Entrance to Math Competition

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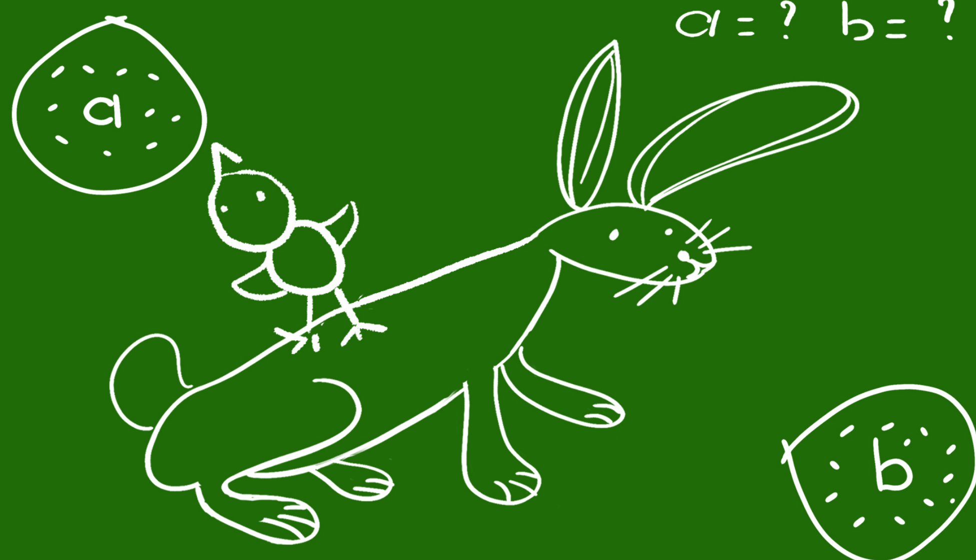

Mathematics

A Magnificent Key to the Future

Textbook for Pre-AMC Course

Last updated: September 2020

Contents

Preface

This entry level book is the first one in the math competition trilogy produced by Morning Star Institute. It covers basic concepts, ideas and techniques in a number of important topics from competition math. This book is suitable for elementary school students who have no previous experience in math competitions. Deeper contents and topics are included in two other books in the trilogy: *Tour Guide of Pre-college math, level A and level B*".

Topics can be put into 5 major categories: Arithmetic, Number theory, Geometry, Counting Problems/Probability and Algebra. Many examples and exercises in book are either chosen from or strongly inspired by problems in AMC (American Mathematics Competition) and various math contests in other countries. Math competition problems usually do not focus on tedious calculations like 1234×4567. Instead, they emphasize on important and interesting ideas and strategies. Let me briefly mention three of them.

(1) **Shortcut for calculations.** For example, the divisibility test to quickly tell whether a number is divisible by 9.

(2) **Consider problems from different perspectives.** For instance, in order to find the number of 4-digit numbers that contain at least one “2", it is more convenient to look at the opposite case: 4-digit numbers that do not contain 2.

(3) **Forward and backward applications of the formula.** Here is one of my favorite examples. Find the height of a right triangle on the hypotenuse with given side lengths. The solution is actually quite simple if we realize that the area of a right triangle can be computed either by the product of two legs or the product of the hypotenuse and its height. However, many kids fail to see this link and do not know how to solve it when they first encounter this type of questions.

It is true that most of techniques that students have learned in math competitions will not be used directly in their future careers. What is really important is the underlying logic thinking and ideas, which will benefit them through lifetimes. So it is extremely important for students to develop the habit of asking “WHY" in their learning experience. For instance, why the divisibility test for 9 is correct, why there is $\frac{1}{2}$ before the area formula of triangles, why the Pythagorean Theorem holds, etc. Of course, to rigorously answer most of such questions (called *"math proofs"*) requires more advanced knowledge in algebra and will be presented in Level A and Level B books.

This book is mainly based on lectures notes of a 30-week Pre-AMC class that I have taught at Morning Star Institute (Irvine) from 2019-2020. The enthusiastic participation of students and support of parents are very encouraging. Moreover, the experience of teaching my own kid (Austin) math at home also helps me a lot in understanding how kids' math ability develops and when they usually get stuck. Like many parents, my main challenge of teaching at home is to control the temper when things do not go as wished.

Here I would like to thank Dr. Hongwei Gao, Ms. Zhen Tian and Mr. Sean Zhang for carefully proofreading the draft. In particular, Dr. Gao has drawn a lot of math figures for it. I also want to thank Dr. Shuhao Cao for help in issues related to latex.

Gratitude also goes to Ms. Van Hai Van who has designed and drawn all covers of the trilogy. In addition, I also want to thank Maggie Shen for drawing some pictures in the book.

It is very likely that this version still contains typos and errors. We will keep updating the book online. Any comment and suggestion to improve the book will be greatly appreciated. Please email comments and suggestion to yifengy@uci.edu.

Moreover, readers may visit http://www.morningstarinstitute.org/ for 8 sets of comprehensive practices. Each set has 20 problems covering various topics from this book. Solution videos are available on the same website, which can be viewed as practice-based crash courses.

When first reading, readers can omit contents and problems marked with ⋆⋆.

Yifeng Yu

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Chapter 1

Arithmetic

1.1 Basic Arithmetic Rules

1.

$$\boxed{a+b=b+a}$$

For example, $5+3=3+5=8$.

2.

$$\boxed{(a+b)+c=a+(b+c)}$$

For example, $(8+4)+6=8+(4+6)=18$.

3.

$$\boxed{a+(b-c)=(a+b)-c}$$

For example, $10+(5-3)=(10+5)-3=12$

4.

$$\boxed{a-(b+c)=a-b-c}$$

For example $10-(5+3)=10-5-3=2$.

5.

$$\boxed{a-(b-c)=a-b+c}$$

For example $10-(5-3)=10-5+3=8$.

6.

$$a\times b=b\times a$$

For example, $5\times 3=3\times 5=15$.

7.

$$\boxed{(a \times b) \times c = a \times (b \times c) = (a \times c) \times b}$$

For example, $(5 \times 3) \times 2 = 5 \times (3 \times 2) = (5 \times 2) \times 3 = 30$.

8.

$$\boxed{(a + b) \times c = a \times c + b \times c}$$

For example, $(5 + 3) \times 2 = 5 \times 2 + 3 \times 2 = 16$.

9.

$$\boxed{(a - b) \times c = a \times c - b \times c}$$

For example, $(5 - 3) \times 2 = 5 \times 2 - 3 \times 2 = 4$.

10.

$$\boxed{(a \times b) \div c = (a \div c) \times b}$$

For example, $(12 \times 15) \div 6 = (12 \div 6) \times 15 = 30$. For convenience, we may also write division as

$$(12 \times 15) \div 6 = \frac{12 \times 15}{6} = \frac{\cancel{12}^{=2} \times 15}{\cancel{6}} = \frac{2 \times 15}{1} = 30$$

11.

$$\boxed{(a \div b) \div c = a \div (b \times c)}$$

For example, $(10 \div 5) \div 2 = 10 \div (5 \times 2) = 10 \div 10 = 1$.

1.2 Some Common Arithmetic Tricks

Example 1 Calculate

$$998 + 1413 + 9989.$$

- **Solution:** We use a shortcut. Note that

$$998 = 1000 - 2 \quad \text{and} \quad 9989 = 10000 - 11.$$

Hence

$$\begin{aligned} 998 + 1413 + 9989 &= 1000 - 2 + 1413 + 10000 - 11 \\ &= 1000 + 1413 + 10000 - 13 \\ &= 12413 - 13 = \boxed{12400.} \end{aligned}$$

Example 2

$$16 + 106 + 1006 + 10006 + 100006.$$

- **Solution:** Note that

$$16 = 10 + 6, \ 106 = 100 + 6, \ 1006 = 1000 + 6$$

$$10006 = 10000 + 6, \ 100006 = 100000 + 6.$$

Since $10 + 100 + 1000 + 10000 + 100000 = 111110$ and $6 \times 5 = 30$, we have that

$$16 + 106 + 1006 + 10006 + 100006 = 111110 + 30 = \boxed{111140.}$$

Example 3 Calculate

$$125 + (375 - 136).$$

- **Solution:** Note that

$$125 + (375 - 136) = (125 + 375) - 136 = 500 - 136 = \boxed{364.}$$

Example 4 Calculate

$$195 + 99 \times 95.$$

- **Solution:** Since

$$195 = 95 + 100,$$

$$\begin{aligned} 195 + 99 \times 95 = 100 + 95 + 99 \times 95 &= 100 + (95 + 99 \times 95) \\ &= 100 + (1 + 99) \times 95 = 100 + 9500 = \boxed{9600.} \end{aligned}$$

Example 5 Let

$$A = 1 + 3 + 5 + 7 + \cdots + 31$$

and

$$B = 2 + 4 + 6 + 8 + \cdots + 32.$$

What is $B - A$?

- **Solution:** Note that

$$\begin{aligned} B - A &= (2 + 4 + 6 + 8 + \cdots + 32) - (1 + 3 + 5 + 7 + \cdots + 31) \\ &= 2 + 4 + 6 + 8 + \cdots + 32 - 1 - 3 - 5 - 7 - \cdots - 31 \\ &= \underbrace{(2 - 1) + (4 - 3) + (6 - 5) + \cdots + \cdot (32 - 31)}_{\text{16 pairs}} \\ &= \boxed{16.} \end{aligned}$$

Example 6 Compute

$$4 \times 13 \times 25.$$

- **Solution:** Note that

$$4 \times 13 \times 25 = (4 \times 25) \times 13 = 100 \times 13 = \boxed{1300.}$$

Example 7 Compute

$$(35 \times 11) \div 7.$$

- **Solution:** Note that

$$(35 \times 11) \div 7 = (35 \div 7) \times 11 = 5 \times 11 = \boxed{55.}$$

Or, we write

$$\frac{35 \times 11}{7} = \frac{\cancel{35}^{=5} \times 11}{\cancel{7}} = \frac{5 \times 11}{1} = \boxed{55.}$$

Example 8: (Gauss' Trick) Calculate

$$S = 1 + 2 + 3 + \cdots + 19 + 20.$$

- **Solution:**

Method 1: We can do paring as (10 pairs)

$$21 = 1 + 20 = 2 + 19 = 3 + 18 = \cdots = 10 + 11.$$

$$S = 1 + 2 + 3 + \cdots + 18 + 19 + 20 = (20 + 1) \times 10 = \boxed{210.}$$

Method 2: Let us view this process from a similar perspective. Note that

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + 18 + 19 + 20 \\ S &= 20 + 19 + 18 + \cdots + 3 + 2 + 1 \end{aligned} \tag{1}$$

Accordingly,

$$2S = \underbrace{(1+20) + (2+19) + (3+18) + \cdots + (18+3) + (19+2) + (20+1)}_{\text{20 pairs}}.$$

Here $2S = 2 \times S$. When letters are involved, we often omit the multiplication sign $\times$. Therefore

$$2S = 21 \times 20 = 420 \quad \Rightarrow \quad \boxed{S = 210.}$$

We will revisit Gauss' trick in Chapter 6.

1.3 Number Puzzles

Example 9: [AJHSME.1986.8] In the product shown, B is a digit. The value of B is

$$\begin{array}{rr} & \text{B2} \\ \times & \text{7B} \\ \hline & 6396 \end{array}$$

- **Solution:** Since the unit digit of $2 \times B$ is 6, we deduce that

$$B = 3 \quad \text{or} \quad 8.$$

It is easy to check that $\boxed{B = 8.}$

Example 10 A number is called a *square* if it is the product of two equal whole numbers. For example, $1 = 1 \times 1$, $4 = 2 \times 2$ and $9 = 3 \times 3$ are all squares. We usually write $1 = 1^2$, $4 = 2^2$ and $9 = 3^2$. The number 1296 is a square. Find the whole number A such that

$$A^2 = A \times A = 1296.$$

- **Solution:** Notice that

$$30 \times 30 = 900 \quad \text{and} \quad 40 \times 40 = 1600.$$

Since

$$900 < 1296 < 1600,$$

A must be a number between 30 and 40. By considering the last digit, A is either 34 or 36 since only 4×4 and 6×6 will give a last digit of 6. We can quickly check that $\boxed{A = 36.}$

Example 11 Austin wants to add 69 and another number. By mistake, he writes 69 as 96 and obtains a sum of 120. What should be the correct sum?

- **Solution:** Since

$$96 + \boxed{\text{the other number}} = 120,$$

the other number is

$$120 - 96 = 24.$$

Hence the correct answer is

$$24 + 69 = \boxed{93.}$$

1.4 Defined Operations

Example 12 Define

$$a \star b = 2 \times a + b = 2a + b.$$

Find $3 \star 7$ and $(1 \star 1) \star 1$.

Solution: By the given definition,

$$3 \star 7 = 2 \times 3 + 7 = \boxed{13.}$$

As for $(1 \star 1) \star 1$, we first calculate $1 \star 1$

$$1 \star 1 = 2 \times 1 + 1 = 3.$$

So

$$(1 \star 1) \star 1 = 3 \star 1 = 2 \times 3 + 1 = \boxed{7.}$$

Example 13 In the above problem, if

$$a \star 3 = 13 \quad \text{and} \quad 4 \star b = 11,$$

Find a and b.

- **Solution:**

1. Since

$$13 = a \star 3 = 2a + 3,$$

we deduce that $\boxed{a = 5.}$

2. Since

$$11 = 4 \star b = 4 \times 2 + b = 8 + b,$$

we have that

$$\boxed{b = 3.}$$

1.5 Exercises

Problem 1 Use short-cut to calculate

$$773 + 368 + 227$$

$$365 - 56 - 44$$

$$623 - 159 - 41 - 23$$

$$125 \times 13 \times 8$$

$$(99 + 999) \times 9$$

$$11 + 22 + 33 + 44 + 55 + 66 + 77 + 88 + 99$$

$$(49 \times 12) \div 7$$

Problem 2 Let

$$A = 25 \times 25$$

and

$$B = 4 \times 11 \times 4.$$

What is

$$A \times B?$$

Problem 3 **[AMC10A.2011.4]**

$$X = 10 + 12 + 14 + \cdots + 100$$

and

$$Y = 12 + 14 + 16 + \cdots + 102.$$

What is

$$Y - X?$$

Problem 4 Observe the pattern

$$2 + 4 = 2 \times 3, \quad 2 + 4 + 6 = 3 \times 4, \quad 2 + 4 + 6 + 8 = 4 \times 5, \quad \cdots$$

Here $(2,3)$, $(3,4)$ and $(4,5)$ are pairs of consecutive positive integers. Can you deduce that

$$2 + 4 + 6 + \cdots + 98 + 100$$

is the product of which two consecutive positive integers?

Problem 5 Use Gauss' trick to compute

$$1+3+5+7+\cdots+23.$$

Problem 6 When Ben was copying down a 5-digit number $\overline{6ab4c}$, he switched 6 and 4 by mistake. What is the difference between these two numbers?

Problem 7 Use 1, 2, 3, 4, 5, 6 to form two 3-digit numbers. Each number is used exactly once. What is the largest possible difference between the larger and the smaller one? What is the least possible difference?

Problem 8 Use 7 number flashcards $\boxed{0}$, $\boxed{0}$, $\boxed{0}$, $\boxed{0}$, $\boxed{6}$, $\boxed{6}$, $\boxed{6}$ to form 7-digit numbers. Arrange them from the largest to the least. What is the fifth largest one ("6" can also be used as "9")?

Problem 9 **[AJHSME.1989.14]** When placing each of the digits $2, 4, 5, 6, 9$ in exactly one of the boxes of this subtraction problem, what is the smallest difference that is possible?

$$\begin{array}{rccc} & \square & \square & \square \\ - & & \square & \square \\ \hline \end{array}$$

Problem 10 If the product of two consecutive odd numbers is 1295, what is the smaller odd number?

Problem 11 What number should we put on the right hand side $\boxed{?}$ to make number sentence true

$$15\times15\times15\times15 = 5\times5\times5\times5\times\boxed{?}\,?$$

Problem 12 For $n = 1, 2, 3, \ldots$, let

$$n! = n\times(n-1)\times(n-2)\times\cdots\times3\times2\times1.$$

For example, $3! = 3\times2\times1 = 6$ and $4! = 4\times3\times2\times1 = 24$. Find the number N such that

$$10! = 8!\times N.$$

Problem 13 Given a 7-digit number

□□□□□□□

From left to right, sums of two adjacent digits are

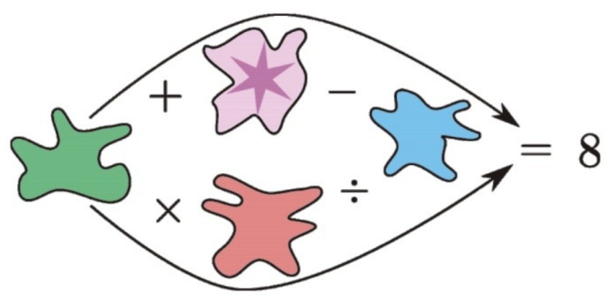
$$6,\ 5,\ 4,\ 3,\ 2,\ 1.$$

What is the sum of all possible 7-digit numbers which satisfy the above requirement?

Problem 14 Write a number in the triangle that will make the answer 45.

$$\triangle ? \to \boxed{\times 3} \to \boxed{\div 2} \to \boxed{+15} = 45.$$

Problem 15 [**Kangaroo Math**] Each of the spots covers one of the numbers 1,2,3,4 or 5 so that both of the calculations following the arrow are correct. What number is covered by the spot with the star?



Problem 16 [**Kangaroo Math**] John does a calculation using the digits A, B, C and D. Which digit is represented by B?

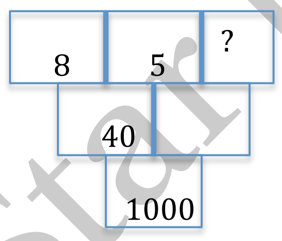
$$\begin{array}{cccc} & A & B & C \\ + & C & B & A \\ \hline D & D & D & D \end{array}$$

Problem 17 [**AMC8.2003.14**] In this addition problem, each letter stands for a different digit.

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

If $T = 7$ and the letter O represents an even number, what is the only possible value for W?

Problem 18 **Modification of [AMC8.2013.6]** The number in each box below is the product of the numbers in the two boxes that touch it in the row above. For example, $40 = 8 \times 5$. What is the missing number in the top row?



Problem 19 [**Australian Math Competition**] When a number has the digit 2 put at both ends, its value increases by 2785. What is the original number?

For example, if the original number is 34, the new number is 2342 and the value increases by $2342 - 34 = 2308$.

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Problem 20 [**Kangaroo Math**] Tim, Tom and Jim are triplets, while their brother Carl is 3 years younger. Which of the following numbers could be the sum of the ages of the four brothers?

(A) 53 (B) 54 (C) 56 (D) 59 (E) 60

Problem 21 **Modification of [AMC8.2000.20]** You have nine coins: a collection of pennies, nickels, dimes, and quarters having a total value of $0.83, with at least one coin of each type. How many dimes must you have?

Problem 22 Use 1,1,2,2,3,3,4,4 to form a 8-digit number so that

(1) there is one digit between two 1;

(2) there are two digits between two 2;

(3) there are three digits between two 3;

(4) there are four digits between two 4.

There are 2 such kind of 8-digit numbers. One of them is 23421314. What is the other one?

Problem 23 A *cubic* number is a number that is the product of three equal whole numbers. For example, $1 = 1\times1\times1$, $8 = 2\times2\times2$ and $27 = 3\times3\times3$ are all cubic numbers. We know 1728 is a cubic number. Find the whole number A such that

$$A \times A \times A = 1728.$$

We usually denote $A \times A \times A$ by A^3.

Problem 24 **[AJHSME.1998.2]** If $\begin{array}{c|c} \mathrm{a} & \mathrm{b} \\ \hline \mathrm{c} & \mathrm{d} \end{array} = \mathrm{a}\cdot\mathrm{d} - \mathrm{b}\cdot\mathrm{c}$, what is the value of $\begin{array}{c|c} 3 & 4 \\ \hline 1 & 2 \end{array}$?

Problem 25 Define

$$a\#b = (a \times b) \div (a+b) = \frac{ab}{a+b}.$$

(1) Calcuate 4#4;

(2) What is the value of x if

$$x\#x = 5.$$

Problem 26 **Modification of [AMC8.2016.10]** Suppose that

$$a * b = 3a - b.$$

(1) What is

$$6 * 2;$$

(2) What is

$$(1 * 2) * 3;$$

(3) Find A such that

$$2 * A = 1;$$

(4) What is the value of x if

$$2 * (5 * x) = 1$$

Problem 27 Define $\boxed{N}$ to be the sum of digits of N. For example, $\boxed{123} = 1+2+3 = 6$. Calculate $\boxed{\boxed{5763}}$.

Chapter 2

Averages

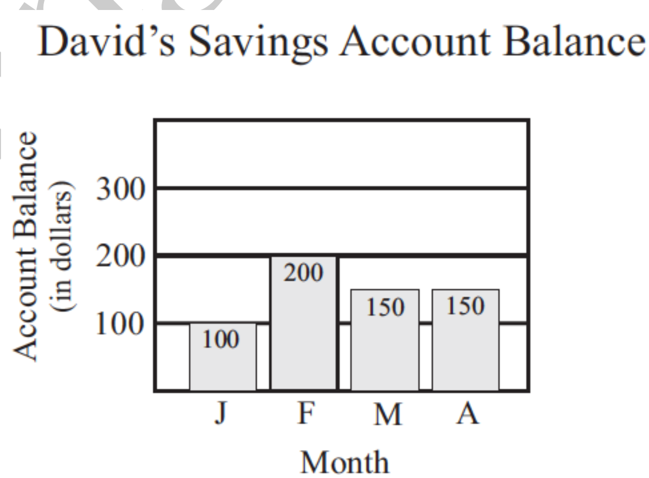
2.1 Basic Concepts

Example 1 Abel received grades of 90, 85, and 74 on three tests. What is his average score?

- **Solution:** The average is

$$\frac{\text{total score}}{\text{the number of tests}} = \frac{90+85+74}{3} = \boxed{83.}$$

Example 2: [Mathcount Problem] According to the graph, what is the average monthly balance, in dollars, of David's savings account during the four-month period shown?



• **Solution:** The average monthly balance is

$$\frac{\text{total balance}}{\text{the number of months}} = \frac{100+200+150+150}{4} = \frac{600}{4} = 150.$$

Example 3 There are 5 kids and 4 adults on a bus. The average age of 5 kids is 12. The average age of the remaining 4 adults is 30. What is the average age of these 9 people?

• **Solution:** Since the average age of kid is

$$12 = \frac{\text{total age of kids}}{\text{the number of kids}} = \frac{\text{total age of kids}}{5},$$

we deduce that the total age of 5 kids is

$$12 \times 5 = 60.$$

Similarly, the total age of 4 adults is

$$30 \times 4 = 120.$$

So the total age of these 9 people is

$$60 + 120 = 180.$$

Accordingly, the average age is

$$\frac{180}{9} = \boxed{20.}$$

Example 4 A math competition was held in Euclid elementary school. The average score of 4 students is 80. After counting one more person, the average score of these 5 students becomes 82. What is the score of the fifth person?

• **Solution:** The total score of the first 4 students is

$$80 \times 4 = 320.$$

The total score of the 5 students is

$$82 \times 5 = 410.$$

Hence the score of the fifth student is

$$410 - 320 = \boxed{90.}$$

Example 5 Elizabeth is learning violin. On the average, she plans to practice at least 30 minutes per day. In the first week of August, She practiced 35 minutes everyday from Monday to Thursday. On Friday, she practiced 30 minutes. On Saturday, she went to visit her grandparents and didn't practice. What is the minimum number of minutes did she need to practice on Sunday in order to meet her goal in this week?

- **Solution:** In total, she plans to practice at least

$$30 \times 7 = 210 \text{ minutes}$$

in a week. From Monday to Saturday, she has practiced

$$35 \times 4 + 30 = 170 \text{ minutes}$$

Hence she needs to practice at least

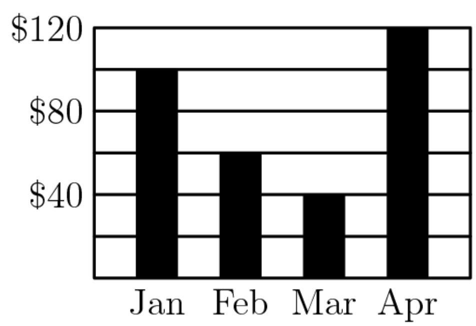
$$210 - 170 = \boxed{40} \text{ minutes}$$

on Sunday.

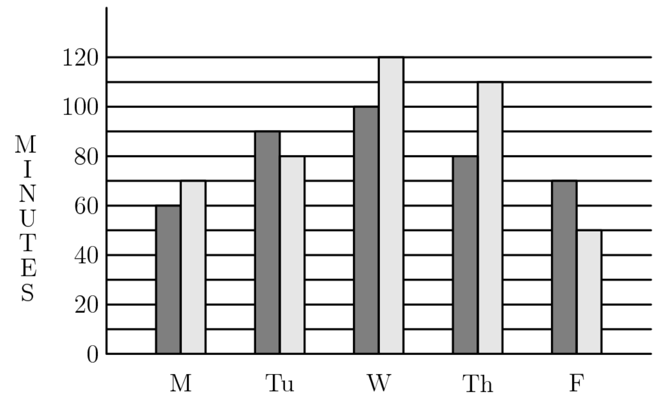
2.2 Exercises

Problem 1 What is the average of the first 10 odd numbers $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$?

Problem 2 **[AMC8.2008.8]** Candy sales from the Boosters Club from January through April are shown. What were the average sales per month in dollars?



Problem 3 **[AMC8.2011.11]** The graph shows the number of minutes studied by both Asha (black bar) and Sasha (grey bar) in one week. On the average, how many more minutes per day did Sasha study than Asha?



Problem 4 **[AMC8.2016.3]** Four students take an exam. Three of their scores are 70, 80, and 90. If the average of their four scores is 70, then what is the remaining score?

?, 70, 80, 90

Average=70

Problem 5 The average (mean) of 20 numbers is 10, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

Problem 6 **[AMC8.2004.9]** The average of the five numbers in a list is 54. The average of the first two numbers is 48. What is the average of the last three numbers?

Problem 7 **[AMC8.2007.7]** The average age of 5 people in a room is 30 years. An 18-year-old person leaves the room. What is the average age of the four remaining people?

Problem 8 **[AMC8.2008.10]** The average age of the 6 people in Room A is 40. The average age of the 4 people in Room B is 25. If the two groups are combined, what is the average age of all the people?

Problem 9 The average of 5 number is 20. If we change one of them to 4, the average becomes 14. What is the number that has been changed?

Problem 10 Mrs. Smith is leading an AMC8 competition team consisting of 4th graders. One day, the team attended a competition. AMC8's full score is 25. After test results came out, Mrs Smith found that the average score of all girls in the team is 20 and the average score of all boys in the team is 16. Which of the following could be the average score of all team members?

(A) 14 (B) 15 (C) 19 (D) 21 (E) 22

Problem 11 In the above problem, find one combination of boys and girls so that all requirements of averages scores (boys, girls and total) are satisfied.

Problem 12 Divide numbers from 1 to 100 into 10 groups. If the average of the numbers in each group is the same, what is the average?

Chapter 3

Travel Problems

3.1 Baisc Concepts

The central formula

$$\boxed{\text{Average speed} = \frac{\text{Total travel distance}}{\text{Total travel time}}.}$$

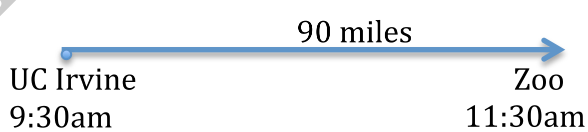
Equivalently,

$$\text{Total travel time} = \frac{\text{Total travel distance}}{\text{Average speed}}$$

and

$$\text{Total travel distance} = (\text{Average speed}) \times (\text{Total travel time}).$$

Example 1 Jack drove from UC Irvine to San Diego Zoo. He left the campus at 9:30 am and arrived at the Zoo at 11:30 am. Given that the distance from UC Irvine to San Diego Zoo is around 90 miles, what is his average speed?



- **Solution:** The total distance is

$$90 \text{ miles}$$

and the total travel time is (from 9:30 am to 11:30am)

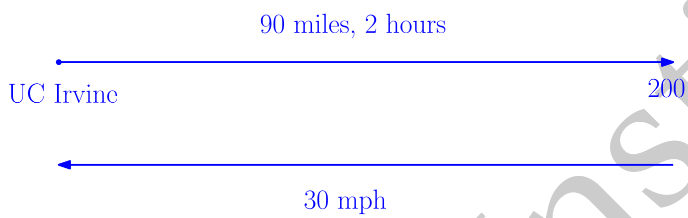
$$2 \text{ hours}.$$

Hence the average speed is

$$\frac{90}{2} = \boxed{45} \text{ mph.}$$

Here *mph* stands for miles per hour.

Example 2 On the return trip, due to heavy traffic, Jack only averaged 30 miles per hour along the same route. What is the average speed for **the round trip**?



• **Solution:** It is tempting to think that the average speed of the round trip is

$$\frac{45+30}{2} = 37.5 \text{ mph.}$$

However,

this is **wrong**!

The correct way is to apply the formula. The total travel distance of the round trip is

$$90+90 = 180.$$

Also, Jack spent

$$\frac{90}{30} = 3 \text{ hours}$$

on his way back. So the total travel time is

$$2+3 = 5 \text{ hours.}$$

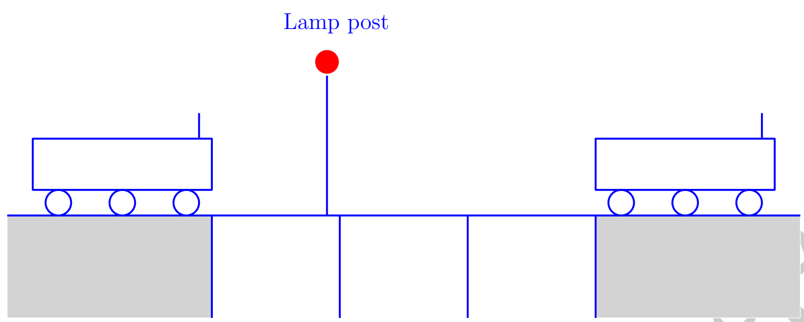
Therefore, the average speed of the round trip is

$$\frac{180}{5} = \boxed{36} \text{ mph.}$$

Example 3 A train crosses a bridge in 30 seconds and a lamp post on the bridge in 10 seconds. The speed of a train is 15m/s. Here m/s: meters per second.

(1) What is the length of the train?

(2) What is the length of the bridge?



• **Solution:**

(1) Since the length of a lamppost can be ignored, the length of the train is

$$15 \times 10 = \boxed{150} \text{ meters.}$$

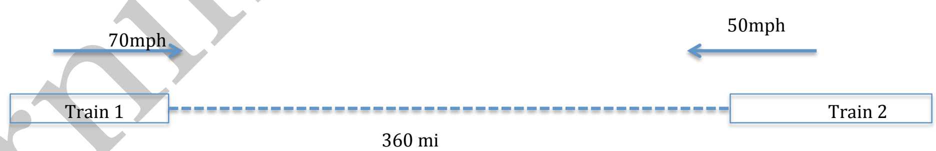
(2) In order to pass the bridge, the train needs to travel

$$\underbrace{150}_{\text{length of the train}} + (\text{the length of the bridge}) = 15 \times 30 = 450 \text{ meters.}$$

Hence the length of the bridge is

$$450 - 150 = \boxed{300} \text{ meters.}$$

Example 4: [Meet Problem] Two trains, 360 miles apart are traveling towards each other with one with a constant speed of 70 mph and other 50 mph. How long does it take for these two trains to meet?



• **Solution:** Clearly, the distance between these two trains decreases by

$$70+50 = 120 \text{ miles}$$

every hour. Note that

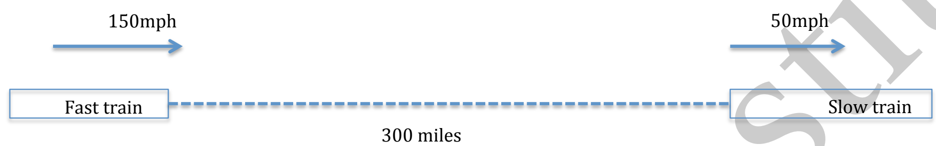
$$360 = 120 \times 3.$$

Hence it takes $\boxed{3}$ hours for them to meet.

Example 5: [Chase Problem] A slow train leaves the train station traveling 50 miles per hour to the east. A fast train leaves 6 hours later traveling 150 miles per hour along the same direction. How long will it take the fast train to catch up with the slow train?

• **Solution:** When the fast train starts, the distance between the fast train and the slow train is

$$6 \times 50 = 300 \text{ miles.}$$



The distance between these two trains decrease by

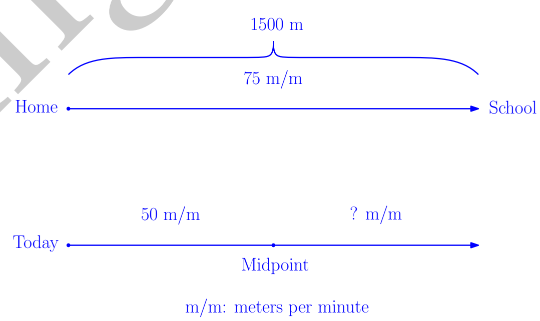
$$150 - 50 = 100 \text{ miles}$$

every hour. Since

$$300 = 100 \times 3,$$

it takes the fast train $\boxed{3}$ hours to catch up with the slow train.

Example 6: Modification of [AMC8.2014.17] George walks 1500 meters to school. He leaves home at the same time each day, walks at a steady speed of 75 meter per minute, and arrives just as school begins. Today he was distracted by the pleasant weather and walked the first 750 meters at a speed of only 50 meters per minute. At how many meters per minute must George run the last 750 meters in order to arrive just as school begins today?



• **Solution:**

Step 1: We deduce that George can spend

$$1500 \div 75 = 20 \text{ minutes}$$

on his way to school in order to arrive on time.

Step 2: For the first 750 meters, he has spent

$$750 \div 50 = 15 \text{ minutes.}$$

Step 3: George has only $20 - 15 = 5$ minutes to finish the remaining 750 minutes. Hence his speed should be

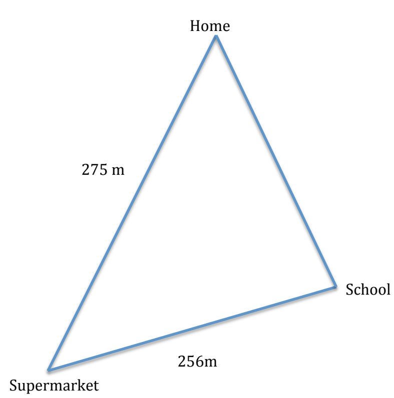
$$750 \div 5 = \boxed{150} \text{ meters per minute.}$$

3.2 Exercises

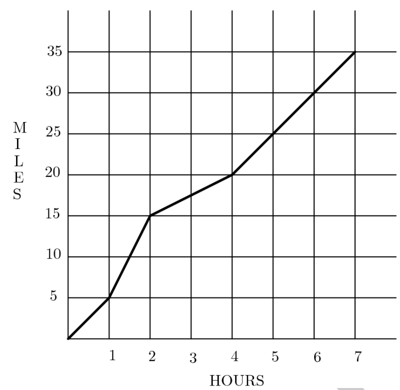
Problem 1 Craig drove from his home to Disneyland. He left home at 9 am and arrived at Disneyland at 10 am. On his way back, due to heavy traffic, his average speed was only 30 mph. If the distance from his home to Disneyland is 60 miles, what is his average speed in this round trip: Home–>Disneyland–>Home ?

Problem 2 Jack went up a hill along a 3-kilometer road and came down along the same road. If his average speed of going up is 1500 meters per hour and his average speed of coming back down is 3 kilometers per hour, what is his average speed (meters per hour) of up and down the hill?

Problem 3 It takes Mary 6 minutes to work from home to school. With the same speed, she needs 9 minutes to go from home to supermarket first and then to school. What is the distance between her home and the school?



Problem 4 **[AMC8.2011.9]** Carmen takes a long bike ride on a hilly highway. The graph indicates the miles traveled during the time of her ride. What is Carmen's average speed for her entire ride in miles per hour?



Problem 5 The speed of a train is 15m/s. If it takes 20 seconds for the train to pass a person standing on the platform, what is the length of the train, in meters?

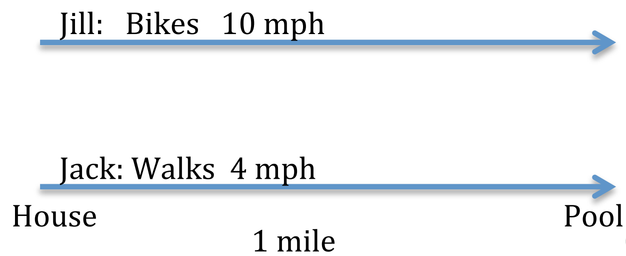
m/s: meters per second.

Problem 6 In the above problem, if the length of the platform is 450 meters, how many seconds does it take the train to completely pass the platform?

Problem 7 It takes a train 1 minute to completely pass a long tunnel. If the length of the train is 200 meters and its speed is 15 meters per second, how long is the tunnel, in meters?

Problem 8 **[AMC 8.2016.4]** When Cheenu was a boy he could run 15 miles in 3 hours and 30 minutes. As an old man he can now walk 10 miles in 4 hours. How many minutes longer does it take for him to walk a mile now compared to when he was a boy?

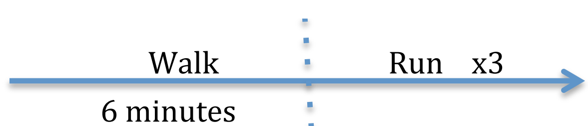
Problem 9 **[AMC 8.2015.3]** Jack and Jill are going swimming at a pool that is one mile from their house. They leave home simultaneously. Jill rides her bicycle to the pool at a constant speed of 10 miles per hour. Jack walks to the pool at a constant speed of 4 miles per hour. How many minutes before Jack does Jill arrive?



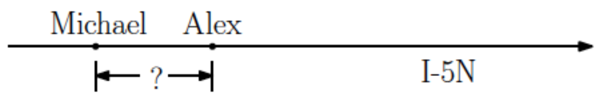
Problem 10 [**Frog in the Well Problem**]: A frog is at the bottom of a 30 meter well. Each day it summons enough energy for one 3 meter leap up the well. Exhausted, then it hangs there for the rest of the day. At night, while it is asleep, it slips 2 meters backwards. How many days does it take the frog to escape from the well?

Figure 3.1: Picture drawn by Maggie Shen

Problem 11 [**AMC8.2005.10**]: Joe had walked half way from home to school when he realized he was late. He ran the rest of the way to school. He ran 3 times as fast as he walked. Joe took 6 minutes to walk half way to school. How many minutes did it take Joe to get from home to school?



Problem 12 Alex and Michael live in two cities within Orange County. One day, they drive separately along Highway I-5 northbound toward San Francisco. They enter I-5 around the same time and Alex is ahead of Michael. The average speed of Alex is 65 mph and the average speed of Michael is 70mph. If Michael catches Alex in 2 hours, what is the distance between them when they enter I-5?



Problem 13 Two cars, car A and car B, are 600 miles apart. They start at the same time and drive toward each other. Car A travels at a speed of 70 mph and car B travels at 80 mph. How far is car A away from its starting point when these two cars meet?

Problem 14 Everyday William walks from home to school which is 800 meters away. The school starts at 8:00am. One day, William left home at 7:40am. For the first 10 minutes, he was distracted by a parade and walked at a speed of only 20 meters per minute. In order to get to school on time, how fast should William walk in the remaining part, in meters per minute?

The following problem 15-18 are based on real stories from the documentary *"Most Dangerous Ways To School"***.**

Problem 15 11 year-old Jack and his family live on a tiny floating island in Lake Titicaca (Peru). Every morning, he rows a small reed boat to school and comes back in the afternoon. On her way to school, his speed is 3mph. he leaves school at 2pm. Due to the bad weather over the lake, he arrives at home at 4pm. If the distance from his home to school is 3 miles, what is his average speed, in mph, during the entire trip?

Figure 3.2: Figure from Internet

Problem 16 In the above problem, Jack leaves home at 7am. The school starts at 8am. After rowing the boat for half an hour, Jack took a detour into a nearby reed field and spent 10 minutes to set up a trap to catch wild ducks before he went to the route to school. In order to arrive school on time, how fast does Jack need to row his boat for the remaining 20 minutes, in meters per minute?

Problem 17 Amanda and her family live a valley in Philippine. Everyday, he has to walk to school. Her speed is 2mph. It usually takes her 3 hours on the way to school. One day, on her way back, she was lucky that an acquaintance of her mother happened to be in front of the school and was willing to give her ride to home on motorcycle. If the speed of motorcycle is 30 mph and they left the school at 2pm, when will Amanda arrive home?

(Hint: 30mph=? miles per minute)

Problem 18 Altantsetseg and her family live in a village in Mongolia. Everyday, her father takes her to school on motorcycle. Due to the icy road in the winter, the speed of the motorcycle is only 18mph. Ganbold lives in the same village. As a boy, he rides a horse to school with a speed of 6mph. If it takes 20 minutes for Altantsetseg and her father to arrive at school, how long will it take Ganbold to reach school?

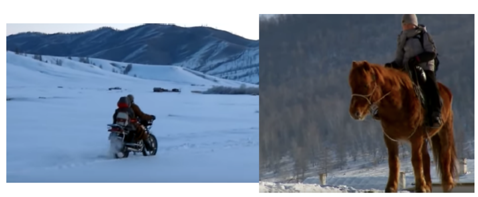


Figure 3.3: Figure from Internet

Problem 19 [**The famous two trains puzzle**]Two trains 100 miles apart are traveling toward each other along the same track, each at speed of 50 miles per hour. A fly is hovering just above the nose of the first train. It buzzes from the first train to the second train, turns around immediately, flies back to the first train, and turns around again. It goes on flying back and forth between the two trains until they collide. If the fly zigzags at 60 miles per hour, how far will it travel, in miles?

(Hint: How many hours has the fly flied when these two trains collide?)

Figure 3.4: Picture drawn by Maggie Shen

Chapter 4

Other Speed Problems

4.1 Basic Concepts

Example 1 Suppose that 100 cows produce 6000 gallons of milk in 10 days. At this rate, how many gallons of milk will 50 cows give in 15 days?

Figure 4.1: Picture drawn by Maggie Shen

- **Solution:** According to the assumption, a cow could produce

$$\frac{6000}{100 \times 10} = 6 \text{ gallons}$$

of milk per day. Therefore, in 15 days, 50 cows will produce

$$\boxed{50 \times 15 \times 6 = 4500 \text{ gallons}}$$

of milk in total.

Example 2: [AMC8.2001.15] Homer began peeling a pile of 44 potatoes at the rate of 3 potatoes per minute. Four minutes later Christen joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Christen peeled?

• **Solution:** The key is figure out how many minutes Christen spent on peeling potatoes. Note that when Christen joined Homer, Homer has already peeled

$$3 \times 4 = 12 \text{ potatoes.}$$

Hence there are

$$44 - 12 = 32 \text{ potatoes}$$

left. Together, Homer and Christen can peel $3 + 5 = 8$ potatoes in one minute. So it took them

$$32 \div 8 = 4 \text{ minutes}$$

to finish the job. Therefore Christen has peeled

$$\boxed{4 \times 5 = 20} \text{ potatoes.}$$

Example 3 Alice can make 300 cookies in 30 minutes and Emily can make the same number of cookies in 60 minutes. If Alice and Emily work together, how many minutes does it take them to make 300 cookies?

• **Solution:** As the first step, we need to figure out how many cookies they can make in one minute. Note that

$$\text{Alice makes } 300 \div 30 = 10 \text{ cookies per minute}$$

and

$$\text{Emily makes } 300 \div 60 = 5 \text{ cookies per minute.}$$

Hence together, they can make $10 + 5 = 15$ cookies per minute. So it takes

$$\boxed{300 \div 15 = 20 \text{ minutes}}$$

to make 300 cookies if Alice and Emily work together.

4.2 Exercises

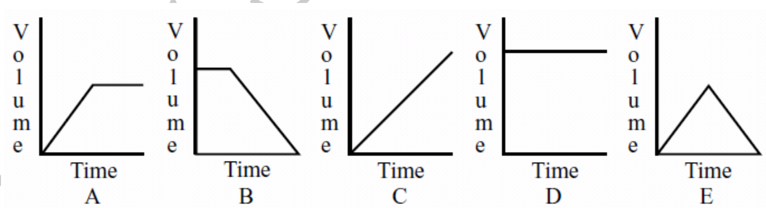
Problem 1 3 monkeys can eat 90 bananas in 5 days. How many bananas can 6 monkeys eat in 10 days?

Problem 2 A computer can do 10, 000 additions per second. How many additions can it do in one hour? Note: 1 million=100,0000

Problem 3 [**AMC8.2009.6**]: Steve's empty swimming pool will hold 24,000 gallons of water when full. It will be filled by 4 hoses, each of which supplies 2.5 gallons of water per minute. How many hours will it take to fill Steve's pool?

Figure 4.2: Figure from Internet

Problem 4 [**AMC8.2002.6**]: A birdbath is designed to overflow so that it will be self-cleaning. Water flows in at the rate of 20 milliliters per minute and drains at the rate of 18 milliliters per minute. One of these graphs shows the volume of water in the birdbath during the filling time and continuing into the overflow time. Which one is it?



Problem 5 [**AMC10A.2017.4**]: Mia is "helping" her mom pick up 30 toys that are strewn on the floor. Mia's mom manages to put 3 toys into the toy box every 30 seconds, but each time immediately after those 30 seconds have elapsed, Mia takes 2 toys out of the box. How much time, in minutes, will it take Mia and her mom to put all 30 toys into the box for the first time?

Problem 6 [**AMC8.2012.2**]: In the country of East Westmore, statisticians estimate there is a baby born every 8 hours and a death every day. How many people are added to the population of East Westmore each year?

Problem 7 A construction company A plans to build a bridge. If 20 workers work on the project, it takes 10 day. How many days will it take if 5 workers work on the project?

Problem 8 Dave and Jack are making paper cranes together for a party. On the average, Dave can make twice as many as Jack in one minute. After 1 hour, they have made 360 paper cranes in total. What is the average number of paper cranes can Dave make in one minute?

Problem 9 Nathan and Matt plan to paint a house. If Nathan works alone, it will take 10 days. If Matt works alone, it will take 15 days. How many days will it take to finish the job if they work together?

Problem 10 Alice can make 300 cookies in 30 minutes and Emily can make the same number of cookies in 60 minutes. If Alice and Emily work together, how many minutes does it take them to make 300 cookies? (Hint: How many cookies can Alice and Emily, together, make in 1 minute?)

Problem 11 George and Mike needed to make 80 paper planes. George started first at the rate of 5 planes per minute. Four minutes later Mike joined him and made at the rate of 7 planes per minute. When they finished, how many planes had Mike made?

Problem 12 Mr. Smith wants to put a turf on his large lawn. An artificial grass company tells him that they have 4 workers and need 6 days to finish the job. Right before the work starts, the company manages to hire two more workers to do Jack's turf. How many days will it take to finish the project?

Problem 13 In the above problem, right before the works starts, a friend tells Mr. Smith that he and his family plan to visit him in three days. If Mr. Smith wants the company to finish the job in 3 days, how many workers are needed?

Problem 14 Jack wants to remodel his home. He finds a contractor who has a team of 6 workers. The contractor tells Jack that they need 8 days to finish the job. However, right before the work starts, two workers fall sick and have to leave. How many days will it take for the remaining 4 workers to remodel Jack's house? (*Hint: You may consider an intermediate step: how many day will it take 2 workers to finish the job?*)

Chapter 5

Quantity Relations

5.1 Chicken-Rabbit Problem

Example 1: [Chicken-Rabbit-Problem] Some chickens and rabbits are in a cage. There are 14 heads and 44 feet inside of the cage. How many rabbits are there?

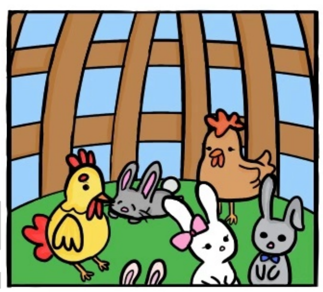


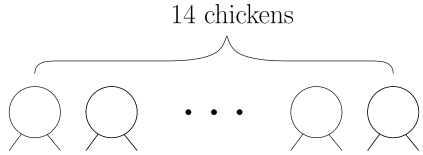
Figure 5.1: Picture drawn by Maggie Shen

• **Solution:** We will try two different ways.

Approach 1: "Guess and Check": try different combinations of 14 to find right numbers.

Approach 2:

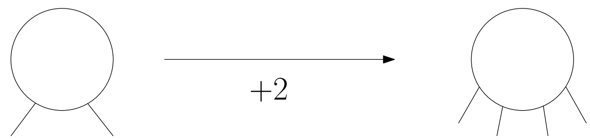
Step 1: If all animals are chickens, there are 28 legs.



Step 2: The difference is

$$44 - 28 = 16.$$

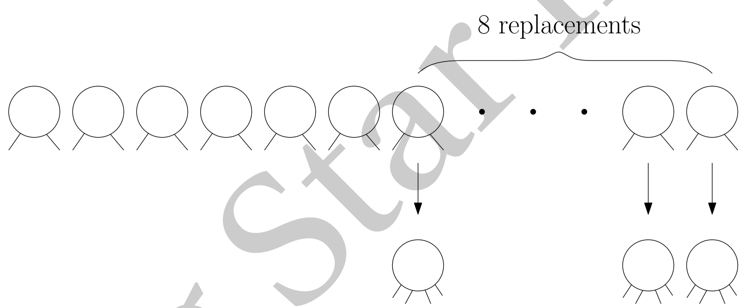
We need 16 more legs. If a chicken is replaced by one rabbit, we get two more legs.



Hence we should replace

$$\frac{16}{2} = 8$$

chicken by $\boxed{8}$ rabbits.



5.2 Other Types of Problems

Example 2 Abel, Ben and Cathy take turns to play tennis. Each round has two players. If Abel played 6 times, Ben played 5 times and Cathy played 3 times, what is the total number of rounds?

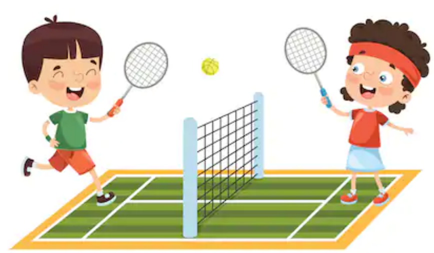


Figure 5.2: Figure from Internet

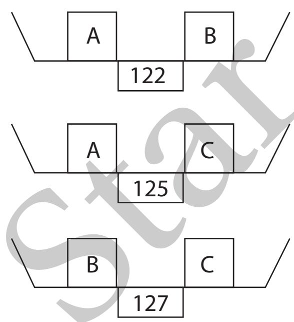
• **Solution:** It is tempting to think that the total number is

$$6+5+3=14.$$

However, this is **wrong** since each game involves two players. So the correct answer is

$$\boxed{\frac{14}{2}=7.}$$

Example 3: [AMC8.2008.13] Mr. Harman needs to know the combined weight in pounds of three boxes he wants to mail. However, the only available scale is not accurate for weights less than 100 pounds or more than 150 pounds. So the boxes are weighed in pairs in every possible way. The results are 122, 125 and 127 pounds. What is the combined weight in pounds of the three boxes?



$$\boxed{A} + \boxed{B} + \boxed{C} = ?$$

• **Solution:** From the above picture, it is easy to see

$$(\boxed{A}+\boxed{B})+(\boxed{A}+\boxed{C})+(\boxed{B}+\boxed{C})=122+125+127=374.$$

This implies that

$$2\left(\boxed{A}+\boxed{B}+\boxed{C}\right)=374.$$

Accordingly,

$$\boxed{A}+\boxed{B}+\boxed{C}=\boxed{374\div 2=187.}$$

Example 4: [Age Problem] This year, Benjamin is 5 years old and his father is 41 years old. After how many years from now will his father be three times Benjamin's age?

• **Solution:** The key is to notice that the age difference between Benjamin and his father remains the same

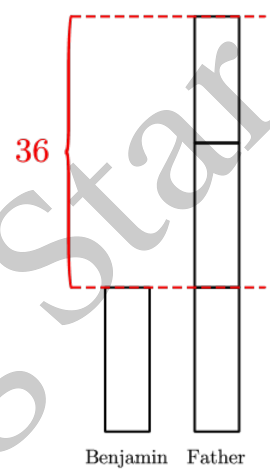
$$41 - 5 = 36.$$

When the father is three times Benjamin's age, Benjamin's age is half of the age difference. Hence Benjamin in that year is

$$36 \div 2 = 18$$

years old. So the answer is

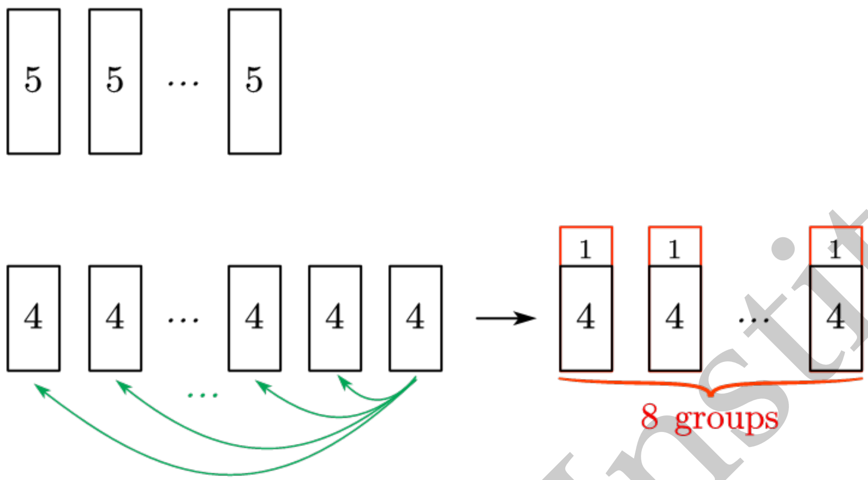
$$18 - 5 = \boxed{13.}$$



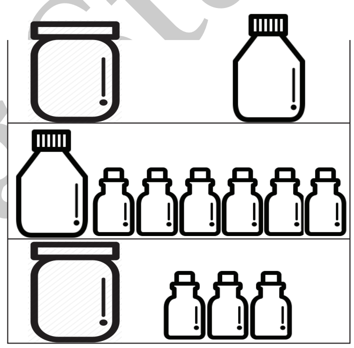
Example 5 Mrs. Bath divided students in her class into groups of equal size. At first, each group had 5 students. Then she reorganized students and tried 4 people per group. If the number of 4-person groups is two more than the number of 5-person groups, how many students are in Mrs. Bath's class?

• **Solution:** We can distribute 8 students from the extra 4-person groups to the other 4-person groups to form 5-person groups. Since the total number of students does not change, there should be 8 5-person groups. Hence the total number of students is

$$8 \times 5 = \boxed{40.}$$



★★ **Example 6** Water bottles of three different size (small, medium and large size) are put a three-tier shelf. Each bottle is filled with water. If the total amout of water on each tier is 900 ml. How much water can each size water bottle hold?



• **Solution:** From the first and second tiers, we have that

 = 

Combining with the third tier, we deduce that

 +  = 900.

Accordingly, each small bottle can hold

$$\frac{900}{9} = \boxed{100 \text{ ml.}}$$

Each large bottle can hold

$$100 \times 6 = \boxed{600 \text{ ml}}$$

and each medium sized bottle can hold

$$900 - 600 = \boxed{300 \text{ ml.}}$$

Remark We would like to point out that many problems in this chapter can be easily solved by setting up linear equations, which will be discussed in the last three chapters.

5.3 Exercises

Problem 1 **[AMC8.2003.4]** A group of children riding on bicycles and tricycles rode past Billy Bob's house. Billy Bob counted 7 children and 19 wheels. How many tricycles were there?

Problem 2 **[AMC8.2012.9]** The Fort Worth Zoo has a number of two-legged birds and a number of four legged mammals. On one visit to the zoo, Margie counted 200 heads and 522 legs. How many of the animals that Margie counted were two-legged birds?

Problem 3 A crab has 10 feet; a dragonfly has 6 feet and two pairs of wings; a praying mantis has 6 feet and one pair of wings. There are currently 37 crabs, dragonfly flies, and praying mantises, with a total of 250 feet and 52 pairs of wings. How many praying mantis are there? (A Chinese 3rd grade math Olympia problem)

Problem 4 **[AMC8.2002.17]** In a mathematics contest with ten problems, a student gains 5 points for a correct answer and loses 2 points for an incorrect answer. If Olivia answered every problem and her score was 29, how many correct answers did she have?

Problem 5 The sum of two numbers is 25. One of the numbers exceeds the other by 9. Find the numbers.

Problem 6 A water tank full of water weighs 250 kilograms. After half of water is used, it weighs 145 kilograms. How much does the empty water tank weigh, in kilograms?

Problem 7 Two pencils have the same length. After the first one is used 14 cm and the second is used 2cm, the length of the second one is 5 times the first one. What is their original length?

Problem 8 Lincoln elementary school has 40 more basket balls than soccer balls. After buying 30 more basketballs, the number of basketballs is three times the number of soccer balls. What is the number of soccer balls?

Problem 9 **[AMC8.2017.13]** Peter, Emma, and Kyler played chess with each other. Peter won 4 games and lost 2 games. Emma won 3 games and lost 3 games. If Kyler lost 3 games, how many games did he win?

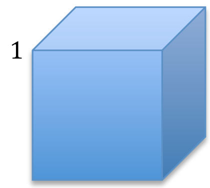
Problem 10 Ben is 4 years old and his dad is 30 years old. After how many years, the age of the father is twice the age of Ben?

Problem 11 When the young elephant Dori becomes as old as her mom, her mom will be 28 years old. When the mom elephant was as old as the young elephant Dori, Doris was only one year old. What is the sum of current ages of the Dori and her mom?

Problem 12 Class A has 170 books and class B has 30 books. After giving how many books to class B, class A will have 20 books more than twice the number of books class B has?

Problem 13 The cost of two tables and three chairs is \$250. If a table costs \$50 more than a chair, find the cost of a table.

Problem 14 ⋆⋆ **[AMC10A.2007.11]** The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum? Also, find one valid arrangement of these numbers.

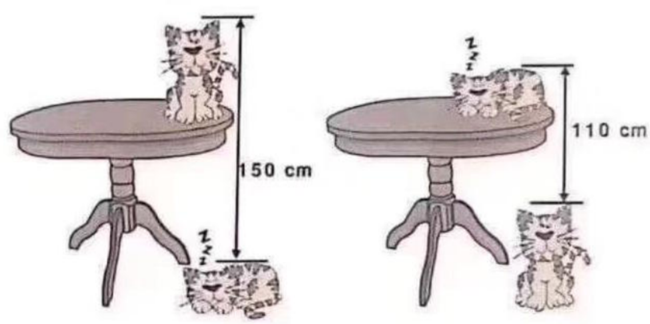


Problem 15 It costs \$ 18 to buy 1 book A and 2 book B. Also, it costs \$ 31 to buy 2 book A and 3 book B. What is the price of book B? (*Hint: How much does it cost to buy 2 book A and 4 book B?*)

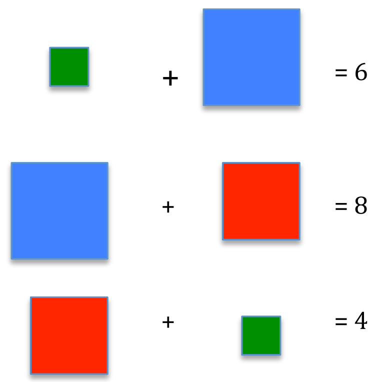
Problem 16 [**AMC8.2014.10**] The first AMC 8 was given in 1985 and it has been given annually since that time. Samantha turned 12 years old the year that she took the seventh AMC 8. In what year was Samantha born?

Problem 17 [**Australian Math Competition**] Helen is adding some numbers and gets the total 157. Then she realizes that she has written one of the numbers as 73 rather than 37. What should the total be?

Problem 18 [**Kangaroo Math**] Determine the height of the table from the following picture, in centimeters (cm).



Problem 19 [**AMC8.2012.19**] In a jar of red, green, and blue marbles, all **but** 6 are red marbles (i.e., blue + green = 6), all **but** 8 are green, and all **but** 4 are blue. How many marbles are in the jar?



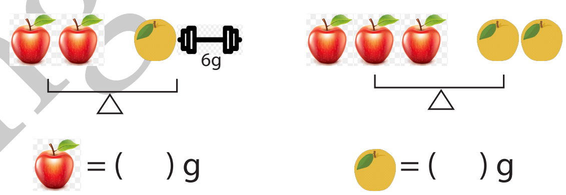
Problem 20 **[Australian Math Competition]**

In these two number sentences

$$♥ + ♥ + ♥ + ☆ = 12$$
$$☆ + ☆ + ☆ + ♥ = 20$$

what is the value of ♥ ?

Problem 21 Here g: gram.



Chapter 6

Sequences

6.1 Arithmetic Sequence

A sequence of numbers is called an *arithmetic sequence* if difference between the consecutive terms is constant. For example,

$$1,\ 2,\ 3,\ 4,\ 5,\ \cdots$$
$$0,\ 2,\ 4,\ 6,\ 8,\ \cdots$$
$$0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ \cdots$$

are all arithmetic sequences since

$$2-1=3-2=4-3=5-4=\cdots=1$$
$$2-0=4-2=6-4=8-6=\cdots=2$$
$$1-0.5=1.5-1=2-1.5=2.5-2=\cdots=0.5$$

Example 1: [Gauss' formula] Calculate

$$S=1+2+3+\cdots+97+98+99+100.$$

• **Solution:** The following approach was discovered by the great mathematician Carl Friedrich Gauss (1777-1855) when he was only 7 years old. Note that

$$\begin{aligned} S &= 1+2+3+\cdots+97+98+99+100 \\ &= 100+99+98+97+\cdots+3+2+1 \end{aligned}$$

So

$$2S = (1+100)+(2+99)+(3+98)+\cdots+(98+3)+(99+2)+(100+1)$$
$$= 101\times 100.$$

Accordingly,

$$\boxed{S = (101\times 100)\div 2 = \frac{101\times 100}{2} = 5050.}$$

For a general arithmetic sequence

$$\underbrace{a_1}_{\text{first term}},\ a_2,\ a_3,\ \cdots,\ \underbrace{a_n}_{\text{last term}}.$$

n: the number of terms in this sequence.

The sum is also given by the Gauss' formula

$$\boxed{S = a_1 + a_2 + a_3 + \cdots + a_n = \frac{(a_1 + a_n)\times n}{2}.}$$

Example 2 How many numbers are in the following arithmetic sequence?

$$3,\ 6,\ 9,\ \cdots,57,\ 60.$$

Find the sum

$$3+6+9+\cdots+57+60.$$

• **Solution:** Note that the difference between consecutive terms is 3

$$3 = 6-3 = 9-6 = \cdots = 60-57.$$

The difference between the first and last terms is

$$60-3 = 57 = 3\times 19.$$

Hence the number of terms is $\boxed{19+1=20}$.

Owing to Gauss' formula,

$$3+6+9+\cdots+57+60 = \boxed{\frac{(3+60)\times 20}{2} = 630.}$$

Example 3 The following is an arithmetic sequence

$$-1, \boxed{?}, \boxed{?}, 5, \cdots.$$

What are the second and 10th terms in this sequence?

- **Solution:** Clearly, the common difference is

$$(5-(-1)) \div 3 = 2.$$

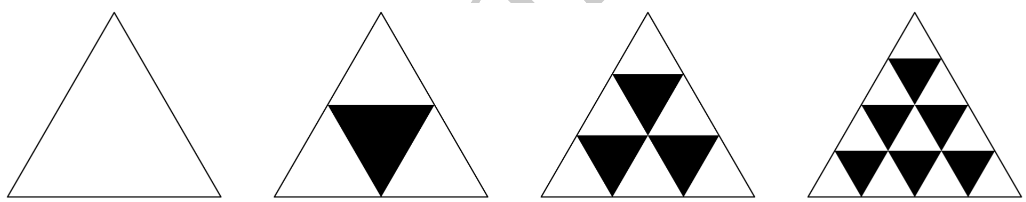
Hence the sequence is

$$-1,\ 1,\ 3,\ 5,\ \cdots$$

So the second term is $\boxed{1}$ and the 10th term is

$$\boxed{-1+9\times 2 = 17.}$$

Example 4: Modification of [AJHSME.1998.23] If the pattern in the diagram continues, what is the number of small black triangles in the eighth triangle?



- **Solution:** From the pattern, it is not hard to see that there are

$$\boxed{1+2+3+4+5+6+7 = \frac{(1+7)\times 7}{2} = 28.}$$

black triangles in the eighth triangle.

6.2 Other Types of Sequences

Example 5: [Geometric sequence] What is the missing number in the following sequence?

$$1,\ 2,\ 4,\ 8,\ 16,\ \boxed{?}.$$

- **Solution:** Notice that

$$2 = 1\times 2, \quad 4 = 2\times 2, \quad ,8 = 4\times 2, \quad ,16 = 8\times 2.$$

By this pattern,

$$\boxed{?} = \boxed{16 \times 2 = 32.}$$

Such a sequence is called a "**geometric sequence**", where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. in the previous example, the common ratio is 2.

Example 6 If the following is a geometric sequence of positive numbers

$$5,\ \boxed{?},\ 45,\ \cdots$$

What is second term $\boxed{?}$?

• **Solution:** Since $45 \div 5 = 9 = 3 \times 3$, the common ratio is 3 and the second number is $\boxed{5 \times 3 = 15}$. The sequence is

$$5,\ 15,\ 45,\ \cdots$$

⋆⋆ **Example 7** Compute the sum of the following geometric sequence.

$$1,\ 2,\ 4,\ 8,\ 16,\ \cdots, 256,\ 512,\ 1024.$$

We may also write this sequence as

$$1,\ 2,\ 2^2,\ 2^3,\ 2^4,\ \cdots, 2^8,\ 2^9,\ 2^{10}.$$

Here $2^n = \underbrace{2 \times 2 \times 2 \times 2 \cdots \times 2}_{n}$.

• **Solution:** Denote

$$S = 1 + 2 + 4 + 8 + \cdots + 256 + 512 + 1024.$$

Then

$$\begin{aligned} 2S &= 2 \times (1 + 2 + 4 + 8 + \cdots + 256 + 512 + 1024) \\ &= 2 \times 1 + 2 \times 2 + 2 \times 4 + \cdots + 2 \times 256 + 2 \times 512 + 2 \times 1024 \\ &= 2 + 4 + 8 + 16 + \cdots 512 + 1024 + 2048. \end{aligned}$$

Hence

$$\begin{aligned} S = 2S - S &= (2 + 4 + 8 + \cdots 512 + 1024 + 2048) - (1 + 2 + 4 + \cdots + 256 + 512 + 1024) \\ &= 2048 - 1 = \boxed{2047.} \end{aligned}$$

Example 8: [Factorial] Find the missing number in the following sequence.

$$1, 1,\ 2,\ 6,\ 24,\ 120,\ \boxed{?}.$$

- **Solution:** Observe that

$$1 = 1\times 1,\quad 2 = 1\times 2,\quad 6 = 2\times 3,\quad 24 = 6\times 4,\quad 120 = 24\times 5.$$

Accordingly,

$$\boxed{?} = 120\times 6 = \boxed{720} = 1\times 2\times 3\times 4\times 5\times 6.$$

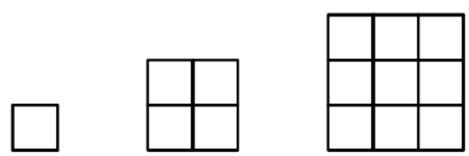
Here we introduce a notation $n!$ (called *"n factorial"*):

$$\boxed{n! = 1\times 2\times 3\times\cdots\times(n-1)\times n}$$

As a convention, we set $0! = 1$. Hence the above sequence can be written as

$$0!,\ 1!,\ 2!,\ 3!,\ 4!,\ 5!,\ 6!,\ \cdots$$

Example 9: [AMC8.2002.11] A sequence of squares is made of identical square tiles. The edge of each square is one tile length longer than the edge of the previous square. The first three squares are shown. How many more tiles does the seventh square require than the sixth?



- **Solution:** Clearly, by this pattern, the number of tiles in each picture is

$$1,\ 2^2,\ 3^2,\ 4^2,\ \cdots$$

Recall that $a^2 = a\times a$. Hence the seventh square has $7^2 = 49$ and the sixth square hs $6^2 = 36$ tiles, respectively. So the difference is

$$\boxed{49 - 36 = 13.}$$

Example 10: [Fibonacci sequence] This famous sequence is named after the Italian mathematician Fibonacci (1175-1250):

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \cdots$$

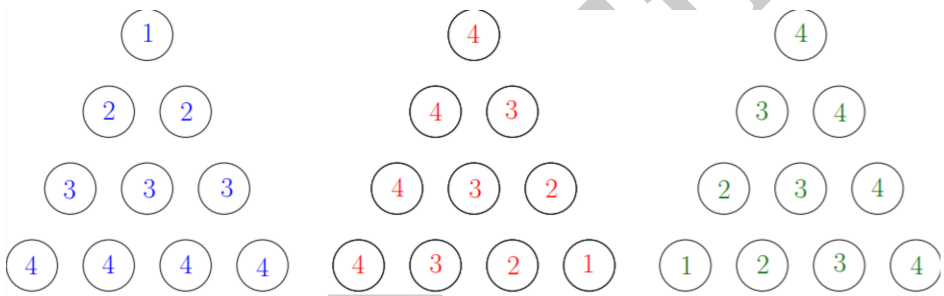
Starting from the third term, each term is the sum of the previous two terms.

$$2 = 1 + 1,\ 3 = 2 + 1,\ 5 = 3 + 2,\ 8 = 5 + 3,\ \cdots.$$

⋆⋆ **Example 11: Sum of squares** Here we introduce a Gauss type method to sum up squares of consecutive positive integers. This beautiful method is from the book "*An Illustrated Theory of Numbers*" by Martin H. Weissman and is brought to our attention by Yunong Gan. Let us use

$$1 + 2^2 + 3^2 + 4^2$$

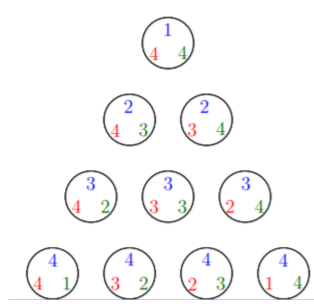
as a special case for demonstration. Consider the following three triangular arrangements of numbers 1,2,3,4.



Clearly, within each arrangment, the sum of numbers is

$$S = 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 = 1 + 2^2 + 3^2 + 4^2.$$

Next we combine these three triangular arrangments into one.



The point is that the sum of three numbers in each circle are the **same**:

$$1 + 4 + 4 = 2 + 4 + 3.$$

By adding all these numbers together, we deduce that

$$\begin{aligned} 3 \times (1 + 2^2 + 3^2 + 4^2) &= (4+4+1) \times \text{the number of circles} \\ &= (4+4+1) \times (1+2+3+4). \end{aligned}$$

Accordingly,

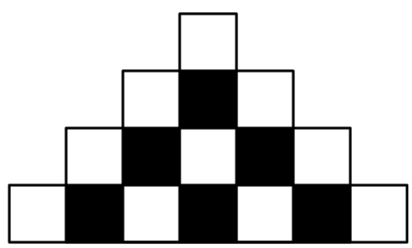
$$\boxed{1 + 2^2 + 3^2 + 4^2 = \frac{(4+4+1) \times (1+2+3+4)}{3} = 30.}$$

Similarly, we can derive that

$$1 + 2^2 + 3^2 + \cdots + 9^2 + 10^2 = \frac{(10+10+1) \times (1+2+3+\cdots+9+10)}{3} = 385.$$

6.3 Exercises

Problem 1 **[AJHSME.1988.7]** A "stair-step" figure is made of alternating black and white squares in each row. Rows 1 through 4 are shown. All rows begin and end with a white square. The number of black squares in the 37th row is



Problem 2 **[AJHSME.1988.19]** What is the 100th number in the arithmetic sequence: $1, 5, 9, 13, 17, 21, 25, \ldots$?

Problem 3 **[AMC10A.2015.7]** How many terms are in the arithmetic sequence 13, 16, 19, ..., 70, 73?

Problem 4 What is the following sum

$$1 + 5 + 9 + \cdots + 97 + 101\ ?$$

Problem 5 An arithmetic sequence starts from 1 and has 6 numbers.

$$1, \boxed{?}, \boxed{?}, \boxed{?}, \boxed{?}, \boxed{?}$$

If the sum of these 6 numbers is 60, what is the last number?

Problem 6 The Incredible Hulk can double the distance he jumps with each succeeding jump. If his first jump is 1 meter, the second jump is 2 meters, the third jump is 4 meters, and so on, then on which jump will he first be able to jump more than 1 kilometer?

Figure 6.1: Figure from Internet

Problem 7 Brent has goldfish that quadruple (become four times as many) every month, and Gretel has goldfish that double every month. If Brent has 4 goldfish at the same time that Gretel has 128 goldfish, then in how many months from that time will they have the same number of goldfish?

Problem 8 A lily pad is growing in a pond and it doubles in size every day. After 30 days it covers the entire pond. On what day does it cover half of the pond?

Problem 9 In Pascal's Triangle, each number is the sum of the number just above it and to the left and the number just above it and to the right. So the middle number in Row 2 is 2 because $1+1=2$. What is the sum of the numbers in Row 8 of Pascal's Triangle? (Mathcounts Problem)

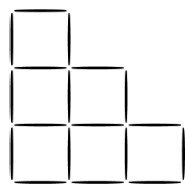
Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

Problem 10 ⋆⋆ Find a short-cut to calculate

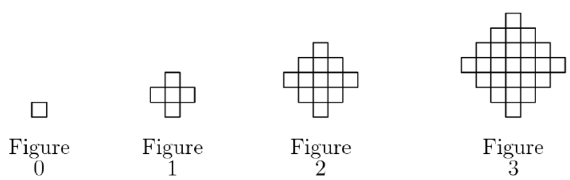
$$S = 1+3+9+27+81+243+729 = 1+3+3^2+3^3+3^4+3^5+3^6.$$

(*Hint: Consider* $3S - S$).

Problem 11 **[AMC10A.2015.3]** Ann made a 3-step staircase using 18 toothpicks as shown in the figure. How many toothpicks does she need to add to complete a 5-step staircase?



Problem 12 **[AMC10.2000.12]** Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in figure 10?



Problem 13 According the following pattern, what should be the number in the box?

$$1,\ 1,\ 1,\ 3,\ 5,\ 9,\ 17,\ \boxed{?}.$$

Problem 14 According to the following pattern, what should be the number in the box?

$$1,\ 1,\ 3,\ 15,\ 105,\ \boxed{?}.$$

Problem 15 According to the following patter, what should be the numbers in the boxes?

$$1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 4,\ 8,\ 5,\ 16,\ \boxed{?},\ \boxed{?}.$$

Problem 16 What is the 15th term in the Fibonacci sequence?

Problem 17 [**AMC10.2000.6**] Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence?

Problem 18 What is the 100th term in the following sequence?

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\cdots$$

Problem 19 What is the number a such that?

$$10! = 8! \times a.$$

Problem 20 ⋆⋆ Use the method in Example 10 to compute

$$1 + 2^2 + 3^2 + \cdots + 19^2 + 20^2.$$

Chapter 7

Number Theory I

7.1 Even Number and Odd Number

7.1.1 Basic Concepts

• Positive integers (Natural numbers): $\{1,2,3,4,\cdots,\}$

• Whole numbers: $\{0,1,2,3,4,\cdots\}$

• Even numbers: $\{0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\cdots\}$

• Odd numbers: $\{1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\cdots\}$

• Even or odd, is called the *parity* of a number. To identify a number as odd or even, we only need to check the last digit. If the last digit ends with 0, 2, 4, 6, or 8, then it is even. If the last digit ends with 1, 3, 5, 7, or 9, then it is odd.

• The sum of first n odd numbers is n^2. For example,

$$1 = 1$$

$$1 + 3 = 2^2$$

$$1 + 3 + 5 = 3^3$$

$$1 + 3 + 5 + 7 = 4^2$$

Example 1 The 5-digit number 12345 is odd since the last digit is 5. The 6-digit number 901678 is even since the last digit is 8.

Example 2 How many even numbers are between 113 and 333?

• **Solution:** Even numbers between 113 and 333 are

$$114,\ 116,\ 118,\ \ldots,\ 330,\ 332.$$

Hence there are

$$\boxed{\frac{332-114}{2}+1=110}$$

even numbers.

7.1.2 Arithmetic Rules

• Even + Even = Even (e.g., 4+10=14)

• Even - Even = Even (e.g., 20-8=12)

• Odd + Odd =Even (e.g., 11+5=16)

• Odd +Even =Odd (e.g. 11+8=19)

• Odd × Odd= Odd (e.g., $3 \times 5 = 15$)

• Even × Any number = Even (e.g., $4 \times 2 = 8$, $4 \times 3 = 12$)

Example 3 If m and n are two odd numbers, is $m^2 + 2n^2$ even or odd?

• **Solution:** Since m is odd, $m^2 = m \times m$ is also odd. On the other hand, $2n^2$ is even. Hence

$$\boxed{m^2 + 2n^2 = \text{Odd} + \text{Even} = \text{Odd.}}$$

Example 4 Determine whether the following number

$$A = 1 + 2^2 + 3^2 + 4^2 + \cdots + 19^2 + 20^2$$

is even or odd.

• **Solution:** Note that

$$1,\ 3^2,\ 5^2,\ 7^2,\ \cdots\ 19^2$$

are 10 odd numbers and

$$2^2,\ 4^2,\ 6^2,\ 8^2,\ \cdots\ 20^2$$

are 10 even numbers. Since

$$\underbrace{O+O+O+\cdots+O}_{\text{10 odd numbers}} = \text{E}$$

and

$$\underbrace{E+E+E+\cdots+E}_{\text{10 even numbers}} = \text{E.}$$

Hence

$$A = 10 \text{ odd numbers} + 10 \text{ even numbers} = \boxed{\text{Even.}}$$

7.2 Divisors, Factors and Multiples

7.2.1 Basic Concepts

Let us use an example to demonstrate these three concepts. Consider

$$12 = 1 \times 12 = 3 \times 4 = 2 \times 6.$$

1, 2, 3, 4, 6, 12 are **divisors (factors)** of 12.

and

12 is a **multiple** of 1, 2, 3, 4, 6, 12.

Or we say that

12 is **divisible by** 1, 2, 3, 4, 6, 12.

Example 5 How many numbers between 1 and 80 are divisible by 3?

• **Solution:** Clearly, the smallest one is $3 = 3 \times 1$ and the largest one is $78 = 3 \times 26$. Hence there are $\boxed{26}$ mutliples of 3 between 1 and 80:

$$3 \times 1,\ 3 \times 2,\ 3 \times 3,\ ...,\ 3 \times 26.$$

7.2.2 Prime Number

A positive integer is called a *prime number* if it is greater than 1 and has no factors other than 1 and itself. For example,

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\cdots$$

- "2" is the ONLY even prime number. All other prime numbers are odd.
- There are infinitely many prime numbers.

Example 6: [AMC8.2014.4] In how many ways can we write 85 as the sum of two prime numbers?

• **Solution:** Since 85 is an odd number, these two prime numbers have to be one even and one odd:

$$85 = \underbrace{E}_{\text{prime}} + \underbrace{O}_{\text{prime}}.$$

Because there is only one even prime number that is 2, the only possible way is

$$85 = 2 + 83.$$

83 is indeed a prime number. So the answer is $\boxed{1}$.

Goldbach's Conjecture

Goldbach's Conjecture is one of the oldest unsolved mathematical problems. It was proposed by German mathematician Christian Goldbach in 1742 AD. The conjecture says that

"Every even number that is greater than 2 can be written as the sum of two prime numbers".

For example, $6 = 3 + 3$, $8 = 3 + 5$, $12 = 5 + 7$.

7.2.3 Prime Factorization

It is very often that we need to express an integer as a product of prime numbers. This is the so called "prime factorization". For example,

$$12 = 2 \times 2 \times 3 = 2^2 \times 3$$

The following *"Ladder method"* provides a systematic way to find all prime factors and the prime factorization.

Example 7 Find the prime factorization of 180.

$$\begin{array}{r|l} 2 & 180 \\ \hline 2 & 90 \\ \hline 3 & 45 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

5 is prime number. So we can not reduce any further. Notice that

$$\boxed{180 = 2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5.}$$

Note that 54 has two prime factors, 2 and 3.

7.2.4 Divisibility Test

The following table provides short-cuts to tell whether a given number is divisible by 2, 3, 4, 5, 8, 9, 11.

Divisibility Tests	Examples
A number is **even** if and only the last digit is **even**	3174 is **even** since the last digit is **even**
A number is divisible by **3** if and only if the sum of its digits is divisible by **3**	3651 is divisible by **3** since the sum of its digits is 3+6+5+1=15 which is divisible by **3**
A number is divisible by **4** if and only if the last two digits form a number that is divisible by **4**	57296 is divisible by **4** since 96 is divisible by **4**
A number is divisible by **5** if and only if the last digit is either **0** or **5**	1265 and 4320 are divisible by **5**
A number is divisible by **8** if and only if the last three digits form a number that is divisible by **8**	562320 is divisible by **8** since 320 is divisible by **8**
A number is divisible by **9** if and only if the sum of its digits is divisible by **9**	45252 is divisible by **9** since the sum of its digits is 4+5+2+5+2=18 which is divisible by **9**
A number is divisible by **11** if and only if the alternating sum of the digits is divisible by **11**	3916 is divisible by **11** since 3-9+1-6=-11 which is divisible by **11**

Example 8 Determine whether the 8-digit number $\overline{12345678}$ is divisible by 4.

- **Solution:** According to the above divisibility test, we only need to look at the last two digits 78 which is not divisible by 4. Hence $\overline{12345678}$ is not divisible by 4.

Example 9 If a 5-digit number $\overline{12A45}$ is divisible by 9, what is A?

- **Solution:** According to the above divisibility test, the sum of its 5 digits

$$1+2+A+4+5=12+A$$

must be divisible by 9 too. Then the only possibility is $\boxed{A=6}$.

7.3 Exercises

Problem 1 **[AMC8.2005.8]** Suppose m and n are positive odd integers. Which of the following must also be an odd integer?

(A) $m+3n$ **(B)** $3m-n$ **(C)** $3m^2+3n^2$ **(D)** $(nm+3)^2$ **(E)** $3mn$

Problem 2 What is the parity of the following number N?

$$N=1+2^2+3^3+4^4+\cdots+2019^{2019}+2020^{2020}.$$

Problem 3 Put 1, 3, 5, 7, 9 on the left 5 boxes and 2, 4, 6, 8 on the right 4 boxes so that calculations of both sides give the same number (no fraction is involved). What is the smallest possible value of that number? Hint: Consider the parity of both sides.

$$\boxed{?} \div \boxed{?} + \boxed{?} + \boxed{?}\boxed{?} = \boxed{?} \div \boxed{?} + \boxed{?}\boxed{?}.$$

Problem 4 **[AMC8.2011.24]** In how many ways can 10001 be written as the sum of two primes?

Problem 5 a, b and c are three prime numbers. If

$$a + b = c \quad \text{and} \quad b + c = 24,$$

what is $b \times c$?

Problem 6 In how many ways can we write 80 as the sum of two primes?

Problem 7 The sum of four consecutive even numbers is 208. What is the smallest one?

Problem 8 What is the sum of the first 20 positive odd numbers?

Problem 9 Victoria added odd numbers starting from 1:

$$1 + 3 + 5 + \cdots$$

At one point, she got a sum of 200 and realized that one number is missing in the addition process. What is the missing number?

Problem 10 How many integers between 1 and 1000 are divisible by 3 but not by 6?

Problem 11 How many numbers between 1 and 20 (inclusive) are divisible by 2 or 5?

Problem 12 How many divisors does 60 have?

Problem 13 If a 5-digit number $\overline{1A2A3}$ is divisible by 9, what is A?

Problem 14 If a 8-digit number $\overline{1234567A}$ is divisible by 8, what is A?

Problem 15 What is the smallest number whose digits consist of 0 and 1 and is divisible by both 3 and 4?

Problem 16 If $360 = 2^x \times 3^y \times 5^z$ for three positive integers x, y and z, what is $x + y + z$?

Problem 17 ⋆⋆ What is the largest positive integer n such that 2^n is a factor of 10!? In other words, $10! = 2^n \times A$ for an odd number A.

Problem 18 ⋆⋆ Find all 4-digit numbers that consist of four different digits 2, 3, 4, 5 and are divisible by 11. Hint: Consider the divisibility test for 11.

Chapter 8

Number Theory II

8.1 Greatest Common Divisor (gcd)

Greatest common divisor (gcd) is also known as the "*greatest common factor*" (gcf)). For example,

$$1,\ 2,\ 3,\ 4,\ 6,\ 12 \text{ are divisors of } 12$$

and

$$1,\ 2,\ 3,\ 6,\ 9,\ 18 \text{ are divisors of } 18.$$

So

$$1,\ 2,\ 3,\ 6 \text{ are common divisors of } 12 \text{ and } 18$$

and

$$\boxed{6} \text{ is the } \textbf{greatest common divisor} \text{ of } 12 \text{ and } 18.$$

We write

$$\boxed{gcd(12,\ 18) = 6.}$$

Two positive integers are called *relatively prime or coprime* if their greatest common divisor is 1. For example, 3 and 4 are relatively prime, i.e., $gcd(3,4) = 1$.

8.2 Least Common Multiple (lcm)

For example,

$$4,\ 8,\ 12,\ 16,\ 20,\ 24,\ 28,\ 32,\ 36,\ \cdots \text{ are all multiples of } 4$$

and

$$6,\ 12,\ 18,\ 24,\ 30,\ 36,\ \cdots \text{ are all multiples of } 6.$$

So

$$\textcolor{red}{12}, \textcolor{red}{24}, \textcolor{red}{36}, \cdots \text{ are common multiples of 4 and 6}$$

and

$\boxed{\textcolor{red}{12}}$ is the **least common multiple** of 4 and 6.

We write

$$\boxed{lcm(4,\ 6) = 12.}$$

Example 1 How many numbers between 1 and 1000 (inclusive) are divisible by both 4 and 6?

• **Solution:** A number is multiple of both 4 and 6 if and only if it is a multiple of their least common multiple 12. Note that

$$1000 = 83 \times 12 + 4.$$

Hence there are $\boxed{83}$ numbers between 1 and 1000 (inclusive)that are divisible by both 4 and 6:

$$1 \times 12,\ 2 \times 2,\ 3 \times 12,\ ...,\ 83 \times 12.$$

We can also talk about lcm and gcd of three or more positive integers.

Example 2 Find gcd(15,20, 25) and lcm(15,20,25).

• **Solution:** Note that

$$\text{divisors of 15: } 1, 3, \textcolor{red}{5}, 15$$

$$\text{divisors of 20: } 1, 4, \textcolor{red}{5}, 20$$

$$\text{divisors of 25: } 1, \textcolor{red}{5}, 25.$$

Hence the greatest common divisor of 15, 20, 25 is $gcd(15,20,25) = \boxed{\textcolor{red}{5}}$.

Next let us find $lcm(15,20,25)$. As a short-cut, we look at multiples of the largest number 25 and find the first one that is also divisible by 15 and 20.

Multiples of 25:

$$25,\ 50,\ 75,\ 100,\ 125,\ 150,\ 175,\ 200,\ 225,\ 250,\ 275,\ \textcolor{red}{300},\ 325,\ 350, \ldots$$

Clearly, 300 is the first number that is divisible by both 15 and 20. So

$$lcm(15,20,25) = \boxed{\textcolor{red}{300}}.$$

Example 3: [AMC8.2003.19] How many integers between 1000 and 2000 have all three of the numbers 15, 20, and 25 as factors?

• **Solution:** A number is divisible by 15, 20 and 25 if and only if it is divisible by $lcm(15,20,25)=300$. Since

$$900<1000<1200<1500<1800<2000<2100,$$

those valid integers are 1200, 1500, 1800. Hence the number is $\boxed{3.}$

8.3 ⋆⋆ Ladder method

Ladder method is an efficient algorithm to calculate lcm and gcd.

Example 4 Use the ladder method to find $gcd(24,108)$ and $lcm(24,108)$.

$$\begin{array}{r|l}
2 & 24,\ 108 \\ \hline
2 & 12,\ 54 \\ \hline
3 & 6,\ 18 \\ \hline
 & 2 \quad 9
\end{array}$$

Notice that 2 and 9 are relatively prime. So we can not reduce any further. Then

$$gcd(24,\ 108)=2\times2\times3=12$$

and

$$lcm(24,\ 108)=2\times2\times3\times2\times9=216.$$

Example 5 Use the ladder method to find $gcd(15,20,25)$ and $lcm(15,20,25)$.

$$\begin{array}{r|l}
5 & 15,\ 20,\ 25 \\ \hline
 & 3 \quad 4 \quad 5
\end{array}$$

Since 3, 4 and 5 are mutually relatively prime, i.e., every two numbers are relatively prime, we can not reduce any further. Then

$$gcd(5,15,20)=5$$

and

$$lcm(5,15,20)=5\times3\times4\times5.$$

Next let us look at a more complicated example.

Example 6 Use the ladder method to find $gcd(48, 72, 108)$ and $lcm(48, 72, 108)$.

$$\begin{array}{r|l} 2 & 48, 72, 108 \\ \hline 2 & 24, 36, 54 \\ \hline 3 & 12, 18, 27 \\ \hline \underline{3} & 4, \underline{6}, \underline{9} \\ \hline \underline{2} & \underline{4}, \underline{2}, 3 \\ \hline & 2, 1, 3 \end{array}$$

There are 5 numbers on the left hand side

$$\{2,\ 2,\ 3,\ \underline{3},\ \underline{2}\}.$$

Group I:

- The first 3 numbers $\{2,\ 2,\ 3\}$ are common divisors of $\{48,\ 72,\ 108\}$.

Group II:

- $\underline{3}$ is the greatest common divisor of $\underline{6}$ and $\underline{9}$.
- $\underline{2}$ is the greatest common divisor of $\underline{4}$ and $\underline{2}$.

Group III:

- After these iterations, $(48,\ 72,\ 108)$ is eventually reduced to 3 numbers

$$\{2,\ 1,\ 3\}$$

which are mutually relatively prime, i.e., every two numbers are relatively prime. Then we have

$$lcm(48, 72, 108) = \underbrace{\boxed{2 \times 2 \times 3}}_{I} \times \underbrace{\boxed{\underline{3} \times \underline{2}}}_{II} \times \underbrace{\boxed{2 \times 1 \times 3}}_{III} = 2^4 \times 3^3 = 432$$

$$gcd(48, 72, 108) = \underbrace{\boxed{2 \times 2 \times 3}}_{I} = 2^2 \times 3 = 12.$$

8.4 Quotient and Remainder

8.4.1 Basic Concepts

Remainder is the minimum “leftover" when one number (dividend) is divided by another (divisor). If the remainder is zero, then the dividend is divisible by the divisor. For example, consider that 14 is divided by 5. 14 is called the *dividend* and 5 the *divisor*. We obtain a quotient of 2 and a remainder of 4. The remainder is a whole number that is always less than the divisor.

$$14 = 5 \times \underbrace{2}_{quotient} + \underbrace{4}_{remainder}.$$

Here is a short hand notation

$$\boxed{14 \equiv 4 \pmod 5.}$$

Below are three more examples.

$$100 = 9 \times \underbrace{11}_{quotient} + \underbrace{1}_{remainder},$$

$$67 = 10 \times \underbrace{6}_{quotient} + \underbrace{7}_{remainder}$$

and

$$32 = 8 \times \underbrace{4}_{quotient} + \underbrace{0}_{remainder}.$$

Here

"2" is the remainder of 100 divided by 9,

"7" is the remainder of 67 divided by 10

and

"0" is the remainder of 32 divided by 8.

- *"Quotient"* and *"remainder"* can be obtained through long division.

$$\begin{array}{r|lll} & & 1 & 5 \\ \hline 7 & 1 & 1 & 1 \\ & & 7 & \\ \hline & & 4 & 1 \\ & & 3 & 5 \\ \hline & & & 6 \end{array}$$

$$111 = 7 \times \underbrace{15}_{quotient} + \underbrace{6}_{remainder}.$$

• **Basic Fact:** After subtracting the remainder, the original number becomes a multiple of the quotient. For example,

$$14 - \underbrace{4}_{remainder} = 5 \times \underbrace{2}_{quotient}.$$

• **Notation:**

$$\boxed{10 \equiv 4 \pmod 3} \tag{1}$$

is also a short-hand expression of the following statement

"10 and 4 have the same remainder when they are divided by 3".

Example 7 What is the smallest positive integer that leaves a remainder of 1 when divided by 3, remainder of 2 when divided by 4?

• **Solution:** We apply the brutal force.

Step 1: List numbers whose remainder is 1 when divided by 3.

$$1,\ 4,\ 7,\ 10,\ 13,\ \cdots$$

Step 2: In the above list, search the first integer whose remainder is 2 when divided by 4. Clearly the answer is $\boxed{10}$.

Example 8: [AMC8.2016.5] The number N is a two-digit number.
When N is divided by 9, the remainder is 1.
When N is divided by 10, the remainder is 3.
What is the remainder when N is divided by 11?

• **Solution:** Since the remainder is 3 when it is divided by 10, the last digit must be 3. So this 2-digit number can be written as

$$\overline{a3}.$$

Since the remainder is 1 when it divided by 9, we have two methods to determine the tens digit a.

Method 1: Brutal force We write out all 2-digit numbers that end with 3.

$$13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93.$$

Apparently, 73 is the only number that has remainder 2 when divided by 9.

Method 2: Use algebra

$$\overline{a3} - 1 = \overline{a2}$$

must be divisible by 9. So

$$\overline{a2} = 72.$$

Hence $\boxed{N = 73.}$ and the remainder when N is divided by 11 is $\boxed{7}$.

Example 9 The Halloween in 2019 is on Thursday. What day is the Halloween in 2020?

• **Solution:** There are 366 days from Oct 31st, 2019 to Oct 31st, 2020 since 2020 is a leap year whose February has 29 days. Note that

$$366 \equiv 2 \pmod 7,$$

i.e., the remainder is 2 when 366 is divided by 7 (the cycle of one week). Hence next year's Halloween is on $\boxed{\text{Saturday}}$.

Example 10 What is the unit digit of 3^{99}?

• **Solution:** Note that unit digits of

$$3,\ 3^2,\ 3^3,\ 3^4,\ 3^5,\ 3^6,\ 3^7,\ 3^8,\ 3^9,\ \ldots$$

are

$$3,\ 9,\ 7,\ 1,\ 3,\ 9,\ 7,\ 1,\ 3,$$

It is a 4-cycle (3, 9, 7, 1) pattern. Since

$$99 = 4 \times 24 + 3,$$

the remainder of 99 when divided by 4 is 3. Hence the unit digit of 3^{99} is $\boxed{7}$.

8.4.2 ⋆⋆ Arithmetic Rules for Remainders

Remainders obey the usual arithmetic rules in addition, difference and multiplication. For example,

$$\begin{cases} 5 \equiv 1 \pmod 4 \\ 6 \equiv 2 \pmod 4. \end{cases}$$

We have that

$$5 + 6 = 11 \equiv 2 + 1 = 3 \pmod 4$$

and

$$5 \times 6 = 30 \equiv 1 \times 2 = 2 \pmod 4.$$

Example 11 What is the remainder of

$$A = 106 \times 215$$

when divided by 4?

- **Solution:** Note that

$$\begin{cases} 106 \equiv 2 \pmod 4 \\ 215 \equiv 3 \pmod 4. \end{cases}$$

Hence

$$106 \times 215 \equiv 2 \times 3 = 6 \equiv 2 \pmod 4.$$

8.5 Exercises

Problem 1 Find $gcd(60, 75)$ and $lcm(60, 75)$.

Problem 2 Find $gcd(12, 16, 20)$ and $lcm(12, 16, 20)$.

Problem 3 Find $gcd(12, 18, 24)$ and $lcm(12, 18, 24)$.

Problem 4 Find the positive integer m such that

$$gcd(m, 12) = 6 \quad \text{and} \quad lcm(m, 12) = 60.$$

Problem 5 How many positive integers between 1 and 30 (inclusive) are relative prime to 10?

Problem 6 **[AMC9.2009.11]** The Amaco Middle School bookstore sells pencils costing a whole number of cents. Some seventh graders each bought a pencil, paying a total of 1.43 dollars. Some of the 30 sixth graders each bought a pencil, and they paid a total of 1.95 dollars. How many more sixth graders than seventh graders bought a pencil?

Problem 7 How many integers between 100 and 500 (inclusive) are divisible by both 12 and 15?

Problem 8 How many integers between 100 and 1000 are divisible by 12, 16 and 20?

Problem 9 Abel, Betty and Calvin walk around a round track along the same direction. It takes Abel 8 minutes, Betty 10 minutes and Calvin 20 minutes respectively to finish one round. If they start at the same time and the same place, after how many minutes will all of them walk back to the starting point for the first time?

Problem 10 How many three-digit numbers are divisible by 6 or 9?

Problem 11 How many positive integers no greater than 100 have a remainder of 2 when divided by both 4 and 5?

Problem 12 [**AMC8.2002.5**] Carlos Montado was born on Saturday, November 9, 2002. On what day of the week will Carlos be 706 days old?

Problem 13 What is the unit digit of 2^{1111}?

Problem 14 What is the tens digit of 7^{2020}?

Problem 15 A group of Alumni from Nankai University held reunion in the March every 5 years. If they had their first reunion on a Monday of 2017, what day of the week will be the next reunion?

Problem 16 If it is 2pm now, what is the time 3300 minutes later?

Problem 17 2020 beads (black and white) are put on a string as follows. The first bead on the left is white. How many black beads are on the string? (Hope Cup Math Competition, China)

Problem 18 ⋆⋆ what is the remainder of

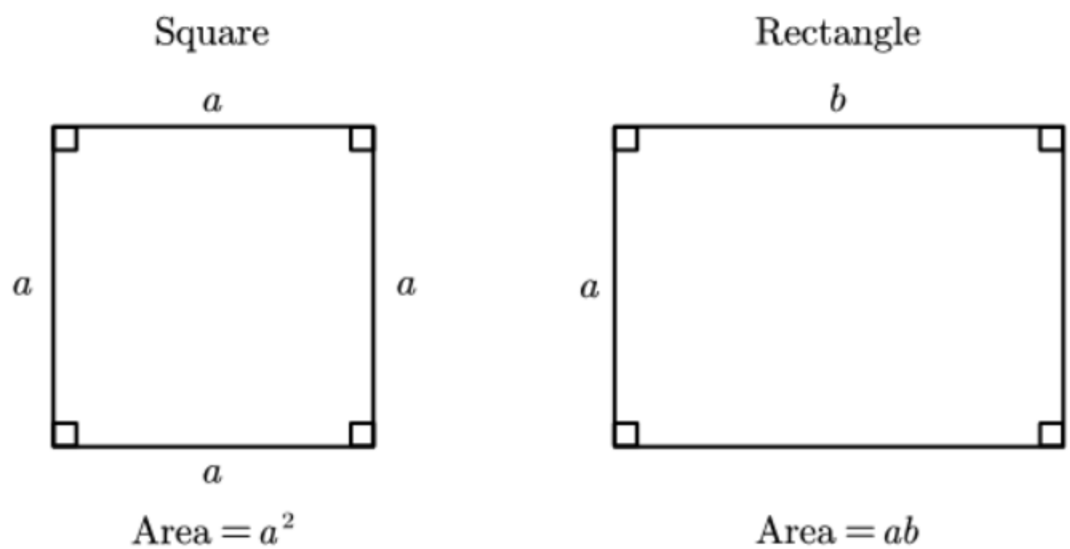
$$N = 101 \times 102 \times 103 \times 105$$

when divided by 4?

Chapter 9

Geometry I

9.1 Area of Square and Rectangle

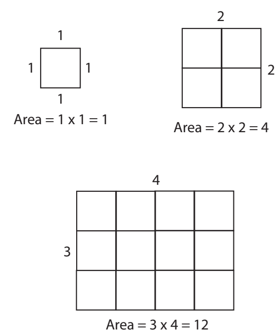


In both the square and rectangle, adjacent sides are perpendicular. In other words,

$$\boxed{\text{each corner angle} = 90^\circ.}$$

See Chapter 11 for more information regarding angles. A square is a special rectangle when $a = b$.

Example 1

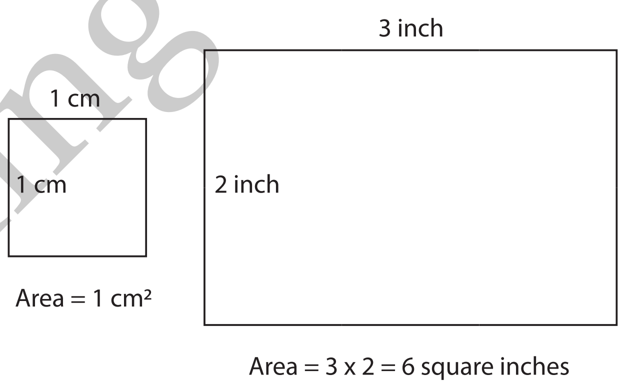


Remark: If side lengths are integers, then

$$\boxed{\text{the area} = \text{the number of } 1 \times 1 \text{ squares needed to cover it}}$$

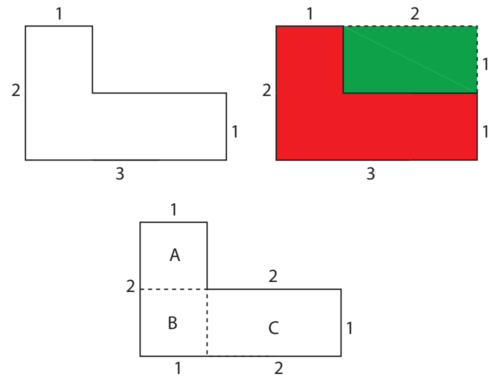
• **Units.** In general, side lengths have units like “cm", “inch", “foot", etc. The unit for corresponding areas are “square cm"(cm^2), “square inch" (in^2), “square feet", etc. Hereafter, for convenience, we often omit units.

Example 2



9.2 The Cut and Paste Technique for Irregular Shapes

Example 3 Calculate the area of the following floor plan.



• **Solution:** We present two methods.

Method 1: (paste) We add the green par to make a large 2×3 rectangle. Since the green part is a 1×2 rectangle,

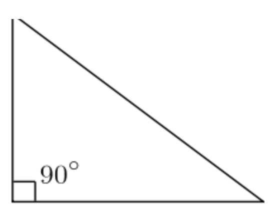
$$\boxed{\text{area of the floor}} = \underbrace{\boxed{\text{area of the rectangle}}}_{3\times 2=6} - \underbrace{\boxed{\text{area of the extra green part}}}_{2\times 1=2}$$
$$= 6 - 2 = 4.$$

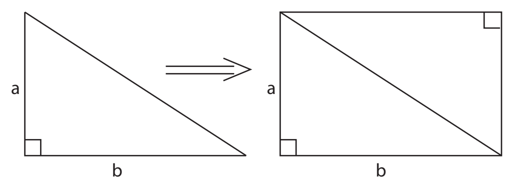
Method 2: (cut) We divide this floor plan into three small rectangles (A,B,C). A and B are actually squares. Then

$$\boxed{\text{area of the floor}} = \underbrace{\boxed{\text{area of A}}}_{1\times 1=1} + \underbrace{\boxed{\text{area of B}}}_{1\times 1=1} + \underbrace{\boxed{\text{area of C}}}_{2\times 1=2}$$
$$= 1 + 1 + 2 = 4.$$

9.3 Area of Right Triangle

A right triangle is a triangle with an angle of $90°$.



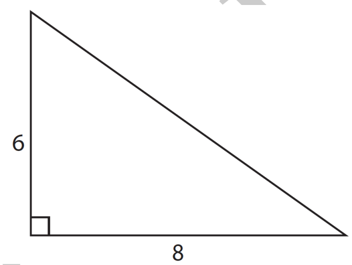


Clearly, a right triangle is half of a rectangle. Hence

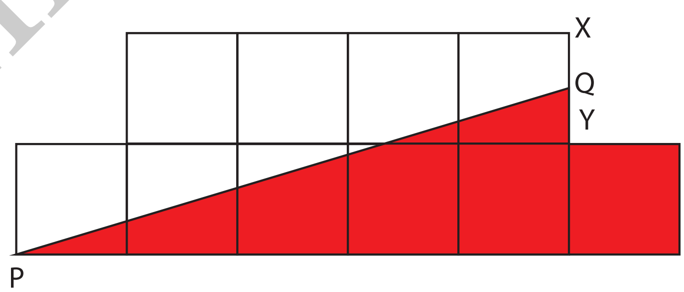
$$\boxed{\text{The area} = (a \times b) \div 2 = \frac{1}{2}ab.}$$

Example 4 The area of the following right triangle is

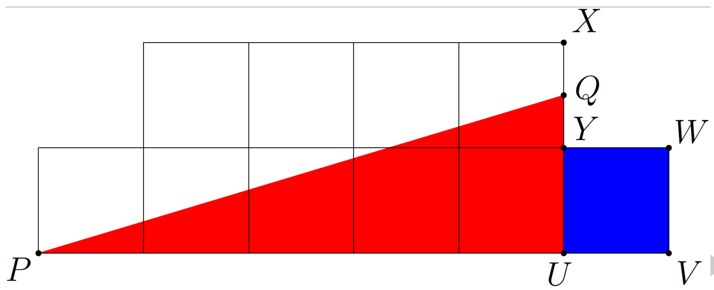
$$\frac{1}{2} \times 6 \times 8 = 24.$$



Example 5: Modification of [AMC8.2010.17] The diagram shows an octagon consisting of 10 squares of side length 2. If $XQ = QY$, Find the area of the red portion.



- **Solution: (cut)** We divide the red portion into two parts:



Part I: The red right triangle ΔQUP. The area is

$$\frac{1}{2} \times 10 \times 3 = 15.$$

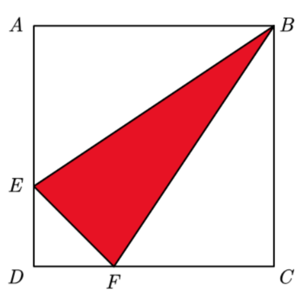
Part II: The blue square $YWUV$. Its area is

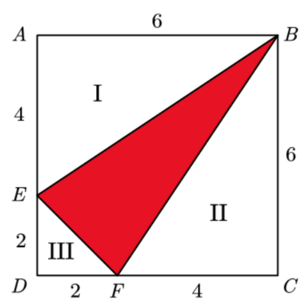
$$2 \times 2 = 4.$$

Hence the total area of the red portion is

$$\boxed{15 + 4 = 19.}$$

Example 6: Modification of [AMC8.2008.23] In square ABCD with side length 6, $AE = CF = 4$ and $ED = FD = 2$. Find the area of the red triangle ΔBEF.





- **Solution: (paste)**

The area of the square $ABCD$ =$6 \times 6 = 36$,

Part I: The area of right triangle ΔABE =$\frac{1}{2} \times 6 \times 4 = 12.$

Part II: The area of right triangle ΔBCF =$\frac{1}{2} \times 6 \times 4 = 12.$

and

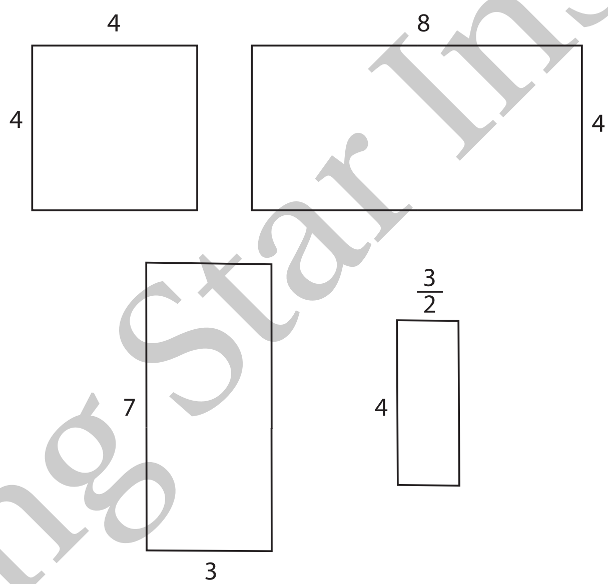
$$\text{Part III: The area of right triangle } \Delta EDF = \frac{1}{2} \times 2 \times 2 = 2.$$

Hence the area of

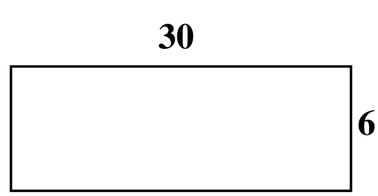
$$\Delta BFD = ABCD - I - II - III = 36 - 12 - 12 - 2 = \boxed{10.}$$

9.4 Exercises

Problem 1 Calculate areas of the following rectangles/squares.

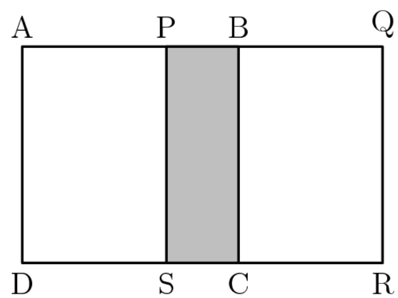


Problem 2 **[Mathstars Problem]** How many 2×5 tiles are needed to cover this floor.

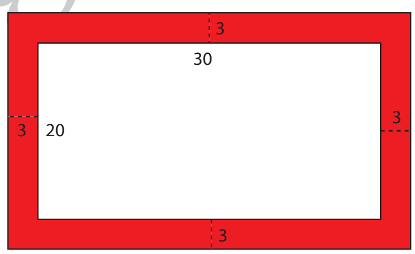


Problem 3 A square-shaped floor is covered with congruent square tiles. If the total number of tiles that lie on the two diagonals is 17, how many tiles cover the floor?

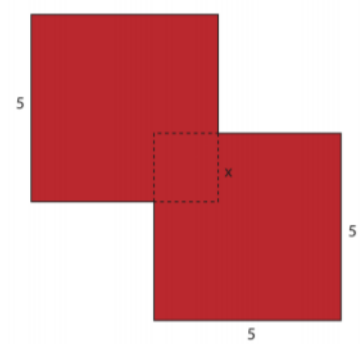
Problem 4 **[AMC8.2011.13]** Two congruent squares, $ABCD$ and $PQRS$, have side length 15. They overlap to form the 15 by 25 rectangle $AQRD$ shown. Find the area of the shaded part?



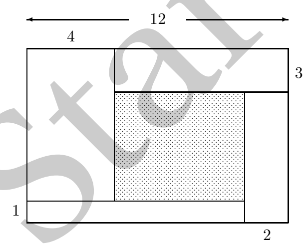
Problem 5 A rectangle lot 30 by 20 is surrounded on all four sides by a concrete walk 3 wide. What is the area of the side walk?



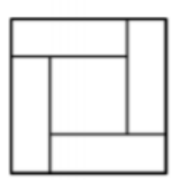
Problem 6 Two squares of side length 5 are positioned with an overlap as follows. If the area of the entire red region is 46. What is side length x of the small square (the overlap part)?



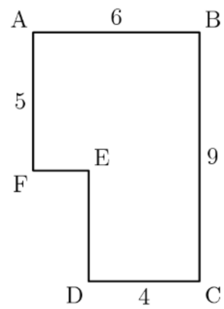
Problem 7 Four rectangular paths of width 1, 2, 3 and 4 metres are arranged as shown in the diagram to form a larger rectangular area, 8m by 12m, which borders a smaller internal rectangular lawn. What is the area, in square metres, of the lawn?



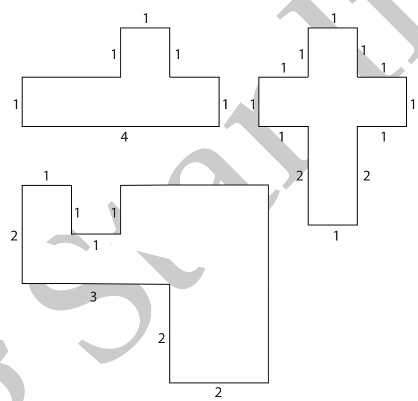
Problem 8 Four identical rectangles are placed as shown. The area of the outer square is 16 and the area of the inner square is 8. What is the area of each rectangle? (Here we omit the unit.)



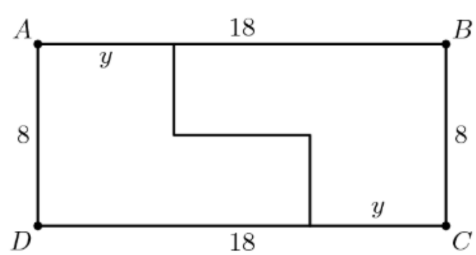
Problem 9 **[AJHSME.1985.4]** The area of polygon $ABCDEF$, in square units, is



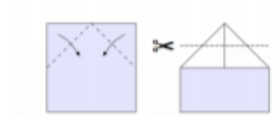
Problem 10 Calculate areas of the following figures.



Problem 11 **[AMC10A.2006.6]** The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be re-positioned without overlap to form a square. What is y?

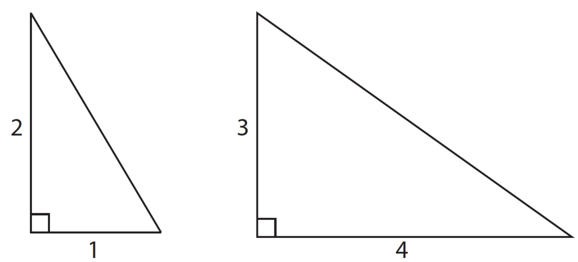


Problem 12 [**Australian Math Competition**] A square piece of paper is folded along the dashed lines shown and then the top is cut off.

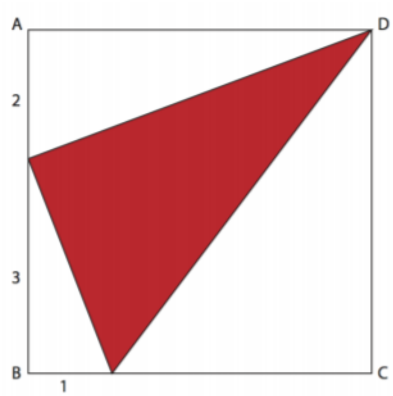


The paper is then unfolded. Which shape shows the unfolded piece?

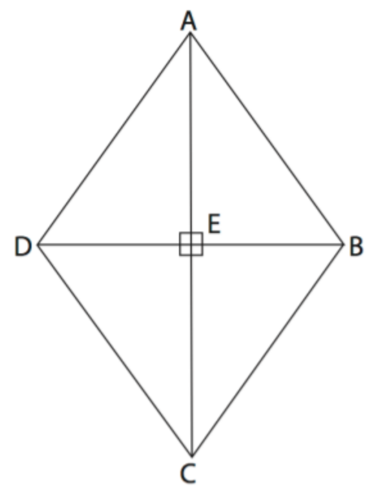
Problem 13 Find the are of the following right triangles.



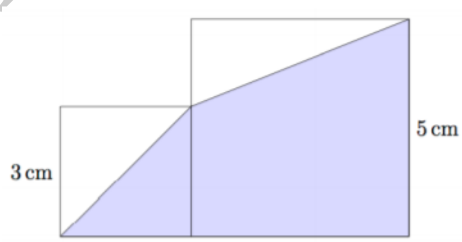
Problem 14 Find the area of the following red triangle inside a 5×5 square.



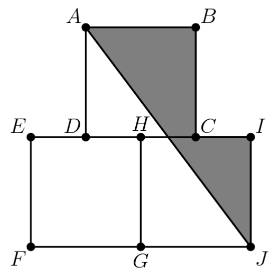
Problem 15 The following parallelogram is a “rhombus". In geometry, a rhombus is a four-sided shape where all sides have equal length. For example, a square is a special rhombus. Two diagonals of a rhombus are perpendicular (i.e., the angle is 90°) and bisect each other (i.e., the intersection point E is the midpoint of both AC and BD). If $AC = 6$ and $BD = 4$, what is the area of $ABCD$?



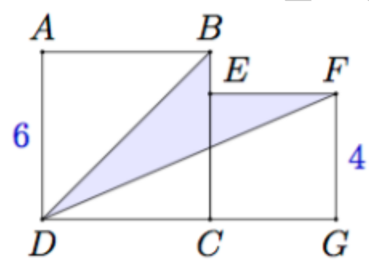
Problem 16 A square of side length 3 cm is placed alongside a square of side 5 cm. Find the area of the shaded region.



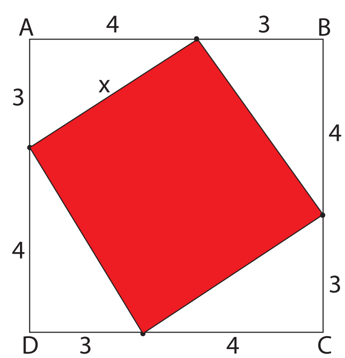
Problem 17 **Modification of [AMC8.2013.24]** $ABCD$, $EFGH$, and $GHIJ$ are squares of side length 2. Points C and D are the midpoints of sides IH and HE, respectively. Find the area of the shaded part.



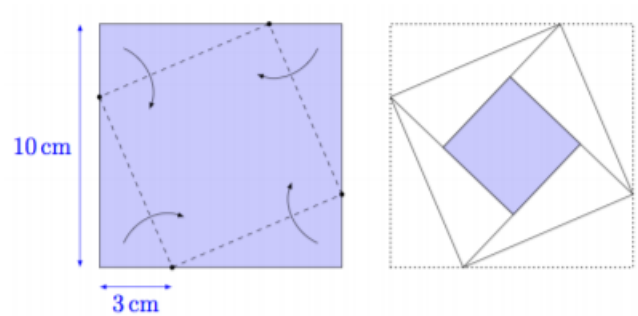
Problem 18 A square $ABCD$ with a side of 6cm is joined with a smaller square $EFGC$ with a side of 4 cm as shown. What is the area of the shaded shape $BDFE$? (omit unit in your answer)



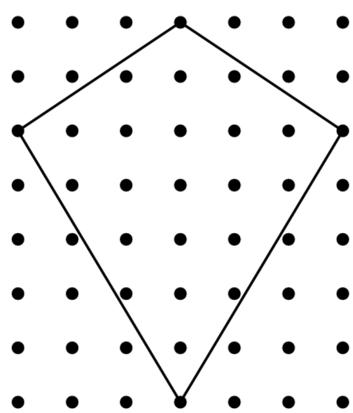
Problem 19 $ABCD$ is a square of side length 7. What is the side length x of the internal red square?



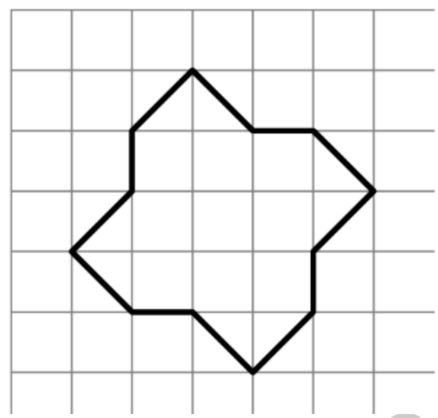
Problem 20 [**Australian Math Competition**] A square of paper has its corners folded in as shown to make a smaller square with an internal square, as shown on the right. What is the area of this internal square?



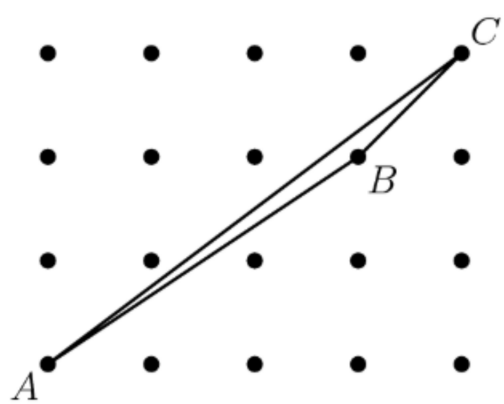
Problem 21 [**AMC8.2001.9**] To promote her school's annual Kite Olympics, Genevieve makes a large kite for a bulletin board display. The kites look like the one on the following one-inch grid. Find the area of the kite.



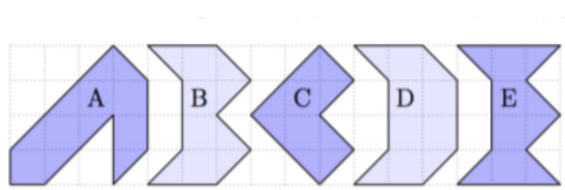
Problem 22 The twelve-sided figure shown has been drawn on 1 cm × 1 cm graph paper. What is the area of the figure in cm^2?



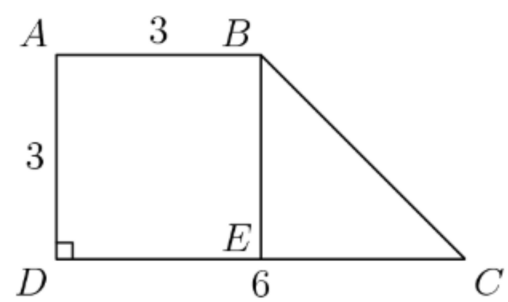
Problem 23 [**AJHSME.1996.22**] The horizontal and vertical distances between adjacent points equal 1 unit. The area of triangle ABC is



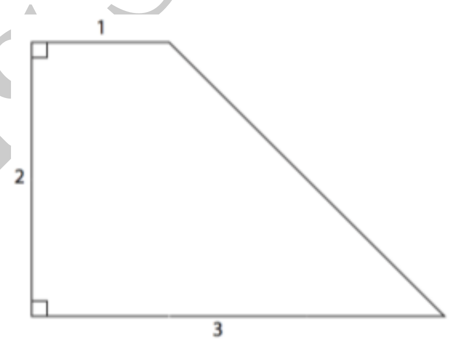
Problem 24 [**Australian Math Competition**] Which of the following has the largest area?



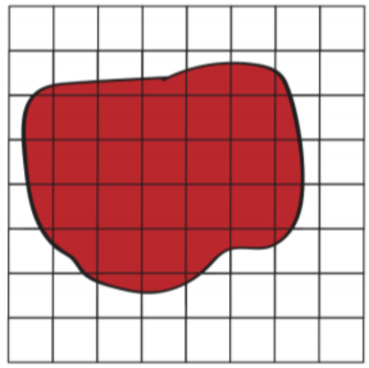
Problem 25 **[AMC8.2007.8]** In trapezoid $ABCD$, AD is perpendicular to DC, $AD = AB = 3$, and $DC = 6$. In addition, E is on DC, and BE is parallel to AD. Find the area of $\triangle BEC$.



Problem 26 What is the area of the following trapezoid?



Problem 27 The following is a 8×8 square. How many 1×1 squares are completely inside the red region? (*Note that this provides a way to estimate the area of an curved region.*)

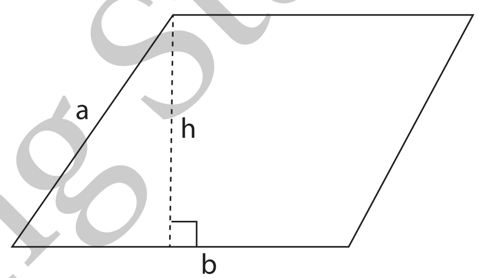


Chapter 10

Geometry II

10.1 Area of Parallelogram

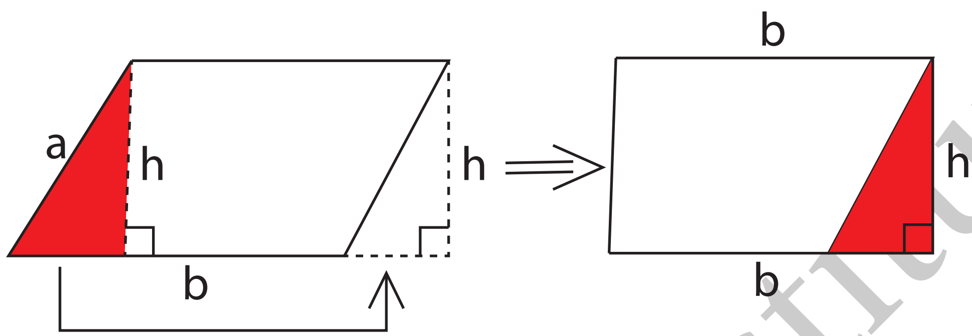
A parallelogram is a quadrilateral with two pairs of parallel sides. Squares and rectangles are two special types of parallelogram.



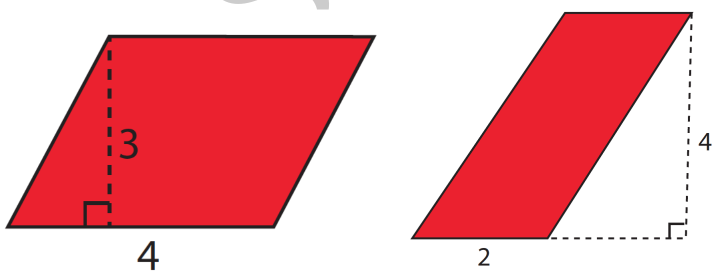
b: one side. h: the corresponding height

$$\boxed{\text{Area} = bh.}$$

By "*cut and paste*", we can transform a parallelogram into a $b \times h$ rectangle.



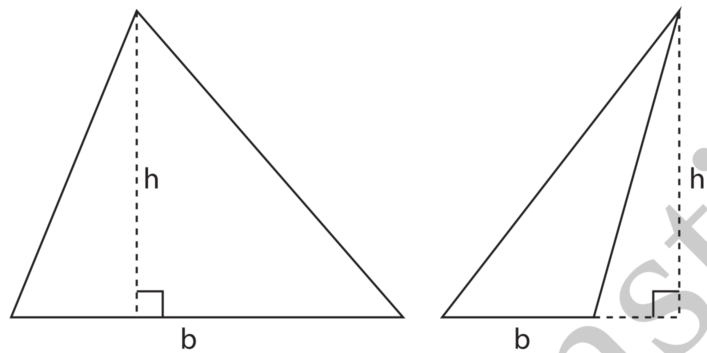
Example 1



Area of the left parallelogram is $\boxed{4 \times 3 = 12}$

Area of the right parallelogram is $\boxed{2 \times 4 = 8}$.

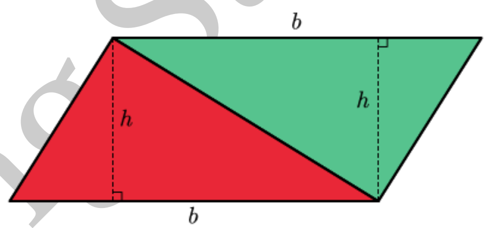
10.2 Area of General Triangle



b: one side (a base); h: the corresponding height

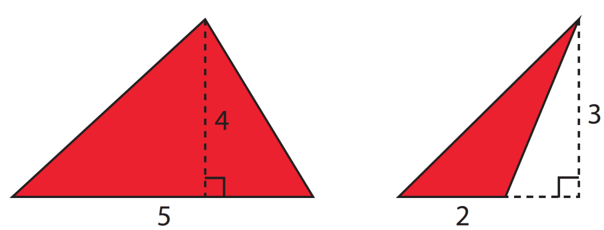
$$\boxed{\text{Area} = \frac{1}{2}bh.}$$

Note that a parallelogram with base b and height h can be split into two identical triangles with the same base and height.



- *The area should be the same no matter which side and its corresponding height are used.*

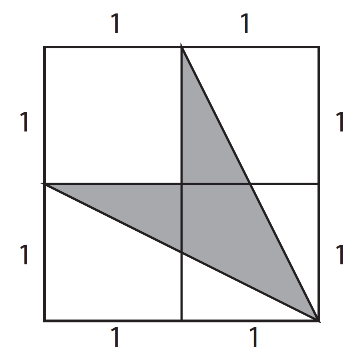
Example 2



Area of the triangle on the left is $\boxed{\frac{1}{2} \times 5 \times 4 = 10}$

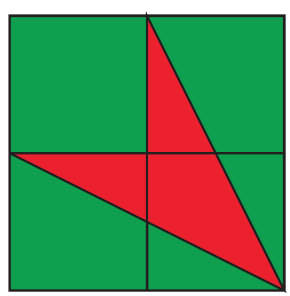
Area of the right on the right is $\boxed{\frac{1}{2} \times 2 \times 3 = 3}$.

Example 3 Use two methods, cut (remove parts) and paste (add parts), to calculate the area of the following shaded region within a 2×2 square.



• **Solution:**

Method 1 (paste):



Paste

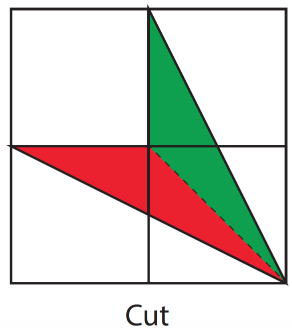
The area of the green part is

$$\underbrace{1 \times 1}_{\text{a } 1 \times 1 \text{ square}} + \underbrace{2 \times \frac{1}{2} \times 1 \times 2}_{\text{two congruent right triangles}} = 3.$$

Hence

the are of the shaded region= $\boxed{2 \times 2 - 3 = 1}$.

Method 2 (cut):

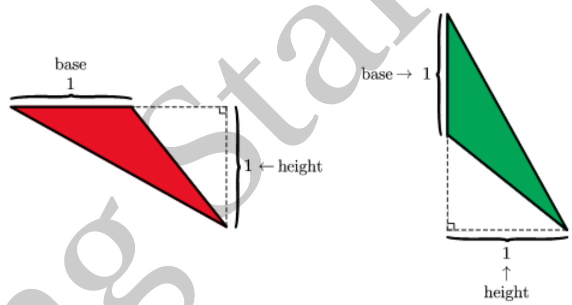


The shaded region can be split into two congruent triangles. The area of the each triangle is

$$\frac{1}{2} \times 1 \times 1 = 0.5.$$

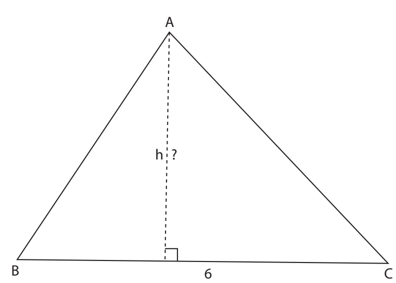
Hence

$$\text{the are of the shaded region} = \boxed{2 \times 0.5 = 1}.$$



10.3 Finding Height from Areas

Example 4 The area of ΔABC is 12. If $BC = 6$, what is the height h on AB?



• **Solution:** Since

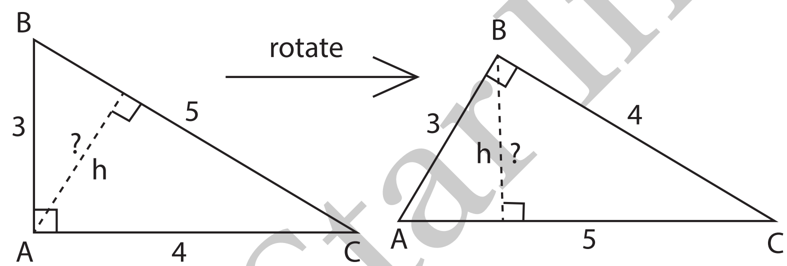
$$12 = \text{area of } \Delta ABC = \frac{1}{2} \times 6 \times h,$$

we derive that

$$12 = 3 \times h.$$

Hence the height $\boxed{h = 4}$.

⋆⋆ **Example 5** ΔABC is a right triangle. $\angle BAC = 90°$. $AB = 3$, $AC = 4$ and $BC = 5$. What is the height h on BC?



• **Solution:** Note that

$$\text{area of } \Delta ABC = \frac{1}{2} \times 3 \times 4 = \underbrace{\frac{1}{2} \times 5 \times h}_{\text{use AC as the base}}.$$

Hence

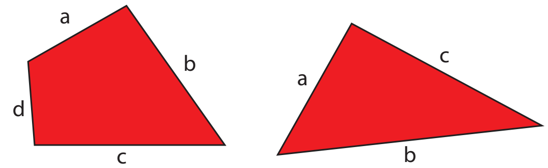
$$5h = 12.$$

So $\boxed{h = 12 \div 5 = 2.4.}$

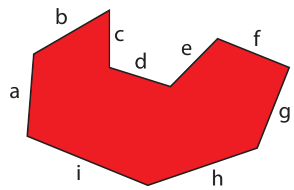
10.4 Perimeters

10.4.1 Basic Concepts

The *perimeter* of a geometric figure is the sum of lengths of its sides.

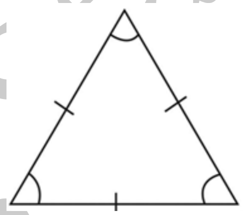


Perimeter=a+b+c+d Perimeter=a+b+c



Perimeter=a+b+c+d+e+f+g+h+i

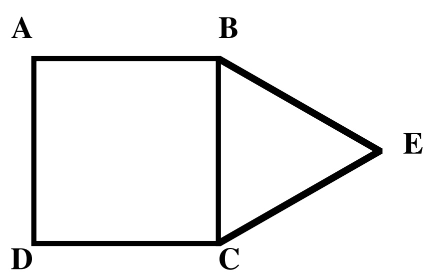
Example 6 A equilateral triangle a triangle in which all three sides are equal. If the perimeter of an equilateral triangle is 18, what is the size of each side?



- **Solution:**

$$\text{The side length is } \frac{18}{3} = \boxed{6}.$$

Example 7 Square ABCD has one side of length 4 cm. Triangle BEC is an equilateral triangle. What is the perimeter of the figure ABECD?

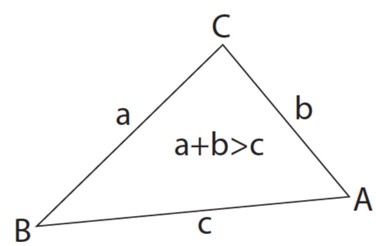


- **Solution:** The perimeter is

$$AB + BE + EC + CD + AD = 4 \times 5 = \boxed{20\ cm}.$$

10.4.2 Side Lengths of a Triangle

In a triangle, the sum of the lengths of any two sides of the triangle is greater than the length of the third side.



Example 8 Find all triangles whose side lengths are integers and perimeter is 12.

- **Solution:** There are three such triangles in terms of side lengths.

$$(5,5,2), \quad (5,4,3), \quad (4,4,4).$$

Note that things like $(7,3,2)$ is not a valid choice since $3+2=5<7$.

Example 9 The sides of a triangle have length 3, 8 and s. If s is a whole number, find all possible values of s?

- **Solution:** In order to form a triangle, we need

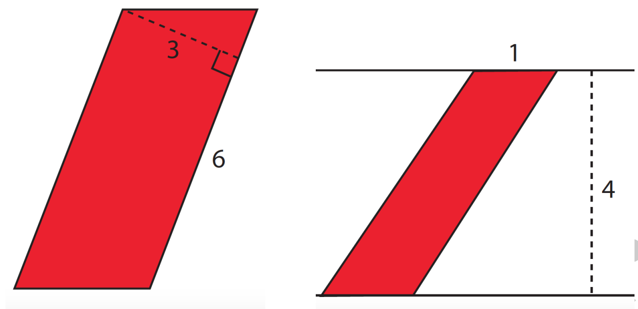
$$s < 3 + 8 = 11 \quad \text{and} \quad s + 3 > 8.$$

Since s is a whole number, all possible values of s are

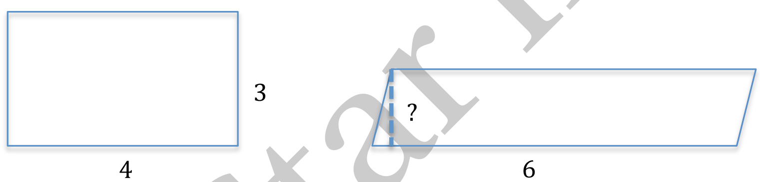
$$6,\ 7,\ 8,\ 9,\ 10.$$

10.5 Exercises

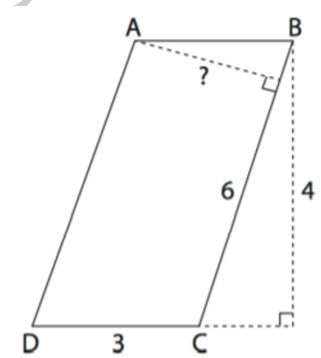
Problem 1 Find the area of the following two parallelograms.



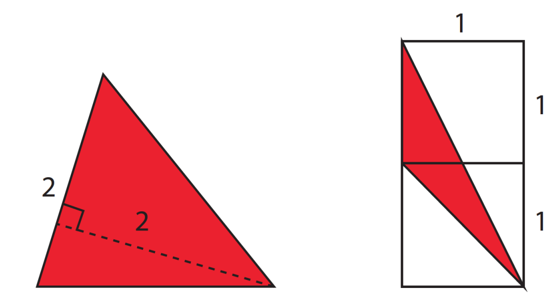
Problem 2 If the following two figures have the same area, find the height of the parallelogram on the right. The left is a rectangle.



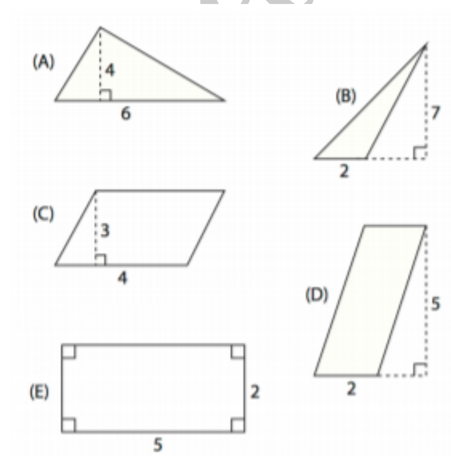
Problem 3 ABCD is a parallelogram. $CD = 3$ and the corresponding height is 4. If $BC = 6$, what is the corresponding height?



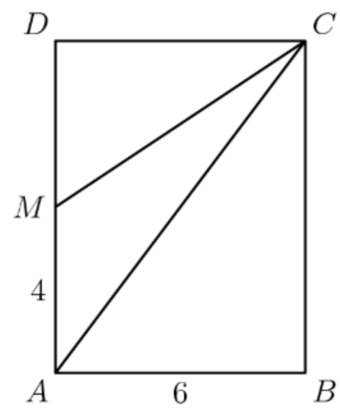
Problem 4 Calculate the areas of the following triangles.



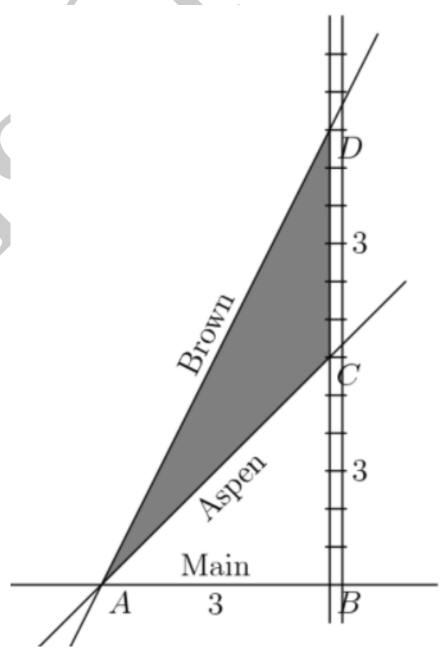
Problem 5 Which of the following two have the largest area?



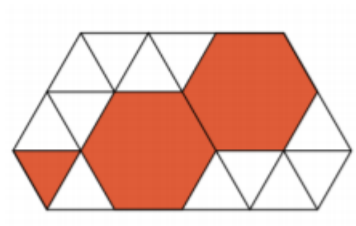
Problem 6 **[AMC8.2016.2]** In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of $\overline{AD}$. What is the area of $\triangle AMC$?



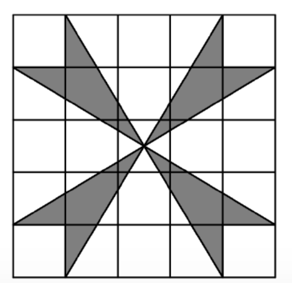
Problem 7 [**AMC8.2009.7**].The triangular plot of ACD lies between Aspen Road, Brown Road and a railroad. Main Street runs east and west, and the railroad runs north and south. The numbers in the diagram indicate distances in miles. The width of the railroad track can be ignored. How many square miles are in the plot of land ACD?



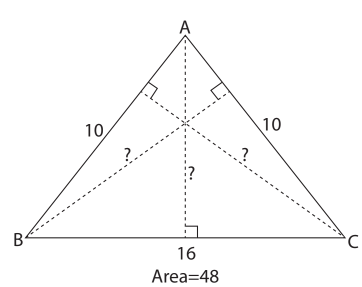
Problem 8 [**Australian Math Competition**] If the area of each small triangle in the following picture is 1, what is the area of the shaded region?



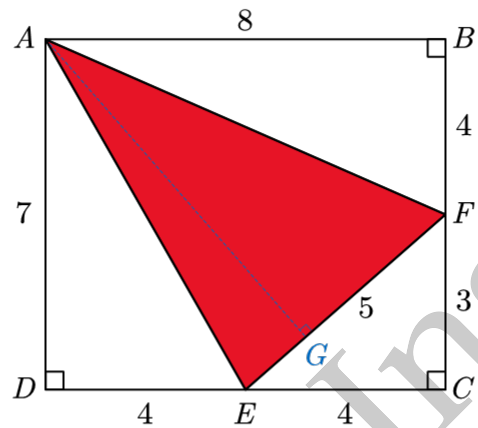
Problem 9 [**AMC8.2007.23**] What is the area of the shaded pinwheel shown in the 5×5 grid?



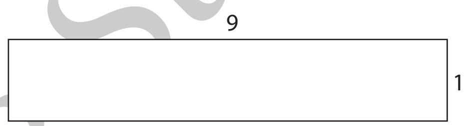
Problem 10 The area of ΔABC is 48. If $AB = AC = 10$ and $BC = 16$, find all corresponding heights.



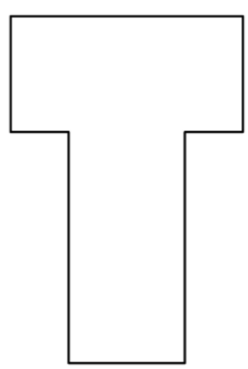
Problem 11 $ABCD$ is 7×8 rectangle. E and F are points on CD and CB respectively. $CE = 4$, $CF = 3$ and $EF = 5$. What is the height AG on EF in ΔAEF?



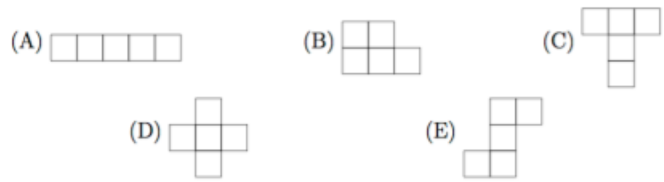
Problem 12 Use a iron wire of 20 cm to make a rectangle with integer side length. Find all possible rectangles (e.g., a 9×1 rectangle is a valid one). Which one has the largest area?



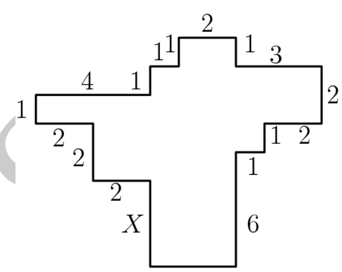
Problem 13 **[AMC8.2006.6]** The letter T is formed by placing two 2×4 inch rectangles next to each other, as shown. What is the perimeter of the T, in inches?



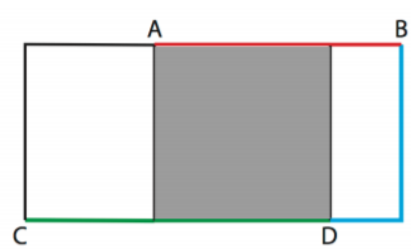
Problem 14 Recall that the perimeter of a shape is the distance around the outside. Which of these shapes has the smallest perimeter?



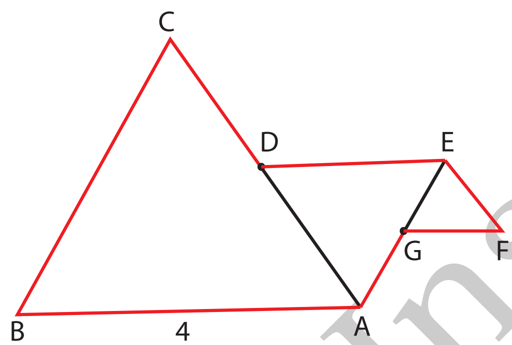
Problem 15 [**AMC8.2012.5**] In the diagram, all angles are right angles and the lengths of the sides are given in centimeters. Note the diagram is not drawn to scale. What is X, in centimeters?



Problem 16 There is a shaded square inside a rectangle as shown. From A to B is 6cm (the red portion) and from C to D is 8cm (the green portion). What is the perimeter of the large rectangle?

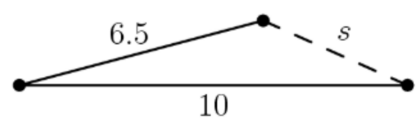


Problem 17 **[AMC8.2000.15]** Triangles ABC, ADE, and EFG are all equilateral. Points D and G are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. If $AB = 4$, what is the perimeter of figure $ABCDEFG$?

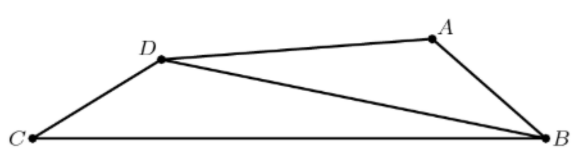


Problem 18 Using 5 sticks of lengths 1, 2, 3, 4 and 5, how many different triangles can we make so that each side consists of only one stick?

Problem 19 **[AJHSME.1992.17]** The sides of a triangle have lengths 6.5, 10, and s, where s is a whole number. What is the smallest possible value of s?



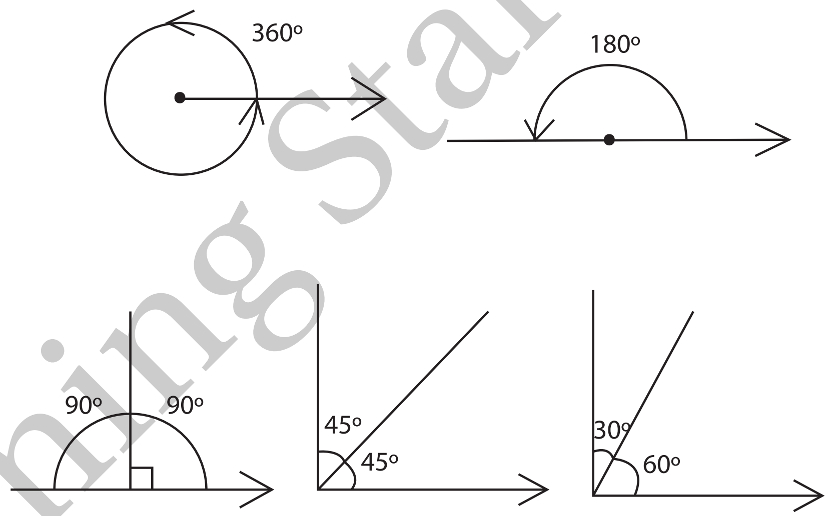
Problem 20 ★★ **[AMC10A.2009.12]** In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



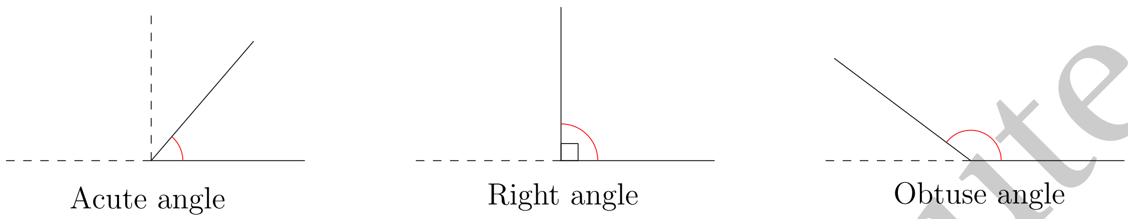
Chapter 11

Geometry III

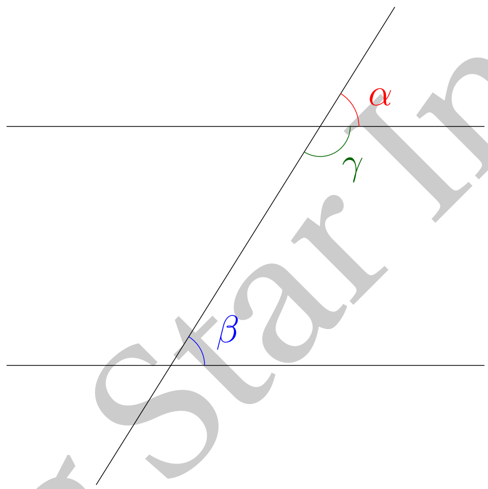
11.1 Basic Concepts of Angles



• A *right* angle measures 90°. An *acute* angle is less than 90°. An *obtuse* angle is greater than 90°, but less than 180°.

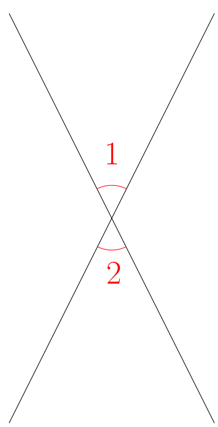


• If two lines are parallel and intersect with the third line, below are the relation between formed angles.



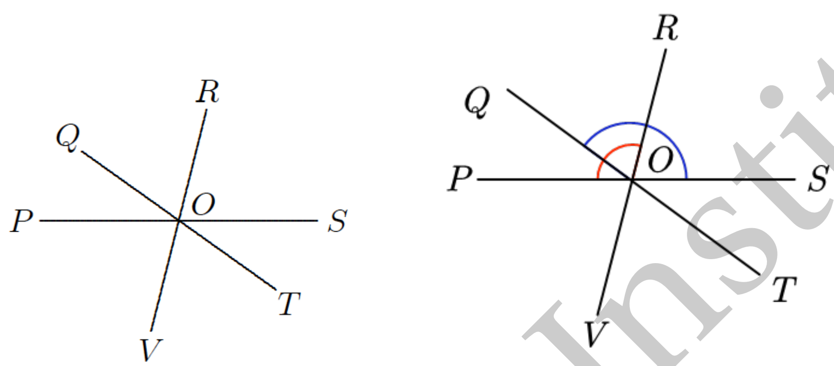
$$\boxed{\angle\alpha = \angle\beta,} \quad \text{equivalently,} \quad \boxed{\angle\beta + \angle\gamma = 180^\circ.}$$

• Two opposite angles are equal.



$$\boxed{\angle 1 = \angle 2.}$$

Example 1: [Australian Math Competition] In the diagram, $\angle POR = 120°$ and $\angle QOS = 145°$. What is the size of $\angle TOV$?



• Solution: Note that

$$\angle POR + \angle QOS = 180^\circ + \angle ROQ.$$

Hence

$$\angle ROQ = 120^\circ + 145^\circ - 180^\circ = 85^\circ$$

and

$$\boxed{\angle TOV = \angle ROQ = 85^\circ.}$$

Example 2 What is the angle ($< 180°$) between the hour hand and minute hand of a clock when the time is $2:30$ pm?

Figure 11.1: Figure from Internet

• **Solution:** Since the full circle on a clock is divided into 12 sections, each section is

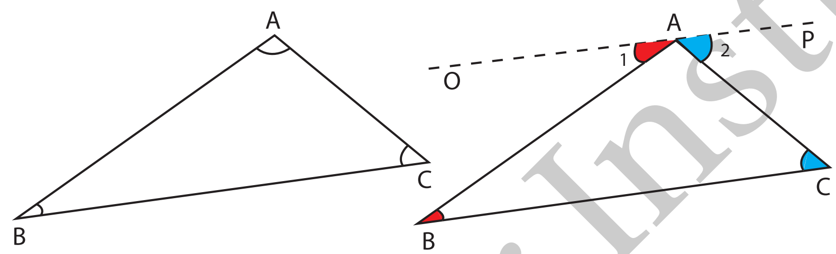
$$360^\circ \div 12 = 30^\circ.$$

At 2:30 pm, there are 3 and half sections between the hour hand and the minute hand. Hence

$$\text{degree } \boxed{= 3 \times 30^\circ + 15^\circ = 105^\circ}.$$

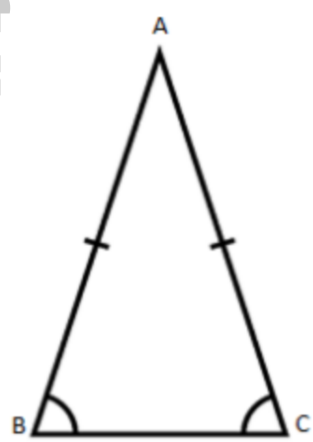
11.2 Sum of Three Angles of a Triangle

The sum of three angles of a triangle is always 180°.



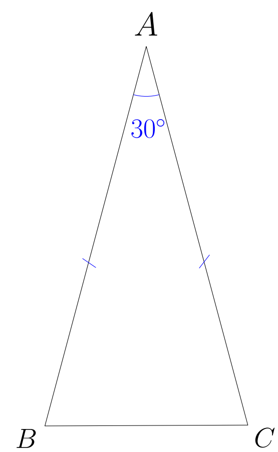
$$\boxed{\angle A + \angle B + \angle C = 180^\circ.} \tag{1}$$

An **"isosceles triangle"** is a triangle with (at least) two equal sides. This is equivalent to saying that two corresponding angles are also equal.



$$\boxed{AB = AC \quad \Leftrightarrow \quad \angle B = \angle C.}$$

Example 3 In ΔABC, $\angle A = 30^\circ$ and $AB = AC$. What is $\angle B$?



• **Solution:** Since $AB = AC$, we have that

$$\angle B = \angle C.$$

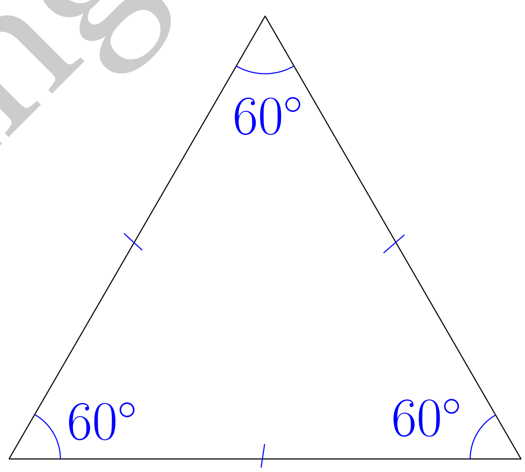
Then

$$\angle A + \angle B + \angle C = 180^\circ \quad \Rightarrow \quad 2\angle B = 180^\circ - 30^\circ = 150^\circ.$$

Hence

$$\boxed{\angle B = 75^\circ.}$$

An *equilateral triangle* is a triangle whose all three sides are equal, equivalently all angles are the same.

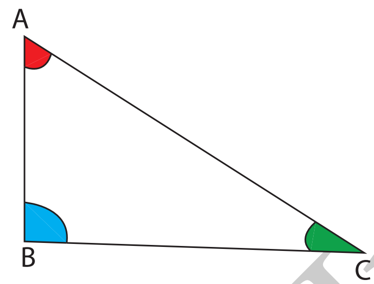


$$\boxed{AB = BC = AC, \quad \angle A = \angle B = \angle C = 60^\circ.}$$

Example 4 In the following triangle ΔABC,

$$\angle C = \frac{\angle A}{2} = \frac{\angle B}{3}.$$

What is $\angle C$?



- **Solution:** Since

$$\underbrace{\angle A}_{=2\angle C} + \underbrace{\angle B}_{=3\angle C} + \angle C = 180^\circ,$$

we have that

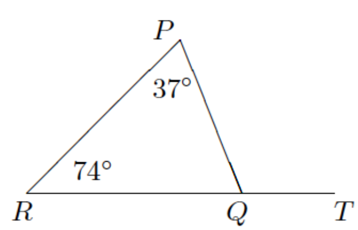
$$6\angle C = 180^\circ.$$

Hence

$$\boxed{\angle C = 30^\circ.}$$

We

Example 5 In the diagram, the size of $\angle PQT$, in degrees, is



- **Solution:** Since

$$\underbrace{\angle P}_{=37^\circ} + \underbrace{\angle R}_{=74^\circ} + \angle PQR = 180^\circ,$$

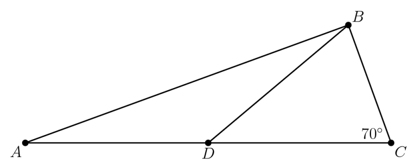
we have that

$$\angle PQR = 69^\circ.$$

Accordingly,

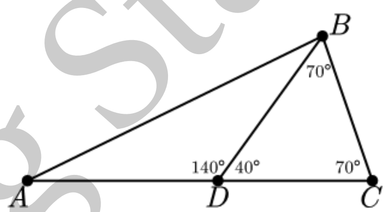
$$\boxed{\angle PQT = 180^\circ - 69^\circ = 111^\circ.}$$

Example 6: [AMC8.2014.9] In ΔABC, D is a point on side $\overline{AC}$ such that $BD = DC$ and $\angle BCD$ measures 70°. What is the degree measure of $\angle ADB$?



• **Solution:** Since $BD = DC$, we have that

$$\angle DBC = \angle C = 70^\circ.$$



Hence

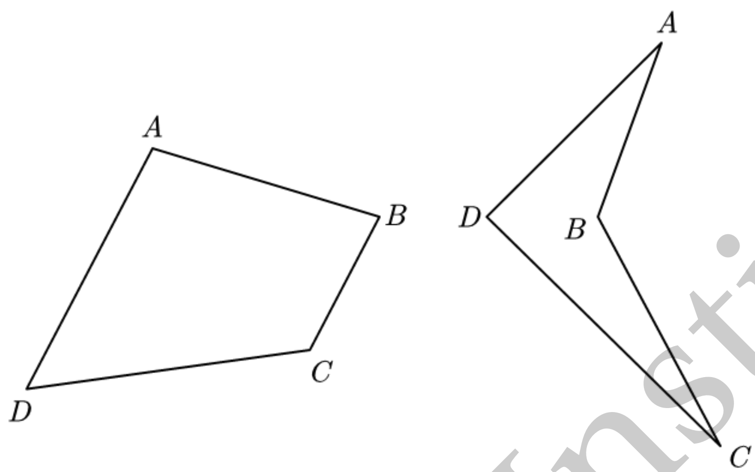
$$\angle BDC = 180^\circ - 70^\circ - 70^\circ = 40^\circ.$$

So

$$\boxed{\angle ADB = 180^\circ - 40^\circ = 140^\circ.}$$

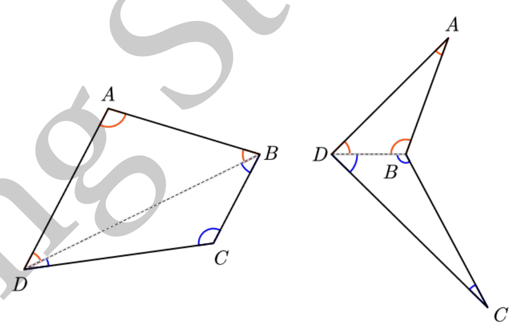
11.3 Sum of Angles of Polygons

• **Quadrilateral**



$$\boxed{\angle A + \angle B + \angle C + \angle D = 360^\circ.}$$

This is true because we can divide a quadrilateral into two triangles.

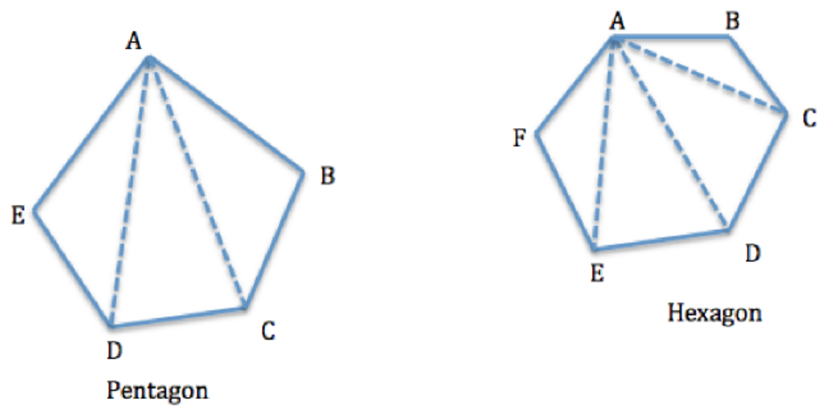


• For a pentagon

$$\boxed{\angle A + \angle B + \angle C + \angle D + \angle E = 3 \times 180^\circ = 540^\circ.}$$

• For a hexagon

$$\boxed{\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 4 \times 180^\circ = 720^\circ.}$$



Example 7 A regular hexagon is a hexagon with equal sides and equal angles. What is the degree of each angle of a regular hexagon? A honeycomb consists of regular hexagons.

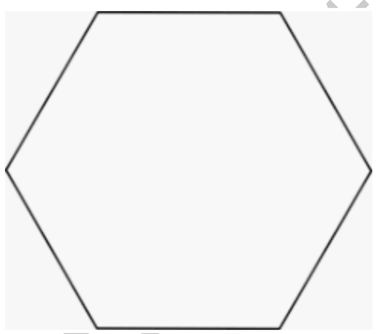
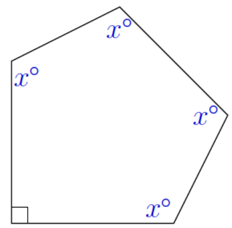


Figure 11.2: Figure from Internet

- **Solution:** Each angle equals

$$\boxed{\frac{720^\circ}{6} = 120^\circ.}$$

Example 8: [Australian Math Competition] What is x in the following figure?



- **Solution:** Since

$$4x + 90 = 540,$$

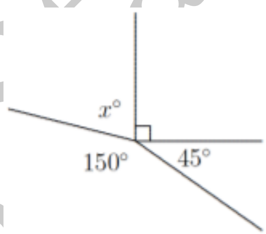
we deduce that

$$4x = 450.$$

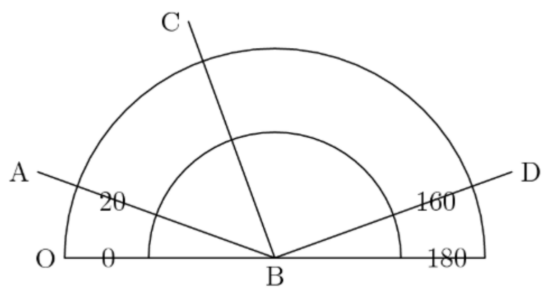
Hence $\boxed{x = 112.5}$.

11.4 Exercises

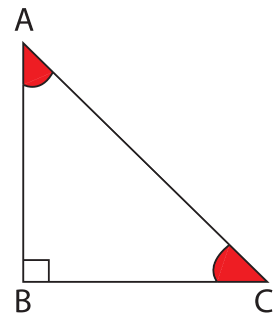
Problem 1 In the diagram, the value of x is



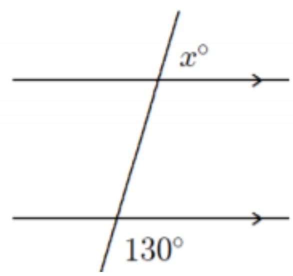
Problem 2 **[AJHSME-1988-5]** If $\angle CBD$ is a right angle, then this protractor indicates that the measure of $\angle ABC$ is



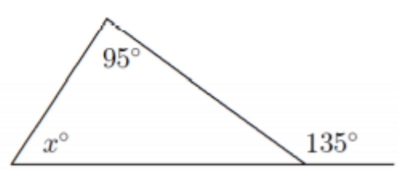
Problem 3 ΔABC is a right triangle with $\angle B = 90°$. If $AB = BC$, what is $\angle A$, n degrees"



Problem 4 In the diagram (two horizontal lines are parallel), the value of x is

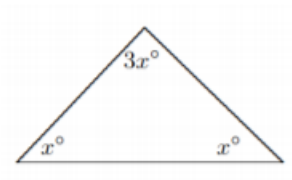


Problem 5 In the following picture, what is x?

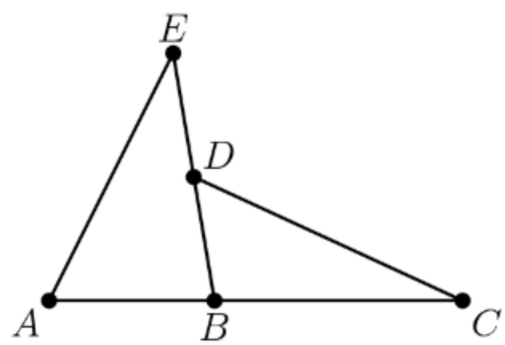


Problem 6 What is the acute angle between hour hand and minute hand of a clock when the time is 3 : 30pm?

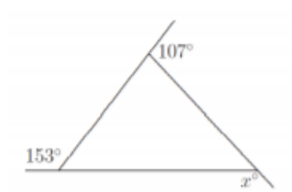
Problem 7 In the diagram, the value of x is



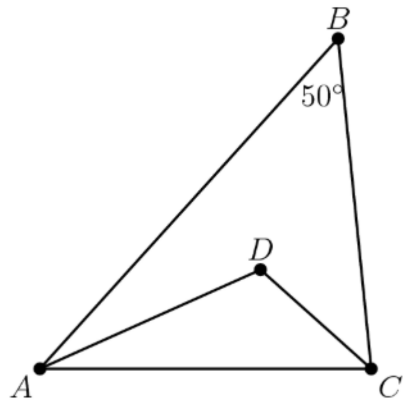
Problem 8 **[AJHSME.1994.7]** If $\angle A = 60°$, $\angle E = 40°$ and $\angle C = 30°$, then $\angle BDC =$



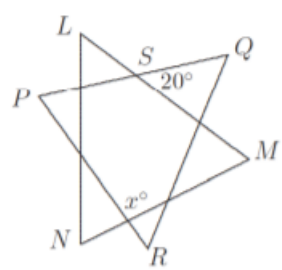
Problem 9 In the diagram, the sides of the triangles are extended and three angles are as shown. The value of x is



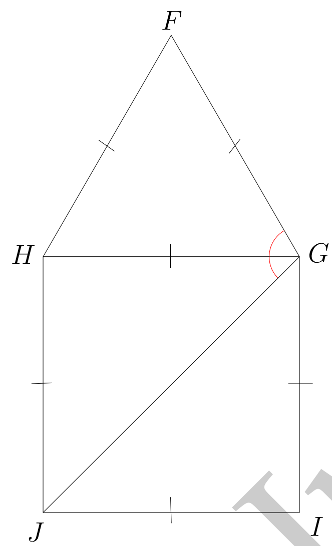
Problem 10 **[AJHSME.1996.24]** The measure of angle ABC is $50°$, $\overline{AD}$ bisects angle BAC, and $\overline{DC}$ bisects angle BCA. The measure of angle ADC is



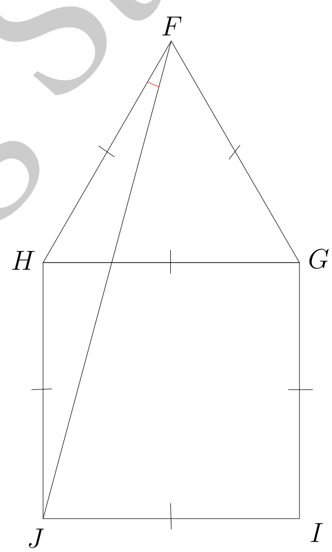
Problem 11 [**Australian Math Competition**] In the diagram, triangles ΔPQR and ΔLMN and are both equilateral and $\angle QSM = 20°$. What is the value of x?



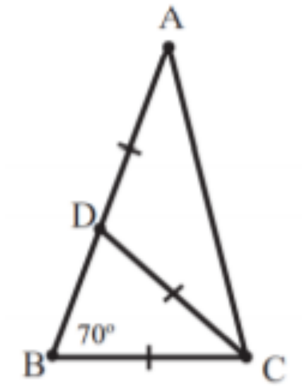
Problem 12 An equilateral triangle ΔFGH sits on top of a square $GIJH$ as shown. The size of the angle $\angle FGJ$ is



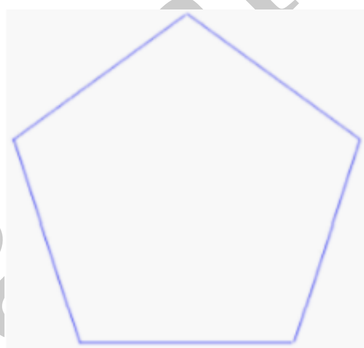
Problem 13 In the above problem, what is $\angle HFJ$?



Problem 14 In ΔABC, D is point on AB. If $AD = CD = BC$ and $\angle B = 70°$, what is $\angle A$?



Problem 15 What is the degree of each angle of a regular pentagon, i.e., a pentagon with equal angles and equal side length?



Problem 16 In quadrilateral $ABCD$, $\angle B = 2\angle A$, $\angle C = 2\angle B$ and $\angle D = 2\angle C$. What is $\angle A$?

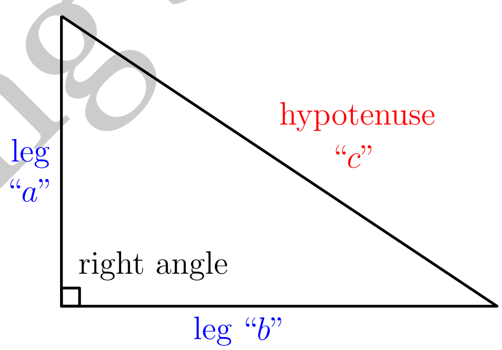
Chapter 12

Geometry IV

12.1 Pythagorean Theorem

Pythagorean Theorem *For a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides (legs).*

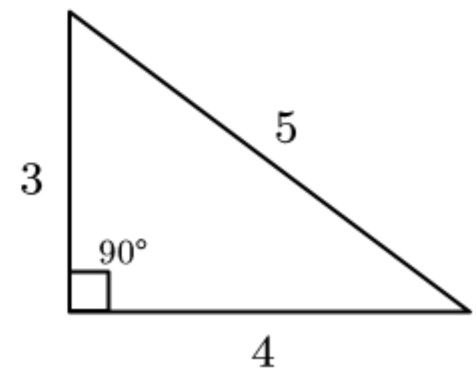
$$a^2 + b^2 = c^2.$$



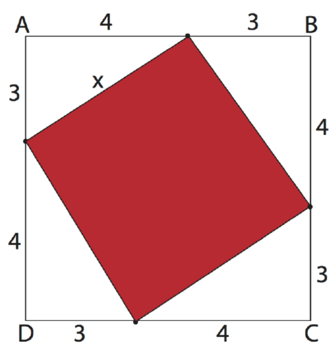
It is named after the ancient Greek philosopher Pythagoras (570 BC–495 BC) .

A typical example is $a = 3$, $b = 4$ and $c = 5$:

$$3^2 + 4^2 = 5^2.$$



Below we demonstrate why the hypotenuse is 5 in the above example by areas.



Area of the red square = (Area of the big square) − (Areas of four identical right triangles)

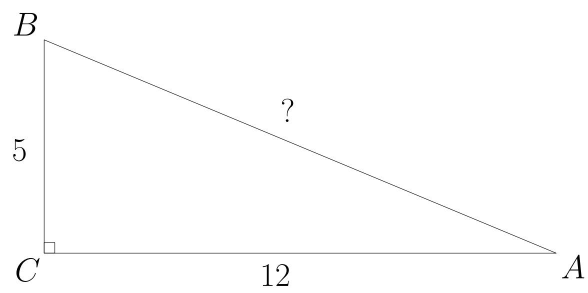
$$= 7^2 - 4 \times \tfrac{1}{2} \times 3 \times 4 = 25.$$

Hence

$$x^2 = \text{area of the red square} = 25.$$

So $\boxed{x = 5}$.

Example 1 What is the hypotenuse AB of a right triangle whose two legs are $BC = 5$ and $AC = 12$?



• **Solution:** According to the Pythagorean theorem,

$$AB^2 = 5^2 + 12^2 = 169.$$

Hence

$$\boxed{c = 13.}.$$

Example 2 What is the hypotenuse of the isosceles right triangle whose two legs are both 1?

• **Solution:** According to the Pythagorean theorem,

$$c^2 = 1^2 + 1^2 = 2.$$

Hence

$$\boxed{c = \sqrt{2}.}$$

Here

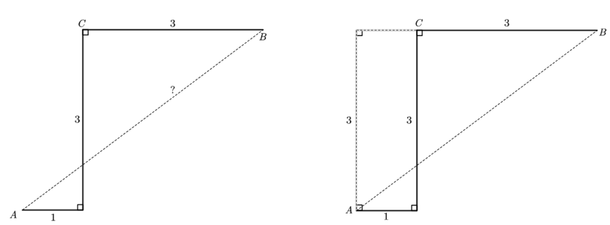
$$\sqrt{a}\text{: the number whose square is } a\text{, i.e., } \sqrt{a} \times \sqrt{a} = a.$$

In particular,

$$\sqrt{2} = 1.41421356237\cdots,$$

where the decimal digits will go forever without repetition.

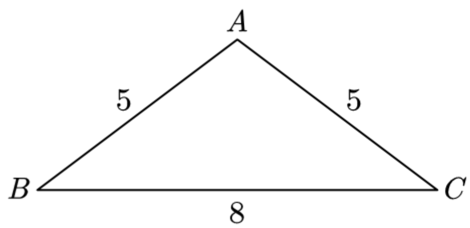
⋆⋆ **Example 3** Starting from location A, a car drives east for one mile, then north for 3 miles, then east for 3 miles and arrives at location B. What is the distance, in miles, from A to B along a straight line?

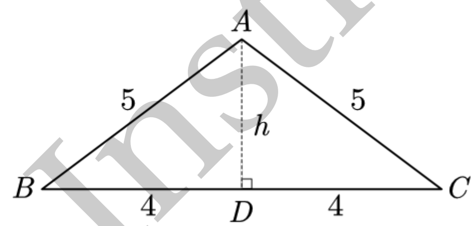


• **Solution:** We may extend BC to form a right triangle whose two legs are 3 and 4 (see the right figure). Then the hypotenuse

$$\boxed{AB = 5.}$$

Example 4 Find the area of the isosceles triangle ΔABC with $AB = AC = 5$ and $BC = 8$.





• **Solution:** Draw the height on the side BC. Due to the symmetry, it bisects BC. Hence by the Pythagorean theorem

$$5^2 = 4^2 + h^2.$$

We obtain the height

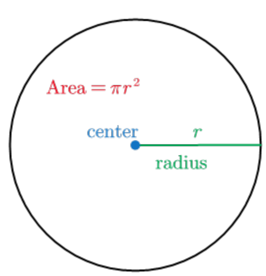
$$h = 3.$$

Hence the area is

$$\boxed{\frac{1}{2} \times 8 \times 3 = 12.}$$

12.2 Circle

12.2.1 Area



Here

$$r\text{: radius of the circle}$$

and

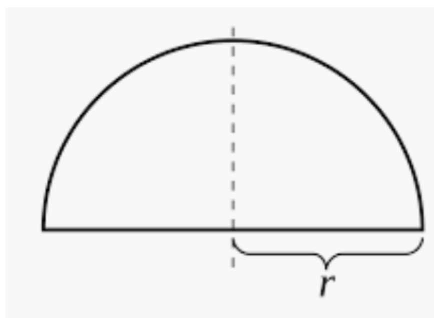
$$\pi = 3.14159265359\cdots$$

Like $\sqrt{2}$, the decimal digits of π go forever without repetition.

$$\pi \approx 3.14$$

is a usual approximation and March 14th is called the "*Pi Day*".

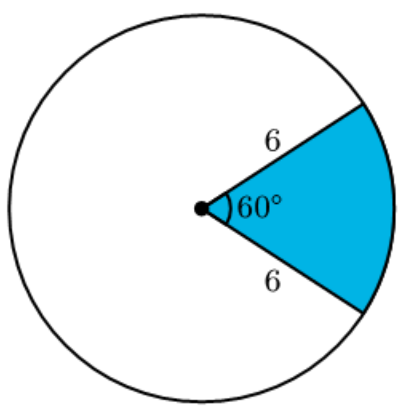
A semi-circle is half of a full circle.



Its area is

$$\boxed{\frac{1}{2} \times \pi r^2.}$$

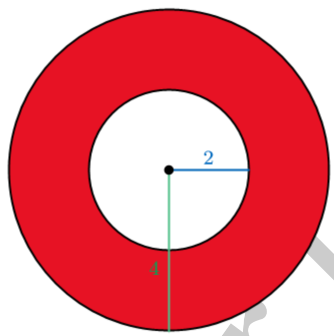
Example 5: [Area of a sector] Find the area of the following shaded region (called "sector").



• **Solution:** Since $360 \div 60 = 6$, this sector is one sixth $(\frac{1}{6})$ of the full circle. Hence its area is

$$\frac{\pi 6^2}{6} = \boxed{6\pi}\, cm^2.$$

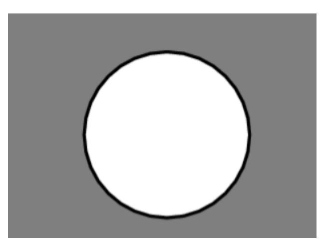
Example 6: [Area of a ring] Find the area of the following shaded region between two circles (called "ring" or "annulus").



• **Solution:** Apparently,

$$\boxed{\text{area of the ring}} = \underbrace{\boxed{\text{area of the large circle}}}_{=16\pi} - \underbrace{\boxed{\text{area of the inner circle.}}}_{=4\pi}$$
$$= \boxed{12\pi\, cm^2.}$$

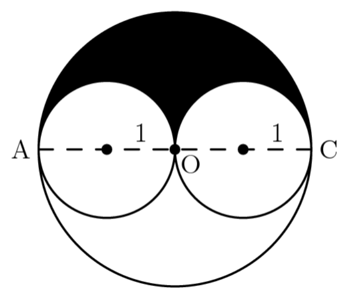
Example 7: Modification of [AJHSME.1992.5] A circle of radius 1 is removed from a 3×4 rectangle, as shown. If we take $\pi = 3$ as an approximation, what is the approximate area of the shaded region?



• **Solution:**

$$\begin{aligned}\text{Area of the shaded region} &= \text{Area of the rectangle} - \text{Area of the circle} \\ &= 3 \times 4 - \pi \approx 12 - 3 = \boxed{9.}\end{aligned}$$

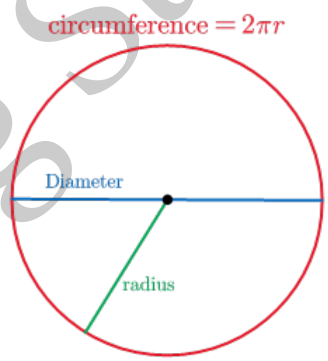
Example 8: [AJHSME.1986.23] The large circle has diameter AC. The two small circles have their centers on AC and just touch at O, the center of the large circle. If each small circle has radius 1, what is area of the shaded region?



• **Solution:** Clearly, the radius of the large circle is $1+1=2$.

$$\begin{aligned}\text{Area of the shaded region} &= \text{Area of large semi-circle} - \text{two small semi-circles}\\ &= \tfrac{1}{2}\times\pi\times 2^2-\pi=\boxed{\pi.}\end{aligned}$$

12.2.2 ⋆⋆ Circumference



Circumference $\boxed{2\pi r}$: the perimeter of a circle with radius r

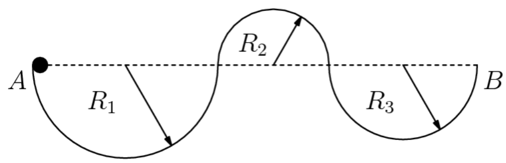
Diameter $2r$: twice of the radius.

Example 9 What is the circumference of a circle with radius 1?

• **Solution:** The circumference is

$$2\pi\times 1=2\pi.$$

Example 10: Simplification of [AMC8.2013.25] A track from A to B is comprised of 3 semicircular arcs whose radii are $R_1 = 100$ inches, $R_2 = 60$ inches, and $R_3 = 80$ inches, respectively. What is the total length of the track?



• **Solution:** The total length is the sum of lengths of three semicircles. Note that

The length of the semicircle with radius R_1 is: 100π

The length of the semicircle with radius R_2 is: 60π

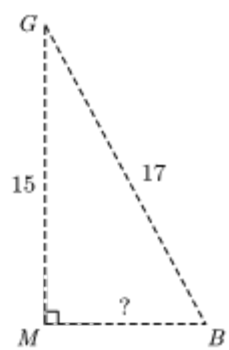
The length of the semicircle with radius R_1 is: 80π.

Hence the total length from A to B is

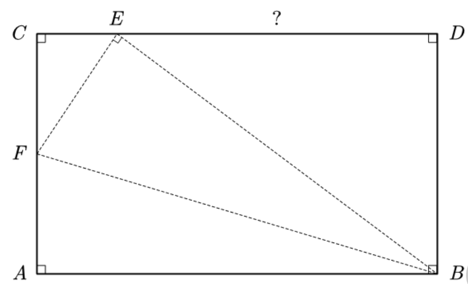
$$\boxed{100\pi + 60\pi + 80\pi = 240\pi.}$$

12.3 Exercises

Problem 1 Three people are sitting on a bus. Miki is directly behind Garrett and directly left of Bonnie. If Garrett and Miki are 15 feet apart, and Bonnie and Garrett are 17 feet apart, what is the distance between Miki and Bonnie, in feet?



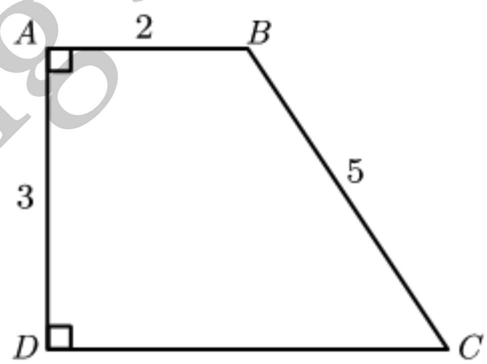
Problem 2 $ABCD$ is a rectangle with $AB = 5$ and $BC = 3$. Fold the rectangle along a line BF so that A falls on point E on CD. What is the length of CE?



Problem 3 $ABCD$ is rectangle with $AB = 3$ and $BC = 2$. What is the square of the diagonal AC?

Problem 4 ΔABC is right triangle with $\angle A = 90°$. If $AB = 6$ and $AC = 8$, what is the hypotenuse BC?

Problem 5 ⋆⋆ $ABCD$ is a trapezoid. $\angle A = \angle D = 90°$. $AB = 2$, $AD = 3$ and $BC = 5$. What is the area of $ABCD$?



Problem 6 What is $\sqrt{1} + \sqrt{4} + \sqrt{9}$?

Problem 7 Find x such that $\sqrt{x} = \sqrt{100} - \sqrt{36}$.

Problem 8 For how many positive integers x, we have that

$$3 < \sqrt{x} < 5\,?$$

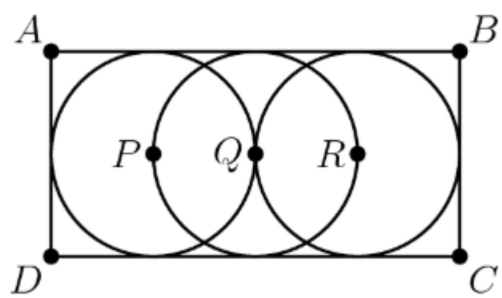
Problem 9 What is

$$\sqrt{3} \times \sqrt{5} \times \sqrt{3} \times \sqrt{5}\,?$$

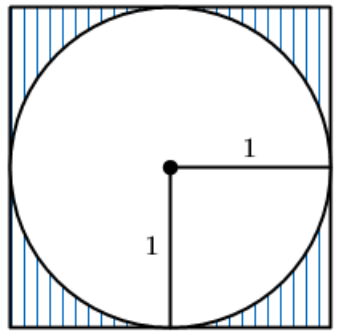
Problem 10 $\sqrt{10}$ represents the number satisfying that $\sqrt{10} \times \sqrt{10} = 10$. What is the integer that is closest to $\sqrt{10}$?

Problem 11 ΔABC is an isosceles triangle. If $AB = AC = 17$ and $BC = 16$, what is the area of ΔABC?

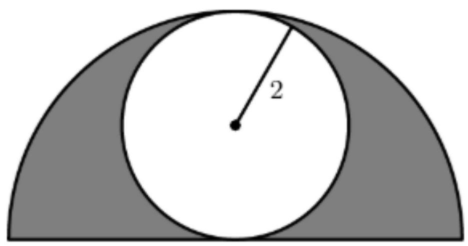
Problem 12 **[AJHSME.1995.9]** Three congruent circles with centers P, Q, and R are tangent to the sides of rectangle $ABCD$ as shown. The circle centered at Q has diameter 4 and passes through points P and R. The area of the rectangle is



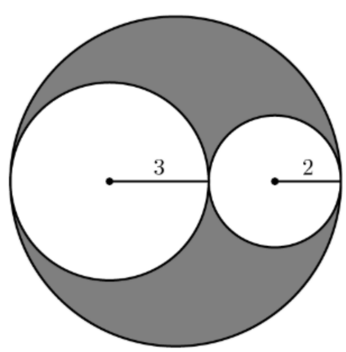
Problem 13 A circle is inscribed in a 2×2 square. What is the area of the shaded region that consists of four identical parts?



Problem 14 **Modification of [AMC10A.2009.6]** A circle of radius 2 is inscribed in a semicircle, as shown. The area inside the semicircle but outside the circle is shaded. What is the area of the shaded part?

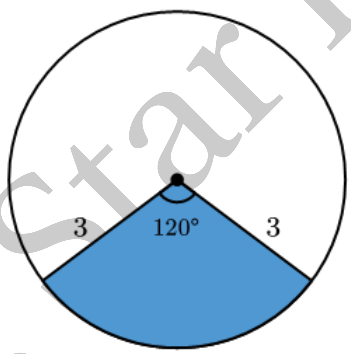


Problem 15 **[AMC10B.2002.5]** Circles of radius 2 and 3 are externally tangent and are circumscribed by a third circle, as shown in the figure. Find the area of the shaded region.

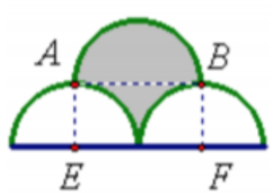


Problem 16 The area of the following blue ring is 3π . If the radius of the outer circle is 2, what is the radius (the green one) of the inner circle?

Problem 17 What is the area of the shaded sector?

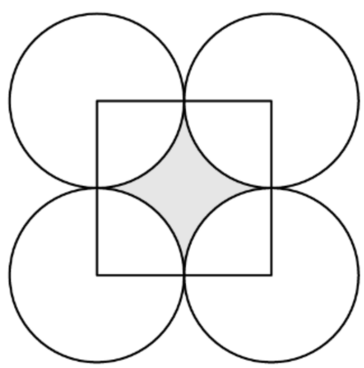


Problem 18 **[Hope Cup Math Competition, China]** 3 semi-circles with radius 1 are put as follows. One is one top of the other two. Points A and B are directly above the centers and of two lower semicircles. The area, in , of the shaded region is



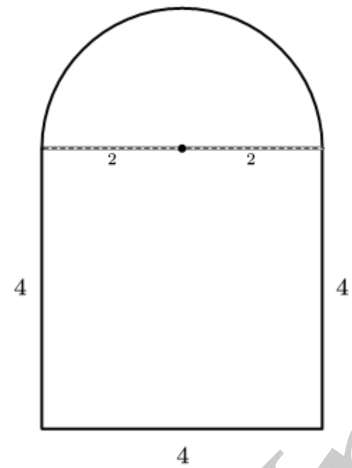
Problem 19 ⋆⋆ Draw two quarter circles centered at two opposite vertices of a square with side length 2. What is the area of the overlap (the shaded region)?

Problem 20 [**AJHSME.1992.24**] Four circles of radius 3 are arranged as shown. Their centers are the vertices of a square. The area of the shaded region is

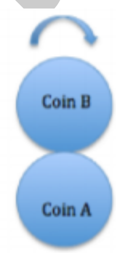


Problem 21 What is the perimeter of a quarter-circle with radius 2?

Problem 22 Put a semi-circle with radius 2 on the top of a square with side length 2. What is the perimeter of the resulting figure?



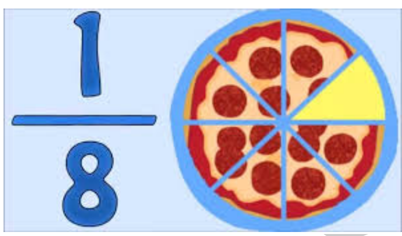
Problem 23 ⋆⋆ **[Coin rotation paradox]** Two US one-penny coins (coin A and coin B) are placed next to each other on the table. Fix coin A and roll coin B completely along the circumference of coin A. How many rotations did coin B make?



Chapter 13

Fractions and Decimals

13.1 Basic Concepts



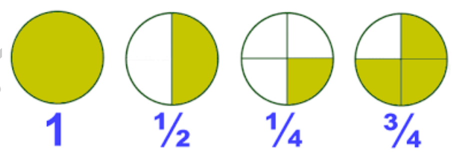


Figure 13.1: Figures are from internet

A fraction has the following form

$$\frac{a}{b}.$$

Here a and b are two numbers. $b \neq 0$. In this chapter, we focus on positive integers a and b.

a: *"numerator"*

b: *"denominator"*.

For example,

$$\frac{1}{2}, \quad \frac{2}{5}, \quad \frac{3}{4}, \quad \frac{8}{12}, \quad \frac{11}{3}, \quad \frac{8}{5}.$$

$$1,\ 2,\ 3,\ 8,\ 11: \quad \text{numerators}$$

$$2,\ 5,\ 4,\ 12,\ 3: \quad \text{denominators.}$$

Fractions can be categorized as proper and improper fractions.

• "*Proper fractions*": the numerator $\boxed{a < b}$ the denominator. Among above examples,

$$\frac{1}{2},\quad \frac{2}{5},\quad \frac{3}{4},\quad \frac{8}{12}$$

are proper fractions.

• "*Improper fractions*": the numerator $\boxed{a \geq b}$ the denominator. Among above examples,

$$\frac{11}{3},\quad \frac{8}{5}$$

are improper fractions. An improper fraction can also be expressed as a "*mixed number*", i.e., a non-zero integer plus a proper fraction. For example,

$$\frac{11}{3} = 3 + \frac{2}{3} = \boxed{3\frac{2}{3}}$$

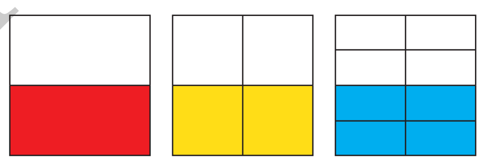
$$\frac{8}{5} = 1 + \frac{3}{5} = \boxed{1\frac{3}{5}}$$

Hereafter, we will focus on proper fractions. It is important to known that a fraction has infinitely many equivalent forms. For example,

$$2 = \frac{2}{1} = \frac{4}{2} = \cdots$$

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \cdots$$

$$\frac{2}{5} = \frac{4}{10} = \frac{8}{20} = \cdots$$



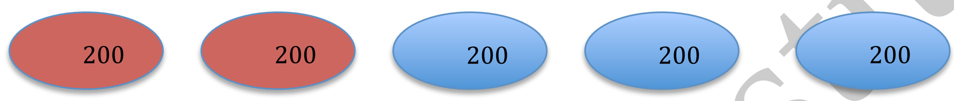
$\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

We can always reduce a fraction to its **simplest form** by dividing the *greatest common divisor* of the numberator and denominator. For example,

$$\boxed{\frac{16}{20} = \frac{4 \times 4}{5 \times 4} = \frac{4 \times \cancel{4}}{5 \times \cancel{4}} = \frac{4}{5}.}$$

Example 1 Euclid elementary school has 1000 students. $\frac{2}{5}$ (two-fifths) of students are boys. What is the number of boys?

Solution: We evenly divided 1000 students into 5 groups. Then each group has $\boxed{1000 \div 5 = 200}$ members.



$$\frac{2}{5}$$

Since $\frac{2}{5}$ (two-fifths) of students are boys, two groups are boys. So the number of boys is

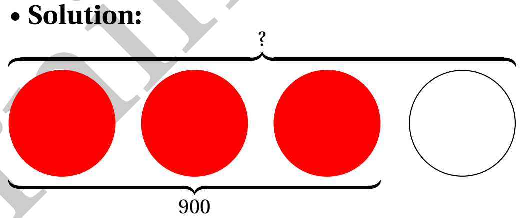
$$200 \times 2 = 400.$$

Another way to formulate the above procedure is:

$$\text{Number of boys} = \boxed{1000 \times \frac{2}{5} = \cancel{1000}^{200} \times \frac{2}{\cancel{5}} = 200 \times 2 = 400.}$$

Example 2 Three-fourths of families in City A have 2 kids. If the number families with 2 kids is 900. What is the total number of families in City A?

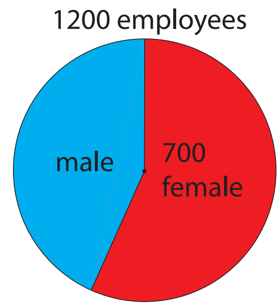
- **Solution:**



Owing to the above picture, it is clear that

$$\text{the total number} = \frac{900}{3} \times 4 = \boxed{1200}.$$

Example 3 A chocolate factory has 1200 employees. 700 of them are female. What fraction of employees are male?



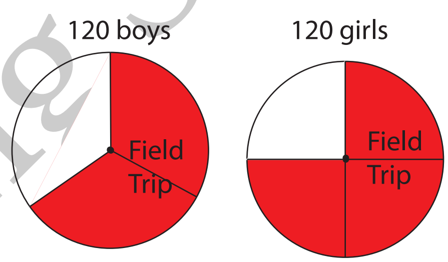
Solution: Clearly, there are

$$1200 - 700 = 500 \quad \text{male employees.}$$

Hence the fraction is

$$\boxed{\frac{500}{1200} \underbrace{=}_{\text{simplify}} \frac{5}{12}.}$$

Problem 4: Modification of [AMC8.2016.12] Jefferson Middle School has 120 boys and 120 girls. Three-fourths of the girls and two-thirds of the boys went on a field trip. What fraction of the students on the trip were girls?



• **Solution:** Note that

$$\text{fraction} = \frac{\text{number of girls (or boys) on the trip}}{\text{total number of kids on the trip}}.$$

Clearly,

$$\text{number of girls on the trip} = 120 \times \tfrac{3}{4} = \cancel{120}^{30} \times \tfrac{3}{\cancel{4}} = 30 \times 3 = 90$$

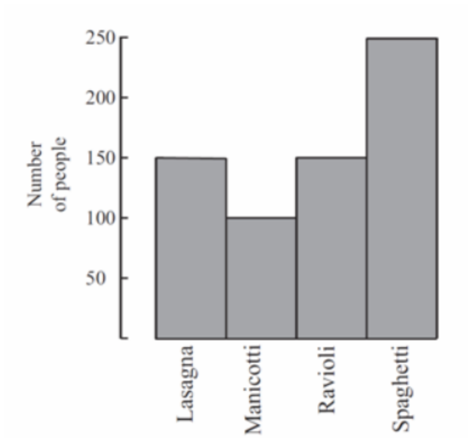
$$\text{number of boys on the trip} = 120 \times \tfrac{2}{3} = \cancel{120}^{40} \times \tfrac{2}{\cancel{3}} = 40 \times 2 = 80.$$

Hence

$$\text{fraction} = \frac{90}{90+80} = \boxed{\frac{9}{17}}.$$

The *ratio* between two numbers A and B can also be expressed in the form of simplest fraction. For example, the ratio of 12 to 10 is $12:10$, or $6:5$ or $\frac{6}{5}$.

Example 5: [AMC8.2007.2] 650 students were surveyed about their pasta preferences. The choices were lasagna, manicotti, ravioli and spaghetti. The results of the survey are displayed in the bar graph. What is the ratio of the number of students who preferred spaghetti to the number of students who preferred manicotti?



• **Solution:** By definition,

$$\boxed{\text{Ratio} = \frac{\text{The number of students who preferred spaghetti}}{\text{The number of students who preferred manicotti}} = \frac{250}{100} = \frac{5}{2} = 5:2.}$$

Example 6: [AJHSME.1997.10] What fraction of this square region is shaded? Stripes are equal in width, and the figure is drawn to scale.

• **Solution:** We may assume that the width of strip is 1. Then the area of the square is

$$6 \times 6 = 36.$$

By cutting shaded regions into rectangles and 1×1 square, we can deduce that the total area of shaded region is

$$3 + (2 \times 3 \times 1 + 1) + (2 \times 5 \times 1 + 1) = 3 + 7 + 11 = 21.$$

Details are left to readers as an exercise. Hence the fraction is

$$\boxed{\frac{21}{36} = \frac{7}{12}.}$$

Example 7: [AMC8.2008.7] If

$$\frac{3}{5} = \frac{M}{45} = \frac{60}{N},$$

what is $M + N$?

• **Solution:** Since

$$45 = 5 \times 9 \quad \text{and} \quad 60 = 3 \times 20,$$

we have that

$$M = 3 \times 9 = 27 \quad \text{and} \quad N = 5 \times 20 = 100,$$

i.e.,

$$\frac{3}{5} = \frac{27}{45} = \frac{60}{100}.$$

So

$$\boxed{M + N = 127.}$$

Example 8: Modification of [AMC10B.2010.10] How many different pairs of positive integers (M, N) satisfy the equation

$$\frac{M}{2} = \frac{3}{N}.$$

• **Solution:** There are two cases.

Case 1: Both $\frac{M}{2} = \frac{3}{N}$ is a proper fraction, which implies that $M < 2$, i.e., $M = 1$. Then we can only have

$$(M, N) = (1, 6).$$

, i.e.,

$$\frac{1}{2} = \frac{3}{6}.$$

Case 2: Both $\frac{M}{2} = \frac{3}{N}$ is an improper fraction, which implies that $N \le 3$, i.e., $N = 1$, 2, or 3. Then we can only have

$$(M,N) = (2,3), \quad (M,N) = (3,2) \quad \text{and} \quad (M,N) = (6,1).$$

i.e,

$$\frac{2}{2} = \frac{3}{3}, \quad \frac{3}{2} = \frac{3}{2} \quad \text{and} \quad \frac{6}{2} = \frac{3}{1}.$$

Hece there are $\boxed{4}$ valid pairs in total.

⋆⋆ Note that we can observe from above valid pairs of (M,N) that

$$\frac{M}{2} = \frac{3}{N} \quad \Leftrightarrow \quad MN = 3 \times 2 = 6.$$

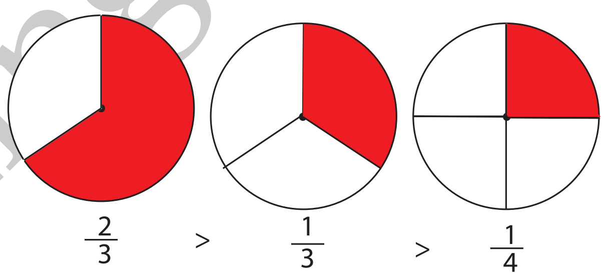
If fact, by using the common denominator $2N$,

$$\frac{M}{2} = \frac{MN}{2N} \quad \text{and} \quad \frac{3}{N} = \frac{6}{2N}.$$

13.2 How to Compare Two Fractions

Some are easy to compare. For example,

$$\frac{2}{3} > \frac{1}{3} \quad \text{and} \quad \frac{1}{3} > \frac{1}{4}.$$



Some are less obvious.

Example 9 Compare

$$\frac{3}{5} \quad \text{and} \quad \frac{5}{9}.$$

- Solution: Below is a common procedure.

Step I: Find a common denominator. The *least common multiple (lcm)* is a popular choice.

$$\boxed{\frac{3}{5}=\frac{27}{45}} \quad \text{and} \quad \boxed{\frac{5}{9}=\frac{25}{45}}.$$

Here

$$45 = lcm(5,9)$$

is a common denominator.

Step II: Now it is clear that

$$\boxed{\frac{3}{5}=\frac{27}{45}>\frac{25}{45}=\frac{5}{9}.}$$

Example 10 Compare

$$\frac{50}{49} \quad \text{and} \quad \frac{51}{50}.$$

• **Solution:** For this problem, instead of implementing the general procedure in the previous example, we use a shortcut by using mixed numbers. Notice that

$$\frac{50}{49}=1+\frac{1}{49}=1\frac{1}{49} \quad \text{and} \quad \frac{51}{50}=1+\frac{1}{50}=1\frac{1}{50}.$$

It is obvious that $\frac{1}{49}>\frac{1}{51}$. We deduce that

$$\boxed{\frac{50}{49}>\frac{51}{50}.}$$

13.3 Addition and Subtraction between Fractions

If the denominator is the same, the Addition and Subtraction are straightforward. For example,

$$\frac{1}{5}+\frac{2}{5}=\frac{1+2}{5}=\frac{3}{5}, \quad \frac{2}{5}-\frac{1}{5}=\frac{2-1}{5}=\frac{1}{5}.$$

If denominators are not the same, it requires more work. For example, we want to compute

$$\frac{1}{3}+\frac{1}{4}$$

and

$$\frac{1}{3}-\frac{1}{4}.$$

The common procedure is

Step 1: Find a common denominator; The *least common multiple (lcm)* is a popular choice. We write

$$\boxed{\frac{1}{3}=\frac{4}{12}} \quad \text{and} \quad \boxed{\frac{1}{4}=\frac{3}{12}.}$$

Here

$$12 = lcm(3,4)$$

is a common denominator.

Step 2: Add or subtract the new numerators

$$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{4+3}{12}=\frac{7}{12},$$

$$\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{4-3}{12}=\frac{1}{12}.$$

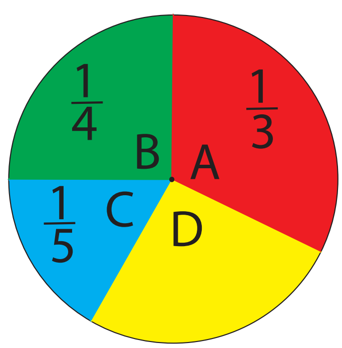
One more example,

$$\frac{3}{8}+\frac{1}{6}=\frac{9}{24}+\frac{4}{24}=\frac{13}{24} \quad \text{“24" is a common denominator,}$$

$$\frac{3}{8}-\frac{1}{6}=\frac{9}{24}-\frac{4}{24}=\frac{5}{24} \quad \text{“24" is a common denominator.}$$

Example 11: Modification of [AMC10B.2015.4] Four siblings ordered an extra large pizza. A ate $\frac{1}{3}$, B $\frac{1}{4}$, and C $\frac{1}{5}$ of the pizza. D got the leftovers.

Who ate more, D or C?



• **Solution:** The total fraction eaten by A, B and C is

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{20}{60}+\frac{15}{60}+\frac{12}{60}=\frac{47}{60}.$$

Here we choose 60 as the common denominator. Hence D ate

$$1-\frac{47}{60}=\frac{13}{60}$$

of the pizza. Since C ate $\frac{1}{5}=\frac{12}{60}$ of the pizza, $\boxed{D}$ ate more.

⋆⋆ **Example 12** Compute

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{8\times 9}+\frac{1}{9\times 10}.$$

• **Solution:** Here the key is to notice that

$$\frac{1}{1\times 2}=1-\frac{1}{2},\quad \frac{1}{2\times 3}=\frac{1}{2}-\frac{1}{3},\quad \cdots,\quad \frac{1}{9\times 10}=\frac{1}{9}-\frac{1}{10}.$$

Accordingly,

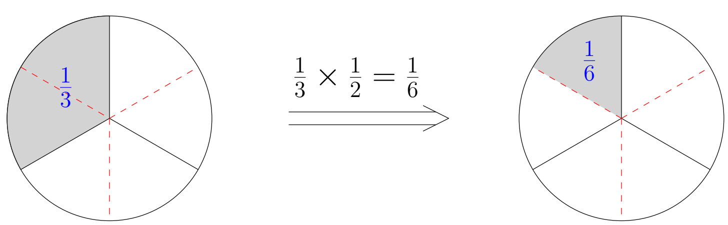
$$\begin{aligned}&\tfrac{1}{1\times2}+\tfrac{1}{2\times3}+\tfrac{1}{3\times4}+\cdots+\tfrac{1}{8\times9}+\tfrac{1}{9\times10}\\&=1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots+\tfrac{1}{8}-\tfrac{1}{9}+\tfrac{1}{9}-\tfrac{1}{10}\\&=1-\tfrac{1}{10}=\boxed{\frac{9}{10}.}\end{aligned}$$

13.4 Multiplication of Fractions

Multiple corresponding numerators and denominators. Simplify the result if needed. For example,

$$\frac{1}{3}\times\frac{1}{2}=\frac{1\times 1}{3\times 2}=\frac{1}{6}.$$

We may say that one half of $\frac{1}{3}$ is $\frac{1}{6}$.



Below are two other examples.

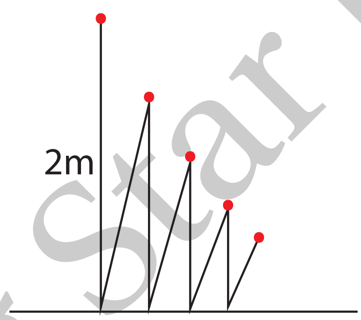
$$\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4} = \frac{6}{20} = \frac{3}{10}.$$

$$\frac{1}{2} \times \frac{3}{4} \times \frac{1}{5} = \frac{1 \times 3 \times 1}{2 \times 4 \times 5} = \frac{3}{40}.$$

• A short cut for multiplying fractions is called "*cancellation*". For example,

$$\frac{2}{5} \times \frac{3}{4} = \frac{\cancel{2}^1}{5} \times \frac{3}{\cancel{4}^2} = \frac{1 \times 3}{5 \times 2} = \frac{3}{10}.$$

Example 13: Modification of [AMC8.2008.12] A ball is dropped from a height of 3 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to $\frac{2}{3}$ of the height it reached in the previous bounce. What is the height does it rise on the fourth bounce?



• **Solution:** After the 1st bouncing, its height is

$$2 \times \frac{2}{3} = \frac{4}{3}.$$

After the 2nd bouncing, its height is

$$\frac{4}{3} \times \frac{2}{3} = \frac{8}{9}.$$

After the 3rd bouncing, its height is

$$\frac{8}{9} \times \frac{2}{3} = \frac{16}{27}.$$

After the 4th bouncing, its height is

$$\frac{16}{27} \times \frac{2}{3} = \boxed{\frac{32}{81}.}$$

Note that the whole process can also be expressed as

$$\boxed{2\times\underbrace{\frac{2}{3}}_{\text{1st bounce}}\times\underbrace{\frac{2}{3}}_{\text{2nd bounce}}\times\underbrace{\frac{2}{3}}_{\text{3rd bounce}}\times\underbrace{\frac{2}{3}}_{\text{4th bounce}}=\frac{32}{81}.}$$

Example 14 What is

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}?$$

- **Solution:** The short-cut is to apply the "cancellation".

$$\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{4}}{\cancel{5}}\times\frac{\cancel{5}}{6}=\boxed{\frac{1}{6}.}$$

13.5 ⋆⋆ Division between Fractions

$$\frac{a}{b}\div\frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\times\underbrace{\frac{d}{c}}_{\text{flip the second}}=\frac{a\times d}{b\times c}.$$

For example,

$$1\div\frac{2}{3}=\frac{1}{\frac{2}{3}}=1\times\underbrace{\frac{3}{2}}_{\text{flip}}=\frac{3}{2}$$

$$\frac{4}{5}\div\frac{3}{7}=\frac{\frac{4}{5}}{\frac{3}{7}}=\frac{4}{5}\times\underbrace{\frac{7}{3}}_{\text{flip}}=\frac{4\times 7}{5\times 3}=\frac{28}{15}.$$

13.6 ⋆⋆ Fractions and Decimal Expression

Example 15: (Simple cases)

$$\frac{3}{10}=0.3,\quad\frac{47}{10}=4.7,\quad\frac{57}{100}=0.57,\quad\frac{125}{100}=1.25,\quad\frac{234}{1000}=0.234,\quad\frac{1234}{1000}=1.234.$$

Example 16: (Other finite decimals)

$$\frac{1}{2}=0.5,\quad\frac{1}{4}=0.25,\quad\frac{1}{8}=0.125$$

$$
\begin{array}{r l}
 & 0.\;1\;2\;5 \\
\cline{2-2}
8 & \big)\;1.\;0\;0\;0 \\
 & \;\;0.\;8 \\
\cline{2-2}
 & \;\;\;\;\;\;2\;0 \\
 & \;\;\;\;\;\;1\;6 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;4\;0 \\
 & \;\;\;\;\;\;\;\;\;4\;0 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;\;\;\;0
\end{array}
$$

Example 17: (Repeating decimals)

$$\frac{1}{3} = 0.333\cdots \underbrace{=}_{\text{write as}} 0.\overline{3}.$$

$$
\begin{array}{r l}
 & 0.\;3\;3\;3\;3\;\cdots \\
\cline{2-2}
3 & \big)\;1.\;0\;0\;0\;0\;\cdots \\
 & \;\;0.\;9 \\
\cline{2-2}
 & \;\;\;\;\;\;1\;0 \\
 & \;\;\;\;\;\;1\;8 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;2\;0 \\
 & \;\;\;\;\;\;\;\;\;0\;9 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;\;\;\;1\;0 \\
 & \;\;\;\;\;\;\;\;\;\;\;\;0\;9 \\
\cline{2-2}
 & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;1 \\
 & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\ddots
\end{array}
$$

$$\frac{3}{11} = 0.272727\cdots \underbrace{=}_{\text{write as}} 0.\overline{27}.$$

$$
\begin{array}{r@{\,}l}
 & 0.\ 2\ 7\ 2\ 7\ \cdots \\
\cline{2-2}
11 \big) & 3.\ 0\ 0\ 0\ 0\ \cdots \\
 & 2.\ 2 \\
\cline{2-2}
 & \quad\ 8\ 0 \\
 & \quad\ 7\ 7 \\
\cline{2-2}
 & \qquad 3\ 0 \\
 & \qquad 2\ 2 \\
\cline{2-2}
 & \qquad\quad 8\ 0 \\
 & \qquad\quad 7\ 7 \\
\cline{2-2}
 & \qquad\qquad 3 \\
 & \qquad\qquad\quad \ddots
\end{array}
$$

$$
\frac{41}{333} = 0.123123123\cdots \underbrace{=}_{\text{denoted as}} 0.\overline{123}
$$

$$
\begin{array}{r@{\,}l}
 & \ \ 0.\ 1\ 2\ 3\ 1\ \cdots \\
\cline{2-2}
333 \big) & 41.\ 0\ 0\ 0\ 0\ \cdots \\
 & 33.\ 3 \\
\cline{2-2}
 & 7\ \ 7\ 0 \\
 & 6\ \ 6\ 6 \\
\cline{2-2}
 & 1\ \ 0\ 4\ 0 \\
 & \quad\ 9\ 9\ 9 \\
\cline{2-2}
 & \qquad 4\ 1\ 0 \\
 & \qquad 3\ 3\ 3 \\
\cline{2-2}
 & \qquad\quad 7\ 7\ \ 0 \\
 & \qquad\qquad\quad \ddots
\end{array}
$$

13.6.1 Arithmetic rule between Decimal Numbers

Adding and subtracting two decimals

This is almost the same as addition and subtraction between whole numbers. The only difference is that we need to make sure that the decimal point ‘‘." is along the same column when we calculate decimals in the vertical formulation.

Example 18

$$3.781 + 4.352 = 8.133 \qquad 1.52 + 2.6 = 4.12$$

$$\begin{array}{cccccc} & 4 & . & 3 & 5 & 2 \\ + & 3 & . & 7 & 8 & 1 \\ \hline & 8 & . & 1 & 3 & 3 \end{array} \qquad \begin{array}{ccccc} & 1 & . & 5 & 2 \\ + & 2 & . & 6 & 0 \\ \hline & 4 & . & 1 & 2 \end{array}$$

$$3.11 - 2.62 = 0.49 \qquad 4 - 1.567 = 2.433.$$

$$\begin{array}{ccccc} & 3 & . & 1 & 1 \\ - & 2 & . & 6 & 2 \\ \hline & 0 & . & 4 & 9 \end{array} \qquad \begin{array}{cccccc} & 4 & . & 0 & 0 & 0 \\ - & 1 & . & 5 & 6 & 7 \\ \hline & 2 & . & 4 & 3 & 3 \end{array}$$

Multiplication between Decimals

Let us use the following example to demonstrate the algorithm.

Example 19 Compute

$$2.3 \times 1.45.$$

This is very similar to multiplication between whole numbers.

Step 1: Pretend that there are no decimal points. We calculate

$$145 \times 23 = 3335.$$

Step 2: Count the total numbers of digits after the decimal in each fact. Here 2.3 has 1 number. 1.45 has 2 numbers. Here the total is 1+2=3.

Step 3: Put a decimal point on 3335 so that there are 3 digits after the decimal point, which leads to $\boxed{3.335.}$

$$\begin{array}{cccc} & 1. & 4 & 5 \\ \times & & 2. & 3 \\ \hline & 4 & 3 & 5 \\ 2 & 9 & 0 & \\ \hline 3. & 3 & 3 & 5 \end{array}$$

Note: The above algorithm can also be understood from the point view of fraction multiplication:

$$2.3 = \frac{23}{10}, \quad 1.45 = \frac{145}{100} \quad \Rightarrow \quad \boxed{2.3 \times 1.45 = \frac{23 \times 145}{10 \times 100} = \frac{3335}{1000} = 3.335.}$$

Division of a Decimal by a Whole Number

It is very similar to division between whole numbers

Example 20

$$1.2 \div 3 = 0.4$$

$$\begin{array}{r|l} & 0.\,4 \\ \hline 3 & 1.\,2 \\ & 1.\,2 \\ \hline & \quad 0 \end{array}$$

$$6.24 \div 4 = 1.56.$$

$$\begin{array}{r|lll} & 1. & 5 & 6 \\ \hline 4 & 6. & 2 & 4 \\ & 4 & & \\ \hline & 2 & 2 & \\ & 2 & 0 & \\ \hline & & 2 & 4 \\ & & 2 & 4 \\ \hline & & & 0 \end{array}$$

13.7 Exercise

Problem 1 **[AJHSME.1998.10]** Each of the letters W, X, Y, and Z represents a different integer in the set $\{1, 2, 3, 4\}$, but not necessarily in that order. If $\frac{W}{X} - \frac{Y}{Z} = 1$, then the sum of W and Y is

Problem 2 If

$$\frac{2}{3} = \frac{M}{12} = \frac{10}{N},$$

what is $N - M$?

Problem 3 Write the fraction

$$\frac{2+4+6+\cdots+22}{3+6+9+\cdots+33}$$

into the simplest form.

Problem 4 Calculate

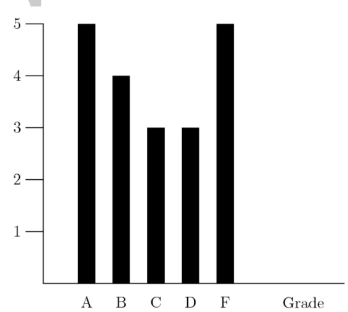
(1) $\frac{1}{3}$ of 90; (2) $\frac{2}{3}$ of 60; (3) $\frac{3}{4}$ of 120; (4) $\frac{4}{11}$ of 33.

(5) $\frac{7}{5}$ of 100.

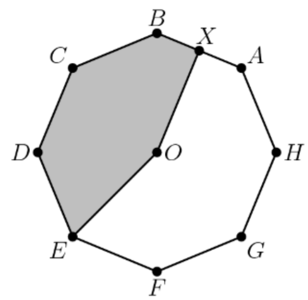
Problem 5 In a library, the ratio of fictions to non-fictions is 2:7. What fraction of the books in the library are non-fictions?

Problem 6 **Modification of [AMC8.1999.10]** A complete cycle of a traffic light takes 60 seconds. During each cycle the light is green for 25 seconds, yellow for 5 seconds, and red for 30 seconds. What fraction of the cycle the light is NOT green?

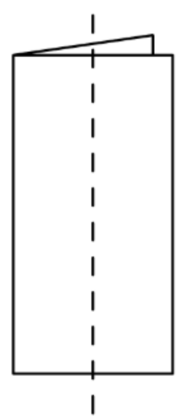
Problem 7 **[AJHSME.1985.5]** The bar graph shows the grades in a mathematics class for the last grading period. If A, B, C, and D are satisfactory grades, what fraction of the grades shown in the graph are satisfactory?



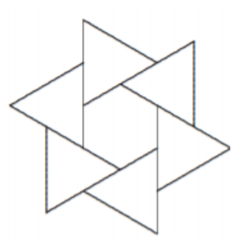
Problem 8 **[AMC8.2015.2]** Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side $\overline{AB}$. What fraction of the area of the octagon is shaded?



Problem 9 [**AMC8.2001.16**] A square piece of paper, 4 inches on a side, is folded in half vertically. Both layers are then cut in half parallel to the fold. Three new rectangles are formed, a large one and two small ones. What is the ratio of the perimeter of one of the small rectangles to the perimeter of the large rectangle?

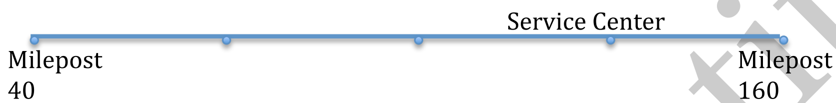


Problem 10 ⋆⋆ The side of each of the equilateral triangles in the figure is twice the side of the central regular hexagon. What is the ratio of the area of the middle hexagon to the total area of the six equilateral triangles?

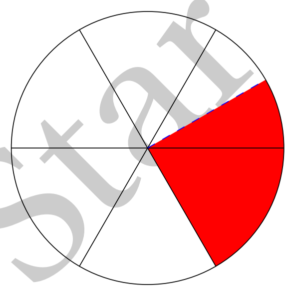


Problem 11 **[AMC8.1999.7]** The third exit on a highway is located at milepost 40 and the tenth exit is at milepost 160. There is a service center on the highway located three-fourths of the way from the third exit to the tenth exit. At what milepost would you expect to find this service center?

(A) 90 (B) 100 (C) 110 (D) 120 (E) 130



Problem 12 **Modification of [AMC8.2012.4]** Peter's family ordered a 6-slice pizza for dinner. Peter ate one slice and shared another slice equally with his brother Paul. What fraction of the pizza did Peter eat?



Problem 13 **[AMC8.2004.16]** Two 600 mL pitchers contain orange juice. One pitcher is 1/3 full and the other pitcher is 2/5 full. Water is added to fill each pitcher completely, then both pitchers are poured into one large container. What fraction of the mixture in the large container is orange juice?

Problem 14 ⋆⋆ **[AMC10B.2012.10]** How many ordered pairs of positive integers (M,N) satisfy the equation $\frac{M}{6}=\frac{6}{N}$?

Problem 15 Calculate

$$\frac{1}{2}+\frac{2}{3},\quad \frac{3}{5}+\frac{2}{15},\quad \frac{5}{12}+\frac{4}{15}$$

and

$$\frac{2}{3}-\frac{1}{2},\quad \frac{3}{5}-\frac{2}{15},\quad \frac{5}{12}-\frac{4}{15}.$$

Write your answers in the simplest form.

Problem 16 Calculate

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$$

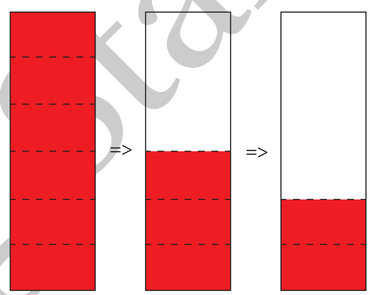
and

$$1-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}.$$

Problem 17 Which fraction is in the midway of $\frac{1}{3}$ and $\frac{1}{5}$?

Problem 18 The triathlon consists of swimming, running, and biking. The biking is three-quarters of the total distance; the running is one-fifth of the total distance and the swimming is 2 km, what is the total distance of this triathlon, in km?

Problem 19 One half of the water is poured out of a full container containing 12 gallons of water. Then one third of the remainder is poured out. What fraction of original water remains?



Problem 20 Compare the following fractions. Put " < " or " > " in ◯

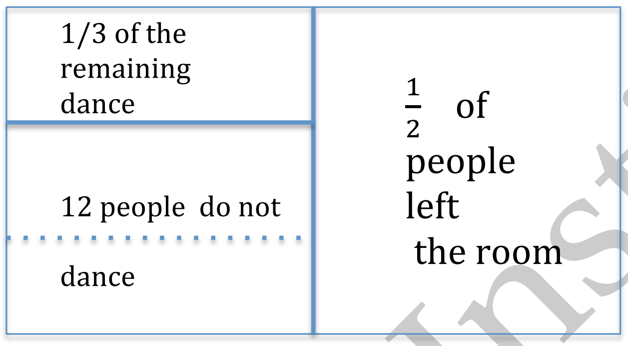
1. $\frac{2}{3} \bigcirc \frac{3}{4}$. 2. $\frac{5}{4} \bigcirc \frac{8}{7}$. 3. $\frac{7}{9} \bigcirc \frac{10}{11}$.

Problem 21 **[AMC8.2019.2]** Order fractions $\frac{15}{11}$, $\frac{19}{15}$, and $\frac{17}{13}$, from least to greatest?

Problem 22 For how many positive integer n,

$$\frac{1}{5}<\frac{n}{8}<\frac{11}{12}?$$

Problem 23 ⋆⋆ **[AJHSME.1987.18]** Half the people in a room left. One third of those remaining started to dance. There were then 12 people who were not dancing. Find the original number of people in the room.



Problem 24 Calculate (Write answers in the simplest form)

(1) $10 \times \frac{2}{5}$; (2) $\frac{2}{3} \times \frac{3}{4}$; (3) $\frac{7}{11} \times \frac{3}{5}$; (4) $\frac{6}{7} \times \frac{2}{15}$; (5) $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$.

Problem 25 **[AMC8.2018.2]** What is the value of the product

$$\left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\left(1+\frac{1}{6}\right)?$$

Problem 26 **[AMC8.2019.17]** What is the value of the product

$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

Problem 27 Calculate

$$\frac{2}{3} \div \frac{3}{4}, \quad \frac{5}{6} \div \frac{2}{9}, \quad \frac{4}{5} \div \frac{8}{5}.$$

Problem 28 Three-fourth of a number A is $\frac{1}{3}$, i.e.,

$$\frac{3}{4} \times A = \frac{1}{3}.$$

What is the number A?

Problem 29 Write out decimal expressions of the following fractions:

$$\frac{2}{5}, \quad \frac{3}{4}, \quad \frac{3}{25}, \quad \frac{1}{9}, \quad \frac{2}{7}.$$

Problem 30 What is the 1000th decimal digit of the following fraction

$$\frac{22}{7} = 3.142857142857142857\cdots = 3.\overline{142857}\ ?$$

Problem 31 Calculate the following

$$3.56+1.24, \quad 4.78+3.69, \quad 5.3-4.1, \quad 4.12-3.65, \quad 1.\overline{2}+3.\overline{4}$$

$$2.3\times 1.2, \quad 3.25\times 1.8, \quad 5.1\div 3, \quad 4.68\div 3.$$

Problem 32 ⋆⋆ What is $1.\overline{3}+2.\overline{7}$?

Problem 33 Abel, Ben and Cathy had dinner together in a restaurant. The bill is:

- *Direct cost:* \$41*;*
- *Tax:* \$3.58*;*
- *Tips:* \$9.5

If they split the bill evenly, how much should each of them pay?

Chapter 14

Percentage

14.1 Basic Concepts

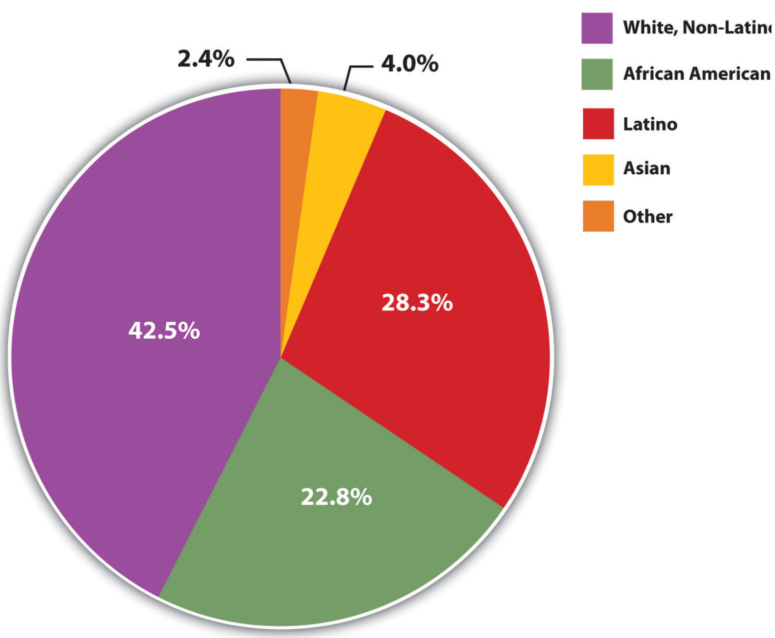


Figure 14.1: Racial Makeup Of The United States. This picture is from gabis.com.

A percentage can be viewed as a special fraction when the denominator is 100. That is,

$$\boxed{a\% = \frac{a}{100}.}$$

For example,

$$30\% = \frac{30}{100} = \frac{3}{10}, \quad 50\% = \frac{50}{100} = \frac{1}{2}, \quad 80\% = \frac{80}{100} = \frac{4}{5},$$

$$\frac{1}{4} = \frac{25}{100} = 25\%, \quad \frac{3}{25} = \frac{12}{100} = 12\%.$$

14.2 Arithmetic with Percentage

Arithmetic with percentages is similar to that with fractions. For example,

$$30\% - 20\% = (30-20)\% = 10\%, \quad 30\% + 20\% = (30+20)\% = 50\%,$$

$$1000 \times 35\% = 1000 \times \frac{35}{100} = 350$$

and

$$20\% \times 30\% = \frac{20}{100} \times \frac{30}{100} = \frac{20 \times 30}{10000} = \frac{6}{100} = 6\%.$$

Example 1 Euclid elementary school has 1000 students. 45% are boys and 55% are girls. How many more girls than boys are there in the school?

- **Solution:** The number of boys is

$$1000 \times 45\% = 1000 \times \frac{45}{100} = 450$$

and the number of girls is

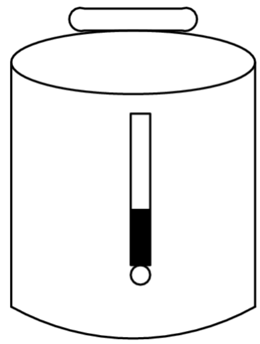
$$1000 \times 55\% = 1000 \times \frac{55}{100} = 550.$$

Hence there are

$$\boxed{550 - 450 = 100}$$

more girls than boys.

Example 2: [AJHSME.1988.20] The glass gauge on a cylindrical coffee maker shows that there are 45 cups left when the coffee maker is 36% full. How many cups of coffee does it hold when it is full?



• **Solution:** Write

$$A = \text{total number of cups when the coffere maker is full.}$$

The problem says that (note that $36\% = \frac{36}{100} = \frac{9}{25}$)

$$45 = 36\% \times A = \frac{9}{25} \times A.$$

This means that if we divide all cups into 25 equal portions, 9 portions have 45 cups.

$$A = \underbrace{\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}}_{45 \text{ cups}} \cdots \boxed{?}\boxed{?} : 25 \text{ portions}$$

Then each portion

$$\boxed{?} = \frac{45}{9} = 5 \,\text{cups}.$$

Hence, in total, there are

$$\boxed{A = 25 \times 5 = 125} \text{ cups}.$$

Note that A can also be solved by quotients of fractions:

$$A = 45 \div \frac{9}{25} = 45 \times \frac{25}{9} = 125.$$

Example 3: [AMC8.2019.9] Ryan got 80% of the problems correct on a 25-problem test, 90% on a 40-problem test, and 70% on a 10-problem test. What percent of all the problems did Ryan answer correctly?

• **Solution:**

Step 1: We find the total number of correct answers.

The number of correct answers in the 25-problem test is

$$25 \times 80\% = 25 \times \frac{80}{100} = 25 \times \frac{4}{5} = 20.$$

The number of correct answers in the 40-problem test is

$$40 \times 90\% = 40 \times \frac{90}{100} = 40 \times \frac{9}{10} = 36.$$

The number of correct answers in the 10-problem test is

$$10 \times 70\% = 10 \times \frac{70}{100} = 10 \times \frac{7}{10} = 7.$$

Hence the total number of correct answer is

$$20 + 36 + 7 = 63.$$

Step 2: Find the fraction

$$\frac{\text{Number of correct answers}}{\text{Total number of problems}} = \frac{63}{75} = \frac{21}{25}.$$

Step 3: Convert the above fraction to the corresponding percentage:

$$\frac{21}{25} = \frac{84}{100} = \boxed{84\%}.$$

Example 4: **[AJHSME.1989.21]** Jack had a bag of 128 apples. He sold 25% of them to Jill. Next he sold 25% of those remaining to June. Of those apples still in his bag, he gave the shiniest one to his teacher. How many apples did Jack have then?

• **Solution:**

Step 1: At first, Jack sold

$$128 \times 25\% = 128 \times \frac{1}{4} = 32$$

to Jill. He has

$$128 - 32 = 96$$

apples left.

Step 2: Then he sold

$$96 \times 25\% = 96 \times \frac{1}{4} = 24$$

apples to June. He has

$$96 - 24 = 72$$

apples left.

Step 3: After giving one apple to his teacher, Jack finally had

$$72 - 1 = \boxed{71}$$

apples left.

14.3 Increase and Decrease in Percentage

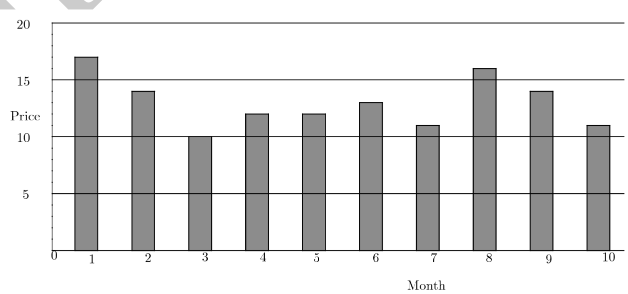
Example 5 School A has 800 students. If school B has 15% more students, what is the number of students in school B?

• **Solution:** The number of students in school B is

(number of students in school A) + (number of students in school A) × 15%

$= 800 + 800 \times 15\% = 800 + 120 = \boxed{920.}$

Example 6: [AMC8.2010.3] The graph shows the price of five gallons of gasoline during the first ten months of the year. By what percent is the highest price more than the lowest price?



• **Solution:** From the graph

the highest price: 17; the lowest price: 10.

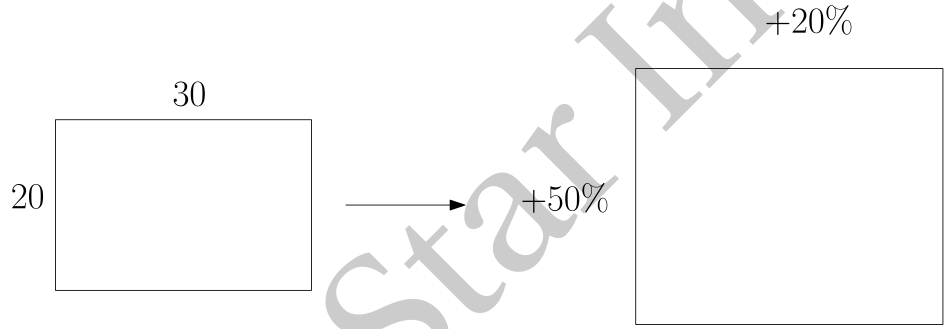
Step 1: We first find the fraction

$$\frac{\text{highest price} - \text{lowest price}}{\text{lowest price}} = \frac{7}{10}.$$

Step 2: Convert the fraction to the corresponding percentage.

$$\frac{7}{10} = \frac{70}{100} = \boxed{70\%.}$$

Example 7: Modification of [AJHSME.1993.21] If the length of a rectangle 30×20 is increased by 20% and its width is increased by 50%, by what percent is the area increased?



• **Solution:**

Step 1: After the increase,

$$\text{the new length} = 30 + 30 \times 20\% = 30 + 6 = 36$$

and

$$\text{the new width} = 20 + 20 \times 50\% = 20 + 10 = 30.$$

So the new area is

$$36 \times 30 = 1080.$$

Step 2: The fraction

$$\frac{\text{new area} - \text{original area}}{\text{original area}} = \frac{1080 - 600}{600} = \frac{480}{600} = \frac{4}{5}.$$

Step 3: Convert the fraction to percentage:

$$\frac{4}{5} = \frac{80}{100} = 80\%.$$

Hence the area is increased by $\boxed{80\%}$.

Example 8 John had opened a saving account in a bank with a fixed interest rate 2% at the beginning of 2019. If he first deposited \$10,000, what will be the total amount of money in his account at the end of 2020? Note

$$\$ \text{ at the end of year} = \$ \text{ at the beginning} + \$ \text{ at the beginning} \times (\text{interest rate}).$$

- **Solution:**

Step 1: At the end of 2019, the amount of money in John's account is

$$10000 + 10000 \times 2\% = 10000 + 200 = 10200.$$

Step 2: At the end of 2020, the amount of money in John's account is

$$10200 + 10200 \times 2\% = 10200 + 204 = \boxed{10404.}$$

Example 9 Jack's annual salary is \$70,000 in 2017. Due to his excellent job performance, he received 5% salary increase in 2018. However, the company that Jack works for encountered a financial crisis at the beginning of 2019 and decided to cut the salary of every employee by 10%. How much money did Jack receive at the end of 2019?

- **Solution:**

Step 1: (increase) Jack's salary in 2018 is

$$70000 + 70000 \times 5\% = 73500.$$

Step 2: (decease) Jack's salary in 2019 is

$$73500 - 73500 \times 10\% = 73500 - 7350 = \boxed{66150.}$$

Example 10: [AJHSME.1990.8] A dress originally priced at 80 dollars was put on sale for 25% off. If 10% tax was added to the sale price, then the total selling price (in dollars) of the dress was

- **Solution:**

Step 1: After 25% off, the price of the dress is

$$80 - 80 \times 25\% = 80 - 20 = 60.$$

Step 2: The amount paid on sale tax is

$$60 \times 10\% = 6.$$

Step 3: The final selling price is

$$60 + 6 = \boxed{66.}$$

14.4 Exercises

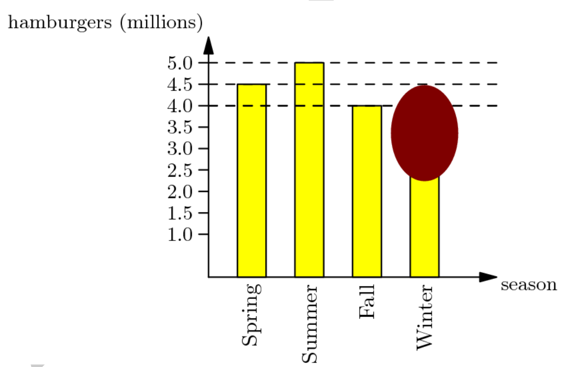
Problem 1 Convert the following fractions to percentage.

$$\frac{2}{5},\quad \frac{3}{20},\quad \frac{3}{4},\quad \frac{11}{25},\quad \frac{9}{10}.$$

Problem 2 Write the following percentage into the simplest form of fractions.

12%, 16%, 28%, 32%, 48%, 64%

Problem 3 **[AJHSME.1986.16]** A bar graph shows the number of hamburgers sold by a fast food chain each season. However, the bar indicating the number sold during the winter is covered by a smudge. If exactly 25% of the chain's hamburgers are sold in the fall, how many million hamburgers are sold in the winter?



Problem 4 **[AJHSME.1989.9]** There are 2 boys for every 3 girls in Ms. Johnson's math class. If there are 30 students in her class, what percent of them are boys?

Problem 5 **[AJHSME.1985.14]** The difference between a 6.5% sales tax and a 6% sales tax on an item priced at $20 before tax is

Problem 6 **Modification of [AMC10B.2007.14]** Some boys and 8 girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive. Now what percent of the group are girls?

Problem 7 **Modification of [AMC10a.2007.5]** Last year Mr. Jon Q. Public received an inheritance of $37500. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. How much did he pay in total for both taxes?

Problem 8 **[AJHSME.1992.4]** During the softball season, Judy had 35 hits. Among her hits were 1 home run, 1 triple and 5 doubles. The rest of her hits were single. What percent of her hits were single?

Problem 9 **[AJHSME.1987.16]** Joyce made 12 of her first 30 shots in the first three games of this basketball game, so her seasonal shooting average was 40%. In her next game, she took 10 shots and raised her seasonal shooting average to 50%. How many of these 10 shots did she make?

Problem 10 **[AJHSME.1990.8]** A dress originally priced at 80 dollars was put on sale for 25% off. If 10% tax was added to the sale price, then the total selling price (in dollars) of the dress was

Problem 11 **[AMC8.1999.16]** Tori's mathematics test had 75 problems: 10 arithmetic, 30 algebra, and 35 geometry problems. Although she answered 70% of the arithmetic, 40% of the algebra, and 60% of the geometry problems correctly, she did not pass the test because she got less than 60% of the problems right. How many more problems would she have needed to answer correctly to earn a 60% passing grade?

Problem 12 **[AMC8.2010.15]** A jar contains 5 different colors of gumdrops. 30% are blue, 20% are brown, 15% are red, 10% are yellow, and other 30 gumdrops are green. If half of the blue gumdrops are replaced with brown gumdrops, how many gumdrops will be brown?

Problem 13 School A has 1000 students. If the number of students in school B is 20% more than the number of students in school A, how many students are in school B?

Problem 14 Jerry got 16 in 2019 AMC 8 and Nick got 20 in 2019 AMC 8. What percent is Nick's score more than Jerry's score?

Problem 15 At the beginning of 2017, Mr. Smith opened a saving account at Bank A and initially deposited $10,000. If the interest rate is 3% , what is amount of money in Mr. Smith's saving account at the end of 2017 and 2018?

Problem 16 Due to Mandy's excellent job performance, her boss decided to give her 5% salary increase every year. If Mandy's initial annual salary is $100K$, what will be her annual salary after two years' salary increases?

Problem 17 If the length of a 30×20 rectangle is decreased by 30% and its width is Increased by 30%, how much has its area been changed?

Problem 18 The stock price of company A was \$100 in 2007. At the beginning of 2018, the stock price went up 20%. At the end of 2018, the stock price went down 20%? What is the stock price at the end of 2018?

Problem 19 Kennith's annual salary was \$70,000 in 2006. Unfortunately, he made a critical mistake in his job at the end of 2006. As a penalty, Jack's salary was cut by 5% in 2007. Due to the financial meltdown in 2008, to survive, the company had to cut the salary of every employee (across the board) by 10%. How much money did Kennith receive at the end of 2008?

Chapter 15

Counting Problems

15.1 Enumeration

Enumeration is to list out all possibilities.

Example 1 How many different three-digit numbers can we form from 1,2,3 without repetition? e.g. 123, 132.

- **Solution:** We may write out all possibilities.

$$\{123,\ 132,\ 213,\ 231,\ 312,\ 321\}.$$

So the answer is $\boxed{6}$.

Example 2 Toss a pair of dice and add up two values. How many different sums can you get? For example, the sum in the following picture is $6+4=10$.

Figure 15.1: Figure from Internet

- **Solution:** We draw a table.

Sum	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

So there are $\boxed{11}$ different sums:

$$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

15.1.1 Casework

To simplify computations, it is a common strategy to divide the problem into different cases, calculate each case separately and then add them up. This is so called "*casework*" or "*addition principle*". Counting a pile of coins is an example in real life. You can choose to count one by one, or first sort coins into groups of penny, nickel, dime and quarter.


Figure 15.2: Figure from Internet

Example 3 How many 3-digit whole numbers whose digit-sum is 24?

- **Solution:** There are three cases:

Case 1:

$$9 + 9 + 6 = 24.$$

There are $\boxed{3}$ different ways to arrange 9, 9 and 6: 996, 969 and 699.

Case 2:

$$9 + 8 + 7 = 24.$$

As in Example 1, there are $\boxed{6}$ different ways to arrange 9, 8 and 7

Case 3:

$$8 + 8 + 8 = 24.$$

Obviously, there is only $\boxed{1}$ number 888 in this case.

Accordingly, in total, there are

$$\boxed{3+6+1=10.}$$

valid 3-digit numbers.

Example 4 What is the number of triples of positive integers (x, y, z) satisfying that

$$x+y+z=6\,?$$

Here the order matters, i.e., $(1,2,3)$ and $(3,2,1)$ are two different triples.

- **Solution:**

x+y+z=6	(y,z)	
Case 1: $x=1$	(1,4), (4,1), (2,3), (3,2)	$y+z=5$, 4 pairs
Case 2: $x=2$	(1,3), (3,1), (2,2)	$y+z=4$, 3 pairs
Case 3: $x=3$	(1,2), (2,1)	$y+z=3$, 2 pairs
Case 4: $x=4$	(1,1)	$y+z=2$, 1 pair

Hence there are

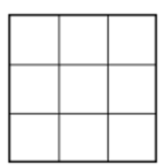
$$\boxed{4+3+2+1=10}$$

valid triples.

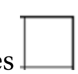
15.1.2 Patterns

Sometimes, we may deduce the pattern in relatively simpler cases and extend to more complicated situations.

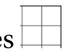
Example 5 How many squares in the following 3×3 squares?



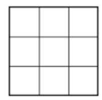
- **Solution:**

Case 1: Number of 1×1 squares 

$$3\times 3=9.$$

Case 2: Number of 2×2 squares 

4.

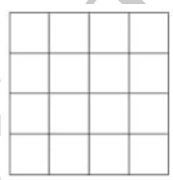
Case 3: Number of 3×3 squares 

1.

Hence the total number of squares is

$$\boxed{1+4+9=1+2^2+3^2=14.}$$

Extension: By observing the pattern, we can deduce that the number of squares in a 4×4 square

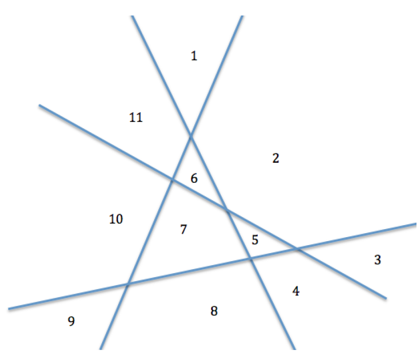


is

$$\boxed{1+2^2+3^2+4^2=30.}$$

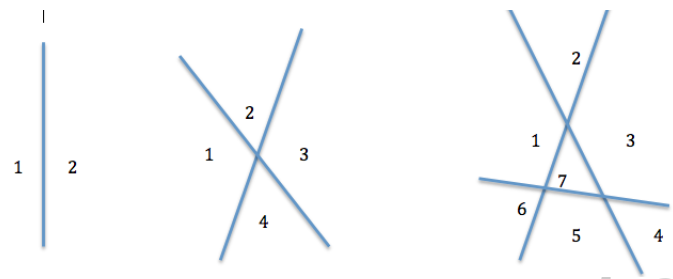
Example 6: [Plane division by lines] What is the maximum number of regions when a plane is divided by 4 lines?

- **Solution:**



From the above the picture, we obtain that the answer is $\boxed{11.}$

Extension: Consider cases where the number of lines is 1, 2 and 3.



We can observe the following pattern:

Number of lines:	1	2	3	4
Number of regions:	2	4	7	11

+2 +3 +4

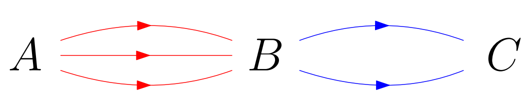
By this pattern, the maximum number of regions when a plane is divided by 7 lines is

$$\boxed{2+2+3+4+5+6+7=29.}$$

15.2 Multiplication Principle

This is a fundamental concept in counting problems. It basically says that, if a problem consists of several steps and each step has multiple choices, then the total number of possible choices is the **multiplication** of choices of each step.

Example 7 There are 3 different ways to go from village A to village B and 2 different ways from village B to village C. How many different ways are there from village A to village C by passing B?

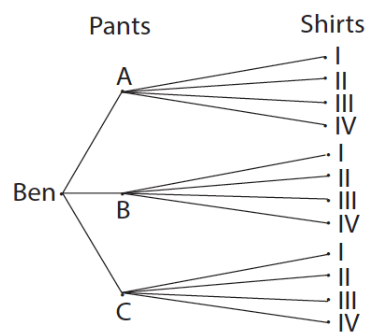


- **Solution:** It is clear that the answer is

$$\boxed{3\times 2=6.}$$

Example 8 Ben has three different pants and 4 different shirts. One day, he needs to wear a pant and a shirt to attend a party. In how many ways can Ben dress himself?

- **Solution:**

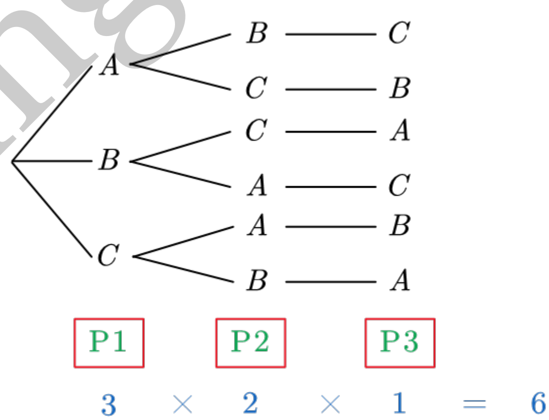


The total number of ways is

$$\boxed{\underbrace{3}_{\#\text{ of pants}} \times \underbrace{4}_{\#\text{ of shirts}} = 12.}$$

Example 9 How many ways can 3 people (Austin, Ben and Cathy) line up in a photo?

- **Solution:** This is essentially the same as Example 1.



Thare are three positions

$$\boxed{P1}, \quad \boxed{P2}, \quad \boxed{P3}.$$

We start from $\boxed{P1}$, there are 3 choices. Then for $\boxed{P2}$, there are 2 choices left. Finall, for $\boxed{P3}$, there is only 1 choice. So the total number is

$$\boxed{3 \times 2 \times 1 = 3! = 6.}$$

All possible combinations are

$$ABC,\ ACB,\ BCA,\ BCA,\ CAB,\ CBA.$$

Extension: If we want to line up 4 people, then the number of different ways is $\boxed{4! = 4 \times 3 \times 2 \times 1 = 24}$. More general, if the number of people is n, then the answer is $\boxed{n! = n \times (n-1) \times \cdots \times 2 \times 1}$.

Example 10 Leo wants to make a 3-character password $\boxed{?}\boxed{?}\boxed{?}$ using characters from the following collection

$$\{1,\ a,\ \$\}$$

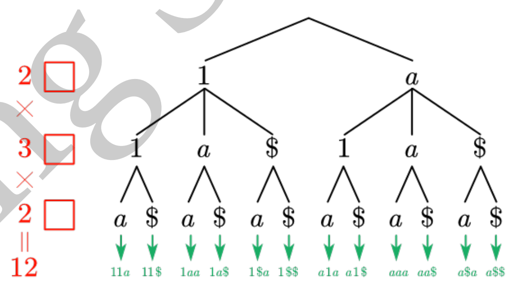
with two requirements:

(1) the first character can not be the special character \$;

(2) the last character can not be the number 1.

How many different passwords does Leo have?

• **Solution:**



The first character has 2 choices (1 or a), the second character has 3 choices (1, a or \$) and the last character has 2 choices (a or \$). Hence by the multiplication principle, in total, there are

$$\boxed{2 \times 3 \times 2 = 12}$$

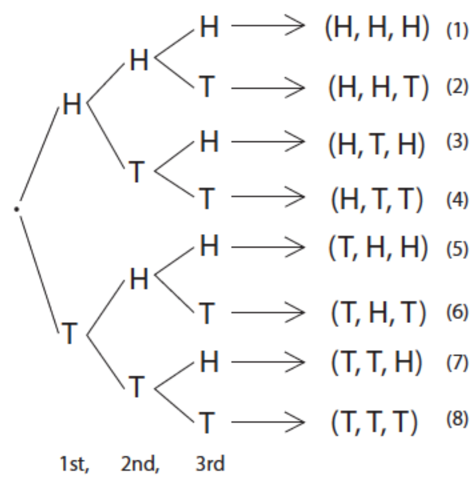
different passwords.

Note: If restrictions (1) and (2) are removed from the above problem, then the total number of possible passwords is $\boxed{3 \times 3 \times 3 = 27}$ since each character has 3 choices (1, a or \$). If no repetition is allowed, then the total number of possible passwords is $\boxed{3 \times 2 \times 1 = 3! = 6}$, which this is equivalent to Example 1 or Example 9.

Example 11 A coin is flipped 3 times where each flip comes up either heads or tails. How many possible outcomes are there in total? For example, *(Head,Head,Tail)* is one outcome.

• **Solution:** Note that each flip has two outcomes: head or tail. By the multiplication principle. Hence the total number is

$$\boxed{2 \times 2 \times 2 = 8}.$$



Example 12 How many two-digit numbers contain at least one 2?

• **Solution:**

Method 1 (Casework): We consider two cases.

Case 1: 2 appears on the unit digit. Clearly, there are 9 such numbers:

$$12,\ 22,\ \ldots,\ 92.$$

Case 2: 2 appears on the tens digit. Clearly, there are 10 such numbers:

$$20,\ 21,\ \ldots,\ 29.$$

Hence, the total number is

$$9 + 10 - \underbrace{1}_{\text{22 is counted twice}} = 18.$$

Method 2: (Opposite case) We consider the opposite case (invalid ones): two-digit numbers that do NOT contain 2

$$\underbrace{\boxed{\text{tens digit}}}_{\text{8 choices}}\underbrace{\boxed{\text{unit digit}}}_{\text{9 choices}}$$

By multiplication principle, there are

$$8 \times 9 = 72$$

such numbers. Accordingly, the number of two-digit numbers that contain at least one 2 is

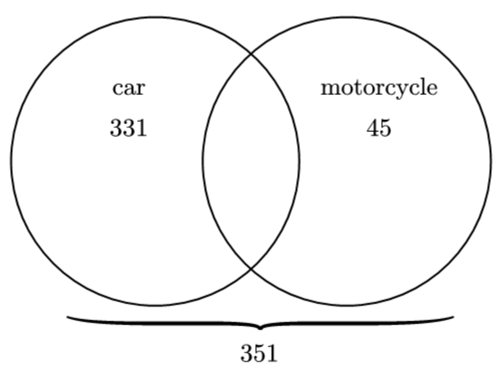
$$\underbrace{90}_{\text{total number of two-digit numbers}} - \underbrace{72}_{\text{number of invalid ones}} = \boxed{18.}$$

15.3 Venn Diagram

We use the following example to demonstrate the application of the Venn diagram, a very convenient tool in handling interactions between different sets.

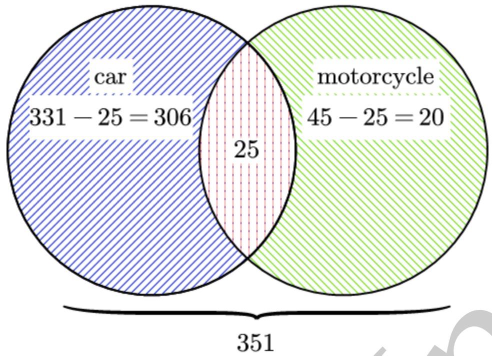
Example 13: [AMC8.2011.6] In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

• **Solution:** Here we have two sets. One set is the collection of adults who own a car. Another set is the collection of adults who own a motorcycle. The intersection (overlap) of these two sets contains adults who own both a car and a motorcycle. Let us draw the following Venn diagram.



Hence the overlap part has

$$331 + 45 - 351 = 25 \quad \text{adults.}$$



So we can see from the above figure (the blue part) that the number of car owners who do not own a motorcycle is

$$331 - 25 = \boxed{306.}$$

15.4 Exercises

Problem 1 **[AJHSME.1992.7]** The digit-sum of 998 is $9+9+8=26$. How many 3-digit whole numbers, whose digit-sum is 26, are even?

Problem 2 How many ordered triples of positive integers (x, y, z) satisfying that

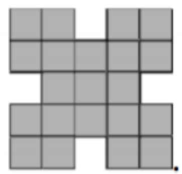
$$x + y + z \le 5\,?$$

Here the order matters, i.e., $(1,1,3)$ and $(1,3,1)$ are two different triples. Also "$\le$" means "no greater than".

Problem 3 Toss a pair of dice and multiply two values. How many different products can you get? For example, in the following picture. The product is $6 \times 4 = 24$.

Figure 15.3: Figure from Internet

Problem 4 [**Kangaroo Math**] How many 2×2 squares  are in the following figure

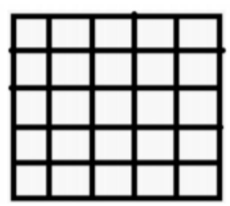


?

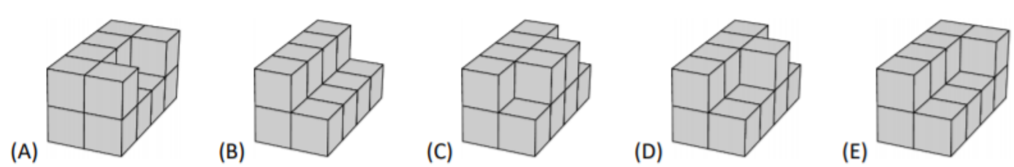
Problem 5 In how many ways can we line up four letters a, a, b, b ?

Problem 6 [**AMC8.200611**] How many two-digit numbers have digits whose sum is a perfect square?

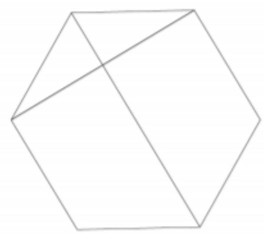
Problem 7 What is the number of squares in the following 5×5 grid?



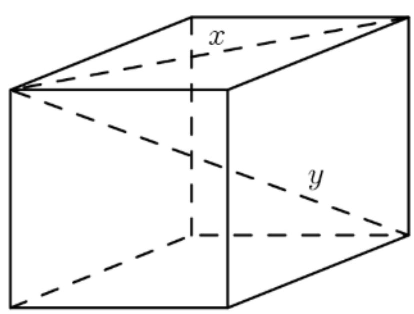
Problem 8 [**Kangaroo Math**] Michael paints the following buildings made up of identical cubes. Their bases are made of 8 cubes. Which building needs the most paint?



Problem 9 A diagonal of a polygon is a segment that connects two non-consecutive vertices. Find the number of diagonals of a hexagon. The following shows two diagonals.



Problem 10 [**AJHSME.1997.17**]A cube has eight vertices (corners) and twelve edges. A segment, such as x, which joins two vertices not joined by an edge is called a diagonal. Segment y is also a diagonal. How many diagonals does a cube have?



Problem 11 How many 2-digit numbers $\overline{ab}$ satisfy that $a < b$? For example, 13 and 45 are such numbers.

Problem 12 Four people A, B, C and D sit around a table. If A can not sit next to B, how many different ways can we sit them?

Figure 15.4: Figure from Internet

Problem 13 **[AMC8.2012.10]** How many 4-digit numbers greater than 1000 are there that use the four digits of 2012?

Problem 14 Larry has 4 different suits, 5 different pants and 3 different pairs of shoes. One day, he needs to wear a suit, a pant and one pair of shoes to attend a party. How many different choices does Larry have?

Problem 15 **[AJHSME.1994.8]** For how many three-digit whole numbers does the sum of the digits equal 25?

Problem 16 William wants to form a 3-character password $\boxed{?}\boxed{?}\boxed{?}$ using the following 4 characters

$$\{a,\ A,\ 1,\ \#\}.$$

Here the repetition is allowed and the password is case sensitive. How many different options does Willam have?

Problem 17 In the above problem, if no repetition is allowed, what is the number of different passwords?

Problem 18 Tyler is in a buffet line to choose one kind of meat, two different vegetables and one dessert. If the order of food items is not important, how many different meals might he choose?

Meat: beef, chicken, pork

Vegetables: corn, potatoes, tomatoes

Dessert: brownies, chocolate cake, chocolate pudding, ice cream

Problem 19 In how many ways can we arrange 1,2,3,4 to form a 4-digit number that is divisible by 4?

Problem 20 How many 3-digit numbers are divisible by 5 and do not contain "1"?

Problem 21 In how many ways can we sit three people A, B, and C in a row of 5 chairs so that no two people can sit next to each other?

Figure 15.5: Figure from Internet

Problem 22 **Modification of [AMC8.1999.15]** Bicycle license plates in Flatville each contain three letters $\boxed{?}\boxed{?}\boxed{?}$. The first is chosen from the set {C,H,L,P,R}, the second from {A,I,O}, and the third from {D,M,N,T}. How many different possible plates does Flatville have?

Problem 23 **[AMC10B.2009.11]** How many 7-digit palindromes (numbers that read the same backward as forward) can be formed using the digits 2, 2, 3, 3, 5, 5, 5? For example, 2355532 is such a number.

Problem 24 ⋆⋆ How many 3-digit numbers contain at least one 2?

Problem 25 How many ways can 4 people line up in a photo?

Figure 15.6: Figure from Internet

Problem 26 In how many ways can we seat two people on a row of 3 chairs?

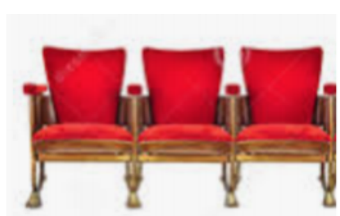
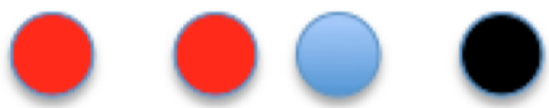


Figure 15.7: Figure from Internet

Problem 27 In how many ways can we line up four balls: two identical red balls, 1 blue ball and 1 black ball?



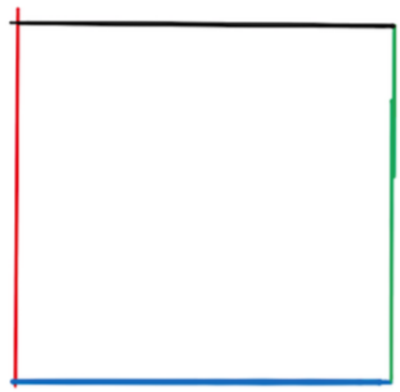
Problem 28 Adam has 2 identical red balls and 4 identical blue balls. In how many ways can he line up these 6 balls so that there are at least two blue balls between two red balls?

Problem 29 Daniel's family has four members: Daniel, his twin brother Eric and their parents. One day, they decide that every week, two of them will clean the house. The family will try all possible combinations and do roations. Daniel and his father start on the first week. On which week will the house-cleaning duty goes to Jack and his father for the second time?

Problem 30 Using the letters A , M , O and S, we can form four-letter "words". If these "words" are arranged in alphabetical order, which position does the "word" SAMO occupies position? For example, *AMOS* and *AMSO* occupy the 1st and 2nd positions respectively.

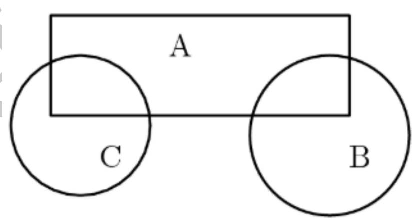
Problem 31 What is the maximum number of regions when a plane is divided by 10 lines?

Problem 32 ⋆⋆ In how many ways can we color four sides of a square with four different colors (red, blue, green, black) so that all sides have different colors? Here two colored squares are considered to be the same if they match after suitable flips and rotations.



Problem 33 Mr. Evans has 36 students in his class. One day, he did a survey about how they like two movies: Frozen 1 and Jurassic world I . He found that 30 of them like Frozen 1, 20 of them like Jurassic world I and 18 of them like both. How many students like neither movie?

Problem 34 [**AMC8.1999.9**] Three flower beds overlap as shown. Bed A has 500 plants, bed B has 450 plants, and bed C has 350 plants. Beds A and B share 50 plants, while beds A and C share 100. The total number of plants is



Chapter 16

Probability

16.1 Basic Concepts

Two basic examples are flipping of fair coins and tossing fair dice. Here the word "fair" means all outcomes have equal chance to appear.

Figure 16.1: Figure from Internet

If we flip a coin, there are two different outcomes:

$$\{\text{Head},\ \text{Tail}\}.$$

The **probability** to get a head (or a tail) is the **fraction**:

$$\boxed{\frac{\text{number of heads}}{\text{number of all possible outcomes}} = \frac{1}{2}.}$$

Similarly, if we toss a die, there are 6 different outcomes:

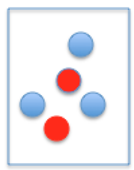
$$\{1, 2, 3, 4, 5, 6\}$$

The **probability** to have the number “4" (or any other number) is the **fraction**:

$$\boxed{\frac{1}{6}.}$$

The probability basically represents the frequency in long-term trend. For example, if a fair coin is tossed 1000 times, we expect to see around $1000 \times \frac{1}{2} = 500$ head.

Example 1 There are 2 red balls and 3 blue balls in a bag. Jack randomly picks a ball from the bag. What is the probablity that he gets a red ball?

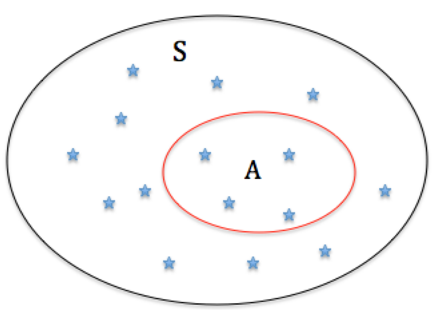


• **Solution:** The **probability** is the fraction of all 5 balls that are red:

$$\boxed{\frac{2}{5}.}$$

In general, if S is the collection of all possible cases and A is the collection of those cases we are looking for. If the number of all possible cases is finite, the **probability** that cases in A happen is the same as the fraction of S that is in A:

$$\boxed{\text{Probability} = \text{Fraction: } \frac{\text{Number of cases in } A}{\text{Number of cases in } S}.}$$



Example 2 A fair coin is tossed 2 times. What is the probability of two heads?

- **Solution:**

Step 1: In total, there are

$$\underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} = 4$$

different situations:

$$(\text{H,H}), \quad (\text{H,T}), \quad (\text{T, H}), \quad (\text{T, T})$$

Step 2: The red one (H,H) is what we are looking are: two heads.

Step 3: Hence the probability is the fraction

$$\boxed{\frac{1}{4}.}$$

Example 3: **[AMC8.2013.8]** A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

- **Solution:**

Step 1: As we discussed in last chapter, due to the multiplication principle, in total, there are

$$\underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} = 8$$

different situations:

$$(H,H,H), \quad (H,H,T), \quad (H,T,H), \quad (H,T,T)$$

$$(T,T,T), \quad (T,T,H), \quad (T,H,T), \quad (T,H,H).$$

Step 2: Those $\boxed{3}$ red ones are what we are looking are: there are at least two consecutive heads.

Step 3: Hence the probability is

$$\boxed{\frac{3}{8}.}$$

Example 4 A fair coin is tossed 4 times. What is the probability there are more heads than tails?

- **Solution:**

Step 1: Owing to the multiplication principle, in total, there are

$$\underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} \times \underbrace{2}_{\text{head/tail}} = 16$$

different situations.

Step 2: Below we list all 5 situations where there are more heads than tails

$$(H,H,H,T), \quad (H,H,T,H), \quad (H,T,H,H), \quad (T,H,H,H)$$
$$(H,H,H,H)$$

Step 3: Hence

$$\boxed{\text{the probability} = \frac{5}{16}.}$$

Example 5 Toss a pair of fair dice and add up two values. What is the probability that the sum is divisible by 5?

Figure 16.2: Figure from Internet

- **Solution:**

Step 1: By the multiplication principle, the total number of pairs $\left(\boxed{?},\boxed{?}\right)$ is

$$\underbrace{6}_{1,2,..,6} \times \underbrace{6}_{1,2,..,6} = 36.$$

Step 2: Let us list all those pairs whose sum is divisible by 5. There are two cases.

Case 1: The sum is 5: $\boxed{?}+\boxed{?}=5.$

$$(1,4),\quad (4,1),\quad (2,3),\quad (3,2).$$

Case 2: The sum is 10: $\boxed{?}+\boxed{?}=10.$

$$(4,6),\quad (6,4),\quad (5,5).$$

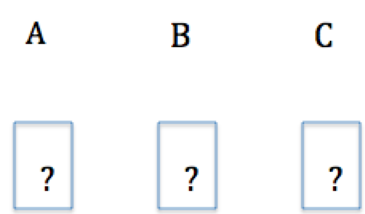
.
So there are $4+3=7$ valid pairs.

Step 3: Accordingly,

$$\boxed{\text{the probability} = \frac{7}{36}.}$$

Example 6: [AMC8.2014.12] A magazine printed photos of three celebrities along with three photos of the celebrities as babies. The baby pictures did not identify the celebrities. Readers were asked to match each celebrity with the correct baby pictures. What is the probability that a reader guessing at random will match all three correctly?

- **Solution:** For convenience, let us label these three celebrities as A, B, C.



Step 1: The total number of ways to randomly match their baby pictures is the same as the number of way of shuffling these pictures. So the number of ways is

$$\boxed{3! = 3\times 2\times 1 = 6.}$$

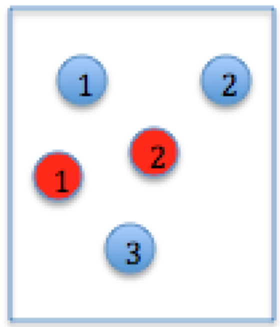
Step 2: Obviously, only one way is correct.

Step 3: Hence

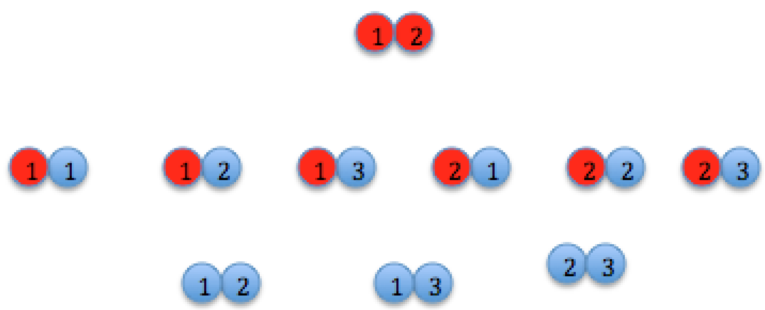
$$\boxed{\text{the probability} = \frac{1}{6}.}$$

Example 7 A bag has 2 red balls and 3 blue balls, Jack randomly draws two balls from the bag. What is the probability that he gets one red and one blue?

- **Solution:** For convenience, let us label red and blue balls respectively



Step 1: It is not hard to see that there are 10 different combinations of two balls.



Step 2: Clearly, $\boxed{6}$ out of these 10 combinations have one red and one blue.

Step 3: Hence

$$\boxed{\text{the probability} = \frac{6}{10} = \frac{3}{5}.}$$

16.2 Exercises

Problem 1 A bag contains 20 number cards:

$$\boxed{1}, \boxed{2}, \boxed{3}, \cdots, \boxed{20}.$$

Cathy randomly picks a card from the bag. What is the probability that she gets a perfect square?

Problem 2 Ben tells Ethan that his score on the past AMC8 test is a whole number between 10 and 25, inclusive. He further gives a hint that his score is divisible by 5 and then asks Ethan to guess his score. What is the probability that Ethan will guess correctly?

Problem 3 Flip a fair coin four times. What is the probability of getting at least 3 consecutive heads?

Problem 4 Roll two fair dice. What is the probability that the sum of two values is a prime number?

Problem 5 A bag contains 2 red balls and 2 blue balls. Tom randomly draws two balls. What is the probability that he gets one blue ball and one red ball?

Problem 6 Darcy randomly lines up three cards: King, Jack and Queen. What is the probability that the Queen is not next to the King? For example, the following is an eligible arrangement.

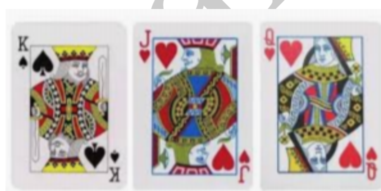


Figure 16.3: Figure from Internet

Problem 7 Alice puts four uno cards (green 5, red 6, wild, wild draw four card) face down on the table. She asks Tom to randomly pick two cards. What is the probability that Tom luckily gets those two special cards (wild, wild draw four card)?

Problem 8 A boy scout group has 4 kids (Abel, Ben, Calvin and David). The scout leader Mr. Smith randomly chooses two of them to recite boy scout oath. What is the probability that Abel is picked?

Problem 9 Elizabeth randomly shuffles numbers 1, 2, 3 to form a 3-digit number. What is the probability that the formed number is divisible by 3? What is the probability that the formed number is divisible by 6?

Problem 10 Natalie and Zac play Rock paper scissors to decide whom to choose the restaurant. They play just once. Before they play, can you calculate the probability that Natalie plays rock and Zac chooses scissor? What is the probability that Natalie wins?

Problem 11 Alex has 5 cards with numbers 1, 2, 3, 4 and 5.

Betty also has 5 cards with numbers 1, 2, 3, 4 and 5.

Alex randomly draws a card from box A and Betty randomly draws a number from box B.

What is the probability that Alex's number is bigger than Betty's number?

Problem 12 Pierre randomly chooses three different numbers from $\{1,2,3,4,5\}$. Here, the order of numbers does not matter. For example, $\{1,2,3\}$ and $\{2,3,1\}$ are considered to be the same three numbers.

What is the probability that these three numbers can be the side lengths of a triangle?

Problem 13 In the above problem, what is the probability that three chosen numbers form an arithmetic sequence?

Problem 14 Roll two dice. What is the probability that the product of two values on the top is a prime number?

Problem 15 Randomly line up 4 people A, B, C and D. What is the probability that A and B occupies the two middle spots? Here positions of A and B can be switched.

Problem 16 Randomly choose two vertices out of 4 vertices of a square and draw a line segment to connect them. What is the probability that that line segment happens to be a diagonal of the square?

Problem 17 Randomly choose two numbers (the order is not important) from the set $\{0,1,2,3,4\}$. What is the probability that their product is 0?

Problem 18 In the above problem, what is the probability that the difference of these two numbers (larger one - smaller one) is 3?

Problem 19 Three men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, she redistributes the hats back to the men at random. What is the probability that at least one man gets his own hat?

Problem 20 A monkey hits keys at random on a typewriter keyboard consisting of 26 capital letters (A-Z). What is the probability that it got the word "AMC" after hitting 3 times?

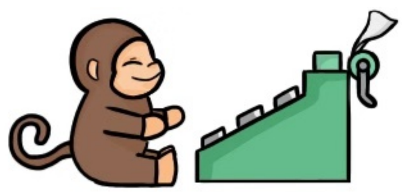


Figure 16.4: Picture drawn by Maggie Shen

Problem 21 Each of two boxes contains three chips numbered 1 , 2 , 3. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?

Problem 22 A bag contains 4 balls, 3 white and 1 black. Austin picks balls from the bag without replacement. Find the probability that the last ball he drew is black.

Chapter 17

Algebra I

17.1 Combine Like Terms

As it was mentioned before, when letters (e.g., x, y, a, b, θ, π, etc) are involved in multiplications, the multiplication sign "$\times$" is often omitted. For example,

$$\boxed{2\times x = 2x, \quad 3\times y = 3y, \quad 4\times a = 4a, \quad 5\times b = 5b.}$$

Basic rules in arithmetic can be extended to operations in algebra. For example,

$$5\times 4 + 3\times 4 = (5+3)\times 4 = 8\times 4$$

and

$$5\times 4 - 3\times 4 = (5-3)\times 4 = 2\times 4.$$

Similarly,

$$\boxed{5x + 3x = (5+3)x = 8x}$$

and

$$\boxed{5x - 3x = (5-3)x = 2x.}$$

This is called "*combining like terms*", which was part of the original meaning of the word "*algebra*" in Arabic.

When brackets are involved in operations, rules to get rid of brackets are similar to those in arithmetic. For example,

$$\boxed{2(x+3) = 2x + 2\times 3 = 2x + 6,}$$

$$\boxed{2(x-3) = 2x - 2\times 3 = 2x - 6,}$$

$$\boxed{2(3-x) = 2\times 3 - 2x = 6 - 2x.}$$

Also, consider

$$5-(2+1)=5-2-1$$

and

$$5-(2-1)=5-2+1.$$

Note that when " $-$ " appears in front of the bracket, " $+$ " and " $-$ " inside the bracket become "$-$" and "$+$", respectively, when the bracket is removed. Similarly,

$$\boxed{5-(x+1)=5-x-1=4-x,}$$

$$\boxed{5-(x-1)=5-x+1=6-x,}$$

$$\boxed{3-(1-x)=3-1+x=2+x.}$$

Example 1 Combine like terms:

$$x+3x+5x.$$

- **Solution:**

$$x+3x+5x=(1+3+5)x=\boxed{9x.}$$

Example 2 Combine like terms

$$x+4(x+2)+3(x-2).$$

- **Solution:** Since

$$4(x+2)=4x+4\times 2=4x+8$$

and

$$3(x-2)=3x-3\times 2=3x-6,$$

we have that

$$\begin{aligned} x+4(x+2)+3(x-2) &= x+4x+8+3x-6 \\ &= \underline{x+4x+3x}+\underline{8-6} \\ &= (1+4+3)x+2=\boxed{8x+2.} \end{aligned}$$

Example 3 Combine like terms:

$$3a+2(a+3)-4(a-2).$$

• **Solution:** Here the letter becomes "a". By similar computations in the above example,

$$\begin{aligned} 3a+2(a+3)-4(a-2) &= 3a+(2a+6)-(4a-8) \\ &= 3a+2a+6-4a+8 \quad \text{Here "}-\text{" becomes "}+\text{"} \\ &= \underline{3a+2a-4a}+\underline{8+6} \\ &= \boxed{a+14.} \end{aligned}$$

Example 4 Combine like terms:

$$10\theta-3(\theta+2)+5(2-\theta).$$

• **Solution:** Here "θ" is a Greek letter.

$$\begin{aligned} 10\theta-3(\theta+2)+5(2-\theta) &= 10\theta-(3\theta+6)+(10-5\theta) \\ &= 10\theta-3\theta-6+10-5\theta \\ &= \underline{10\theta-3\theta-5\theta}+\underline{10-6} \\ &= \boxed{2\theta+4.} \end{aligned}$$

In the above calculation, note that "$-(3\theta+6)$" becomes "$-3\theta-6$" when the bracket is removed.

When fractions or percentages are involved, the calculations are similar.

Example 5 Combine like terms

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{6}.$$

• **Solution:** We have that

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{6}=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)x=x.$$

Similar procedures also apply to expressions with two or more variables.

Example 6 Combine like terms

$$x+4y+1+2(x+y)+3(x+y+1).$$

• **Solution:** Here we have two letters "x" and "y". We need to combine terms associated with x and y separately.

$$
\begin{aligned}
x+4y+1+2(x+y)+3(x+y+1) &= x+4y+1+(2x+2y)+(3x+3y+3)\\
&= x+4y+1+2x+2y+3x+3y+3\\
&= \underline{x+2x+3x}+\underline{4y+2y+3y}+\underline{1+3}\\
&= (1+2+3)x+(4+2+3)y+4\\
&= \boxed{6x+9y+4.}
\end{aligned}
$$

Example 7 n is an integer. What is the average of the following three numbers

$$n+1,\ 2n+2,\ 3n+3?$$

• **Solution:** The average is

$$\frac{n+1+(2n+2)+(3n+3)}{3} \underbrace{=}_{\text{combine like terms}} \frac{6n+6}{3} = \frac{6n}{3}+\frac{6}{3} = \boxed{2n+2.}$$

17.2 Solving Linear Equations with One Variable

Example 8 Solve the following equation, i.e., find the number x such that

$$2x+4=36.$$

x is called an *"unknown variable"*.

• **Solution:**

$$
\begin{aligned}
2x+4=36 \quad &\Rightarrow \quad 2x=36-4=32\\
&\Rightarrow \quad \boxed{x=16.}
\end{aligned}
$$

Example 9 Find the number x such that

$$3x+2x+5=20.$$

• **Solution:** First, we combine like terms

$$3x + 2x + 5 = 5x + 5.$$

Accordingly,

$$\begin{aligned} 3x + 2x + 5 = 20 \quad &\Rightarrow \quad 5x + 5 = 20 \\ &\Rightarrow \quad 5x = 15 \\ &\Rightarrow \quad \boxed{x = 3.} \end{aligned}$$

Example 10 Find the number x such that

$$7(x+2) - 3(x-1) - 6 = 15.$$

• **Solution:** First, we combine like terms

$$\begin{aligned} 7(x+2) - 3(x-1) - 6 \quad &= 7x + 14 - (3x - 3) - 6 \\ &= 7x + 14 - 3x + 3 - 6 \quad \text{note the sign change} \\ &= 4x + 11. \end{aligned}$$

Accordingly,

$$7(x+2) - 3(x-1) - 6 = 15 \quad \Rightarrow \quad 4x + 11 = 15.$$

Hence

$$4x = 4.$$

This leads to

$$\boxed{x = 1.}$$

Example 11 Solve

$$\frac{x}{2} + \frac{x}{8} + x = 26.$$

• **Solution:** First, we combine like terms

$$\frac{x}{2} + \frac{x}{8} + x = \left(\frac{1}{2} + \frac{1}{8} + 1\right) x = \frac{13}{8} x$$

Then

$$\frac{13}{8} x = 26$$

implies that

$$13x = 26 \times 8.$$

Hence $\boxed{x = \frac{26 \times 8}{13} = 16.}$.

If the variable appears on both side, we can move all variables to one side.

Example 12 Solve the following equation

$$4x - 5 = 2x + 1.$$

Solution: We first move all x terms to one side.

$$4x - 5 = 2x + 1 \quad \Rightarrow \quad 4x - 5 - 2x = 2x + 1 - 2x.$$

Combining like terms,

$$2x - 5 = 1.$$

Hence

$$\boxed{x = 3.}$$

Example 13 Solve the following equation

$$3x - 12 = 33 - 2x.$$

- **Solution:** First, we move all x terms to the left

$$3x - 12 = 33 - 2x \quad \Rightarrow \quad 3x - 12 + 2x = 33 - 2x + 2x$$

Combining like terms leads to

$$5x - 12 = 33.$$

Hence

$$5x = 45,$$

which leads to

$$\boxed{x = 9.}$$

17.3 Other Types of Equations

Example 14 Find x such that

$$\frac{x-1}{x-4} = 2.$$

• **Solution:** Note that

$$\frac{A}{B} = 2 \quad \Leftrightarrow \quad A = 2B.$$

Hence we have that

$$x - 1 = 2(x-4).$$

This implies that

$$x = 7.$$

Example 15 Find all real numbers x such that

$$(x-3)(x-8) = 0.$$

• **Solution:** If the product of two numbers are zero, at least one of them has to be zero. Therefore, we deduce that

$$x - 3 = 0 \quad \text{or} \quad x - 8 = 0.$$

So all valid x are

$$\{3,\ 8\}.$$

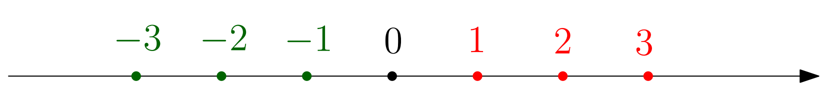
17.4 Negative Numbers

All real Numbers can be categorized as

1. **Positive numbers:** Numbers that are greater than 0. For example, 0.5, 0.6, 1, 1.5, 2, 3, 4, $\cdots$,

2. **Zero:** 0

3. **Negative numbers:** Number that are less than 0. For example, -0.5, -1, -2, -2.5, -3, ... “-" is called the negative sign.



Negative numbers appear often in our daily lives.

• (Overdraft): If you have \$1000 in your bank account and withdraw \$1200 from your account, your bank account balance will show

$$\$1000 - \$1200 = -\$200$$

that is called *overdraft*. In this situation, you owe the bank \$200.

• (Negative temperature). If the temperature drops blow 0°, it becomes negative. For example, the lowest air temperature ever recorded is -89° Celsius (or -128° Fahrenheit) in Russia.

• (Loss of money) Negative numbers can also be viewed as loss of money (negative gain). For example, Jesse has \$100 and plays a game in a casino. According to the rule, if he wins the game, he earns \$10. If he loses the game, he has to pay \$8. Jesse plays three times. He win once and loses twice. Then his net gain is negative

$$10 - 8 - 8 = -6.$$

The total amount of his money becomes

$$\$100 + (-\$6) = \$100 - \$6 = \$94.$$

17.4.1 Arithmetic Rules for Negative Numbers

Let a and b are two positive numbers.

1. If $a < b$, then $a - b$ is negative and

$$a - b = -(b - a).$$

For example,

$$2 - 6 = -(6 - 2) = -4, \quad 1 - 4 = -(4 - 1) = -3, \quad 0 - 2 = -2, \quad \frac{1}{2} - 1 = -\frac{1}{2};$$

2. Ordering.

$$a < b \quad \Rightarrow \quad -b < -a.$$

For example,

$$\cdots > 5 > 4 > 3 > 2 > 1 > 0 > -1 > -2 > -3 > \cdots;$$

3.

$$(-a) + (-b) = -(a + b), \quad a - (-b) = a + b, \quad -a - (-b) = -a + b = b - a.$$

For example,

$$(-2) + (-3) = -5, \quad 2 - (-3) = 2 + 3 = 5, \quad (-2) - (-3) = -2 + 3 = 3 - 2 = 1.$$

4. The product of two negative numbers is positive $(-) \times (-) = +$, i.e., the negative sign is eliminated.

$$(-a) \times (-b) = ab.$$

For example,

$$(-2) \times (-3) = 6, \quad (-2)^2 = (-2) \times (-2) = 4.$$

Hence, equation $x^2 = 1$ has two solutions: $x = 1$ and $x = -1$.

5. The product of one negative and one positive in still negative: $(+) \times (-) = -$

$$a \times (-b) = -ab.$$

For example,

$$2 \times (-3) = -6, \quad (-2)^3 = \underbrace{(-2) \times (-2)}_{=4} \times (-2) = -8.$$

6. All those arithmetic rules and computation tricks mentioned at the beginning of Chapter 1 also apply to all real numbers.

Example 16 Calculate

$$(-2) \times (-4) + 5 \times (-3) + 10.$$

• **Solution:** We have

$$(-2) \times (-4) = 8 \quad \text{and} \quad 5 \times (-3) = -15.$$

Hence

$$(-2) \times (-4) + 5 \times (-3) + 10 = 8 - 15 + 10 = \boxed{3.}$$

Example 17 Calculate

$$4 \times 23 \times (-25) + 275 - (451 - 25).$$

• **Solution:** Notice that

$$4 \times 23 \times (-25) = 4 \times (-25) \times 23 = (-100) \times 23 = -2300.$$

Also,

$$275-(451-25)=275-451+25=275+25-451=300-451=-151.$$

Hence

$$4\times 23\times(-25)+275-(451-25)=-2300-151=\boxed{-2451.}$$

Example 18 Is

$$N=1\times(-2)\times 3\times(-4)\times\cdots\times(-2020)$$

positive or negative?

• **Solution:** Since

$$(-)\times(-)=+ \quad \text{and} \quad (-)\times(+)=-$$

and there are 1010 (even number) of negative numbers in the product, we deduce that $\boxed{N \text{ is positive.}}$

Example 19 For how many integers x,

$$(x-1)(x-6)$$

is negative?

• **Solution:** In order for a product of two numbers to be negative, we need one of them is negative and the other is positive. Hence there are 4 valid integers

$$2,\ 3,\ 4,\ 5.$$

For example, when $x=2$, we have that

$$(x-1)(x-6)=(2-1)(2-6)=-4.$$

Example 20 Solve

$$3x+10=x+2.$$

• **Solution:**

$$3x+10=x+2 \quad \Rightarrow \quad 2x=2-10=-8.$$

Hence $\boxed{x=-4.}$

Example 21 Find all real numbers x such that

$$(x^2-4)(x^2-9)=0.$$

• **Solution:** In order for the product of two numbers to be zero, at least one of the them has to zero. Accordingly,

$$x^2 = 4 \quad \text{or} \quad x^2 = 9.$$

Note that

$$x^2 = 4 \quad \Rightarrow \quad x = 2 \text{ or } x = -2$$

and

$$x^2 = 9 \quad \Rightarrow \quad x = 3 \text{ or } x = -3.$$

Hence valid x are

$$\boxed{\{-2,\ 2,\ -3,\ 3\}.}$$

17.5 Exercises

Problem 1 Combine like terms of the following expressions.

1. $x + 2x + 3x + 4x$;
2. $x + 2x + 3x + 4x + \cdots + 10x$;
3. $2x - (+1) + 3(x + 2) - 2(x + 3)$;
4. $2x + 3x + 2(x + y) - (y + 2) + 2(y + 3)$;
5. $2(a + 1) + 3(a + 1) + 4(a - 1) - (a + 2)$;
6. $10r - 5(r + 2) + 4(2r + 1)$
7. $5\pi - 2(\pi - 2) + 4(1 - \pi) + \pi - 8$;
8. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4}$.

Problem 2 Let

$$A = x + 3x + 5x + 7x + \cdots + 49x$$

and

$$B = 3x + 5x + 7x + \cdots + 51x.$$

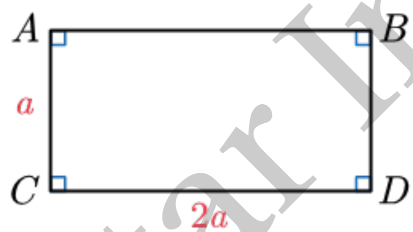
What is $B - A$? Combine like terms in your answer.

Problem 3 What should be the 6th number in the following sequence

$$x,\ 2x,\ 4x, \dots,\ ?$$

Problem 4 ΔABC is a right triangle with $\angle C = 90°$. If $AC = 2$ and $BC = x$, express the area of ΔABC by x and combine like terms if needed.

Problem 5 $ABCD$ is a rectangle with $AB = a$ and $BC = 2a$. Express the perimeter of $ABCD$ by a and combine like terms if needed.



Problem 6 A 2-digit number $\overline{ab}$ can be written as $10a + b$. For example, $23 = 10 \times 2 + 3$. How to express a 3-digit number $\overline{abc}$ as

$$\boxed{?}\,a + \boxed{?}\,b + c\ ?$$

Problem 7 David has x adventure books. Olivia has 4 more adventure books than David. Vivian has twice the number of adventure books as Olivia does. What is the total number of adventure books these three people have? Express your answer in x and combine like terms.

Problem 8 The first number in an arithmetic sequence is a. If the second number is $a + 2$, what is the sum of the first 10 numbers in the sequence? Express your answer in a and combine like terms.

$$a,\ a+2,\ \cdots$$

Problem 9 What is the average of the following 5 numbers

$$x,\ 2x+1,\ 4x-10,\ 5x+30,\ 8x\,?$$

Express your answer in x and combine like terms.

Problem 10 $ABCD$ is a rectangle with length x and width 2. $CDEF$ is a rectangle with length $x+2$ and width 3. What is the

$$\text{area of } CDEF - \text{the area of } ABCD\,?$$

Combine like terms of your answer.

Problem 11 Solve the following equations.

1. $3x = 18$;

2. $4x + 10 = 62$;

3. $2x + 6 = x + 10$;

4. $5x - 11 = 3x + 17$;

5. $5x - 2 = 2x + 19$;

6. $10x + 8 = 5x + 98$;

7. $x - 7 = 9 - x$;

8. $x - 5 = 95 - 4x$;

9. $3x - 6 = 2(x + 2) + 10$;

10. $3x + 9x - 10 = 26$;

11. $5 + x = 4(x - 10)$;

12. $x + 3x + 5x + \cdots + 11x + 13x = 72$;

13. $2x - 4 = 4(8 - x)$;

14. $x + 2x + 3x + 4x = 2x + 3x + 4x + 100$;

15. $\frac{x}{4} + \frac{x}{5} = 18$;

16. $\frac{x}{3}+\frac{x}{6}=x-10.$

17. $x+2x+5x+20=15(x-1);$

18. $x-1+2(x-2)+3(x-3)=4;$

19. $x-4(2-x)+3(x+1)=x+37;$

20 $10(x+1)+5(x-6)=20(x-2);$

21. $100x+200x+300=101x+202x;$

22. $6+7(x-7)+8(x+8)=9(x+9)+10(x-10);$

23. $x+2-2(x-1)+3(x+1)-4(x-1)+5(x+1)-\cdots+9(x+1)-10(x-1)=0;$

24. ★★ $3x+x^2+x^3=x^2+x^3+x+10$

Problem 12 ★★ Define a function

$$f(x)=ax+b.$$

Here a and b are two given numbers to be determined. x is the input and $f(x)$ is the output. For example, if $a=3$ and $b=2$, then the function is

$$f(x)=3x+2.$$

So

$$f(1)=3\times 1+2=5.$$

Now, for another pair of (a,b) , if $f(0)=1$ and $f(1)=5$, find out a and b?

Problem 13 Solve the following equations.

1.
$$\frac{x-3}{x-2}=0;$$

2.
$$\frac{2x+1}{x-10}=5;$$

3.
$$\frac{x}{5x+10}=\frac{1}{3}$$

4.
$$x(x-4)=0;$$

5.
$$(x-1)(x-2)(x-3)=0.$$

Problem 14 Solve the following equations.

1.
$$3x-10=6-x.$$

2.
$$5x+10=-25.$$

3.
$$2(x-1)+3(2-x)=8.$$

4.
$$6x+7x+10(10-x)=220.$$

5.
$$x^2=9.$$

Problem 15 Find all real numbers x such that
$$(2x+1)^2=81.$$

Problem 16 Find all integers x such that
$$(x+1)(x-1)(x-4)(x-9)$$
are negative.

Problem 17 ⋆⋆ **[AMC8.2019.20]** How many different real numbers x satisfy the equation
$$(x^2-5)^2=16?$$

Chapter 18

Algebra II

18.1 Word Problems and Setting up Equations

A milestone in the process of learning algebra is able to solve problems through setting up equations. Below are three main steps:

- **Step 1:** Introduce a suitable unknown variable x;
- **Step 2:** Set up the right equation based on assumptions in the problem;
- **Step 3:** Solve the equation.

Example 1 If the average of the following 4 numbers

$$x-5,\ x+1,\ 2x,\ 3x$$

is 13. What is x?

• Solution: According to the assumption, we have that

$$\frac{x-5+x+1+2x+3x}{4}=13.$$

Hence

$$x-5+x+1+2x+3x=52.$$

Combining terms leads to

$$7x=56.$$

Hence $\boxed{x=8}$.

Example 2 The length of a rectangle is twice its width. If the perimeter is 72 meters, find the length and width of the rectangle.

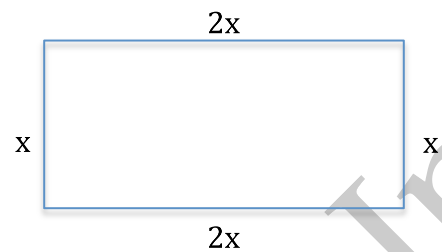
Solution:

Step 1: Introduce an unknown variable x. Denote

$$x: \quad \text{width of the rectangle}$$

Then

$$2x: \quad \text{length of the rectangle.}$$



Step 2: Set up an equation based on assumptions in the problem

$$72 = \text{ the perimeter } = x + x + 2x + 2x = 6x.$$

Step 3: Solve the resulting equation

$$6x = 72 \quad \Rightarrow \quad \boxed{x = 12.}$$

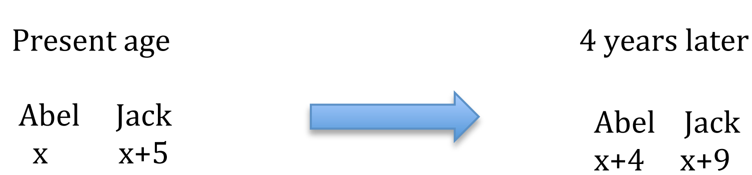
Example 3 Abel is 5 years younger than his brother Jack. Four years later, Jack will be twice as old as Abel. What is Abel's present age?

Solution:

Step 1: Introduce an unknown variable x. Denote

$$\boxed{x: \quad \text{the present age of Abel.}}$$

Then we have the following



Step 2: Set up an equation based on assumptions in the problem

$$\boxed{x+9=2(x+4).}$$

Step 3: Solve this equation

$$2(x+4)=x+9$$

$$\Rightarrow \quad 2x+8=x+9$$

$$\Rightarrow \quad \boxed{x=1.}$$

Example 4: Chicken-Rabbit Problem In a cage there are chickens and rabbits. The total number of heads is 60 and the total number of legs is 170. How many chickens are there?

• **Solution:** Here we will present how to solve this kind of problems by equations.
Step 1: Introduce an unknown variable x. Denote

$$\boxed{x: \quad \text{number of chickens}.}$$

Then

$$\boxed{\text{number of rabbits} = 60-x}$$

Step 2: Set up an equation based on number of legs. Notice that

$$2x: \quad \text{total number of chicken feet}$$

and

$$4(60-x): \quad \text{total number of legs of rabbits.}$$

Then

$$\boxed{2x+4(60-x)=170.}$$

Step 3: Solve this equation. Combining like terms, we deduce that

$$240-2x=170 \quad \Rightarrow \quad 2x=70 \quad \Rightarrow \boxed{x=35.}$$

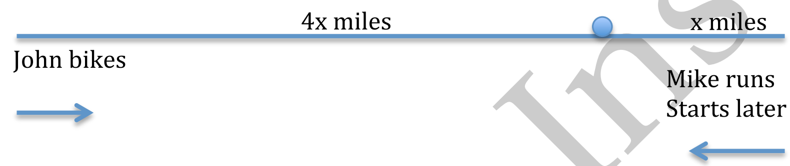
In essence, this is the same as the trick we used before: assume all animals are rabbits. Then there are $60 \times 4 = 240$ legs. The differece is $240-170=70$. Hence we need to replace $\frac{70}{2}=35$ rabbits by chicken since each rabbit has two more legs than a chicken.

Example 5: [**Modification of AMC10 Problem**] Josh and Mike live 5 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to run toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike had run and at twice of Mike's speed. How many miles had Mike run when they met?

Solution:

Step 1: Introduce an unknown variable x. Denote

$$\boxed{x: \quad \text{miles Mike had run when they met .}}$$



Then

$$\boxed{\text{miles Josh had ridden when they met} = x \times \underbrace{2}_{\text{twice time}} \times \underbrace{2}_{\text{twice speed}} = 4x.}$$

Step 2: Set up an equation based on assumptions in the problem.

$$\boxed{x + 4x = 5.}$$

Step 3: Solve this equation

$$x + 4x = 5 \quad \Rightarrow \quad 5x = 5 \quad \Rightarrow \boxed{x = 1.}$$

Example 6 Due to Jack's excellent performance in his workplace, his boss decides to increase his annual salary by 25% next year, which will lead to $ 100,000 per year. What is his present annual salary?

Step 1: Introduce an unknown variable x. Denote

$$\boxed{x: \quad \text{Jack's current annual salary .}}$$

Step 2: Set up an equation based on assumptions in the problem.

$$\boxed{x + x \times 25\% = 100,000.}$$

Step 3: Solve this equation

$$x + x \times 25\% = 100,000 \quad \Rightarrow \quad x + \frac{x}{4} = 100,000$$

$$\Rightarrow \frac{5x}{4} = 100,000$$

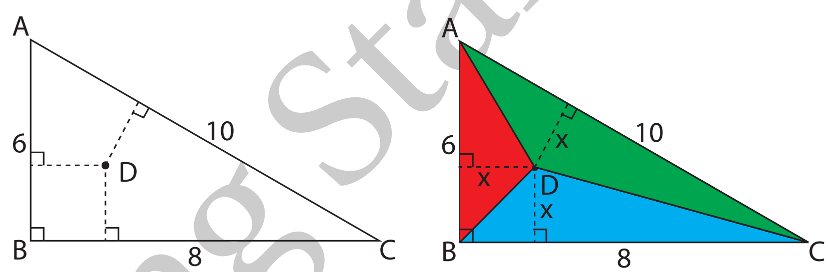
$$\Rightarrow 5x = 400,000$$

$$\Rightarrow \boxed{x = 80,000.}$$

⋆⋆ **Example 7** ΔABC is a right triangle with $AB = 6$, $BC = 8$ and $AC = 10$. D is a point inside ΔABC. If the distance from D to three sides is the same, find the distance.

Step 1: Introduce an unknown variable x. Denote

$$\boxed{x: \quad \text{the common distance} .}$$



Step 2: Set up an equation based on assumptions in the problem. Note that in terms of areas

$$\underbrace{\Delta ABC}_{area=\frac{1}{2}\times 6\times 8=24} = \underbrace{\text{red triangle}}_{\text{area} = \frac{1}{2}\times 6\times x = 3x} + \underbrace{\text{blue triangle}}_{\text{area} = \frac{1}{2}\times 8\times x = 4x} + \underbrace{\text{green triangle}}_{\text{area} = \frac{1}{2}\times 10\times x = 5x} .$$

So the right equation reads

$$\boxed{24 = 3x + 4x + 5x.}$$

Step 3: Solve this equation

$$24 = 3x + 4x + 5x \quad \Rightarrow \quad 24 = 12x \quad \Rightarrow \quad \boxed{x = 2}.$$

18.2 Exercises

Problem 1 If the average of the following 4 numbers

$$x-5,\ x,\ x+6,\ x+19$$

is 30, what is x?

Problem 2 If the average of the following 5 numbers

$$x-3,\ 2x-2,\ 2x+2,\ x+10,\ 3x+3$$

is 20, what is x?

Problem 3 If the average of the following 6 numbers

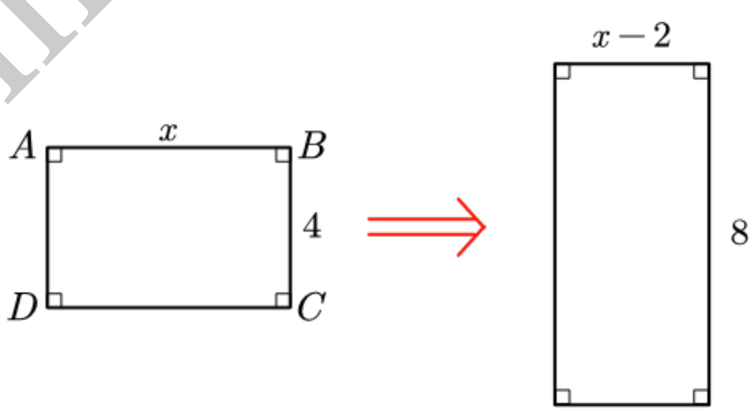
$$x+1,\ 2x+2,\ 3x+3,\ 4x+4,\ 5x+5,\ 6x+6$$

is 126, what is x?

Problem 4 In triangle ΔABC, $\angle B = 2\angle A$ and $\angle C = 3\angle B$, what is $\angle A$?

Problem 5 In a pentagon, degrees of five angles is an arithmetic sequence with common difference of 20°. What is the degree of the smallest angle? (Hint: Denote by x the smallest angle. Express the other angles by x and set up an equation based on the sum of angles equals 540°)

Problem 6 $ABCD$ is a rectangle with $BC = 4$. If BC becomes 8 and the length AB decreases by 2, the area increases by 16. What is the original length of AB?



Let x be the original length of x.

(1) Express the original area of $ABCD$ by x;

(2) Express the new area of $ABCD$ by x;

(3) Set up an equation based on the assumption that *"the area increases by 16"*.

(4) Find x.

Problem 7 On May 5th 2020, Ethan's father is 30 years older than Ethan. Ten years later, his father's age is three times Ethan's age. Let

$$x: \text{ Ethan's age on May 5th 2020.}$$

(1) Express father's age on May 5th 2020 by x;

(2) Express Ethan and father's age ten years later by x;

(3) Set up an equation based on the assumption *"Ten years later, his father's age is three times Ethan's age"*.

(4) What is x?

Problem 8 Frank's grandpa is 60 years older than Ethan. Ten years later, grandpa's age will be 5 times Frank's age. What is the current age of Frank?

Problem 9 Larry has a total of 160 dollars in bills of denomination 1 dollar , 2 dollar and 5 dollar. The number of 1 dollar bills is 4 times the number of 2 dollar bills and 2 times the number of 5 dollar bills. Let

$$x: \text{ the number of 2 dollar bills.}$$

(1) Express the number of 1 dollar bills by x;

(2) Express the number of 5 dollar bills by x;

(3) Set up an equation of x based on *"the total amount is* 160 *dollars"* .

(4) What is x?

Problem 10 Kevin has a total of \$200 in bills of 1 dollar, 2 dollar and 5 dollar. The number of 1 dollar bills is 2 times the number of 2 dollar bills. There are 5 more 2 dollar bills than 5 dollar bills. What is the number of 5 dollar bills?

Problem 11 Two planes, which are 2400 miles apart, fly toward each other. Their speeds differ by 60 miles per hour. They pass each other after 5 hours. Find the speed of the faster plane.

Problem 12 Everyday, a group of ants collect 500 gram food and consume 300 gram. In a rainy day, they stay in the colony but still consume the same amount of food. In seven days, their food storage has been increased by 400 gram. Let

$$x: \text{ number of rainy days in these 7 days.}$$

(1) How many grams of food have these ants consumed in this week?

(2) Express the grams of food have they collected in this week by x?

(3) Set up an equation of x based on the change in food storage.

(4) Find x.

Problem 13 Alice walks from her home to the library to meet Ben at 10am. If she walks 1 meter per second, she will arrive there at 10:05 am. If she speeds up and walks 2 meters per second, she will arrive there at 9:55 am. Let

$$x: \text{ the distance, in meters, from Alice's home to the library.}$$

(1) Express the number minutes Alice needs to walk from home to library with a speed of 1 meter per second by x;

(2) Express the number minutes Alice needs to walk from home to library with a speed of 2 meter per second by x;

(3) Set up an equation of x based on time difference;

(4) Find x.

Problem 14 In the above problem, if Alice arrives at the library right on time after she speeds up, what is the distance, in meters, from her home to library?

Problem 15 A group of students from Euclid middle school take train to New York. In total, they buy 95 tickets which cost 410 dollars. Depending where they sit, there are two kind of tickets (type A and type B). Type A costs 4 dollars per ticket and Type B costs 5 dollars per ticket. What is the number of type A tickets among these 95 tickets they bought?

Problem 16 A pile of 100 apples are divided among three monkeys. What monkey A got is one more than twice of what monkey B got. What monkey B got is two less than twice of what monkey C got. How many apples did monkey C receive?

Problem 17 A factory receives an emergency order to produce certain number of ventilators during this pandemic. If it produces 10 ventilators per day, it needs 3 more days than the scheduled time to finish the job. If it produces 11 ventilators per day, it can finish the job one day earlier than the original plan. What is the originally scheduled number of days to finish the job?

Problem 18 Two groups in a factory are making the same kind of toys. The number of toys that group A can make in 3 days is the same as the number of toys that group B can make in 5 days. If group A produces 18 more toys than group B each day, how many toys does group B produce each day?

Problem 19 A boat travels between City A and City B along a river. It takes the boat 2 hours to go from City A to city B (downstream) and 3 hours to return from City B to City A (up the river). The speed of the water is 2km per hour. What is the speed of the boat in still water, in km/hr?

Upstream speed of a boat = boat speed in still water − water speed.

Downstream speed of a boat = boat speed in still water + water speed.

Think of walking on a moving walkway in an airport.

Hint: Let

$$x: \text{ the speed of the boat in still water, in km/h.}$$

Set up an equation of x based on the distance between A and B: the traveling distance from A to B equals the traveling distance from B to A. Moreover, distance = speed × time.

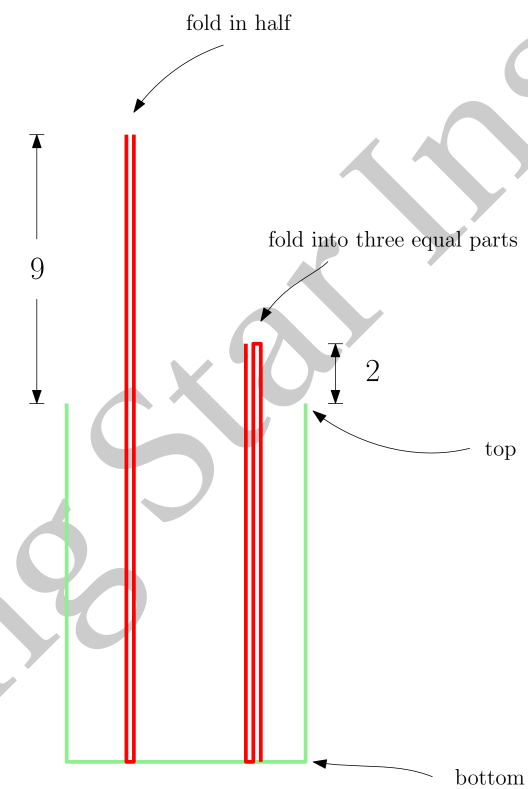
Problem 20 Patrick wants to withdraw \$10000 from bank of America. He requires to have bills of 20 dollar, 50 dollar and 100 dollar. The total number of bills is 178 and the number of 20 dollar bills is the same as the the number of 50 dollar bills. What is the number of 20 dollar bills that Patrick got?

Problem 21 Tom uses a long rope to measure the depth of a deep empty well. He first folds the rope in half and put one end down to bottom of the well. He found that the other end is 9 meters above the top of the well. He then folded it into three equal parts and again put one end down to the bottom

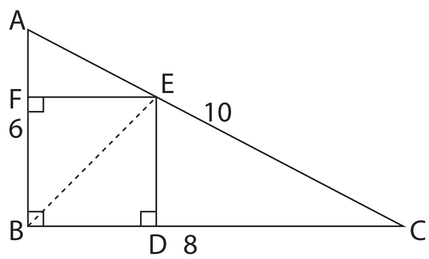
of the well. The other end is 2 meters above the top of the well. What is the depth of the well? (This problem is from a Chinese math Olympia book)
Hint: Let

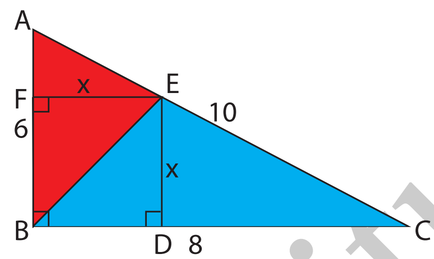
$$x: \text{ the length of the rope, in meters.}$$

Set up an equation of x.



Problem 22 ⋆⋆ ΔABC is a right triangle with $AB = 6$, $BC = 8$ and $AC = 10$. $BDEF$ is a square inscribed inside ΔABC. What is the side length?





Hint: Let

x : the side length of the square $BDEF$.

Set up an equation based on

$$\text{Area of } \Delta ABC = \text{Area of } \Delta AEB + \text{Area of } \Delta CEB.$$

Chapter 19

$\star\star$ Algebra III

19.1 Linear Equations with Two Variables

In this section, we discuss how to solve linear equations with two variables. There are two common methods: **substitution** and **elimination**. Both aim to reduce the number of variables from two to one. Sometimes, one could be more convenient than the other.

Example 1 Solve

$$\begin{cases} x + 2y = 55 & (1) \\ 2x + y = 65 & (2). \end{cases}$$

- **Solution:**

(1) **Method of substitution:** From the second equation (2)

$$2x + y = 65 \quad \Rightarrow \quad y = 65 - 2x.$$

Then we substiute y in the first equation (1) by "$65 - 2x$"

$$x + 2y = 55 \quad \Rightarrow \quad x + 2(65 - 2x) = 55.$$

This leads to a single equation with only one variable x:

$$x + 2(65 - 2x) = 55 \quad \Rightarrow \quad \boxed{x = 25.}$$

Accordingly, $\boxed{y = 15.}$

(2) **Method of elimination:** We choose to eliminate the y variable. Multiplying the second equation (2) by 2, we deduce that

$$2(2x + y) = 2 \times 65 = 130 \quad \Rightarrow \quad 4x + 2y = 130 \quad (3).$$

Subtract equation (1) from equation (3):

$$(3)-(1): \quad 4x+2y-(x+2y)=130-55=75.$$

Hence

$$3x=4x+2y-(x+2y)=75.$$

So $\boxed{x=25}$ and $\boxed{y=15}$.

Example 2 Solve

$$\begin{cases} 3x-2y=38 & (1) \\ 5y-2x=15 & (2). \end{cases}$$

• **Solution:** To avoid working with fractions, we will employ the method of elimination.

Let us eliminate x variable. Multiplying first equation (1) by 2 and the second equation (2) by 3, we obtain

$$\begin{cases} 2(3x-2y)=2\times 38=76 \\ 3(5y-2x)=3\times 15=45. \end{cases}$$

This leads to

$$\begin{cases} 6x-4y=76 & (3) \\ 15y-6x=45 & (4). \end{cases}$$

Equation (3) + equation (4) implies that

$$6x-4y+(15y-6x)=76+45 \quad \Rightarrow \quad \boxed{11y=121.}$$

Then we can deduce that $\boxed{y=11}$ and $\boxed{x=20.}$

We can also apply the same methods to equations having more than 2 variables.

Example 3 a, b and c are three numbers satisfying that

$$\begin{cases} 2a-b+c=100 & (1) \\ a+4b+2c=80 & (2). \end{cases}$$

What is $a+b+c$?

• **Solution:** Adding equation (1) and equation (2) leads to

$$(2a-b+c)+(a+4b+2c)=100+80=180.$$

Since
$$(2a-b+c)+(a+4b+2c)=3a+3b+3c=3(a+b+c),$$
we deduce that
$$a+b+c=180\div 3=60.$$

Let us revisit the Chicken-Rabbit problem.

Example 4: Chicken-Rabbit Problem In a cage there are chickens and rabbits. The total number of heads is 60 and the total number of legs is 170. How many chickens are there?

- **Solution:** Denote

$$x: \text{number of chickens}, \quad y: \text{number of rabbits}.$$

Then
$$\begin{cases} x+y=60 & (1) \\ 2x+4y=170 & (2). \end{cases}$$

Method 1: Substitution

$$x+y=60 \quad \Rightarrow \quad y=60-x.$$

Substitute y in the second equation (2) by $60-x$:

$$2x+4y=170 \quad \Rightarrow \quad 2x+4(60-x)=170.$$

This leads to
$$240-2x=170 \quad \Rightarrow \quad \boxed{x=35}.$$

Method 2: Elimination Multiply the first equation (1) by 4:

$$4(x+y)=4\times 60=240 \quad (3).$$

Notice that this is the number of feet if all animals are rabbits. Then subtract equation (2) from equation (3)

$$(3)-(2)=4(x+y)-(2x+4y)=240-170=70,$$

which leads to
$$2x=70 \quad \Rightarrow \quad \boxed{x=35}.$$

Remark As we have seen in the previous chapter, the chicken-rabbit problem can also be solved using only one variable. Nevertheless, like many other examples, using two variables can make the set-up more symmetric, natural and easier.

Example 5: [Hope Cup Math Competition, China] Certain amount of feed can feed 10 ducks and 15 chickens for 6 days or 12 ducks and 6 chickens for 7 days. How many ducks can the same amount of feed feed for 21 days?

- **Solution:** Denote

$$x: \text{the amount of feed eaten by a duck every day}$$

$$y: \text{the amount of feed eaten by a chicken every day}$$

Then

$$\boxed{6(10x+15y) = 7(12x+6y) = \text{the total amount of feed.}}$$

Hence

$$60x+90y = 84x+42y,$$

which implies that

$$90y-42y = 84x-60x.$$

So

$$\boxed{x = 2y}$$

This is to say that

$$\text{what is eaten by one duck} = \text{what is eaten by two chickens.}$$

Hence the total amount of feed can feed

$$12+\frac{6}{2} = 15 \text{ ducks for 7 days.}$$

Since $21 = 7\times 3$, the same amount can feed

$$\boxed{\frac{15}{3} = 5 \text{ ducks}}$$

in 21 days.

19.2 Expansion of $(a+b)^2$ and Proof of Pythagorean Theorem

For every two numbers a and b, the following identity holds.

$$\boxed{(a+b)^2 = a^2+2ab+b^2.} \tag{1}$$

For example

$$49 = (3+4)^2 = 3^2+2\times 3\times 4+4^2.$$

Let us demonstrate this from two perspectives.

1.Geometric Approach: Assume that a and b are both positive. Decompose a square $ABCD$ with side length $(a+b)$ into four parts:

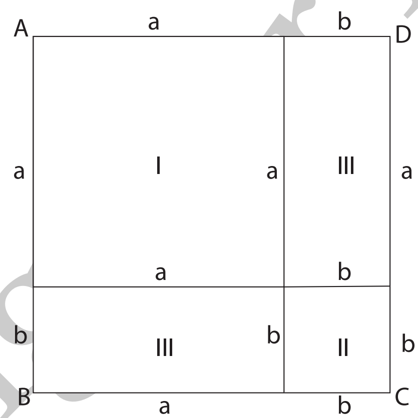
I: a square with side length a;

II: a square with side length b;

III: two identical rectangles with side lengths a and b.

Apparently, in terms of areas,

$$\underbrace{\text{the square } ABCD}_{(a+b)^2} = \underbrace{\text{I}}_{a^2} + \underbrace{\text{III}}_{2ab} + \underbrace{\text{II}}_{b^2}.$$



2.Algebraic Approach: By distribution law,

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a(a+b) + b(a+b) \\ &= (aa+ab) + (ba+bb) \\ &= \underbrace{aa}_{=a^2} + \underbrace{ab+ba}_{=2ab} + \underbrace{bb}_{=b^2} \quad \text{since } ab = ba \\ &= a^2 + 2ab + b^2.\end{aligned}$$

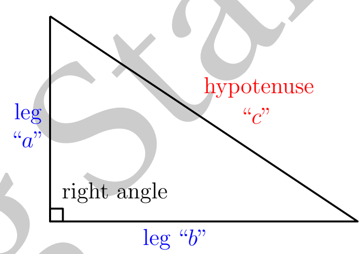
For example,

$$
\begin{aligned}
(3+4)^2 = (3+4)(3+4) \quad &= 3(3+4) + b(3+4) \\
&= (3\times 3 + 3\times 4) + (4\times 3 + 4\times 4) \\
&= \underbrace{3\times 3}_{=3^2} + \underbrace{3\times 4 + 4\times 3}_{=2\times 3\times 4} + \underbrace{4\times 4}_{=4^2} \quad \text{since } ab = ba \\
&= 3^2 + 2\times 3\times 4 + 4^2.
\end{aligned}
$$

Next we will use (1) to prove the Pythagorean Theorem which has been mentioned in a previous chapter.

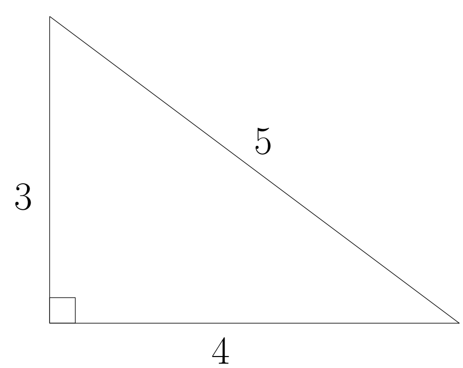
Pythagorean Theorem: *For a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides (legs).*

$$\boxed{a^2 + b^2 = c^2.}$$



The most famous example is $a = 3$, $b = 4$ and $c = 5$:

$$3^2 + 4^2 = 5^2.$$



This theorem was known to mathematicians in other ancient civilizations (e.g. Babylonian, China, India, etc.) long before Pythagoras (a ancient Greek mathematician, 570–495 BC). However, the first recorded **proof** is attributed to Pythagoras. Below is the argument due to Pythagoras.

Proof: Consider the following square and a decomposition.

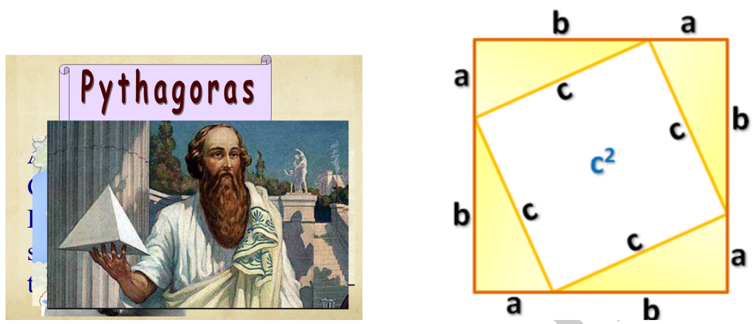


Figure 19.1: Figure is from Internet.

The area of the big square is

$$(a+b)^2.$$

The big square is partitioned into 5 pieces:
(I) Four congruent right triangles.

$$\text{Their total area} = 4 \times \frac{1}{2} \times a \times b = 2ab.$$

(II) An inscribed smaller square with side length c.

$$\text{Its area} = c^2.$$

Hence

$$(a+b)^2 = 2ab + c^2.$$

Since

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2,}$$

we derive that

$$a^2 + \cancel{2ab} + b^2 = \cancel{2ab} + c^2.$$

This leads to

$$\boxed{a^2 + b^2 = c^2.}$$

□

19.3 Exercises

Problem 1 Solve following equations.

1.
$$\begin{cases} x = 2y \\ 2x + y = 125. \end{cases}$$

2.
$$\begin{cases} y = 2x + 1 \\ y = 3x - 10. \end{cases}$$

3.
$$\begin{cases} 2x + 3y = 58 \\ 3x + 4y = 81. \end{cases}$$

4
$$\begin{cases} 3x + 2y = 57 \\ 2x + 7y = 89. \end{cases}$$

5.
$$\begin{cases} 5x + 2y = 100 \\ 2x + 3y = 84 \end{cases}$$

6.
$$\begin{cases} 11x = 113 - 3y \\ 7x - 4y = 1 \end{cases}$$

7.
$$\begin{cases} x - y = 10 \\ 2x + 3y = 95. \end{cases}$$

8.
$$\begin{cases} 3x - 2y = 1 \\ x + 2y = 27. \end{cases}$$

9.
$$\begin{cases} x - 2y = 4 \\ 3x + y = 68 \end{cases}$$

10.
$$\begin{cases} 5x + 2y = 67 \\ 3x - 2y = 5 \end{cases}$$

11.
$$\begin{cases} \frac{x}{2} + y = 16 \\ x + \frac{y}{2} = 20 \end{cases}$$

12.
$$\begin{cases} 3x + 2y = 450 \\ 3y - 2x = 25. \end{cases}$$

13.
$$\begin{cases} 10x + 8y = 141 \\ 2x - y = 9. \end{cases}$$

14.
$$\begin{cases} x + y = 100 \\ y + z = 112 \\ z + x = 126. \end{cases}$$

Problem 2 If
$$\begin{cases} a + b + c = 40 \\ 2a + 3b - c = 10, \end{cases}$$
what is $3a + 4b$?

Problem 3 If
$$\begin{cases} a + b + c = 20 \\ a + b + 2c = 35, \end{cases}$$
what is c?

Problem 4 Robert's father is 4 times as old as Robert. After 5 years, father will be three times as old as Robert. Find present ages of Robert and his father.

Problem 5 Mr. Lee wanted to give a number candies to students in his class. If each student got 6 candies, Mr.Lee needed 6 more candies. If each student got 5 candies, Mr. Lee had 5 candies left. What are the numbers of students and candies?

Problem 6 Henry and Kevin are making paper cranes together. In the morning, Henry worked for 20 minutes and Kevin worked for 15 minutes. In total, they have made 425 plane cranes. In the afternoon, Henry worked for 10 minutes and Kevin worked for 20 minutes. In total, hey have made 400 paper cranes. How many paper cranes can Henry and Kevin make in one minute respectively?

Problem 7 The sum of the digits of a two-digit number is 11. When the digits are reversed, the number is increased by 27. What is this number?

Problem 8 Miller and Nancy have a number of scientific books. If Miller gives 5 books to Nancy, Nancy will have three times scienfict books as many as Miller has. If Nancy gives 5 books to Miller, they will have the same number of books. How many books does each of them have?

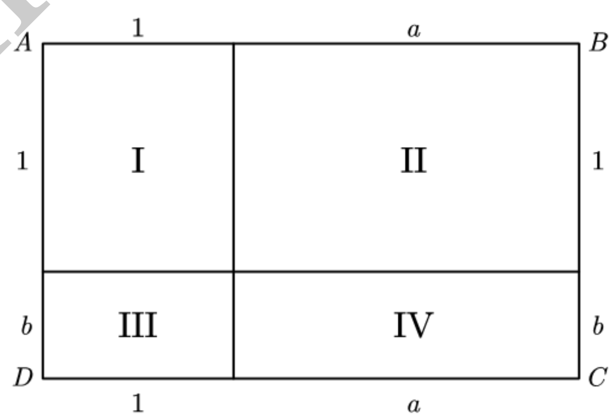
Problem 9 Austin and his brother Benjamin want to collect \$20 to buy a new compute game. If both of them contribute half of their own savings, they will have 2 dollars more than needed. If Austin contributes half of his total saving and Benjamin contributes one third of his saving, they will need 2 more dollars to buy the game. How much money, in dollars, do Austin and his brother have respectively?

Problem 10 A hotel has two different types of rooms, type A and type B. Type A is \$100 per night and can accommodate 3 people. Type B is \$80 per night and can accommodate 2 people. One night, a group of 60 people stay in this hotel. Their rooms consist of both type A and type B. If all their rooms are full and the total cost is \$2120, how many type A and type B rooms do they have respectively?

Problem 11 Use the following geometric picture to put correct numbers in $\boxed{?}$.

$$(1+a)\times(1+b) = \boxed{?} + \boxed{?}\,a + \boxed{?}\,b + \boxed{?}\,ab$$

for every two positive numbers a and b.



Problem 12 $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions. Explain why the following statement is correct.

$$\text{If} \quad \frac{a}{b} = \frac{c}{d}, \quad \text{then } ad = bc.$$

About the Author

Dr. Yu won the first prize (national level) in China's National High School Math Competition in 1994. He graduated form Nankai University's special math class founded by the great mathematician Shiing-Shen Chern; received his PH.D degree from University of California, Berkeley. Dr. Yu is a math professor at University of California, Irvine, and recipient of the prestigious NSF CAREER award (2012).

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