DISC Course: Nonlinear Control Systems

Assignment 2

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Exercise 1. (ISS & sector-bound nonlinearity)

Solution: (a) Let $x = [x_1, x_2]^T$, then the nonlinear system can be described as follows:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_1 - 2x_2 - f(x_1) + d \\ y &= x_1 \end{aligned} \tag{1}$$

Take the following Lyapunov function candidate

$$V(x) = (x_1 + x_2)^2 + 2\int_0^{x_1} f(\tau) d\tau \tag{2}$$

Then we have

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x_1} + 2x_2\dot{x_2} + 2\dot{x_1}x_2 + 2x_1\dot{x_2} + 2f(x_1)\dot{x_1} \\ &= 2x_1^2 - 2x_2^2 - 2x_1 f(x_1) + 2(x_1 + x_2)d \end{aligned} \tag{3}$$

Since f is a smooth function satisfying the sector-bound condition

$$c_1 y^2 < f(y)y < c_2 y^2 \tag{4}$$

where $c_1 = 2$ and $c_2 = 6$, we can conclude that $V(x)$ is positive definite and radially bounded. That is, there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{5}$$

Futhermore,

$$\begin{aligned} \dot{V}(x) &= 2x_1^2 - 2x_2^2 - 2x_1 f(x_1) + 2(x_1 + x_2)d \\ &< -2x_2^2 + 2(1 - c_1)x_1^2 + 2(x_1 + x_2)d \\ &= -2x_1^2 - 2x_2^2 + 2(x_1 + x_2)d \\ &\leq -x_1^2 - x_2^2 + 2d^2 \\ &= -\gamma_1(\|x\|) + \gamma_2(\|d\|) \end{aligned} \tag{6}$$

where $\gamma_1 = r^2$ and $\gamma_2 = 2r^2$, $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. Hence, the closed-loop nonlinear system is ISS with respect to d.

(b) Use the same Lyapunov function as (a) and then we have

$$\begin{aligned} \dot{V}(x) &= 2x_1^2 - 2x_2^2 - 2x_1 f(x_1) + 2(x_1 + x_2)d \\ &< -2x_2^2 + 2(1 - c_1)x_1^2 + 2(x_1 + x_2)d \\ &\le -2x_2^2 + 2(1 - c_1)x_1^2 + \varepsilon_1 x_1^2 + \frac{d^2}{\varepsilon_1} + \varepsilon_2 x_2^2 + \frac{d^2}{\varepsilon_2} \\ &= (2 - 2c_1 + \varepsilon_1)x_1^2 + (\varepsilon_2 - 2)x_2^2 + (\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2})d^2 \\ &\le \max 2 - 2c_1 + \varepsilon_1, \varepsilon_2 - 2 \left\| x \right\|^2 + (\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2})d^2 \end{aligned} \tag{7}$$

where $\varepsilon_1, \varepsilon_2 > 0$. Thus, to make the nonlinear system ISS with respect to d, the following inequalities are required

$$2 - 2c_1 + \varepsilon_1 < 0 \tag{8}$$

$$\varepsilon_2 - 2 < 0 \tag{9}$$

Therefore, we have

$$c_1 > 1 + \frac{\varepsilon_1}{2} > 1 \tag{10}$$

Hence, the lower bound of c_1 is 1.

Exercise 2. (Counter-examples)

Solution: (a) Consider a scalar dynamical system

$$\dot{x} = -x + x^2 u \tag{11}$$

where $x \in \mathbb{R}$ and $x \neq c$, c is some constant. If $u = 0$, then system becomes $\dot{x} = -x$. Take $V(x) = \frac{1}{2}x^2$ the Lyapunov function candidate, then we have $\dot{V}(x) = -x^2$, which is negative definite. Therefore, the system is globally asymptotically stable with zero input.

However, if we take a bounded signal $u \equiv 1$, we can find the solution of the system as follows

$$x = \frac{1}{1 - e^{t - \ln \frac{x(0)}{x(0)-1}}} \tag{12}$$

where $x(0)$ is the initial condition. Since $x(t)$ is not a constant, it is obvious that $x(0) \neq 1$ in the above equation. Hence, the system with the above solution has a finite-escape time $\ln \frac{x(0)}{x(0)-1}$.

(b) Consider the following state equation

$$\dot{x}_1 = x_1 + x_2 + u \tag{13}$$

$$\dot{x}_2 = -x_2 + x_2^2 d \tag{14}$$

If $d = 0$, we can choose $u = -2x_1 - x_2$ as the state feedback law. Then the state equation becomes

$$\dot{x}_1 = -x_1 \tag{15}$$

$$\dot{x}_2 = -x_2 \tag{16}$$

Take $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ the Lyapunov function candidate, then we have $\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 - x_2^2$, which is negative definite. Therefore, the system is globally asymptotically stable with zero disturbance and the state feedback law.

However, according to the results in (a), if $d \equiv 1$, then $x_2(t)$ is unbounded and has a finite escape time. Futhermore, note that the state feedback law $u = \alpha(x_1, x_2)$ cam only affect x_1. Thus, no matter what state feedback law is chosen, x_2 is unbounded and hence, the system cannot be made ISS respect to the disturbance signal d.