DISC Course: Nonlinear Control Systems

Assignment 4

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Exercise 1

Solution: (a) Let $x = [x_1, x_2]^T = [q, \dot{q}]^T$, then we have

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = \frac{1}{m}(-dx_2 - g + u) \tag{2}$$

where $m = m(x_1)$, $d = d(x_1, x_2)$, $g = g(x_1)$ are functions describing the mass, damping and spring. The above equation can write in the following state-space form

$$\dot{x} = f(x) + k(x)u = \begin{bmatrix} x_2 \\ -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{m(x_1)} \end{bmatrix} u \tag{3}$$

$$y = h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{4}$$

Lie derivatives

$$\mathcal{L}_f h(x) = x_2 \tag{5}$$

$$\mathcal{L}_k h(x) = 0 \tag{6}$$

$$\mathcal{L}_f^2 h(x) = -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) \tag{7}$$

$$\mathcal{L}_k \mathcal{L}_f h(x) = -\frac{1}{m(x_1)} \tag{8}$$

Therefore, the system has uniform relative degree 2. Let

$$\xi_1 = h(x) = x_1 \tag{9}$$

$$\xi_2 = \mathcal{L}_f h(x) = x_2 \tag{10}$$

Find a coordinate $\eta_1 = \phi_1(x)$ such that $\mathcal{L}_k \phi_1(x) = 0$, i.e.,

$$\frac{\partial \phi_1}{\partial x}(x)k(x) = -\frac{\partial \phi_1}{\partial x_2}\frac{1}{m(x_1)} = 0 \tag{11}$$

A solution is $\eta_1 = \phi_1(x) = x_1$. Coordinate transformation

$$T(x) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \phi_1(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \tag{12}$$

Normal form

$$\dot{\xi}_1 = \xi_2 \tag{13}$$

$$\dot{\xi}_2 = -\frac{1}{m(\xi_1)}(d(\xi_1, \xi_2) + g(\xi_1)) + \frac{1}{m(\xi_1)}u \tag{14}$$

$$\dot{\eta_1} = \xi_1 \tag{15}$$

(b) In equation (1), let

$$x_2 = -x_1 := k_1(x_1) \tag{16}$$

Take $V_1(x_1) = \frac{1}{2}x_1^2$ as the Lyapunov function candidate, then $\dot{V}_1(x_1) = x_1\dot{x}_1 = -x_1^2$. Thus, the feedback control law stabilizes the system described by equation (1).

Let $w_1 = x_2 - k_1(x_1)$, then

$$\dot{x}_1 = x_2 = w_1 + k_1(x_1) = -x_1 + w_1 \tag{17}$$

$$\dot{w}_1 = \dot{x}_2 - \dot{k}_1(x_1) = -\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2 \tag{18}$$

Take $V(x) = V(x_1, w_1) = V_1(x_1) + \frac{1}{2}w_1^2 = \frac{1}{2}x_1^2 + \frac{1}{2}w_1^2$, then

$$\begin{aligned} \dot{V}(x_1, w_1) &= x_1\dot{x}_1 + w_1\dot{w}_1 \\ &= x_1(-x_1 + w_1) + w_1(-\frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2) \\ &= -x_1^2 + w_1(x_1 - \frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) + \frac{1}{m(x_1)}u + x_2) \end{aligned} \tag{19}$$

Let

$$\begin{aligned} u &= m(x_1)(-x_2 + \frac{1}{m(x_1)}(d(x_1, x_2)x_2 + g(x_1)) - x_1 - c_1 w_1) \\ &= -m(x_1)(c_1 + 1)(x_1 + x_2) + d(x_1, x_2)x_2 + g(x_1) \\ &:= \alpha(x) \end{aligned} \tag{20}$$

where $c_1 > 0$. Thus

$$\dot{V}(x_1, w_1) = -x_1^2 - c_1 w_1^2 \leq 0 \tag{21}$$

Therefore, the feedback law (20) stabilizes the origin of the nonlinear system.

Exercise 2

Solution: (a) The nonlinear system is described as

$$\dot{x}_1 = x_1^3 + x_2 \tag{22}$$
$$\dot{x}_2 = x_2 + x_1 + u \tag{23}$$

In equation (22), Let

$$x_2 = -2x_1^3 := k_1(x_1) \tag{24}$$

Take $V_1(x_1) = \frac{1}{4}x_1^4$ as the Lyapunov function, then

$$\dot{V}_1(x_1) = x_1^3\dot{x}_1 = x_1^3(x_1^3 - 2x_1^3) = -x_1^6 \leq 0 \tag{25}$$

Thus, the feedback law (24) stabilize the system (22).

Let $w_1 = x_2 - k_1(x_1) = x_2 + 2x_1^3$, then we have

$$\dot{x}_1 = x_1^3 + x_2 = -x_1^3 + w_1 \tag{26}$$
$$\dot{w}_1 = \dot{x}_2 + 6x_1^2\dot{x}_1 = -6x_1^5 + 6x_1^2 w_1 + x_2 + x_1 + u \tag{27}$$

Take $V(x) = V(x_1, w_1) = V_1(x_1) + \frac{1}{2}w_1^2 = \frac{1}{4}x_1^4 + \frac{1}{2}w_1^2$ as the Lyapunov function candidate, then

$$\begin{aligned}\dot{V}(x_1, w_1) &= x_1^3\dot{x}_1 + \dot{w}_1 w_1 \\ &= x_1^3(-x_1^3 + w_1) + w_1(-6x_1^5 + 6x_1^2 w_1 + x_2 + x_1 + u) \\ &= -x_1^6 + w_1(x_1^3 - 6x_1^5 + 6x_1^2 w_1 + x_1 + x_2 + u)\end{aligned} \tag{28}$$

Let

$$\begin{aligned}u &= -x_1^3 + 6x_1^5 - 6x_1^2 w_1 - x_1 - x_2 - w_1 \\ &= -6x_1^5 - 3x_1^3 - 6x_1^2 x_2 - x_1 - 2x_2 \\ &:= \alpha(x)\end{aligned} \tag{29}$$

then

$$\dot{V}(x_1, w_1) = -x_1^6 - w_1^2 \leq 0 \tag{30}$$

which implies that feedback law (29) stabilizes the given nonlinear system.

(b) Let $d = [d_1, d_2]^T$, considering the uncertainty in the system and base on the results in (a), we have

$$\begin{aligned}\dot{V}(x_1, w_1) &= x_1^3\dot{x}_1 + \dot{w}_1 w_1 \\ &= -x_1^6 - w_1^2 + x_1^3 d_1 + w_1(d_2 + 6x_1^2 d_1) \\ &= -x_1^6 - w_1^2 + x_1^3 d_1 + w_1 d_2 + 6x_1^2 w_1 d_1 \\ &\leq -x_1^6 - w_1^2 + (\frac{x_1^6}{2} + \frac{d_1^2}{2}) + (\frac{w_1^2}{4} + d_2^2) + (\frac{w_1^2}{2} + \frac{x_1^6}{3} + 6^5 d_1^6) \\ &= -\frac{1}{6}x_1^6 - \frac{1}{4}w_1^2 + (\frac{1}{2} + 6^5)d_1^2 + d_2^2 \\ &\leq -\gamma_1(\|x\|) + \gamma_2(\|d\|)\end{aligned} \tag{31}$$

where $\gamma_1(z) = \frac{1}{6}z^6$, $\gamma_2(z) = (\frac{1}{2} + 6^5)z^2$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$.

Hence, with the feedback control law $u = \alpha(x)$, the closed-loop system is ISS with respect to d_1 and d_2.