DISC Course: Mathematical Models of Systems

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1 Homework 1

1.1 Chapter 1

1.1.1 Exercise 1.9

Solution: The equations which constitute the full behavioral equations of the circuit are as follows:

Constitutive equations:

$$L_1 \frac{dI_{L_1}}{dt} = V_{L_1}, \quad L_2 \frac{dI_{L_2}}{dt} = V_{L_2}, \quad C_1 \frac{dV_{C_1}}{dt} = I_{C_1}, \quad C_2 \frac{dV_{C_2}}{dt} = I_{C_2}; \tag{1.1}$$

Kirchhoff 's current laws:

$$I = I_{L_1} + I_{L_2}, \quad I_{L_1} = I_{C_1}, \quad I_{L_2} = I_{C_2}, \quad I_{C_1} + I_{C_2} = I; \tag{1.2}$$

Kirchhoff 's voltage laws:

$$V = V_{L_1} + V_{C_1}, \quad V = V_{L_2} + V_{C_2}, \quad V_{L_1} + V_{C_1} = V_{L_2} + V_{C_2}. \tag{1.3}$$

Next, we attempt to eliminate the latent variables in order to come up with an explicit relation involving V and I only. For simplicity, we denote $\dot{x} = \frac{dx}{dt}$ the derivative of variable x. Thus equation (1.1) can be written as

$$L_1 \dot{I}_{L_1} = V_{L_1}, \quad L_2 \dot{I}_{L_2} = V_{L_2}, \quad C_1 \dot{V}_{C_1} = I_{C_1}, \quad C_2 \dot{V}_{C_2} = I_{C_2} \tag{1.4}$$

First, we use equation (1.1) to eliminate $V_{L_1}, V_{L_2}, I_{C_1}$ and I_{C_2}. Then we obtain

$$V = L_1\dot{I}_{L_1} + V_{C_1}, \tag{1.5}$$
$$V = L_2\dot{I}_{L_2} + V_{C_2}, \tag{1.6}$$
$$I_{L_1} = C_1\dot{V}_{C_1}, \tag{1.7}$$
$$I_{L_2} = C_2\dot{V}_{C_2}, \tag{1.8}$$
$$I = I_{L_1} + I_{L_2}. \tag{1.9}$$

Next, we further eliminate V_{C_1} and V_{C_2}, which gives

$$\dot{V} = L_1\ddot{I}_{L_1} + \frac{I_{L_1}}{C_1}, \tag{1.10}$$
$$\dot{V} = L_2\ddot{I}_{L_2} + \frac{I_{L_2}}{C_2}, \tag{1.11}$$
$$I_{L_1} = I - I_{L_2}. \tag{1.12}$$

Use equation (1.12) in (1.10) to obtain

$$\dot{V} = L_1\ddot{I} - L_1\ddot{I}_{L_2} + \frac{1}{C_1}(I - I_{L_2}) \tag{1.13}$$

Combining equation (1.11) and (1.13) to eliminate $\ddot{I}_{L_2}$, we can obtain

$$\left(\frac{L_1}{C_2} - \frac{L_2}{C_1}\right)I_{L_2} = (L_1 + L_2)\dot{V} - L_1L_2\ddot{I} - \frac{L_2}{C_1}I \tag{1.14}$$

Now, we should consider two cases:

Case 1: $\frac{L_1}{C_2} = \frac{L_2}{C_1}$. Then equation (1.14) immediately yields

$$(L_1 + L_2)\dot{V} = L_1L_2\ddot{I} + \frac{L_2}{C_1}I \tag{1.15}$$

as the relation between V and I.

Case 2: $\frac{L_1}{C_2} \neq \frac{L_2}{C_1}$. Solve (1.14) for I_{L_2} and substitute into (1.11). Then we can obtain

$$C_1C_2(L_1 + L_2)\dddot{V} + (C_1 + C_2)\dot{V} = L_1L_2C_1C_2\ddddot{I} + (L_1C_1 + L_2C_2)\ddot{I} + I \tag{1.16}$$

as the relation between V and I.

We claim that equations (1.15, 1.16) specify the manifest behavior defined by the full behavioral equations (1.1, 1.2, 1.3).

1.2 Chapter 2

1.2.1 Exercise 2.3

Solution: (a) Recall that the differential equation relating the port voltage V to the port current I is

$$V + CR_1\frac{d}{dt}V = (R_0 + R_1)I + CR_0R_1\frac{d}{dt}I \tag{1.17}$$

Let $R_0 = R_1 = 1$ and $C = 1$. Then the differential equation becomes

$$V + \frac{d}{dt}V = 2I + \frac{d}{dt}I \tag{1.18}$$

Integrate both sides of the above differential equation and we can obtain

$$(\int V) + V = 2(\int I) + I \tag{1.19}$$

where $\int \cdot$ is defined by $(\int w)(t) = \int_0^t w(\tau)d\tau$. With the port voltage

$$V(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1.20}$$

and the current

$$I(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} + \frac{1}{2}e^{-2t} & t \geq 0 \end{cases} \tag{1.21}$$

which are not differentiable at $t = 0$. However, it is easy to validate that they satisfy the integration equation (1.19). Hence, They yield a weak solution of (1.18).

(b) With the current

$$I(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1.22}$$

we can obtain from equation (1.19) that

$$(\int V) + V = \begin{cases} 0 & t < 0 \\ 2t + 1 & t \geq 0 \end{cases} \tag{1.23}$$

Analogously, assume the port voltage is in the following form

$$V(t) = \begin{cases} 0 & t < 0 \\ a + be^{ct} & t \geq 0 \end{cases} \tag{1.24}$$

Then when $t \geq 0$ we have

$$\int V = at + \frac{b}{c}e^{ct} - \frac{b}{c} \tag{1.25}$$

which yields

$$at + \frac{b}{c} - \frac{b}{c}e^{ct} + a + be^{ct} = 2t + 1 \tag{1.26}$$

Therefore,

$$\begin{cases} a = 2 \\ \frac{b}{c} + b = 0 \\ -\frac{b}{c} + a = 1 \end{cases} \tag{1.27}$$

Solving the above equation yields

$$\begin{cases} a = 2 \\ b = -1 \\ c = -1 \end{cases} \tag{1.28}$$

Hence, then the port voltage can be easily obtained as follows

$$V(t) = \begin{cases} 0 & t < 0 \\ 2 - e^{-t} & t \geq 0 \end{cases} \tag{1.29}$$

1.2.2 Exercise 2.5

Solution: (a) Let $V(\xi)$ be a unimodular matrix such that

$$\begin{bmatrix} 2 - 3\xi + \xi^2 & 6 - 5\xi + \xi^2 & 12 - 7\xi + \xi^2 \end{bmatrix} V(\xi) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{1.30}$$

Then we can obtain

$$V(\xi) = \begin{bmatrix} -\frac{1}{2}\xi^2 + \frac{3}{2}\xi & \xi - 3 & \frac{1}{2} \\ \xi^2 - 3\xi + 2 & 5 - 2\xi & -1 \\ -\frac{1}{2}\xi^2 + \frac{3}{2}\xi - 1 & \xi - 2 & \frac{1}{2} \end{bmatrix} \tag{1.31}$$

Hence, we have

$$U(\xi) = V^{-1}(\xi) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 - 3\xi + \xi^2 & 6 - 5\xi + \xi^2 & 12 - 7\xi + \xi^2 \end{bmatrix} \tag{1.32}$$

(b) From equation (1.30), we have

$$\begin{bmatrix} r_1(\xi) & r_2(\xi) & r_3(\xi) \end{bmatrix} \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = 1 \tag{1.33}$$

Hence, the polynomials to be determined are

$$a_1(\xi) = \frac{1}{2}, \quad a_2(\xi) = -1, \quad a_3(\xi) = -\frac{1}{2} \tag{1.34}$$

1.2.3 Exercise 2.7

Solution: The corresponding polynomial matrix of the differential equations is

$$P(\xi) = \begin{bmatrix} -1 + \xi^2 & 1 + \xi \\ -\xi + \xi^2 & \xi \end{bmatrix} \tag{1.35}$$

Multiplying by ξ and then subtracting $\xi+1$ times the second row from the first yields

$$\begin{bmatrix} 0 & 0 \\ -\xi+\xi^2 & \xi \end{bmatrix} \tag{1.36}$$

Therefore, the original differential system is not a full rank representation. The equivalent full rand representation can be obtained from (1.36) as follows

$$-\frac{d}{dt}w_1 + \frac{d^2}{dt^2}w_1 + \frac{d}{dt}w_2 = 0. \tag{1.37}$$

1.2.4 Exercise 2.12

Solution:

1.2.5 Exercise 2.25

Solution: Consider $v_1(\xi) = \begin{bmatrix}\xi & \xi^2\end{bmatrix}^T$, $v_2(\xi) = \begin{bmatrix}1+\xi & \xi+\xi^2\end{bmatrix}^T$ and $a_1(\xi) = \xi+1$, $a_2(\xi) = -\xi$. Then it is trivial to obtain that

$$a_1(\xi)v_1(\xi) + a_2(\xi)v_2(\xi) = 0 \tag{1.38}$$

Thus, the two polynomial vectors $v_1(\xi)$ and (ξ) are linear dependent. However, there is no common factor except 1 of the two vectors. Therefore, one of them cannot be written as a linear combination of another. Hence, this example indicates that the property for real vectors is not true for polynomials.

1.3 Chapter 3

1.3.1 Exercise 3.1

Solution: The corresponding polynomial of the differential equation is

$$P(\xi) = -32 + 22\xi^2 + 9\xi^3 + \xi^4 = (\xi+4)^4(\xi+2)(\xi-1) \tag{1.39}$$

Hence, the solution of (1.39) can be written as

$$w(t) = r_{10}e^t + r_{20}e^{-2t} + r_{40}e^{-4t} + r_{41}e^{-4t} \tag{1.40}$$

where the coefficients are completely free, $r_{ij} \in \mathbb{C}$.

1.3.2 Exercise 3.6

Solution: (a) The matrix $P(\xi)$ is given by

$$P(\xi) = \begin{bmatrix} 1+\xi^2 & -3-\xi+\xi^2+\xi^3 \\ 1-\xi & -1+\xi \end{bmatrix} \tag{1.41}$$

(b) We have

$$\det P(\xi) = (\xi-1)^2(\xi+1)(\xi+2) \tag{1.42}$$

Hence, the root of $\det P(\xi)$ are $1, -1, -2$ with multiplicities $2,\ 1$ and 1 respectively.

(c) According to Theorem 3.2.16, every (strong) solution of the differential system is of the following form

$$w(t) = B_{10}e^t + B_{11}te^t + B_{20}e^{-t} + B_{30}e^{-2t} \tag{1.43}$$

where the coefficients $B_{ij} \in \mathbb{C}^2$ satisfy the relations

$$P(1)B_{10} + P^{(1)}(1)B_{11} = 0 \tag{1.44}$$
$$P(1)B_{11} = 0 \tag{1.45}$$
$$P(-1)B_{20} = 0 \tag{1.46}$$
$$P(-2)B_{30} = 0 \tag{1.47}$$

According to (1.45), we have

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} B_{11} = 0 \tag{1.48}$$

which yields $B_{11} = \begin{bmatrix} \alpha_2 & \alpha_2 \end{bmatrix}^T$. Substituting B_{11} into (1.44) we can obtain $B_{10} = \begin{bmatrix} \alpha_1 - 3\alpha_2 & \alpha_1 \end{bmatrix}^T$. Similarly, it is easy to solve equation (1.46) and (1.47) to obtain $B_{30} = \begin{bmatrix} \gamma & \gamma \end{bmatrix}^T$ and $B_{40} = \begin{bmatrix} \beta & \beta \end{bmatrix}^T$. Thus, every (strong) solution can be written as

$$w(t) = \begin{bmatrix} \alpha_1 - 3\alpha_2 \\ \alpha_1 \end{bmatrix} e^t + \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix} te^t + \begin{bmatrix} \beta \\ \beta \end{bmatrix} e^{-t} + \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} e^{-2t} \tag{1.49}$$

where $\alpha_1, \alpha_2, \beta, \gamma \in \mathbb{C}$.

1.3.3 Exercise 3.20

Solution: (a) The roots of $p(\xi) = \xi(\xi-1)^2$ are 1 and 0 with multiplicities 2 and 1 respectively. Therefore, according to Theorem 3.3.7, we have

$$\frac{q(\xi)}{p(\xi)} = \frac{a_{11}}{\xi} + \frac{a_{21}}{\xi-1} + \frac{a_{22}}{(\xi-1)^2} \tag{1.50}$$

where the coefficients a_{ij} can be calculated as follows

$$a_{11} = \lim_{\xi \to 0} \xi \frac{q(\xi)}{p(\xi)} = \lim_{\xi \to 0} \frac{-1+\xi^2}{(\xi-1)^2} = -1 \tag{1.51}$$

$$a_{21} = \lim_{\xi \to 1} (\xi-1) \frac{q(\xi)}{p(\xi)} = \lim_{\xi \to 1} \frac{-1+\xi^2}{\xi(\xi-1)} = 2 \tag{1.52}$$

$$a_{22} = \lim_{\xi \to 1} (\xi-1)^2 \frac{q(\xi)}{p(\xi)} = \lim_{\xi \to 1} \frac{-1+\xi^2}{\xi} = 0 \tag{1.53}$$

Hence, the partial fraction expansion of $\frac{q(\xi)}{p(\xi)}$ is

$$\frac{q(\xi)}{p(\xi)} = -\frac{1}{\xi} + \frac{2}{\xi-1} \tag{1.54}$$

(b) According to Theorem 3.3.19, we have

$$\mathcal{B} = \mathcal{B}_{\text{i/o}} + \mathcal{B}_{\text{hom}} \tag{1.55}$$

where $\mathcal{B}_{\text{hom}} = (0, y_{\text{hom}})$ and

$$y_{\text{hom}} = B_{10} + B_{20}e^t + B_{21}te^t, \quad t \in \mathbb{R} \tag{1.56}$$

in which $B_{ij} \in \mathbb{C}$. Further, the explicit characterization of $\mathcal{B}_{\text{i/o}}$ is

$$\begin{aligned} y_{\text{i/o}} &= A_0 u(t) + A_{11} \int_0^t u(\tau) d\tau + A_{21} \int_0^t e^{t-\tau} u(\tau) d\tau + A_{22} \int_0^t (t-\tau) e^{t-\tau} u(\tau) d\tau \\ &= -\int_0^t u(\tau) d\tau + 2 \int_0^t e^{t-\tau} u(\tau) d\tau, \quad t \in \mathbb{R} \end{aligned} \tag{1.57}$$

Hence, the explicit characterization of $\mathcal{B}$ is

$$y(t) = -\int_0^t u(\tau) d\tau + 2 \int_0^t e^{t-\tau} u(\tau) d\tau + B_{10} + B_{20}e^t + B_{21}te^t, \quad t \in \mathbb{R} \tag{1.58}$$

(c) Similar with (a), we can determine the the partial fraction expansion of $\frac{\tilde{q}(\xi)}{\tilde{p}(\xi)}$, which is

$$\frac{\tilde{q}(\xi)}{\tilde{p}(\xi)} = -\frac{1}{\xi} + \frac{2}{\xi-1} \tag{1.59}$$

Note that it is the same as $\frac{q(\xi)}{p(\xi)}$. That is, $\frac{\tilde{q}(\xi)}{\tilde{p}(\xi)} = \frac{q(\xi)}{p(\xi)}$.

(d) Similar with (b), the explicit characterization of the behavior $\tilde{\mathcal{B}}$ is given as

$$y(t) = -\int_0^t u(\tau) d\tau + 2 \int_0^t e^{t-\tau} u(\tau) d\tau + \tilde{B}_{10} + \tilde{B}_{20}e^t, \quad t \in \mathbb{R} \tag{1.60}$$

where $\tilde{B}_{10}, \tilde{B}_{20} \in \mathbb{C}$.

(e) Explicitly, in terms of the y_{hom} characterization in $\mathcal{B}_{\text{hom}}$ and $\tilde{B}_{\text{hom}}$, $\mathcal{B}_{\text{hom}}$ consists of an extra term te^t.

(f) According to Theorem 3.5.2, the convolution system is given by

$$y(t) = \int_{-\infty}^{t} H(t-\tau)u(\tau)d\tau \tag{1.61}$$

where

$$H(t) = \begin{cases} 0 & t < 0 \\ -1 + 2e^t & t \geq 0 \end{cases} \tag{1.62}$$

1.3.4 Exercise 3.22

Solution: (a) The corresponding polynomial matrix of the set of differential equations is given by

$$P(\xi) = \begin{bmatrix} 6 - 5\xi + \xi^2 & -3 + \xi \\ 2 - 3\xi + \xi^2 & -1 + \xi \end{bmatrix} \tag{1.63}$$

Thus, we have

$$\det P(\xi) = (6 - 5\xi + \xi^2)(-1 + \xi) - (-3 + \xi)(2 - 3\xi + \xi^2) = 0 \tag{1.64}$$

Hence, the set of differential equations does NOT define an autonomous system.

(b) By performing elementary row operations, we can obtain

$$U(\xi)P(\xi) = \begin{bmatrix} -2 + \xi & 1 \\ 0 & 0 \end{bmatrix} \tag{1.65}$$

where the unimodular matrix $U(\xi)$ is

$$U(\xi) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2}\xi & \frac{3}{2} - \frac{1}{2}\xi \end{bmatrix} \tag{1.66}$$

Denote $P'(\xi) = \begin{bmatrix} -2 + \xi & 1 \end{bmatrix}$. Then the minimal representation of the differential system is $P'(\frac{d}{dt})w = 0$. Further, let $\tilde{P}(\xi) = -2 + \xi$ and $\tilde{Q}(\xi) = -1$. Then we can write the input/out representation of the original system as

$$\tilde{P}(\frac{d}{dt})w_1 = \tilde{Q}(\frac{d}{dt})w_2 \tag{1.67}$$

Hence, the explicit characterization of the behavior is

$$w_1(t) = -\int_0^t e^{2(t-\tau)} w_2(\tau)d\tau, \quad t \in \mathbb{R}. \tag{1.68}$$

1.3.5 Exercise 3.36

Solution: (a) Denote $x(\xi) \in \mathbb{R}^g[\xi]$ an arbitrary polynomial vector, then we have

$$x^T(\xi)R_2(\xi)R_2^T(\xi)x(\xi) = (R_2^T(\xi)x(\xi))^T(R_2^T(\xi)x(\xi)) \geq 0 \tag{1.69}$$

where the equality to zero is satisfied if and only if

$$(R_2^T(\xi)x(\xi))^T(R_2^T(\xi)x(\xi)) = 0 \iff R_2^T(\xi)x(\xi) = 0 \tag{1.70}$$

In addition, since $R_2(\xi)$ is full of row rank, $R_2^T(\xi)$ is full of column rank and the above equations yield to

$$x^T(\xi)R_2(\xi)R_2^T(\xi)x(\xi) = 0 \iff x(\xi) = 0 \tag{1.71}$$

which indicates that $R_2(\xi)R_2^T(\xi)$ is non-singular. Hence, $\det R_2(\xi)R_2^T(\xi) \neq 0$, and $R_2(\xi)R_2^T(\xi)$ is invertible as a rational matrix.

(b) Since $R_2(\xi)R_2^T(\xi)$ is invertible as a rational matrix, we have

$$R_2(\xi)R_2^T(\xi)(R_2(\xi)R_2^T(\xi))^{-1} = I_g \tag{1.72}$$

Thus,

$$R_2(\xi)[R_2^T(\xi)(R_2(\xi)R_2^T(\xi))^{-1}] = I_g \tag{1.73}$$

Denote $R_2^\star(\xi) = R_2^T(\xi)(R_2(\xi)R_2^T(\xi))^{-1}$, then we have $R_2(\xi)R_2^\star(\xi) = I_g$.

(c) Since $\mathcal{B}_1 = \mathcal{B}_2$, there exists a polynomial unimodular matrix $U(\xi) \in \mathbb{R}^{g\times g}[\xi]$ such that $U(\xi)R_2(\xi) = R_1(\xi)$. Therefore,

$$R_1(\xi)R_2^\star(\xi) = U(\xi)R_2(\xi)R_2^\star(\xi) = U(\xi) \tag{1.74}$$

which shows that $R_1(\xi)R_2^\star(\xi)$ is a polynomial unimodular matrix.

(d) Let

$$R_1(\xi) = \begin{bmatrix}\xi & 1-\xi^2\end{bmatrix}, \quad R_2(\xi) = \begin{bmatrix}\xi & 1\end{bmatrix} \tag{1.75}$$

then we have

$$R_1(\xi)R_2^\star(\xi) = R_1(\xi)R_2^T(\xi)(R_2(\xi)R_2^T(\xi))^{-1} = 1 \tag{1.76}$$

is a polynomial unimodular matrix. However, $\mathcal{B}_1 \neq \mathcal{B}_2$. This simple example shows that if $R_1(\xi)R_2^\star(\xi)$ is a polynomial unimodular matrix, then we need not have that $\mathcal{B}_1 \neq \mathcal{B}_2$.

(e) Sufficiency: If $R_1(\xi)R_2^\star(\xi)$ is a polynomial unimodular matrix $U(\xi)$ and $R_1(\xi) = R_1(\xi)R_2^\star(\xi)R_2(\xi)$, we have

$$R_1(\xi) = R_1(\xi)R_2^\star(\xi)R_2(\xi) = U(\xi)R_2(\xi) \tag{1.77}$$

According to Theorem 3.6.3, we have $\mathcal{B}_1 = \mathcal{B}_2$.

Necessity: If $\mathcal{B}_1 = \mathcal{B}_2$, then the result in (c) shows that $R_1(\xi)R_2^\star(\xi)$ is a polynomial unimodular matrix $U(\xi)$. In addition, we have

$$R_1(\xi)R_2^\star(\xi)R_2(\xi) = U(\xi)R_2(\xi) = R_1(\xi) \tag{1.78}$$

which completes the proof.

(f) First, we compute

$$R_1(\xi)R_2^\star(\xi) = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \tag{1.79}$$

which is a polynomial unimodular matrix. In addition, we have

$$R_1(\xi)R_2^\star(\xi)R_2(\xi) = \begin{bmatrix} 1+\xi^2 & \xi & 1+\xi \\ \xi & 0 & 1 \end{bmatrix} = R_1(\xi) \tag{1.80}$$

Hence, according to the result of (e), we prove that $\mathcal{B}_1 = \mathcal{B}_2$.

1.4 Additional Exercise

Solution: (a) According to Theorem 2.5.14, we can assume $P(\xi)$ in the following form

$$P(\xi) = \begin{bmatrix} \alpha_1(\xi) & \alpha_2(\xi) \\ 0 & \alpha_3(\xi) \end{bmatrix} \tag{1.81}$$

where $\alpha_i \in \mathbb{R}[\xi],\ i = 1, 2, 3$. Further, we have, according to Theorem 3.2.15,

$$\det P(\xi) = \alpha_1(\xi)\alpha_3(\xi) = (\xi-1)(\xi-2) \tag{1.82}$$

$$P(1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \tag{1.83}$$

$$P(-2)\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0 \tag{1.84}$$

which can be satisfied given $\alpha_1(\xi) = 1$ and $\alpha_3(\xi) = (\xi-1)(\xi-2)$. Assume $\alpha_2(\xi) = a + b\xi$, then we can determine a and b by the above equations, which are $a = -\frac{7}{12}$ and $b = \frac{1}{12}$. Hence, we have

$$P(\xi) = \begin{bmatrix} 1 & -\frac{7}{12} + \frac{1}{12}\xi \\ 0 & -2 + \xi + \xi^2 \end{bmatrix} \tag{1.85}$$

whose degree is $n = 2$. In addition, we can easily check that the behavior defined by (1.90) is already as small as possible; therefore, it is the Most Powerful Unfalsified Model.

(b) Similarly, by assuming $P(\xi)$ the same form as in (a), we have

$$\det P(\xi) = \alpha_1(\xi)\alpha_3(\xi) = (\xi-1)(\xi-2)(\xi+2) \tag{1.86}$$

$$P(1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \tag{1.87}$$

$$P(-2)\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 0 \tag{1.88}$$

$$P(2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \tag{1.89}$$

which can be satisfied given $\alpha_1(\xi) = \xi - 2$ and $\alpha_3(\xi) = (\xi-1)(\xi+2)$. Similarly, assume $\alpha_2(\xi) = a + b\xi$, then we can determine a and b by the above equations, which are $a = \frac{4}{3}$ and $b = -\frac{5}{6}$. Hence, we have

$$P(\xi) = \begin{bmatrix} \xi - 2 & \frac{4}{3} - \frac{5}{6}\xi \\ 0 & -2 + \xi + \xi^2 \end{bmatrix} \tag{1.90}$$

which is the Most Powerful Unfalsified Model.

1.5 Simulation Exercise

1.5.1 A.3 Autonomous Dynamics of Coupled Masses

Solution: (1) The equations that describe the behavior of the system are

$$M\ddot{w}_1 + k_1 w_1 + k_2(w_1 - w_2) = 0 \tag{1.91}$$

$$M\ddot{w}_2 + k_1 w_2 + k_2(w_2 - w_1) = 0 \tag{1.92}$$

Let $M = 1$, then can rewrite the above equations in the form $P(\frac{d}{dt})w = 0$ as follows

$$\begin{bmatrix} \xi^2 + k_1 + k2 & -k_2 \\ -k_2 & \xi^2 + k_1 + k_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \tag{1.93}$$

Hence, we have

$$P(\xi) = \begin{bmatrix} \xi^2 + k_1 + k2 & -k_2 \\ -k_2 & \xi^2 + k_1 + k_2 \end{bmatrix} \tag{1.94}$$

(2) According to (1), we have

$$\begin{aligned} \det P(\xi) &= \xi^4 + 2(k_1+k_2)\xi^2 + k_1^2 + 2k_1k_2 \\ &= (\xi + i\sqrt{k_1})(\xi - i\sqrt{k_1})(\xi + i\sqrt{k_1+2k_2})(\xi - \sqrt{k_1+2k_2}) \end{aligned} \tag{1.95}$$

which is the characteristic polynomial of the system. Let $\det P(\xi) = 0$, then we can solve to obtain

$$\lambda_1 = i\sqrt{k_1}, \qquad n_1 = 1 \tag{1.96}$$

$$\lambda_2 = -i\sqrt{k_1}, \qquad n_2 = 1 \tag{1.97}$$

$$\lambda_3 = i\sqrt{k_1+2k_2}, \qquad n_3 = 1 \tag{1.98}$$

$$\lambda_4 = -i\sqrt{k_1+2k_2}, \qquad n_4 = 1 \tag{1.99}$$

which are the characteristic values of the system.

(3) According to Theorem 3.3.16, the trajectories in the behavior of the system can be written in the following general form

$$w(t) = \sum_{i=1}^{N} \sum_{j=0}^{n_i-1} B_{ij} t^j e^{\lambda_i t} \tag{1.100}$$

where $B_{ij} \in \mathbb{C}^2$ satisfy the relations

$$\sum_{j=l}^{n_i-1} \binom{j}{l} P^{(j-l)}(\lambda_i) B_{ij} = 0, i = 1, \ldots, N; l = 0, \ldots, n_i - 1 \tag{1.101}$$

Substituting the results in (2) into the above two equations, we can obtain

$$B_{10} = [r_1, r_1]^T \tag{1.102}$$
$$B_{20} = [r_2, r_2]^T \tag{1.103}$$
$$B_{30} = [r_3, -r_3]^T \tag{1.104}$$
$$B_{40} = [r_4, -r_4]^T \tag{1.105}$$

where $r_i \in \mathbb{C}$ are constants. Further, according to Euler's formula, we have

$$e^{\lambda_1 t} = e^{i\sqrt{k_1}t} = \cos\sqrt{k_1}t + \sin\sqrt{k_1}t \tag{1.106}$$
$$e^{\lambda_2 t} = e^{-i\sqrt{k_1}t} = \cos\sqrt{k_1}t - \sin\sqrt{k_1}t \tag{1.107}$$
$$e^{\lambda_3 t} = e^{i\sqrt{k_1+2k_2}t} = \cos\sqrt{k_1+2k_2}t + \sin\sqrt{k_1+2k_2}t \tag{1.108}$$
$$e^{\lambda_4 t} = e^{-i\sqrt{k_1+2k_2}t} = \cos\sqrt{k_1+2k_2}t - \sin\sqrt{k_1+2k_2}t \tag{1.109}$$

Combing those equations, we can obtain

$$w_1(t) = (r_1 + r_2)\cos\sqrt{k_1}t + (r_1 - r_2)\sin\sqrt{k_1}t \tag{1.110}$$
$$+(r_3 + r_4)\cos\sqrt{k_1+2k_2}t + (r_3 - r_4)\sin\sqrt{k_1+2k_2}t \tag{1.111}$$

$$w_2(t) = (r_1 + r_2)\cos\sqrt{k_1}t + (r_1 - r_2)\sin\sqrt{k_1}t \tag{1.112}$$
$$-(r_3 + r_4)\cos\sqrt{k_1+2k_2}t - (r_3 - r_4)\sin\sqrt{k_1+2k_2}t \tag{1.113}$$

Let $\alpha = r_1 + r_2, \beta = r_1 - r_2, \gamma = r_3 + r_4, \delta = r_3 - r_4 \in \mathbb{C}$, then we can obtain the form in trigonometric as follows

$$w_1(t) = \alpha\cos\sqrt{k_1}t + \beta\sin\sqrt{k_1}t + \gamma\cos\sqrt{k_1+2k_2}t + \delta\sin\sqrt{k_1+2k_2}t \tag{1.114}$$
$$w_2(t) = \alpha\cos\sqrt{k_1}t + \beta\sin\sqrt{k_1}t - \gamma\cos\sqrt{k_1+2k_2}t - \delta\sin\sqrt{k_1+2k_2}t \tag{1.115}$$

(4) Take $k_1 = 25$ and $k_2 = 1$, then the differential equations of the system are

$$\ddot{w}_1 + 26w_1 - w_2 = 0 \tag{1.116}$$
$$\ddot{w}_2 + 26w_2 - w_1 = 0 \tag{1.117}$$

Based on the results in (3), we have

$$w_1(t) = \alpha\cos 5t + \beta\sin 5t + \gamma\cos\sqrt{27}t + \delta\sin\sqrt{27}t \tag{1.118}$$
$$w_2(t) = \alpha\cos 5t + \beta\sin 5t - \gamma\cos\sqrt{27}t - \delta\sin\sqrt{27}t \tag{1.119}$$

Combining with the boundary conditions

$$\dot{w}_1(0) = 5\beta + \sqrt{27}\delta = 0 \tag{1.120}$$
$$\dot{w}_2(0) = 5\beta - \sqrt{27}\delta = 0 \tag{1.121}$$
$$w_1(0) = \alpha + \gamma = 1 \tag{1.122}$$
$$w_2(0) = \alpha - \gamma = 0 \tag{1.123}$$

and we can obtain $\beta = \delta = 0$, $\alpha = \gamma = \frac{1}{2}$. Thus, the behavior of the system is

$$w_1(t) = \frac{1}{2}\cos 5t + \frac{1}{2}\cos\sqrt{27}t \tag{1.124}$$
$$w_2(t) = \frac{1}{2}\cos 5t - \frac{1}{2}\cos\sqrt{27}t \tag{1.125}$$

(5) Since $\cos p + \cos q = 2\cos\frac{p-q}{2}\cos\frac{p+q}{2}$ and $\cos p - \cos q = -2\cos\frac{p-q}{2}\cos\frac{p+q}{2}$. we have

$$w_1(t) = \frac{1}{2}\cos 5t + \frac{1}{2}\cos\sqrt{27}t = \cos\frac{5+\sqrt{27}}{2}t\cos\frac{5-\sqrt{27}}{2}t \tag{1.126}$$
$$w_2(t) = \frac{1}{2}\cos 5t - \frac{1}{2}\cos\sqrt{27}t = -\sin\frac{5+\sqrt{27}}{2}t\sin\frac{5-\sqrt{27}}{2}t \tag{1.127}$$

Hence, the fast and slow frequencies are

$$f_{\text{fast}} = \frac{5+\sqrt{27}}{2}/2\pi = \frac{5+\sqrt{27}}{4\pi} \tag{1.128}$$
$$f_{\text{slow}} = \frac{-5+\sqrt{27}}{2}/2\pi = \frac{\sqrt{27}-5}{4\pi} \tag{1.129}$$

(6) $w_1(t)$ and $w_2(t)$ are said to be in antiphase if their phase difference is $\Delta\phi = (2n+1)\pi,\ n \in \mathbb{N}$, which is obvious for the results in (4) and (5).

(7) Similar with (4), we can solve for the behavior of (w_1, w_2):

$$w_1(t) = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{51}t \tag{1.130}$$
$$w_2(t) = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{51}t \tag{1.131}$$

(8) Similar with (5), we have

$$w_1(t) = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{51}t = \cos\frac{1+\sqrt{51}}{2}t\cos\frac{1-\sqrt{51}}{2}t \tag{1.132}$$
$$w_2(t) = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{51}t = -\sin\frac{1+\sqrt{51}}{2}t\sin\frac{1-\sqrt{51}}{2}t \tag{1.133}$$

Hence, the fast and slow frequencies can be determined as follows

$$f_{\text{fast}} = \frac{1+\sqrt{51}}{2}/2\pi = \frac{1+\sqrt{51}}{4\pi} \tag{1.134}$$
$$f_{\text{slow}} = \frac{-1+\sqrt{51}}{2}/2\pi = \frac{\sqrt{51}-1}{4\pi} \tag{1.135}$$