Assignment 1

DISC Nonlinear Control Systems 2017-2018

Due date: Monday 12 February 2018

Note

- Please send the solutions to `{b.besselink, b.jayawardhana}@rug.nl` with the email subject "DISC NCS: Assignment 1".

Exercises

1. Exercise 4.15 from [1].
2. Exercise 4.21 from [1].
3. Exercise 4.25 from [1].

For convenience, the pages from [1] containing the relevant exercises are copied on the next pages.

References

[1] H.K. Khalil. *Nonlinear systems.* Prentice Hall, Upper Saddle River, USA, third edition, 2002.

4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\ dy + x_2^2/2 + \int_0^{x_3} h_2(y)\ dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

4.17 ([77]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + (1/2)x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat part (b) using $V(x) = (1/2)\left[x_2 + \int_0^{x_1} h(y)\ dy\right]^2 + \int_0^{x_1} g(y)\ dy$.

4.18 The mass–spring system of Exercise 1.12 is modeled by

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

Show that the system has a globally asymptotically stable equilibrium point.

4.19 Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q) = 0$ has an isolated root at $q = 0$.

(a) With $u = 0$, use the total energy $V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ as a Lyapunov function candidate to show that the origin ($q = 0,\ \dot{q} = 0$) is stable.

(b) With $u = -K_d \dot{q}$, where K_d is a positive diagonal matrix, show that the origin is asymptotically stable.

(c) With $u = g(q) - K_p(q - q^*) - K_d \dot{q}$, where K_p and K_d are positive diagonal matrices and q^* is a desired robot position in R^m, show that the point $(q = q^*,\ \dot{q} = 0)$ is an asymptotically stable equilibrium point.

4.20 Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $\lim_{t\to\infty} x(t)$ exists and equals one of these points.

4.21 ([81]) A gradient system is a dynamical system of the form $\dot{x} = -\nabla V(x)$, where $\nabla V(x) = [\partial V/\partial x]^T$ and $V : D \subset R^n \to R$ is twice continuously differentiable.

(a) Show that $\dot{V}(x) \leq 0$ for all $x \in D$, and $\dot{V}(x) = 0$ if and only if x is an equilibrium point.

(b) Take $D = R^n$. Suppose the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is compact for every $c \in R$. Show that every solution of the system is defined for all $t \geq 0$.

(c) Continuing with part (b), suppose $\nabla V(x) \neq 0$, except for a finite number of points $p_1, \ldots, p_r$. Show that for every solution $x(t)$, $\lim_{t\to\infty} x(t)$ exists and equals one of the points $p_1, \ldots, p_r$.

4.22 Consider the Lyapunov equation $PA + A^T P = -C^T C$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C), the vector $C \exp(At)x \equiv 0\ \forall\ t$ if and only if $x = 0$.

4.23 Consider the linear system $\dot{x} = (A - BR^{-1}B^T P)x$, where $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^T P x$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

(1) $Q > 0$.

(2) $Q = C^T C$ and (A, C) is observable; see the hint of Exercise 4.22.

4.24 Consider the system[36]

$$\dot{x} = f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$$

[36]This is a closed-loop system under optimal stabilizing control. See [172].

where $V(x)$ is a continuously differentiable, positive definite function that satisfies the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial x}f(x) + q(x) - \tfrac{1}{4}\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T = 0$$

$q(x)$ is a positive semidefinite function, $R(x)$ is a nonsingular matrix, and k is a positive constant. Using $V(x)$ as a Lyapunov function candidate, show that the origin is asymptotically stable when

(1) $q(x)$ is positive definite and $k \geq 1/4$.

(2) $q(x)$ is positive semidefinite, $k > 1/4$, and the only solution of $\dot{x} = f(x)$ that can stay identically in the set $\{q(x) = 0\}$ is the trivial solution $x(t) \equiv 0$.

When will the origin be globally asymptotically stable?

4.25 Consider the linear system $\dot{x} = Ax + Bu$, where (A, B) is controllable. Let $W = \int_0^\tau e^{-At}BB^Te^{-A^Tt}\,dt$ for some $\tau > 0$. Show that W is positive definite and let $K = B^TW^{-1}$. Use $V(x) = x^TW^{-1}x$ as a Lyapunov function candidate for the system $\dot{x} = (A - BK)x$ to show that $(A - BK)$ is Hurwitz.

4.26 Let $\dot{x} = f(x)$, where $f : R^n \to R^n$. Consider the change of variables $z = T(x)$, where $T(0) = 0$ and $T : R^n \to R^n$ is a diffeomorphism in the neighborhood of the origin; that is, the inverse map $T^{-1}(\cdot)$ exists, and both $T(\cdot)$ and $T^{-1}(\cdot)$ are continuously differentiable. The transformed system is

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x}f(x)\right|_{x=T^{-1}(z)}$$

(a) Show that $x = 0$ is an isolated equilibrium point of $\dot{x} = f(x)$ if and only if $z = 0$ is an isolated equilibrium point of $\dot{z} = \hat{f}(z)$.

(b) Show that $x = 0$ is stable (asymptotically stable or unstable) if and only if $z = 0$ is stable (asymptotically stable or unstable).

4.27 Consider the system

$$\dot{x}_1 = -x_2x_3 + 1, \qquad \dot{x}_2 = x_1x_3 - x_2, \qquad \dot{x}_3 = x_3^2(1 - x_3)$$

(a) Show that the system has a unique equilibrium point.

(b) Using linearization, show that the equilibrium point asymptotically stable. Is it globally asymptotically stable?

4.28 Consider the system

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 + x_1^2)x_2$$