MATHCOUNTS® COMPETITION SERIES

EST. 1983

2017-2018 SCHOOL HANDBOOK

Check out this year's math problems on PG. 11!

2017-2018 MATHCOUNTS®
SCHOOL HANDBOOK

2018 MATHCOUNTS
National Competition Sponsor

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Change the Equation has recognized MATHCOUNTS as having one of the nation's most effective STEM learning programs, listing the Math Video Challenge as an Accomplished Program in STEMworks.

The National Association of Secondary School Principals has placed all three MATHCOUNTS programs on the NASSP Advisory List of National Contests and Activities for 2017-2018.

HOW TO USE THIS SCHOOL HANDBOOK

If You're a New Coach

Welcome! We're so glad you're a coach this year.
Check out the **Guide for New Coaches**
starting on the next page.

If You're a Returning Coach

Welcome back! Thank you for coaching again.
Get the **2018-2018 Handbook Materials**
starting on page 8.

GUIDE FOR NEW COACHES

Welcome to the MATHCOUNTS® Competition Series! Thank you so much for serving as a coach this year. Your work truly does make a difference in the lives of the students you mentor. We've created this Guide for New Coaches to help you get acquainted with the Competition Series and understand your role as a coach in this program.

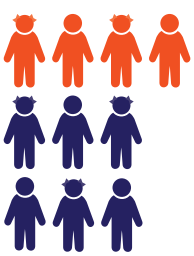
If you have questions at any point during the program year, please feel free to contact the MATHCOUNTS national office at (703) 299-9006 or info@mathcounts.org.

THE MATHCOUNTS COMPETITION SERIES IN A NUTSHELL

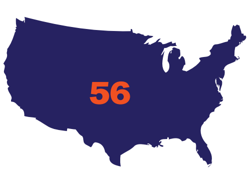
The **MATHCOUNTS Competition Series** is a national program that provides students the opportunity to compete in live, in-person math contests against and alongside their peers. Created in 1983, it is the longest-running MATHCOUNTS program and is open to all sixth-, seventh- and eighth-grade students.

HOW DOES IT WORK? The Competition Series has 4 levels of competition—school, chapter, state and national. Here's what a typical program year looks like.

Schools register in the fall and work with students during the year. Coaches administer the School Competition, usually in January. Any number of students from your school can participate in your team meetings and compete in the School Competition. MATHCOUNTS provides the School Competition to coaches in November. Many coaches use this to determine which student(s) will advance to the Chapter Competition.



Between 1 and 10 students from each school advance to the local Chapter Competition, which takes place in February. Each school can send a team of 4 students plus up to 6 individual competitors. All chapter competitors—whether they are team members or individuals—participate in the individual rounds of the competition; then just the 4 team members participate in the team round. Schools also can opt to send just a few individual competitors, rather than forming a full team. Over 500 Chapter Competitions take place across the country.



Top students from each Chapter Competition advance to their State Competition, which takes place in March. Your school's registration fees cover your students as far as they get in the Competition Series. If your students make it to one of the 56 State Competitions, no additional fees are required.

Top 4 individual competitors from each State Competition receive an all-expenses-paid trip to the National Competition, which takes place in May. These 224 students combine to form 4-person state teams, while also competing individually for the title of National Champion.

WHAT DOES THE TEST LOOK LIKE? Every MATHCOUNTS competition consists of 4 rounds—Sprint, Target, Team and Countdown Round. Altogether the rounds are designed to take about 3 hours to complete. Here's what each round looks like.

Sprint Round
40 minutes
30 problems total
no calculators used
focus on speed and accuracy

Target Round
Approx. 30 minutes
8 problems total
calculators used
focus on problem-solving and mathematical reasoning

The problems are given to students in 4 pairs. Students have 6 minutes to complete each pair.

Team Round
20 minutes
10 problems total
calculators used
focus on problem-solving and collaboration

Only the 4 students on a school's team can take this round officially.

VS

Countdown Round
Maximum of 45 seconds per problem
no calculators used
focus on speed and accuracy

Students with highest scores on Sprint and Target Rounds compete head-to-head. This round is optional at the school, chapter and state level.

HOW DO I GET MY STUDENTS READY FOR THESE COMPETITIONS? What specifically you do to prepare your students will depend on your schedule as well as your students' schedules and needs. But in general, working through lots of different MATHCOUNTS problems and completing practice competitions is the best way to prepare to compete. Each year MATHCOUNTS provides the *School Handbook* to all coaches, plus lots of additional free resources online.

The next sections of this Guide for New Coaches will explain the layout of the *MATHCOUNTS School Handbook* and other resources, plus give you tips on structuring your team meetings and preparation schedule.

THE ROLE OF THE COMPETITION COACH

Your role as the coach is such an important one, but that doesn't mean you need to know everything, be a math expert or treat coaching like a full-time job. Every MATHCOUNTS coach has a different coaching style and you'll find the style that works best for you and your students. But in general **every good MATHCOUNTS coach must do the following**.

- Schedule and run an adequate number of practices for participating students.
- Help motivate and encourage students throughout the program year.
- Select the 1-10 student(s) who will represent the school at the Chapter Competition in February.
- Take students to the Chapter Competition or make arrangements with parents and volunteers to get them there.

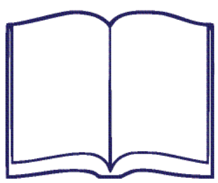
LOOKING FOR TOOLS TO HELP YOU BECOME A TOP-NOTCH COACH? CHECK OUT OUR VIDEOS AT THE COACH SECTION OF THE MATHCOUNTS WEBSITE!

You don't need to know how to solve every MATHCOUNTS problem to be an effective coach. In fact, many coaches have told us that they themselves improved in mathematics through coaching. Chances are, you'll learn with and alongside your students throughout the program year.

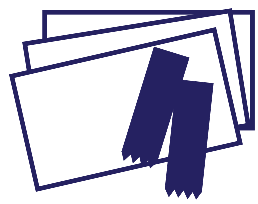
You don't need to spend your own money to be an effective coach. You can prepare your students using solely the free resources and this handbook. We give coaches numerous detailed resources and recognition materials so you can guide your Mathletes® to success even if you're new to teaching, coaching or competition math, and even if you use only the free resources MATHCOUNTS provides all competition coaches.

MAKING THE MOST OF YOUR RESOURCES

As the coach of a registered competition school, you already have received what we at MATHCOUNTS call the **School Competition Kit**. Your kit includes the following materials for coaches.



2017-2018 MATHCOUNTS School Handbook
The most important resource included in the School Competition Kit. Includes 250 problems.



Student Recognition Ribbons and Certificates
10 participation certificates and 1 ribbon for each registered chapter competitor.

You'll also get access to electronic resources. The following resources are available to coaches online at **www.mathcounts.org/coaches**. This section of the MATHCOUNTS website is restricted to coaches and you already should have received an email with login instructions. *If you have not received this email, please contact us at info@mathcounts.org to make sure we have your correct email address.*

Official 2018 MATHCOUNTS School Competition
Released in November 2017
Includes all 4 test rounds and the answer key

2017 MATHCOUNTS School, Chapter + State Competitions
Released by mid-April 2018
Each level includes all 4 test rounds and the answer key

MATHCOUNTS Problem of the Week
Released each Monday
Each multi-step problem relates to a timely event

You can use the **2018 MATHCOUNTS School Competition** to choose the students who will represent your school at the Chapter Competition. Sometimes coaches already know which students will attend the Chapter Competition. If you do not need the School Competition to determine your chapter competitors, then we recommend using it as an additional practice resource for your students.

The **2017-2018 MATHCOUNTS School Handbook** will be your primary resource for the Competition Series this year. It is designed to help your students prepare for each of the 4 rounds of the test, plus build critical thinking and problem-solving skills. This section of the Guide for New Coaches will focus on how to use this resource effectively for your team.

CHECK OUT OUR ONLINE COACH RESOURCE VIDEOS: MAKING THE MOST OF YOUR COACHING RESOURCES

HOW TO USE THE HANDBOOK

WHAT'S IN THE HANDBOOK? There is a lot included in the *School Handbook*, and you can find a full table of contents on pg. 8 of this book, but below are the sections that you'll use the most when coaching your students.

- **Handbook Problems:** 250 math problems divided into Warm-Ups, Workouts and Stretches. These problems increase in difficulty as the students progress through the book. (pg. 11)

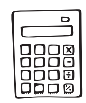
- **Solutions to Handbook Problems:** complete step-by-step explanations for how each problem can be solved. These detailed explanations are only available to registered coaches. (pg. 56)
- **Answers to Handbook Problems:** key available to the general public. Your students can access this key, but not the full solutions to the problems. (pg. 49)
- **Problem Index + Common Core State Standards Mapping:** catalog of all handbook problems organized by topic, difficulty rating and mapping to Common Core State Standards. (pg. 53)

There are 3 types of handbook problems to prepare students for each of the rounds of the competition. You'll want to have your students practice all of these types of problems.

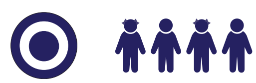
Warm-Ups
14 Warm-Ups in handbook
10 questions per Warm-Up
no calculators used

Warm-Ups prepare students particularly for the Sprint and Countdown Rounds.

Workouts
8 Workouts in handbook
10 questions per Workout
calculators used

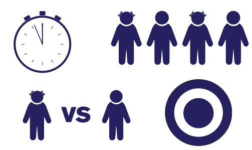


Workouts prepare students particularly for the Target and Team Rounds.



Stretches
3 Stretches in handbook
Number of questions and use of calculators vary by Stretch

Each Stretch covers a particular math topic that could be covered in any round. These help prepare students for all 4 rounds.



IS THERE A SCHEDULE I SHOULD FOLLOW FOR THE YEAR? On average coaches meet with their students for an hour once a week at the beginning of the year, and more often as the competitions approach. Practice sessions may be held before school, during lunch, after school, on weekends or at other times, coordinating with your school's schedule and avoiding conflicts with other activities.

Designing a schedule for your practices will help ensure you're able to cover more problems and prepare your students for competitions. We've designed the *School Handbook* with this in mind. Below is a suggested schedule for the program year that mixes in Warm-Ups, Workouts and Stretches from the *School Handbook*, plus free practice competitions from last year. This schedule allows your students to tackle more difficult problems as the School and Chapter Competition approach.

<table>
<tr><td>Mid-August – September 2017
Warm-Ups 1, 2 + 3
Workouts 1 + 2</td><td>October 2017
Warm-Ups 4, 5 + 6
Workout 3
Probability Stretch</td><td>November 2017
Warm-Ups 7 + 8
Workouts 4 + 5
Patterns Stretch</td><td>December 2017
Warm-Ups 9, 10 + 11
Workout 6
Travel Stretch</td></tr>
<tr><td colspan="2">January 2018
Warm-Ups 12, 13 + 14
Workouts 7 + 8
2018 MATHCOUNTS School Competition
Select chapter competitors (optional at this time)</td><td colspan="2">February 2018
Practice Competition: 2017 School Competition
Practice Competition: 2017 Chapter Competition
Select chapter competitors (required by this time)
2018 MATHCOUNTS Chapter Competition</td></tr>
</table>

You'll notice that in January or February you'll need to select the 1-10 student(s) who will represent your school at the Chapter Competition. This must be done before the start of your local Chapter Competition. You'll submit the names of your chapter competitors either online at **www.mathcounts.org/coaches** or directly to your local Chapter Coordinator.

It's possible you and your students will meet more frequently than once a week and need additional resources. If that happens, don't worry! You and your Mathletes can work together using the **Interactive MATHCOUNTS Platform**, powered by NextThought. This free online platform contains numerous *MATHCOUNTS School Handbooks* and past competitions, not to mention lots of features that make it easy for students to collaborate with each other and track their progress. You and your Mathletes can sign up for free at mathcounts.nextthought.com.

And remember, just because you and your students will meet once a week doesn't mean your students can only prepare for MATHCOUNTS one day per week. Many coaches assign "homework" during the week so they can keep their students engaged in problem solving outside of team practices. Here's one example of what a 2-week span of practices in the middle of the program year could look like.

CHECK OUT THE INTERACTIVE MATHCOUNTS PLATFORM TO GET EVEN MORE HANDBOOK PROBLEMS + PAST COMPETITIONS!

Monday	Tuesday	Wednesday (Weekly Team Practice)	Thursday	Friday
-Students continue to work individually on Workout 4, due Wednesday	-Students continue to work on Workout 4 -Coach emails team to assign new Problem of the Week, due Wednesday	-Coach reviews solutions to Workout 4 -Coach gives Warm-Up 7 to students as timed practice and then reviews solutions -Students discuss solutions to Problem of the Week in groups	-Coach emails math team to assign Workout 5 as individual work, due Wednesday	-Students continue to work individually on Workout 5
-Students continue to work individually on Workout 5, due Wednesday	-Students continue to work on Workout 5 -Coach emails team to assign new Problem of the Week, due Wednesday	-Coach reviews solutions to Workout 5 -Coach gives Warm-Up 8 to students as timed practice and then reviews solutions -Students discuss solutions to Problem of the Week in groups	-Coach emails math team to assign Workout 6 as group work, due Wednesday	-Students work together on Workout 6 using online Interactive Platform

WHAT SHOULD MY TEAM PRACTICES LOOK LIKE? Obviously every school, coach and group of students is different, and after a few practices you'll likely find out what works and what doesn't for your students. Here are some suggestions from veteran coaches about what makes for a productive practice.

- Encourage discussion of the problems so that students learn from each other
- Encourage a variety of methods for solving problems
- Have students write math problems for each other to solve
- Use the **Problem of the Week** (posted online every Monday)
- Practice working in groups to develop teamwork (and to prepare for the Team Round)
- Practice oral presentations to reinforce understanding

On the following page is a sample agenda for a 1-hour practice session. There are many ways you can structure math team meetings and you will likely come up with an agenda that works better for you and your group. It also is probably a good idea to vary the structure of your meetings as the program year progresses.

MATHCOUNTS Team Practice Sample Agenda – 1 Hour

Review Problem of the Week (20 minutes)

- Have 1 student come to the board to show how s/he solved the first part of the problem.
- Discuss as a group other strategies to solve the problem (and help if student answers incorrectly).
- Have students divide into groups of 4 to discuss the solutions to the remaining parts of the problem.
- Have 2 groups share answers and explain their solutions.

Timed Practice with Warm-Up (15 minutes)

- Have students put away all calculators and have one student pass out Warm-Ups (face-down).
- Give students 12 minutes to complete as much of the Warm-Up as they can.
- After 12 minutes is up, have students hold up pencils and stop working.

Play Game to Review Warm-Up Answers (25 minutes)

- Have students divide into 5 groups (size will depend on number of students in meeting).
- Choose a group at random to start and then rotate clockwise to give each group a turn to answer a question. When it is a group's turn, ask the group one question from the Warm-Up.
- Have the group members consult their completed Warm-Ups and work with each other for a maximum of 45 seconds to choose the group's official answer.
- Award 2 points for a correct answer on questions 1-3, 3 points for questions 4-7 and 5 points for questions 8-10. The group gets 0 points if they answer incorrectly or do not answer in 45 seconds.
- Have all students check their Warm-Up answers as they play.
- Go over solutions to select Warm-Up problems that many students on the team got wrong.

GET MORE RESOURCES + ACTIVITIES TO MAKE TEAM MEETINGS FUN AT THE COACH SECTION OF THE MATHCOUNTS WEBSITE!

OK I'M READY TO START. HOW DO I GET STUDENTS TO JOIN? Here are some tips given to us from successful competition coaches and club leaders for getting students involved in the program at the beginning of the year.

- Ask Mathletes who have participated in the past to talk to other students about participating.
- Ask teachers, parent volunteers and counselors to help you recruit.
- Reach parents through school newsletters, PTA meetings or Back-to-School-Night presentations.
- Advertise around your school by:
  1. posting intriguing math questions (specific to your school) and referring students to the first meeting for answers.
  2. designing a bulletin board or display case with your MATHCOUNTS poster (included in your School Competition Kit) and/or photos and awards from past years.
  3. attending meetings of other extracurricular clubs (such as honor society) so you can invite their members to participate.
  4. adding information about the MATHCOUNTS team to your school's website.
  5. making a presentation at the first pep rally or student assembly.

Good luck in the competition! If you have any questions during the year, please contact the MATHCOUNTS national office at (703) 299-9006 or info@mathcounts.org.

COACH RESOURCES: WWW.MATHCOUNTS.ORG/COACHES

2017-2018 HANDBOOK MATERIALS

Thank you for being a coach in the MATHCOUNTS Competition Series this year!
We hope participating in the program is meaningful and enriching for you and your Mathletes.
Don't forget to log in at **www.mathcounts.org/coaches** for additional resources!

WHAT'S IN THIS YEAR'S HANDBOOK

HIGHLIGHTED RESOURCES

Also access resources at www.mathcounts.org/coaches!

C Great for Coaches | M Great for Mathletes | A Advanced Level Book | Free Resource

OPLET

Online database of over 13,000 problems and over 5,000 step-by-step solutions. Create personalized quizzes, flash cards, worksheets and more!

Save $25 when you buy your subscription by Oct. 13, 2017
Renewers: use code RENEW18
First-Time Subscribers: use code NEW18

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www.mathcounts.org/myoplet

Practice Competitions for MATHCOUNTS, Vol. I & II

Practice books written by repeat national-level coach Josh Frost. Each volume includes 4 complete mock-competitions plus solutions.

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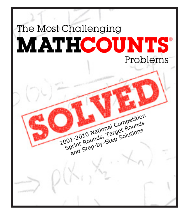
www.mathcounts.org/store

Most Challenging MATHCOUNTS Problems Solved



Advanced level practice book with 10 years of national-level Sprint Rounds, plus detailed step-by-step solutions to each problem.

C M A

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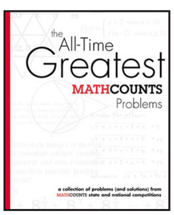
All Time Greatest MATHCOUNTS Problems



A collection of some of the most creative, interesting and challenging MATHCOUNTS competition problems.

C M A

www.mathcounts.org/store

Interactive MATHCOUNTS Platform

Online platform of past and current handbook and competition problems. Interactive features make collaboration easy and fun!

C M

mathcounts.nextthought.com

MATHCOUNTS Trainer App

Train your Mathletes with this fun app, featuring real-time leaderboards and lots of past MATHCOUNTS problems.

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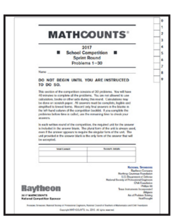
aops.com/mathcounts_trainer or download at the App Store

Past Competitions



Last year's School, Chapter and State competitions are free online! Other years' competitions can be purchased.

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www.mathcounts.org/pastcompetitions
www.mathcounts.org/store

Problem of the Week

A new, multi-step problem every week! Each problem focuses on a particular set of math skills and coincides with a timely event, holiday or season. Get the problem at the beginning of the week and the step-by-step solution the following week.

C M

www.mathcounts.org/potw

MATHCOUNTS Minis

A fun monthly video series featuring Richard Rusczyk from Art of Problem Solving. Each video looks at a particular math skill and walks through how to solve different MATHCOUNTS problems using creative problem-solving strategies.

C M

www.mathcounts.org/minis

CRITICAL 2017-2018 DATES

2017

Aug. 15 – Dec. 15	Submit your school's registration to participate in the Competition Series and receive this year's School Competition Kit, which includes a hard copy of the *2017-2018 MATHCOUNTS School Handbook*. Kits are shipped on an ongoing basis between mid-August and December 31. The fastest way to register is online at **www.mathcounts.org/compreg**. You also can download the MATHCOUNTS Competition Series Registration form and mail or email it with payment to: MATHCOUNTS Foundation – Competition Series Registrations 1420 King Street, Alexandria, VA 22314 *Email*: reg@mathcounts.org To add students to your school's registration, log in at **www.mathcounts.org/coaches** to access the Dashboard. **Questions?** Call the MATHCOUNTS national office at (703) 299-9006 or email us at info@mathcounts.org.
Nov. 1	The 2018 School Competition will be available online. All registered coaches can log in at www.mathcounts.org/coaches to download the competition.
Nov. 3 (postmark)	**Deadline to register for the Competition Series at reduced registration rates** ($30 per student, $300 for full registration of 10 students). After November 3, registration rates will be $35 per student, $350 for full registration.
Dec. 15 (postmark)	**Competition Series Registration Deadline** In some circumstances, late registrations might be accepted at the discretion of MATHCOUNTS and the local coordinator. *Late fees will apply. Register on-time to ensure your students' participation.*

2018

Early Jan.	If you have not been contacted with details about your upcoming competition, call your local or state coordinator. Coordinator contact information is available at www.mathcounts.org/findmycoordinator.
Late Jan.	If you have not received your School Competition Kit, contact the MATHCOUNTS national office at (703) 299-9006 or info@mathcounts.org.
Feb. 1-28	**Chapter Competitions**
March 1 – Apr. 1	**State Competitions**
May 13-14	**2018 Raytheon MATHCOUNTS National Competition in Washington, DC**

THIS YEAR'S HANDBOOK PROBLEMS

You and your students might notice something special about some of the problems in this year's Warm-Ups and Workouts...

Throughout this handbook are names—first and/or last—of people who donated to the MATHCOUNTS Foundation's Giving Tuesday campaign last year to help us cover half the cost of registering for the Competition Series for Mathletes from low-income schools. These donors help make this program possible for students across the country, so we decided it was fitting to include them in the primary preparation resource for participants in this program.

To all of our 2016 Giving Tuesday Donors (whether or not you chose to be featured in this handbook)

THANK YOU!

Probability Stretch

1. ________ % Petra randomly selects a card from a standard deck of 52 playing cards. What is the percent probability that the card shows a red number greater than 6? Express your answer to the nearest hundredth.

2. ________ Max has eight identical cups. Each cup contains a different combination of nickels, dimes and quarters, each totaling 45 cents. Max randomly selects a cup. What is the probability that the cup he selects contains at least three dimes? Express your answer as a common fraction.

3. ________ A bag contains five chips numbered 2 through 6. Danya draws chips from the bag one at a time and sets them aside. After each draw, she totals the numbers on all the chips she has already drawn. What is the probability that at any point in this process her total will equal 10? Express your answer as a decimal to the nearest tenth.

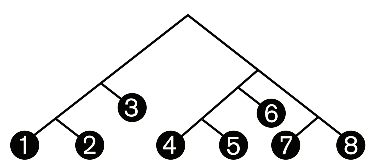
4. ________ A drawer contains five socks: two green and three blue. What is the probability that two socks pulled out of the drawer at random will match? Express your answer as a common fraction.

5. ________ A penny, a nickel and a dime are flipped. What is the probability that at least two coins land heads up and one of them is the nickel? Express your answer as a common fraction.

6. ________ % When the circuit containing blinking lights A and B is turned on, lights A and B blink together. Then A blinks once every 5 seconds and B blinks once every 11 seconds. Lindsey looks at the two lights just in time to see A blink alone. What is the percent probability that the next light to blink will be A blinking alone?

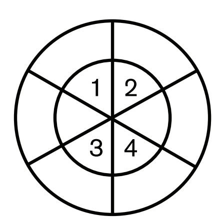
7. ________ % What is the percent probability that a randomly selected multiple of 3 less than or equal to 3000 is also a multiple of 5?

8. ________ Starting at the top and selecting paths randomly as you move downward, what is the probability of ending at an odd number? Express your answer as a common fraction.



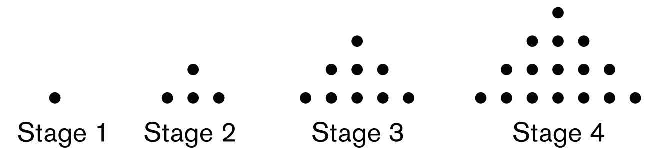
9. ________ A five-digit number is made by randomly ordering the digits 1, 2, 3, 4 and 5. What is the probability that this number is divisible by 4? Express your answer as a common fraction.

10. ________ Pierre throws darts that land randomly in the dartboard shown here. The dartboard is a circle of radius 2 units, with an inner circle of radius 1 unit. Both circles are divided into six congruent sectors. What is the probability that a dart Pierre throws will land in one of the four inner numbered sectors? Express your answer as a decimal to the nearest hundredth.



Patterns Stretch

11.________ dots The first four stages of a dot pattern are shown. How many more dots are in the figure at Stage 47 than in the figure at Stage 27?



12.________ The first three terms of a sequence are 1, 2 and 3. Each subsequent term is the sum of the three previous terms. What is the 11th term of this sequence?

13.________ What is the sum of the terms in the arithmetic series 2 + 5 + 8 + 11 + 14 + ... + 89 + 92?

14.________ Three consecutive terms in an arithmetic sequence are x, $2x + 11$ and $4x - 3$. What is the constant difference between consecutive terms in this sequence?

15.________ What is the sum of the terms in the geometric series 1 + 4 + 16 + ... + 1024?

16.________ What is the sum of the first 51 consecutive odd positive integers?

17.________ What is the sum of the terms in the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$?

18.________ What is the sum of the terms in the infinite series $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$? Express your answer as a common fraction.

19.________ Let $f(x) = 2x + 3$ and $f^2(x) = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$. If $f^5(x) = ax + b$, what is the value of $a + b$?

20.________ degrees The degree measures of the interior angles of a quadrilateral form a geometric sequence whose terms have integer values and are all integer multiples of the first term. What is the largest possible degree measure of an angle in this quadrilateral?

Travel Stretch

speed = $\frac{\text{distance}}{\text{time}}$	distance = speed × time	time = $\frac{\text{distance}}{\text{speed}}$

21. ________ mi/h Jack and Jill travel up a hill at a speed of 2 mi/h. They travel back down the hill at a speed of 4 mi/h. What is their average speed for the entire trip? Express your answer as a mixed number.

22. ____:____ p.m. At 2:20 p.m., Jack is at the top of the hill and starts walking down at the exact same time that Jill, who is at the bottom of the hill, starts walking up. If they maintain the same uphill and downhill speeds from the previous problem, and the distance from the bottom to the top of the hill is 1.5 miles, at what time will Jack and Jill meet?

23. ________ yards When Jack and Jill meet, as described in the previous problem, how many yards will they be from the bottom of the hill?

24. ________ minutes Alysha's average speed when walking from home to the market is 5 mi/h, and it takes her 21 minutes longer than when she drives to the market. If Alysha drives to the market, along the same route, at an average speed that is eight times her average walking speed, how many minutes does it take her to drive from home to the market?

25. ________ miles Based on problem 24, how many miles does Alysha travel to get from home to the market?

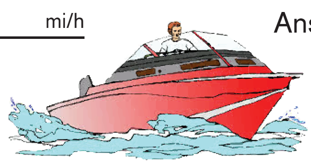
26. ________ minutes

Jana begins jogging along a path and, 5 minutes later, Zhao begins riding his bicycle along the same path, which has a length of 2 miles. Zhao rides his bicycle at a speed of 10 mi/h, and Jana's jogging speed is 6 mi/h. If they both begin at one end of the path and end at the other, how many minutes after Zhao reaches the end of the path will Jana reach the end of the path?

27. ________ minutes Based on problem 26, how many minutes after Zhao begins riding will he catch up with Jana? Express your answer as a mixed number.

28. ________ miles Again, based on problem 26, how many miles will Jana have traveled when Zhao catches up with her? Express your answer as a mixed number.

29. ________ mi/h



Ansel left the dock in his motorboat, traveled 10 miles, and then returned to the dock along the same route. On the return trip, Ansel was traveling against the current of the river, and his average speed relative to the water was 20 mi/h. If the round-trip took Ansel 64 minutes, what is the speed of the river's current?

30. ________ Based on problem 29, what fraction of Ansel's total travel time was spent traveling upstream? Express your answer as a common fraction.

Warm-Up 1

31.__________ What is the value of $5 - 5 \times 5 + 5 \div 5$?

32.______diagonals How many diagonals are in a convex heptagon?

33.__________ What is the first year after 2018 that is a palindrome?

34.__________ A standard 52-card deck of playing cards includes four aces. What is the probability that two cards selected randomly, without replacement, will both be aces? Express your answer as a common fraction.

35.__________ What is the value of $\sqrt{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 10}$? Express your answer in simplest radical form.

36.__________°F The temperature dropped from 13 °F to −5 °F. How many degrees Fahrenheit is the absolute value of the change in temperature?

37.__________ What is the value of $1 \times 2 + 3 \div 6 \times 5 - 4$? Express your answer as a common fraction.

38.__________ If x ® y is defined as $x^2 - y^2$, what is the value of 3 ® (2 ® 1)?

39.__________ If the digits 7, 8, 2, 3 and 0 are used, each exactly once, to form a three-digit positive integer and a two-digit positive integer that differ by exactly 288, what is the sum of the three-digit integer and the two-digit integer?

40.______degrees In rectangle ABCD, point P lies on side BC and point Q lies in the interior of the rectangle so that triangle APQ is equilateral. If the measure of angle PAB is 17 degrees, what is the measure of angle QPC?

Warm-Up 2

41. ________ balls Kim is knitting a baby blanket that requires 750 meters of yarn. There are 180 meters of yarn in each ball. How many balls of yarn must Kim buy to ensure she has enough yarn to complete her blanket?

42. ________ years old On Chris' birthday in 1992, he was half the age of his brother Joseph. On Chris' birthday in 1998, he was two-thirds the age of Joseph. How old will Chris be on his birthday in 2018?

43. ________ degrees On a standard 12-hour clock, the minute hand moves continuously, at a constant rate, making one full revolution every hour, and the hour hand moves similarly, making one full revolution every 12 hours. What is the measure of the smaller of the two angles between the minute hand and the hour hand, in degrees, when the clock reads 5:42?

44. ________ What is the value of the expression $12 \times 37 + 12 \times 7 + 12 \times 6$?

45. ________ factors How many distinct positive factors does 2018 have?

46. ________ Two fair six-sided dice, with sides numbered 1 through 6, are rolled. What is the probability that the values on the two top faces add to at least 9? Express your answer as a common fraction.

47. ________ If the graph of the equation $y = mx + b$ is a line passing through the points (6, 13) and (10, 31), what is the value of m? Express your answer as a common fraction.

48. ________ Dewey buys soda in 12-ounce cans that cost \$1.00 each. Peppar buys soda in 20-ounce bottles that cost \$1.25 each. If Dewey and Peppar buy the same volume of soda in one week, then Peppar pays P% less than Dewey. What is the value of P?

49. ________ logs Gerald Scheetz is building a log cabin. If each log is 9 inches in diameter, how many logs must be stacked on top of one another to create a wall that has a height of 12 feet?

50. ________ units^2 A square with area 8 units^2 is inscribed in a circle. What is the area of the circle? Express your answer in terms of π.

Warm-Up 3

51.__________ If y is a number such that $y^2 = (y + 2018)^2$, what is the value of y?

52.__________ years old Maura is 5 years younger than her sister Cara. Seven years ago, Maura was half as old as her sister. How old is Maura now?

53.__________ A dartboard consists of three concentric circles with radii 10, 5 and 1, respectively, measured in inches. The area between the largest and middle circles is colored green, the area between the middle and smallest circles is colored yellow, and the area within the smallest circle, the bull's-eye, is colored red. If a thrown dart is guaranteed to hit the board, but its position on the board is uniformly random, what is the probability that it lands in the yellow portion of the board? Express your answer as a common fraction.

54.__________ days The 1990 and 2018 calendars are identical in the number of days in each month and the day of the week on which each day of each month occurs. In fact, the calendar repeats in these ways every 28 years until the year 2100. How many days are there in the 28 years preceding 2018?

55.__________ cm A right triangle has legs with lengths of 5 cm and 10 cm. What is the length of the altitude drawn to the hypotenuse of this triangle? Express your answer in simplest radical form.

56.__________ Min Zhang wrote down all of the two-digit multiples of 5. What is the probability that one of these numbers, chosen at random, has exactly two distinct primes that are factors? Express your answer as a common fraction.

57.__________ Given a set of numbers with median m, the median of all the numbers less than m is the called the *lower quartile*. The median of all the numbers greater than m is called the *upper quartile*. The absolute difference between the lower and upper quartiles is called the *interquartile range*. What is the interquartile range for the numbers in the stem-and-leaf plot shown?

0	2 4 6 8 9
1	1 1 3 8 9 9
2	0 4 6 8
3	0 3 5 7 7
4	1 5 7 7

58.__________

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

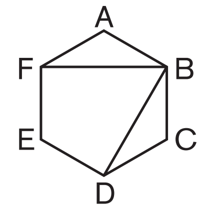
Positive integers 1 to 36 are written in rows in a six-by-six array as shown. Each prime number is crossed off, as well as all the numbers in the diagonal extending up and to the right from that prime. For example, 11 is prime and is crossed off along with the 6 above and to the right. What is the sum of the remaining values after all the primes and associated diagonals have been eliminated?

59.__________ What is the value of 1,000,000! ÷ 999,999!?

60.__________ inches A cubic yard of topsoil is to be spread evenly in the garden at Prove It! Math Academy. The garden measures 10 feet by 8 feet. How many inches deep will the topsoil be? Express your answer as a mixed number.

Warm-Up 4

61. __________ Diagonals FB and BD are drawn in regular hexagon ABCDEF. What is the ratio of the sum of the areas of triangles ABF and BCD to the area of quadrilateral BDEF? Express your answer as a common fraction.



62. __________ What is the value of $\dfrac{11! - (9+1)(9!)}{8(7!)}$?

63. __________ times David's optometrist sold him a bottle of eyeglass cleaner containing 30 mL of glass-cleaning solution. Assuming there are 20 drops per milliliter, and assuming proper cleaning requires 3 drops of glass cleaner on each side of each lens, what is the maximum number of times David can properly clean his glasses before he must buy a new bottle of eyeglass cleaner?

64. __________ combinations The lunch-ordering app for Pete's Pizza Parlor requires you to choose two distinct meats from among pepperoni, Canadian bacon and sausage; or choose two distinct vegetables from among mushrooms, onions, green peppers and black olives; or choose one meat and one vegetable from among the same choices. How many different pizza combinations are possible using the lunch-ordering app?

65. __________ pounds Kathy Beckhardt weighs four of her sheep at the fair. She can weigh two of them at a time on the big scale. Sheep A and sheep B have an average weight of 150 pounds, sheep B and sheep C have an average weight of 127 pounds, and sheep C and sheep D have an average weight of 168 pounds. What is the average weight of sheep A and sheep D?

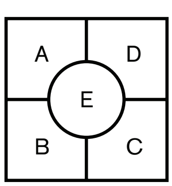
66. __________ meters In circle O, the lengths of chords AB and BC are equal and $m\angle ABC = 90$ degrees. Given that circle O has a radius of 3 meters, what is the length of arc ABC? Express your answer in terms of π.

67. __________ tiles How many 4-inch square tiles are needed to cover a wall that measures 6 feet by 8 feet?

68. __________ What is the units digit of $2^{2017} \times 7^{2017}$?

69. __________ integers How many integers between 100 and 1000 contain no digits other than 3, 4 or 5?

70. __________ paths



The square shown is divided into five cells. How many paths can be drawn that start at any cell, move only to adjacent cells and visit each of the five cells exactly once?

Warm-Up 5

71. ________ paper clips A pencil and 5 paper clips weigh the same as 2 erasers. A pencil weighs the same as 29 paper clips. How many paper clips weigh the same as an eraser?

72. ________ points What is the maximum number of points of intersection of a right triangle with a square, assuming no side of the triangle is collinear with any side of the square?

73. ________ If $p(x) = ax^2 + bx + c$ is a quadratic polynomial satisfying $p(0) = 4$, $p(1) = 15$, $p(2) = 36$, what is the value of the product abc?

74. ________ units A certain sphere has a volume that is numerically equal to three times its surface area. What is the radius of this sphere?

75. ________ candles A layered candle is made with 5 colors, shown here as candle A. How many different candles can be made using the same 5 colors, with BLUE as the middle layer, shown as candle B, and with no color next to a color that it touched in candle A?

A	B
BLUE	
GREEN	
RED	BLUE
ORANGE	
YELLOW	

76. ________ Suppose Luke spins the pointer on a fair 3-color spinner twice. What is the probability that the pointer lands on the same color twice? Express your answer as a common fraction.

77. ________ shots Kevin is playing basketball and up to now made $\frac{1}{3}$ of his attempted shots. If he makes his next 5 shots, he will improve his shooting percent to 50%. How many shots has Kevin attempted up to now, when he has a $\frac{1}{3}$ success rate?

78. ________ base eight What is 110011_2 when rewritten in base eight?

79. ________ points If the point (8, 9) is the center of a circle of radius 10 units, at how many points does the circle intersect the coordinate axes?

80. ________ If $x + \frac{1}{y} = \frac{1}{5}$ and $y + \frac{1}{x} = 20$, what is the value of the product xy?

Warm-Up 6

81.__________ If $3x + 5 = 13$, what is the value of the expression $(3x + 2)(3x + 3)(3x + 4)$?

82.__________ units^2 What is the maximum area of a rectangle with a diagonal of length 16 units?

83.__________ pairs How many pairs of numbers (a, b) satisfy rules I and II shown here?

I. $a = 0$ or $b = -1$ or $b = 1$
II. $a = -1$ or $a = 1$ or $b = 0$

84.__________ If each letter in the sum A.BC + D.EF represents a different nonzero digit, what is the least possible value of the sum? Express your answer as a decimal to the nearest hundredth.

85.__________ ways In the hardware store, Matt goes to the fastener aisle, which has wood screws, sheet metal screws, hex bolts, carriage bolts and lag bolts. How many ways can he choose 10 fasteners if he needs at least one of each kind?

86.__________ A family farm is equally divided among three heirs: Jim, Jan and John. John's share of the farm is then equally divided among his three heirs: Peter, Paul and Patricia. Paul decides to sell his share of the farm, and then later the family decides to sell the remainder of the farm all at once. What portion of the proceeds from the most recent sale should Jim receive? Express your answer as a common fraction.

87.__________ Olivia Justynski earned scores of 82, 86 and 92 on her first three tests. What score does she need on her fourth test to achieve an average score of 90 on the four tests?

88.__________ hours For each child, Kiddie Day Care charges \$330 per month for preschool and \$5.50 per hour for each hour of after-school care. If Cody's cost was \$770 for his son's child care last month, how many hours did his son spend in after-school care?

89.__________ When three consecutive positive integers are multiplied, the product is 16 times the sum of the three integers. What is the difference of the product minus the sum?

90.__________ The lift force on an airplane during flight is directly proportional to the surface area of the wing. Orville builds a model airplane and goes outside to play. Orville's little brother, Wilbur, builds a mini replica of Orville's plane that is half as long in every linear dimension. What is the ratio of the lift force on Wilbur's plane to that on Orville's plane? Express your answer as a common fraction.

Warm-Up 7

91.__________ What whole number n makes $\frac{6}{78} < \frac{1}{n} < \frac{5}{55}$ true?

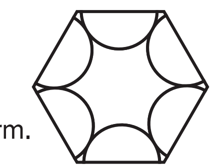
92.__________ In 2016 the Flying Turtles finished their baseball season with a record of 95 wins and 67 losses. The Dolphins finished the season with 84 wins and 78 losses. The Flying Turtles and Dolphins played each other 19 times during the season. If the Flying Turtles had F wins against teams other than the Dolphins, and the Dolphins had D wins against teams other than the Flying Turtles, what is the value of $F + D$?

93.__________ units A point D is placed on the segment with endpoints (0, 8) and (8, 0), and a point E is placed on the segment with endpoints (−3, 0) and (0, −2). What is the shortest possible distance between D and E? Express your answer in simplest radical form.

94.__________ terms In the arithmetic sequence 1, 3, 5, 7, 9, 11, …, how many terms appear after the term 315 but before the term 639?

95.__________ Allen Zhang rolls two fair 6-sided dice with faces numbered 1 through 6. What is the probability that the sum of his two rolls has an odd number of positive integer divisors? Express your answer as a common fraction.

96.__________ Six semicircles, each of radius r, are constructed inside a regular hexagon of side length s, one on each side, so that each semicircle is tangent to two others. What is the ratio of r to s? Express your answer in simplest radical form.



97.__________ Gaylon starts writing down dates from January 1, 2018 onward as follows: 01012018, 01022018, 01032018, etc. What is the 2018th digit Gaylon writes down?

98.__________ lightning bolts Zeus threw, on average, 12 lightning bolts per day in the month of March. During the first week in April, he averaged 15 lightning bolts per day. How many lightning bolts does Zeus need to throw per day on average for the rest of April to maintain a 12-bolt-per-day average over March and April? Express your answer to the nearest integer.

99.__________ For what positive value of x is the equation $9^{2x^2-6} = 27^{x^2-1}$ true?

100.__________ times The decibel is a unit used to describe the loudness of a sound. For every 20-decibel increase, a sound gets 10 times as loud. Normal conversation is about 60 decibels, and a loud rock concert is about 120 decibels. How many times as loud is a rock concert compared to normal conversation?

Warm-Up 8

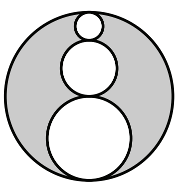
101.________ Pamela Wickham writes a sequence of four consecutive integers on a sheet of paper. The sum of three of these integers is 206. What is the other integer?

102.________ seconds Benjamin starts walking up on an escalator that moves down one flight of stairs every 20 seconds. Benjamin takes 10 seconds to walk up a single flight of stairs on the adjacent stationary staircase. Assuming Benjamin walks at the same speed on the escalator and stairs, how many seconds does it take him to walk up two flights on this escalator?

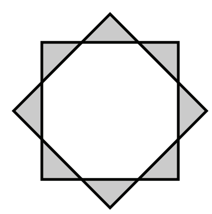
103.________ divisors Let $K = 168 \times 900 = 151{,}200$. How many positive integer divisors does K have?

104.________ Emma Kerwin creates a custom six-sided die by randomly choosing six different integers between 1 and 7, inclusive, to paint on the sides of a blank cube. What is the probability that the faces of her die sum to 24? Express your answer as a common fraction.

105. $________ The owners of two food carts calculate their weekly profits for three weeks. The medians and the highest weekly profit values are the same for the two carts. The mean weekly profit of Cart A is $27 more than that of Cart B. What is the absolute difference between the lowest weekly profit values of Cart A and Cart B?

106.________  Each of the circles in the figure is tangent to exactly two others. The centers of all four lie on a line. If the diameters of the three inner circles are in a ratio of 1:2:3, what fraction of the largest circle is shaded? Express your answer as a common fraction.

107.________ Two congruent squares overlap to form a regular octagon as shown. What is the ratio of the shaded area to the area of the regular octagon? Express your answer in simplest radical form.



108.________ hours It takes Avi one half-hour longer to make a basket than it takes Markus. After 28 hours, Markus has made one more basket than Avi has made. How many hours does it take Avi to make one basket?

109.________ Suppose N is a positive integer such that $N - 1$ is even, $N - 2$ is divisible by 3, $N - 3$ is divisible by 5, and $N - 5$ is divisible by 7. What is the least possible value of N?

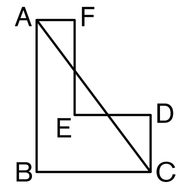
110.________ What fraction of the positive integer factors of 1000^3 are perfect squares? Express your answer as a common fraction.

Warm-Up 9

111.________ Sola's lucky numbers are 7 and 11. So he decides his lucky common fraction, f, will be formed by the repeating decimal $f = 0.\overline{711}$. What is the value of f as a common fraction?

112.________ Suppose m is the line given by the equation $6x - 3y = 7$, and suppose n is the line perpendicular to m and passing through the point (6, 2). If k is the line of slope 5 and y-intercept 1, what is the x-coordinate of the intersection of n and k? Express your answer as a common fraction.

113.________ In hexagon ABCDEF, shown here, adjacent sides are perpendicular. If AB = 8, BC = 6, CD = 3 and DE = 4, what fraction of the segment AC lies inside of the hexagon? Express your answer as a common fraction.



114.________ times stronger The Richter scale is used to describe the strength of an earthquake. An increase of 1 point on the Richter scale represents a tenfold increase in the strength of an earthquake. How many times stronger is an earthquake rated 7.5 on the Richter scale compared to an earthquake rated 5? Express your answer in simplest radical form.

115.________ The third term of a geometric sequence of integers is 45. The seventh term of the sequence is 3645. What is the least possible sum of the first five terms of the sequence?

116.________ words In a new version of Scrabble, a sequence of letters is considered a word if the first and last letters are consonants and every letter in between is a vowel. In this game, how many four-letter words can be formed using each of the letters M, A, T, H, R, U, L, E and S no more than once?

117.________ mi/h Rebecca and Susan live at opposite ends of a 2-mile-long street. At 8:00 a.m., Rebecca starts jogging from her house toward Susan's end of the street. At 8:06 a.m., Susan starts jogging from her apartment toward Rebecca's end of the street. They pass each other at exactly 8:13 a.m. If Rebecca and Susan jog at the same constant speed, what is this speed, in miles per hour?

118.________ units^2 We define a *Heronian triangle* to be a triangle with three integer side lengths and integer area. What is the least possible positive area of a Heronian triangle whose longest side has a length of 17 units?

119.________ For each of the first eight prime numbers, Brian Edwards writes down all the number's positive factors. What is the sum of all the numbers Brian writes down?

120.________ If $\sqrt{x} - \sqrt{y} = 10$ and $\sqrt{x} + \sqrt{y} = 14$, what is the value of $x + y$?

Warm-Up 10

121. ________ What is the greatest prime factor of $(1!)! \times (2!)! \times (3!)! \times (4!)!$?

Method	Hours (to fully charge)
Wall Charger	1.5
Computer	3
Portable Charger	5

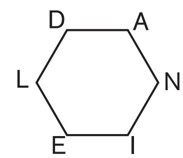
122. ________ minutes The table shows how long it takes Anita's fully discharged cell phone battery to fully charge using three methods. When her phone battery fully discharged, Anita charged the phone for half an hour using the wall charger, and now she will continue charging it for 1 hour using her computer. How many minutes are required to fully charge the phone battery using the portable charger, if the phone is not used during or between chargings?

123. ________ In equilateral triangle ABC, M is the midpoint of side AB. If CMN is also an equilateral triangle, what fraction of the area of triangle $\triangle ABC$ lies inside of $\triangle CMN$? Express your answer as a common fraction.

124. ________ What is the greatest prime factor of $3^7 - 27$?

125. ________ ways In how many ways can eight differently colored balls, including one red, one green and one yellow, be ordered left to right so that the green ball is to the right of the red ball (not necessarily adjacent) and the yellow ball is to the right of the green ball (not necessarily adjacent)?

126. ________ units Sides DL and AN in a regular hexagon DANIEL, shown here, are extended until they intersect at a point F. If the sides of the hexagon have length 6 units, what is the length of segment FE? Express your answer as a radical in simplest form.



127. ________ baskets Annette, Mary and Lynn team up to pick apples. Annette can pick 4 baskets of apples per hour, and Mary can pick 5 baskets of apples per hour. Annette, Mary and Lynn work together to pick 6 baskets of apples in half an hour. How many baskets of apples can Lynn pick by herself in 3 hours?

128. ________ Kayla Straub starts with a pile of 15 stones. She divides the pile into two new piles and finds the product of the numbers of stones in the two new piles. Kayla then divides one of the existing piles into two new piles. She finds the product of the numbers of stones in the two new piles and adds it to the previous product. Kayla continues this process, each time adding the product of the numbers of stones in the two new piles to the previous total, until she has 15 piles with one stone each. What is the greatest possible ending total?

129. ________ A sphere is inscribed in a cube. What is the ratio of the volume of the cube to that of the sphere? Express your answer as a common fraction in terms of π.

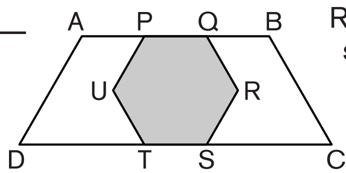
130. ________ Let $\#x$ represent the greatest even integer less than x. If $20 < x < 30$, what is the maximum possible value of $\#(5x) - \#(4x)$?

Warm-Up 11

131.______ points If two distinct ellipses and a square are drawn, what is the maximum possible number of points at which at least two of the three planar figures intersect?

132.________ Isosceles triangles ABC and DEF have six interior angles altogether, but these six angles have only three different measures among them. If the sum of these three different measures is 156 degrees, and both triangles have at least one angle of measure m degrees, what is the value of m?

133.________

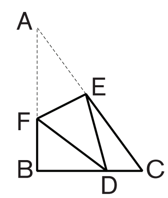


Regular hexagon PQRSTU lies inside of trapezoid ABCD, as shown, so that vertices P and Q trisect the base AB, S and T lie on the base CD, and sides PU and QR are parallel to sides AD and BC, respectively. The shaded area is what fraction of the area of trapezoid ABCD? Express your answer as a common fraction.

134.______ integers How many positive integers in the set of numbers from 1 to 1000, inclusive, are multiples of 2, 3 and 5 but not 8?

135.________ In the sum ABCD + EFGH, each letter represents a digit selected independently at random from the set {1, 2, 3, 4}. What is the probability that the sum of the two four-digit numbers contains the digit 5 at least once? Express your answer as a common fraction.

136.______ cm^2 In right triangle ABC, with AB = 44 cm and BC = 33 cm, point D lies on side BC so that BD:DC = 2:1. If vertex A is folded onto point D to create quadrilateral BCEF, as shown, what is the area of triangle CDE?



137.______ passes After the first eight games of the football season, Jason Doan had completed 70% of his passes. During the ninth game, he completed 49 of his 50 passes, raising his season pass completion rate to 74%. How many total passes did he throw during the first nine games?

138.________ The mean of seven distinct positive integers is 20. What is the difference between the greatest and least possible medians of the seven integers?

139.______ integers How many two-digit positive integers have a units digit that is equal to the product of its two digits?

140.________ Colleen Kipfstuhl rolls a standard fair six-sided die. If she rolls a number with an odd number of positive integer divisors, she steps 1 meter to her right. Otherwise, she steps 1 meter to her left. After four rolls of the die, what is the probability Colleen ends up right where she started? Express your answer as a common fraction.

Warm-Up 12

141. ________ What is the absolute difference between the sum of the multiples of 2, from 1 to 100, inclusive, and the sum of the multiples of 3, from 1 to 100, inclusive?

142. ________ If $p(x)$ is a cubic polynomial with $p(0) = 4$, $p(1) = 10$, $p(-1) = 2$ and $p(2) = 26$, what is the value of $p(3)$?

143. ________ inches What is the greatest possible perimeter of an obtuse triangle, each of whose side lengths is a whole number of inches less than or equal to 100?

144. ________ fist bumps After playing a math game, each member of the MATHCOUNTS national office staff gives a fist bump to every coworker. If 25 members of the national office staff participate as described, how many total fist bumps occur?

145. ________ students Several students were trying out for a class play. When asked which roles they were willing to play, 12 of them were willing to play the knight, 15 were willing to play the princess and 6 were willing to play the sorcerer. Of these students, 8 were willing to play either the knight or the princess, 5 were willing to play the knight or the sorcerer, and 4 were willing to play the princess or the sorcerer. Exactly 3 of these students were willing to play any of the roles. How many students were willing to play the sorcerer but no other role?

146. ________ ways Frankie the frog stands at the number 0 on a number line and wants to hop to the number 8. He can hop 1, 2 or 3 units forward in a single jump. How many different ways are there for Frankie to reach the number 8?

147. ________ The median and the mean of the five integers 10, 12, 26, x, x are equal. What is the sum of all possible values of x?

148. ________ ordered pairs How many ordered pairs of prime numbers (a, b) are there such that $a + b = 100$?

149. ________ cm What is the perimeter of a right triangle with an area of 10 cm^2 and a hypotenuse of length 10 cm? Express your answer in simplest radical form.

150. ________ If $\frac{2}{x+1} + \frac{8}{y-3} = \frac{10}{3}$ and $\frac{4}{x+1} - \frac{2}{y-3} = \frac{2}{3}$, what is the value of $x + y$?

Warm-Up 13

151. ________ ordered triples — How many ordered triples of integers (m, n, p) exist such that $mn = p$, $np = m$ and $mp = n$?

152. ________ cm^2 — What is the least possible area of a rectangle that can enclose an equilateral triangle with side length 6 cm? Express your answer in simplest radical form.

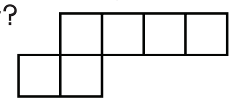
153. ________ integers — How many of the first 2018 positive integers are either perfect squares or perfect cubes?

154. ________ assortments — Alexander Clifton visits Sweet Dreams bakery, which sells three kinds of cookies. How many unique assortments of a dozen cookies can Alexander buy?

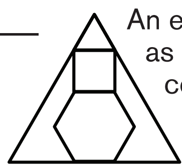
155. ________ dimes — Gabriel and Isabel each start with a pile of 20 coins consisting of nickels, dimes and quarters. After Gabriel gives Isabel 2 coins, and Isabel gives Gabriel 5 coins, Gabriel's pile is worth twice the value of Isabel's pile. If Gabriel and Isabel have the greatest possible combined value of coins, what is the least number of dimes Isabel could end up with?

156. ________ — What is the smallest positive integer multiple of 130 that is divisible by 365?

157. ________ hexominoes — A *hexomino* is a planar figure formed by connecting six unit squares so that adjacent squares have a common side. One possible hexomino is shown. How many distinct hexominoes can be drawn that have exactly four squares in a row? Two hexominoes are distinct if one cannot be reflected or rotated to form the other hexomino.



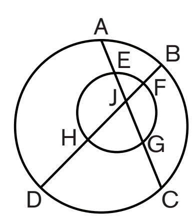
158. ________



An equilateral triangle, a square and a regular hexagon with side length 6 are stacked as pictured. A larger equilateral triangle is then drawn around the stack of polygons, completely enclosing it. The area outside the polygon stack but inside the larger triangle can be expressed in the form $a + b\sqrt{c}$, where a, b and c are integers and $b\sqrt{c}$ is in simplest radical form. What is $a + b + c$?

159. ________ assignments — Abhi, Bryan, Meghna and Noreen are each assigned a different integer from 1 to 10, inclusive. Abhi's number is prime and Noreen's number is a perfect square. Bryan's number is half of the value assigned to another person, while Meghna's number is the sum of two other assigned values. The ordered quadruple (2, 1, 5, 4) is one possible assignment. How many such assignments are there?

160. ________ degrees — For the two circles shown, the measures of arcs CD, GH and EF are 83 degrees, 98 degrees and 10 degrees, respectively. What is the measure of minor arc AB?



Warm-Up 14

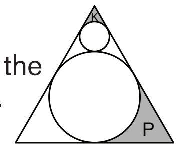
161. ________ If n is the product of three consecutive positive integers and $n = 22 \times 14 \times k$, what is the least possible value of k?

162. ________

23	12	1		9
4	18	7	21	
10	24	13	2	16
	5	19	8	22
17		25	14	3

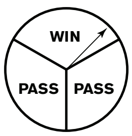
A regular 5 × 5 *magic square* contains a permutation of the integers from 1 through 25, such that every row, every column, and the two main diagonals sum to the same value. What is the sum of the numbers missing from the magic square shown?

163. ________ Two circles are inscribed in an equilateral triangle as shown. What is the ratio of the areas of the shaded regions K to P? Express your answer as a common fraction.



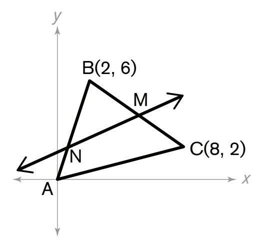
164. ________ What is the sum of the integers strictly between 1 and 100 that are multiples of neither 2 nor 3?

165. ________



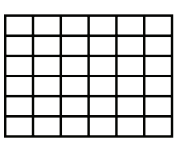
James and John take turns spinning the pointer of a fair spinner that is divided into three congruent sectors. The first player whose spin lands on the WIN sector is the winner of the game. If James goes first, what is the probability that he wins the game? Express your answer as a common fraction.

166. ________ units Triangle ABC has vertices at (0, 0), (2, 6) and (8, 2). The line $x - 3y = -7$ intersects two sides of the triangle at points M and N, as shown. What is the length of segment MN? Express your answer as a common fraction in simplest radical form.



167. ________ bottles Edna enters a room with 1000 bottles lined up in a row left to right. One bottle contains a tasteless magic potion. All bottles to the left of the magic potion contain tasteless water. All bottles to the right of the magic potion contain a bitter poison. Edna can drink from no more than two bottles containing poison before becoming sick and being unable to drink anything else. She can take an unlimited number of drinks from any other bottle. What is the minimum number of bottles from which Edna may need to drink to ensure she can identify the bottle containing the magic potion no matter where it is in the lineup?

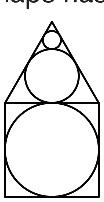
168. ________ silver rectangles



In origami, a *silver rectangle* is any rectangle such that the ratio of the length of the short side to the length of the long side is exactly $1:\sqrt{2}$. Each of the small rectangles in the figure shown is a silver rectangle. How many silver rectangles of any size can be found in the figure?

169. ________ laps Priya, Amanda and Du simultaneously begin jogging in the same direction from the same point on a circular track. Amanda's speed is the average of Priya's and Du's speeds. Du passes Priya for the first time at the moment when Du completes his fourth lap. How many laps has Amanda completed at the moment when she passes Priya for the first time?

170. ________



An equilateral triangle is stacked above a square as shown, with a circle inscribed inside the square and two stacked circles inscribed in the triangle so that they are tangent to each other. What is the ratio of the area of the smallest circle to the area of the largest circle? Express your answer as a common fraction.

Workout 1

171. ________ If three fair coins are simultaneously flipped, what is the probability that exactly two heads will be showing? Express your answer as a fraction in simplest form.

172. ________ m/s^2 An Euler Airline flight is getting ready to take off. Gary McDonald, the pilot, starts from rest at the edge of the runway. He needs to accelerate to a speed of 300 km/h in 30 seconds. Acceleration is defined as the change in speed per unit time. What is Gary's average acceleration, in meters per second per second, which is equivalent to meters per second squared, during takeoff? Express your answer as a decimal to the nearest tenth.

173. ________ pounds An object's weight on a planet is directly proportional to the mass of that planet and inversely proportional to the square of the radius of the planet. Jupiter is 318 times as massive as Earth and has a radius 11 times as large as that of Earth. If Gordon weighs 100 pounds on Earth, how many pounds would he weigh on Jupiter? Express your answer to the nearest whole number.

174. ________ million dollars The 1998 film *Armageddon* had a production budget of \$140 million. The domestic box office gross was about \$200 million, and the international box office gross was about \$350 million. The studio considers a film a financial success if the worldwide (domestic plus international) gross is at least double the sum of the production budget and the advertising budget. In millions of dollars, what was the greatest advertising budget the film could have had to be considered a financial success? Express your answer to the nearest whole number.

175. ________ days What is the mean number of days per month among all months in the year 2018? Express your answer as a decimal to the nearest tenth.

176. ________ If $A = x^2 - 2x + 6$ and $B = \dfrac{5x^2 - 1}{x + 3}$, what is $A + B$ if $x = -2$?

177. ________ cm What is the height of a cone with a volume of 1187.5 cm^3 and a base of diameter 18 cm? Express your answer to the nearest whole number.

178. ________ degrees What is the degree measure of each interior angle of a regular decagon?

179. \$ ________ Elliott's stock portfolio was valued at \$5000 on January 1. Its value decreased by 20% during January but then increased by 25% during February. What was the value of his stock portfolio at the end of February?

180. ________ hours How many hours are in the decade from January 1, 2011, through December 31, 2020?

Workout 2

181. ________ inches If a television screen with a length-to-height ratio of 16:9 has an area of 576 in^2, what is its perimeter?.

182. ________ miles Alex Zhu bikes between home and school every day. He uses the same route to go to and from school, but it takes him 20 minutes to bike to school and only 15 minutes to bike back. If his average biking pace for the whole round-trip is 7 minutes per mile, how many miles long is the trip from home to school? Express your answer as a decimal to the nearest tenth.

183. ________ m/s Bruce and Lawson are playing ice hockey. Bruce shoots the puck at the goal 40 meters directly in front of him at a speed of 50 m/s. If Lawson is standing exactly 30 meters to the left of Bruce, what is the minimum speed at which he must skate, to reach the goal when the puck does? Express your answer as a decimal to the nearest tenth.

184. $ ________ In 2015 the average two-adult family in a particular town paid $619 per month for groceries, excluding sales tax. If groceries in this town were subject to a 9% sales tax, how much sales tax was paid by the average two-adult family in one month?

185. $ ________ Linda makes 6 cakes per hour. Sara makes 4 cakes per hour. If Linda gets paid $11 less than Sara for each cake, how much in dollars should Sara be paid for each cake for Linda and Sara to earn the same amount each hour?

186. ________ yards In a golf long-drive competition, Jason Zuback hits his first five drives 394, 401, 387, 414 and 421 yards, respectively. How long must he hit his sixth drive to ensure that the mean of his six drives is at least 400 yards?

187. ________ km A pilot flying due east is forced to make a detour from her original route to avoid turbulent weather. The pilot turns 30 degrees north of east. After traveling some distance, she turns and rejoins her original route and is 1000 km away from where she took the detour. The turn back to her original route put her at a 45 degree angle to that route. How much farther did the pilot travel due to her detour? Express your answer to the nearest whole kilometer.

188. ________ What is 12% of $\frac{3}{4}$ of 1.8? Express your answer to the nearest thousandth.

189. ________ ways In how many different ways can four people be seated around a circular table so that no one ever has the same two neighbors more than once?

190. ________ fluid ounces A cylindrical container holds 20 fluid ounces. It has a radius of 3 inches and a height of 12 inches. How many fluid ounces will a similar container with a radius of 4.5 inches hold? Express your answer as a decimal to the nearest tenth.

Workout 3

191. ______ seats The number of seats per row in an auditorium increases from the front to the back. The first row has 15 seats, the second row has 2 more seats than the first row, the third row has 3 more seats than the second row, the fourth row has 2 more seats than the third row, the fifth row has 3 more seats than the fourth row. This pattern continues, with successive rows alternating between 2 more seats and then 3 more seats than the previous row. How many seats are in the auditorium if there are 30 rows total?

192. ______ degrees Three interior angles of a pentagon measure 110, 120 and 130 degrees, respectively. Of the two remaining interior angles, one is three times the measure of the other. What is the measure of the pentagon's smallest interior angle?

193. ______ Hisham Dimashkieh chooses four distinct positive integers a, b, c and d, each less than or equal to 10. He chooses the numbers so that a is prime, b is composite, c is a perfect square and d is a perfect cube. What is the greatest possible sum of the four numbers?

194. ______ If p, q and r are prime numbers such that $pq + r = 73$, what is the least possible value of $p + q + r$?

195. ______ % If Bella runs 40% as fast as Thomas and 35% as fast as Tam, what percent faster than Thomas is Tam? Express your answer to the nearest percent.

196. ______ % The probability that it will rain today is 50%. The probability that it will rain tomorrow is 40%. Assuming today and tomorrow's precipitation outcomes are independent from one another, what is the percent probability that it will rain on at least one of the two days? Express your answer as a whole number.

197. ______ ft^2 What is the area of a 60 degree sector of a circle with radius 30 feet? Express your answer in terms of π.

198. ______ kg One year, the U.S. government printed \$700 million worth of paper money every day, for 365 days. Half of the total value came from \$1 bills. If a new \$1 bill weighs exactly 1 gram, what was the weight, in kilograms, of all the \$1 bills printed that year?

199. ______ trees The number of bushels of apples, $B(n)$, that can be harvested from an acre of land is a function of n, the number of trees planted per acre, where $B(n) = 2025n - n^3$. How many trees planted per acre will produce the greatest harvest? Express your answer as a whole number.

200. ______ Three days ago, there were p cupcakes on the counter. Two days ago, exactly 20% of the cupcakes were eaten. Today, there are 30% fewer cupcakes than yesterday and half as many as there were three days ago. If a whole number of cupcakes were eaten every day, what is the least possible value of p?

Workout 4

201. ________ The sum of the digits of a two-digit integer is 12. The integer is equal to 15 times its units digit. What is the integer?

202. ________ If $\dfrac{x^2+8x+15}{x+5} = 4.01$, then what is the value of x? Express your answer as a decimal to the nearest hundredth.

203. ________ units^2 A triangle has three vertices given by coordinates (2, 2), (2, −6) and (−5, −9). What is the area of the triangle?

204. ________ In a regular octagon, the diagonals have three possible lengths —"short," "medium," and "long." What is the ratio of the length of the medium diagonal to the long diagonal? Express your answer as a decimal to the nearest thousandth.

205. ________ % Not realizing that an 18% tip had already been added to the cost of a meal, Emalee added another 15% to the total bill. Given that there is no sales tax, what percent tip did Emalee actually pay? Express your answer as a percent to the nearest tenth.

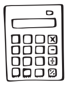
206. ________ Penner has a deck of 40 cards composed of four suits (red, blue, green, and yellow) and cards numbered 1 through 10 in each suit. Tell secretly chooses a card. Penner then chooses the following 4 cards from the deck: Red-2, Blue-3, Green-5 and Yellow-7. For each card Penner chooses, Tell says "yes" if his secret card is of the same color or shares a common factor greater than 1 with Penner's card. Otherwise Tell says "no." Tell says "no," "yes," "no," and "yes," respectively, in response to Penner's cards. With this information, what is the best possible probability Penner has of guessing Tell's secret card? Express your answer as a common fraction.

207. ________ Henry Flannigan chooses a two-digit positive integer at random. What is the probability that the two digits have an absolute difference greater than 1? Express your answer as a common fraction.

208. ________ mi^3 Ngorongoro Crater is shaped approximately like a cylinder that is 10 miles across and 2000 feet deep. How many cubic miles of water would it take to fill the crater? Express your answer to the nearest whole number.

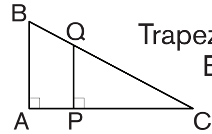
209. ________ Mady distributes w candies evenly among 20 bags. The next day, she discovers 5 more empty bags and decides to redistribute the w candies evenly into all of the bags. On the third day, Mady finds 1 more bag and redistributes the w candies evenly again. There are 2 fewer candies on the third day in each of the bags than there were in the bags on the second day. What is the value of w?

210. ________ If $f(x) = ax^2 + bx + c$, with $f(0) = 4$, $f(2) = 2$ and $f(4) - f(3) = 4$, what is the value of $f(1)$?



Workout 5

211. ________ units^2



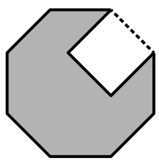
Trapezoid APQB lies inside of right triangle ABC, as shown. If AP = 30, BQ = 34 and AB = 60, what is the area of triangle ABC?

212. ________ There are five two-digit positive integers arranged in decreasing order. Each digit is unique. What is the absolute difference between the greatest possible range and the least possible range of such a set of integers?

213. ________ ways Six different donuts are lined up in a box. Six different cookies are lined up in another box. Mackenzie wants to alternate the donuts and cookies in one big long box. How many ways are there to arrange them?

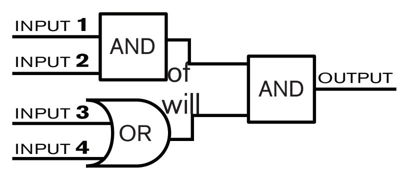
214. ________ workers For a certain crew of workers, it takes the *n*th worker exactly *n* hours to complete a certain job alone. What is the least number of workers needed to complete an identical job in under 20 minutes by working together?

215. ________ cm^2



A regular octagon has a perimeter of 64 cm. A square with one side along an edge of the octagon has been cut out of the octagon, as shown. What is the remaining area of the octagon? Express your answer in simplest radical form.

216. ________ inputs The function machine shown here consists of three logic functions. A Boolean is a member of the set {0, 1}. For each function, the inputs are on the left and the output is on the right. The output of each function is a Boolean. The input of each function is a pair of Booleans. For the AND function, the output is 1 if and only if both inputs are 1. For the OR function, the output is 0 if and only if both inputs are 0. Among the 16 distinct sets of inputs that can be applied on the far left, how many will produce a 1 as the final output on the far right?

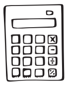


217. $________ Joe, Bob and Randell split a restaurant bill that totaled \$80 before the tip. The group tipped 25%, Joe paid twice as much as Bob, and Randell paid the same amount as Joe. How much did Bob pay?

218. ________ pairs How many pairs of positive integers a and b exist such that $a^2 - b^2 = 144$?

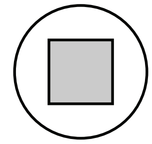
219. ________ cm^3 A rectangular box measures 5 mm × 10 mm × 1 m. What is the volume of the box in cubic centimeters?

220. ________ A college has been trying to decrease the number of cars on campus and increase the number of bicycles. The price of a parking permit was tripled, and the number of cars on campus decreased 30%. Student tuition was decreased, and the number of bicycles on campus increased by 20%, producing a car to bicycle ratio of 1:3. What was the ratio of cars to bicycles before the changes occurred? Express your answer as a common fraction.



Workout 6

221.________ in² The gasket shown consists of a circular disk with a square removed. The square and the disk have the same center. If each corner of the square is exactly 1 inch away from the boundary of the disk, and the midpoint of each side of the square is exactly 2 inches away from the boundary of the disk, what is the area of the top face of the gasket? Express your answer as a decimal to the nearest tenth.

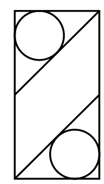


222.________ % Jeffrey Pribble needs to buy 6 pairs of socks. The Sock Shop is running a limited time promotion: buy 3 pairs of socks and get 3 pairs at half off the regular price. What percent savings does Jeffrey get with the promotion compared to the regular price without the promotion?

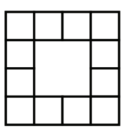
223.________ What is the greatest possible absolute difference between the median and mean of a list of 10 positive integers that are at most 20? Express your answer as a decimal to the nearest tenth.

224.________ tourists The tour from Ajim to the Mos Eisley Cantina can take at most 36 tourists. The price for the tour is \$520 per person until at least 15 people have signed up. After that, the price for each person, including the first 15, drops \$5 per additional tourist. If the total amount the tourists paid was \$12,740, how many tourists signed up?

225.________ inches A rectangle has a width of 1 inch and a height 2 inches. There are two lines drawn, each connecting a vertex to the midpoint of the opposite side, and circles are inscribed in the triangles created, as shown. How far apart are the centers of the circles? Express your answer as a decimal to the nearest hundredth.



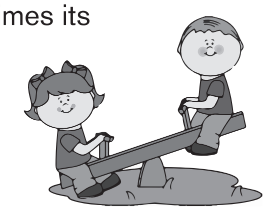
226.________ rect-angles



How many rectangles of any size are in the figure shown?

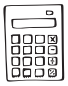
227.________ Let S be the set of all integers N such that both N and the number formed by reversing the digits of N are three-digit perfect squares. What is the sum of the integers in S?

228.________ feet A seesaw is in balance when the weight on one side of the fulcrum times its distance from the fulcrum is equal to the weight on the other side of the fulcrum times its distance from the fulcrum. Shandra weighs 96 pounds. Her little sister weighs 72 pounds. The seesaw at their playground has a beam with seats 14 feet apart. The position of the fulcrum can be adjusted as required. If each girl sits in her seat, how far should the fulcrum be from Shandra's seat to achieve perfect balance with her sister?



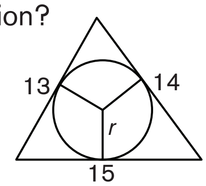
229.________ Jennie Weiner has p pennies, n nickels, d dimes and q quarters with a total value of \$1.08. If the numbers p, n, d and q are distinct and positive, and the greatest common divisor of each pair of these numbers is 1, what is the least possible value of $p + n + d + q$?

230.________ ft² What is the total surface area of a right square pyramid with a height of 12 feet and a base with side length 10 feet?



Workout 7

231. \$________ Rahul Ilangovan can arrange his dad's collection of quarters as a rectangular array with 10 equal rows, 12 equal rows or 18 equal rows, using all the quarters in each arrangement. What is the least possible monetary value in dollars of the quarter collection?



232. ________ units What is the radius of the largest circle that can be inscribed in an acute triangle with sides 13, 14 and 15 units?

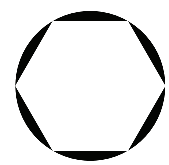
233. ________ Francisco is born at 1:00 a.m. on a Tuesday and gets married exactly 2^{18} hours later. On what day of the week does Francisco get married?

234. ________ Erica drew a 4 of hearts out of a standard 52-card deck, without replacement. If she draws a second card from the deck, what is the probability that her two cards will show consecutive numbers? Express your answer as a common fraction.

235. ________ A set *S* contains some, but not all, of the positive integers from 3 to 7. Some statements describing *S* are given below. The statement numbered *n* is true if the number *n* is in *S* and false if *n* is not in *S*. What is the product of the numbers that are in *S*?

3. The sum of the numbers in *S* is odd.
4. The sum of the numbers in *S* is less than 15.
5. *S* contains exactly one composite number.
6. *S* contains exactly one prime number.
7. The product of the numbers in *S* is odd.

236. ________ m^2 The figure shows a regular hexagon of side length 12 meters inscribed in a circle. What is the total area of the shaded regions between the hexagon and the circle? Express your answer to the nearest whole number.



237. ________ Kendra starts at a positive integer *k* and counts up by 4s until she hits exactly 200. Mason starts at a positive integer *m* and counts up by 6s until he hits exactly 200. If it takes Kendra half as many steps to reach 200 as it takes Mason, what is the greatest possible value of $k - m$?

238. ________ The graph of the line $3x - 4y = 13$ is translated 2018 units to the right. What is the *y*-intercept of the translated line? Express your answer as a decimal to the nearest hundredth.

239. \$________ Ron works five days a week selling wallets in a booth at the mall. He earns a salary of \$215 per week plus 15% of his weekly sales. If he earned \$383.75 this week, what was the amount of his average daily sales for the week?

240. ________ inches A regular hexagon is inscribed in a circle. If the area of the hexagon is $216\sqrt{3}$ in^2, what is the circumference of the circle? Express your answer in terms of π.

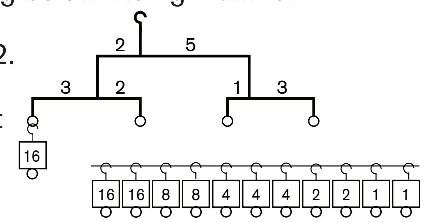
Workout 8

241. ________ Suppose that $A \& B = k \times A^m \times B^n$, where k, m and n are constants. Suppose that $5 \& 3 = 18$, $10 \& 3 = 72$ and $5 \& 6 = 36$. What is the value of $10 \& 6$?

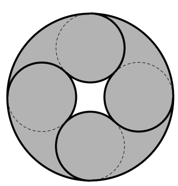
242. ________ combinations How many different combinations of pennies, nickels, dimes and quarters are possible in the cup holder of Terry's car if he counts 15 coins total?

243. ________ in^3 If the average length of the edges of a right rectangular prism is 13 inches, and the dimensions of the prism are distinct integers in geometric progression, what is the sum of the volumes of the distinct prisms that meet these criteria?

244. ________ weights In the mobile shown, a beam is *in balance* when the length l_L of the left arm of the beam times the total weight w_L hanging below the left arm of the beam is equal to the length l_R of the right arm of the beam times the total weight w_R hanging below the right arm of the beam. Every beam must be in balance for the mobile to be in balance. Decorative weights are available only in powers of 2. Multiple weights can be hung in a vertical chain, one below another. If the existing weight in the figure is not removed, what is the minimum number of weights that must be added to bring the mobile shown into balance?



245. ________ %



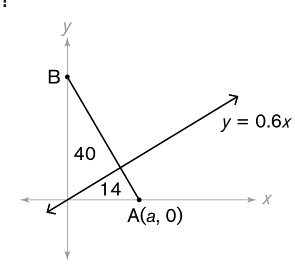
Four non-overlapping congruent circles are inscribed in a larger circle. Each small circle is shaded. Each region between two adjacent small circles and the enclosing large circle is also shaded. What percent of the figure is not shaded? Express your answer as a decimal to the nearest tenth of a percent.

246. ________ Maxine secretly chooses a positive integer between 1 and 2018, inclusive. Martin wants to identify her number with a series of guesses. Each time Martin makes a guess, Maxine tells him whether his number is correct, too high or too low. With an appropriate strategy, Martin can always identify Maxine's number after at most n guesses. What is the least value of n for which Martin can correctly guess Maxine's number?

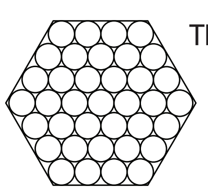
247. ________ novels Ms. Ault's reading list has novels by only three authors: Mark Twain, Ernest Hemingway and John Steinbeck. For a summer reading assignment, Austen Mazenko must pick one of the authors and read two of that author's novels on the list. If there are exactly 100 ways for Austen to pick two novels that satisfy this requirement, what is the greatest possible total number of novels on the reading list?

248. ________ What is the sum of all prime numbers p less than 60 such that there exists a right triangle whose side lengths are all integers and whose hypotenuse has length p?

249. ________ A line segment AB from the positive x-axis to the positive y-axis cuts off a triangle of area 54 square units in the first quadrant of the coordinate plane. The line $y = 0.6x$ divides this triangle into two triangles of areas 40 units^2 and 14 units^2. If the point A has coordinates $(a, 0)$, what is the value of a? Express your answer in simplest radical form.



250. ________ %



The Solar Sunflower is made up of super-efficient circular solar panels within a hexagonal frame. Using a two-dimensional diagram of the Solar Sunflower, as shown, with a hexagonal frame with side length 10 meters, what percent of the hexagon is covered in solar discs? Express your answer to the nearest whole number.

OFFICIAL RULES + PROCEDURES

The following rules and procedures govern all MATHCOUNTS competitions. The MATHCOUNTS Foundation reserves the right to alter these rules and procedures at any time. **Coaches are responsible for being familiar with the rules and procedures outlined in this handbook.** Coaches should bring any difficulty in procedures or in student conduct to the immediate attention of the appropriate chapter, state or national official. Students violating any rules may be subject to immediate disqualification.

Any questions regarding the MATHCOUNTS Competition Series Official Rules + Procedures articulated in this handbook should be addressed to the MATHCOUNTS national office at (703) 299-9006 or info@mathcounts.org.

REGISTRATION

The fastest and easiest way to register for the MATHCOUNTS Competition Series is online at www.mathcounts.org/compreg.

For your school to participate in the MATHCOUNTS Competition Series, a school representative is required to complete a registration form and pay the registration fees. A school representative can be a teacher, administrator or parent volunteer who has received expressed permission from his/her child's school administration to register. By completing the Competition Series Registration Form, the coach attests to the school administration's permission to register students for MATHCOUNTS.

School representatives can register online at www.mathcounts.org/compreg or download the Competition Series Registration Form and mail or email a scanned copy of it to the MATHCOUNTS national office. Refer to the Critical 2017-2018 Dates on pg. 10 of this handbook for contact information.

WHAT REGISTRATION COVERS: Registration in the Competition Series entitles a school to:

1) send 1-10 student(s)—depending on number registered—to the Chapter Competition. *Students can advance beyond the chapter level, but this is determined by their performance at the competition.*
2) receive the School Competition Kit, which includes the *2017-2018 MATHCOUNTS School Handbook*, one recognition ribbon for each registered student, 10 student participation certificates and a catalog of additional coaching materials. *Mailings of School Competition Kits will occur on a rolling basis through December 31, 2017.*
3) receive online access to the 2018 School Competition, along with electronic versions of other competition materials at www.mathcounts.org/coaches. *Coaches will receive an email notification no later than November 1, 2017 when the 2018 School Competition is available online.*

Your state or chapter coordinator will be notified of your registration, and then you will be informed of the date and location of your Chapter Competition. **If you have not been contacted by mid-January with competition details, it is your responsibility to contact your local coordinator** to confirm that your registration has been properly routed and that your school's participation is expected. Coordinator contact information is available at www.mathcounts.org/findmycoordinator.

DEADLINES: The sooner your Registration Form is received, the sooner you will receive your preparation materials. To guarantee your school's participation, submit your registration by one of the following deadlines:

Early Bird Discount Deadline: November 3, 2017	Online registrations: submitted by 11:59 PST Emailed forms: received by 11:59 PST Mailed forms: postmarked by November 3, 2017
Regular Registration Deadline: December 15, 2017*	Online registrations: submitted by 11:59 PST Emailed forms: received by 11:59 PST Mailed forms: postmarked by December 15, 2017

*Late Registrations may be accepted at the discretion of the MATHCOUNTS national office and your local coordinators, but are not guaranteed. If a school's late registration is accepted, an additional $20 processing fee will be assessed.

REGISTRATION FEES: The cost of your school's registration depends on when your registration is postmarked/emailed/submitted online. The cost of your school's registration covers the students for the entire Competition Series; there are no additional registration fees to compete at the state or national level. Title I schools (as affirmed by a school's administration) receive a 50% discount off the total cost of their registration.

Early Bird Registrations (by November 3, 2017)	**$30 per student** $120 for 1 team of 4 | $300 for 1 team of 4 + 6 individuals
Regular Registrations (by December 15, 2017)	**$35 per student** $140 for 1 team of 4 | $350 for 1 team of 4 + 6 individuals
Late Registrations (after December 15, 2017)	**$35 per student + $20 late fee on entire order** $160 for 1 team of 4 | $370 for 1 team of 4 + 6 individuals

CANCELLATION FEES: Registered schools that need to cancel their Competition Series registration must notify the MATHCOUNTS national office in writing via email or mail. Schools may request and receive a full refund minus a $30 non-refundable cancellation fee to cover refund processing and the cost of materials shipped to the school. MATHCOUNTS will verify a school's non-participation with local coordinators and reserves the right to refuse a refund request. No cancellations or refund requests will be processed after February 1, 2018. *This fee does not apply to schools that reduce their number of registered students but remain registered with at least one student.*

ELIGIBILITY REQUIREMENTS

Eligibility requirements for the MATHCOUNTS Competition Series are different from other MATHCOUNTS programs. Eligibility for the National Math Club or the Math Video Challenge does not guarantee eligibility for the Competition Series.

WHO IS ELIGIBLE:

- U.S. students enrolled in the 6th, 7th or 8th grade can participate in MATHCOUNTS competitions.
- Schools that are the students' official school of record can register.
- Any type of school, of any size, can register—public, private, religious, charter, virtual or homeschools—but virtual and homeschools must fill out additional forms to participate (see pgs. 39-40).
- Schools in 50 U.S. states, District of Columbia, Guam, Puerto Rico and Virgin Islands can register.
- Overseas schools that are affiliated with the U.S. Departments of Defense and State can register.

WHO <u>IS NOT</u> ELIGIBLE:

- Students who are not full-time 6th, 7th or 8th graders cannot participate, even if they are taking middle school math classes.
- Academic centers, tutoring centers or enrichment programs that do not function as students' official school of record cannot register. *<u>If it is unclear whether your educational institution is considered a school, please contact your local Department of Education for specific criteria governing your state.</u>*
- Schools located outside of the U.S. states and territories listed on the previous page cannot register.
- Overseas schools not affiliated with the U.S. Departments of Defense or State cannot register.

NUMBER OF STUDENTS ALLOWED: A school can register a maximum of one team of four students and six individuals; these 1-10 student(s) will represent the school at the Chapter Competition. Any number of students can participate at the school level. Prior to the Chapter Competition, coaches must notify their chapter coordinator of which students will be team members and which students will compete as individuals.

NUMBER OF YEARS ALLOWED: Participation in MATHCOUNTS competitions is limited to 3 years for each student, but there is no limit to the number of years a student may participate in school-based coaching.

WHAT TEAM REGISTRATION MEANS: Members of a school team will participate in the Target, Sprint and Team Rounds. Members of a school team also will be eligible to qualify for the Countdown Round (where conducted). Team members will be eligible for team awards, individual awards and progression to the state and national levels based on their individual and/or team performance. It is recommended that your strongest four Mathletes form your school team. Teams of fewer than four will be allowed to compete; however, the team score will be computed by dividing the sum of the team members' scores by four (see pg. 43), meaning, teams of fewer than four students will be at a disadvantage. *<u>Only one team (of up to four students) per school is eligible to compete.</u>*

WHAT INDIVIDUAL REGISTRATION MEANS: Students registered as individuals will participate in the Target and Sprint Rounds, but not the Team Round. Individuals will be eligible to qualify for the Countdown Round (where conducted). Individuals also will be eligible for individual awards and progression to the state and national levels. A student registered as an "individual" may not help his/her school's team advance to the next level of competition. *<u>Up to six students may be registered in addition to or in lieu of a school team.</u>*

HOW STUDENTS ENROLLED PART-TIME AT TWO SCHOOLS PARTICIPATE: *<u>A student may compete only for his/her official school of record.</u>* A student's school of record is the student's base or main school. A student taking limited course work at a second school or educational center may not register or compete for that second school or center, even if the student is not competing for his/her school of record. MATHCOUNTS registration is not determined by where a student takes his or her math course. If there is any doubt about a student's school of record, the chapter or state coordinator must be contacted for a decision before registering.

HOW SMALL SCHOOLS PARTICIPATE: MATHCOUNTS does not distinguish between the sizes of schools for Competition Series registration and competition purposes. Every "brick-and-mortar" school will have the same registration allowance of up to one team of four students and/or up to six individuals. A school's participants may not combine with any other school's participants to form a team when registering or competing.

HOW HOMESCHOOLS PARTICIPATE: Homeschools and/or homeschool groups in compliance with the homeschool laws of the state in which they are located are eligible to participate in MATHCOUNTS competitions in accordance with all other rules. Homeschool coaches must complete the 2017-2018 Homeschool + Virtual School Participation Form, verifying that students from the homeschool or homeschool group are in the 6th, 7th or 8th grade and that each homeschool complies with applicable state laws. Forms can be downloaded at www.mathcounts.org/competition and must be submitted to the MATHCOUNTS national office in order for registrations to be processed.

HOW VIRTUAL SCHOOLS PARTICIPATE: Virtual schools that want to register must contact the MATHCOUNTS national office by December 8, 2017 for specific registration details. Any student registering as a virtual school student must compete in the MATHCOUNTS Chapter Competition assigned according to the student's home address. Additionally, virtual school coaches must complete the 2017-2018 Homeschool + Virtual School Participation Form, verifying that the students from the virtual school are in the 6th, 7th or 8th grade and that the virtual school complies with applicable state laws. Forms must be submitted to the national office in order for registrations to be processed; forms can be downloaded at www.mathcounts.org/competition.

WHAT IS DONE FOR SUBSTITUTIONS OF STUDENTS: Coaches determine which students will represent the school at the Chapter Competition. Coaches cannot substitute team members for the State Competition unless a student voluntarily releases his/her position on the school team. Additional requirements and documentation for substitutions (such as requiring parental release or requiring the substitution request be submitted in writing) are at the discretion of the State Coordinator. A student being added to a team need not be a student who was registered for the Chapter Competition as an individual. Coaches cannot make substitutions for students progressing to the State Competition as individuals. At all levels of competition, student substitutions are not permitted after on-site competition registration has been completed.

WHAT IS DONE FOR RELIGIOUS OBSERVANCES: A student who is unable to attend a competition due to religious observances may take the written portion of the competition up to one week in advance of the scheduled competition. In addition, all competitors from that student's school must take the Sprint and Target Rounds at the same earlier time. If the student who is unable to attend the competition due to a religious observance: (1) is a member of the school team, then the team must take the Team Round at the same earlier time; (2) is not part of the school team, then the team has the option of taking the Team Round during this advance testing or on the regularly scheduled day of the competition with the other school teams. The coordinator must be made aware of the team's decision before the advance testing takes place. *Advance testing will be done at the discretion of the chapter and state coordinators. If advance testing is deemed possible, it will be conducted under proctored conditions.* Students who qualify for an official Countdown Round but are unable to attend will automatically forfeit one place standing.

WHAT IS DONE FOR STUDENTS WITH SPECIAL NEEDS: Reasonable accommodations may be made to allow students with special needs to participate. However, many accommodations that are employed in a classroom or teaching environment cannot be implemented in the competition setting. Accommodations that are not permissible include, but are not limited to: granting a student extra time during any of the competition rounds or allowing a student to use a calculator for the Sprint or Countdown Rounds. *A request for accommodation of special needs must be directed to chapter or state coordinators in writing at least three weeks in advance of the Chapter or State Competition.* This written request should thoroughly explain a student's special need, as well as what the desired accommodation would entail. In conjunction with the MATHCOUNTS Foundation, coordinators will review the needs of the student and determine if any accommodations will be made. In making final determinations, the feasibility of accommodating these needs at the National Competition will be taken into consideration.

LEVELS OF COMPETITION

There are four levels in the MATHCOUNTS Competition Series: school, chapter (local), state and national. Competition questions are written for 6th, 7th and 8th graders. The competitions can be quite challenging, particularly for students who have not been coached using MATHCOUNTS materials. All competition materials are prepared by the national office.

SCHOOL COMPETITIONS (TYPICALLY HELD IN JANUARY 2018): After several months of coaching, schools registered for the Competition Series should administer the 2018 School Competition to all interested students. The School Competition should be an aid to the coach in determining competitors for the Chapter Competition. *Selection of team and individual competitors is entirely at the discretion of the coach and does not need to be based solely on School Competition scores.* School Competition materials are sent to the coach of a school, and it may be used by the teachers and students only in association with that school's programs and activities. The current year's School Competition questions must remain confidential and may not be used in outside activities, such as tutoring sessions or enrichment programs with students from other schools. For updates or edits, please check www.mathcounts.org/coaches before administering the School Competition.

It is important that the coach look upon coaching sessions during the academic year as opportunities to develop better math skills in all students, not just in those students who will be competing. Therefore, it is suggested that the coach postpone selection of competitors until just prior to the Chapter Competition.

CHAPTER COMPETITIONS (HELD FEB. 1–28, 2018): The Chapter Competition consists of the Sprint, Target and Team Rounds. The Countdown Round (official or just for fun) may or may not be conducted. The chapter and state coordinators determine the date and location of the Chapter Competition in accordance with established national procedures and rules. Winning teams and students will receive recognition. The winning team will advance to the State Competition. Additionally, the two highest-ranking competitors not on the winning team (who may be registered as individuals or as members of a team) will advance to the State Competition. This is a minimum of six advancing Mathletes (assuming the winning team has four members). Additional teams and/or individuals also may progress at the discretion of the state coordinator, but the policy for progression must be consistent for all chapters within a state.

STATE COMPETITIONS (HELD MAR. 1– APR. 1, 2018): The State Competition consists of the Sprint, Target and Team Rounds. The Countdown Round (official or just for fun) may or may not be included. The state coordinator determines the date and location of the State Competition in accordance with established national procedures and rules. Winning teams and students will receive recognition. The four highest-ranked Mathletes and the coach of the winning team from each State Competition will receive an all-expenses-paid trip to the National Competition.

2018 RAYTHEON MATHCOUNTS NATIONAL COMPETITION (HELD MAY 13–14 IN WASHINGTON, DC): The National Competition consists of the Sprint, Target, Team and Countdown Rounds (conducted officially). Expenses of the state team and coach to travel to the National Competition will be paid by MATHCOUNTS. The national program does not make provisions for the attendance of additional students or coaches. All national competitors will receive a plaque and other items in recognition of their achievements. Winning teams and individuals also will receive medals, trophies and college scholarships.

COMPETITION COMPONENTS

The four rounds of a MATHCOUNTS competition, each described below, are designed to be completed in approximately three hours:

TARGET ROUND (approximately 30 minutes): In this round eight problems are presented to competitors in four pairs (six minutes per pair). The multi-step problems featured in this round engage Mathletes in mathematical reasoning and problem-solving processes. *Problems assume the use of calculators.*

SPRINT ROUND (40 minutes): Consisting of 30 problems, this round tests accuracy, with the time period allowing only the most capable students to complete all of the problems. *Calculators are not permitted.*

TEAM ROUND (20 minutes): In this round, interaction among team members is permitted and encouraged as they work together to solve 10 problems. *Problems assume the use of calculators.*

Note: The order in which the written rounds (Target, Sprint and Team) are administered is at the discretion of the competition coordinator.

COUNTDOWN ROUND: A fast-paced oral competition for top-scoring individuals (based on scores on the Target and Sprint Rounds), this round allows pairs of Mathletes to compete against each other and the clock to solve problems. *Calculators are not permitted.*

At Chapter and State Competitions, a Countdown Round (1) may be conducted officially, (2) may be conducted unofficially (for fun) or (3) may be omitted. However, the use of an official Countdown Round must be consistent for all chapters within a state. In other words, *all* chapters within a state must use the round officially in order for *any* chapter within a state to use it officially. All students, whether registered as part of a school team or as individual competitors, are eligible to qualify for the Countdown Round.

An official Countdown Round determines an individual's final overall rank in the competition. If a Countdown Round is used officially, the official procedures as established by the MATHCOUNTS Foundation must be followed, as described below.*

- The top 25% of students, up to a maximum of 10, are selected to compete. These students are chosen based on their Individual Scores.
- The two lowest-ranked students are paired; a question is read and projected, and students are given 45 seconds to solve the problem. A student may buzz in at any time, and if s/he answers correctly, a point is scored. If a student answers incorrectly, the other student has the remainder of the 45 seconds to answer.
- Three total questions are read to the pair of students, one question at a time, and the student who scores the higher number of points (not necessarily 2 out of 3) progresses to the next round and challenges the next-higher-ranked student.
- If students are tied in their matchup after three questions (at 1-1 or 0-0), questions should continue to be read until one is successfully answered. The first student who answers an additional question correctly progresses to the next round.
- This procedure continues until the 4th-ranked Mathlete and his/her opponent compete. For the final four matchups, the first student to correctly answer three questions advances.
- The Countdown Round proceeds until a 1st place individual is identified. More details about Countdown Round procedures are included in the 2018 School Competition.

**Rules for the Countdown Round change for the National Competition.*

An unofficial Countdown Round does not determine an individual's final overall rank in the competition, but is done for practice or for fun. The official procedures do not have to be followed. Chapters and states choosing not to conduct the round officially must determine individual winners solely on the basis of students' scores in the Target and Sprint Rounds of the competition.

SCORING

MATHCOUNTS Competition Series scores do not conform to traditional grading scales. Coaches and students should view an Individual Score of 23 (out of a possible 46) as highly commendable.

INDIVIDUAL SCORE: calculated by taking the sum of the number of Sprint Round questions answered correctly and twice the number of Target Round questions answered correctly. There are 30 questions in the Sprint Round and eight questions in the Target Round, so the maximum possible Individual Score is $30 + 2(8) = 46$. If used officially, the Countdown Round yields final individual standings.

TEAM SCORE: calculated by dividing the sum of the team members' Individual Scores by four (even if the team has fewer than four members) and adding twice the number of Team Round questions answered correctly. The highest possible Individual Score is 46. Four students may compete on a team, and there are 10 questions in the Team Round. Therefore, the maximum possible Team Score is $((46 + 46 + 46 + 46) \div 4) + 2(10) = 66$.

TIEBREAKING ALGORITHM: used to determine team and individual ranks and to determine which individuals qualify for the Countdown Round. In general, questions in the Target, Sprint and Team Rounds increase in difficulty so that the most difficult questions occur near the end of each round. In a comparison of questions to break ties, generally those who correctly answer the more difficult questions receive the higher rank. The guidelines provided below are very general; competition officials receive more detailed procedures.

- *Ties between individuals:* the student with the higher Sprint Round score will receive the higher rank. If a tie remains after this comparison, specific groups of questions from the Target and Sprint Rounds are compared.
- *Ties between teams:* the team with the higher Team Round score, and then the higher sum of the team members' Sprint Round scores, receives the higher rank. If a tie remains after these comparisons, specific questions from the Team Round will be compared.

RESULTS DISTRIBUTION

Coaches should expect to receive the scores of their students, as well as a list of the top 25% of students and top 40% of teams, from their competition coordinators. In addition, single copies of the blank competition materials and answer keys may be distributed to coaches after all competitions at that level nationwide have been completed. *Before distributing blank competition materials and answer keys, coordinators must wait for verification from the national office that all such competitions have been completed.* Both the problems and answers from Chapter and State Competitions will be posted on the MATHCOUNTS website following the completion of all competitions at that level nationwide, replacing the previous year's posted tests.

Student competition papers and answers will not be viewed by or distributed to coaches, parents, students or other individuals. Students' competition papers become the confidential property of MATHCOUNTS.

ADDITIONAL RULES

All answers must be legible.

Pencils and paper will be provided for Mathletes by competition organizers. However, students may bring their own pencils, pens and erasers if they wish. They may not use their own scratch paper or graph paper.

Use of notes or other reference materials (including dictionaries and translation dictionaries) is prohibited.

Specific instructions stated in a given problem take precedence over any general rule or procedure.

Communication with coaches is prohibited during rounds but is permitted during breaks. All communication between guests and Mathletes is prohibited during competition rounds. Communication between teammates is permitted only during the Team Round.

Calculators are not permitted in the Sprint and Countdown Rounds, but they are permitted in the Target, Team and Tiebreaker (if needed) Rounds. When calculators are permitted, students may use any calculator (including programmable and graphing calculators) that does not contain a QWERTY (typewriter-like) keypad. Calculators that have the ability to enter letters of the alphabet but do not have a keypad in a standard typewriter arrangement are acceptable. Smart phones, laptops, tablets, iPods®, personal digital assistants (PDAs) and any other "smart" devices are not considered to be calculators and may not be used during competitions. Students may not use calculators to exchange information with another person or device during the competition.

Coaches are responsible for ensuring their students use acceptable calculators, and students are responsible for providing their own calculators. Coordinators are not responsible for providing Mathletes with calculators or batteries before or during MATHCOUNTS competitions. Coaches are strongly advised to bring backup calculators and spare batteries to the competition for their team members in case of a malfunctioning calculator or weak or dead batteries. Neither the MATHCOUNTS Foundation nor coordinators shall be responsible for the consequences of a calculator's malfunctioning.

Pagers, cell phones, tablets, iPods® and other MP3 players should not be brought into the competition room. Failure to comply could result in dismissal from the competition.

Should there be a rule violation or suspicion of irregularities, the MATHCOUNTS coordinator or competition official has the obligation and authority to exercise his/her judgment regarding the situation and take appropriate action, which might include disqualification of the suspected student(s) from the competition.

FORMS OF ANSWERS

The following rules explain acceptable forms for answers. Coaches should ensure that Mathletes are familiar with these rules prior to participating at any level of competition. Competition answers will be scored in compliance with these rules for forms of answers.

Units of measurement are not required in answers, but they must be correct if given. When a problem asks for an answer expressed in a specific unit of measure or when a unit of measure is provided in the answer blank, equivalent answers expressed in other units are not acceptable. For example, if a problem asks for the number of ounces and 36 oz is the correct answer, 2 lb 4 oz will not be accepted. If a problem asks for the number of cents and 25 cents is the correct answer, \$0.25 will not be accepted.

All answers must be expressed in simplest form. A "common fraction" is to be considered a fraction in the form $\pm\frac{a}{b}$, where a and b are natural numbers and GCF(a, b) = 1. In some cases the term "common fraction" is to be considered a fraction in the form $\frac{A}{B}$, where A and B are algebraic expressions and A and B do not have a common factor. A simplified "mixed number" ("mixed numeral," "mixed fraction") is to be considered a fraction in the form $\pm N\frac{a}{b}$, where N, a and b are natural numbers, $a < b$ and GCF(a, b) = 1. Examples:

Problem: What is 8 ÷ 12 expressed as a common fraction? *Answer:* $\frac{2}{3}$ *Unacceptable:* $\frac{4}{6}$

Problem: What is 12 ÷ 8 expressed as a common fraction? *Answer:* $\frac{3}{2}$ *Unacceptable:* $\frac{12}{8}$, $1\frac{1}{2}$

Problem: What is the sum of the lengths of the radius and the circumference of a circle of diameter $\frac{1}{4}$ unit expressed as a common fraction in terms of π? *Answer:* $\frac{1+2\pi}{8}$

Problem: What is 20 ÷ 12 expressed as a mixed number? *Answer:* $1\frac{2}{3}$ *Unacceptable:* $1\frac{8}{12}$, $\frac{5}{3}$

Ratios should be expressed as simplified common fractions unless otherwise specified. Examples:

Acceptable Simplified Forms: $\frac{7}{2}$, $\frac{3}{\pi}$, $\frac{4-\pi}{6}$ *Unacceptable:* $3\frac{1}{2}$, $\frac{\frac{1}{4}}{3}$, 3.5, 2:1

Radicals must be simplified. A simplified radical must satisfy: 1) no radicands have a factor which possesses the root indicated by the index; 2) no radicands contain fractions; and 3) no radicals appear in the denominator of a fraction. Numbers with fractional exponents are *not* in radical form. Examples:
Problem: What is $\sqrt{15} \times \sqrt{5}$ expressed in simplest radical form? *Answer:* $5\sqrt{3}$ *Unacceptable:* $\sqrt{75}$

Answers to problems asking for a response in the form of a dollar amount or an unspecified monetary unit (e.g., "How many dollars...," "How much will it cost...," "What is the amount of interest...") should be expressed in the form (\$) *a.bc*, where *a* is an integer and *b* and *c* are digits. The *only* exceptions to this rule are when a is zero, in which case it may be omitted, or when b and c are both zero, in which case they both may be omitted. Answers in the form (\$) *a.bc* should be rounded to the nearest cent, unless otherwise specified. Examples:
Acceptable Forms: 2.35, 0.38, .38, 5.00, 5 *Unacceptable:* 4.9, 8.0

Do not make approximations for numbers (e.g., π, $\frac{2}{3}$, $5\sqrt{3}$) in the data given or in solutions unless the problem says to do so.

Do not do any intermediate rounding (other than the "rounding" a calculator performs) when calculating solutions. All rounding should be done at the end of the calculation process.

Scientific notation should be expressed in the form $a \times 10^n$ where a is a decimal, $1 \leq |a| < 10$, and n is an integer. Examples:
Problem: What is 6895 expressed in scientific notation? *Answer:* 6.895×10^3
Problem: What is 40,000 expressed in scientific notation? *Answer:* 4×10^4 *or* 4.0×10^4

An answer expressed to a greater or lesser degree of accuracy than called for in the problem will not be accepted. Whole-number answers should be expressed in their whole-number form. Thus, 25.0 will not be accepted for 25, and 25 will not be accepted for 25.0.

The plural form of the units will always be provided in the answer blank, even if the answer appears to require the singular form of the units.

COMPETITION COACH TOOLKIT

This is a collection of lists, formulas and terms that Mathletes frequently use to solve problems like those found in this handbook. There are many others we could have included, but we hope you find this collection useful.

Fraction	Decimal	Percent
$\frac{1}{2}$	0.5	50
$\frac{1}{3}$	$0.\overline{3}$	$33.\overline{3}$
$\frac{1}{4}$	0.25	25
$\frac{1}{5}$	0.2	20
$\frac{1}{6}$	$0.1\overline{6}$	$16.\overline{6}$
$\frac{1}{8}$	0.125	12.5
$\frac{1}{9}$	$0.\overline{1}$	$11.\overline{1}$
$\frac{1}{10}$	0.1	10
$\frac{1}{11}$	$0.\overline{09}$	$9.\overline{09}$
$\frac{1}{12}$	$0.08\overline{3}$	$8.\overline{3}$

Common Arithmetic Series

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

$$2 + 4 + 6 + 8 + \cdots + 2n = n^2 + n$$

Prime Numbers

2	43
3	47
5	53
7	59
11	61
13	67
17	71
19	73
23	79
29	83
31	89
37	97
41	

Combinations & Permutations

$${}_nC_r = \frac{n!}{r!(n-r)!} \qquad {}_nP_r = \frac{n!}{(n-r)!}$$

n	n^2	n^3
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000
11	121	1331
12	144	1728
13	169	2197
14	196	2744
15	225	3375

Divisibility Rules

2: units digit is 0, 2, 4, 6 or 8

3: sum of digits is divisible by 3

4: two-digit number formed by tens and units digits is divisible by 4

5: units digit is 0 or 5

6: number is divisible by both 2 and 3

8: three-digit number formed by hundreds, tens and units digits is divisible by 8

9: sum of digits is divisible by 9

10: units digit is 0

Geometric Mean

$$\frac{a}{x} = \frac{x}{b} \quad \text{and} \quad x = \sqrt{ab}$$

Equation of a Line

Standard Form

$$\boxed{Ax + By = C}$$

Slope-Intercept Form

$$\boxed{y = mx + b}$$

m = slope $\qquad$ b = y-intercept

Point-Slope Form

$$\boxed{y - y_1 = m(x - x_1)}$$

m = slope $\qquad$ (x_1, y_1) = point on the line

Distance Traveled

Distance = Rate × Time

Quadratic Formula

For $ax^2 + bx + c = 0$, where $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Pythagorean Triples

(3, 4, 5)	(5, 12, 13)	(7, 24, 25)
(8, 15, 17)	(9, 40, 41)	(12, 35, 37)

Difference of Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Sum and Difference of Cubes

$$a^3 + b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 - b^3 = (a + b)(a^2 - ab + b^2)$$

Circles

Circumference	Area
$2 \times \pi \times r = \pi \times d$	$\pi \times r^2$

For radius r

Arc Length	Sector Area
$\frac{x}{360} \times 2 \times \pi \times r$	$\frac{x}{360} \times \pi \times r^2$

For central angle of x degrees

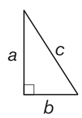
Given $A(x_1, y_1)$ and $B(x_2, y_2)$

Distance from A to B $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint of $\overline{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

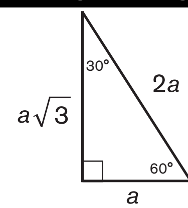
Slope of $\overline{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

Pythagorean Theorem

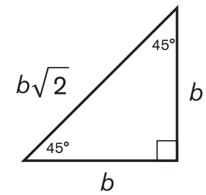


$a^2 + b^2 = c^2$

Special Right Triangles



30-60-90 Right Triangle



45-45-90 Right Triangle

Area of Polygons

Square	side length s	s^2
Rectangle	length l, width w	$l \times w$
Parallelogram	base b, height h	$b \times h$
Trapezoid	bases b_1, b_2, height h	$\frac{1}{2}(b_1 + b_2) \times h$
Rhombus	diagonals d_1, d_2	$\frac{1}{2} \times d_1 \times d_2$
Triangle	base b, height h	$\frac{1}{2} \times b \times h$
Triangle	semi-perimeter s, side lengths a, b, c	$\sqrt{s(s-a)(s-b)(s-c)}$
Equilateral Triangle	side length s	$\frac{s^2\sqrt{3}}{4}$

Polygon Angles (*n* sides)

Sum of the interior angle measures:

$$180 \times (n - 2)$$

Central angle measure of a regular polygon:

$$\frac{360}{n}$$

Interior angle measure of a regular polygon:

$$\frac{180 \times (n - 2)}{n} \quad \text{or} \quad 180 - \frac{360}{n}$$

Solid	Dimensions	Surface Area	Volume
Cube	side length s	$6 \times s^2$	s^3
Rectangular Prism	length l, width w, height h	$2 \times (l \times w + w \times h + l \times h)$	$l \times w \times h$
Circular Cylinder	base radius r, height h	$2 \times \pi \times r \times h + 2 \times \pi \times r^2$	$\pi \times r^2 \times h$
Circular Cone	base radius r, height h	$\pi \times r^2 + \pi \times r \times \sqrt{r^2 + h^2}$	$\frac{1}{3} \times \pi \times r^2 \times h$
Sphere	radius r	$4 \times \pi \times r^2$	$\frac{4}{3} \times \pi \times r^3$
Pyramid	base area B, height h		$\frac{1}{3} \times B \times h$

Vocabulary & Terms

The following list is representative of terminology used in the problems but **should not** be viewed as all-inclusive. It is recommended that coaches review this list with their Mathletes.

absolute difference
absolute value
acute angle
additive inverse *(opposite)*
adjacent angles
apex
arithmetic mean
arithmetic sequence
base 10
binary
bisect
box-and-whisker plot
center
chord
circumscribe
coefficient
collinear
common divisor
common factor
common fraction
complementary angles
congruent
convex
coordinate plane/system
coplanar
counting numbers
counting principle
diagonal of a polygon
diagonal of a polyhedron
digit-sum
direct variation
divisor
domain of a function
edge
equiangular
equidistant
expected value
exponent
exterior angle of a polygon
factor
finite
frequency distribution
frustum
function
GCF
geometric sequence
hemisphere
image(s) of a point(s) *(under a transformation)*
improper fraction
infinite series
inscribe
integer
interior angle of a polygon
intersection
inverse variation
irrational number
isosceles
lateral edge
lateral surface area
lattice point(s)
LCM
median of a set of data
median of a triangle
mixed number
mode(s) of a set of data
multiplicative inverse *(reciprocal)*
natural number
obtuse angle
ordered pair
origin
palindrome
parallel
Pascal's Triangle
percent increase/decrease
perpendicular
planar
polyhedron
polynomial
prime factorization
principal square root
proper divisor
proper factor
proper fraction
quadrant
quadrilateral
random
range of a data set
range of a function
rate
ratio
rational number
ray
real number
reciprocal *(multiplicative inverse)*
reflection
regular polygon
relatively prime
revolution
right angle
right polyhedron
rotation
scalene triangle
scientific notation
sector
segment of a circle
segment of a line
semicircle
semiperimeter
sequence
set
significant digits
similar figures
slope
space diagonal
square root
stem-and-leaf plot
supplementary angles
system of equations/inequalities
tangent figures
tangent line
term
transformation
translation
triangular numbers
trisect
twin primes
union
unit fraction
variable
whole number
y-intercept

ANSWERS

In addition to the answer, we have provided a difficulty rating for each problem. Our scale is 1-7, with 7 being the most difficult. These are only approximations, and how difficult a problem is for a particular student will vary. Below is a general guide to the ratings:

Difficulty 1/2/3 - One concept; one- to two-step solution; appropriate for students just starting the middle school curriculum.
4/5 - One or two concepts; multistep solution; knowledge of some middle school topics is necessary.
6/7 - Multiple and/or advanced concepts; multistep solution; knowledge of advanced middle school topics and/or problem-solving strategies is necessary.

Probability Stretch

	Answer	Difficulty			
1.	15.38	(2)	6.	50	(4)
2.	1/4	(3)	7.	20	(2)
3.	0.2	(4)	8.	9/16	(4)
4.	2/5	(3)	9.	1/5	(3)
5.	3/8	(3)	10.	0.17	(4)

Patterns Stretch

	Answer	Difficulty			
11.	1480	(4)	16.	2601	(3)
12.	423	(3)	17.	2	(4)
13.	1457	(3)	18.	4/3	(4)
14.	36	(4)	19.	125	(4)
15.	1365	(3)	20.	243	(5)

Travel Stretch

	Answer	Difficulty			
21.	$2\frac{2}{3}$	(5)	26.	3	(3)
22.	2:35	(4)	27.	$7\frac{1}{2}$	(5)
23.	880	(3)	28.	$1\frac{1}{4}$	(3)
24.	3	(4)	29.	5	(4)
25.	2	(4)	30.	5/8	(3)

Warm-Up 1

	Answer	Difficulty			
31.	−19	(1)	36.	18	(2)
32.	14	(2)	37.	1/2	(2)
33.	2112	(2)	38.	0	(3)
34.	1/221	(3)	39.	452	(4)
35.	$60\sqrt{(14)}$	(3)	40.	47	(4)

Warm-Up 2

	Answer	Difficulty			
41.	5	(2)	46.	5/18	(3)
42.	32	(4)	47.	9/2	(4)
43.	81	(4)	48.	25	(3)
44.	600	(2)	49.	16	(3)
45.	4	(3)	50.	4π	(4)

Warm-Up 3

	Answer	Difficulty			
51.	−1009	(3)	56.	13/18	(3)
52.	12	(3)	57.	25	(4)
53.	6/25	(5)	58.	348	(3)
54.	10,227	(3)	59.	1,000,000	(3)
55.	$2\sqrt{5}$	(4)	60.	$4\frac{1}{20}$	(3)

** The plural form of the units is always provided in the answer blank, even if the answer appears to require the singular form of the units.*

Warm-Up 4

Answer	Difficulty			
61. 1/2	(3)	66.	3π	(4)
62. 900	(4)	67.	432	(3)
63. 50	(3)	68.	4	(4)
64. 21	(4)	69.	27	(3)
65. 191	(4)	70.	48	(5)

Warm-Up 5

Answer	Difficulty			
71. 17	(3)	76.	1/3	(2)
72. 6	(2)	77.	15	(4)
73. 120	(5)	78.	63	(3)
74. 9	(3)	79.	4	(3)
75. 4	(3)	80.	1	(5)

Warm-Up 6

Answer	Difficulty			
81. 1320	(2)	86.	3/8	(4)
82. 128	(4)	87.	100	(2)
83. 5	(3)	88.	80	(2)
84. 3.81	(2)	89.	315	(4)
85. 126	(5)	90.	1/4	(3)

Warm-Up 7

Answer	Difficulty			
91. 12	(2)	96.	$\sqrt{3}/4$	(5)
92. 160	(4)	97.	9	(4)
93. $5\sqrt{2}$	(5)	98.	11	(4)
94. 161	(3)	99.	3	(4)
95. 7/36	(3)	100.	1000	(3)

Warm-Up 8

Answer	Difficulty			
101. 68	(3)	106.	11/18	(4)
102. 40	(3)	107.	$\sqrt{2} - 1$	(5)
103. 144	(4)	108.	4	(5)
104. 1/7	(3)	109.	173	(5)
105. 81	(3)	110.	1/4	(5)

Warm-Up 9

Answer	Difficulty			
111. 79/111	(2)	116.	180	(3)
112. 8/11	(5)	117.	6	(4)
113. 17/24	(5)	118.	36	(7)
114. $100\sqrt{10}$	(4)	119.	85	(2)
115. 305	(4)	120.	148	(4)

Warm-Up 10

Answer	Difficulty		
121. 23	(3)	126. $6\sqrt{7}$	(6)
122. 100	(3)	127. 9	(4)
123. 3/8	(5)	128. 105	(5)
124. 5	(3)	129. $6/\pi$	(6)
125. 6720	(5)	130. 30	(4)

Warm-Up 11

Answer	Difficulty		
131. 20	(4)	136. 132	(6)
132. 76	(4)	137. 350	(4)
133. 3/8	(5)	138. 28	(4)
134. 25	(4)	139. 18	(3)
135. 175/256	(5)	140. 8/27	(5)

Warm-Up 12

Answer	Difficulty		
141. 867	(3)	146. 81	(5)
142. 58	(5)	147. 58	(4)
143. 241	(5)	148. 12	(3)
144. 300	(3)	149. $10+2\sqrt{35}$ or $2\sqrt{35}+10$	(6)
145. 0	(3)	150. 8	(5)

Warm-Up 13

Answer	Difficulty		
151. 5	(4)	156. 9490	(5)
152. $18\sqrt{3}$	(3)	157. 13	(4)
153. 53	(3)	158. 105	(5)
154. 91	(4)	159. 17	(5)
155. 3	(4)	160. 25	(4)

Warm-Up 14

Answer	Difficulty		
161. 30	(4)	166. $(11\sqrt{10})/8$	(5)
162. 52	(2)	167. 45	(6)
163. 1/9	(5)	168. 140	(4)
164. 1632	(4)	169. 7	(5)
165. 3/5	(5)	170. 1/27	(5)

Workout 1

Answer	Difficulty		
171. 3/8	(2)	176. 33	(3)
172. 2.8	(4)	177. 14	(3)
173. 263	(5)	178. 144	(4)
174. 135	(3)	179. 5000 or 5000.00	(3)
175. 30.4	(2)	180. 87,672	(3)

Workout 2

Answer	Difficulty		
181. 100	(3)	186. 383	(2)
182. 2.5	(5)	187. 250	(5)
183. 62.5	(5)	188. 0.162	(2)
184. 55.71	(2)	189. 3	(4)
185. 33	(4)	190. 67.5	(4)

Workout 3

Answer	Difficulty		
191. 1530	(3)	196. 70	(3)
192. 45	(3)	197. 150π	(3)
193. 34	(3)	198. 127,750,000	(2)
194. 44	(4)	199. 26	(4)
195. 14	(3)	200. 70	(3)

Workout 4

Answer	Difficulty		
201. 75	(2)	206. 1/3	(4)
202. 1.01	(3)	207. 32/45	(3)
203. 28	(3)	208. 30	(2)
204. 0.924	(5)	209. 1300	(3)
205. 35.7	(3)	210. 2	(4)

Workout 5

Answer	Difficulty		
211. 3375	(5)	216. 3	(3)
212. 57	(4)	217. 20 or 20.00	(3)
213. 1,036,800	(5)	218. 4	(4)
214. 11	(5)	219. 50	(2)
215. $64 + 128\sqrt{2}$ or $128\sqrt{2} + 64$	(5)	220. 4/7	(4)

Workout 6

Answer	Difficulty		
221. 37.9	(5)	226. 52	(5)
222. 25	(2)	227. 2996	(3)
223. 7.6	(4)	228. 6	(3)
224. 28	(3)	229. 11	(4)
225. 1.47	(5)	230. 360	(4)

Workout 7

Answer	Difficulty		
231. 45	(4)	236. 78	(5)
232. 4	(6)	237. 128	(5)
233. Thursday	(4)	238. −1516.75	(4)
234. 8/51	(3)	239. 225 or 225.00	(3)
235. 72	(5)	240. 24π	(5)

Workout 8

Answer	Difficulty		
241. 144	(4)	246. 11	(5)
242. 816	(5)	247. 26	(5)
243. 1729	(6)	248. 195	(6)
244. 5	(4)	249. $3\sqrt{7}$	(6)
245. 4.7	(6)	250. 87	(6)

MATHCOUNTS Problems Mapped to Common Core State Standards (CCSS)

Forty-two states, the District of Columbia, four territories and the Department of Defense Education Activity (DoDEA) have voluntarily adopted the Common Core State Standards (CCSS). As such, MATHCOUNTS considers it beneficial for teachers to see the connections between the *2017-2018 MATHCOUNTS School Handbook* problems and the CCSS. MATHCOUNTS not only has identified a general topic and assigned a difficulty level for each problem but also has provided a CCSS code in the Problem Index (pages 54-55). A complete list of the Common Core State Standards can be found at www.corestandards.org.

The CCSS for mathematics cover K-8 and high school courses. MATHCOUNTS problems are written to align with the NCTM Standards for Grades 6-8. As one would expect, there is great overlap between the two sets of standards. MATHCOUNTS also recognizes that in many school districts, algebra and geometry are taught in middle school, so some MATHCOUNTS problems also require skills taught in those courses.

In referring to the CCSS, the Problem Index code for each of the Standards for Mathematical Content for grades K-8 begins with the grade level. For the Standards for Mathematical Content for high school courses (such as algebra or geometry), each code begins with a letter to indicate the course name. The second part of each code indicates the domain within the grade level or course. Finally, the number of the individual standard within that domain follows. Here are two examples:

- *6.RP.3 → Standard #3 in the Ratios and Proportional Relationships domain of grade 6*
- *G-SRT.6 → Standard #6 in the Similarity, Right Triangles and Trigonometry domain of Geometry*

Some math concepts utilized in MATHCOUNTS problems are not specifically mentioned in the CCSS. Two examples are the Fundamental Counting Principle (FCP) and special right triangles. In cases like these, if a related standard could be identified, a code for that standard was used. For example, problems using the FCP were coded 7.SP.8, S-CP.8 or S-CP.9 depending on the context of the problem; SP → Statistics and Probability (the domain), S → Statistics and Probability (the course) and CP → Conditional Probability and the Rules of Probability. Problems based on special right triangles were given the code G-SRT.5 or G-SRT.6, explained above.

There are some MATHCOUNTS problems that either are based on math concepts outside the scope of the CCSS or based on concepts in the standards for grades K-5 but are obviously more difficult than a grade K-5 problem. When appropriate, these problems were given the code SMP for Standards for Mathematical Practice. The CCSS include the Standards for Mathematical Practice along with the Standards for Mathematical Content. The SMPs are (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure and (8) Look for and express regularity in repeated reasoning.

PROBLEM INDEX

It is difficult to categorize many of the problems in the *MATHCOUNTS School Handbook*. It is very common for a MATHCOUNTS problem to straddle multiple categories and cover several concepts. This index is intended to be a helpful resource, but since each problem has been placed in exactly one category and mapped to exactly one Common Core State Standard (CCSS), the index is not perfect. In this index, the code **9 (3) 7.SP.3** refers to problem 9 with difficulty rating 3 mapped to CCSS 7.SP.3. For an explanation of the difficulty ratings refer to page 49. For an explanation of the CCSS codes refer to page 53.

Number Theory

45	(3)	4.OA.4
56	(3)	4.OA.4
59	(3)	SMP
68	(4)	N-RN.1
69	(3)	S-CP.9
78	(3)	SMP
84	(2)	SMP
109	(5)	SMP
110	(5)	8.EE.2
111	(2)	7.NS.2
119	(2)	4.OA.4
121	(3)	6.NS.4
130	(4)	SMP
134	(4)	6.NS.4
139	(3)	SMP
148	(3)	7.NS.3
151	(4)	SMP
153	(3)	6.EE.2
156	(5)	6.NS.4
161	(4)	F-BF.2
193	(3)	SMP
194	(4)	SMP
201	(2)	7.NS.3
227	(3)	8.EE.2
229	(4)	6.NS.4
231	(4)	6.NS.4
233	(4)	SMP
237	(5)	6.NS.4
241	(4)	N-RN.2
246	(5)	SMP

Logic

83	(3)	S-CP.9
92	(4)	SMP
131	(4)	SMP
145	(3)	SMP
167	(6)	SMP
216	(3)	S-CP.9
235	(5)	SMP

Plane Geometry

32	(2)	4.G.2
43	(4)	7.G.6
53	(5)	7.G.4
61	(3)	7.G.6
66	(4)	G-C.2
82	(4)	7.G.6
106	(4)	7.G.4
107	(5)	7.G.6
113	(5)	8.G.8
118	(7)	G-SRT.4
123	(5)	G-SRT.6
133	(5)	7.G.6
136	(6)	G-SRT.5
149	(6)	A-REI.4
152	(3)	G-SRT.6
158	(5)	G-SRT.6
160	(4)	G-C.2
163	(5)	G-SRT.5
170	(5)	G-SRT.5
178	(4)	8.G.5
187	(5)	G-SRT.6
192	(3)	8.G.5
197	(3)	G-C.2
204	(5)	7.G.1
211	(5)	G-SRT.5
215	(5)	G-SRT.6
221	(5)	G-SRT.6
225	(5)	G-SRT.6
232	(6)	G-C.2
236	(5)	G-C.2
240	(5)	7.G.6
245	(6)	G-SRT.6
250	(6)	G-SRT.6

Measurement

40	(4)	8.G.5
49	(3)	5.MD.1
50	(4)	G-SRT.5
55	(4)	8.G.8
67	(3)	6.G.1
96	(5)	7.G.6
126	(6)	8.G.8
132	(4)	8.G.5
143	(5)	8.G.7
155	(4)	7.RP.3
172	(4)	6.RP.3
181	(3)	6.EE.7
183	(5)	8.G.7
198	(2)	6.RP.3

Solid Geometry

60	(3)	7.G.6
74	(3)	G-GMD.3
129	(6)	G-GMD.3
177	(3)	8.G.9
190	(4)	8.G.9
208	(2)	8.G.9
219	(2)	7.G.6
230	(4)	G-GMD.3

Coordinate Geometry

79	(3)	G-C.2
93	(5)	8.G.8
112	(5)	8.F.3
166	(5)	8.F.3
203	(3)	6.G.1
238	(4)	8.G.1
249	(6)	8.G.8

Algebraic Expressions & Equations

38	(3)	6.EE.2
42	(4)	6.EE.9
47	(4)	8.F.3
51	(3)	A-REI.4
52	(3)	6.EE.9
71	(3)	6.EE.7
73	(5)	A-REI.4
80	(5)	8.EE.8
81	(2)	6.EE.3
89	(4)	N-RN.1
91	(2)	7.EE.4
99	(4)	N-RN.1
101	(3)	A-CED.1
120	(4)	N-RN.1
142	(5)	N-RN.1
150	(5)	8.EE.8
176	(3)	6.EE.2
185	(4)	6.EE.7
199	(4)	A-SSE.3
202	(3)	A-REI.4
209	(3)	A-CED.1
210	(4)	A-SSE.3
217	(3)	7.EE.4
218	(4)	A-REI.4
224	(3)	A-CED.1

Statistics

57	(4)	6.SP.5
65	(4)	6.SP.5
87	(2)	6.SP.5
98	(4)	6.SP.5
105	(3)	6.SP.5
138	(4)	SMP
147	(4)	7.EE.4
175	(2)	6.SP.2
186	(2)	6.SP.2
223	(4)	6.SP.5

Problem Solving (Misc.)

54	(3)	SMP
58	(3)	4.OA.4
70	(5)	S-CP.9
72	(2)	SMP
128	(5)	SMP
157	(4)	G-CO.3
162	(2)	A-CED.1
180	(3)	7.NS.3
212	(4)	SMP

General Math

31	(1)	3.OA.8
33	(2)	SMP
35	(3)	6.EE.2
36	(2)	4.OA.2
37	(2)	3.OA.8
39	(4)	SMP
44	(2)	5.OA.1
62	(4)	SMP
88	(2)	7.NS.3
100	(3)	N-RN.2
114	(4)	N-RN.2
124	(3)	6.NS.4
174	(3)	6.NS.3

Probability, Counting & Combinatorics

Probability Stretch[1]

34	(3)	7.SP.8
46	(3)	7.SP.8
64	(4)	S-CP.9
75	(3)	S-CP.9
76	(2)	7.SP.8
85	(5)	S-CP.9
95	(3)	7.SP.8
103	(4)	8.EE.1
104	(3)	7.SP.7
116	(3)	S-CP.9
125	(5)	S-CP.9
135	(5)	7.SP.8
140	(5)	7.SP.7
144	(3)	S-CP.9
146	(5)	S-CP.9
154	(4)	S-CP.9
159	(5)	SMP
168	(4)	S-CP.9
171	(2)	7.SP.7
189	(4)	S-CP.9
196	(3)	7.SP.8
206	(4)	7.SP.8
207	(3)	7.SP.7
213	(5)	S-CP.9
226	(5)	SMP
234	(3)	7.SP.8
242	(5)	S-CP.9
247	(5)	S-CP.9
248	(6)	8.G.7

[1] CCSS 7.SP.8 & S-CP.9

Percents & Fractions

48	(3)	6.RP.3
77	(4)	7.RP.3
86	(4)	7.NS.3
137	(4)	A-CED.1
179	(3)	7.RP.3
184	(2)	6.RP.3
188	(2)	6.NS.1
195	(3)	6.NS.1
200	(3)	A-CED.1
205	(3)	7.RP.3
222	(2)	7.RP.3
239	(3)	7.NS.3

Proportional Reasoning

21	(5)	6.RP.3
22	(4)	6.EE.7
23	(3)	6.RP.3
24	(4)	6.EE.7
25	(4)	6.RP.3
26	(3)	6.RP.3
27	(5)	6.RP.3
28	(3)	6.RP.3
29	(4)	6.RP.3
30	(3)	6.RP.3
41	(2)	5.MD.1
63	(3)	4.OA.2
90	(3)	7.RP.2
102	(3)	6.RP.3
108	(5)	6.RP.3
117	(4)	6.RP.3
122	(3)	7.RP.3
127	(4)	7.RP.3
169	(5)	SMP
173	(5)	6.RP.3
182	(5)	7.RP.2
220	(4)	7.EE.4
228	(3)	7.RP.2
244	(4)	SMP

Sequences, Series & Patterns

Patterns Stretch[2]

94	(3)	F-BF.2
97	(4)	F-BF.2
115	(4)	F-BF.2
141	(3)	F-LE.1
164	(4)	F-LE.2
165	(5)	F-LE.2
191	(3)	F-LE.2
214	(5)	F-LE.2
243	(6)	F-BF.2

[2] CCSS F-BF.2

SOLUTIONS

The solutions provided here are only *possible* solutions. It is very likely that you or your students will come up with additional—and perhaps more elegant—solutions. Happy solving!

Probability Stretch

1. In a standard deck of 52 playing cards, the red number cards greater than 6 are the 7, 8, 9 and 10 in the suits of diamonds and hearts. That's a total of 8 cards. The percent probability that Perta randomly selects one of these 8 cards, then, is $8/52 \approx$ **15.38**%.

quarters	0					1		
dimes	**4**	**3**	2	1	0	2	1	0
nickels	1	3	5	7	9	0	2	4

2. The table shows all the ways to make 45 cents from nickels, dimes and quarters. Only two of the cups contain three or more dimes. Therefore, the probability that Max randomly selects one of these cups is $2/8 =$ **1/4**.

3. Danya can get a total of 10 with two or three chips if the first two chips drawn are 4 and 6, which can occur in 2 ways, or if the first three chips drawn are 2, 3 and 5, which can occur in 6 ways. There are $5 \times 4 = 20$ ways to randomly select two chips, and there are $5 \times 4 \times 3 = 60$ ways to randomly select three chips. The probability that Danya's total will equal 10 at some point is $2/20 + 6/60 = 1/10 + 1/10 = 2/10 =$ **0.2**.

4. The probability of pulling out two green socks is $2/5 \times 1/4 = 1/10$. The probability of pulling out two blue socks is $3/5 \times 2/4 = 3/10$. The probability, therefore, of randomly pulling out a matching pair of socks is $1/10 + 3/10 = 4/10 =$ **2/5**.

5. The probability that the nickel comes up heads is 1/2. The probability that one or more of the other two coins comes up heads is 3/4. The probability that at least two heads come up, with one of them being the nickel, is $1/2 \times 3/4 =$ **3/8**.

6. After that circuit is turned on, lights A and B will blink together every 55 seconds. Lindsey sees light A blink alone. Because light B blinks once every 11 seconds, and it did not blink this time, it will blink in one of the 10 following seconds, with equal likelihood. In 5 of those cases, it will blink before or at the same time as light A; and in the other 5 cases, light A will blink alone first. Therefore the probability that the next light to blink will be light A blinking alone is $5/10 = 0.50 =$ **50**%. *Alternative solution:* Whether or not light B blinks with A, A will always blink alone twice before the next time light B blinks. When she sees light A blink alone, it could be either the first or the second time, so the probability that it's the first time is **50**%.

7. Multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, … . It appears that every fifth multiple of 3 is also a multiple of 5. The percent probability that a randomly selected multiple of 3 is also a multiple of 5 is $1/5 =$ **20**%.

8. Starting at the top of the figure and at each subsequent junction, there is a 1/2 probability of choosing a path to the right and a 1/2 probability of choosing a path to the left. The number of junctions leading to each of the eight endpoints varies. There are only two junctions in the path ending at ❸, for a probability of $1/2 \times 1/2 = 1/4$. There are three junctions in each of the paths ending at ❶ and ❼, so there is a $1/2 \times 1/2 \times 1/2 = 1/8$ probability of ending at either. There are four junctions in the path ending at ❺, for a probability of $1/2 \times 1/2 \times 1/2 \times 1/2 = 1/16$. Therefore, the probability of randomly choosing a path that ends at an odd number is $1/8 + 1/4 + 1/16 + 1/8 =$ **9/16**.

9. Recall that a number is divisible by 4 if the two-digit number formed by the tens and ones digits is divisible by 4. Using the digits 1, 2, 3, 4 and 5, the following two-digit multiples of four can be formed: 12, 24, 32 and 52. There are $5 \times 4 = 20$ permutations of two digits chosen from the digits 1, 2, 3, 4 and 5. Therefore, the probability that the five-digit number is divisible by 4 is $4/20 =$ **1/5**.

10. The inner circle of radius 1 has an area of $\pi \times 1^2 = \pi$ units^2. The combined area of the four numbered sectors is $4/6 = 2/3$ the area of the inner circle, or $2\pi/3$ units^2. The area of the dartboard is $\pi \times 2^2 = 4\pi$ units^2. The probability of a dart landing in one of the numbered sectors is $(2\pi/3)/(4\pi) = 1/6 \approx$ **0.17**.

Patterns Stretch

stage	1	2	3	4
dots	1	4	9	16

11. The table shows the number of dots at each of the first four stages of the dot pattern. We can see that the number of dots at Stage n is n^2. If we let $n = 27$, then the difference between the numbers of dots in the figures at Stage 27 and Stage 47 is $(n + 20)^2 - n^2 \rightarrow n^2 + 40n + 400 - n^2 \rightarrow 40n + 400$ dots, or $40(27) + 400 = 1080 + 400 =$ **1480** dots. *Alternative solution:* If we recognize that the difference between the numbers of dots in the figures at Stage 27 and Stage 47 is the difference of two squares, then we get a difference of $47^2 - 27^2 = (47 + 27)(47 - 27) = (74)(20) =$ **1480** dots.

12. The terms in the sequence are 1, 2, 3, 6, 11, 20, 37, 68, 125, 230, 423, …. The 11th term is **423**.

13. The series $2 + 5 + 8 + 11 + 14 + \ldots + 89 + 92$ has 31 terms, and the average value of a term is $(2 + 92)/2 = 47$. The sum of the terms is then $31 \times 47 =$ **1457**.

14. The difference between the first and second terms is $2x + 11 - x = x + 11$. The difference between the second and third terms is $4x - 3 - (2x + 11) = 4x - 3 - 2x - 11 = 2x - 14$. Since the difference between consecutive terms is constant, we set these two differences equal to each other and get $x + 11 = 2x - 14 \rightarrow x = 25$. Substituting, we see that the difference between consecutive terms is $x + 11 = 25 + 11 =$ **36**.

15. Filling in the missing terms, we get the series $1 + 4 + 16 + 64 + 256 + 1024 =$ **1365**.

16. The nth consecutive odd positive integer has a value of $2n - 1$. For example, we know that 31 is the 16th consecutive odd positive integer since when $n = 16$, $2n - 1 = 31$. The sum of the 16 terms is then $1 + 3 + 5 + 7 + ... + 29 + 31 = 32/2 \times 16 = 16^2 = 256$. In general, the sum of the first n consecutive odd positive integers is n^2. Therefore, the sum of the first 51 consecutive odd positive integers is $51^2 =$ **2601**.

17. Let $x = 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$. So, $x/2 = 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots$. Substituting this back into the first equation yields $x = 1 + x/2$. Solving for x, we get $2x = 2 + x$ and $x =$ **2**.

18. Let $x = 1 + 1/4 + 1/16 + 1/64 + 1/256 + \ldots$. So, $x/4 = 1/4 + 1/16 + 1/64 + 1/256 + 1/1024 + \ldots$. Substituting this back into the first equation yields $x = 1 + x/4$. Solving for x, we get $4x = 4 + x \rightarrow 3x = 4$ and $x =$ **4/3**.

19. Based on the information provided, we can find $f^5(x)$ in a number of ways. For example, $f^5(x) = f^2(f^3(x)) = f^3(f^2(x)) = f^4(f(x)) = f(f^4(x)) = ax + b$. We can easily derive either $f^3(x)$ or $f^4(x)$ using the information provided in the problem. We have $f^3(x) = f^2(f(x)) = 4(2x + 3) + 9 = 8x + 12 + 9 = 8x + 21$. So, $f^5(x) = f^2(f^3(x)) = 4(8x + 21) + 9 = 32x + 84 + 9 = 32x + 93$. We see that $a = 32$, $b = 93$ and $a + b = 32 + 93 =$ **125**.

20. Let x, rx, r^2x, and r^3x represent the degree measures of the angles of this quadrilateral. If $r = 2$, our angle measures are x, $2x$, $4x$ and $8x$. Then $x + 2x + 4x + 8x = 360 \rightarrow 15x = 360 \rightarrow x = 24$. The largest angle has degree measure $8 \times 24 = 192$ degrees. If $r = 3$, our angle measures are x, $3x$, $9x$ and $27x$. Then $x + 3x + 9x + 27x = 360 \rightarrow 40x = 360 \rightarrow x = 9$. The largest angle has degree measure $27 \times 9 = 243$ degrees. Letting $r = 4$, or any higher value, yields non-integer angle measures. So, the largest possible degree measure of an angle in this quadrilateral is **243** degrees.

Travel Stretch

21. To determine the average speed for the entire round-trip, we need the total distance and the total time. If Jack and Jill travel d miles uphill, then they travel another d miles downhill, for a total distance of $2d$ miles. Since time = distance/speed, it follows that the time to travel uphill is $d/2$ hours and the time to travel downhill is $d/4$ hours, for a total time of $d/2 + d/4 = (2d + d)/4 = 3d/4$ hours. Now we can use these values in the formula speed = distance/time to get $2d/(3d/4) = 2d \times 4/(3d) = 8/3 =$ $\mathbf{2\frac{2}{3}}$ mi/h.

22. Since distance = speed × time, we know that when they meet in t hours, Jack will have traveled $4t$ miles and Jill will have traveled $2t$ miles. We also know that the total distance traveled is 1.5 miles. Therefore, $4t + 2t = 1.5 \rightarrow 6t = 1.5 \rightarrow t = 1/4$ hour, or 15 minutes. They will meet at **2:35** p.m.

23. We need to determine how far Jill has traveled up the hill when she meets Jack. Jill traveled at a speed of 2 mi/h for 1/4 hour. That's a total distance of $2 \times 1/4 = 1/2$ mile $= 1/2 \times 5280 \times 1/3 = 1760/2 =$ **880** yards.

24. Let t represent the time it takes Alysha to drive from home to the market. We know that her walking speed is 5 mi/h and it takes her $t + 21$ minutes to walk to the market from home. Converting her speed to miles per minute, we get $5 \div 60 = 1/12$ mile per minute. We are told that she drives eight times her walking speed, so her driving speed is $8 \times 1/12 = 8/12 = 2/3$ mile per minute. Since Alysha takes the same route to the store whether she walks or drives, we have the equation $(2/3)t = (1/12)(t + 21)$. Solving for t, we have $8t = t + 21 \rightarrow 7t = 21 \rightarrow t = 3$. So, the time it takes Alysha to drive to the market is **3** minutes.

25. We know that Alysha drives to the market at a speed of 2/3 mile per minute and it takes her 3 minutes. That means the distance from her home to the market is $2/3 \times 3 =$ **2** miles.

26. Running at a speed of 6 mi/h, Jana's total time to complete the 2-mile path is $2 \div 6 = 1/3$ hour, or 20 minutes. Riding his bicycle at a speed of 10 mi/h, Zhao's total time to complete the path is $2/10 = 1/5$ hour, or 12 minutes. That's a difference of $20 - 12 = 8$ minutes. Since Jana starts running 5 minutes before Zhao starts riding, she will reach the end of the path $8 - 5 =$ **3** minutes after Zhao does.

27. Jana runs for 5 minutes, or 1/12 hour, before Zhao begins riding. During that time she travels a distance of $6 \times 1/12 = 1/2$ mile. Let t represent the number of hours it will take Zhao to catch up to Jana. When Zhao catches up to her, Jana will have traveled a total of $6t + 1/2$ miles, while Zhao will have traveled $10t$ miles. They will have traveled the same distance, so we can set these two quantities equal to each other to get the equation $6t + 1/2 = 10t$. Solving for t, we get $1/2 = 4t \rightarrow t = 1/8$ hour, or $\mathbf{7\frac{1}{2}}$ minutes. *Alternative solution:* Let t be the number of minutes it takes Zhao to catch up with Jana. He is traveling at 10 mi/h, or 1/6 mile per minute, so he will travel $t \times 1/6$ miles before he catches up with her. She is traveling at 6 mi/h, or 1/10 mile per minute, and will have traveled $t + 5$ minutes before he overtakes her. Therefore $t \times 1/6 = (t + 5) \times 1/10 \rightarrow 5t = 3t + 15 \rightarrow 2t = 15 \rightarrow t =$ $\mathbf{7\frac{1}{2}}$ minutes.

28. Recall from the previous problem that Jana will have traveled $6t + 1/2$ miles when Zhao catches up with her. That's 6(1/8) + 1/2 = 3/4 + 1/2 = (3 + 2)/4 = 5/4 = $\mathbf{1\frac{1}{4}}$ miles. *Alternative solution:* Recall from the previous problem that Jana will have jogged for 5 + 7 1/2 = 12 1/2 minutes before Zhao catches up with her. Since she is traveling at 1/10 mile per minute, the distance she jogged will be 25/2 × 1/10 = 5/4 = $\mathbf{1\frac{1}{4}}$ miles.

29. Let c represent the speed of the river's current. Then Ansel's speed was $20 + c$ downstream and $20 - c$ upstream. The time to travel the 10 miles downstream and then back upstream was $10/(20 + c) + 10/(20 - c)$. Since we know that the entire round-trip took 64 minutes, or 16/15 hours, we have $10/(20 + c) + 10/(20 - c) = 16/15$. Solving for c, we find that $(200 + 10c + 200 - 10c)/(400 - c^2) = 16/15 \rightarrow 400/(400 - c^2) = 16/15 \rightarrow 15(400) = 16(400 - c^2) \rightarrow 6000 = 6400 - 16c^2 \rightarrow 16c^2 = 400 \rightarrow c^2 = 25 \rightarrow c = 5$. Thus, the speed of the river's current is **5** mi/h.

30. The time to travel upstream for 10 miles was 10/(20 − 5) = 10/15 = 2/3 hour, or 40 minutes. So, of the entire 64-minute round-trip, the fraction that was spent traveling upstream was 40/64 = **5/8**.

Warm-Up 1

31. We evaluate the expression according to the Order of Operations as follows: 5 − 5 × 5 + 5 ÷ 5 = 5 − 25 + 1 = −20 + 1 = **−19**.

32. First we note that a convex heptagon is a figure with 7 sides and 7 vertices. Each vertex has an interior angle measure less than 180 degrees. Also, a diagonal is any line segment that connects two vertices that are not connected by a side. A diagonal can be drawn from each vertex of the heptagon to 4 other vertices, excluding itself and its two neighbors. If we multiply 7 × 4 = 28, we have counted each diagonal at both ends, so a convex heptagon actually has a total of 28 ÷ 2 = **14** diagonals.

33. A palindrome is a number (or word) that reads the same forward and backward. Since we are looking for the first palindrome year after 2018, we'll start by considering years that begin and end with 2 and have two middle digits that are the same. We have 2002, 2112, 2222, 2332, etc. The first year after 2018 that is a palindrome, then, is **2112**.

34. There are four aces in a deck of 52 cards, so the probability that the first card selected at random is an ace is 4/52 = 1/13. If an ace is indeed chosen, then there are three aces left in the remaining 51 cards. The probability that the second card selected at random is an ace is 3/51 = 1/17. The probability that these two events both occur is 1/13 × 1/17 = **1/221**. *Alternative solution:* There are "4 choose 2" or ${}_4C_2 = 4!/(2! \times 2!) = (4 \times 3)/2 = 6$ ways to pick two aces out of "52 choose 2" or ${}_{52}C_2 = 52!/(50! \times 2!) = (52 \times 51)/2 = 1326$ ways to pick any two cards. That's a probability of 6/1326 = **1/221**.

35. To simplify this radical expression, we will rewrite the product in terms of prime factors. Then we will take the square root of all the perfect square factors. We have $\sqrt{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 10} = \sqrt{2 \times 3 \times (2 \times 2) \times 5 \times (2 \times 3) \times 7 \times (2 \times 5)} = \sqrt{2^5 \times 3^2 \times 5^2 \times 7} = 2^2 \times 3 \times 5 \times \sqrt{2 \times 7} = \mathbf{60\sqrt{14}}$.

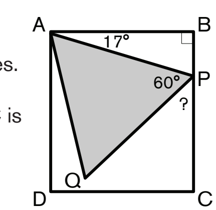
36. The temperature went from 13 °F to −5 °F, which is a drop of 13 − (−5) = 13 + 5 = **18** °F.

37. Using the Order of Operations, we get 1 × 2 + 3 ÷ 6 × 5 − 4 = 2 + (1/2) × 5 − 4 = 2 + 5/2 − 4 = 5/2 − 4/2 = **1/2.**

38. First, we evaluate 2 ® 1 inside the parentheses to get $2^2 - 1^2 = 4 - 1 = 3$. Using the result to evaluate 3 ® 3, we get $3^2 - 3^2 =$ **0**.

39. The three-digit number and the two-digit number differ by 288, so let's start by filling in the blanks of the subtraction problem __ __ __ − __ __ = 288 with the digits 7, 8, 2, 3, and 0. In the ones place, 0 − 2 leaves an 8 (after borrowing). In the tens place of the three-digit number we had to reduce the digit by one to give 10 to the ones place, so the 7 could become a 6 and 16 − 8 = 8. That leaves the 3 for the hundreds place of the three-digit number. This works since 370 − 82 = 288. So, the three-digit number is 370 and the two-digit number is 82, and the desired sum is 370 + 82 = **452**.

40. The figure shows equilateral triangle APQ inside rectangle ABCD. In right triangle PAB, the measure of angle PAB is 17 degrees. Since the two acute angles in a right triangle are complementary, we know that the measure of angle APB must be 90 − 17 = 73 degrees. We also know that the measure of angle APQ is 60 degrees. Angle BPC is a straight angle, so the measure of angle QPC is 180 − 73 − 60 = **47** degrees.



Warm-Up 2

41. Kim will have to buy a whole number of balls of yarn, so we need to round 750 ÷ 180 = 25 ÷ 6 = 4 1/6 balls to **5** balls of yarn.

42. Let's say C is Chris' age in 1992 and J is Joseph's age in 1992. The first sentence translates to $C = (1/2)J$ or $2C = J$, and the second sentence translates to $C + 6 = (2/3)(J + 6)$. Substituting $2C$ for J in the second equation, we get $C + 6 = (2/3)(2C + 6) \rightarrow C + 6 = (4/3)C + 4 \rightarrow 2 = (1/3)C \rightarrow C = 6$. So, Chris must have been 6 in 1992. In 2018, Chris will be 2018 − 1992 = 26 years older, which will make him **32** years old.

43. At 5:42, the time elapsed is 42 of the 60 minutes in the 5 o'clock hour. Since the minute hand will make the complete revolution during that hour, at 5:42 it has traveled 42/60 = 7/10 of the full 360 degrees, or (7/10) × 360 = 252 degrees. The hour hand makes 1/12 of a complete revolution every hour. So, from 12:00 to 5:00, it travels 5/12 of the full 360 degrees, or (5/12) × 360 = 150 degrees. By 5:42, it has traveled another 7/10 of 1/12 of the full 360 degrees, or (7/10) × (1/12) × 360 = 21 degrees, for a total of 150 + 21 = 171 degrees. The measure of the angle between the hands at 5:42 is 252 − 171 = **81** degrees.

44. The expression 12 × 37 + 12 × 7 + 12 × 6 can be rewritten as 12 × (37 + 7 + 6), which equals 12 × 50 = **600**.

45. The prime factorization of 2018 is 2 × 1009, since 1009 is prime. Thus, 1, 2, 1009 and 2018 are the **4** positive factors of 2018.

46. The possible values for the sum of the two top faces that are *at least* 9 are 9, 10, 11 and 12. There are 4 ways to roll a 9, 3 ways to roll a 10, 2 ways to roll an 11 and only 1 way to roll a 12. That's 4 + 3 + 2 + 1 = 10 rolls out of the 6 × 6 = 36 possible rolls, for a probability of 10/36 = **5/18**.

47. The letter m in the equation $y = mx + b$ represents the slope of the line. We need to calculate the ratio of the change in the y-values to the change in the x-values. Using the points (6, 13) and (10, 31), we have a slope of m = (31 − 13)/(10 − 6) = 18/4 = **9/2**.

48. The least common multiple (LCM) of 12 and 20 is 60. So let the number of 12-ounce cans of soda that Dewey buys be 5, and let the number of 20-ounce bottles of soda that Peppar buys be 3. They each get a total of 60 ounces, but Dewey spends 5 × 1.00 = \$5.00, and Peppar spends 3 ×1.25 = \$3.75. Peppar spends 5.00 − 3.75 = \$1.25 less than Dewey, which is 1.25 ÷ 5.00 × 100 = 25**,** so P = **25**.

49. A diameter of 9 inches is 3/4 of one foot. We want to know how many 3/4 of a foot there are in 12 feet. Dividing, we get 12 ÷ (3/4) = 12 × 4/3 = 16. So, to create the 12-foot wall, Gerald will have to stack **16** logs.

50. The area of a square of side length s is s^2. So, a square of area 8 units^2 has side length $s = \sqrt{8} = 2\sqrt{2}$ units. By properties of 45-45-90 right triangles, we know that a square of side length s has diagonal length $s\sqrt{2}$. So, a square of side length $2\sqrt{2}$ units, has diagonal length $2\sqrt{2} \times \sqrt{2}$ = 4 units. Since the inscribed square's diagonal is a diameter of the circle, it follows that the circle has radius r = 2 units and area $\pi r^2 = \pi \times 2^2$ = **4π** units^2.

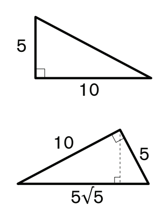
Warm-Up 3

51. If we take the square root of each side we see that $y = \pm(y + 2018)$. So $y = y + 2018$ or $y = -y - 2018$. Solving the first equation leads to the false statement 0 = 2018. Solving the second equation yields $2y = -2018 \rightarrow y =$ **−1009**, which is the only valid answer.

52. Let M and C represent Maura's and Cara's current ages, respectively. The first sentence translates to $M + 5 = C$, and the second sentence translates to $C - 7 = 2(M - 7)$. Substituting $M + 5$ for C in the second equation, we get $M + 5 - 7 = 2(M - 7) \rightarrow M - 2 = 2M - 14 \rightarrow M = 12$. So, Maura is **12** years old.

53. The area of the entire dartboard is the area of the circle of radius 10 inches, or $\pi \times 10^2 = 100\pi$ in^2. The area of the yellow region of the dart board is the difference between the areas of the circle of radius 5 inches and the circle of radius 1 inch, or $\pi \times 5^2 - \pi \times 1^2 = 25\pi - \pi = 24\pi$ in^2. Thus, 24π/(100π) = **6/25** of the dartboard's total area is colored yellow, and that fraction is the desired probability.

54. In the 28 years preceding 2018, there are 7 leap years, which each contain 1 more day than a non-leap year. Therefore, the total number of days in those 28 years is 365 × 28 + 7 = 10,220 + 7 = **10,227** days.



55. If the triangle is positioned as shown in the top figure, it has base length 10 cm and height 5 cm, and it has area (1/2) × b × h = (1/2) × 5 × 10 = 25 cm^2. Using the Pythagorean Theorem, we can determine that a right triangle with legs of length 5 cm and 10 cm has a hypotenuse of length $\sqrt{(5^2 + 10^2)} = \sqrt{(25 + 100)} = \sqrt{125} = 5\sqrt{5}$ cm. Now, if the triangle is positioned as shown in the bottom figure, it has base length $5\sqrt{5}$ cm. We can determine the height of the triangle in this position since we know that (1/2) × b × h = (1/2) × $5\sqrt{5}$ × h = 25. Solving for h, we get $h = 50/(5\sqrt{5}) = 50\sqrt{5}/25$ = **2√5** cm.

56. Of the 18 two-digit multiples of 5 from 10 to 95, only 13 have exactly two distinct prime factors, as shown. (Note that some of these numbers have more than one copy of their distinct prime factors.) The probability of choosing one of these 13 at random, then, is **13/18**.

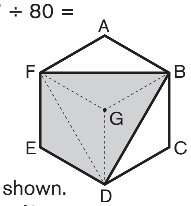
$10 = 2 \times 5$	$25 = 5^2$	$40 = 2^3 \times 5$	$55 = 5 \times 11$	$70 = 2 \times 5 \times 7$	$85 = 5 \times 17$
$15 = 3 \times 5$	$30 = 2 \times 3 \times 5$	$45 = 3^2 \times 5$	$60 = 2^2 \times 3 \times 5$	$75 = 3 \times 5^2$	$90 = 2 \times 3^2 \times 5$
$20 = 2^2 \times 5$	$35 = 5 \times 7$	$50 = 2 \times 5^2$	$65 = 5 \times 13$	$80 = 2^4 \times 5$	$95 = 5 \times 19$

57. In a stem-and-leaf plot, commonly the numbers to the left of the vertical line are the tens digits and the numbers to the right of the vertical line are the corresponding ones digits of the numbers in the data set. There are 24 numbers in this set. Recall that in a set of data with an even number of data points, the median is the mean of the two middle numbers when the data points have been arranged in ascending or descending order. In this case, the median is (20 + 24) ÷ 2 = 22, and there are 12 numbers less than 22 and 12 greater than 22. Next, to determine the lower quartile, we find the median of the 12 numbers less than 22. In this case, the lower quartile is (11 + 11) ÷ 2 = 11. To determine the upper quartile, we find the median of the 12 numbers greater than 22. In this case, the upper quartile is (35 + 37) ÷ 2 = 36. The interquartile range, then, is 36 − 11 = **25**.

58. Locating the primes in each row (starting from 36 and working backward), numbers are crossed off in the following order: 31, 26, 21, 16, 11, 6, 29, 24, 23, 18, 19, 14, 9, 4, 17, 12, 13, 8, 3, 7, 2, 5. The sum of the 14 remaining numbers is 1 + 10 + 15 + 20 + 22 + 25 + 27 + 28 + 30 + 32 + 33 + 34 + 35 + 36 = **348**.

59. In mathematics, the exclamation point indicates the "factorial" of a number, which is the product of the positive integers less than or equal to the number. We are asked to evaluate 1,000,000! ÷ 999,999! = (1,000,000 × 999,999 × 999,998 × ⋯ × 2 × 1)/(999,999 × 999,998 × ⋯ × 2 × 1). Once we get rid of the common factors in the numerator and denominator, the only number that remains is 1,000,000 in the numerator. Therefore, the answer is **1,000,000**.

60. One cubic yard of topsoil is 3 × 3 × 3 = 27 ft^3 of topsoil. The garden is 10 × 8, which is 80 ft^2, so the depth of the soil will be 27 ÷ 80 = 27/80 feet, which is equivalent to 27/80 × 12 = 81/20 = $\mathbf{4\frac{1}{20}}$ inches.



Warm-Up 4

61. We will draw a few more line segments, between vertices D and F and between each of the vertices B, D and F and center G, as shown. We now have six congruent obtuse triangles, four of which are in the interior of quadrilateral BDEF. The desired ratio of areas is 2/4 = **1/2**.

62. The expression can be evaluated as follows: $\frac{11!-(9+1)(9!)}{8(7!)} = \frac{11!-(10)(9!)}{8!} = \frac{11\times 10!-10!}{8!} = \frac{10!(11-1)}{8!} = \frac{10!(10)}{8!} = \frac{10\times 9\times 8!\times 10}{8!} = 10^2 \times 9 =$ **900**.

63. There are 30 × 20 = 600 drops of eyeglass cleaner in the bottle. David needs 3 × 4 = 12 drops to clean both sides of both lenses, so he can clean his glasses 600 ÷ 12 = **50** times.

64. There are ${}_3C_2$ = 3!/(1! × 2!) = 3 ways to choose 2 meats. There are ${}_4C_2$ = 4!/(2! × 2!) = (4 × 3)/(2 × 1) = 6 ways to choose vegetables. There are 3 × 4 = 12 ways to choose 1 meat and 1 vegetable. That's 3 + 6 + 12 = **21** possible combinations. *Alternative solution:* The rules really amount simply to choosing 2 of the 7 toppings, and this can be done in ${}_7C_2$ = 7!/(5! × 2!) = (7 × 6)/(2 × 1) = **21** ways.

65. If we add the 150 pounds of the first weighing and the 168 pounds of the last weighing, we have the sum of the average of sheep A and B and the average of sheep C and D. We can now subtract the 127 pounds of the second weighing, which is the average of sheep B and C, to get the average of sheep A and D. That's 150 + 168 − 127 = **191** pounds.

66. The measure of an angle inscribed in a circle is half the angle measure of the arc it intercepts. Since $m\angle ABC$ = 90 degrees, it follows that the measure of arc ABC is 180 degrees, and it has length equal to half the circumference of the circle. The circle has radius 3 meters, so the length of arc ABC is **3π** meters.

67. The wall has an area of 6 × 8 = 48 ft^2. Each square foot will require 3 × 3 = 9 of the 4-inch square tiles. Thus, 9 × 48 = **432** tiles are needed.

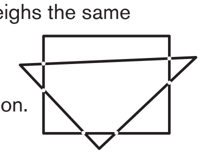
68. To find the units digit of a product, multiply the units digits of the factors and take the units digit of the result. Each of the factors in the problem is quite large, but there is a cycle to the units digits of both the powers of 2 and the powers of 7. In the powers of 2, we get units digits 2, 4, 8, 6 and then the pattern repeats. Since 2017 is one more than a multiple of 4 (the number of digits in the pattern), the 2017th power of 2 will have a units digit of 2. Similarly, in the powers of 7, we get units digits 7, 9, 3, 1 and then the pattern repeats. The units digit of the 2017th power of 7 is 7. Finally, the units digit of the product of these powers is just the units digit of 2 × 7 = 14, which is **4**.

69. Each of the hundreds, tens and units digits of these three-digit numbers can be a 3, a 4 or a 5. That's a total of 3 × 3 × 3 = **27** integers.

70. The 10 paths that start at cell A and move only to adjacent cells and include all five cells are ABCDE, ABCED, ABECD, ABEDC, ADCBE, ADCEB, ADEBC, ADECB, AEBCD and AEDCB. By symmetry, there are also 10 paths that start at cells B, C and D. The 8 paths that start at E are EABCD, EADCB, EBADC, EBCDA, ECBAD, ECDAB, EDABC and EDCBA. In all, there are 10 × 4 + 8 = **48** paths.

Warm-Up 5

71. Let p, c and e represent the weights of a pencil, paper clip and eraser, respectively. From the information given, we have $p + 5c = 2e$ and $p = 29c$. Substituting the $29c$ for p in the first equation, we get $29c + 5c = 2e$. This simplifies to $34c = 2e \rightarrow e = 17c$. So, an eraser weighs the same as **17** paper clips.



72. Each of the triangle's three sides can intersect twice with a side of the square, as shown, for a maximum of **6** points of intersection.

73. Substituting $x = 0$ into the function, we get $p(0) = a(0)^2 + b(0) + c = 4 \rightarrow c = 4$. Substituting $x = 1$ and $c = 4$ into the function, we get $p(1) = a(1)^2 + b(1) + 4 = 15 \rightarrow a + b = 11$. Substituting $x = 2$ and $c = 4$ into the function, we get $p(2) = a(2)^2 + b(2) + 4 = 36 \rightarrow 4a + 2b = 32 \rightarrow 2a + b = 16$. Subtracting the second and third equations yields $(2a + b) - (a + b) = 16 - 11 \rightarrow 2a + b - a - b = 5 \rightarrow a = 5$. Substituting back into the second equation, we see that $5 + b = 11 \rightarrow b = 6$. So, $a = 5$, $b = 6$, $c = 4$ and $abc = 5 \times 6 \times 4 =$ **120**.

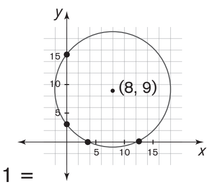
74. A sphere of radius r has volume $(4/3) \times \pi \times r^3$ and surface area $4 \times \pi \times r^2$. Because the given sphere's volume is numerically three times its surface area, we have $(4/3) \times \pi \times r^3 = 3 \times 4 \times \pi \times r^2$. Solving for r, we get $r^3/r^2 = (3 \times 4 \times \pi)/[(4/3) \times \pi] \rightarrow r =$ **9** units.

75. We'll use B, G, R, O and Y to represent the colors Blue, Green, Red, Orange and Yellow, respectively. We know that B must be the middle layer. If subsequent candles are made with no color next to a color it touched in the original candle, then G must be the top or bottom layer, since it touched B in the original candle. If G is the top layer, then the two possible color arrangements are: GOBRY and GOBYR. If G is the bottom layer we have the reverse of those two arrangements: YRBOG and RYBOG. That makes **4** different candles.

76. No matter what color the pointer lands on with the first spin, the probability is **1/3** that he will match that color on his second spin.

77. Let m represent the number of shots Kevin has made and a represent the number of shots he has attempted. Currently, his ratio of made shots to attempted shots is $m/a = 1/3 \rightarrow 3m = a$. If he makes the next 5 shots, the new ratio will be $(m + 5)/(a + 5) = 1/2 \rightarrow 2(m + 5) = a + 5 \rightarrow 2m + 10 = a + 5 \rightarrow 2m + 5 = a$. Setting these two expressions for a equal to each other, we get $3m = 2m + 5 \rightarrow m = 5$. This accounts for 1/3 of the initial $3 \times 5 =$ **15** shots Kevin has attempted up to now.

78. The binary number 110011_2 is $2^5(1) + 2^4(1) + 2^3(0) + 2^2(0) + 2^1(1) + 2^0(1) = 32 + 16 + 2 + 1 = 51$ in base ten. Since $51 = 8^1(6) + 8^0(3)$, it follows that 110011_2 is **63** base eight.

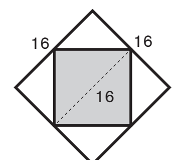


79. Since the circle's radius of 10 units is greater than both 8 and 9, the circle intersects each axis at two points, as shown, for a total of **4** points.

80. If we multiply the equations, we get $(x + (1/y)) \times (y + (1/x)) = (1/5) \times 20 \rightarrow xy + 2 + 1/(xy) = 4 \rightarrow xy + 1/(xy) = 2 \rightarrow (xy)^2 + 1 = 2xy \rightarrow (xy)^2 - 2xy + 1 = 0 \rightarrow (xy - 1)^2 = 0 \rightarrow xy - 1 = 0 \rightarrow xy =$ **1**. This is confirmed by substituting $1/y$ for x in the equation $x + (1/y) = 1/5$ to get $(1/y) + (1/y) = 1/5 \rightarrow 2/y = 1/5 \rightarrow y = 10$, and $x = 1/y = 1/10$.

Warm-Up 6

81. We know that $3x + 5 = 13$. Since $3x + 4$ is one less than $3x + 5$, we have $3x + 4 = 12$. Similarly, $3x + 3 = 11$ and $3x + 2 = 10$. Thus, $(3x + 2)(3x + 3)(3x + 4) = 10 \times 11 \times 12 =$ **1320**.

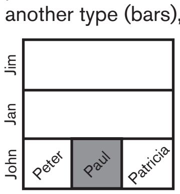


82. We will get the greatest possible area when the rectangle is a square. Since the diagonal of our square is 16 units, the area of the square is half the area of a 16 by 16 square, as shown. The maximum area, then, is $(1/2) \times 16^2 = (1/2) \times 256 =$ **128** units^2.

83. In order to satisfy rules I *and* II, at least one of the options in each rule must be true. The **5** pairs of numbers that work are (0, 0), (1, 1), (−1, 1), (1, −1), (−1, −1).

84. Since we are looking for the least possible sum, let's consider using the digits 1 through 6. We'll use the 1 and 2 as the units digits, so we have 1.BC + 2. EF. Now we need to arrange the digits 3, 4, 5 and 6 to get the smallest sum. Assigning the digits in any of the following ways gives us the smallest sum: 1.35 + 2.46 = 1.46 + 2.35 = 1.36 + 2.45 = 1.45 + 2.36 = **3.81**.

85. The problem requires that Matt choose one of each kind of fastener. So let's imagine that Matt first selects one of each kind and sets them aside. The question now is how many ways he can choose the other 5 fasteners. To make an organized list of all possibilities is a challenging and worthwhile exercise. We will take a shortcut known to some as "stars and bars." Five stars will represent the 5 fasteners, and 4 bars will separate the fasteners into the 5 different categories in the given order: wood screws, sheet metal screws, hex bolts, carriage bolts and lag bolts. The arrangement *****|||| represents 5 wood screws and none of the other types. The arrangement *|*|*|*|* represents one of each type of fastener. All possible combinations can be represented with this notation. The number of ways to arrange 9 items, with 5 being of one type (stars) and 4 being of another type (bars), is $9!/(5! \times 4!) = (9 \times 8 \times 7 \times 6)/(4 \times 3 \times 2 \times 1) =$ **126** ways.



86. The figure shows the share of the farm each heir is given. Jim gets a third of the farm in the first division. When John's third is then divided among his heirs, Peter, Paul and Patricia, each gets a third of his third, which is a ninth of the farm. After Paul sells his ninth of the farm, 8/9 of the farm remains. Jim, who still owns 1/3 = 3/9 of the original farm, owns 3/8 of the remaining 8/9 of the farm. Therefore, from the most recent sale, Jim should receive **3/8** of the proceeds.

87. For Olivia to achieve an average score of 90 on the four tests, the sum of all four scores would need to be $4 \times 90 = 360$. The sum of her first three test scores is $82 + 86 + 92 = 260$. The score Olivia needs to get on the fourth test is $360 - 260 =$ **100**. *Alternative solution:* Olivia's first test was 8 points below her desired average, the second was 4 points below, and the third was 2 points above. Thus, she has a deficit of $8 + 4 - 2 = 10$ points to make up and therefore needs to score 10 points above 90, or **100**.

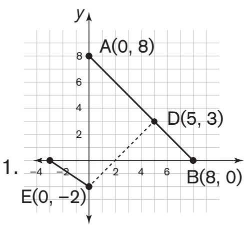
88. After the preschool fee of \$330, Cody paid an additional 770 − 330 = \$440 for after-school care. Dividing \$440 by the hourly rate of \$5.50, we find that Cody's son must have spent **80** hours in after-school care.

89. Let $n - 1$, n and $n + 1$ be positive consecutive integers such that $(n - 1) \times (n) \times (n + 1) = 16 \times [(n - 1) + (n) + (n + 1)]$. Simplifying and solving for n, we have $n(n^2 - 1) = 16 \times 3n \to n^3 - n = 48n \to n^3 - 49n = 0 \to n(n^2 - 49) = 0$. So, $n = 0$ or $n^2 - 49 = 0 \to (n + 7)(n - 7) = 0 \to n = -7$ or $n = 7$. Since n is a positive integer, it follows that the three consecutive numbers are 6, 7 and 8. The difference between their product and sum is $6 \times 7 \times 8 - (6 + 7 + 8) = 336 - 21 =$ **315**.

90. The ratio of the surface area of Wilbur's plane to the surface area of Orville's plane is the square of the scale factor of Wilbur's mini replica of Orville's plane. Since Wilbur's mini replica has linear dimensions that are 1/2 the size of Orville's model airplane, it follows that the ratio between the lift forces on Wilbur's and Orville's planes is $(1/2) \times (1/2) =$ **1/4**.

Warm-Up 7

91. Simplifying, we have $6/78 < 1/n < 5/55 \to 1/13 < 1/n < 1/11$. This compound inequality is true when $n =$ **12**.

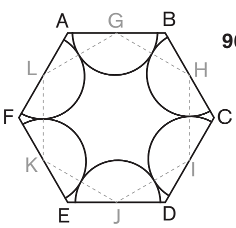


92. If we add the wins of both Flying Turtles and Dolphins, we have to then subtract the 19 wins for the games they played against each other regardless of who won those games. The value of $F + D$, then, is $95 + 84 - 19 =$ **160**.

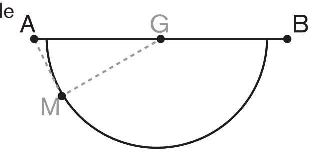
93. The shortest distance between the two lines is the segment from E(0, −2) drawn perpendicular to the segment with endpoints A(0, 8) and B(8, 0). Since segment AB has slope $(0 - 8)/(8 - 0) = -1$, the desired segment must have a slope of 1. It intersects segment AB at D(5, 3). As the figure shows, segment DE has length $\sqrt{(5^2 + 5^2)} = \sqrt{50} =$ **$5\sqrt{2}$** units.

94. We are just counting the numbers determined by the sequence 317, 319, ..., 637. We do so as follows: $(317 - 315)/2 = 2/2 = 1$ term, $(319 - 315)/2 = 4/2 = 2$ terms, and so forth. So, $(637 - 315)/2 = 322/2 =$ **161** terms.

95. Only perfect square numbers have an odd number of factors. Though there are $6 \times 6 = 36$ possible outcomes when two standard dice are rolled, there are only 11 possible sums: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,12. Of these, only 4 and 9 are perfect squares. There are 3 ways to roll a sum of 4: (1, 3), (2, 2) and (3, 1). There are 4 ways to roll a sum of 9: (3, 6), (4, 5), (5, 4) and (6, 3). That's a total of $3 + 4 = 7$ sums with an odd number of factors, for a probability of **7/36**.



96. If we draw a second regular hexagon, with vertices at the midpoints of the sides of the original hexagon, then the midpoints of the sides of this smaller hexagon are the points of tangency between adjacent semicircles. If we zoom in on one part of this figure, we see a 30-60-90 triangle, labeled AGM in the figure shown. Side AG of triangle AGM is half of side AB of the original hexagon and has length $s/2$. Side GM of triangle AGM is a radius of a semicircle and the long leg of the 30-60-90 triangle. The ratio of GM = r to AG = $s/2$, which is the ratio of the long leg to the hypotenuse of a 30-60-90 triangle, is $r/(s/2) = \sqrt{3}/2$. So, the ratio of r to s is **$\sqrt{3}/4$**.



97. Gaylon uses 8 digits for every date he writes down. If we divide 2018 by 8, we get 252 with a remainder of 2. The 253rd day of the year occurs in the month of September, so Gaylon writes 09 for the month of September. The 2018th digit he writes down is **9**.

98. April has 30 days, so Zeus needs to throw a total of $30 \times 12 = 360$ lightning bolts in all of April. So far he has thrown $7 \times 15 = 105$ lightning bolts. In the next $30 - 7 = 23$ days, he needs to throw $360 - 105 = 255$ lightning bolts. That's an average of about $255 \div 23 \approx$ **11** lightning bolts per day.

99. Both 9 and 27 are powers of 3, so we can rewrite the equation as $(3^2)^{2x^2-6} = (3^3)^{x^2-1}$. Using the laws of exponents, we can rewrite this as $3^{2(2x^2-6)} = 3^{3(x^2-1)} \to 3^{4x^2-12} = 3^{3x^2-3}$. Since the base is 3 on both sides of the equation, the expressions will be equal when their exponents are equal. To determine what positive value of x makes this equation true, we need only solve the equation $4x^2 - 12 = 3x^2 - 3$ to see that $x^2 = 9$, so $x =$ **3**.

100. The increase from 60 to 120 decibels is 60 decibels, which is three increases of 20 decibels. Since every increase of 20 decibels corresponds to sound becoming 10 times as loud, the rock concert must be $10 \times 10 \times 10 =$ **1000** times as loud. as a conversation

Warm-Up 8

101. The sum of any three consecutive integers is divisible by 3. Since 206 is not, we know it is either the sum of the first, second and fourth terms or of the first, third and fourth terms. If we call the first term in Pamela's sequence x, then the four consecutive terms are x, $x + 1$, $x + 2$ and $x + 3$. Based on the two options above, we can have either $x + x + 1 + x + 3 = 206 \to 3x = 206 - 4 \to 3x = 202$ or $x + x + 2 + x + 3 = 206 \to 3x = 206 - 5 \to 3x = 201$. The latter is the only one that yields an integer value for x. Solving, we find $x = 67$. The other integer is $x + 1 =$ **68**.

102. Benjamin is walking up at a rate of 1 flight per 10 seconds, and the escalator is moving down at a rate of 1 flight per 20 seconds. So, his overall rate of progress is $1/10 - 1/20 = 1/20$ flight per second. It takes him 20 seconds to walk up one flight and **40** seconds to walk up two flights on the escalator.

103. The prime factorization of K is equal to the product of the prime factorizations of factors 168 and 900, which is $K = 168 \times 900 = (2^3 \times 3 \times 7) \times (2^2 \times 3^2 \times 5^2) = 2^5 \times 3^3 \times 5^2 \times 7^1$. If we list the available powers of each prime, *including the zero power*, as shown, we can create all the positive divisors of K by multiplying one number from each column in all possible ways. This means there are $6 \times 4 \times 3 \times 2 =$ **144** divisors of K.

1	1	1	1
2	3	5	7
4	9	25	
8	27		
16			
32			

104. The sum of the 7 integers from 1 to 7 is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. Since $28 - 4 = 24$, to get a sum of 24, Emma needs to choose every integer except the 4. By symmetry, the probability that she leaves out the 4 is the same as the probability that she leaves out any one of the 7 numbers, which is **1/7**.

105. Each data set is only three numbers, the profits for three weeks. Since the medians and largest numbers of the two sets are the same, the difference in the means of the sets must depend entirely on the difference in their lowest numbers. Cart A's mean for the three weeks is \$27 more, so the difference in the lowest week's profit values is $3 \times \$27 =$ **\$81**.

106. Since this question is about a ratio, we have the freedom to assign convenient values to the radii. Let's choose radii of 1, 2 and 3 units for the inner circles. This choice makes the radius of the largest circle 6 units and gives it an area of $\pi \times 6^2 = 36\pi$ units^2. The total area of the inner circles is $\pi \times 1^2 + \pi \times 2^2 + \pi \times 3^2 = \pi + 4\pi + 9\pi = 14\pi$ units^2. The shaded area is the difference $36\pi - 14\pi = 22\pi$ units^2. Finally, the fraction that is shaded is $22\pi/36\pi =$ **11/18**.

107. As in Problem 106, since the question is about a ratio, we have the freedom to assign convenient values to the lengths. If we say that the side length of the regular octagon is 1 unit, then four of the shaded isosceles right triangles can be rearranged to form a unit square. Since there are eight of those triangles, the total area of the shaded regions is 2 units^2. The side length of the congruent squares is $1/\sqrt{2} + 1 + 1/\sqrt{2} = \sqrt{2}/2 + 1 + \sqrt{2}/2 = 1 + \sqrt{2}$ units, so the area of one of them is $(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$ units^2. The area of the octagon is 1 unit^2 less than the area of the square, so it's $2+2\sqrt{2}$ units^2. The ratio of the shaded area to the area of the octagon is $2/(2 + 2\sqrt{2}) = 1/(1 + \sqrt{2})$, but this answer is not in simplest radical form. To "rationalize the denominator," we multiply this fraction by a well-chosen form of the number 1, known as the conjugate as follows: $1/(1 + \sqrt{2}) \times (1 - \sqrt{2})/(1 - \sqrt{2}) = (1 - \sqrt{2})/(-1) =$ $\mathbf{\sqrt{2} - 1}$.

108. If we let M be the number of hours it takes Markus to make a basket, then it takes Avi $M + 1/2$ hour to make a basket. After 28 hours, Markus has made $28/M$ baskets and Avi has made $28/(M + 1/2)$ baskets, which is one fewer basket. Therefore, we can write the following equation: $28/M = 28/(M + 1/2) + 1$. We now multiply both sides of this equation by $M(M + 1/2)$, which gives us $28(M + 1/2) = 28M + M(M + 1/2)$. Distributing on both sides, we get $28M + 14 = 28M + M^2 + (1/2)M$, which we simplify to $14 = M^2 + (1/2)M$, which we further simplify to $2M^2 + M - 28 = 0$. The trinomial expression on the left can be rewritten as the following product of two binomials: $(2M - 7)(M + 4) = 0$. This product will equal zero if $2M - 7 = 0$ or $M + 4 = 0$. The solution to the first possibility is $M = 7/2 = 3\ 1/2$ hours, which makes sense. The solution to the second possibility is $M = -4$ hours, which does not make sense. Since it takes Markus 3 1/2 hours to make a basket, it must take Avi **4** hours to make a basket.

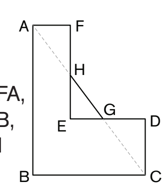
109. We will restate the given properties of N using modular arithmetic. The statement "$N \equiv 1 \pmod{2}$" means that N leaves a remainder of 1 when divided by 2. We know the following: $N \equiv 1 \pmod{2}$, $N \equiv 2 \pmod{3}$, $N \equiv 3 \pmod{5}$ and $N \equiv 5 \pmod{7}$. The first two statements can be rewritten as: $N \equiv -1 \pmod{2}$ and $N \equiv -1 \pmod{3}$. Since 2 and 3 are relatively prime and the remainder is now the same, we can say that $N \equiv -1 \pmod{6}$, which means that N is 1 less than a multiple of 6. Similarly, the last two original statements can be rewritten as $N = -2 \pmod{5}$ and $N = -2 \pmod{7}$. Now since 5 and 7 are relatively prime, we can write $N \equiv -2 \pmod{35}$. We can now just consider a list of numbers that are 2 less than a multiple of 35: 33, 68, 103, 138 and finally 173. Only this last number, **173**, is also 1 less than a multiple of 6, so this is our least possible value of N. (Incidentally, the next value of N would be $173 + 6 \times 35 = 173 + 210 = 383$.)

110. The prime factorization of 1000^3, or 1,000,000,000, is $2^9 \cdot 5^9$. This number has $10 \times 10 = 100$ positive factors. To combine these prime factors in ways that produce perfect square factors, we calculate that each of 5 perfect square powers of 2 (namely 2^0, 2^2, 2^4, 2^6 and 2^8) can be multiplied by each of 5 perfect square powers of 5 (namely 5^0, 5^2, 5^4, 5^6 and 5^8). That's $5 \times 5 = 25$ perfect square factors, which is $25/100 =$ **1/4** of the factors of 1000^3.

Warm-Up 9

111. Since Sola's lucky common fraction is $f = 0.\overline{711}$, $1000 \times f = 711.\overline{711}$. Subtracting the first equation from the second, we get $999 \times f = 711 \rightarrow f = 711/999 =$ **79/111**.

112. The equation for line m can be rewritten in slope-intercept form as $y = 2x - 7/3$. Since line n is perpendicular to m, its slope is the negative reciprocal of 2, which is $-1/2$. Given this slope and the point (6, 2), we can write the equation for line n in point-slope form as $y - 2 = (-1/2)(x - 6) \rightarrow y - 2 = (-1/2)x + 3 \rightarrow y = (-1/2)x + 5$. Using the information given, we can write the equation for line k as $y = 5x + 1$. We want to find the x-coordinate of the point where lines n and k intersect, so we set those two expressions for y equal to each other to get $(-1/2)x + 5 = 5x + 1 \rightarrow -x + 10 = 10x + 2 \rightarrow 8 = 11x \rightarrow x =$ **8/11**.



113. Triangle ABC has side lengths that form the Pythagorean Triple 6-8-10. So AC = 10. If we label the intersection of segment AC with side FE as H and the intersection of segment AC with side DE as G, we see that three triangles all similar to ABC are formed – HFA, HEG and CDG. We can find the lengths of segments AH and GC by setting up ratios using the known lengths. Since HA/AF = AC/CB, we have $HA/2 = 10/6 \rightarrow HA = 20/6 = 10/3$. And since CG/CD = AC/AB, we have $CG/3 = 10/8 \rightarrow CG = 30/8 = 15/4$. The desired fraction is $(10/3 + 15/4) \div 10 = 1/3 + 3/8 =$ **17/24** of the segment AC.

114. We can interpret the reading of the Richter scale as the exponent to which the base 10 is raised. Since 7.5 − 5 = 2.5, a reading of 7.5 is $10^{2.5}$ times stronger than a reading of 5 on the Richter scale. In simplest radical form, that's $10^{2.5} = 10^{5/2} = \sqrt{(10^5)} = \mathbf{100\sqrt{10}}$ times stronger.

115. In a geometric sequence with first term a and common ratio between consecutive terms r, the third term can be expressed as ar^2, and the seventh term can expressed as ar^6. Since we know that $ar^2 = 45$ and $ar^6 = 3645$, we have $ar^6/(ar^2) = 3645/45 \rightarrow r^4 = 81 \rightarrow r = 3$ or $r = -3$. Substituting either value for r yields $a = 5$. The two possible sequences are 5, 15, 45, 135, 405, 1215, 3645 and 5, −15, 45, −135, 405, −1215, 3645. The least possible sum of the first five terms, then, is 5 − 15 + 45 − 135 + 405 = **305**.

116. There are 6 consonant tiles available for the first letter of the word. For each consonant that may be chosen for the first letter, there are 3 vowel tiles available for the second letter, 2 vowel tiles available for the third letter and 5 consonant tiles available for the last letter. The number of four-letter words that can be formed, then, is 6 × 3 × 2 × 5 = **180** words.

117. Let s represent the constant speed at which the girls run. Rebecca runs for 13 minutes, or 13/60 hour at speed s, and Susan runs for 7 minutes, or 7/60 hour at speed s. Rebecca covers $(13/60)s$ miles, and Susan covers $(7/60)s$ miles, so they run a total distance of $(20/60)s = (1/3)s$ miles. When they meet, they will have covered a total of 2 miles. Therefore, $(1/3)s = 2 \rightarrow s = \mathbf{6}$ mi/h.

118. Some Mathletes may find the 8-15-17 right triangle that has an area of (1/2) × 8 × 15 = 60 units^2. But there is a Heronian triangle with a smaller area. Heron's formula for the area of a triangle with side lengths a, b and c and semiperimeter s is $A = \sqrt{[s(s - a)(s - b)(s - c)]}$. We can let $a = 17$ and try positive integer values for b and c that are less than 17 and have a sum greater than 17. The first sum greater than 17 is 18, but this would give us a semiperimeter of 35/2. This won't work since we are looking for an integer area. The next sum greater than 17 is 19. This gives us a semiperimeter of 36/2 = 18. This might work. What integer values could we use for the two sides that sum to 19? There is only one combination that will give us an integer area. The triangle with sides 17, 9 and 10 has an area of $\sqrt{[18(18 - 17)(18 - 9)(18 - 10)]} = \sqrt{(18 \times 1 \times 9 \times 8)} = \sqrt{(1296)} = \mathbf{36}$ units^2

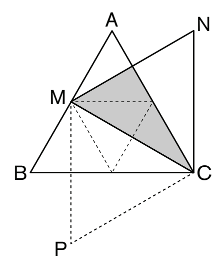
119. The first eight primes are 2, 3, 5, 7, 11, 13, 17 and 19. Each of these primes has two positive factors, 1 and itself. So, the sum of all the numbers Brian writes down is 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + (1 × 8) = 77 + 8 = **85**.

120. If we add the two equations, we get $2\sqrt{x} = 24 \rightarrow \sqrt{x} = 12 \rightarrow x = 144$. Substituting into the first equation, we get $12 - \sqrt{y} = 10 \rightarrow -\sqrt{y} = -2 \rightarrow y = 4$. The sum is $x + y = 144 + 4 = \mathbf{148}$.

Warm-Up 10

121. Since (1!)! × (2!)! × (3!)! × (4!)! = 1! × 2! × 6! × 24!, there is no need to expand this product further. The greatest prime factor of the product is a factor of 24! = 24 × 23 × 22 × ⋯ × 2 × 1. It is **23**.

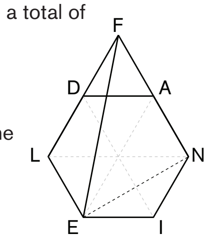
122. In the half hour with the wall charger, Anita's cell phone battery charged 0.5/1.5 = 1/3 of a complete charge. In the 1 hour with her computer, the phone battery charged an additional 1/3 of a complete charge, bringing the total to 1/3 + 1/3 = 2/3 of a complete charge. To charge the remaining 1/3, it will take 1/3 of the 5 hours required to fully charge her phone battery using the portable charger. That's 5/3 hours, which is 5/3 × 60 = **100** minutes.



123. We can solve this problem geometrically if we draw the additional equilateral triangle CMP and introduce a few more lines. Triangle ABC is now subdivided into 8 congruent 30-60-90 triangles, 3 of which are inside of △CMN, so the fraction is **3/8**. For an algebraic solution, we could assign a side length of 2 units to △ABC. Then segment AM would be 1 unit, segment MC would be $\sqrt{3}$ units, and the area of △ABC would be $1/2 \times 2 \times \sqrt{3} = \sqrt{3}$ units^2. The side length of △CMN would be $\sqrt{3}$ units, its altitude would be $\sqrt{3} \times \sqrt{3}/2 = 3/2$, and its area would be $1/2 \times \sqrt{3} \times 3/2 = (3\sqrt{3})/4$ units^2. Half of triangle CMN is in △ABC, which is an area of $(3\sqrt{3})/8$ units^2. The fraction of the area of △ABC inside △CMN, then, is $(3\sqrt{3})/8 \div \sqrt{3} = \mathbf{3/8}$.

124. Factoring, we get $3^7 - 27 = 3^7 - 3^3 = 3^3(3^4 - 1) = 27(81 - 1) = 27 \times 80 = 2^4 \times 3^3 \times 5$. The greatest prime factor is **5**.

125. There are 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40,320 ways to arrange 8 differently covered balls with no restrictions. Since we only want one of the 3 × 2 × 1 = 6 ways that the red, green and yellow balls can be ordered, we divide this 40,320 by 6 to get **6720** ways. *Alternative solution:* We could, instead, choose the 3 spots for the red, green and yellow balls as a group, out of the 8 positions. This can be done in ${}_8C_3 = 8!/(5! \times 3!) = (8 \times 7 \times 6)/(3 \times 2 \times 1) = 8 \times 7 = 56$ ways. Since the remaining 5 balls can be ordered in 5! ways, we have a total of 56 × 5! = **6720** ways.



126. This regular hexagon can be subdivided into six equilateral triangles, each of side length 6 units, as shown. Triangle DAF is congruent to one of these six triangles. So, triangle DAF has side length 6 units and FN = 2 × 6 = 12 units. Segment EN is twice the altitude of one of the equilateral triangles. So, $EN = 2 \times (6\sqrt{3})/2 = 6\sqrt{3}$ units. We can calculate the length of segment FE using the Pythagorean Theorem to get $FE = \sqrt{[12^2 + (6\sqrt{3})^2]} = \sqrt{(144 + 108)} = \sqrt{(252)} = \mathbf{6\sqrt{7}}$ units.

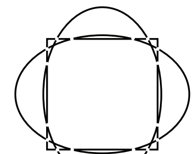
127. Since Annette can pick 4 baskets in one hour, she must have picked 2 baskets in the half hour the three girls worked together. Similarly, Mary must have picked 2.5 baskets in the half hour. That means Lynn picked the remaining 6 − 2 − 2.5 = 1.5 baskets in the half hour. By herself, Lynn must be able to pick 2 × 1.5 = 3 baskets per hour, so she can pick 3 × 3 = **9** baskets in 3 hours.

128. No matter how Kayla divides the stones, it will take 14 divisions to end up with 15 piles of one stone, and the sum of all 14 products will always be the same. (This is worth trying to prove!) We will systematically separate 1 stone at a time to get the following sum of products: $14 + 13 + 12 + \cdots + 2 + 1 = (14 \times 15) \div 2 = 7 \times 15 =$ **105**.

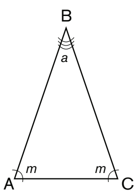
129. The diameter of the sphere is equal to the side length s of the cube. The volume of the cube is s^3, and the volume of the sphere is $4/3 \times \pi \times (s/2)^3 = \pi s^3/6$. The ratio of the volume of the cube to the volume of the sphere is $s^3/(\pi s^3/6) =$ **6/π**. Note that because the sphere is inside the cube, the ratio has to be greater than 1, of course. It is interesting that the ratio is almost 2 (because $\pi \approx 3.1$), meaning the sphere takes up only a little more than half the space inside the cube.

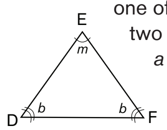
130. To get the maximum possible value of $\#(5x) - \#(4x)$, we will use the greatest possible value of x, which is something slightly less than 30. If x were equal to 30, then we would have $\#(5x) = \#(5 \times 30) = \#150$. The greatest even integer less than 150 is 148. If x is some real number slightly less than 30, we will still get 148. Similarly, for $\#(4x)$, we get $\#(4 \times 30) = \#120 = 118$. The difference is $148 - 118 =$ **30**.

Warm-Up 11

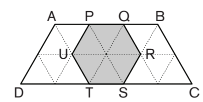


131. Each ellipse can intersect the square in 8 different places, and the two ellipses can intersect each other in 4 different places, so there are a maximum of $2 \times 8 + 4 =$ **20** points of intersection.





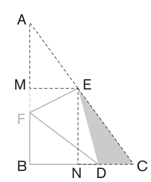
132. In order to have only three different angle measures in the two triangles, there must be two angles of degree measure m in one of the triangles and one angle of degree measure m in the other triangle, as shown. Let a and b represent the other two angle measures. Since the sum of the three different angle measures is 156, we can write the following equations: $a + 2m = 180$, $m + 2b = 180$, $a + m + b = 156$. If we subtract the third equation from the first, we get $m - b = 24 \rightarrow m = b + 24$. Substituting this value for m in the second equation yields $3b + 24 = 180 \rightarrow 3b = 156 \rightarrow b = 52$. We can now substitute this value of b back into the second equation to get $m = 180 - 2 \times 52 =$ **76** degrees. We should also note that $a + 2(76) = 180 \rightarrow a + 152 = 180 \rightarrow a = 28$.



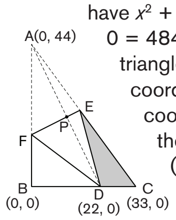
133. As the figure shows, trapezoid ABCD can be subdivided into 16 congruent equilateral triangles. The hexagon PQRSTU accounts for 6/16 = **3/8** of the area of the trapezoid.

134. Since 2, 3 and 5 are relatively prime, the least common multiple is $2 \times 3 \times 5 = 30$. From 30 up to 990 there are $960 \div 30 = 32$ multiples of 30, 33 multiples if we include 990. These are the only integers that are multiples of 2, 3 and 5. From this list, we need to eliminate the multiples of $4 \times 30 = 120$ that are divisible by 8. There are 8 of them. That leaves $33 - 8 =$ **25** integers.

135. Since the individual digits are small, we don't have to worry about the sum in any place value "carrying over" to the next place value. For any given value of A, there is exactly 1 out of the 4 values for E that will make a sum of 5. The same reasoning applies to each of the four place values in the sum ABCD + EFGH. Since the probability in each place value is 1/4 that we get a sum of 5, the probability is 3/4 that we won't get a sum of 5. The probability that we will not get any sums of 5 is $3/4 \times 3/4 \times 3/4 \times 3/4 = 81/256$. So, the probability that we will get a sum of 5 must be $1 - 81/256 =$ **175/256**.



136. Right triangle ABC has sides in the ratio 3:4:5. We can draw two similar triangles by adding two additional segments to our figure. From point E, extend a line parallel to side BC to intersect with AB at point M. Then again from point E, extend a line parallel to side AB to intersect with side BC at point N. Triangle AME is similar to triangle ABC. If we label the length ME as x, then AM is $(4/3)x$. The length EN is $44 - (4/3)x$ and ND is $22 - x$. We know that AE and ED are equal since segment ED was created from folding triangle AEF down. We can use the Pythagorean Theorem for triangles AME and END and solve for x. We have $x^2 + ((4/3)x)^2 = (22 - x)^2 + (44 - (4/3)x)^2 \rightarrow x^2 + (16/9)x^2 = 484 - 44x + x^2 + 1936 - (352/3)x + (16/9)x^2 \rightarrow 0 = 484 - 44x + 1936 - (352/3)x \rightarrow (484/3)x = 2420 \rightarrow x = 15$. Now that we have x, we know that $EN = 44 - 4/3 \times 15 = 24$. So triangle CDE, with base 11 cm, has altitude 24 cm and area $1/2 \times 11 \times 24 =$ **132** cm^2. *Alternative solution:* If we place the figure on a coordinate plane, the coordinates of the various points are A(0,44), B(0,0), C(33,0), D(22,0). Let P be the midpoint of AD; then P has coordinates that are the average of the coordinates of A and the coordinates of D, so P = (11, 22). The slope of line AD is −2, so the slope of line FE is 1/2. Because line FE passes through P, we can write its equation using point-slope form as $y - 22 = (1/2)(x - 11)$. We can also write the equation of side AC as $x/33 + y/44 = 1$. This system can be solved to get the intersection point E as (15, 24). So the area of triangle CDE is $(1/2) \times 11 \times 24 =$ **132** cm^2.



137. Using p for the number of passes after the first eight games, we have $(0.7p + 49)/(p + 50) = 0.74 \rightarrow 0.7p + 49 = 0.74p + 37 \rightarrow 0.04p = 12 \rightarrow p = 300$. So, in the first eight games Jason threw 300 passes, and we are told that, in the ninth game, he threw 50 passes. In total, Jason threw $300 + 50 =$ **350** passes.

138. The seven numbers must be distinct positive integers with a mean of 20. That means the sum of the seven integers must be 140. If the set of integers is arranged in ascending order, the median is the fourth number. The set {1, 2, 3, 4, 14, 16, 100} works, with the least possible value of the fourth number being 4, which is the least possible median. Now to maximize the fourth number, we make the first three numbers as small as possible, giving them values of 1, 2 and 3. That leaves a sum of $140 - (1 + 2 + 3) = 140 - 6 = 134$ to be split among the remaining four integers in the set of seven integers. Since the integers all must be distinct, let's consider four consecutive integers with a sum of 134. We have $n + (n + 1) + (n + 2) + (n + 3) = 134 \rightarrow 4n + 6 = 134 \rightarrow 4n = 128 \rightarrow n = 32$. The set {1, 2, 3, 32, 33, 34, 35} works, and the greatest possible median, then, is 32. That's a difference of $32 - 4 =$ **28**.

139. The product of the digits will equal the units digit if and only if the tens digit is a 1 or the units digit is a zero. The **18** integers that meet this criterion are 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 40, 50, 60, 70, 80, and 90.

140. Only perfect squares have an odd number of positive integer divisors. The only perfect squares she could roll are 1 and 4. So, on any roll of the die, the probability is 2/6 = 1/3 that she will step 1 meter to the right and 2/3 that she will step 1 meter to the left. There are six ways that Colleen could end up right where she started after four rolls: RRLL, RLRL, RLLR, LRRL, LRLR, LLRR. In each case the probability is 1/3 × 1/3 × 2/3 × 2/3 = 4/81, for a total probability of 6 × 4/81 = **8/27**.

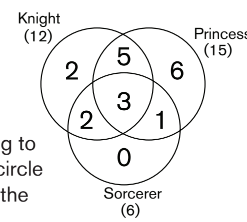
Warm-Up 12

141. There are 50 multiples of 2 from 1 to 100, inclusive. The sum of these numbers is 2 + 4 + 6 + ⋯ + 98 + 100 = (2 + 100) × 50 ÷ 2 = 102 × 25 = 2550. There are 33 multiples of 3 from 1 to 100, inclusive. The sum of these numbers is 3 + 6 + 9 + ⋯ + 96 + 99 = (3 + 99) × 33 ÷ 2 = 51 × 33 = 1683. The absolute difference is 2550 − 1683 = **867**.

142. The general form of a cubic polynomial is $p(x) = ax^3 + bx^2 + cx + d$. We have four specific values for $p(x)$, so we can set up a system of four equations and solve for a, b, c and d. First, we have $p(0) = 4$, so $a(0^3) + b(0^2) + c(0) + d = 4 \rightarrow d = 4$. Then, we have $p(1) = 10$, so $a(1^3) + b(1^2) + c(1) + 4 = 10 \rightarrow a + b + c = 6$. Next, we have $p(-1) = 2$, so $a(-1)^3 + b(-1)^2 + c(-1) + 4 = 2 \rightarrow -a + b - c = -2$. Finally, we have $p(2) = 26$, so $a(2^3) + b(2^2) + c(2) + 4 = 26 \rightarrow 8a + 4b + 2c = 22$. Our system of equations includes $a + b + c = 6$, $-a + b - c = -2$ and $8a + 4b + 2c = 22$. If we add the first and the second equations, we get $2b = 4 \rightarrow b = 2$. Substituting 2 into the first and third equations, we now have $a + 2 + c = 6 \rightarrow a + c = 4$ and $8a + 4(2) + 2c = 22 \rightarrow 8a + 8 + 2c = 22 \rightarrow 8a + 2c = 14 \rightarrow 4a + c = 7$. Subtracting these two resulting equations yields $4a + c - (a + c) = 7 - 4 \rightarrow 3a = 3 \rightarrow a = 1$. Substituting, we see that $1 + c = 4 \rightarrow c = 3$. We can now substitute these values for a, b, c and d in the general cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ to get $p(x) = x^3 + 2x^2 + 3x + 4$. So, $p(3) = (3^3) + 2(3^2) + 3(3) + 4 = 27 + 2(9) + 9 + 4 = 27 + 18 + 13 =$ **58**.

143. The Pythagorean Theorem tells us that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. Since we are dealing with an obtuse triangle, the sum of the squares of the shortest two sides will be less than the square of the longest side. In other words, we want $a^2 + b^2 < c^2$. To achieve the greatest possible perimeter for our obtuse triangle, we will make $c = 100$ and find the greatest integer values of a and b that satisfy this inequality. The square of 100 is 10,000. We want a and b to be nearly equal, so we are looking for a perfect square near 5000. Since $70^2 = 4900$, we'll let $a = 70$ and $b = 71$. Then $70^2 + 71^2 = 4900 + 5041 = 9941$, which is less than 10,000. Since $71^2 + 72^2 = 5041 + 5184 = 10,225$, which is greater than 10,000, we see that the greatest possible perimeter is 70 + 71 + 100 = **241** inches.

144. Each of the 25 staff members gives a fist bump to 24 coworkers, but this counts each fist bump twice, so the total is 25 × 24 ÷ 2 = **300** fist bumps.



145. A Venn diagram can help us solve this problem. We draw three overlapping circles, one labeled for each of the three roles knight, princess and sorcerer. Let's start with the 3 students who were willing to play any of the three roles. This number goes in the center, where all three circles overlap. Since there are 8 students willing to play either the knight or the princess, there must be 8 − 3 = 5 who were willing to play the knight or the princess but not the sorcerer. By similar reasoning, we find that there are 5 − 3 = 2 who were willing to play the knight or the sorcerer but not the princess, and 4 − 3 = 1 who was willing to play the princess or the sorcerer but not the knight. Finally, we can fill in the remaining numbers in the regions that are in one circle only. That's 12 − (3 + 5 + 2) = 12 − 10 = 2 willing to play only the knight, 15 − (3 + 5 + 1) = 15 − 9 = 6 willing to play only the princess, and 6 − (3 + 2 + 1) = 6 − 6 = **0** willing to play only the sorcerer.

hop combinations	hop sequences	# ways
1-1-1-1-1-1-1-1	7!/(7!)	1
1-1-1-1-1-1-2	7!/(6! × 1!)	7
1-1-1-1-1-3	6!/(5! × 1!)	6
1-1-1-1-2-2	6!/(4! × 2!)	15
1-1-1-2-3	5!/(3! × 1! × 1!)	20
1-1-2-2-2	5!/(2! × 3!)	10
1-1-3-3	4!/(2! × 2!)	6
1-2-2-3	4!/(1! × 2! × 1!)	12
2-2-2-2	4!/(4!)	1
2-3-3	3!/(1! × 2!)	3

146. The table shown demonstrates a method of counting all the combinations and sequences of 1, 2 or 3 hops Frankie can use to jump 8 units forward. This is calculated using the formula for the number of permutations of n objects with n_1 identical objects of type 1, n_2 identical objects of type 2, …, and n_k identical objects of type k, which is $n!/[(n_1)! \times (n_2)! \times \cdots \times (n_k)!]$. In total, he can hop from 0 to 8 in 1 + 7 + 6 + 15 + 20 + 10 + 6 + 12 + 1 + 3 = **81** ways.

147. The sum of the five integers is $10 + 12 + 26 + x + x = 48 + 2x$. If the mean is 10, then we have $(48 + 2x)/5 = 10 \rightarrow 48 + 2x = 50 \rightarrow x = 1$. The five numbers, in ascending order, are 1, 1, 10, 12 and 26, which satisfies the condition that the median also be 10. If the mean is 12, then we have $(48 + 2x)/5 = 12 \rightarrow 48 + 2x = 60 \rightarrow x = 6$. The five numbers, in ascending order, are 6, 6, 10, 12 and 26, which does not give us a median of 12. So, x cannot be 6. If the mean is 26, then we have $(48 + 2x)/5 = 26 \rightarrow 48 + 2x = 130 \rightarrow x = 41$. The five numbers, in ascending order, are 10, 12, 26, 41 and 41, which satisfies the condition that the median also be 26. Finally, if the mean is x, then we have $(48 + 2x)/5 = x \rightarrow 48 + 2x = 5x \rightarrow 48 = 3x \rightarrow x = 16$. The five numbers, in ascending order, are 10, 12, 16, 16 and 26, which satisfies the condition that the median also be 16. The sum of the possible values of x is 1 + 41 + 16 = **58**.

(3, 97)	(17, 83)	(41, 59)
(11, 89)	(29, 71)	(47, 53)

148. Each of the 6 ordered pairs shown can be reversed for a total of **12** ordered pairs of primes with a sum of 100.

149. Using the given information, we can set up two equations. Because we know the area is 10 cm², we can write $1/2 \times b \times h = 10 \rightarrow bh = 20$. We also know this is a right triangle with hypotenuse length 10 cm. This gives us the equation $b^2 + h^2 = 10^2$. If we double the first equation and add it to the second, we have $b^2 + 2bh + h^2 = 140 \rightarrow (b + h)^2 = 140 \rightarrow b + h = \sqrt{140} = 2\sqrt{35}$. The perimeter of the triangle is $\mathbf{10 + 2\sqrt{35}}$ cm.

150. If we multiply the second equation by 4 and add it to the first equation, we eliminate the y term completely and get the much simpler equation $18/(x + 1) = 18/3$. Since the numerators are equal, the denominators must also be equal. So, $x + 1 = 3 \rightarrow x = 2$. Going back to the original equations, we can double the first equation and subtract it from the second equation to eliminate x completely and get $(-18)/(y - 3) = (-18)/3$. Again, since the numerators are equal, the denominators must also be equal. So, $y - 3 = 3 \rightarrow y = 6$. Therefore, $x + y = 2 + 6 =$ **8**. *Alternative solution:* Let $u = 3/(x + 1)$ and $v = 3/(y - 3)$. That yields the much simpler system of equations $2u + 8v = 10$ and $4u - 2v = 2$. Solving this system leads to $u = v = 1$ and then $x = 2$ and $y = 6$. Again, the result is $x + y = 2 + 6 =$ **8**.

Warm-Up 13

151. The numbers could all be 1, resulting in the ordered triple (1, 1, 1). The numbers could all be 0, resulting in the ordered triple (0, 0, 0). Since if one of the numbers is 0, they all must be 0 and otherwise they are all nonzero, the first two equations imply that $n^2 = 1$ and $n = 1$ or $n = -1$. Two of the numbers could be -1 and the third could be 1, resulting in the ordered triples $(-1, -1, 1)$, $(-1, 1, -1)$ and $(1, -1, -1)$. These are the **5** possible ordered triples.

152. By the properties of 30-60-90 right triangles, we know that an equilateral triangle with side length 6 cm has altitude $3\sqrt{3}$ cm. The rectangle with the least area that can enclose this equilateral triangle would have dimensions $a = 6$ cm and $b = 3\sqrt{3}$ cm. The product of the side lengths, then, is $ab = 6 \times 3\sqrt{3} = \mathbf{18\sqrt{3}}$ cm^2.

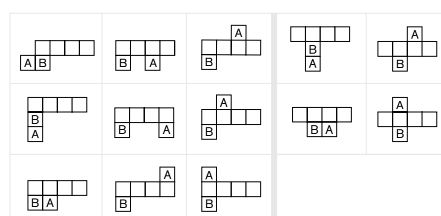
153. Since $44^2 = 1936 < 2018 < 2025 = 45^2$, we know that there are 44 positive integers less than 2018 that are perfect squares. Since $12^3 = 1728 < 2018 < 2197 = 13^3$, we know that there are 12 positive integers less than 2018 that are perfect cubes. If we were to list all these numbers, those that are perfect sixth powers would appear on both lists. Since $3^6 = 729 < 2018 < 4096 = 4^6$, we know that there are 3 positive integers less than 2018 that are perfect sixth powers. Therefore, the total number of positive integers less than 2018 that are perfect squares or perfect cubes is $44 + 12 - 3 =$ **53** integers.

154. Let's try to count all the possibilities in an organized manner. Alexander can get 12 of one type of cookie in 3 ways. He can get 11 of one type and 1 of another type in 3 ways. If there are 12 glazed, then there is only 1 way to complete the order with chocolate and cherry. If there are 11 glazed, then there are 2 ways (1 chocolate or 0 chocolate). If there are 10 glazed, then there are 3 ways. And so on, until if there are 0 glazed, then there are 13 ways (anywhere from 12 to 0 chocolate). So the answer is $1 + 2 + \ldots + 13 = 13 * 14 / 2 =$ **91** assortments. *Alternative solution:* Let's, instead, use the counting technique known as "stars and bars." The idea is that we arrange 12 stars to represent the cookies and two bars to separate the cookies into the three different categories. For example, the arrangement ****|**|****** represents the possibility that Alexander buys 4 of the first type of cookie, 2 of the second type, and 6 of the third type. Our question now becomes: How many ways can we arrange the 12 stars and 2 bars? Thus, the number of assortments of a dozen cookies he can buy is ${}_{14}C_2 = 14!/(12! \times 2!) = (14 \times 13)/(2 \times 1) = 7 \times 13 =$ **91** assortments.

155. That Gabriel and Isabel each start with 20 coins is not important. After the exchange, Gabriel has 23 coins and Isabel has 17 coins, and Gabriel has twice as much money as Isabel. Since we are looking for the greatest possible combined value, let's suppose that Gabriel has 23 quarters. This would be \$5.75 which is not an even number, so we can swap one quarter for a dime, which gives Gabriel \$5.60. If this works, Isabel would have to have \$2.80. She could have 8 quarters, 7 dimes and 2 nickels, which is 17 coins, but we want the fewest dimes Isabel could have. If she has 9 quarters, 3 dimes and 5 nickels, that's 17 coins worth \$2.80. We can't do any better than that, so **3** dimes is the fewest dimes Isabel could have.

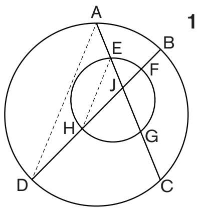
156. We are looking for the least common multiple of 130 and 365. The prime factorization of 365 is 5×73. We know 130 has 5 as a factor, but not 73. The LCM is $130 \times 73 =$ **9490**.

157. The **13** distinct hexominoes that have exactly four squares in a row are shown. To find them all, we systematically "moved" square A around the figure, leaving square B in place for the first 9 hexominoes. Then we shifted square B to the right one unit and "moved" square A around the figure, keeping only the new arrangements that were not duplicates.



158. Since the side length of the hexagon is 6, we know that the height of the square is 6. By properties of 30-60-90 right triangles, we know that the heights of the hexagon and small equilateral triangle are $6\sqrt{3}$ and $3\sqrt{3}$, respectively. The height of the large equilateral triangle, then, is $6\sqrt{3} + 6 + 3\sqrt{3} = 6 + 9\sqrt{3}$. To find the base length of the large equilateral triangle, we divide its height by $\sqrt{3}/2$ to get $(6 + 9\sqrt{3}) \div (\sqrt{3}/2) = (6 + 9\sqrt{3}) \times 2/\sqrt{3} = 12/\sqrt{3} + (18\sqrt{3})/\sqrt{3} = 4\sqrt{3} + 18$. The area of the small equilateral triangle is $1/2 \times 6 \times 3\sqrt{3} = 9\sqrt{3}$. The area of the square is $6 \times 6 = 36$. The area of the hexagon is six times the area of the small equilateral triangle, or $6 \times 9\sqrt{3} = 54\sqrt{3}$. The area of the large equilateral triangle is $1/2 \times (4\sqrt{3} + 18) \times (6 + 9\sqrt{3}) = 1/2 \times (24\sqrt{3} + 108 + 108 + 162\sqrt{3}) = 1/2 \times (216 + 186\sqrt{3}) = 108 + 93\sqrt{3}$. Now we subtract from this the sum of the areas of the three small shapes to get $(108 + 93\sqrt{3}) - (9\sqrt{3} + 36 + 54\sqrt{3}) = 72 + 30\sqrt{3} = a + b\sqrt{c}$. Thus, $a + b + c = 72 + 30 + 3 =$ **105**.

159. We will write our ordered pairs in alphabetical order as (A, B, M, N). Abhi can have any of the prime numbers 2, 3, 5 or 7, and Noreen can have any of the perfect squares 1, 4 or 9. If Bryan has half of Abhi's number, then Abhi has to have 2 and Bryan has 1. That gives us (2, 1, 3, 4), (2, 1, 5, 4), (2, 1, 6, 4), (2, 1, 3, 9) and (2, 1, 10, 9), where Meghna gets the sum of two other values, as long as the sum does not exceed 10. If Bryan has half of Noreen's number, then Noreen has to have 4 and Bryan has 2. That gives us (3, 2, 5, 4), (3, 2, 6, 4), (3, 2, 7, 4), (5, 2, 6, 4), (5, 2, 7, 4), (5, 2, 9, 4), (7, 2, 6, 4) and (7, 2, 9, 4). Finally, if Bryan has half of Meghna's value, then Meghna's value has to be the sum of Abhi's and Noreen's values and this sum must be even. The possibilities are (2, 3, 6, 4), (3, 2, 4, 1), (5, 3, 6, 1) and (7, 4, 8, 1). That's 5 + 8 + 4 = **17** assignments.

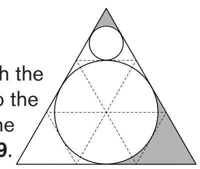


160. The measure of an inscribed angle is half the measure of the arc that it intercepts. Since arc EF has measure 10 degrees, the measure of angle EHF is 10 ÷ 2 = 5 degrees. By the same reasoning, the measure of angle GEH is 98 ÷ 2 = 49 degrees. These two angles are two of the three angles in triangle EJH, so the measure of the third angle, EJH, must be 180 – (5 + 49) = 180 – 54 = 126 degrees. This is the same angle that is in triangle AJD, so the measure of angle AJD is also 126 degrees. Angle JAD is the same as angle CAD, which has measure 83 ÷ 2 = 41.5 degrees. The remaining angle in triangle AJD must have measure 180 – (126 + 41.5) = 180 – 167.5 = 12.5 degrees. This angle, ADJ, is half the measure of the arc AB, so the measure of arc AB is 2 × 12.5 = **25** degrees.

Warm-Up 14

161. The prime factors of *n* must include the prime factors of 22 and 14, which include 11 and 7. These two primes are not close enough together to be among three consecutive integers. Since we know that *n* is a multiple of 22, let's consider 21 and 22, which are consecutive and have factors of 7 and 11. We are looking for the least possible value of *k*, a factor of *n*, so let the three consecutive positive integers be 20, 21 and 22. Then their product is 20 × 21 × 22 = 9240, and *k* would have to be 9240 ÷ (22 × 14) = **30**.

162. In the middle column and middle row, 1 + 7 + 13 + 19 + 25 = 10 + 24 + 13 + 2 + 16 = 65. Since 65 must also be the sum of the numbers in every other row and column, we can determine that the missing numbers are 20, 15, 11 and 6. Their sum is **52**. *Alternative solution:* The sum of the numbers in the third row of the magic square is 10 + 24 + 13 + 2 + 16 = 65. The sums of the numbers in the first, second, fourth and fifth rows, respectively, are 45, 50, 54, 59. Since each of these rows should sum to 65, the sum of the missing numbers is 4 × 65 – (45 + 50 + 54 + 59) = **52**.



163. The larger equilateral triangle can be subdivided into 9 equilateral triangles that are all congruent to the equilateral triangle with the smaller circle inscribed in it, as shown in the figure. The centroid of the larger equilateral triangle is 1/3 of the way up the triangle, so the top of the larger circle is 2/3 of the way up. That means the horizontal tangent at the top of the circle is 1/3 of the way down from the top. The ratio of the areas of shaded regions K and P is the same as the ratio of the area of the smaller circle to the larger circle, **1/9**.

164. We will start with the sum of all the numbers, subtract the sum of the multiples of 2 and 3 and then add back the multiples of 6, which will have been subtracted twice. The sum of all the numbers from 2 to 99 is (2 + 99) × 98 ÷ 2 = 101 × 49 = 4949. The sum of the multiples of 2 strictly between 1 and 100 is 2 + 4 + 6 + ⋯ + 96 + 98 = (2 + 98) × 49 ÷ 2 = 50 × 49 = 2450. The sum of the multiples of 3 between 1 and 100 is 3 + 6 + 9 + ⋯ + 96 + 99 = (3 + 99) × 33 ÷ 2 = 51 × 33 =1683. The sum of the multiples of 6 between 1 and 100 is 6 + 12 + 18 + ⋯ + 90 + 96 = (6 + 96) × 16 ÷ 2 = 51 × 16 = 816. The sum we want is 4949 – 2450 – 1683 + 816 = **1632**.

165. There is a 1/3 probability that James will win on his first spin. For him to win on his second spin, James would have to get a PASS, then John would have to get a PASS and then James would have to get WIN, so the probability is 2/3 × 2/3 × 1/3 = 4/27. If we continue this reasoning, we get an infinite sum that looks like this: 1/3 + [$(2/3)^2$ × 1/3] + [$(2/3)^4$ × 1/3] + [$(2/3)^6$ ×1/3] + ⋯. This is an infinite geometric series, and there is a formula that many Mathletes know and can use to calculate the answer. Let's look for a simpler way to think about it. In any given round, there is a 1/3 = 3/9 chance that James wins with his spin and a 2/3 × 1/3 = 2/9 chance that John wins. There is a 2/3 × 2/3 = 4/9 chance that neither of them wins and they play again. The insight is to realize that we can completely ignore the double-pass situations and just look at the odds of James winning to John winning in any round, which is 3/9 to 2/9. Since the odds are 3 to 2, then the probability that James wins is **3/5** and the probability that John wins is 2/5.

166. Rewriting the equation of the line in slope-intercept form, we get $y_1 = (1/3)x + 7/3$. Side AB, with slope (6 – 0)/(2 – 0) = 3 and *y*-intercept 0 is given by $y_2 = 3x$. We can set the expressions for y_1 and y_2 equal to each other to determine the *x*-coordinate of the point of intersection N. We get $(1/3)x + 7/3 = 3x \rightarrow x + 7 = 9x \rightarrow 7 = 8x \rightarrow x = 7/8$. Substituting back into the equation for y_2, we see that $y_2 = 3(7/8) = 21/8$. So the line intersects side AB at N(7/8, 21/8). Side BC, with slope (2 – 6)/(8 – 2) = −4/6 = −2/3 and endpoint (2, 6), is given by the equation $y_3 - 6 = (-2/3)(x - 2) \rightarrow y_3 - 6 = (-2/3)x + 4/3 \rightarrow y_3 = (-2/3)x + 22/3$. We can set the expressions for y_1 and y_3 equal to each other to determine the *x*-coordinate of the point of intersection M. We get $(1/3)x + 7/3 = (-2/3)x + 22/3 \rightarrow x = 15/3 = 5$. Substituting back into the equation for y_3, we see that $y_3 = (-2/3)(5) + 22/3 = -10/3 + 22/3 = 12/3 = 4$. So the line intersects side BC at M(5, 4). The length of segment MN is $\sqrt{[(5 - 7/8)^2 + (4 - 21/8)^2]} = \sqrt{[(33/8)^2 + (11/8)^2]} = \sqrt{[(1089 + 121)/64]} = \sqrt{(1210)}/8 =$ $\mathbf{(11\sqrt{10})/8}$ units.

167. If Edna starts with the first bottle on the left and samples every bottle, she will eventually taste a bitter poison and know that the previous bottle contains the magic potion. If Edna samples every even-numbered bottle, she will have to check the odd-numbered bottle right before the first bitter poison she gets. If the odd bottle is tasteless, then it contains the magic potion; if it's bitter, then she will become sick and unable to drink anything else, but she will have succeeded in being able to identify the bottle that contains the magic potion. Likewise, if Edna samples every third bottle, she can go back after she gets the first bitter bottle, but she has to go back to check the bottle after the last tasteless bottle. If we continue with this line of reasoning, we realize that Edna can skip-count by any number she wants. As soon as she gets to a bitter bottle, she has to go back to the bottle that comes right after the last tasteless bottle and check every bottle until she gets to another bitter one. At this point, some students will arrive at the idea of using the square root of 1000, which is a little more than 31. Edna could set out to check multiples of 31 until she comes to a bitter bottle. Edna would check bottles 31, 62, 93, …, up until 992, which is 32 × 31. In the worst-case scenario, bottle 991 would contain the magic potion. Edna would find bottle 992 to be the first bitter one she samples and she would have to go back to check bottles 962 through 991, which would all be tasteless. She would then know that bottle 991 contains the magic potion, but she would have sampled 32 + 30 = 62 bottles. (The total is the same if she counts by 32s instead of 31s.) This is pretty good, but it turns out that Edna can do better. She can take bigger steps at the beginning and reduce the size of her jump as she goes. If she reduces the jump by 1 each time, then what we want is a triangular number that is close to 1000. The 45th triangular number is 45 × 6 ÷ 2 = 1035, so Edna should first sample bottle 45, then bottle 45 + 44 = 89, then bottle 89 + 43 = 132, etc. At any point, when she samples a bitter bottle, she will have to go back and check the bottles in between, and the total number of bottles sampled is always **45** bottles.

short side	long side	qty
1	$\sqrt{2}$	36
2	$2\sqrt{2}$	25
3	$3\sqrt{2}$	16
4	$4\sqrt{2}$	9
5	$5\sqrt{2}$	4
6	$6\sqrt{2}$	1
$\sqrt{2}$	2	30
$2\sqrt{2}$	4	15
$3\sqrt{2}$	6	4

168. It's fairly easy to see the silver rectangles in the figure that have the following dimensions: $1\times\sqrt{2}$, $2\times2\sqrt{2}$, $3\times3\sqrt{2}$, $4\times4\sqrt{2}$, $5\times5\sqrt{2}$ and $6\times6\sqrt{2}$. Less obvious, however, are the silver rectangles in the figure that are $\sqrt{2}\times2$, $2\sqrt{2}\times4$ and $3\sqrt{2}\times6$. For all these rectangles, the ratio of the length of the short side to the length of the long side is exactly $1:\sqrt{2}$. You might miss the last three sizes if you fail to realize that $\sqrt{2}/2 \times \sqrt{2}/\sqrt{2} = 2/(2\sqrt{2}) = 1/\sqrt{2}$, $(2\sqrt{2})/4 \times \sqrt{2}/\sqrt{2} = 4/(4\sqrt{2}) = 1/\sqrt{2}$ and $(3\sqrt{2})/6 \times \sqrt{2}/\sqrt{2} = 6/(6\sqrt{2}) = 1/\sqrt{2}$. The table shows the number of silver rectangles in each of these nine sizes. There are a total of 36 + 25 + 16 + 9 + 4 + 1 + 30 + 15 + 4 = **140** silver rectangles.

169. Since Du passes Priya just as he completes his 4th lap, Priya must be finishing her 3rd lap. At the same moment, Amanda must be finishing 3 1/2 laps, which is the average of 3 and 4 laps. If we double the time, then Du will complete his 8th lap, Amanda will complete her 7th lap, and Priya will complete her 6th lap at the same moment. Since the difference between Amanda and Priya is 1 lap, it must be that the first time she passes Priya, Amanda has completed **7** laps.

170. We saw in problem 163 that the ratio of the areas of the circles inside the equilateral triangle is 1 to 9. The radius of the middle circle in the current problem is 1/3 of the altitude of the equilateral triangle, so if we call the radius of the middle circle 3 units, then the radius of the smallest circle will be 1 unit and the radius of the largest circle will be $3\sqrt{3}$ units. The ratio of the areas of these circles is equal to the square of the ratio of their radii. So the ratio we want is $1^2/(3\sqrt{3})^2 = 1/(9 \times 3) =$ **1/27**.

Workout 1

171. The 8 possible equally-likely outcomes when flipping three coins are HHH, HHT, HTH, HTT, THH, THT, TTH and TTT. Three of the 8 outcomes have exactly two heads, so the probability is **3/8**.

172. Since 300 km/h = 300 ÷ 60 × 1000 ÷ 60 = 250/3 m/s, it follows that Gary's average rate of acceleration is 250/3 ÷ 30 ≈ **2.8** m/s^2.

173. Gordon's weight on Jupiter would be $100 \times 318 \div 11^2 \approx$ **263** pounds.

174. Half of the sum of the domestic gross and international gross was 550 ÷ 2 = 275, so the greatest advertising budget for financial success was 275 − 140 = **135** million dollars.

175. The mean number of days per month in 2018 is 365 ÷ 12 ≈ **30.4** days.

176. The value of A is $(-2)^2 - 2(-2) + 6 = 4 + 4 + 6 = 14$, and the value of B is $(5 \times (-2)^2 - 1)/(-2 + 3) = (20 - 1)/1 = 19$. Therefore, $A + B = 14 + 19 =$ **33**.

177. With a base diameter of 18 cm, the radius is 18 ÷ 2 = 9 cm, and we are told that the volume of the cone is 1187.5 cm^3. The formula for the volume of a cone is $V = 1/3 \times \pi \times r^2 \times h$. So, we have $1/3 \times \pi \times (9)^2 \times h = 1187.5 \rightarrow 27\pi h = 1187.5 \rightarrow h = 1187.5/(27\pi) \approx$ **14** cm.

178. A regular decagon has exterior angles of 360 ÷ 10 = 36 degrees. So, the interior angles must be 180 − 36 = **144** degrees.

179. A decrease of 20% means the portfolio's value decreased to 80%, or 4/5, of its original value. An increase of 25% means the portfolio's value increased to 125%, or 5/4, of its previous value. The value of Elliott's portfolio at the end of February, then, was 5000 × 4/5 × 5/4 = $**5000**.

180. The decade from January 1, 2011, through December 31, 2020, includes 3 leap days, for the years 2012, 2016 and 2020. The number of days is 10 × 365 + 3 = 3653 days. The total number of hours in the decade is 3653 × 24 = **87,672** hours.

Workout 2

181. If we call the length $16x$ and the height $9x$, then the equation for the area of the screen is $16x \times 9x = 576 \rightarrow 144x^2 = 576 \rightarrow x^2 = 576/144 \rightarrow x^2 = 4 \rightarrow x = 2$. The length of the screen must be 16 × 2 = 32 inches, and its height must be 9 × 2 = 18 inches. The perimeter is 2(32 + 18) = 2(50) = **100** inches.

182. Using Alex's biking pace of 7 minutes per mile and the total round-trip time of 20 + 15 = 35 minutes, we can set up the proportion $7/1 = 35/x$ and solve to see that $7x = 35 \rightarrow x = 5$. So, the round-trip distance is 5 miles, and the trip from home to school is 5 ÷ 2 = **2.5** miles.

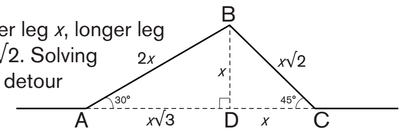
183. Bruce, Lawson and the goal are located at the vertices of a 3-4-5 right triangle. The distance from Lawson to the goal is 50 meters. The puck will reach the goal in 40 ÷ 50 = 0.8 second. To reach the goal at the same time as the puck, Lawson must skate at 50 ÷ 0.8 = **62.5** m/s.

184. To calculate a tax of 9% on a bill of $619, we multiply to get 0.09 × 619 = $**55.71**.

185. If we say Sara gets x dollars per cake, then Linda gets $x - 11$ dollars per cake. For them to earn the same amount each hour, the following must be true: $4x = 6(x - 11)$. Solving for x, we get $4x = 6x - 66 \rightarrow 2x = 66 \rightarrow x = 33$. Sara must be paid $**33** per cake.

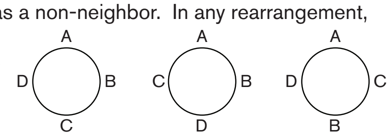
186. If Jason wants the mean of his six drives to be 400 yards, the sum of his drives must be 6 × 400 = 2400 yards. So far the sum of his drives is 394 + 401 + 387 + 414 + 421 = 2017 yards. The last drive must be at least 2400 − 2017 = **383** yards. *Alternative solution:* If we consider each distance relative to 400 yards, then the result of the first five drives is −6 + 1 − 13 + 14 + 21 = 17 yards. To ensure a mean of at least 400 yards for all six drives, the sixth drive must be at least −17, or 400 − 17 = **383** yards.

187. The distance from A to C along the original route is 1000 km. The 30-60-90 right triangle ABD has shorter leg x, longer leg $x\sqrt{3}$ and hypotenuse $2x$. The 45-45-90 right triangle BCD has two legs of length x and hypotenuse of length $x\sqrt{2}$. Solving the equation $x\sqrt{3} + x = 1000$, we get $x(\sqrt{3} + 1) = 1000 \rightarrow x = 1000/(\sqrt{3} + 1)$. So the difference between the detour and the intended route is $2(1000/(\sqrt{3} + 1)) + (1000/(\sqrt{3} + 1))\sqrt{2} - 1000 \approx$ **250** km.



188. This is calculated as follows: 0.12 × 3/4 × 1.8 = **0.162**.

189. When four people sit around a circular table, each person has two neighbors, leaving only one person as a non-neighbor. In any rearrangement, there must be at least one repeat neighbor. As the figure shows, from the perspective of person A, we basically give each of persons B, C and D a turn at being the non-neighbor across from A. Thus, four people can be seated in **3** ways.



190. The ratio of the larger radius to the smaller radius is 4.5/3 = 3/2. This is the scale factor between the two similar containers. The volume of the larger container will be greater by a factor that is the cube of this scale factor. We are told that the volume of the smaller container is 20 fluid ounces, so the volume of the larger container will be $20 \times (3/2)^3 = 20 \times 27/8 = 135/2 =$ **67.5** fluid ounces.

Workout 3

191. The number pattern is 15, 17, 20, 22, 25, 27, …. We can separate this into the number pattern of the odd rows, which is 15, 20, 25, …, and the pattern of the even rows, which is 17, 22, 27, …. In both cases, the terms increase by 5 each time, so the last term will be 14 × 5 = 70 more than the first term. The number of seats in the 15 odd rows is 15 × (15 + 85) ÷ 2 = 15 × 100 ÷ 2 = 15 × 50 = 750 seats, and the number of seats in the 15 even rows is 15 × (17 + 87) ÷ 2 = 15 × 104 ÷ 2 = 15 × 52 = 780 seats. The total number of seats in all 30 rows is 750 + 780 = **1530** seats.

192. A pentagon can be subdivided into three triangles, so the interior angles' sum must be 3 × 180 = 540 degrees. The three known angles have a sum of 110 + 120 + 130 = 360, so the remaining two angles must have a sum of 540 − 360 = 180. Since one of the remaining angles is three times the measure of the other, we can divide the 180 degrees by 4 to get the smallest angle of **45** degrees. The other angle is 180 − 45 = 135 degrees.

193. The possible prime values for a are 2, 3, 5 and 7. The possible composite values for b are 4, 6, 8, 9 and 10. The possible perfect square values for c are 1, 4 and 9. The possible perfect cube values for d are 1 and 8. If we choose the greatest from each of these lists, we get the distinct numbers $a = 7$, $b = 10$, $c = 9$ and $d = 8$. The greatest possible sum of four numbers chosen as described is 7 + 10 + 9 + 8 = **34**.

194. If p and q are both odd primes, then the product pq would also be odd, and r would have to be even. Since the only even prime is 2, the product pq would have to be 73 − 2 = 71, but 71 is prime. This means that either p or q will have to be 2 so that we get an even product pq. We'll let $p = 2$. Now, in order to minimize the sum $p + q + r$, we want q, which gets doubled, to be a large prime so that r can be small. If $q = 31$, we get $2 \times 31 + 11 = 73$, and $p + q + r = 2 + 31 + 11 =$ **44**.

195. If we let B be Bella's speed, T be Thomas's speed and M be Tam's speed, then we have the two equations $B = 0.4T$ and $B = 0.35M$, which means that $0.4T = 0.35M \rightarrow M = (0.4/0.35)T \rightarrow M = (8/7)T$. Tam is 1/7 ≈ 0.14 = **14**% faster than Thomas.

196. There are three possibilities that qualify as rain on at least one of the two days. First, the probability that there is rain on both days is 0.5 × 0.4 = 0.20. Second, the probability that it rains on the first day but not on the second day is 0.5 × 0.6 = 0.30. Third, the probability that it does not rain on the first day but does rain on the second day is 0.5 × 0.4 = 0.20. The combined probability for rain on at least one day, then, is 0.2 + 0.3 + 0.2 = 0.7, which is **70%**. *Alternative solution:* We could have calculated the probability that it does not rain on either day, which is 0.5 × 0.6 = 0.3, and subtracted this from 1 to get 1 − 0.3 = 0.7, or **70**%.

197. A 60 degree sector of a circle accounts for 60/360 = 1/6 of the circle's area. So, the area of the 60 degree sector of a circle of radius 30 feet is $(1/6) \times \pi \times 30^2 =$ **150π** ft^2.

198. Half of $700 million is $350 million, and this amount was made 365 times. The weight of that many $1 bills was 350,000,000 × 365 = 127,750,000,000 grams. Since 1000 grams is equivalent to a kilogram, that's 127,750,000,000 ÷ 1000 = **127,750,000** kg.

199. The graph of the given cubic function is an "S" shape rotated 90 degrees. If we factor $B(n)$, we get $B(n) = n(45 - n)(45 + n)$. The zeros of this function will be −45, 0 and 45. So the maximum we are looking for occurs somewhere between $x = 0$ and $x = 45$. We would expect the maximum to occur close to the middle of the interval. Since 22.5 is the exact middle, we can start by looking at 22 and 23. We see that the function value increases from 22 to 23, so let's continue until we see a decrease. The function begins to decrease between 26 and 27. The greatest harvest value is at **26** trees per acre as shown in the table.

n	$B(n)$
22	33,902
23	34,408
24	34,776
25	35,000
26	35,074
27	34,992

200. There were p cupcakes three days ago. Two days ago, 20% of the cupcakes were eaten, so $(4/5) \times p$ cupcakes remained. We don't know how many cupcakes there were yesterday, but let's say that the number from two days ago got multiplied by some factor x, so that there were $(4/5) \times xp$ cupcakes yesterday. Today, there are 30% fewer cupcakes than yesterday, or $(7/10) \times (4/5) \times xp = (14/25) \times xp$ cupcakes. We are told that this quantity is equal to $(1/2) \times p$. We can solve the equation $(14/25) \times xp = (1/2) \times p$ to get $(14/25)x = 1/2 \rightarrow x = 25/28$. Thus, we need to find a number that is a multiple of 5 (so that there were a whole number of cupcakes three days ago) and 4/5 of which is a multiple of 28, which means that it is a multiple of $28 \div 4 = 7$. The smallest number that is a multiple of both 5 and 28 is $5 \times 28 = 70$. If $p = 70$, then the numbers of cupcakes on the last few days were: 70, 56, 50, 35. We get half of p for today, so it works. The least possible value of p is **70**.

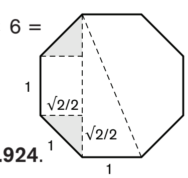
Workout 4

201. Multiplying 15 times the units digits 1, 2, 3, 4, 5 or 6 will result in a two-digit number. Here are the first six multiples of 15: 15, 30, 45, 60, 75 and 90. Since $75 = 15 \times 5$ and $7 + 5 = 12$, we conclude that the two-digit integer in question is **75**.

202. The trinomial $x^2 + 8x + 15$ can be factored into the product of two binomials $(x + 3)(x + 5)$. So, $(x^2 + 8x + 15)/(x + 5) = 4.01 \rightarrow$ $(x + 3)(x + 5)/(x + 5) = 4.01 \rightarrow x + 3 = 4.01 \rightarrow x =$ **1.01**.

203. Since the first and second ordered pairs both have 2 as the x-coordinate, we can use the side that connects these two points as a base. It has length $2 - (-6) = 8$ units. The altitude from the base to the third point has length $2 - (-5) = 7$ units. The area of the triangle is $(1/2) \times 8 \times 7 =$ **28** units^2.

204. The sum of the interior angles of a regular n-gon is $180 \times (n - 2)$. So the sum of the interior angles of a regular octagon is $180 \times 6 = 1080$. Each interior angle has degree measure $1080 \div 8 = 135$. In the figure, the two shaded triangles are 45-45-90 right triangles with hypotenuse length 1 unit. Using the properties of 45-45-90 right triangles, we can determine that the length of each leg is $(1/\sqrt{2}) \times (\sqrt{2}/\sqrt{2}) = \sqrt{2}/2$ units. As the figure shows, the medium diagonal has length $\sqrt{2}/2 + 1 + \sqrt{2}/2 = \sqrt{2} + 1$ units. To find the length of the long diagonal, we use the Pythagorean Theorem as follows: $\sqrt{(1^2 + (\sqrt{2} + 1)^2)} = \sqrt{(1 + (2 + 2\sqrt{2} + 1))} = \sqrt{(4 + 2\sqrt{2})}$ units. The ratio of the length of the medium diagonal to the length of the long diagonal, then, is $(\sqrt{2} + 1)/(\sqrt{(4 + 2\sqrt{2})}) \approx$ **0.924**.



205. If the cost of the meal was x dollars, then the meal plus the first 18% tip was $1.18x$. The cost after the second tip was added was $1.15 \times 1.18x = 1.357x$. Emalee actually paid a **35.7%** tip. Note that the answer is not 18 + 15 = 33%, because the second tip was 15% of the larger amount $1.18x$.

206. From the first "no," Penner learns that the card is neither red nor a multiple of 2. That eliminates all 10 red cards and the 5 even cards from each of the other colors. The 15 remaining cards are odd numbers that are blue, green or yellow. From the next answer, "yes," Penner learns that the card is either blue or a multiple of 3. The 9 remaining cards are odd blues, the green 3 and 9 and the yellow 3 and 9. From the next answer, "no," Penner learns that the card is neither green nor a multiple of 5. That eliminates all the green cards and the blue 5, leaving the 1, 3, 7 and 9 that are blue and the 3 and 9 that are yellow. Finally, from the last answer, "yes," Penner learns that the card is either yellow or a multiple of 7. That leaves the 7 that is blue and the 3 and 9 that are yellow. With these 3 cards to choose from, the probability that Penner guesses Tell's secret card is **1/3**.

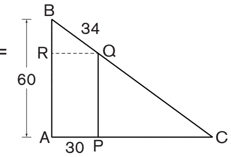
207. There are 90 two-digit numbers, so the denominator of our probability is 90. There are 9 two-digit numbers with digits that have an absolute difference of 0, namely 11, 22, 33, 44, 55, 66, 77, 88 and 99. Now we just need to count the numbers with digits that have an absolute difference of 1, such as 10 and 12. There are 2 of these in every "decade" except the 90s, so there must be $2 \times 9 - 1 = 17$ of them. Since there are $9 + 17 = 26$ numbers that have digits with an absolute difference of 0 or 1, there must be $90 - 26 = 64$ numbers that have digits with an absolute difference greater than 1. The probability is, thus, $64/90 =$ **32/45**.

208. Ngorongoro Crater is 10 miles across, so its radius is 5 miles. Its depth of 2000 feet is $2000/5280 = 25/66$ mile. The formula for the volume of a cylinder of radius r and height h is $\pi \times r^2 \times h$. This particular cylindrical crater has a volume of $\pi \times 5^2 \times (25/66)$. So, it will take $625\pi/66 \approx$ **30** mi^3 of water to fill the crater.

209. If there are x candies in each of the 25 bags on the second day, that would be $25x$ candies in all, which equals the original w candies. When Mady redistributes the candies into 26 bags on the third day, there are $x - 2$ candies in each bag, so we have the equation $26(x - 2) = 25x \rightarrow 26x - 52 = 25x \rightarrow x = 52$. The value of w must be $25 \times 52 =$ **1300**.

210. If $f(x) = ax^2 + bx + c$ and $f(0) = 4$, then $a(0)^2 + b(0) + c = 4 \rightarrow c = 4$. We are told that $f(2) = 2$. Substituting 4 for c, we get $a(2)^2 + b(2) + 4 = 2 \rightarrow 4a + 2b + 4 = 2 \rightarrow 4a + 2b = -2 \rightarrow 2a + b = -1$. Similarly, since $f(4) - f(3) = 4$, we have $(a(4)^2 + b(4) + 4) - (a(3)^2 + b(3) + 4) = 4 \rightarrow 16a + 4b + 4 - 9a - 3b - 4 = 4 \rightarrow 7a + b = 4$. Subtracting the two equations yields $(7a + b) - (2a + b) = 4 - (-1) \rightarrow 7a + b - 2a - b = 4 + 1 \rightarrow 5a = 5 \rightarrow a = 1$. Substituting this back into $2a + b = -1$ yields $2(1) + b = -1 \rightarrow b = -3$. So, $f(x) = x^2 - 3x + 4$ and $f(1) = (1)^2 - 3(1) + 4 = 1 - 3 + 4 =$ **2**.

Workout 5

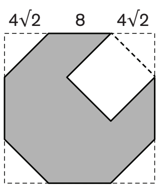


211. If we draw segment RQ parallel to AC, as shown, then we create triangle RBQ, which is similar to triangle ABC. We know that BQ = 34 and RQ = AP = 30. Using the Pythagorean Theorem, we find that BR = $\sqrt{(34^2 - 30^2)} = \sqrt{(1156 - 900)} = \sqrt{256}$ = 16. Since AB = 60, it follows that AR = PQ = 60 − 16 = 44. Triangles ABC and PQC are similar, so the ratios of corresponding sides are equal. In particular, AB/PQ = AC/PC → 60/44 = (30 + PC)/PC → 15/11 = (30 + PC)/PC → 15 × PC = 11 × (30 + PC) → 15PC = 330 + 11PC → 4PC = 330 → PC = 82.5. So, AC = 30 + 82.5 = 112.5, and triangle ABC has area 1/2 × 112.5 × 60 = **3375** units^2.

212. Since all ten digits must be used in the five two-digit numbers, the greatest range between the first and the last number will occur in a list such as 98, 76, 54, 32, 10, and the range is 98 − 10 = 88. The least positive range will occur in a list such as 50, 46, 37, 28, 19, and the range is 50 − 19 = 31. The difference between the greatest and the least possible ranges is 88 − 31 = **57**.

213. There are 6! = 720 ways to arrange the donuts and 6! = 720 ways to arrange the cookies. Mackenzie also has to decide whether to start with a donut or a cookie, so there are 2 × 720 × 720 = **1,036,800** ways to create an arrangement with alternating donuts and cookies.

214. Since the *n*th worker takes *n* hours to complete the job alone, that worker completes exactly 1/*n* of the job each hour. Collectively, *k* people must complete 1 + 1/2 + 1/3 + ⋯ + 1/*k* of the job each hour. The reciprocal of this sum is the fraction of an hour it would take the group to complete 1 job. We want to find the value of *k* for which the previous sum is first greater than 3, since its reciprocal would be less than 1/3 of an hour, or 20 minutes. That occurs when *k* = 11 and 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + 1/11 ≈ 3.02. This means 11 people could do more than 3 of the jobs in an hour, so each job would take less than 20 minutes. The fewest workers needed, then, to complete the job in under 20 minutes, working together, is **11** workers.



215. Since the perimeter of the octagon is 64 cm, each side of the octagon has length 64 ÷ 8 = 8 cm. If we draw a square around the octagon, as shown, the 45-45-90 triangles in the four corners have hypotenuse length 8 cm and side length $8/\sqrt{2} \times \sqrt{2}/\sqrt{2} = 8\sqrt{2}/2 = 4\sqrt{2}$ cm. The enclosing square has side length $4\sqrt{2} + 8 + 4\sqrt{2} = 8 + 8\sqrt{2}$ cm and area $(8 + 8\sqrt{2})^2 = 64 + 128\sqrt{2} + 128 = 192 + 128\sqrt{2}$ cm^2. The four triangles in the corners can be combined to make a single 8-by-8 square, so we need to subtract 64 cm^2 from the area of the square to get the area of the octagon, which is $192 + 128\sqrt{2} - 64 = 128 + 128\sqrt{2}$ cm^2. Finally, we need to subtract the area of the 8-by-8 cutout square to get a total area of $128 + 128\sqrt{2} - 64$ = **$64 + 128\sqrt{2}$** cm^2.

216. Of the 16 distinct sets of inputs that can be applied on the far left of the function machine, (1, 1, 1, 1), (1, 1, 1, 0) and (1, 1, 0, 1) are the only **3** inputs that result in an output of 1.

217. If the amount Bob paid was *x*, then Joe and Randell each paid 2*x*. Twenty-five percent, or 1/4, of $80 is 80 ÷ 4 = $20. So, the group paid a total of 80 + 20 = $100. We have 2*x* + 2*x* + *x* = 100 → 5*x* = 100 → *x* = 20. Bob must have paid $**20**, and Joe and Randell each paid $40.

218. The left side of the equation $a^2 - b^2$ is called a difference of two squares and can be factored so that we get the equivalent equation $(a - b)(a + b) = 144$. We now consider the following factor pairs of 144: 1 × 144, 2 × 72, 3 × 48, 4 × 36, 6 × 24, 8 × 18, 9 × 16, 12 × 12. We are looking for the pairs of factors that are the difference and sum of two positive integers *a* and *b*. Consider $a - b = 1$ and $a + b = 144$. If we add these equations, we get $2a = 145 \rightarrow a = 72.5$, and $b = 144 - 72.5 = 71.5$. The table shows the values of *a* and *b* for each factor pair of 144. There are **4** positive integer pairs whose squares have a difference of 144

$a - b$	$a + b$	a	b
1	144	72.5	71.5
2	72	37	35
3	48	25.5	22.5
4	36	20	16
6	24	15	9
8	18	13	5
9	16	12.5	3.5
12	12	12	0

219. The formula for the volume of a rectangular prism is $V = l \times w \times h$. Since 1 m = 100 cm and 10 mm = 1 cm, the volume of this box is 0.5 × 1 × 100 = **50** cm^3.

220. Let *c* represent the number of cars there used to be on campus, and *b* represent the number of bicycles there used to be. Then the current number of cars is 0.7*c*, and the current number of bicycles is 1.2*b*. We are told that there is a 1:3 ratio between the numbers of cars and bicycles. So, 1/3 = 0.7*c*/1.2*b* → *c*/*b* = 1.2/2.1 = 12/21 = **4/7**.

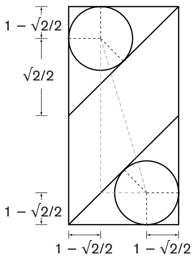
Workout 6

221. If we let 2*x* inches equal the side length of the square, then the distance from the center of the square to the midpoint of a side is *x* inches. Using the properties of 45-45-90 right triangles, we can determine that the distance from the center to a corner of the square is $x\sqrt{2}$ inches. A radius of the circle passes through each corner of the square and has length $x\sqrt{2} + 1$ inches. A radius that passes through the midpoint of a side of the square is *x* + 2 inches. Since all radii are equal, it follows that $x\sqrt{2} + 1 = x + 2 \rightarrow x\sqrt{2} - x = 1 \rightarrow x(\sqrt{2} - 1) = 1 \rightarrow x = 1/(\sqrt{2} - 1) \times (\sqrt{2} + 1)/(\sqrt{2} + 1) = \sqrt{2} + 1$. The circle has radius $x + 2 = \sqrt{2} + 1 + 2 = \sqrt{2} + 3$ inches and area $\pi \times (\sqrt{2} + 3)^2 = (11 + 6\sqrt{2})\pi$ in^2. The square has side length $2x = 2 \times (\sqrt{2} + 1) = 2\sqrt{2} + 2$ inches and area $(2\sqrt{2} + 2)^2 = 12 + 8\sqrt{2}$ in^2. The area of the gasket is the difference $((11 + 6\sqrt{2})\pi) - (12 + 8\sqrt{2})$ ≈ **37.9** in^2.

222. Getting half off the regular price on half the socks is the same as getting one quarter off the regular price on all the socks. Jeffrey paid 3/4, or 75% of the regular price for all the socks. That's a savings of **25**%.

223. The median of 10 numbers is the average of the 5th and 6th numbers when they are listed in order. The lowest possible median of a list of positive integers is 1. We can get a median of 1 if we include six 1s. We then want to raise the mean as much as possible by including four 20s. Our data set is {1, 1, 1, 1, 1, 1, 20, 20, 20, 20}, with a median of 1 and a mean of 86 ÷ 10 = 8.6. The absolute difference between the median and the mean is 8.6 – 1 = **7.6**. We get the same answer with the set {1, 1, 1, 1, 20, 20, 20, 20, 20, 20}, which has a median of 20 and a mean of 124 ÷ 10 = 12.4. The absolute difference is again 20 – 12.4 = **7.6**.

224. Let x represent the number of tourists beyond the first 15 tourists. From the information provided, we have $(x + 15)(520 - 5x) = 12{,}740 \rightarrow 520x - 5x^2 + 7800 - 75x = 12{,}740 \rightarrow 5x^2 - 445x + 4940 = 0 \rightarrow x^2 - 89x + 988 = 0$. Factoring, we see that $(x - 76)(x - 13) = 0 \rightarrow x - 76 = 0 \rightarrow x = 76$ or $x - 13 = 0 \rightarrow x = 13$. Since the maximum number of tourists is 36, our solution is $x = 13$, meaning there were 15 + 13 = **28** tourists.



225. The two lines drawn to the midpoint of each side of the rectangle create two 45-45-90 right triangles. As the figure shows, radii drawn to each circle's points of tangency with the sides of the right triangle form a square and two congruent kites. Each isosceles right triangle has leg length 1 inch and hypotenuse length $\sqrt{2}$ inches. The long side of each kite has length $\sqrt{2}/2$ inch, making the radius of each circle $1 - \sqrt{2}/2$ inch. The vertical distance between the centers of the circles is $2 - 2(1 - \sqrt{2}/2) = 2 - 2 + \sqrt{2} = \sqrt{2}$ inches. The horizontal distance between the centers of the circles is $1 - 2(1 - \sqrt{2}/2) = 1 - 2 + \sqrt{2} = \sqrt{2} - 1$ inch. Using the Pythagorean Theorem, we see that the distance between the centers of the circles is $\sqrt{((\sqrt{2})^2 + (\sqrt{2} - 1)^2)} = \sqrt{(2 + 2 - 2\sqrt{2} + 1)} = \sqrt{(5 - 2\sqrt{2})} \approx$ **1.47** inches.

226. If we consider the smallest rectangle to be a 1×1 square, we can make an organized list to count the number of rectangles in the figure. For each size, the number of rectangles is as follows: 1×1 - <u>12</u>, 1×2 - <u>6</u>, 2×1 - <u>6</u>, 1×3 - <u>4</u>, 3×1 - <u>4</u>, 1×4 - <u>2</u>, 4×1 - <u>2</u>, 2×2 - <u>1</u>, 2×3 - <u>2</u>, 3×2 - <u>2</u>, 2×4 - <u>1</u>, 4×2 - <u>1</u>, 3×3 - <u>4</u>, 3×4 - <u>2</u>, 4×3 - <u>2</u>, 4×4 - <u>1</u>. That's a total of 12 + 6 + 6 + 4 + 4 + 2 + 2 + 1 + 2 + 2+ 1 + 1 + 4 + 2 + 2 + 1 = **52** rectangles.

227. The set of all three-digit integers that are perfect squares and whose digits, when reversed, also form a perfect square is {121, 144, 169, 441, 484, 676, 961}. The sum of these integers is 121 + 144 + 169 + 441 + 484 + 676 + 961 = **2996**.

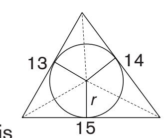
228. The ratio of Shandra's weight to her little sister's weight is 96/72 = 4/3. For the seesaw to be perfectly balanced, the fulcrum should be positioned so that the 14-foot beam is split in a 3 to 4 ratio. Thus the fulcrum should be placed at a distance of 3/7 ×14 = **6** feet from Shandra, which results in a distance of 4/7 × 14 = 8 feet from her sister. (Notice that 96 × 6 = 72 × 8 = 576.)

229. Since we know that the quantities are distinct and pairwise relatively prime, the least we can hope for is 1 + 2 + 3 + 5 = 11. Let's see if we can find a scenario that works with those quantities and totals \$1.08. We can start by letting $p = 3$. That leaves 1.08 – 0.03 = \$1.05 left to make with only nickels, dimes and quarters. If we then let $n = 1$, $d = 5$ and $q = 2$, our total is \$1.08. Thus, the least possible value of $p + n + d + q$ is **11**.

230. The 12-foot altitude of this right square pyramid meets the square base at its center, which is 5 feet away from the midpoint of a side of the square base. This forms a 5-12-13 triangle, which is a Pythagorean Triple. The 13-foot side is the slant height of each of the four triangular faces of the pyramid. The combined area of the four triangular faces is 4 × 1/2 × 10 × 13 = 260 ft^2. The area of the 10-foot by 10-foot square base is 10 × 10 = 100 ft^2. The total surface area of the pyramid, then, is 260 + 100 = **360** ft^2.

Workout 7

231. The least common multiple (LCM) of 10, 12 and 18 is 180. The monetary value of 180 quarters is 180 ÷ 4 = **\$45**.



232. We can use Heron's formula to find the area of the triangle given its three side lengths. A triangle with side lengths a, b and c and with semiperimeter $s = (a + b + c)/2$ has area $A = \sqrt{(s(s - a)(s - b)(s - c))}$. This triangle has side lengths 13, 14 and 15 units. So, $s = (13 + 14 + 15)/2 = 42/2 = 21$ units. The area of the triangle, then, is $\sqrt{(21(21 - 13)(21 - 14)(21 - 15))} = \sqrt{(21 \times 8 \times 7 \times 6)} = \sqrt{(7 \times 3 \times 2 \times 2 \times 2 \times 7 \times 3 \times 2)} = \sqrt{(7^2 \times 3^2 \times 4^2)} = \sqrt{(7 \times 3 \times 4)^2} = 7 \times 3 \times 4 = 84$ units^2. As the figure shows, the total area of this triangle is the sum of the areas of the three triangles with bases of lengths 13, 14 and 15 units, each with altitude equal to the radius of the circle. So, we have $1/2 \times 13 \times r + 1/2 \times 14 \times r + 1/2 \times 15 \times r = 84 \rightarrow (1/2)r \times 42 = 84 \rightarrow (1/2)r = 2 \rightarrow r =$ **4** units.

233. There are 24 hours in a day, so 2^{18} hours from any day is $2^{18} \div 24 = 32{,}768/3 = 10{,}922\ 2/3$ days later. Since Francisco is born at 1:00 a.m., the extra 2/3 of a day is still the same day, 2/3 × 24 = 16 hours later, at 5:00 p.m. We just need to determine the value of 10,922 (mod 7), which is the remainder when 10,922 is divided by 7. Since 10,920 is a multiple of 7, it follows that 10,922 (mod 7) ≡ 2. Since Francisco is born on Tuesday, he gets married on the day of the week that is 2 days after Tuesday, which is **Thursday**.

234. Erica will have two consecutive numbers if and only if she draws a 3 or a 5 of any suit. There are four suits, so Erica can draw any of 8 cards out of the remaining 51 cards in the deck. The probability is **8/51**.

235. There are 5 numbers from 3 to 7, namely 3, 4, 5, 6 and 7. A set of 5 elements has $2^5 = 32$ subsets. Excluding the null set and the complete set, there are at most 30 possibilities to test. If we consider the clues carefully, we should not have to try all 30 subsets. We might notice that statements 5 and 7 cannot both be true. The only composite numbers available are even, and any even factor will make the product even. This means we cannot have both 5 and 7 in the set. Furthermore, if statement 3 is correct, then we can't have either 5 or 7 since two odds would make the sum even. If we left 3 out of the set and just included 5 or 7, then statement 3 would be correct, in which case 3 should be included. If we try to exclude all the odd numbers, then statement 6 would be false and the set would contain only the number 4. But that would make statement 5 true, which is a contradiction. All this means that 3 must be the only odd and only prime number in the set. If 3 is the only prime, then statement 6 is correct and we include 6. Now we have to include 4 also, so that statement 5 will be false. The set must be $S = \{3, 4, 6\}$, the statements are true, true, false, true, false in that order, and the product of the numbers in S is $3 \times 4 \times 6 =$ **72**.

236. We can divide the circle into 6 congruent sectors, each composed of an equilateral triangle and a shaded segment. The area of the shaded segment is the difference between the area of the sector, which is 1/6 the area of the circle, and the area of the equilateral triangle. Since the regular hexagon is composed of 6 congruent equilateral triangles of side length 12 meters, it follows that the radius of the circle also is 12 meters. Thus, the area of each sector is $1/6 \times 12^2 \times \pi = 24\pi$ m^2. Using properties of 30-60-90 right triangles, we see that the area of one equilateral triangle is $1/2 \times 12 \times 6\sqrt{3} = 36\sqrt{3}$ m^2. If follows, then, that the area of one segment is $24\pi - 36\sqrt{3}$ m^2. The total area of the shaded regions is $6 \times (24\pi - 36\sqrt{3}) = 144\pi - 216\sqrt{3} \approx$ **78** m^2.

237. To get the greatest possible value of $k - m$, we want to make m as small as possible. Working backward, we can subtract 6 from 200 a total of 33 times and get 2 (because $200 - 33 \times 6 = 2$). If we continue to subtract 6, the result is a negative number. But Mason must take an even number of steps, since Kendra takes only half as many steps. Let's have Mason start at 8 and count up by 6 a total of 32 times. Then Kendra would have to start at $200 - 16 \times 4 = 136$. This gives us the greatest possible difference $k - m = 136 - 8 =$ **128**.

238. If we rewrite the equation $3x - 4y = 13$ in slope-intercept form, we get $y = 0.75x - 3.25$. The y-intercept is -3.25. To translate the line 2018 units to the right, we substitute $x - 2018$ for x and get $y = 0.75(x - 2018) - 3.25 \rightarrow y = 0.75x - 1513.5 - 3.25 \rightarrow y = 0.75x - 1516.75$. So, the y-intercept of the translated line is **−1516.75**.

239. If we subtract Ron's base salary from this week's earnings, we get $383.75 - 215 = \$168.75$, which must have been 15% of his sales this week. Now we can set up the percent proportion $168.75/x = 15/100 \rightarrow 15x = 168.75 \times 100 \rightarrow 15x = 16{,}875 \rightarrow x = 1125$. This means that Ron sold \$1125 worth of wallets during his five days of work this week. His average daily sales must have been $1125 \div 5 =$ **\$225**.

240. A regular hexagon is made up of six equilateral triangles, so the area of each triangle must be $216\sqrt{3} \div 6 = 36\sqrt{3}$ in^2. The area of an equilateral triangle of side length s is $\sqrt{3}/4 \times s^2$. We have $36\sqrt{3} = \sqrt{3}/4 \times s^2 \rightarrow s^2 = 144 \rightarrow s = 12$ inches. This is also the radius of the circle that circumscribes the hexagon. A circle with a radius of 12 inches has a diameter of 24 inches and a circumference of **24π** inches.

Workout 8

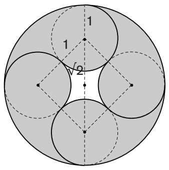
241. Comparing 5 & 3 = 18 and 10 & 3 = 72, we see that the result of (2×5) & 3 is $72 \div 18 = 4$ times the result of 5 & 3. This suggests that the first number might get squared, which means that $m = 2$. Next, comparing 5 & 3 and 5 & 6, we see that the result of 5 & (2×3) is $36 \div 18 = 2$ times as great as 5 & 3. This suggests that the second number might be raised to the first power, which means that $n = 1$. Substituting 2 for m and 1 for n in the expression $k \times A^m \times B^n$ for $A = 5$ and $B = 3$ yields $k \times 5^2 \times 3^1 = 18 \rightarrow 75k = 18 \rightarrow k = 18/75 = 6/25$. Let's confirm this with 10 & 3 and 5 & 6. We have $k \times 10^2 \times 3^1 = 72 \rightarrow 300k = 72 \rightarrow k = 72/300 = 6/25$ and $k \times 5^2 \times 6^1 = 36 \rightarrow 150k = 36 \rightarrow k = 36/150 = 6/25$. Now that we've confirmed that $k = 6/25$, we can calculate the value of 10 & 6 $= (6/25) \times 10^2 \times 6^1 = 3600/25 =$ **144**.

242. This is another problem that can be solved using the "stars and bars" technique. We imagine a line of 15 stars that represent the coins and 3 bars that represent partitions that separate the coins into the 4 groups of coins in the order stated: pennies, nickels, dimes, quarters. We now determine the number of permutations of 18 objects with 15 identical stars and 3 identical bars. There are $18!/(15! \times 3!) = (18 \times 17 \times 16)/(3 \times 2 \times 1) = 3 \times 17 \times 16 =$ **816** possible combinations.

a	r	ar	ar^2	$a + ar + ar^2$
1	2	2	4	7
1	3	3	9	13
1	4	4	16	21
1	5	5	25	31
1	6	6	36	43
2	2	4	8	14
2	3	6	18	26
2	4	8	32	42
3	2	6	12	21
3	3	**9**	**27**	**39**
4	3/2	6	9	19
4	2	8	16	28
4	3	12	36	52
4	5/2	**10**	**25**	**39**
5	3	15	45	65

243. There are 12 edges on a rectangular prism, but there are only three different lengths, so the sum of these three different lengths must be $3 \times 13 = 39$. We need to find three distinct positive integers that are in geometric progression and have a sum of 39. If we let a represent the least integer, then the next two integers are ar and ar^2. We know that $a + ar + ar^2 = 39$. The table shows the results of our systematic guess-and-check strategy. It turns out that, although the three dimensions have to be integers, the value of r does not have to be an integer. In particular, when a is a perfect square, we can divide by a factor twice, so r can be a rational number. In any case, the two possible sets of dimensions are (3, 9, 27) and (4, 10, 25). The prism has volume $a \times ar \times ar^2 = (ar)^3$. So, in the first case, with $a = 3$ and $r = 3$, the prism has volume $(3 \times 3)^3 = 9^3 = 729$ in^3. In the second case, with $a = 4$ and $r = 5/2$, the prism has volume $(4 \times 5/2)^3 = 10^3 = 1000$ in^3. The desired sum is **1729** in^3.

244. The length of the lower left beam is divided so that the right and left sides are in the ratio 2:3, with a total of 16 pounds hanging from the left side. To maintain balance, the weight associated with each side is inversely proportional to the length of its beam. It follows, then, that the total weight on the right side must be 3/2 the total weight on the left, or $3/2 \times 16 = 24$ pounds. This can be accomplished using the fewest weights with a 16-pound weight and an 8-pound weight. Now let's consider the overall mobile, whose top beam is divided so that the right and left sides are in the ratio 5:2, with a total of $16 + 24 = 40$ pounds hanging from the left side. To maintain balance, the total weight on the right side must be 2/5 the total weight on the left side, or $2/5 \times 40 = 16$ pounds. We now need to consider the lower right beam, whose length is divided so that the right and left sides are in the ratio 3:1, with a combined 16 pounds hanging from the pair of sides. To maintain balance, $1/4 \times 16 = 4$ pounds needs to hang from the right side and $3/4 \times 16 = 12$ pounds needs to hang from the left. This can be accomplished using the fewest weights by hanging an 8-pound weight and a 4-pound weight on the left and hanging a 4-pound weight on the right. With the weights as described, the mobile is in balance by adding the fewest weights possible, which is **5** weights.



245. Suppose the radius of each of the four non-overlapping circles is 1 unit. Then the square created by connecting the centers of these four circles has side length $1 + 1 = 2$ units and area $2^2 = 4$ units^2. By properties of 45-45-90 right triangles, we know that the diagonal of the square has length $2\sqrt{2}$ units, making the larger circle's radius $1 + \sqrt{2}$ units and its area $\pi \times (1 + \sqrt{2})^2 = (3 + 2\sqrt{2})\pi$ units^2. The area of the region that is not shaded is the difference between the area of the square and the combined area of four quarter circles of radius 1 unit, which is $\pi \times 1^2 = \pi$ units^2. The unshaded region has area $4 - \pi$ units^2. So, $(4 - \pi)/[(3 + 2\sqrt{2})\pi] \approx 0.047$ of the figure is not shaded. That accounts for about **4.7**%.

246. The best strategy for Martin to follow can be found by looking at the problem backward and asking "How wide a range of numbers can Martin handle with g guesses?" Suppose he has just 1 guess. Then clearly he cannot handle any range with more than 1 number. If he has 2 guesses, then he can handle the range from 1 to 3, inclusive, because he can first guess the middle number, 2, and if that is not correct, then by knowing whether his guess is too high or too low, he will be able to guess correctly the second time. If he has 3 guesses, then he can handle the range from 1 to 7 by initially guessing 4, again the middle number. If that is wrong, then he has narrowed the range to 3 numbers and can apply his strategy on that range with his remaining 2 guesses. The range pattern is 1, 3, 7, 15, ... With g guesses, Martin can handle a range of $2^g - 1$ numbers. Because $2^{10} - 1 = 1023$ and $2^{11} - 1 = 2047$, 10 guesses will not be enough to handle the range from 1 to 2018, but 11 guesses will be enough. The strategy is to make a guess right in the middle of the narrowed range at each stage until the range has been narrowed to one number. The answer is n = **11**.

247. Suppose the book list contains a novels by Twain, b novels by Hemingway and c novels by Steinbeck. If there are exactly 100 ways for Austen to select two of an author's novels, then we have ${}_aC_2 + {}_bC_2 + {}_cC_2 = 100$. If we were to calculate the values of ${}_nC_2$ for n = 2, 3, 4, 5, 6, …, we would get the list of triangular numbers, namely 1, 3, 6, 10, 15, …. The table shows the values of n for which ${}_nC_2$ is a triangular number less than 100. We are looking for groups of three triangular numbers with a sum of 100. Since we want to maximize $a + b + c$, we should look for three numbers that are close in value. The group of three that works is 36, 36 and 28, which corresponds to 9, 9 and 8 novels. The greatest possible number of novels on the list is **26** novels.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
${}_nC_2$	1	3	6	10	15	21	28	36	45	55	66	78	91

m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	**5**
3	2	5	12	**13**
4	1	15	8	**17**
4	3	7	24	25
5	2	21	20	**29**
5	4	9	40	**41**
6	1	35	12	**37**
6	5	11	60	61
7	2	45	28	**53**

248. The table shows a way to generate Pythagorean Triples by choosing positive integer values m and n such that $m > n$, GCF(m, n) = 1, and one of them is even. Then we compute $m^2 - n^2$, $2mn$ and $m^2 + n^2$, which is a primitive Pythagorean Triple. If we systematically assign values of m and n, we can catch all primitive Pythagorean Triples. There are 7 primes less than 60 that can be the length of the hypotenuse of a right triangle. The sum of these primes is $5 + 13 + 17 + 29 + 41 + 37 + 53 =$ **195**.

249. We are given the point A(a, 0). So, let's say that the coordinates of point B are (0, b). Since the area of the whole triangle is 54 units^2, the product ab must be twice as much, so we have $ab = 108$. Suppose, then, that the coordinates of the intersection of the line $y = 0.6x$ and segment AB are (x, $0.6x$). This allows us to write two more equations like the previous one: $a \times 0.6x = 28$ and $bx = 80$. We now have a system of three equations with three unknowns. We'll rewrite the second equation as $ax = 140/3$. We only need the value of a, so we don't necessarily need to find b and x. If we multiply the first and second equations, we get $ab \times ax = 108 \times 140/3 \rightarrow a^2bx = 5040$. Now we can divide by the third equation to eliminate the bx on the left. The result is $a^2 = 5040/80 = 63 \rightarrow a = \sqrt{63} =$ $\mathbf{3\sqrt{7}}$.

250. The hexagon can be subdivided into 6 equilateral triangles. The base of each triangle is 10 meters. By the properties of 30-60-90 right triangles, we know that the altitude of each equilateral triangle is $5\sqrt{3}$ meters. The area of the whole hexagon, then, is $6 \times 1/2 \times 10 \times 5\sqrt{3} = 150\sqrt{3}$ m^2. Although there are seven solar discs across a long diagonal of the hexagon, the diameter of those discs is not 20/7 since the discs on the ends do not intersect the vertices of the hexagon. To find the radius of the solar discs, we draw a perpendicular segment from the center of the center disc to the midpoint of one side of the hexagon. The length of this altitude is $5\sqrt{3}$ meters, and it represents $3\sqrt{3} + 1$ radii of the discs. The radius of each disc is $(5\sqrt{3})/(3\sqrt{3} + 1)$, so the area of the 37 discs is $37 \times \pi \times [(5\sqrt{3})/(3\sqrt{3} + 1)]^2$. This accounts for $[37 \times \pi \times [(5\sqrt{3})/(3\sqrt{3} + 1)]^2]/[150\sqrt{3}] \approx 0.87$ of the hexagon, which is **87**%.

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MATHCOUNTS was founded in 1983 as a way to provide new avenues of engagement in math for middle school students. MATHCOUNTS began solely as a competition, but has grown to include 3 unique but complementary programs: the **MATHCOUNTS Competition Series**, the **National Math Club** and the **Math Video Challenge**. Your school can participate in all 3 MATHCOUNTS programs!

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The **National Math Club** is a free enrichment program that provides teachers and club leaders with resources to run a math club. The materials provided through the National Math Club are designed to engage students of all ability levels—not just the top students—and are a great supplement for classroom teaching. This program emphasizes collaboration and provides students with an enjoyable, pressure-free atmosphere in which they can learn math at their own pace.

Active clubs also can earn rewards by having a minimum number of club members participate (based on school/organization/group size). **There is no cost to sign up for the National Math Club**, and registration is open to schools, organizations and groups that consist of at least 4 students in 6th, 7th and/or 8th grade and have regular in-person meetings. More information can be found at **www.mathcounts.org/club**, and the 2017-2018 School Registration Form is included on the next page.

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! The fastest way to register is online at **www.mathcounts.org/clubreg**!

THE NATIONAL **MATHCLUB**®
powered by **MATHCOUNTS**®

2017-2018 **SCHOOL** REGISTRATION FORM

This registration form is for U.S. middle schools only. To register a non-school group (such as a Girl Scout troop, Boys and Girls Club Chapter or math circle) for the National Math Club, please go to www.mathcounts.org/club to review eligibility requirements and register.

*indicates required information

STEP 1
Tell Us About Your School

☒ **U.S. School with Students in Grades 6-8**
One school can have multiple clubs, as long as each club has a different club leader.

School Name* ______________________________

School Type (check one)* ☐ Public ☐ Charter ☐ Private ☐ Homeschool ☐ Virtual

Title I School? (check one)* ☐ Yes ☐ No

Overseas U.S. schools must provide additional information below:

My school is sponsored by the U.S. Department of: ☐ *Defense (DoDDS)* ☐ *State*

Country ______________________________

Approximate Total Number of Students Participating in Club (Minimum 4)***** __________

STEP 2
Tell Us About You

Club Leader First & Last Name* ______________________________

Club Leader Email* ______________________________

Club Leader Alternate Email* ______________________________

Your email and alternate email address will not be made public or shared and will only be used by MATHCOUNTS.

Club Street Address* ______________________________

City, State and ZIP Code* ______________________________

☐ **I am a MATHCOUNTS alumnus/a. ⟶ Please tell us more information below!**

When? (for example, 1999-02) __________________ Where? (state/territory) __________________

Which Program(s)? ☐ MATHCOUNTS Competition Series ⟶ Highest Level Reached: __________________
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☐ Math Video Challenge (formerly Reel Math Challenge)

STEP 3
Turn in Your Form

! **IMPORTANT!** *By submitting this form you attest your group consists of at least 4 U.S. students in grades 6-8 who meet in person regularly, and is therefore eligible to participate in the National Math Club. The club leader will receive an emailed confirmation once this registration has been processed.*

Mail or email a scanned copy of this completed form to:

Address: MATHCOUNTS Registration | 1420 King Street | Alexandria, VA 22314 *Email:* reg@mathcounts.org

! The fastest way to add students to your school's registration is by logging in to your Coach Dashboard at **www.mathcounts.org/coaches**!

MATHCOUNTS® COMPETITION SERIES

2017–2018 ADDITIONAL STUDENTS REGISTRATION FORM

Step 1: Tell us about your school so we can find your original registration (please print legibly).

Coach Name ____________________ School Name ____________________

Coach Email Address ____________________

School Address ____________________ City, State ZIP ____________________

Step 2: Tell us how many students you are adding to your school's registration. Following the instructions below.

Please circle the *number of additional students* you will enter in the Chapter Competition and the *associated cost* below (depending on the date your registration is postmarked).

*# of Students You Are **Adding***	1	2	3	4	5	6	7	8	9
Early Bird Rate (*postmarked **by** Nov. 3, 2017*)	**$30**	**$60**	**$90**	**$120**	**$150**	**$180**	**$210**	**$240**	**$270**
Regular Rate (*postmarked **by** Dec. 15, 2017*)	**$35**	**$70**	**$105**	**$140**	**$175**	**$210**	**$245**	**$280**	**$315**
Late Registration (*postmarked **after** Dec. 15, 2017*)	**$55**	**$90**	**$125**	**$160**	**$195**	**$230**	**$265**	**$300**	**$335**

☐ My school qualifies for the **50% Title I discount**, so the Amount Due in Step 4 will be half the amount I circled above. Principal signature required below to verify Title I eligibility.

Principal Name ____________________ Principal Signature ____________________

Step 3: Tell us what your school's FINAL registration should be (including all changes/additions).

***Total** # of Registered Students*	1 (1 individual)	2 (2 ind)	3 (3 ind)	4 (1 team)	5 (1 tm, 1 ind)	6 (1 tm, 2 ind)	7 (1 tm, 3 ind)	8 (1 tm, 4 ind)	9 (1 tm, 5 ind)	10 (1 tm, 6 ind)

Step 4: Almost done... just fill in payment information and turn in your form!

Amount Due $________ ☐ Credit Card ☐ Check (*payable to MATHCOUNTS Foundation*) ☐ Money Order ☐ Purchase Order #__________ (*must include copy of P.O.*)

Do NOT include any credit card information on this form. *Within 5 business days you will receive an email invoice enabling you to pay by credit card.*

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Mail or email a scanned copy of this completed form to: MATHCOUNTS Registration | 1420 King Street | Alexandria, VA 22314 **Email:** reg@mathcounts.org

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