Multi-agent Network Dynamics and Games – Assignments on Game theory and Network Systems

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I. Assignments

Solve 2 exercises on game theory (§I-A) and 2 exercises on network systems (§I-B). Submit the solutions in PDF via the DISC course platform, `http://disc-courseplatform.nl`, by March 11th, 2018.

A. *Exercises on game theory*

E1.01 Determine a game triplet $\mathcal{G}$ that defines the rock-paper-scissors game.

E1.02 Prove the Banach–Picard theorem.

E1.03 Show that uniqueness of Nash equilibrium does not necessarily hold in monotone games.

E1.04 Consider a (unconstrained, strongly-convex-quadratic) game with $\mathcal{I} = \{1, \ldots, N\}$, $J_i(x_i, \boldsymbol{x}_{-i}) = x_i^\top Q_i x_i + (C_i\, \boldsymbol{x}_{-i})^\top x_i$, $Q_i \succ 0$, and $\mathcal{X}_i(x_{-i}) = \mathbb{R}^{n_i}$, for all $i \in \mathcal{I}$.

  i) Provide (matrix inequality) conditions on the data, $\{Q_i, C_i\}_{i\in\mathcal{I}}$, such that the game is monotone.

  ii) Derive the parallel best-response dynamics in closed form.

E1.05 Derive the parallel best-response dynamics in closed form for (unconstrained, strongly-convex-quadratic, aggregative) games with $\mathcal{I} = \{1, \ldots, N\}$, $J_i(x_i, \boldsymbol{x}_{-i}) = x_i^\top Q_i x_i + \left(C_i\, \frac{1}{N}\sum_{j=1}^N x_j\right)^\top x_i$, $Q_i \succ 0$, and $\mathcal{X}_i(\boldsymbol{x}_{-i}) = \mathbb{R}^{n_i}$, for all $i \in \mathcal{I}$.

E1.06 Let $A = [a_{i,j}] \in \mathbb{R}^{N\times N}$ be a row-stochastic matrix. Derive the parallel best-response dynamics in closed form for (unconstrained, strongly-convex-quadratic, network) games with $\mathcal{I} = \{1, \ldots, N\}$, $J_i(x_i, \boldsymbol{x}_{-i}) = x_i^\top Q_i x_i + \left(C_i \sum_{j=1}^N a_{i,j}\, x_j\right)^\top x_i$, $Q_i \succ 0$, and $\mathcal{X}_i(x_{-i}) = \mathbb{R}^{n_i}$, for all $i \in \mathcal{I}$.

E1.07 Show via an example that the parallel best-response dynamics for jointly-convex games do not necessarily converge.

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B. Exercises on network systems

E1.08 Let $A \in \mathbb{R}^{n\times n}$ be such that $\rho(A) = 1$. Show that the following statements are equivalent:

i) A is semi-convergent;

ii) there exist $T \in \mathbb{R}^{n\times n}$ non-singular and $k \in \{1, \dots, n\}$ such that

$$A = T^{-1} \begin{bmatrix} I_k & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & B \end{bmatrix} T,$$

for some convergent matrix $B \in \mathbb{R}^{(n-k)\times(n-k)}$, i.e., $\rho(B) < 1$.

E1.08 Let $A \in \mathbb{R}^{n\times n}$ be doubly-stochastic. Show that:

i) $I_n - A^\top A \succcurlyeq 0$;

ii) $0 \in \Lambda\left(I_n - A^\top A\right)$.

E1.09 Recall that, $\forall x \in [0, 1)$, $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Let $A \in \mathbb{R}^{n\times n}$ be convergent, i.e., $\rho(A) < 1$. Show that

$$(I_n - A)^{-1} = \textstyle\sum_{k=0}^{\infty} A^k.$$

Let $B \in \mathbb{R}^{n\times n}_{\geq 0}$ be primitive and let $\lambda \in \mathbb{R}$ be a constant. Show that the following statements are equivalent:

i) $\lambda I_n - B$ is invertible and $(\lambda I_n - B)^{-1} \in \mathbb{R}^{n\times n}_{\geq 0}$;

ii) $\lambda > \rho(B)$.

E1.10 Let $A \in \mathbb{R}^{n\times n}$ be row-stochastic. Show that the following statements are equivalent:

i) $1 \in \Lambda(A)$ is simple and $\max_{\lambda\in\Lambda(A)\setminus\{1\}} |\lambda| < 1$;

ii) $\lim_{k\to\infty} A^k = \mathbf{1}_n w^\top$, for some $w \in \mathbb{R}^n_{\geq 0}$ such that $\mathbf{1}_n^\top w = 1$.

E1.11 Let $\mathcal{C}(n)$ be the complete, undirected, non-weighted graph with n nodes. Determine:

i) the adjacency matrix of $\mathcal{C}(n)$ and its eigenvalues;

ii) the Laplacian matrix of $\mathcal{C}(n)$ and its eigenvalues.

E1.12 Show that if a digraph G is weight-balanced, then its Laplacian L is such that $L + L^\top \succcurlyeq 0$.

E1.13 Let $A \in \mathbb{R}^{n\times n}$ be a row-stochastic adjacency matrix and let L be the Laplacian matrix of the digraph associated with A. Characterize $\Lambda(L)$ as a function of $\Lambda(A)$.