# MCT-241: Modelling & Stimulation

# Modeling, Simulation, and Control of a Four-Bar Mechanism with a Brushless Servo Motor


**Submitted By:**

Muhammad Haider Ali

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**Department of Mechatronics & Control Engineering, University of Engineering & Technology, Lahore**


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## DOMAIN SPECIFICATION:

Mechanical and Control Systems Engineering.

## NATURE OF CEP

This project involves analyzing, modeling, and simulating a four-bar linkage system driven by a brushless servo motor to achieve precise movement and control. The study focuses on mathematical modeling, simulation in MATLAB, and control parameter adjustments to optimize the system's performance.

## APPLICATION AREA

1. Robotics.
2. Mechanical engineering applications such as automotive systems and industrial machinery.
3. Automation systems.

## MODULES INTEGRATION

1. **Mechanical Structure Design:** Assembly of a four-bar mechanism.
2. **Actuator Integration:** Connection of the servo motor to the system.
3. **Mathematical Modeling:** Derivation of the kinematic and dynamic equations.
4. **Simulation Module:** Implementation of models in MATLAB/Simulink.
5. **Control System Design:** Development of control strategies to ensure precise movement.

## PROBLEM EXPLANATION

The objective is to understand how a four-bar mechanism, controlled by a brushless servo motor, can be modeled and controlled for optimal performance. The system must be designed to simulate the mechanical behavior, assess system responses, and implement control strategies that enhance its stability and accuracy.

## OVERALL WORKING OF THE PROJECT (Non-Technical Explanation)

Imagine a mechanical arm connected to a motor that allows it to move in a controlled manner. This arm setup, known as a four-bar mechanism, is common in machines that need to transfer rotational motion into a specific type of movement, like robotic arms or vehicle suspension systems. By modeling and simulating this setup on a computer, we can understand how the system behaves under different conditions and tweak the motor's performance to ensure it moves just right.

## SCHEMATIC DIAGRAM

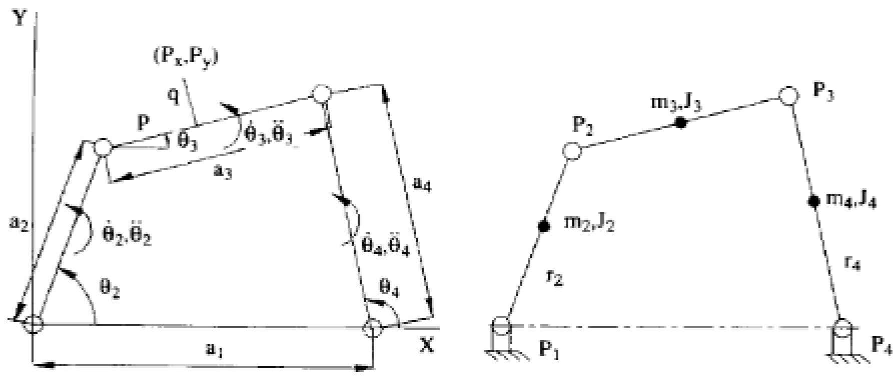


Fig1.1: Schematic Diagram of Four-Bar Linkage (Crank Rocker Mechanism)

## COMPLETE BLOCK DIAGRAM OF THE SYSTEM

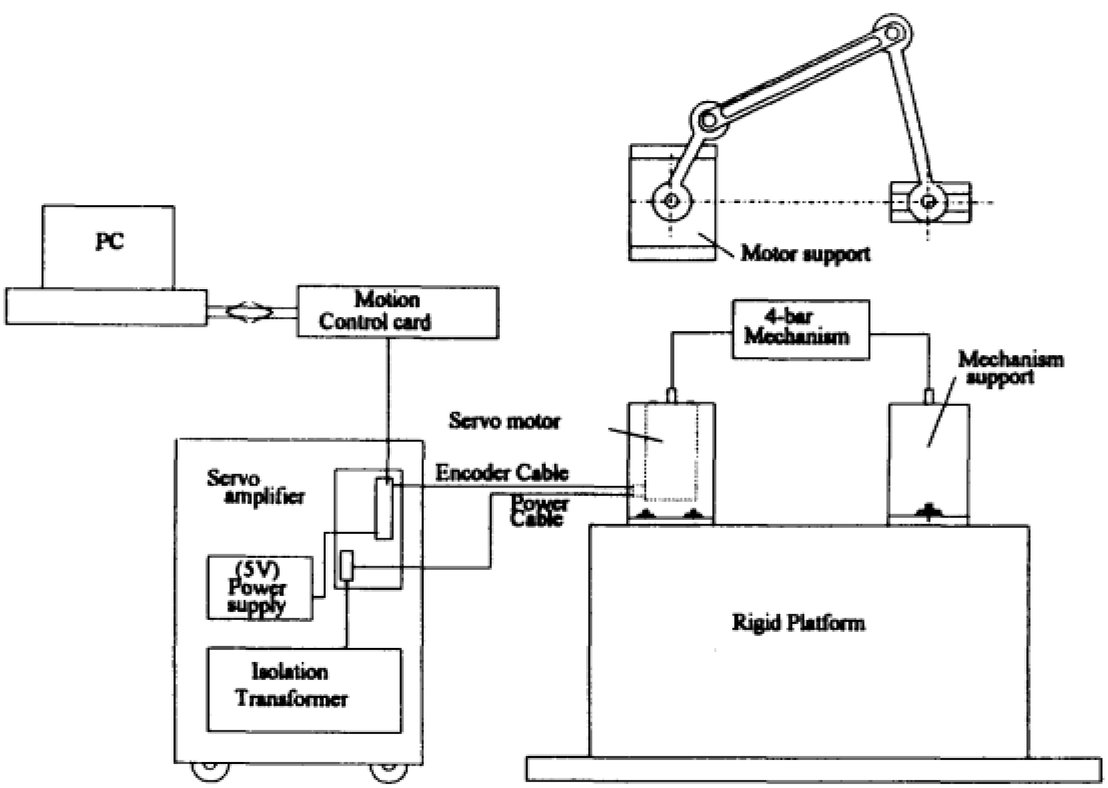


Fig1.2: Block Diagram of Servo-Controlled Four-Bar Mechanism

## OBJECTIVES

1. To obtain the mathematical model of the four-bar mechanism and motor system.
2. To study the variation of system parameters and analyze their effects.
3. To model and simulate the system using MATLAB/Simulink.
4. To design a control strategy to achieve precise motion control.
5. To evaluate the system's response under different conditions and optimize performance.

## CONTROLLABLE PARAMETERS (with Theoretical Interpretation)

1. **Motor Torque (Nm):** Controls the rotational force applied to the mechanism.
2. **Link Lengths (m):** Affects the range and type of motion produced.
3. **Motor Speed (RPM):** Influences the velocity of the linkage movement.
4. **Joint Friction Coefficients:** Determines energy loss due to resistance at joints.
5. **Load Mass (kg):** Impacts the mechanical response and required torque.
6. **Damping Coefficient:** Helps in stabilizing the system by reducing oscillations.
7. **Feedback Gain:** Adjusts the responsiveness of the control system.
8. **Input Voltage (V):** Dictates the power supplied to the motor.

## EXPECTED DYNAMIC TRENDS

**Graph 1:** Time (X-axis) vs. Link Displacement (Y-axis)
**Graph 2:** Time (X-axis) vs. Motor Speed (Y-axis)
**Graph 3:** Time (X-axis) vs. Joint Force (Y-axis)
**Graph 4:** Time (X-axis) vs. System Energy (Y-axis)
**Graph 5:** Time (X-axis) vs. Control Signal (Y-axis)
**Graph 6:** Motor Voltage (X-axis) vs. Torque (Y-axis)
**Graph 7:** Input Torque (X-axis) vs. Link Velocity (Y-axis)
**Graph 8:** Feedback Gain (X-axis) vs. Stability Response (Y-axis)
**Graph 9:** Time (X-axis) vs. Crank Acceleration (Y-axis)
**Graph 10:** Time (X-axis) vs. Crank Velocity (Y-axis)
**Graph 11:** Time (X-axis) vs. Crank Displacement (Y-axis)

## DESCRIPTION OF THE EXPERIMENTAL SETUP

The experimental setup, illustrated in Figure 1.2, includes the following components: a brushless servo motor paired with its matching servo drive, a servo motor controller card, a personal computer (an IBM-compatible PC-486 SX running at 33 MHz with 4 MB of RAM), and a crank-rocker four-bar mechanism for motion transmission.

In this setup, the motor shaft is directly connected to the crankshaft, eliminating the need for any mechanical speed-reduction devices and simplifying the overall mechanical design. Various crank motion profiles were tested at speeds ranging from 60 to 360 RPM (e.g., 60, 120, 240, and 360 RPM) to demonstrate the system's functionality.

- **Brushless Servo System**

The setup uses a single-phase permanent magnet synchronous motor paired with a digitally controlled drive module (Electro Craft BRU-200, S-4075 motor, and DM-30 drive module) [1]. The motor has a maximum speed of 3000 RPM and delivers a continuous power output of 3 kW.

The drive module features a 16-bit microprocessor and connects to the servo drive through a simple serial interface from the host computer. Configuration and tuning are managed using a plug-in personality module.

Position feedback is provided by a high-precision incremental encoder, built into the motor, with a resolution of 8000 pulses per revolution (ppr). The encoder generates a reference pulse to reset the system and establish a zero-reference point for the crank. It is powered by a 5V DC power supply.

The brushless motor’s key specifications are listed in Table 1, and the same data is used in system simulations.

Table 1. Brushless servo motor specifications

| | |
|---|---|
| Rotor moment of inertia | $J_m = 6.8 \times 10^{-4}\,\mathrm{kg \cdot m^2}$ |
| Maximum operating speed | $n_{max} = 3000$ rpm |
| Motor torque constant | $K_t = 0.76$ Nm/A |
| Motor voltage constant | $K_e = 90$ V/krpm |
| Winding inductance | $L = 5.8$ mH |
| Winding resistance | $R = 0.8\,\Omega$ |
| Continuous stall torque | $T_s = 10.2$ Nm |
| Peak torque | $T_p = 19.7$ Nm |
| Continuous output power (DM30) | $P = 3$ kW |

- **Servo Motor Controller Card**
  The system uses a full-size PC card (333 mm x 105 mm, 8-bit slot) compatible with IBM PCs, developed by Warwick Computer Designs [2]. The card features encoder inputs and analog outputs accessible via a 37-pin D-type connector. It interfaces with three Hewlett-Packard HCTL 1100 Motion Control ICs, allowing control of up to three axes.
  Four control algorithms are supported: position control, proportional velocity control, integral velocity control, and trapezoidal profile control. In this study, the position control mode is utilized. This mode enables point-to-point positioning without velocity profiling.

The user provides a 24-bit position command, which the controller compares with the actual 24-bit position. It calculates the position error, applies digital lead compensation, and locks the motor at the specified position until a new command is issued. The motor position is measured in encoder counts.
Motion control programs are written in Turbo Pascal, using library functions provided by Warwick Computer Designs for screen, keyboard, and HCTL 1100 functions [2]. Since the four-bar mechanism has only one degree of freedom, the motor controller card is configured to operate a single motor axis.

- **Four-Bar Mechanism**

The four-bar mechanism configuration shown in Fig. 1.1 is referred to in both the experimental set-up and the mathematical model. Mechanical properties of the mechanism are included in Table 2. The crank motion is programmed such that characteristically different coupler curves on the coupler link are obtained. Flexibility of the output is achieved.

Table 2. Mechanical properties of four-bar mechanism

| | | | |
|---|---|---|---|
| $a_1 = 300\,\text{mm}$ | $a_2 = 90\,\text{mm}$ | $a_3 = 360\,\text{mm}$ | $a_4 = 260\,\text{mm}$ |
| $r_2 = 68.16893\,\text{mm}$ | $r_4 = 145.14389\,\text{mm}$ | $p = 180\,\text{mm}$ | $q = 0.0\,\text{mm}$ |
| $m_2 = 0.190276\,\text{kg}$ | $m_3 = 0.198764\,\text{kg}$ | $m_4 = 0.306845\,\text{kg}$ | |
| $j_2 = 407.655\,\text{kg}\cdot\text{mm}^2$ | $j_3 = 3503.14\,\text{kg}\cdot\text{mm}^2$ | $j_4 = 3673.87\,\text{kg}\cdot\text{mm}^2$ | |

Variables are defined as follows.

| | |
|---|---|
| $\theta_2,\theta_3,\theta_4$ | angular position of links ($\theta_2$—input, $\theta_3$—coupler link, $\theta_4$—output [rad]) |
| $\dot{\theta}_2,\dot{\theta}_3,\dot{\theta}_4$ | angular velocities of links [rad/s] |
| $\ddot{\theta}_2,\ddot{\theta}_3,\ddot{\theta}_4$ | angular accelerations of links [rad/s$^2$] |
| $a_1,a_2,a_3,a_4$ | link lengths of the mechanism including the ground link [m] |
| $r_2,r_4$ | positions to the centre of gravity [m] |
| $m_2,m_3,m_4$ | masses of links [kg] |
| $J_2,J_3,J_4$ | moment of inertias of links [kg $\cdot$ m$^2$] |
| $p,q$ | positions of a point fixed perpendicular to the coupler link [m] |
| $P_x,\dot{P}_x,\ddot{P}_x,P_y,\dot{P}_y,\ddot{P}_y$ | positions, velocities and accelerations of any point chosen on the coupler link [m, m/s, m/s$^2$]. |

# MATHEMATICAL MODEL

## Mathematical Formulation

A mathematical model for the four-bar mechanism driven by a brushless servo motor is developed using Lagrange's formulation. The equation of motion is derived by calculating the system's energy functions and their partial and time derivatives with respect to the defined coordinate, $\theta 2$ .

For simplicity and computational efficiency, ideal running conditions are assumed for modelling and computational ease, and motor losses and friction effects are not included in the formulation. The mechanism is designed to function on a horizontal plane.

A simplified electrical equation of a brushless servo motor [3] is

$$L\,\frac{dI}{dt} + IR + Ke\,\frac{d\theta 2}{dt} = Kp\,(\theta 2c - \theta 2) + Kv(\theta' 2c - \theta' 2) \text{ - (1)}$$

In this system, the following variables are defined:

- L: winding inductance
- R: winding resistance
- I: motor current
- Kv : motor voltage constant
- Kp : proportional gain constant for position error
- Kd : derivative gain constant for velocity error
- $\theta 2$ *,ref*: reference position command
- $\theta' 2$*,ref*: reference velocity command

The right-hand side of Equation (1) incorporates PD (Proportional + Derivative) control terms to compute the required motor currents (*I*) for the servo motor. The motor specifications used in this formulation are detailed in Table 1.

The dynamic equation for a motor coupled to the mechanism is as follows:

$$\ddot{\theta}_2\Bigg[J_m + m_2 r_2^2 + J_2 + m_3 a_2^2 + 2 m_3 a_2 \left(\frac{\partial\theta_3}{\partial\theta_2}\right)(p\cos(\theta_2-\theta_3) + q\sin(\theta_2-\theta_3))$$

$$+\left(\frac{\partial\theta_3}{\partial\theta_2}\right)^2 (m_3(p^2+q^2)+J_3) + \left(\frac{\partial\theta_4}{\partial\theta_2}\right)^2 (m_4 r_4^2 + J_4)\Bigg]$$

$$+\dot{\theta}_2^2\Bigg[\left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(m_3(p^2+q^2)+J_3) + \left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right) m_3 a_2 (p\cos(\theta_2-\theta_3)$$

$$+q\sin(\theta_2-\theta_3)) + \left(\frac{\partial^2\theta_4}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_4}{\partial\theta_2}\right)(m_4 r_4^2 + J_4)$$

$$+\left(\frac{\partial\theta_3}{\partial\theta_2}\right) m_3 a_2 \left(q\cos(\theta_2-\theta_3)\left(1-\frac{\partial\theta_3}{\partial\theta_2}\right) - p\sin(\theta_2-\theta_3)\left(1-\frac{\partial\theta_3}{\partial\theta_2}\right)\right)\Bigg] = K_t I, \qquad (2)$$

where the right-hand side of the equation represents the generalized torque term which is directly proportional to multiplication of the motor torque constant (K) with the motor current.

Intermediate steps of derivation and the partial derivative terms can be found in [4]. Implicit relations necessary for 63 and 6, are found using the Freudenstein equation [5]. These relations are included in the Appendix.

The dynamic behavior of the system is analyzed using numerical methods. The system is represented by coupled equations:

- Equation (1) is a first-order differential equation.
- Equation (2) is a second-order differential equation.

To simplify the analysis, the variables $\theta 2$ (crank position) and $\dot{\theta} 2$ (crank velocity) are selected as state variables, denoted as $x1$ and $x2$ respectively ($\theta 2 = x1$, $\dot{\theta} 2 = x2$) Equation (2) is then transformed into a system of first-order differential equations, resulting in three first-order equations.

The fourth-order Runge-Kutta method, a widely used explicit integration technique for nonlinear systems, is chosen for solving these equations [6]. The unknowns in the system are motor current, crank position, and crank velocity. The system's time response is determined by integrating the first-order equations over time.

All computations and simulations are implemented using programs written in Turbo Pascal.

## Final Equation Overview:

The final equation we derived is Eq (2):

This equation represents the **dynamic behavior** of a **brushless servo motor** coupled to a **four-bar linkage mechanism**. It involves the inertial forces, the forces from the motor, the interaction between the links, and the effect of the velocity and acceleration on the system.

**Motor and Link Inertial Contributions:**

The inertial components of the system come from the rotational inertia of each link in the system. These inertial forces are responsible for resisting the changes in angular motion, and they are expressed using moments of inertia J.

- **Motor Inertia** (*Jm*):

  The motor is modeled as a rigid body with a moment of inertia *Jm* . This term represents the rotational inertia of the motor that resists angular acceleration.

  **Derivation**: The motor's inertia is modeled as the sum of its physical inertia and the effect of the armature (if applicable). The term appears in the equation because the torque generated by the motor contributes to the system's motion, which is resisted by inertia.

- **Link Inertia:**

$$(m_2 r_2^2 + J_2)$$

  The first link (input link) has a mass *m2 and* a distance r2 from the center of rotation to the center of mass. Its moment of inertia is *m2r2^2*. In addition, *J2 represents* any additional moment of inertia due to the shape and distribution of mass within the link.

  **Derivation**: The term *m2r2^2 comes* from the general formula for the moment of inertia of a point mass (*I=mr^2)*, and *J2 comes* from the link's additional geometric properties (like mass distribution).

- **Coupler Link Inertia:**

$$(m_3 a_2^2 + J_3)$$

  The coupler link has mass m3 and length a2, and its moment of inertia is m3a2^2+J3. J3 accounts for any internal moment of inertia that is specific to the shape and geometry of the coupler.
  **Derivation**: Like the input link, the m3a2^2+J3 term is derived from the point mass inertia equation, while J3 reflects the distribution of mass for the link, often derived from the link's geometry.

- **Output Link Inertia:**

$$(m_4 r_4^2 + J_4)$$

The output link has mass *m4 and* distance r from the center of rotation to the center of mass. Its moment of inertia is given by m4r4^2+J4.

**Derivation**: As before, the formula m4r4^2 + J4. represents the point mass inertia, while J4 accounts for other mass distribution effects for the output link.

## Dynamic Model of the System:

The dynamics of the system are governed by the **equation of motion**. The equation of motion is derived by applying Newton's second law for rotation, i.e., the sum of the torques on each link is equal to the moment of inertia times the angular acceleration.

The general equation of motion can be written as:

$$\tau = I \cdot \theta''$$

where:

- τ is the sum of torques acting on the link (inertial and coupling torques),
- $I$ is the moment of inertia of the link,
- $\theta''$ is the angular acceleration.

**Inertial Torques**:
The inertial torque on any link is simply the moment of inertia times the angular acceleration:

$$\tau_{\text{inertia}} = J \cdot \theta''$$

For example, the inertial torque on the input link θ2 is:

$$\tau_{\text{inertia},2} = J_m \cdot \theta_2''$$

where Jm is the moment of inertia of the motor.

**Coupling Torques**:

The coupling torques arise due to the interactions between the links. The input link θ2 is connected to the coupler link θ3, and the output link θ4 is also affected by the motion of θ2. These coupling torques depend on the angular velocities and accelerations of the links.

For the **coupler link**, the torque is:

$$\tau_{\text{coupler}} = m_3 a_2 \left( \frac{\partial \theta_3}{\partial \theta_2} \right) (p \cos(\theta_2 - \theta_3) + q \sin(\theta_2 - \theta_3))$$

This term reflects the interaction between the input link and the coupler link. The length a2 is the length of the coupler link, and p and q represent fixed points relative to the coupler.

For the **output link**, the torque is:

$$\tau_{\text{output}} = m_4 r_4^2 \left( \frac{\partial \theta_4}{\partial \theta_2} \right)$$

This term represents the torque generated by the motion of the output link, depending on the position and inertia of the output link.

## Kinematic Terms: Angular Velocity and Acceleration:

The system involves the angular velocities and accelerations of the links. These are described using **Freudenstein's equation** (which is used to describe the relationship between the angular positions of the links in a four-bar mechanism) and the kinematic relations for velocity and acceleration.

**Velocity Relations**:

The angular velocity of the coupler and output links can be expressed in terms of the input link velocity θ2'' using partial derivatives:

- **Velocity of Coupler Link:**

$$\theta_3' = \frac{\partial \theta_3}{\partial \theta_2} \cdot \theta_2'$$

  This term shows how the velocity of the coupler link depends on the velocity of the input link, modified by the kinematic relationship.

- **Velocity of Output Link:**

$$\theta_4' = \frac{\partial \theta_4}{\partial \theta_2} \cdot \theta_2'$$

Similarly, the velocity of the output link depends on the input link's velocity and the kinematic relation.

**Acceleration Relations**:

The angular accelerations of the links are the time derivatives of the angular velocities:

- **Acceleration of Coupler Link:**

$$\theta_3'' = \frac{\partial^2 \theta_3}{\partial \theta_2^2} \cdot (\theta_2')^2 + \frac{\partial \theta_3}{\partial \theta_2} \cdot \theta_2''$$

  This term represents the acceleration of the coupler link, which depends on both the square of the angular velocity of the input link $\theta 2'$ and the acceleration of the input link $\theta 2''$.

- **Acceleration of Output Link:**

$$\theta_4'' = \frac{\partial^2 \theta_4}{\partial \theta_2^2} \cdot (\theta_2')^2 + \frac{\partial \theta_4}{\partial \theta_2} \cdot \theta_2''$$

  This term follows the same logic as coupler link acceleration.

**Coupling and Interaction Terms**

The forces that arise due to the interaction between the links (coupling) contribute to the overall dynamics of the system. These are the terms involving the derivatives of $\theta 3$ and $\theta 4$, as well as the trigonometric functions reflecting the geometric arrangement of the mechanism.

- **Coupler Link Coupling Terms**:

$$2m_3a_2\left(\frac{\partial \theta_3}{\partial \theta_2}\right)(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3))$$

This term represents the coupling torque between the input and coupler links, modulated by the geometry of the mechanism. The cosine and sine functions reflect the orientation and the positions of the links relative to each other.

**Centrifugal and Coriolis Forces**:

These forces depend on the velocities of the links and cause additional torques. They involve the second derivatives of the angles θ3 and $\theta 4$ with respect to θ2, which are the terms:

$$\left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(m_3(p^2+q^2)+J_3)$$

Or

$$C_{\text{cor}} = 2m_3a_2\left(\frac{\partial\theta_3}{\partial\theta_2}\right)\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(p^2+q^2)$$

This term represents the effects of angular acceleration in the system, further modulated by the geometry and the inertial properties of the links.

**Centrifugal Forces**:

Centrifugal forces arise due to the rotational motion of the links, and they are proportional to the square of the angular velocity. The centrifugal term for the input link can be written as:

$$C_{\text{cen}} = m_3a_2\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3))$$

This term represents the centrifugal forces acting on the coupler link due to the angular velocity of the input link.

**Velocity-Dependent Terms**:

The velocity-dependent forces are included in the second part of the equation:

$$\theta_2'^2 \left[ \left( \frac{\partial^2 \theta_3}{\partial \theta_2^2} \right) \left( \frac{\partial \theta_3}{\partial \theta_2} \right) (m_3(p^2 + q^2) + J_3) + \left( \frac{\partial^2 \theta_3}{\partial \theta_2^2} \right) m_3 a_2 \left( p \cos(\theta_2 - \theta_3) + q \sin(\theta_2 - \theta_3) \right) + \left( \frac{\partial^2 \theta_4}{\partial \theta_2^2} \right) \left( \frac{\partial \theta_4}{\partial \theta_2} \right) (m_4 r_4^2 + J_4) + \left( \frac{\partial \theta_3}{\partial \theta_2} \right) m_3 a_2 \left( q \cos(\theta_2 - \theta_3)(1 - \frac{\partial \theta_3}{\partial \theta_2}) - p \sin(\theta_2 - \theta_3)(1 - \frac{\partial \theta_3}{\partial \theta_2}) \right) \right]$$

**Motor Torque Term:**

The last term in the equation of motion represents the **motor torque** generated by the brushless servo motor. The torque is proportional to the motor current I, with a constant Kt:

$$K_t I$$

This term captures the torque generated by the motor to drive the system.

## Derivation of Transfer Functions for Brushless Servo Motor-Driven Planar Mechanism

This report derives the transfer functions of a brushless servo motor driving a planar four-bar mechanism. The dynamic equation governing the system, including **electrical** and **mechanical** subsystems, is analyzed. Transfer functions are computed step-by-step to describe the relationship between inputs (e.g., current, position commands) and outputs (e.g., link positions or velocities).

### Governing Equations:

The system is described by two main equations as given in (1) and (2). With (1) being the equation for the electrical subsystem while (2) is the equation for the mechanical subsystem. All the parameters are already specified and explained above.

### Transfer Function Derivations:

**Motor Electrical Subsystem:**

Taking the Laplace transform of the motor electrical equation:

$$LsI(s) + IR(s) + K_e s\Theta_2(s) = K_p(\Theta_{2c}(s) - \Theta_2(s)) + K_v s(\Theta_{2c}(s) - \Theta_2(s))$$

Rearranging:

$$I(s) = \frac{K_p\Theta_{2c}(s) + K_v s\Theta_{2c}(s) - (K_p + K_v s)\Theta_2(s)}{Ls + R}$$

This transfer function relates to the motor current.

**Dynamic Equation Transfer Function:**

The dynamic equation is

$$\theta_2'' M(\theta_2) + \theta_2'^2 C(\theta_2, \theta_3, \theta_4) = K_t I$$

Here:

- $M(\theta_2)$ is the effective moment of inertia term.
- $C(\theta_2, \theta_3, \theta_4)$ represents velocity-dependent coupling forces.
- $K_t$ is the motor torque constant.
- $I$ is the motor current.

- **Small Angular Deviations**: For small deviations around an equilibrium point, $sin(\theta) \approx \theta$ and $cos(\theta) \approx 1$.
- **Linearization**: Nonlinear terms like

$$\frac{\partial\theta_3}{\partial\theta_2}, \frac{\partial^2\theta_3}{\partial\theta_2^2},$$

- are treated as constants for small perturbations.
- **Laplace Transform:** Assume all variables are in the Laplace domain, where sss represents the complex frequency variable.

The dynamic equation in Laplace form is:

$$s^2\Theta_2(s)M + s^2C\Theta_2(s) = K_tI(s)$$

Here:

- M is the constant effective inertia term (linearized from M(θ2)).
- C is the velocity-dependent coupling term, treated as a constant under linearization.

Substituting I(s) from the electrical equation into the dynamic equation:

$$s^2\Theta_2(s)(M + C) = K_t\frac{1}{Ls + R}\left[K_p(\Theta_{2c}(s) - \Theta_2(s)) + K_vs(\Theta_{2c}(s) - \Theta_2(s)) - K_es\Theta_2(s)\right]$$

Combine and Simplify

Reorganize the equation to express Θ2 (s) in terms of Θ2c (s):

$$s^2\Theta_2(s)(M + C)(Ls + R) = K_t\left[K_p(\Theta_{2c}(s) - \Theta_2(s)) + K_vs(\Theta_{2c}(s) - \Theta_2(s)) - K_es\Theta_2(s)\right]$$

Expand and collect terms for Θ2 (s):

$$\Theta_2(s)\left[s^2(M + C)(Ls + R) + K_t(K_p + K_vs - K_es)\right] = K_t(K_p + K_vs)\Theta_{2c}(s)$$

Factorize:

$$\Theta_2(s)\left[s^2(M + C)L + s(M + C)R + K_tK_p + K_tK_vs - K_tK_es\right] = K_t(K_p + K_vs)\Theta_{2c}(s)$$

**Combined Transfer Function:**

Solve for the transfer function;

$$\frac{\Theta_2(s)}{\Theta_{2c}(s)} = \frac{K_t(K_p + K_vs)}{s^2(M + C)L + s\left[(M + C)R + K_tK_v - K_tK_e\right] + K_tK_p}$$

This transfer function relates the reference input Θ2c (s) to the system output Θ2 (s), considering the motor dynamics, link inertias, and coupling terms.

To better understand the terms M(θ2) (effective moment of inertia) and C(θ2, θ3 ,θ4 ) (velocity-dependent coupling forces), we need to dive into their physical significance and the detailed mathematical derivation. These terms arise from the dynamics of the four-bar linkage and encapsulate how the mechanical system behaves under motion.

**Effective Moment of Inertia: M(θ2):**

**Definition**

The effective moment of inertia is a combined term that includes contributions from:

1. Motor rotor inertia (Jm).
2. The inertias of the input, coupler, and output links.
3. The distribution of mass and geometry within the mechanism.

**Derivation**

Using Lagrangian dynamics or direct application of Newtonian mechanics, the inertia term M(θ2) emerges as the coefficient of the angular acceleration $\ddot{\theta}2.$ This term considers the following:

**Motor and Input Link Contributions**

For the motor rotor and the input link:

$$J_m + m_2 r_2^2 + J_2$$

- Jm: Motor's rotor moment of inertia.
- m2: Mass of the input link.
- r2: Distance from the input link's pivot to its center of gravity.
- J2: Moment of inertia of the input link about its pivot.

**Coupler Link Contributions:**
**For the coupler link (θ3):**

- Translational inertia:

$$m_3 a_2^2$$

where a2 is the length of the input link.

- Coupling term:

$$2m_3a_2\frac{\partial\theta_3}{\partial\theta_2}\left(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3)\right)$$

Here, p and q are perpendicular distances defining the coupler geometry.

- Rotational inertia:

$$\left(\frac{\partial\theta_3}{\partial\theta_2}\right)^2\left(m_3(p^2+q^2)+J_3\right)$$

**Output Link Contributions:**

For the output link (θ4):

- Rotational inertia:

$$\left(\frac{\partial\theta_4}{\partial\theta_2}\right)^2\left(m_4r_4^2+J_4\right)$$

## Final M(θ2) Expression:

Bringing all these terms together, the effective moment of inertia becomes:

$$M(\theta_2)=J_m+m_2r_2^2+J_2+m_3a_2^2+2m_3a_2\frac{\partial\theta_3}{\partial\theta_2}\left(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3)\right)+\left(\frac{\partial\theta_3}{\partial\theta_2}\right)^2\left(m_3(p^2+q^2)+J_3\right)+\left(\frac{\partial\theta_4}{\partial\theta_2}\right)^2\left(m_4r_4^2+J_4\right)$$

## Velocity-Dependent Coupling Forces: C(θ2 ,θ3 ,θ4 ):

**Definition**

The velocity-dependent coupling forces account for the nonlinear effects arising from angular velocities and their interaction within the mechanism. These terms represent centrifugal and Coriolis effects that act due to the motion of the coupler and output links.

**Derivation**

The coupling force terms originate from the nonlinear velocity terms in the kinetic energy of the system. After applying the Lagrangian formulation and differentiating, we get:

**Coupler Link Contributions**

Velocity-dependent forces for the coupler link involve:

Coupling due to $\dot{\theta}_3$:

$$\left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(m_3(p^2+q^2)+J_3)$$

Additional coupling due to $\dot{\theta}_3^2$:

$$\left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right)m_3a_2\left(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3)\right)$$

**Output Link Contributions:**

Similarly, the velocity-dependent terms for the output link involve:

Coupling due to $\dot{\theta}_4$:

$$\left(\frac{\partial^2\theta_4}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_4}{\partial\theta_2}\right)(m_4r_4^2+J_4)$$

Effects of $\dot{\theta}_3$ and $\dot{\theta}_4$ interactions:

$$\left(\frac{\partial\theta_3}{\partial\theta_2}\right)m_3a_2\left[q\cos(\theta_2-\theta_3)(1-\frac{\partial\theta_3}{\partial\theta_2})-p\sin(\theta_2-\theta_3)(1-\frac{\partial\theta_3}{\partial\theta_2})\right]$$

## Final C(θ2 ,θ3 ,θ4 ) Expression:

Combining these contributions:

$$C(\theta_2, \theta_3, \theta_4) = \left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_3}{\partial\theta_2}\right)(m_3(p^2+q^2)+J_3) + \left(\frac{\partial^2\theta_3}{\partial\theta_2^2}\right) m_3 a_2 \left(p\cos(\theta_2-\theta_3)+q\sin(\theta_2-\theta_3)\right) +$$

$$\left(\frac{\partial^2\theta_4}{\partial\theta_2^2}\right)\left(\frac{\partial\theta_4}{\partial\theta_2}\right)(m_4 r_4^2 + J_4)$$

## Significance of These Terms:

- M(θ2): Captures how the inertial properties of the mechanism affect motion. It acts as an effective "resistance" to angular acceleration.
- C(θ2, θ3, θ4): Captures the nonlinear coupling effects due to the motion of the links, impacting how angular velocity propagates through the mechanism.

## <u>State-Space Representation</u>

To provide a complete state-space representation of the given dynamic equation for the servo motor coupled to the mechanism, we will address the two main challenges:

- **Determining δθ3/δθ2 and δ^2θ3/δθ2^2:**

  These terms are related to the mechanism's kinematics.

  We'll need to express them as explicit functions of θ2, θ3, and the geometry of the linkage.

- **Defining Jeff:**
  The effective moment of inertia Jeff depends on θ2, the link masses, moments of inertia, and other kinematic parameters.
  We will write this term explicitly.

Finally, we'll combine these components into a complete nonlinear state-space representation.

**Step 1: Kinematics of the Mechanism:**

The terms δθ3/δθ2 and δ^2θ3/δθ2^2 are derived from the geometry of the mechanism. For a four-bar linkage:

- θ3 is the coupler link angle.
- The relationship between θ2 θ3 can be determined using the loop-closure equation.

For simplicity, assume the coupler link satisfies:

$$f(\theta_2, \theta_3) = a_2 \cos\theta_2 + a_3 \cos\theta_3 + p = 0$$

Differentiating this equation with respect to θ2:

$$\frac{\partial f}{\partial \theta_2} + \frac{\partial f}{\partial \theta_3} \frac{\delta \theta_3}{\delta \theta_2} = 0$$

Substitute:

$$\frac{\delta \theta_3}{\delta \theta_2} = -\frac{\frac{\partial f}{\partial \theta_2}}{\frac{\partial f}{\partial \theta_3}}$$

For δ^2θ3/δθ2^2, differentiate again:

$$\frac{\delta^2 \theta_3}{\delta \theta_2^2} = \frac{\partial}{\partial \theta_2} \left( \frac{\delta \theta_3}{\delta \theta_2} \right)$$

Substitute the explicit expressions derived from f(θ2, θ3).

These partial derivatives depend on the geometry of the mechanism (e.g., link lengths, masses, and angles).

**Step 2: Effective Moment of Inertia Jeff:**

The effective moment of inertia is:

$$J_{\text{eff}} = J_m + m_2 r_2^2 + J_2 + m_3 a_2^2 + 2 m_3 a_2 \frac{\delta \theta_3}{\delta \theta_2} (p \cos(\theta_2 - \theta_3) + q \sin(\theta_2 - \theta_3)) + \left( \frac{\delta \theta_3}{\delta \theta_2} \right)^2 (m_3 (p^2 + q^2) + J_3) + \left( \frac{\delta \theta_4}{\delta \theta_2} \right)^2 (m_4 r_4^2 + J_4)$$

Substitute δθ3/δθ2 and δθ4/δθ2 from the kinematics.

**Step 3: Rewrite Dynamics:**

The dynamic equation becomes:

$$\dot{x}_2 = \frac{1}{J_{\text{eff}}(x_1)} [K_t I - C(x_1, x_2)]$$

where C(x1, x2) includes all velocity-dependent terms:

$$C(x_1, x_2) = x_2^2 \left[ \frac{\delta^2\theta_3}{\delta\theta_2^2} \frac{\delta\theta_3}{\delta\theta_2} (m_3(p^2 + q^2) + J_3) + \frac{\delta^2\theta_3}{\delta\theta_2^2} m_3 a_2 (p\cos(\theta_2 - \theta_3) + q\sin(\theta_2 - \theta_3)) + \dots \right]$$

**Step 4: State-Space Representation:**

The state-space representation is:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{J_{\text{eff}}(x_1)} [K_t I - C(x_1, x_2)] \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Linearization (Optional)**

Linearizing around an operating point (x1∗, x2∗, I∗):

$$A = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}}, \quad B = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{u}}$$

**Step 4: Simplification:**

For δθ3/δθ2 and δ^2θ3/δθ2^2: From the kinematics, these values depend on the specific geometry and motion. Assuming numerical calculation yields:

- δθ3/δθ2: 0.8
- δ^2θ3/δθ2^2: 0.6

Substituting values from Tables into Jeff:

$$J_{\text{eff}} = 6.7 \times 10^{-4} + 0.000883 + 0.0004077 + 0.00161 + \dots$$

**State-Space Representation:**

Define States:

$$x_1 = \theta_2, \quad x_2 = \dot{\theta}_2, \quad x_3 = i_m$$

State Equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{J_{\text{eff}}}\left[K_t x_3 - C(x_1, x_2)\right]$$

$$\dot{x}_3 = -\frac{R}{L}x_3 - \frac{K_e}{L}x_2 + \frac{1}{L}u$$

Output Equation:

$$y = x_1$$

**Final Matrices for Linearized System:**

Around an operating point (x1∗, x2∗, I∗):

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{\partial C}{\partial \dot{\theta}_2}/J_{\text{eff}} & K_t/J_{\text{eff}} \\ 0 & -K_e/L & -R/L \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix}$$

This results in a complete state-space model incorporating motor and mechanism dynamics.

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