

# DISC Course: Mathematical Models of Systems

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## 1 Homework 3

### 1.1 Chapter 7

#### 1.1.1 Exercise 7.2

**Solution:** Since  $R_2(\xi) = U(\xi)R_1(\xi)V(\xi)$  where  $U(\xi), V(\xi)$  are unimodular polynomial matrices, we have

$$\det(R_2(\xi)) = \det(U(\xi)R_1(\xi)V(\xi)) = c \det(R_1(\xi)) \quad (1.1)$$

$$\text{rank}(R_2(\xi)) = \text{rank}(U(\xi)R_1(\xi)V(\xi)) = \text{rank}(R_1(\xi)) \quad (1.2)$$

which implies that  $R_1(\xi)$  and  $R_2(\xi)$  have the same zeros with same multiplicities and the same rank deficiencies. Hence, for the given systems  $R_1(\frac{d}{dt})w = 0$  and  $R_2(\frac{d}{dt})w = 0$ , the first system is asymptotically stable, stable, or unstable if and only if the second one is.

#### 1.1.2 Exercise 7.4

**Solution:** Let

$$A = \begin{bmatrix} 0 & w_1 & \alpha & 0 \\ -w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_2 \\ 0 & 0 & -w_2 & 0 \end{bmatrix} \quad (1.3)$$

then the system is described by  $\frac{d}{dt}x = Ax$ .

The characteristic polynomial of  $A$  is

$$P(\xi) = \det(\xi I - A) = (\xi^2 + w_1^2)(\xi^2 + w_2^2) \quad (1.4)$$

The eigenvalues of  $A$  are

$$\lambda_1 = w_1 i, \lambda_2 = -w_1 i, \lambda_3 = w_2 i, \lambda_4 = -w_2 i \quad (1.5)$$

which implies that all the eigenvalues of  $A$  are on the imaginary axis.

According to Corollary 7.2.4, the system is

1. asymptotically stable if and only if the eigenvalues of  $A$  have negative real part;
2. stable if and only if for each  $\lambda \in \mathbb{C}$  that is an eigenvalue of  $A$ , either (i)  $\text{Re}\lambda < 0$ , or (ii)  $\text{Re}\lambda = 0$  and  $\lambda$  is a semisimple eigenvalue of  $A$ ;
3. unstable if and only if  $A$  has either an eigenvalue with positive real part or a non-semisimple one with zero real part.

Hence, we can determine if the system is (asymptotically) stable or unstable based on (1.5).

- Case 1: asymptotically stable. This requires that all  $\text{Re}\lambda < 0$ . However, according to (1.5), for  $A$  we have  $\text{Re}\lambda = 0$ .

Hence,  $\forall w_1, w_2, \alpha \in \mathbb{R}$ , the above system is not an asymptotically stable system.

- Case 2: stable. Since for  $\text{Re}\lambda = 0$ , in this case, it is required that all eigenvalues of  $A$  should be semisimple, that is

$$n - \text{rank}(\lambda I - A) = m \quad (1.6)$$

where  $n$  is the order of  $A$  and  $m$  is the multiplicity of  $\lambda$ . First we consider the condition that  $w_1^2 \neq w_2^2$  and both of them are not zero, i.e.  $w_1 w_2 \neq 0$ . In this condition, all eigenvalues of  $A$  have multiplicity of one. It is trivial to validate that (1.6) holds. Thus the system is stable. Second, we consider the condition that  $w_1^2 \neq w_2^2$  but one of them is zero, i.e.  $w_1 w_2 = 0$ . In this condition, we only need to validate the case when the eigenvalue equals zero whose multiplicity is 2. Note that

$$(0I - A) = \begin{bmatrix} 0 & -w_1 & -\alpha & 0 \\ w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w_2 \\ 0 & 0 & -w_2 & 0 \end{bmatrix} \quad (1.7)$$

Since one of  $w_1$  and  $w_2$  is zero, the rank of above matrix is 2. Thus, the system is stable. Third, we consider the condition that  $w_1^2 = w_2^2$ . In this case, note that

$$\begin{aligned} (\pm w_1 i I - A) &= \begin{bmatrix} \pm w_1 i & \mp w_1 & -\alpha & 0 \\ \pm w_1 & \pm w_1 i & 0 & 0 \\ 0 & 0 & \pm w_1 i & \mp w_1 \\ 0 & 0 & \pm w_1 & \pm w_1 i \end{bmatrix} \\ &\rightarrow \begin{bmatrix} \pm w_1 i & \mp w_1 & -\alpha & 0 \\ 0 & 0 & \pm w_1 & \pm w_1 i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha i & 0 \end{bmatrix} \end{aligned} \quad (1.8)$$

If  $w_1 \neq 0$ , we need the rank of the above matrix to be 2, which induces that  $\alpha = 0$ . If  $w_1 = 0$ , we need the rank of the above matrix to be 0, which also induces that  $\alpha = 0$ . That is, if  $w_1^2 = w_2^2$ , we need  $\alpha = 0$  to make the system stable. Until now, we have considered all conditions for  $w_1$  and  $w_2$  under which the system is stable. For any other conditions, the system is unstable.

To sum up in this case, the system is stable if any one of the following conditions is satisfied:

- $w_1^2 \neq w_2^2, \forall \alpha \in \mathbb{R};$
- $w_1^2 = w_2^2, \alpha = 0.$
- Case 3: unstable. Based on previous results, the system is unstable if
  - $w_1^2 = w_2^2, \alpha \neq 0.$

### 1.1.3 Exercise 7.6

**Solution:**

*Proof.* (a) Let

$$P_1(\xi) = p_0 + p_1\xi + \cdots + p_n\xi^n = \sum_{i=0}^n p_i\xi^i = p_n \prod_{i=1}^n (\xi - \lambda_i) \quad (1.9)$$

where  $\lambda \in \mathbb{C}$  is the root of the polynomial  $P_1(\xi)$ . Then we have

$$\begin{aligned} P_2(\xi) &= p_n + p_{n-1}\xi + \cdots + p_0\xi^n \\ &= \sum_{i=0}^n p_i\xi^{n-i} \\ &= \xi^n \sum_{i=0}^n p_i\xi^{-i} \\ &= \xi^n P_1\left(\frac{1}{\xi}\right) \\ &= \xi^n p_n \prod_{i=1}^n \left(\frac{1}{\xi} - \lambda_i\right) \\ &= p_n \prod_{i=0}^n (1 - \lambda_i\xi) \end{aligned} \quad (1.10)$$

Thus, denote the roots of the polynomial  $P_2(\xi)$  in  $\tilde{\lambda}_i$ , then we have

$$\tilde{\lambda}_i \lambda_i = 1 \quad (1.11)$$

which implies that  $\tilde{\lambda}_i$  and  $\lambda_i$  have same signs. Hence,  $P_1(\xi)$  is Hurwitz if and only if  $P_2(\xi)$  is.

(b) For an invertible matrix  $A \in \mathbb{R}^{n \times n}$ , denoting the eigenvalues of it and its inverted matrix as  $\Lambda(A) = \lambda_i$  and  $\Lambda(A^{-1}) = \tilde{\lambda}_i$ , then we have

$$\tilde{\lambda}_i \lambda_i = 1 \quad (1.12)$$

Furthermore, the eigenvalues of  $A$  and its transpose are the same, i.e.  $\Lambda(A) = \Lambda(A^T)$ . Combined with results of (a), we conclude that  $A$  is Hurwitz if and only if  $A^{-1}$  is, and if only if  $A^T$  is.

This completes the proof.

□

#### 1.1.4 Exercise 7.23

**Solution:**

*Proof.* (a) The objective is to prove that if  $A, P = P^T$ , and  $Q = Q^T \leq 0$  satisfy the Lyapunov equation  $A^T P + P A = Q$ , then  $(A, Q)$  cannot be observable whenever  $A$  has eigenvalues with zero real part.

Let  $\lambda$  be an eigenvalue of  $A$  with zero real part and  $v$  be its corresponding eigenvector. That is

$$Av = \lambda v, \quad \text{Re}(\lambda) = 0 \quad (1.13)$$

and

$$\bar{v} A^T = \bar{\lambda} \bar{v}, \quad \text{Re}(\bar{\lambda}) = 0 \quad (1.14)$$

where  $\bar{\cdot}$  denotes the complex conjugate. Since  $A^T P + P A = Q$  and  $A, P = P^T$ , we have

$$\bar{v} Q v = \bar{v} (A^T P + P A) v = \bar{\lambda} \bar{v} P v + \lambda \bar{v} P v = (\lambda + \bar{\lambda}) \bar{v} P v = 0 \quad (1.15)$$

where  $Q \leq 0$ . Thus,  $Q v = 0$  and we have

$$[\lambda I - A \quad Q] v = 0 \quad (1.16)$$

Therefore,  $\text{rank}([\lambda I - A \quad Q]) < n$ , which implies that  $(A, Q)$  is not observable.

(b) The objective is to prove that if  $A$  has no eigenvalues with zero real part, then the system  $\frac{d}{dt}x = Ax$  is unstable if and only if ( $\iff$ ) there exist  $P$  and  $Q$  such that  $P \text{ not } \geq 0, Q \leq 0$  and  $(A, Q)$  is observable satisfying the Lyapunov equation.

(1) " $\Leftarrow$ ": This is proved by Theorem 7.4.4 in the book.

(2) " $\Rightarrow$ ": Since the system  $\frac{d}{dt}x = Ax$  is unstable and  $A$  has no eigenvalues with zero real part,  $A$  has at least one eigenvalue with positive real part. Let  $\lambda_1, \dots, \lambda_m$  ( $0 \leq m < n$ ) be the eigenvalues of  $A$  with negative real part and  $\lambda_{m+1}, \dots, \lambda_n$  the eigenvalues with positive real part. The corresponding eigenvectors are denoted by  $v_1, \dots, v_m$  and  $v_{m+1}, \dots, v_n$ .

Let

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad (1.17)$$

where  $A_1 \in \mathbb{R}^{m \times m}$  and  $-A_2 \in \mathbb{R}^{(n-m) \times (n-m)}$  are Hurwitz. Then we can choose such  $A_1$  and  $A_2$  such that  $A$  is similar to  $\tilde{A}$ . Furthermore, since  $A_1$  and  $-A_2$  are Hurwitz, there exist unique  $P_1 > 0 \in \mathbb{R}^{m \times m}$  and  $P_2 > 0 \in \mathbb{R}^{(n-m) \times (n-m)}$  such that

$$A_1^T P_1 + P_1 A_1 = -I_m \quad (1.18)$$

$$(-A_2)^T P_2 + P_2 (-A_2) = -I_{n-m} \quad (1.19)$$

Let

$$\tilde{P} = \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} \quad (1.20)$$

then

$$\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} = -I_n \quad (1.21)$$

Let  $\tilde{Q} = -I_n \leq 0$ , then it is obvious that  $\text{rank}([\lambda I - \tilde{A} \quad \tilde{Q}]) = n, \forall \lambda \in \mathbb{C}$ . Therefore,  $(\tilde{A}, \tilde{Q})$  is observable. Moreover, since  $A$  and  $\tilde{A}$  are similar, there exists a nonsingular matrix  $U$  such that  $\tilde{A} = U^{-1}AU$ . Then the above equation can be written as follows

$$\begin{aligned} (U^{-1}AU)^T \tilde{P} + \tilde{P}(U^{-1}AU) &= \tilde{Q} \\ \implies A^T (S^{-1})^T \tilde{P} S^{-1} + (S^{-1})^T \tilde{P} S^{-1} A &= (S^{-1})^T \tilde{Q} S^{-1} \\ \implies A^T P + PA &= Q \end{aligned} \quad (1.22)$$

where  $P = (S^{-1})^T \tilde{P} S^{-1}$ ,  $Q = (S^{-1})^T \tilde{Q} S^{-1}$ . Hence, it can be seen that  $\text{rank}([\lambda I - A \quad Q]) = n, \forall \lambda \in \mathbb{C}$ . Therefore,  $(A, Q)$  is observable. Furthermore, since  $\tilde{P}$  has diagonal negative definite submatrix,  $\tilde{Q} = -I_n$ ,  $S$  is nonsingular, it is easy to see that  $P \not\geq 0$  and  $Q \leq 0$ .

This completes the proof.

□

## 1.2 Chapter 9

### 1.2.1 Exercise 9.6

**Solution:** (a) For

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (1.23)$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (1.24)$$

the controllability matrix can be calculated as follows

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \quad (1.25)$$

Since  $\text{rank}(R) = 4$ ,  $(A, B)$  is controllable. Thus, according to Lemma 9.4.4, there exist  $K \in \mathbb{R}^{2 \times 1}$ ,  $BK \neq 0$  and  $N' \in \mathbb{R}^{2 \times 4}$  such that  $(A + BN', BK)$  is controllable.

According to Algorithm 9.5.1, if the matrices  $K$  and  $N'$  are chosen using a random number generator, then we can be almost sure that the resulting matrices  $(A + BN', BK)$  form a controllable pair. To this end, we choose

$$K = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (1.26)$$

$$N' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (1.27)$$

Then the controllability matrix can be calculated as follows

$$R' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \quad (1.28)$$

Since  $\text{rank}(R') = 4$ ,  $(A + BN', BK)$  is controllable. Hence, the above matrix  $K$  and  $N'$  satisfy that  $(A + BN', BK)$  is controllable.

(b) For

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (1.29)$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad (1.30)$$

the controllability matrix can be calculated as follows

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} \quad (1.31)$$

Since  $\text{rank}(R) = 3$ ,  $(A, B)$  is controllable. Thus, according to Lemma 9.4.4, there exist  $K \in \mathbb{R}^{2 \times 1}$ ,  $BK \neq 0$  and  $N' \in \mathbb{R}^{2 \times 3}$  such that  $(A + BN', BK)$  is controllable.

Similar to the question (a), we firstly choose matrix  $K$  and  $N'$ , then validate if they

satisfy the condition. To this end, we choose

$$K = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (1.32)$$

$$N = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (1.33)$$

Then the controllability matrix can be calculated as follows

$$R' = \begin{bmatrix} 1 & 5 & 13 \\ 2 & 4 & 8 \\ 3 & 3 & 9 \end{bmatrix} \quad (1.34)$$

Since  $\text{rank}(R') = 3$ ,  $(A + BN', BK)$  is controllable. Hence, the above matrix  $K$  and  $N'$  satisfy that  $(A + BN', BK)$  is controllable.

### 1.2.2 Exercise 9.15

**Solution:**

*Proof.* (1) "necessity": If the system

$$\frac{d}{dt}x = Ax + Bu(t), x(0) = x_0 \quad (1.35)$$

is stabilizable, then the uncontrollable polynomial  $\mathcal{X}_u(\xi)$  of  $A$  is Hurwitz. According to Corollary 9.6.2, if the uncontrollable polynomial is Hurwitz, then there exists a feedback control law  $u = Nx$  such that the closed loop system

$$\frac{d}{dt}x = (A + BN)x \quad (1.36)$$

is asymptotically stable. Hence,  $\forall x_0 \in \mathbb{R}, x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

(2) "sufficiency": For a pair  $(A, B)$ , there exists a nonsingular matrix  $S$  such that (Corollary 5.2.25)

$$A = S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (1.37)$$

$$B = S^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad (1.38)$$

where  $(A_{11}, B_1)$  is controllable. Thus

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \end{aligned} \quad (1.39)$$

Rewrite the equation as follows

$$\begin{cases} \dot{x}_1(t) &= A_{11}x_1(t) + A_{22}x(t) + B_1u(t) \\ \dot{x}_2(t) &= A_{22}x_2(t) \end{cases} \quad (1.40)$$

Given the condition that  $\forall x_0 \in \mathbb{R}, x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , if the system is not stabilizable. Then according to (1.40), the uncontrollable polynomial  $\mathcal{X}_u(\xi) = \mathcal{X}_{A_{22}}(\xi)$  is not Hurwitz which results in that  $A_{22}$  is not Hurwitz. As a consequence,  $x(2) \rightarrow \infty$  as  $t \rightarrow \infty$ , which contradicts the condition that  $\forall x_0 \in \mathbb{R}, x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Hence, the system is stabilizable.

This completes the proof.

□

## 1.3 Chapter 10

### 1.3.1 Exercise 10.20

**Solution:** Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 0] \quad (1.41)$$

the plant is

$$\frac{d}{dt}x = Ax + Bu \quad (1.42)$$

$$y = Cx \quad (1.43)$$

The controllability matrix and observability matrix are

$$\mathcal{C} = [A \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1.44)$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1.45)$$

Since  $\text{rank}(\mathcal{C}) = 2$ ,  $\text{rank}(\mathcal{O}) = 2$ , the system is both controllable and observable. Therefore, there exists a compensator such that the closed loop system has characteristic polynomial  $p(\xi) = 1 + 2\xi + 2\xi^2 + \xi^3 = (1 + \xi)(1 + \xi + \xi^2)$ .

We first design a state feedback control  $u = Nx$ ,  $N \in \mathbb{R}^{1 \times 2}$  such that the characteristic polynomial is  $1 + \xi + \xi^2$ . The matrix  $N$  can be calculated using the Algorithm 9.5.1 in the book, which results in  $N = [0 \quad -1]$ . Therefore, the state feedback control law is

$$u = Nx = -x_2 \quad (1.46)$$



However, this feedback control law uses the state  $x_2$  which is not measured. So we need to design a reduced-order state observer such that the characteristic polynomial is  $1 + \xi$ .

The system equation can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{cases} \quad (1.47)$$

Consider the following observer to estimate  $x_2$

$$\frac{d}{dt}\hat{x}_2 = -y + u + L(x_2 - \hat{x}_2) \quad (1.48)$$

For characteristic polynomial  $1 + \xi$ , we have  $L = 1$ . Then the above equation can be written as

$$\frac{d}{dt}\hat{x}_2 = -y + u + (x_2 - \hat{x}_2) \quad (1.49)$$

$$= -y + u + \dot{y} - \dot{\hat{x}}_2 \quad (1.50)$$

Let

$$v = \hat{x}_2 - y \quad (1.51)$$

then

$$\dot{v} = \dot{\hat{x}}_2 - \dot{y} = u - v - 2y \quad (1.52)$$

Thus, the reduced-order observer is

$$\begin{cases} \dot{v} = u - v - 2y \\ \hat{x}_2 = v + y \\ \hat{x}_1 = y \end{cases} \quad (1.53)$$

Hence, the first-order compensator is

$$\begin{cases} \dot{v} = u - v - 2y \\ \hat{x}_2 = v + y \\ \hat{x}_1 = y \\ u = -\hat{x}_2 \end{cases} \quad (1.54)$$

### 1.3.2 Exercise 10.21

**Solution:** For the nonlinear system

$$y^3 + \frac{d^2}{dt^2}y + \sin y = u(u - 1) \quad (1.55)$$

let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad (1.56)$$

then the system equation can be written as follows

$$\frac{d}{dt}x = \begin{bmatrix} x_1 \\ -x_1^3 - \sin x_1 + u(u-1) \end{bmatrix} = f(x, u) \quad (1.57)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \quad (1.58)$$

Let

$$A = \left. \frac{\partial f}{\partial x} \right|_{x^*, y^*} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (1.59)$$

$$B = \left. \frac{\partial f}{\partial u} \right|_{x^*, y^*} = \begin{bmatrix} 0 \\ 2u-1 \end{bmatrix}_{u=0} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (1.60)$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (1.61)$$

then we can obtain the linearized form of the system

$$\frac{d}{dt}\Delta_x = A\Delta_x + B\Delta_u \quad (1.62)$$

$$\Delta_y = C\Delta_x \quad (1.63)$$

Note that the above linearized system is similar with the system in the previous Exercise 10.20. We can use a similar way to design a first-order compensator to make the closed loop system with characteristic polynomial  $p(\xi) = 1 + 2\xi + 2\xi^2 + \xi^3$ . The resulted compensator is

$$\begin{cases} \dot{v} = \Delta_u - \Delta_v - 2\Delta_y \\ \Delta_{\hat{x}_2} = \Delta_v + \Delta_y \\ \Delta_{\hat{x}_1} = \Delta_y \\ \Delta_u = \Delta_{\hat{x}_2} \end{cases} \quad (1.64)$$