DISC Course: Multi-agent Network Dynamics and Games

Assignment 1

Hai Zhu

Delft University of Technology

h.zhu@tudelft.nl

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E1.01 Determine a game triplet $\mathcal{G}$ that defines the rock-paper-scissors game.

Solution: The rock-paper-scissors game is a hand game usually played between two people, in which each player simultaneously forms one of three shapes with an outstretched hand. These shapes are "rock", "paper" and "scissors". Let $x_i \in \Omega_i = \{1, 2, 3\}$ $(i = 1, 2)$. We denote $x_i = 1 =$ "rock", $x_i = 2 =$ "scissors" and $x_i = 3 =$ "paper". In the game, we assume one score is obtained by the winner and minus one for the loser. If there is a tie, then both of the two players get zero scores. Then according to the rules of the game, we have

P1 \ P2	1	2	3
1	(0, 0)	(1, -1)	(-1,1)
2	(-1, 1)	(0, 0)	(1, -1)
3	(1, -1)	(-1, 1)	(0, 0)

Now we can define the game by a triplet $\mathcal{G} = (\mathcal{I}, \{J_i\}_{i\in\mathcal{I}}, \{\mathcal{X}_i\}_{i\in\mathcal{I}})$

$$\mathcal{I} = \{1, 2\}; \tag{1}$$
$$J_1(1,1) = 0,\ J_1(1,2) = 1,\ J_1(1,3) = -1, \tag{2}$$
$$J_1(2,1) = -1,\ J_1(2,2) = 0,\ J_1(2,3) = 1, \tag{3}$$
$$J_1(3,1) = 1,\ J_1(3,2) = -1,\ J_1(3,3) = 0, \tag{4}$$
$$J_2(1,1) = 0,\ J_2(1,2) = -1,\ J_2(1,3) = 1, \tag{5}$$
$$J_2(2,1) = 1,\ J_2(2,2) = 0,\ J_2(2,3) = -1, \tag{6}$$
$$J_2(3,1) = -1,\ J_2(3,2) = 1,\ J_2(3,3) = 0; \tag{7}$$
$$\mathcal{X}_1 = \{1, 2, 3\}, \tag{8}$$
$$\mathcal{X}_2 = \{1, 2, 3\}. \tag{9}$$

E1.02 Prove the Banach–Picard theorem.

Proof. Recall the Banach–Picard theorem: Let the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ be ℓ-Lipschitz continuous, with $\ell \in [0, 1)$. Set $x(k+1) = T(x(k))$, for some $x(0) \in \mathbb{R}^n$. The following hold:

i) $\exists! \bar{x} \in \mathrm{fix}(T)$;

ii) $\|x(k) - \bar{x}\| \leq \ell^k \|x(0) - \bar{x}\|$;

iii) $\lim_{k\to\infty} x(k) = \bar{x}$.

We now prove the theorem.

Let (M, d) be a metric space where M is a set and d is a metric on M. In this case, $M = \mathbb{R}^n$, so we have

$$d(x, y) = \|y - x\| \tag{10}$$

where $x, y \in \mathbb{R}^n$ and $d(x, y)$ indicates the Euclidean distance.

Denote $x_k = x(k)$. First we prove that $\forall k \in \mathbb{N}, d(x_{k+1}, x_k) \leq \ell^k d(x_1, x_0)$ using mathematical induction.

When $k = 1$, since $T : \mathbb{R}^n \to \mathbb{R}^n$ is ℓ-Lipschitz continuous with $\ell \in [0, 1)$, we have the following inequality holds

$$d(x_2, x_1) = d(T(x_1), T(x_0)) \leq \ell d(x_1, x_0) \tag{11}$$

Suppose the inequality holds for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_{k+2}, x_{k+1}) &= d(T(x_{k+1}), T(x_k)) \\ &\leq \ell d(x_{k+1}, x_k) \\ &\leq \ell\ell^k d(x_1, x_0) \\ &= \ell^{k+1} d(x_1, x_0) \end{aligned} \tag{12}$$

Hence, the inequality holds for $\forall k \in \mathbb{N}$.

Next, we prove that $\{x_k\}$ is a Cauchy sequence in $(\mathbb{R}^n, d)$. Let $m, n \in \mathbb{N}$ and $m > n$, then we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \ell^{m-1} d(x_1, x_0) + \ell^{m-2} d(x_1, x_0) + \cdots + \ell^n d(x_1, x_0) \\ &= \ell^n \sum_{i=0}^{m-n-1} \ell^i d(x_1, x_0) \\ &\leq \ell^n \sum_{i=0}^{\infty} \ell^i d(x_1, x_0) \\ &= \frac{\ell^n}{1-\ell} d(x_1, x_0) \end{aligned} \tag{13}$$

Since $\ell \in [0, 1)$, the above expression for $d(x_m, x_n)$ can be arbitrary small by choosing a large n. Hence, $\{x_k\}$ is a Cauchy sequence and therefore it converges to some point $\bar{x} \in \mathbb{R}^n$.

Next, we prove that $\bar{x}$ is a fixed point, that is, $\bar{x} \in \text{fix}(T)$. Since $x_{k+1} = T(x_k)$, we have

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} (x_{k-1}) = T(\lim_{k\to\infty} x_{k-1}) \tag{14}$$

$$\implies \bar{x} = T(\bar{x}) \tag{15}$$

Hence, $\bar{x}$ is a fixed point of T.

Then, we prove that $\bar{x}$ is unique. Suppose there exists $y \in \mathbb{R}^n$ which is also a fixed point of T, then we have

$$d(\bar{x}, y) = d(T(\bar{x}), T(y)) \leq \ell d(\bar{x}, y) \tag{16}$$

Since $d(\bar{x}, y) \geq 0$ and $\ell \in [0, 1)$, the above equation indicates that $0 \leq (1-\ell)d(\bar{x}, y) \leq 0$, which results in $d(\bar{x}, y) = 0 \implies \bar{x} = y$. Hence, $\bar{x}$ is the unique fixed point of T. This completes the proof for i).

Note that for $\forall k \in \mathbb{N}$

$$d(x_{k+1}, \bar{x}) = d(T(x_k), T(\bar{x})) \leq \ell d(x_k, \bar{x}) \tag{17}$$

That is,

$$\|x_{k+1} - \bar{x}\| \leq \ell \|x_k - \bar{x}\| \tag{18}$$

Hence, we have

$$\begin{aligned} \|x_k - \bar{x}\| &\leq \ell \|x_{k-1} - \bar{x}\| \\ &\leq \ell^2 \|x_{k-1} - \bar{x}\| \\ &\leq \cdots \\ &\leq \ell^k \|x_0 - \bar{x}\| \end{aligned} \tag{19}$$

This completes the proof for ii).

We have proved that x_k is a Cauchy sequence in equation (13) and converges to an unique fixed point $\bar{x}$ in (15) and (16). That is

$$\lim_{k\to\infty} x(k) = \bar{x} \tag{20}$$

This completes the proof of the theorem. $\square$

E1.08 Let $A \in \mathbb{R}^{n\times n}$ be doubly-stochastic. Show that

i) $I_n - A^T A \succeq 0$;

ii) $0 \in \Lambda(I_n - A^T A)$

Proof. i) Since A is doubly-stochastic, we have

$$A\mathbf{1} = [\sum_{j=1}^{n} a_{ij}] = \mathbf{1} \tag{21}$$

$$\mathbf{1}^T A = [\sum_{i=1}^{n} a_{ij}]^T = \mathbf{1}^T \tag{22}$$

Thus,

$$A^T A\mathbf{1} = A^T\mathbf{1} = \mathbf{1} \tag{23}$$

$$\mathbf{1}^T A^T A = \mathbf{1}^T A = \mathbf{1}^T \tag{24}$$

which implies that $A^T A$ is also doubly-stochastic. Furthermore, $A^T A$ is symmetric and thus all its eigenvalues are real number. According to Gershgorin's theorem, we have $\rho(A^T A) \leq 1$. As a result, it is true that

$$x^T A^T Ax \leq \rho(A^T A)x^T x \leq x^T x \tag{25}$$

for any $x \in \mathbb{R}^n$. Rewrite the above equation as follows

$$x^T(I_n - A^T A) = x^T x - x^T A^T Ax \geq 0 \tag{26}$$

which implies that $I_n - A^T A$ is semi-positive definite. That is, $I_n - A^T A \succeq 0$.

ii) Based on the results of i), we have

$$(I_n - A^T A)\mathbf{1} = \mathbf{1} - A^T A\mathbf{1} = \mathbf{1} - \mathbf{1} = \mathbf{0} = 0 \cdot \mathbf{1} \tag{27}$$

Hence, $I_n - A^T A$ has a zero eigenvalue,

$$0 \in \Lambda(I_n - A^T A) \tag{28}$$

This completes the proof.

□

E1.12 Show that if a digraph G is weighted-balanced, then its Laplacian L is such that $L + L^T \succeq 0$.

Proof. Assume that the number of vertices of the digraph G is N. Since G is weighted-balanced, we have

$$d_{in}(i) = d_{out}(i), \quad i = 1, \ldots, N \tag{29}$$

where d_{in} and d_{out} are in-degree and out-degree respectively. Denote A the weighted adjacency matrix, $D_{out} = diag[d_{out}(1), \ldots, d_{out}(N)]$ the out-degree matrix, then

$$d_{out}(i) = \sum_{k\neq i}^{N} a_{ik} \tag{30}$$

$$d_{in}(i) = \sum_{k\neq i}^{N} a_{ki} \tag{31}$$

$$\tag{32}$$

Thus

$$\sum_{k\neq i}^{N} a_{ik} = \sum_{k\neq i}^{N} a_{ki} \tag{33}$$

Denote L the Laplacian matrix of G,

$$L = D_{out} - A = diag[d_{out}(1), \dots, d_{out}(N)] - A \tag{34}$$

We have

$$\begin{aligned} L + L^T &= 2 \times diag[d_{out}(1), \dots, d_{out}(N)] - A - A^T \\ &= \begin{bmatrix} 2\sum_{k\neq 1}^{N} a_{1k} & -a_{12} - a_{21} & \cdots & -a_{1N} - a_{N1} \\ -a_{21} - a_{12} & 2\sum_{k\neq 2}^{N} a_{2k} & \cdots & -a_{2N} - a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} - a_{1N} & -a_{N2} - a_{2N} & \cdots & 2\sum_{k\neq N}^{N} a_{Nk} \end{bmatrix} \end{aligned} \tag{35}$$

Let $P = L + L^T$, and

$$R_i = \sum_{k\neq i}^{N} |p_{ik}| = \left| \sum_{k\neq i}^{N} a_{ik} + \sum_{k\neq i}^{N} a_{ki} \right| = 2\sum_{k\neq i}^{N} a_{ik} \tag{36}$$

$$p_{ii} = 2\sum_{k\neq i}^{N} a_{ik} \tag{37}$$

According to the Gershgorin theorem, every eigenvalue of P lies within at least one of the Gershgorin discs $D_i(p_{ii}, R_i)$. Note that P is a symmetric matrix, so all its eigenvalues are real. Thus,

$$\Lambda(P) \subset \cup_{i\in\{1,\dots,n\}} \{x \in \mathbb{R} \mid \|x - p_{ii}\| \leq R_i\} \tag{38}$$

which implies that all eigenvalues of P are non-negative. Hence, $P = L + L^T$ is semi-positive definite, that is, $L + L^T \succeq 0$.

$\square$