DISC Course: Multi-agent Network Dynamics and Games

Homework 3

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Problem 1

Solution: **a)** For different values of x,

- Case 1: $x = 0$.
  The payoff matrix is

1,1	2,0
0,2	3,3

  There are two pure strategy Nash equilibria: (e_1, e_1) and (e_2, e_2).

  Apparently, (e_2, e_2) is a strict symmetric Nash equilibrium. Thus, e_2 is an evolutionary stable strategy (Corollary 5.53 in [1]). For e_1, since we have $\pi(e_1, e_1) > \pi(e_2, e_1)$, thus e_1 is also an evolutionary stable strategy.

- Case 2: $x = 1$.
  The payoff matrix is

1,1	2,1
1,2	3,3

  There are two pure strategy Nash equilibria: (e_1, e_1) and (e_2, e_2).

  Similarly, since (e_2, e_2) is a strict symmetric Nash equilibrium. Thus, e_2 is an evolutionary stable strategy (Corollary 5.53 in [1]). For e_1, we have $\pi(e_1, e_1) = \pi(e_2, e_1)$ but $\pi(e_1, e_2) < \pi(e_2, e_2)$. So e_1 is not an evolutionary stable strategy.

- Case 3: $x = 2$.
  The payoff matrix is

1,1	2,2
2,2	3,3

There is only one pure strategy Nash equilibria: (e_2, e_2).

Note that the only one pure strategy Nash equilibrium is strict symmetric. Thus e_2 is the evolutionary stable strategy (Corollary 5.53 in [1]).

b) Recall the definition of weakly dominated strategy [1]:

Definition 1. *A strategy s_i of player i is termed weakly dominated if there exists another strategy t_i of player i satisfying the following two conditions:*

(i) For every strategy vector s_{-i} of other players,

$$\pi(s_i, s_{-i}) \leq \pi(t_i, s_{-i}) \tag{1}$$

(ii) There exists a strategy vector t_{-i} of other players such that

$$\pi(s_i, t_{-i}) < \pi(s_i, t_{-i}) \tag{2}$$

If the equal condition in equation (1) is removed, then the above definition is refined to "strictly dominated".

In the given game, since X is weakly dominated and (X, X) is a Nash equilibrium, we have

$$c = a \tag{3}$$
$$d > b \tag{4}$$

We can compute

$$\pi(X, X) = a \tag{5}$$
$$\pi(Y, X) = c \tag{6}$$

We further compute that

$$\pi(X, Y) = b \tag{7}$$
$$\pi(Y, Y) = d \tag{8}$$

Thus we have $\pi(X, X) = \pi(Y, X)$ but $\pi(X, Y) < \pi(Y, Y)$. Hence, X is not an evolutionary stable strategy.

Problem 2

Solution: a) Let $s_1 = e_1$ the first pure strategy of player 1. We can suppose $t_i = e_2, e_3, e_4$ and check if it satisfies the above conditions. The result they all do not satisfy the conditions since

$$\begin{aligned}\pi(e_1, e_2) &> \pi(e_2, e_2)\\ \pi(e_1, e_1) &> \pi(e_3, e_1)\\ \pi(e_1, e_3) &> \pi(e_4, e_3)\end{aligned} \tag{9}$$

which contradicts with equation (1). Hence, the first pure strategy of player 1 is not dominated by a pure strategy.

b) We suppose that it is weakly dominated by a mixed strategy $p = [p_1, p_2, p_3, p_4]^T$. Then the payoff of player 1 by choosing this mixed strategy is

$$\pi(p, q) = p^T A q \tag{10}$$

where $q = [q_1, q_2, q_3, q_4]$ is a strategy of the other player. Then according to the definition of "weakly dominated", we can get the following condition

$$p^T A q \geq e_1^T A q, \ \ \forall q \in Q \tag{11}$$
$$p^T A q > e_1^T q, \ \ \exists q \in Q \tag{12}$$

where Q is the strategy space of the other player. The above conditions can be written more clearly as follows

$$[p_1, p_2, p_3, p_4]A \geq [1, 2, 0, -2] \tag{13}$$
$$\|[p_1, p_2, p_3, p_4]A\| > \|[1, 2, 0, -2]\| \tag{14}$$
$$p_1 + p_2 + p_3 + p_4 = 1 \tag{15}$$

To solve the above underdetermined equation, we can use optimization based method such as linear programming. Here is a solution $p = [0, \frac{2}{3}, \frac{1}{3}, 0]^T$. Furthermore, we can valid that this mixed strategy actually strictly dominants the first pure strategy.

c) Yes. We have shown that the mixed strategy $p = [0, \frac{2}{3}, \frac{1}{3}, 0]^T$ strictly dominants the first pure strategy in previous question.

d) No, it is impossible. According to the Theorem 5.20 in [1], in every Nash equilibrium of a game in strategic form, the pure strategy strictly dominated by a mixed strategy is chosen by the player with probability 0. Since we have shown that the first pure strategy of player 1 is strictly dominated by a mixed strategy, it is impossible that it is at a Nash equilibrium.

e) Use the definition of strictly dominated, we can valid the following statements:

- The first pure strategic e_1 of player 1 is strictly dominated by a mixed strategy $p = [0, \frac{2}{3}, \frac{1}{3}, 0]^T$.
- The forth pure strategic e_4 of player 1 is strictly dominated by a mixed strategy $p = [0, \frac{2}{3}, \frac{1}{3}, 0]^T$.

Therefore, the set of Nash equilibrium is confined to the space of strategies e_2 and e_3, esulting in the following two-player payoff table:

1,1	1,4
4,1	3,3

It can be verified that

$$\pi(e_3, e_3) > \pi(e_2, e_3) \tag{16}$$
$$\pi(e_3, e_3) > \pi(e_3, e_2) \tag{17}$$

Hence, (e_3, e_3) is a pure strategy Nash equilibrium of the symmetric game.

f) According to the Theorem 5.51 in [1], for any two-player symmetric game, an evolutionary stable equilibrium is also a Nash equilibrium. Since we have shown that there is only one Nash equilibrium (e_3, e_3) of the given game in previous questions, we only need to check if it is evolutionary stable.

Please observe equation (16) and (17). It shows that (e_3, e_3) is a strict symmetric Nash equilibrium, then the conditions for ESS hold (Corollary 5.53 [1]). Hence, e_3 is an evolutionary stable strategy in this game.

Problem 3

Proof. To show that "can be", I only need to give an example to support the statement. Consider the following payoff matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

The replicator dynamics are

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} -x_1(x_1x_2 - x_2 + x_3(2x_2 + x_3)) \\ -x_2(x_1x_2 - 2x_3 + x_3(2x_2 + x_3)) \\ -x_3(x_1x_2 - x_3 + x_3(x_2 + x_3)) \end{bmatrix} \tag{19}$$

A ω-limit point is a point where the dynamics are zero. For the replicator dynamics (RP) means that $\pi(e^i, x) = \pi(x, x)$. Furthermore, if $\hat{x} \in \Delta$ and is a ω-limit point then $\hat{x} \in \Delta^{NE}$ (Preposition 3.5 in [2]).

It can be verified that in this game, (e_1, e_1) is the unique Nash equilibrium. However, e_1 is not Lyapunov stable. The proof is given below: Define a ball around e^1 with $r > 0$ as $B_r = \{x \in \Delta| \; ||x - e_1|| \leq r\}$. Furthermore, let $V(x) = \frac{1}{2}^T x$ be a Lyapunov function candidate. Then we have $\dot{V(x)} = x^T \dot{x}$. According to the Theorem 4.3 in [4] for $\forall x^* \in B_r$ if $\dot{V(x)} > 0$ then e^1 is unstable. If we check for the neighbor mixed strategy of e^1 $x_{test} = [1 - \epsilon \;\; \epsilon \;\; 0]^T | \epsilon > 0$ we get $\dot{V(x_{test})} = e^2 * (2 * e^2 - 3 * e + 1) > 0$ with ϵ sufficiently small. So e_1 is unstable. Futhermore, since all ω-limit point of all orbits x(t) in the interior of simplex converge to e_1 because it is the strongly dominant strategy. Hence, The state is unstable. This completes the proof.

□

Problem 4

Proof. Recall the definition of asymptotically stable set (Theorem 6.3 [2]):

Definition 2. *Suppose that $A \in C$ is a closed set. Then A is asymptotically stable if and only if there exists a neighborhood D of A and a continuous function $v : D \to R_+$ satisfying the following conditions:*

$$v(x) = 0 \quad \text{if and only if } x \in A, \tag{20}$$

$$v(\xi(t,x)) < v(x) \quad \text{if } x \notin A, t > 0, \text{and } \xi(s,x) \in D \ \ \forall s \in [0,t]. \tag{21}$$

For this theorem, there is a proof in [2] (Proposition 3.13). The main idea goes as follows:

Consider that $X \subset \delta$ is a evolutionary stable set (ES set). Then $\forall x \in X$, there exists W_x be a neighbor of x such that:

$$\pi(x,y) > \pi(y,y) \tag{22}$$

$\forall y \in W_x \setminus \{x\}$. Let $Q_x \in \Delta$ be the set of mixed strategies $y \in \Delta$ that assign positive probabilities to all pure strategies with positive probabilities assigned by x:

$$Q_x = \{y \in \Delta : C(x) \subset C(y)\} \tag{23}$$

Then $V_x = W_x \cap Q_x$ is a (relative) neighborhood of x on which the entropy function H_x is defined. After, we identify a neighborhood P of X which is a basin of attraction for X. Hence, $\forall x \in \Delta \exists \alpha_x \in \mathbb{R} : \alpha_x > 0$ such that the lower contour set $P_x = \{y \in Q_x : H_x(y) < \alpha_x\}$ is contained in the above neighborhood V_x. Let P be the union of all P_x. Then $P \subset \Delta$ is a neighborhood of X (relative to Δ). Furthermore, if $y \in P \setminus \{x\}$, then $y \in P_x$ for some $x \in X$, and $\dot{H}_x(y < 0)$ for each such x.

For each $y \in P$ let $X(y) = \{x \in X : C(x) \subset C(y)\}$ with the function H defined as:

$$H(y) = min(H_x(y)) \tag{24}$$

By Berge's maximum theorem, H is continuous. Furthermore, $H(y) \geq 0 \forall y \in P$ if and only if $y \in X$. As we defined previously $y \in P \setminus \{x\}$, consequently $\xi(x_0, t) \in P$ and $H(\xi(x_0,t)) < H(x_0 \forall t > 0)$ Hence, X is asymptotically stable.

□

References

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