1(y)dy$ and $\int_0^{x_3} h_2(y)dy$ equals to 0 only at $x_1 = 0$ and $x_3 = 0$, respectively. Besides, the term $x_2^2/2$ equals to 0 only at $x_2 = 0$ as well. Thus, we have $V(x) = 0 \Rightarrow x = 0$, which gives that $V(x) > 0, \forall x \in \mathbb{R}^3, x \neq 0$. Hence, $V(x)$ is positive definite for all $x \in \mathbb{R}^3$.

(c) Taking $V(x)$ the Lyapunov function candidate, from (b) we show that $V(x)$ is positive definite. This is equivalent to the existence of $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{3}$$

Furthermore, since h_2 is locally Lipschitz functions, there exists a constant $L_2 > 0$ and a domain $D_2 \subset \mathbb{R}^3$ such that

$$\|h_2(x) - h_2(y)\| \leq L_2 \|x - y\|, \forall x, y \in \mathcal{B}_r(x_0), \forall x_0 \in D_2 \tag{4}$$

Letting $x = 0$ in the above equations and $L = \max\{L_2, 1\}$, we have $\|h_2(y)\| \leq L \|y\|$. Thus we have

$$\begin{aligned}
\frac{\partial V}{\partial x} f(x) &= h_1(x_1)\dot{x}_2 + x_2\dot{x}_2 + h_3(x_3)\dot{x}_3 \\
&= h_1(x_1)x_2 + x_2(-h_1(x_1) - x_2 - h_2(x_3)) + h_3(x_3)(x_2 - x_3) \\
&= -x_2^2 - x_3 h_2(x_3) \\
&\leq -Lx^2 \\
&\leq -\alpha_3(\|x\|)
\end{aligned} \tag{5}$$

with $\alpha_3(r) = Lr^2 \in \mathcal{K}$. Hence, the origin is (locally) asymptotically stable.

(d) Note that we already have $\alpha_3 \in \mathcal{K}$. To show that the origin is globally asymptotically stable, we need $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. That is, $V(x)$ should be radially unbounded. Hence, the two integral terms $\int_0^{x_1} h_1(y)dy$ and $\int_0^{x_3} h_2(y)dy$ should tend to infinity $x_1 \to \infty$ and $x_3 \to \infty$ respectively. To conclude, the conditions on h_1 and h_2 are

$$\lim_{y \to \infty} \int_0^y h_i(y)dy = \infty, \quad i = 1, 2 \tag{6}$$

Exercise 2 (4.21 from [1]): A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Take $D = \mathbb{R}^n$. Suppose the set $\Omega_c = \{x \in \mathbb{R}^n | V(x) \leq c\}$ is compact for every $c \in \mathbb{R}$. Show that every solution of the system is defined for all $t \geq 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t \to \inf} x(t)$ exits and equals to one of the points $p_1, \ldots, p_r$.

Solution: (a) According to the definition, we have

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = -(\nabla V)^T(\nabla V) \leq 0 \tag{7}$$

Thus

$$\dot{V}(x) = 0 \iff \nabla V = 0 \iff \dot{x} = 0 \tag{8}$$

Hence, $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Denote $\dot{x} = f(x)$. Since $V(x)$ is twice continuously differentiable in $D = \mathbb{R}^n$, we have

$$\frac{\partial f(x)}{\partial x} = -\frac{\partial^2 V}{\partial x} \tag{9}$$

is continuous in $\mathbb{R}^n$. Thus, $f(x)$ is locally Lipschitz in $\mathbb{R}^n$. Further, for every solution $x_0 \in \Omega_c$, since $\dot{V}(x) \leq 0$ in $D = \mathbb{R}^n$, we have $V(x) \leq V(x_0) \leq c$. That is, $V(x)$ lies in the same compact Ω_c. Hence, according to Theorem 3.3 in [1], there is a unique solution that is defined for all $t \geq 0$.

(c) Combining the results in (b) with LaSalle's invariance principle, $x(t)$ approaches $M = \{p_1, \ldots, p_r\}$ as $t \to \infty$. That is, given $\varepsilon > 0$, there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and $x(t_k) \in N(p, 2\varepsilon)$, where $p \in M$. Since ε can be sufficiently small in the statement and $p_1, \ldots, p_r$ are isolated points, it implies that $x(t)$ approaches p as $t \to \infty$ for some $p \in M$.

Exercise 3 (4.25 from [1]): Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At}BB^Te^{-A^Tt}dt$ for some $\tau > 0$. Show that W is positive definite and let $K = B^TW^{-1}$. Use $V(x) = x^TW^{-1}x$ as a Lyapunov function candidate for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz.

Solution: (a) The controllability Gramian of the linear system is defined as

$$W_c = \int_0^\tau e^{At}BB^Te^{A^Tt}dt \tag{10}$$

for some $\tau > 0$. Since the pair (A, B) is controllable, according to Theorem 6.1 in [2], W_c is positive definite. Define $\xi = \tau - t$, then we have

$$\begin{aligned} W_c &= \int_{\xi=\tau}^0 e^{A(\tau-\xi)}BB^Te^{A^T(\tau-\xi)}(-d\xi) \\ &= \int_0^\tau e^{A\tau}e^{-A\xi}BB^Te^{-A^T\xi}e^{A^T\tau}d\xi \\ &= e^{A\tau}We^{A^T\tau} \end{aligned} \tag{11}$$

Hence, $W = e^{-A\tau}W_ce^{-A^T\tau}$ is positive definite.

(b) To show $(A - BK)$ is Hurwitz, we need to show that all eigenvalues of $(A - BK)$ satisfy Re$[\lambda] < 0$, or equivalently, to show that the origin of the system $\dot{x} = (A - BK)x$ is asymptotically stable.

Note that

$$\begin{aligned} AW + WA^T &= \int_0^\tau \{Ae^{-At}BB^Te^{-A^Tt} + e^{-At}BB^Te^{-A^Tt}A^T\}dt \\ &= \int_0^\tau \frac{d}{dt}\{-e^{-At}BB^Te^{-A^Tt}\}dt \\ &= -e^{-A\tau}BB^Te^{-A^T\tau} + BB^T \end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}(A - BK)W + W(A - BK)^T &= AW + WA^T - BKW - WK^TB^T \\ &= AW + WA^T - 2BB^T \\ &= -e^{-A\tau}BB^Te^{-A^T\tau} - BB^T\end{aligned} \tag{13}$$

Use $V(x) = x^TW^{-1}x$ as a Lyapunov function, we have

$$\begin{aligned}\dot{V}(x) &= x^T(A - BK)^TW^{-1}x + x^TW^{-1}(A - BK)x \\ &= x^T[(A - BK)^TW^{-1} + W^{-1}(A - BK)]x \\ &= x^TW^{-1}[(A - BK)W + W(A - BK)^T]W^{-1}x \\ &= -x^TW^{-1}(e^{-A\tau}BB^Te^{-A^T\tau} + BB^T)W^{-1}x\end{aligned} \tag{14}$$

Since W is positive definite, W^{-1} is also positive definite. Let $G = W^{-1}(e^{-A\tau}BB^Te^{-A^T\tau} + BB^T)W^{-1}$. Since the pair (A, B) is controllable, it is trivial that BB^T is definite positive. Thus, the matrix G is positive definite. Hence, we have

$$\dot{V}(x) = -x^TGx < 0, \quad \forall x \neq 0 \tag{15}$$

Therefore, the origin of the system $\dot{x} = (A - BK)x$ is asymptotically stable. Hence, the matrix $(A - BK)$ is Hurwitz.

References

[1] H.K. Khalil. *Nonlinear systems*. Prentice Hall, Upper Saddle River, USA, third edition, 2002.

[2] C.T. Chen. *Linear system theory and design*. Oxford University Press, Inc., USA, 1998.