Mathematical models of systems (2017– 2018)

Homework 4

Hand out: March 20, 2018

Hand in: April 15, 2018

- Cooperation in groups of two people of approximately the same level is allowed.
- Everybody hands in his or her own version.
- It is not allowed to copy from each other.
- The exercise numbers refer to the Springer edition of the book which can be downloaded from the website.
- Please typeset your reports by using LaTeX. Sharing LaTeX files is strictly forbidden. Please send your reports by email to `h.l.trentelman@rug.nl`

1

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a full plant behavior, with to be controlled variable w and control variable c. Let $\mathtt{v}$ be a given non-negative integer. A controller is a system behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}+\mathtt{v}}$ with system variable (c, v). The behaviors $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ have the variable c in common, through which they can be interconnected, leading to the interconnected system $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$, with full controlled behavior

$$\mathcal{K}_{\text{full}} := \{ \begin{pmatrix} w \\ c \\ v \end{pmatrix} \mid \begin{pmatrix} w \\ c \end{pmatrix} \in \mathcal{P}_{\text{full}}, \begin{pmatrix} c \\ v \end{pmatrix} \in \mathcal{C} \}.$$

Now, let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ be an arbitrary behavior with system variable (w, v). We will say that the controller $\mathcal{C}$ *implements* $\mathcal{K}$ if

$$\mathcal{K} = \{ \begin{pmatrix} w \\ v \end{pmatrix} \mid \text{ there exists } c \text{ such that } \begin{pmatrix} w \\ c \\ v \end{pmatrix} \in \mathcal{K}_{\text{full}} \}.$$

If for a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ there exists a controller $\mathcal{C}$ that implements it, then we call $\mathcal{K}$ *implementable with respect to* $\mathcal{P}_{\text{full}}$.

In this problem we will derive conditions under which a given $\mathcal{K}$ is implementable with respect to $\mathcal{P}_{\text{full}}$. For this, let $\mathcal{N}$ and $\mathcal{P}$ be the hidden behavior and manifest plant behavior, respectively, associated with $\mathcal{P}_{\text{full}}$. In addition, define similar behaviors associated with $\mathcal{K}$:

$$\mathcal{N}(\mathcal{K}) := \{ w \mid \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathcal{K} \},$$

and

$$\mathcal{P}(\mathcal{K}) := \{ w \mid \text{ there exists } v \text{ such that } \begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{K} \}.$$

a. Prove that if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ is implementable then $\mathcal{N} \subseteq \mathcal{N}(\mathcal{K})$ and $\mathcal{P}(\mathcal{K}) \subseteq \mathcal{P}$.

b. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$. Define a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}+\mathtt{v}}$ by $\mathcal{C} := (\mathcal{P}_{\text{full}} \wedge_w \mathcal{K})_{(c,v)}$. Prove that if $\mathcal{N} \subseteq \mathcal{N}(\mathcal{K})$ and $\mathcal{P}(\mathcal{K}) \subseteq \mathcal{P}$, then $\mathcal{C}$ implements $\mathcal{K}$. (This controller is called the *canonical controller*).

c. What do the results under a. and b. say about the case that $\mathtt{v} = 0$?

2

Let $\mathcal{P} \in \mathfrak{L}^q$ be a linear differential system. We consider the control problem of *stabilization by regular full interconnection*, i.e. the problem to find conditions on $\mathcal{P}$ under which there exists a system $\mathcal{C} \in \mathfrak{L}^q$ (a controller) such that the full interconnection $\mathcal{P} \cap \mathcal{C}$ is regular, and the behavior $\mathcal{P} \cap \mathcal{C}$ is stable.

a. Let $R \in \mathbb{R}^{g \times q}[\xi]$ have full row rank. Let $(D \quad 0)$ be its Smith form, with $D(\xi) = \operatorname{diag}(d_1(\xi), d_2(\xi), \ldots, d_g(\xi))$. Show that $\operatorname{rank}(R(\lambda)) = \operatorname{rank}(R)$ for all λ satisfying $\operatorname{Re}(\lambda) \geq 0$ if and only if all roots λ of the polynomials d_i satisfy $\operatorname{Re}(\lambda) < 0$.

b. Use this to show that if $R \in \mathbb{R}^{g \times q}[\xi]$ is such that $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal kernel representation of $\mathcal{P}$, then $\mathcal{P}$ is stabilizable (in the sense of behaviors) if and only if there exists $R' \in \mathbb{R}^{(q-g) \times q}[\xi]$ such that

$$\begin{bmatrix} R \\ R' \end{bmatrix}$$

is Hurwitz.

c. Use the above to prove the following theorem: Given $\mathcal{P} \in \mathfrak{L}^q$, there exists $\mathcal{C} \in \mathfrak{L}^q$ such that the full interconnection $\mathcal{P} \cap \mathcal{C}$ is stable and regular if and only if $\mathcal{P}$ is stabilizable.

3

Consider the plant behavior $\mathcal{P}_{\text{full}}$ with manifest variable $w = (w_1, w_2)$ and control variable $c = (c_1, c_2)$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

a. Compute the manifest plant behavior $(\mathcal{P}_{\text{full}})_w$ and the hidden behavior $\mathcal{N}$.

b. Show that the desired behavior $\mathcal{K}$ given by $\mathcal{K} = \{(w_1, w_2) \mid w_1 + \dot{w}_2 = 0\}$ is regularly implementable.

c. Show that the controller $\mathcal{C} = \{(c_1, c_2) \mid \dot{c}_1 + c_2 = 0\}$ regularly implements $\mathcal{K}$.

4

Consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0 \\ w_2 + c_1 + c_2 &= 0 \\ \dot{c}_1 + c_1 + \dot{c}_2 + c_2 &= 0 \end{aligned}$$

a. Compute the manifest plant behavior $(\mathcal{P}_{\text{full}})_w$ and the hidden behavior $\mathcal{N}$.

b. Show that there exists a stabilizing controller for $\mathcal{P}_{\text{full}}$.

c. Prove that $\mathcal{C} = \{(c_1, c_2) \mid \dot{c}_2 + 2c_1 + c_2 = 0\}$ is a stabilizing controller.

5

In this problem we consider the issue of regular implementability by full interconnection. Let $\mathcal{P}, \mathcal{K} \in \mathfrak{L}^q$. Let $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{K}$, respectively.

a. Let F be a polynomial matrix. Show that there exists a polynomial matrix W such that

$$\begin{pmatrix} F \\ W \end{pmatrix}$$

is unimodular if and only if $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

b. Use the result under a. to prove that $\mathcal{K}$ is regularly implementable by full unterconnection with $\mathcal{P}$ if and only if there exists a polynomial matrix F such that $R = FK$ and $F(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

6

In this problem we study systems represented in what is called *descriptor representation*. In classical systems and control theory, systems in descriptor representation are also often called *singular systems*. A descriptor representation is a representation of the form

$$E\frac{dx}{dt} = Ax + Bu, \tag{1}$$

$$y = Cx + Du, \tag{2}$$

where $E \in \mathbb{R}^{k\times n}$, $A \in \mathbb{R}^{k\times n}$, $B \in \mathbb{R}^{k\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$.

Consider the linear differential system with behavior

$$\mathcal{P}_{\text{full}} = \{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{n+m+p}) \mid \text{ the equations (1), (2) hold} \}.$$

Suppose now we want to control the system $\mathcal{P}_{\text{full}}$. As interconnection variable we take $c = (u, y)$. The variable to be controlled will be $w = x$. Our aim is to find controllers that in the controlled system achieve stability or pole placement. To be more precise, a controller $\mathcal{C}$ represented by $C_1(\frac{\mathrm{d}}{\mathrm{d}t})u = C_2(\frac{\mathrm{d}}{\mathrm{d}t})y$ (with C_1 and C_2 polynomial matrices) is called a *stabilizing controller* if the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular and if for all x in the controlled system we have $x(t) \to 0$ as $t \to \infty$.

a. Determine the hidden behavior $\mathcal{N}$ and the manifest plant behavior $\mathcal{P}$.

b. Find necessary and sufficient conditions on E, A, B, C and D for the existence of a stabilizing controller for $\mathcal{P}_{\text{full}}$.

c. Carefully formulate and prove a similar result for pole placement.

7

Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{q}}$ be a controllable plant. Let $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a minimal kernel representation of $\mathcal{P}$, R is a polynomial matrix with g rows and q columns. In this problem we want to find a *parametrization* of all polynomial matrices C such that the controller $\mathcal{C}$: $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ stabilizes $\mathcal{P}$ by regular full interconnection. Let C_0 be a polynomial matrix such that

$$\begin{pmatrix} R \\ C_0 \end{pmatrix}$$

is unimodular.

a. Let F be an arbitrary $(q-g) \times g$ polynomial matrix, and let D be a $(q-g) \times (q-g)$ Hurwitz polynomial matrix. Define $C = FR + DC_0$. Show that the controller $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ stabilizes $\mathcal{P}$ by regular full interconnection.

b. Assume now that C is a polynomial matrix. Prove that there exist polynomial matrices G_{11}, G_{12}, G_{21} and G_{22} such that

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} R \\ C_0 \end{pmatrix}.$$

c. Prove that, in fact, $G_{11} = I$ and $G_{12} = 0$.

d. Now assume that the controller $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ stabilizes $\mathcal{P}$ by regular full interconnection. Prove that G_{22} is Hurwitz.

e. Conclude that if $C(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ stabilizes $\mathcal{P}$ by regular full interconnection, then there exists a polynomial matrix F and a Hurwitz polynomial matrix D such that $C = FR + DC_0$.