DISC Course: Nonlinear Control Systems

Assignment 3

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Exercise 1 (13.12 from [1])

Solution: (a) The system dynamics can be written in the form $\dot{x} = f(x) + g(x)u$ where $x = [x_1, x_2, x_3]^T$ and

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 + x_3 \\ \delta(x) \end{bmatrix} \tag{1}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

Let $h(x) = y = x_1 + x_2$, then the system can be described as follows

$$y = x_1 + x_2 \tag{3}$$

$$\dot{y} = \dot{x}_1 + \dot{x}_2 = -x_1 + x_1 x_2 + x_2 + x_3 \tag{4}$$

$$\ddot{y} = -\dot{x}_1 + \dot{x}_1 x_2 + x_1 \dot{x}_2 + \dot{x}_2 + \dot{x}_3 \tag{5}$$

$$= (-1 + x_2)(-x_1 + x_1 x_2) + (x_1 + 1)(x_2 + x_3) + \delta(x) + u \tag{6}$$

It is obvious that the system has a relative degree of 2 in $\mathbb{R}^3$ ($L_g L_f h(x) = 1$). Hence, the system is input-output linearizable.

(b) Consider the following change of variables

$$\eta = \phi(x) \tag{7}$$

$$\xi_1 = h(x) = x_1 + x_2 \tag{8}$$

$$\xi_2 = L_f h(x) = -x_1 + x_1 x_2 + x_2 + x_3 \tag{9}$$

where $\phi(x)$ satisfies

$$L_g \phi(x) = \frac{\partial \phi(x)}{\partial x} g(x) = 0 \tag{10}$$

which gives a solution $\phi(x) = x_1$. Thus, consider the following coordinate transformation matrix

$$T(x) = \begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ -x_1 + x_1x_2 + x_2 + x_3 \end{bmatrix} \tag{11}$$

which is a diffeomorphism in the domain $\{x \in \mathbb{R}^3 | x \neq 0\}$. Furthermore, it transforms the system into the following normal form

$$\dot{\eta} = -\eta + \eta(\xi_1 - \eta) \tag{12}$$

$$\dot{\xi}_1 = \xi_2 \tag{13}$$

$$\dot{\xi}_2 = (-1 + x_2)(-x_1 + x_1x_2) + (x_1 + 1)(x_2 + x_3) + \delta(x) + u \tag{14}$$

(c) Consider the zero dynamics of the system $y(t) = 0$, which results in $\xi_1(t) = 0$ and $\xi_2(t) = 0$. Therefore, the dynamics of η is described as follows

$$\dot{\eta} = -\eta - \eta^2 \tag{15}$$

Let $\eta(0) = \eta_0 \neq -1$, then we can obtain the solution of the above differential equation

$$\eta(t) = -\frac{e^{C-t}}{e^{C-t} - 1}, \quad C = \log \frac{y_0}{y_0 + 1} \tag{16}$$

Thus, $\lim_{t\to\infty} \eta(t) = 1$. Therefore, the origin of the zero dynamics is asymptotically stable. Hence, the system is minimum phase.

(d) According to the system dynamics, we have

$$ad_f g(x) = -\begin{bmatrix} -1 + x_2 & x_1 & 0 \\ 0 & 1 & 1 \\ \frac{\partial \delta(x)}{\partial x_1} & \frac{\partial \delta(x)}{\partial x_2} & \frac{\partial \delta(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -\frac{\partial \delta(x)}{\partial x_3} \end{bmatrix} \tag{17}$$

$$\begin{aligned} ad_f^2 g(x) = &-\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial^2 \delta(x)}{\partial x_1^2} & \frac{\partial^2 \delta(x)}{\partial x_2^2} & \frac{\partial^2 \delta(x)}{\partial x_3^2} \end{bmatrix} \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 + x_3 \\ \delta(x) \end{bmatrix} \\ &- \begin{bmatrix} -1 + x_2 & x_1 & 0 \\ 0 & 1 & 1 \\ \frac{\partial \delta(x)}{\partial x_1} & \frac{\partial \delta(x)}{\partial x_2} & \frac{\partial \delta(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ -\frac{\partial \delta(x)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_1 \\ \star \\ \star \end{bmatrix} \end{aligned} \tag{18}$$

Consider the matrix $M = [g(x) \;\; ad_f g(x) \;\; ad_f^2 g(x)]$

$$M = \begin{bmatrix} 0 & 0 & x_1 \\ 0 & -1 & \star \\ 1 & -\frac{\partial \delta(x)}{\partial x_3} & \star \end{bmatrix} \tag{19}$$

whose rank is 3 if $x_1 \neq 0$ and we have $\det(M) = x_1$. Hence, the system is feedback linearizable in the domain $\{x \in \mathbb{R}^3 | x \neq 0\}$.

(e) The objective is to find an output $h(x)$ which satisfies

$$h(0) = 0 \tag{20}$$

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x) = 0 \tag{21}$$

$$L_g L_f h(x) = \frac{\partial L_f h(x)}{\partial x} g(x) = 0 \tag{22}$$

$$L_g L_f^2 h(x) = \frac{\partial L_f^2 h(x)}{\partial x} g(x) \neq 0 \tag{23}$$

It can be verified that $h(x) = x_1$ is a solution of the above equations, which results in

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = -x_1 + x_1 x_2 \tag{24}$$

$$L_f^2 h(x) = \frac{\partial L_f h(x)}{\partial x} f(x) = x_1(-1 + x_2)^2 + x_1(x_2 + x_3) \tag{25}$$

Thus, consider the following transformation matrix

$$T(x) = [h(x), L_f h(x), L_f^2 h(x)]^T = \begin{bmatrix} x_1 \\ -x_1 + x_1 x_2 \\ x_1(-1 + x_2)^2 + x_1(x_2 + x_3) \end{bmatrix} \tag{26}$$

which is a diffeomorphism in the domain $\{x \in \mathbb{R}^3 | x \neq 0\}$. Taking $z = T(x)$, the resulting transformed system is

$$\dot{z}_1 = z_2 \tag{27}$$

$$\dot{z}_2 = z_3 \tag{28}$$

$$\dot{z}_3 = x_1(u - \alpha(x)) \tag{29}$$

and the feedback control law is taken as $u = \alpha(x) + \frac{v}{x_1}$.

Exercise 2 (13.17 from [1])

Solution: The system dynamics can be written in the form $\dot{x} = f(x) + g(x)u$ where $x = [x_1, x_2, x_3]^T$ and

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 - x_1 x_3 \\ x_1 + x_1 x_2 - 2x_3 \end{bmatrix} \tag{30}$$

$$g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{31}$$

Thus, we can calculate

$$ad_f g(x) = [f(x), g(x)] = \begin{bmatrix} -1 \\ 1 \\ -x_1 \end{bmatrix} \tag{32}$$

$$[g(x), ad_f g(x)] = [0\ 0\ 0]^T \tag{33}$$

and

$$ad_f^2 g(x) = [f(x), ad_f g(x)] = \begin{bmatrix} -2 \\ 2 - x_3 - x_1^2 \\ 1 - 2x_1 \end{bmatrix} \tag{34}$$

Consider the matrix $M = [g(x)\ \ ad_f g(x)\ \ ad_f^2 g(x)]$

$$M = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 2 - x_3 - x_1^2 \\ 0 & -x_1 & 1 - 2x_1 \end{bmatrix} \tag{35}$$

where $\det(M) = 1$. Therefore, the system is feedback linearizable. The objective is to find an output $h(x)$ which satisfies

$$h(0) = 0 \tag{36}$$

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x) = 0 \tag{37}$$

$$L_g L_f h(x) = \frac{\partial L_f h(x)}{\partial x} g(x) = 0 \tag{38}$$

$$L_g L_f^2 h(x) = \frac{\partial L_f^2 h(x)}{\partial x} g(x) \neq 0 \tag{39}$$

It can be verified that $h(x) = x_1^2 - 2x_3$ is a solution of the above equations, which results in

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = -2(x_1^2 + x_1 - 2x_3) \tag{40}$$

$$L_f^2 h(x) = \frac{\partial L_f h(x)}{\partial x} f(x) = 4x_1^2 + 6x_1 - 2x_2 - 8x_3 \tag{41}$$

Thus, consider the following transformation matrix

$$T(x) = [h(x), L_f h(x), L_f^2 h(x)]^T = \begin{bmatrix} x_1^2 - 2x_3 \\ -2(x_1^2 + x_1 - 2x_3) \\ 4x_1^2 + 6x_1 - 2x_2 - 8x_3 \end{bmatrix} \tag{42}$$

which can be verified is a global diffeomorphism. Taking $z = T(x)$, we can obtain

$$\dot{z}_1 = z_2 \tag{43}$$

$$\dot{z}_2 = z_3 \tag{44}$$

$$\dot{z}_3 = -8x_1^2 + 2x_1x_3 - 16x_1 + 8x_2 + 16x_3 + 2u \tag{45}$$

$$= -4z_2 - 4z_3 + 2(u + x_1x_3) \tag{46}$$

Thus, the above equations can be written in the following form

$$\dot{z} = Az - 2B(u - \alpha(x)) \tag{47}$$

Hence, the feedback control law to globally stabilize the origin can be taken as $u = \alpha(x) + \frac{1}{2}K$, where K is a matrix such that $A - BK$ is Hurwitz.

Exercise 3 (13.24 from [1])

Solution: Since $V(\eta,\xi) = V_0(\eta) + \lambda\sqrt{\xi^T P\xi}$, we have

$$\dot{V}(\eta,\xi) = \frac{V_0(\eta)}{\partial\eta}\dot{\eta} + \frac{\lambda}{2\sqrt{\xi^T P\xi}}(\dot{\xi}^T P\xi + \xi^T P\dot{\xi}) \tag{48}$$

Note that P satisfies

$$P(A-BK) + (A-BK)^T P = -I \tag{49}$$

from which we can conclude that P is symmetric. Hence, we have

$$\begin{aligned}\dot{V}(\eta,\xi) &= \frac{V_0(\eta)}{\partial\eta}f(\eta,\xi) + \frac{\lambda}{2\sqrt{\xi^T P\xi}}(-\xi^T\xi + 2\xi^T PB\delta(z))\\ &= \frac{V_0(\eta)}{\partial\eta}f(\eta,0) + \frac{V_0(\eta)}{\partial\eta}[f(\eta,\xi) - f(\eta,0)] + \frac{\lambda}{2\sqrt{\xi^T P\xi}}(-\xi^T\xi + 2\xi^T PB\delta(z))\end{aligned} \tag{50}$$

Note that $f_0(\eta,\xi)$ is local Lipschitz and $\frac{V_0(\eta)}{\partial\eta}$ is bounded in some neighborhood of $\eta = 0$. We have

$$\frac{V_0(\eta)}{\partial\eta}[f(\eta,\xi) - f(\eta,0)] \leq c_1\|\xi\| \tag{51}$$

where c_1 is a positive constant. Furthermore, since

$$\frac{V_0(\eta)}{\partial\eta}f(\eta,0) \leq -W(\eta) \tag{52}$$

and

$$\|\delta(z)\| \leq k(\|\xi\| + W(\eta)) \tag{53}$$

We have

$$\begin{aligned}\dot{V}(\eta,\xi) &= \frac{V_0(\eta)}{\partial\eta}f(\eta,0) + \frac{V_0(\eta)}{\partial\eta}[f(\eta,\xi) - f(\eta,0)] + \frac{\lambda}{2\sqrt{\xi^T P\xi}}(-\xi^T\xi + 2\xi^T PB\delta(z))\\ &\leq -W(\eta) + c_1\|\xi\| - c_2\lambda\|\xi\| + c_3\lambda\|\delta\|\\ &\leq -W(\eta) + c_1\|\xi\| - c_2\lambda\|\xi\| + kc_3\lambda\|\xi\| + kc_3\lambda W(\eta)\\ &= -(1 - kc_3\lambda)W(\eta) - (c_2\lambda - c_1 - kc_3\lambda)\|\xi\|\end{aligned} \tag{54}$$

where c_1, c_2, c_3 are positive constants.

Now, in the Lyapunov function candidate $V(\eta,\xi)$, we take $\lambda = 2c_1/c_2$ and let $c^\star = \frac{\min\{1,c_1\}}{2c_3\lambda}$. Then, for $\forall k < c^\star$ we have

$$\begin{aligned}\dot{V}(\eta,\xi) &\leq -(1 - kc_3\lambda)W(\eta) - (c_2\lambda - c_1 - kc_3\lambda)\|\xi\|\\ &\leq -\frac{1}{2}W(\eta) - \frac{1}{2}c_1\|\xi\|\end{aligned} \tag{55}$$

Hence, the origin $z = 0$ is asymptotically stable for sufficiently small k.

References

[1] H.K. Khalil. *Nonlinear systems*. Prentice Hall, Upper Saddle River, USA, third edition, 2002.