solution of problem 6 kinetic theory

Part 1, Question 1: Linearized Vlasov-Maxwell Equations

Vlasov Equation for Electrons

The Vlasov equation describes the evolution of the electron distribution function $f(\vec{r}, \vec{v}, t)$. For electrons in electromagnetic fields, it is:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0$$

where $q = -e$ for electrons (so $q/m = -e/m$), $\vec{E}$ is the electric field, and $\vec{B}$ is the magnetic field.

Maxwell's Equations

Maxwell's equations govern the electromagnetic fields:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

where $\rho = q \int f \, d^3v$ is the charge density and $\vec{J} = q \int \vec{v} f \, d^3v$ is the current density.

Equilibrium and Perturbations

The equilibrium state has:

- A uniform static magnetic field $\vec{B}_0 = B_0 \hat{z}$.
- No electric field $\vec{E}_0 = 0$.
- A Maxwellian electron distribution function:

$$f_0(v) = \frac{n_0}{(2\pi v_{th}^2)^{3/2}} \exp\left(-\frac{v^2}{2v_{th}^2}\right), \quad v_{th} = \sqrt{\frac{k_B T}{m}}$$

- Ions are stationary and neutralize the plasma, so $\rho = 0$ and $\vec{J} = 0$ in equilibrium.

We introduce small perturbations:

- $f = f_0 + f_1$
- $\vec{E} = \vec{E}_1$ (since $\vec{E}_0 = 0$)
- $\vec{B} = \vec{B}_0 + \vec{B}_1$

Linearization

We assume wave-like perturbations: $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, so all first-order quantities have this form. For example, $f_1(\vec{r}, \vec{v}, t) = f_1(\vec{v}) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.

Linearized Vlasov Equation:
Substitute into the Vlasov equation and keep only first-order terms. The linearized equation becomes:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla_r f_1 + \frac{q}{m}(\vec{v} \times \vec{B}_0) \cdot \nabla_v f_1 = -\frac{q}{m}(\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \nabla_v f_0$$

Using the wave-like dependence, the derivatives become:

- $\frac{\partial}{\partial t} \to -i\omega$
- $\nabla_r \to i\vec{k}$

Thus:

$$-i\omega f_1 + i\vec{k} \cdot \vec{v} f_1 + \frac{q}{m}(\vec{v} \times \vec{B}_0) \cdot \nabla_v f_1 = -\frac{q}{m}(\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \nabla_v f_0$$

Linearized Maxwell's Equations:
Similarly, for the fields, we linearize Maxwell's equations. Since $\rho_1 = q \int f_1 \, d^3v$ and $\vec{J}_1 = q \int \vec{v} f_1 \, d^3v$, we have:

$$i\vec{k} \cdot \vec{E}_1 = \frac{\rho_1}{\epsilon_0}$$

$$i\vec{k} \cdot \vec{B}_1 = 0$$

$$i\vec{k} \times \vec{E}_1 = i\omega\vec{B}_1$$

$$i\vec{k} \times \vec{B}_1 = \mu_0\vec{J}_1 - i\omega\mu_0\epsilon_0\vec{E}_1$$

These equations describe how the perturbations $\vec{E}_1$ and $\vec{B}_1$ are related to f_1.

Appendix : The Maxwillian fontion

1. What is it?

The provided equation is the **Maxwell-Boltzmann distribution** (or Maxwellian) for particle velocities in a plasma (or gas) in thermal equilibrium:

$$f_0(v) = \frac{n_0}{(2\pi v_{th}^2)^{3/2}} \exp\left(-\frac{v^2}{2v_{th}^2}\right)$$

Where:

- $f_0(v)$ is the equilibrium distribution function.
- n_0 is the particle number density (e.g., electrons/m³).
- $v_{th} = \sqrt{\frac{k_B T}{m}}$ is the **thermal speed** (a characteristic speed of particles at temperature T).
- $v^2 = v_x^2 + v_y^2 + v_z^2$ is the square of the speed.
- k_B is Boltzmann's constant.
- T is the temperature.
- m is the particle mass.

2. Where does it come from? (Physics Behind It)

The Maxwellian distribution emerges naturally from **statistical mechanics**:

- It describes the most probable distribution of particle speeds in a system that has reached **thermal equilibrium**.
- It is derived by maximizing the entropy of the system subject to the constraints of constant total energy and particle number.
- In the context of plasma, it represents a plasma where collisions (or other processes) have randomized the particle velocities, resulting in no net energy transfer in any direction—a state of maximum disorder.

3. When is it used? For what?

It is used as the **background** or **equilibrium distribution** function f_0 in the linearization of the Vlasov equation:

- **Equilibrium Assumption**: The plasma is assumed to be in a steady state with no initial electric or magnetic fields (other than the constant $\vec{B}_0$).
- **Perturbation Theory**: When studying waves or instabilities, we linearize the Vlasov equation around this equilibrium state. We write the total distribution as $f = f_0 + f_1$, where f_1 is a small perturbation. The Maxwellian f_0 serves as the reference state.
- **Zero Current/Charge in Equilibrium**: For a pure Maxwellian, the average velocity $\langle \vec{v} \rangle = 0$, so the equilibrium current density $\vec{J}_0 = 0$. Similarly, the charge density is zero if ions provide a neutralizing background.

4. Difference from a Gaussian

- A **Gaussian** in one dimension has the form $e^{-x^2/(2\sigma^2)}$.
- The Maxwellian is essentially a **product of three Gaussians** (for v_x, v_y, v_z), normalized so that its integral over all velocity space equals the density n_0:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(\vec{v}) dv_x dv_y dv_z = n_0$$

- So, it is a **multivariate Gaussian distribution in velocity space**. The term "Maxwellian" specifically refers to this Gaussian distribution applied to particle velocities in thermal equilibrium.

Appendix : the vlasov eqt's

1. Where does it come from? The Physics & Relation to

BBGKY

let's recall the **BBGKY hierarchy** (Bogoliubov–Born–Green–Kirkwood–Yvon). This is its formal origin.

- **The Full Problem:** In a plasma with N particles, to know everything, you'd need to solve for the N-particle distribution function, which depends on $3N$ position and $3N$ velocity coordinates. This is impossible.
- **The BBGKY Hierarchy:** This is a set of coupled equations that describes the evolution of s-particle distribution functions (f_1, f_2, f_3, ...). The equation for f_1 depends on f_2, which depends on f_3, and so on. It's an infinite, intractable chain.
- **The Closure Assumption (The Key Step):** To break the chain, we make a physical assumption. The Vlasov equation is derived by assuming that **the two-particle correlation function f_2 is negligible**. This is known as the **"mean-field"** approximation.

What does this mean physically?
It means an individual particle doesn't feel the specific, discrete force from every other nearby particle (a two-body collision). Instead, it only feels the **smooth, averaged collective force** generated by the total charge and current densities in the plasma. This is an excellent approximation for most plasmas because the long-range nature of the Coulomb force means a particle interacts with a vast number of others simultaneously, making the collective effect dominate over short-range, binary collisions.

In short: The Vlasov equation is the BBGKY hierarchy truncated at the first level, using the mean-field approximation. It's a collisionless kinetic equation.

2. The Vlasov Equation and its Components

The Vlasov equation for a species of charged particles (e.g., electrons) is:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0$$

This is a equation in **6-dimensional phase space** ($\vec{r}$, $\vec{v}$). The distribution function $f(\vec{r}, \vec{v}, t)$ tells you the density of particles in a small volume $d^3r d^3v$ around the point ($\vec{r}$, $\vec{v}$) at time t.

Let's give each term a simple, physical meaning. Imagine you are riding along on a tiny spaceship through phase space, following a group of particles with a specific position and velocity.

Term 1: $\frac{\partial f}{\partial t}$ (The "Time Derivative")

- **Physical Meaning:** The local rate of change of the particle density *at a fixed point* in phase space.
- **Easy to Remember:** It's like a photographer standing in one spot and taking pictures of the crowd density at that spot over time.

Term 2: $\vec{v} \cdot \nabla_r f$ (The "Spatial Flow" or "Convection" Term)

- **Physical Meaning:** This term accounts for particles with a velocity $\vec{v}$ *flowing into* or *out of* the spatial volume d^3r due to their motion.
- **Easy to Remember:** This is the $\vec{v}$ part. If particles are moving, they naturally carry their distribution function with them. A gradient ($\nabla_r f$) means the density is different from one point to the next, so flow will change the local density.

Term 3: $\frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f$ (The "Acceleration" or "Force" Term)

- **Physical Meaning:** This is the most crucial term for plasmas. It accounts for particles with a position $\vec{r}$ *flowing into* or *out of* the velocity volume d^3v due to forces accelerating them.
- **Easy to Remember:** This is the $\vec{a}$ part.
- $\frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$ is the acceleration $\vec{a}$ of a particle due to the Lorentz force.
- $\nabla_v f$ is the gradient of the distribution in *velocity space*. If f changes with velocity (e.g., more slow particles than fast ones), acceleration will move particles along the velocity axis, changing the density at a specific velocity $\vec{v}$.

The Sum ($= 0$): The "Total Derivative"

The whole equation states that the **total derivative** of f is zero:

> $$\frac{Df}{Dt} = 0$$
>
> **Profound Physical Meaning:** As you move along with a group of particles in 6D phase space, the density of that group *does not change*. Particles are neither created nor destroyed (it's a continuity equation in phase space). They just move around in phase space due to their velocity (Term 2) and due to forces (Term 3). This is a beautiful and powerful result.

Summary and Connection to Your Problem

In your problem, you **linearize** this equation. You assume:

1. **Equilibrium (f_0):** The smooth, unchanging background state (the Maxwellian you asked about earlier).
2. **Perturbation (f_1):** A very small wave-like ripple on top of that background ($f = f_0 + f_1$).
3. **Perturbed Fields ($\vec{E}_1, \vec{B}_1$):** The electromagnetic fields created by the perturbations in charge and current density ($\rho_1, \vec{J}_1$).

Appendix: Maxwell's_and_vlasov_eqt

the concept of **self-consistency**.

The Vlasov equation and Maxwell's equations are not just used together; they are **coupled**. They form a closed, self-consistent system that describes how particles create fields and how fields, in turn, govern the motion of particles.

Here's the purpose and the physics behind their relationship, broken down into a simple cause-and-effect loop.

The Self-Consistency Loop

The core idea is that **particles generate fields** and **fields move particles**. This creates a feedback loop:

1. **Particles → Fields (Maxwell's Equations):**

- The charge density (ρ) and current density ($\vec{J}$) of the particles are the **sources** for the electromagnetic fields in Maxwell's equations.
- $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$: Electric charges are the source of electric field divergence.
- $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$: Electric currents are the source of magnetic field curl.

2. **Fields → Particles (Vlasov Equation):**

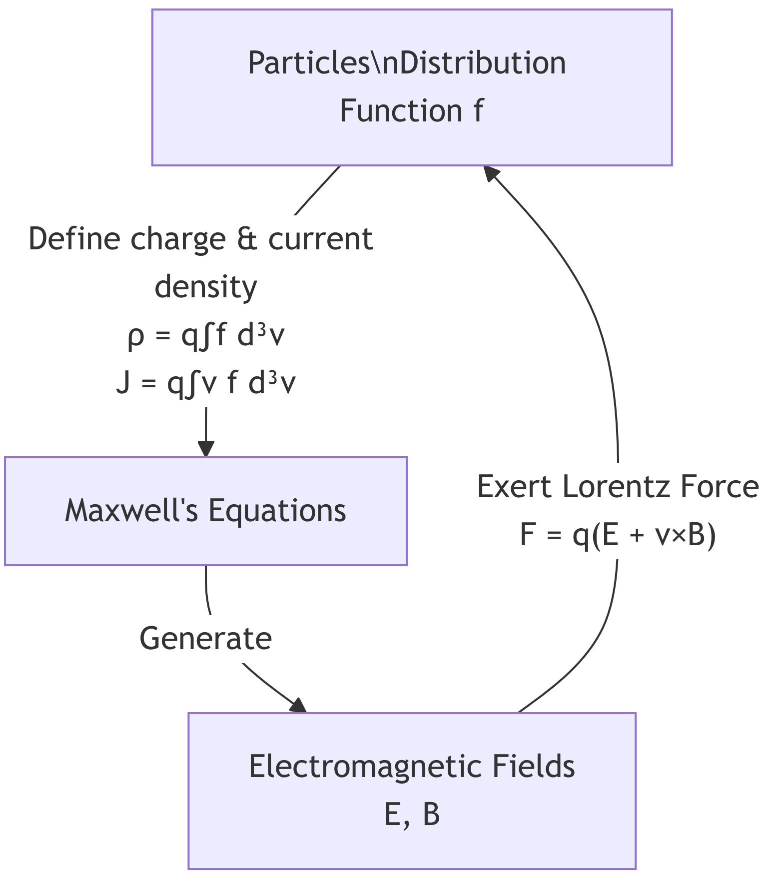
- The electromagnetic fields ($\vec{E}$ and $\vec{B}$) determine the force on every particle via the Lorentz Force Law: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

- This force dictates how particles **move through phase space** (change their position and velocity), which is exactly what the Vlasov equation describes: $\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0$.

3. **The Loop Closes:**

- The motion of the particles (from step 2) changes their spatial distribution and flow, which **alters the charge density ρ and current density $\vec{J}$**.
- These new ρ and $\vec{J}$ become the new sources for Maxwell's equations (back to step 1), which modify the $\vec{E}$ and $\vec{B}$ fields.
- The modified fields then again alter the motion of the particles, and so on.

This loop happens continuously in time. To describe a plasma, you **must** solve both sets of equations simultaneously because each one depends on the solution of the other.



Why is this so crucial for your problem? (Waves and Instabilities)

You are studying waves. In a plasma, waves are **self-consistent perturbations** that propagate through this coupled system.

> 1. **The Perturbation:** You start with an equilibrium (uniform f_0, constant $\vec{B}_0$, $\vec{E}_0 = 0$). Then you "wiggle" the particles a little bit, creating a small perturbed distribution f_1.
> 2. **Particles → Fields:** This "wiggle" in the particles creates oscillating, wave-like perturbations in the charge and current density (ρ_1, $\vec{J}_1$).
> 3. **Fields → Particles:** These oscillating sources, via Maxwell's equations, generate wave-like electromagnetic fields ($\vec{E}_1$, $\vec{B}_1$).
> 4. **The Key Question:** Do these newly created fields $\vec{E}_1$ and $\vec{B}_1$ reinforce the original "wiggle" (f_1), causing it to grow (an **instability**), or do they oppose it, causing it to oscillate (a **wave**) or damp out?
> 5. **The Dispersion Relation:** The condition for a self-sustaining wave (where the fields created by the particle motion are just the right strength and phase to sustain that motion) leads to a mathematical equation called the **dispersion relation**, which relates the wave's frequency (ω) to its wavelength (k). Solving this equation tells you everything about the wave: its existence, its propagation, and its damping.

Part 1, Question 2: Method of Characteristics and Unperturbed Orbits

The linearized Vlasov equation can be solved using the **method of characteristics**. This involves integrating along the **unperturbed orbits** of particles—the trajectories they follow under only the static magnetic field $\vec{B}_0$.

Unperturbed Orbits

For a particle in $\vec{B}_0 = B_0\hat{z}$, the equation of motion is:

$$\dot{\vec{v}} = \frac{q}{m}\vec{v} \times \vec{B}_0$$

This results in **helical motion**: circular gyration in the plane perpendicular to $\vec{B}_0$ and uniform motion along $\vec{B}_0$. The gyrofrequency is $\omega_c = \frac{eB_0}{m}$ (for electrons, ω_c is positive since $q = -e$, but the sense of rotation is opposite to that for ions).

Expressing f_1 as an Integral

The left side of the linearized Vlasov equation represents the derivative of f_1 along the unperturbed orbits. Thus, we can write:

$$\left.\frac{df_1}{dt}\right|_{\text{orbit}} = -\frac{q}{m}(\vec{E}_1 + \vec{v} \times \vec{B}_1) \cdot \nabla_v f_0$$

Integrating along the orbit from the infinite past to time t, we obtain:

$$f_1(\vec{r}, \vec{v}, t) = -\frac{q}{m} \int_{-\infty}^{t} dt' \left[(\vec{E}_1 + \vec{v}' \times \vec{B}_1) \cdot \nabla_v f_0 \right]_{(\vec{r}'(t'), \vec{v}'(t'), t')}$$

Here:

- $\vec{r}'(t')$ and $\vec{v}'(t')$ are the position and velocity along the unperturbed orbit that reaches $\vec{r}$ and $\vec{v}$ at time t.
- The integral sums up the effects of the perturbed fields along the particle's history.

Effect of Magnetic Field

The magnetic field $\vec{B}_0$ affects the orbits by causing gyration, which makes the integral more complex. Specifically:

- The helical motion introduces **cyclotron harmonics** into the response, as the particle experiences the wave fields at multiples of the gyrofrequency.
- This leads to resonances when $\omega - k_\parallel v_\parallel - n\omega_c = 0$ for integer n, which are crucial for wave-particle interactions.

This integral form is key for deriving the dielectric tensor in Part 2.