# **Exercise Session: Game Theory I**

# COMP4418 Knowledge Representation and Reasoning

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These slides are based on lecture slides by Prof. Felix Brandt.

a) Consider the following game. Which outcomes are Pareto-optimal? Can the game be solved by iterated strict dominance?

	V	/	×	(	У	/	Z	:
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	V	/	>	(	У	′	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	V	/	×	(	У	/	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	×	(	У	/	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	W	1	×	(	У	′	Z	
a	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	×	(	У	/	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	X		у		Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	X		у		Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	×	X		/	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	X		у		Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	·	/	×	X		y z		
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	W	1	×	x y		/	Z	
а	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

Can the game be solved by iterated strict dominance?

	W	/	×	(	у	•	Z	
a	0	0	3	2	2	1	1	0
b	3	1	0	1	5	1	3	2
С	3	2	2	1	0	5	0	1
d	5	1	1	4	4	0	0	0

	W	/	×	(	У	/	Z		$\frac{1}{3}x + \frac{1}{3}$	$y + \frac{1}{3}z$
а	0	0	3	2	2	1	1	0	2	1
b	3	1	0	1	5	1	3	2	8 3	<del>4</del> / <del>3</del>
С	3	2	2	1	0	5	0	1	<u>2</u> 3	<del>7</del> /3
d	5	1	1	4	4	0	0	0	<u>2</u> 3	4/3

	W	/	×	(	У	,	Z		$\frac{1}{3}x + \frac{1}{3}$	$y + \frac{1}{3}z$
a	0	0	3	2	2	1	1	0	2	1
b	3	1	0	1	5	1	3	2	8 3	<del>4</del> / <del>3</del>
С	3	2	2	1	0	5	0	1	<u>2</u> 3	<del>7</del> /3
d	5	1	1	4	4	0	0	0	<u>2</u> 3	4/3

	X		У	,	Z		
a	3	2	2	1	1	0	
b	0	1	5	1	3	2	
С	2	1	0	5	0	1	
d	1	4	4	0	0	0	

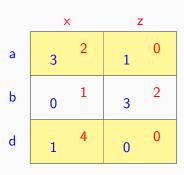
	Х		У	,	Z	
a	3	2	2	1	1	0
b	0	1	5	1	3	2
С	2	1	0	5	0	1
d	1	4	4	0	0	0

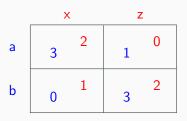
	X		У	/	Z	
a	3	2	2	1	1	0
b	0	1	5	1	3	2
d	1	4	4	0	0	0

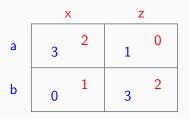
	×	(	У	,	Z		$\frac{2}{3}x +$	$-\frac{1}{3}y$
а	3	2	2	1	1	0	7/3	<del>4</del> / <del>3</del>
b	0	1	5	1	3	2	1	<del>4</del> <del>3</del>
d	1	4	4	0	0	0	<u>2</u> 3	8/3

	Х	y	y z	
a	3 2	2 1	1 0	$\frac{7}{3}$ $\frac{4}{3}$
b	0 1	5 1	3 2	$1^{\frac{4}{3}}$
d	1 4	4 0	0 0	2 8 3 3

	×	(	Z		
a	3	2	1	0	
b	0	1	3	2	
d	1	4	0	0	







None of the remaining actions is dominated.

b) Consider the following game. Which outcomes are Pareto-optimal? Can the game be solved by iterated strict dominance?

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	
<i>a</i> <sub>1</sub>	(2, <mark>3</mark> , 2)	(0, 5, 2)		(4, <mark>5</mark> , 1)	
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	a <sub>2</sub>	(2, 0, 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4,</b> 2)	<i>a</i> 3	(1, 2, 0)	(2, 2, 1)
	-	1		C	2

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, 5, 1)	(1, 0, 1)
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, <mark>0</mark> , 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
	C	1		C	2

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, 5, 1)	(1, 0, 1)
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, <mark>0</mark> , 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4, 2)</b>	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
		1		C	2

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, 5, 1)	(1, <mark>0</mark> , 1)
<b>a</b> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, 0, 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4, 2)</b>	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
<i>c</i> <sub>1</sub>				-	

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, 5, 1)	(1, 0, 1)
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, 0, 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)		(1, 2, 0)	
$c_1$				C	2

	<i>b</i> <sub>1</sub>	<i>b</i> <sub>2</sub>		$b_1$	
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)		(4, 5, 1)	
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, 0, 3)	(1, <b>5</b> , 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$c_1$				C	· · · · · · · · · · · · · · · · · · ·

	<i>b</i> <sub>1</sub>	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)		(4, 5, 1)	
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	a <sub>2</sub>	(2, 0, 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
c <sub>1</sub>			•	C	2

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)		(4, 5, 1)	
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, 0, 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$c_1$				C	<u> </u>

	<i>b</i> <sub>1</sub>	<i>b</i> <sub>2</sub>		$b_1$	
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)		(4, <b>5</b> , 1)	
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	<i>a</i> <sub>2</sub>	(2, 0, 3)	(1, <mark>5</mark> , 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4,</b> 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$c_1$				C	2

	<i>b</i> <sub>1</sub>	$b_2$		$b_1$	
<i>a</i> <sub>1</sub>	(2, 3, 2)	(0, 5, 2)	<i>a</i> <sub>1</sub>	(4, 5, 1)	(1, 0, 1)
<i>a</i> <sub>2</sub>	(1, 4, 1)	(2, 1, 1)	<i>a</i> <sub>2</sub>	(2, <b>0</b> , 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$c_1$				C	2

Can the game be solved by iterated strict dominance?

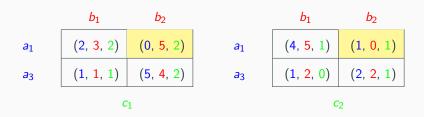
	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	$b_2$
<i>a</i> <sub>1</sub>	(2, <mark>3</mark> , 2)	(0, 5, 2)	a <sub>1</sub>	(4, 5, 1)	(1, <mark>0</mark> , 1)
a <sub>2</sub>	(1, 4, 1)	(2, <b>1</b> , 1)	a <sub>2</sub>	(2, <mark>0</mark> , 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4, 2)</b>	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)

	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	$b_2$
$a_1$	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, <b>5</b> , 1)	(1, <mark>0</mark> , 1)
a <sub>2</sub>	(1, 4, 1)	(1, 1, 1)	a <sub>2</sub>	(2, <mark>0</mark> , 3)	(1, 5, 3)
a <sub>3</sub>	(1, 1, 1)	(5, 4, 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{3}{2}, 2, \frac{3}{2})$	$(\frac{5}{2}, \frac{9}{2}, 2)$	$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$\left(\frac{5}{2}, \frac{7}{2}, \frac{1}{2}\right)$	$(\frac{3}{2}, 1, 1)$
	C	i		C	······································

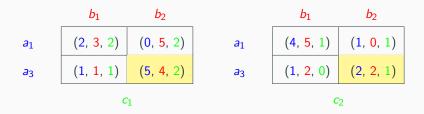
	$b_1$	<i>b</i> <sub>2</sub>		$b_1$	<i>b</i> <sub>2</sub>
	(2, 3, 2)		a <sub>1</sub>	(4, <b>5</b> , 1)	(1, <mark>0</mark> , 1)
<i>a</i> <sub>2</sub>	(1, 4, 1)	(1, <b>1</b> , 1)	a <sub>2</sub>	(2, <mark>0</mark> , 3)	(1, 5, 3)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4, 2)</b>	a <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{3}{2}, 2, \frac{3}{2})$	$\left(\frac{5}{2}, \frac{9}{2}, 2\right)$	$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$\left(\frac{5}{2}, \frac{7}{2}, \frac{1}{2}\right)$	$(\frac{3}{2}, 1, 1)$
				_	

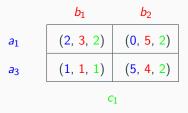
	$b_1$	$b_2$		$b_1$	<i>b</i> <sub>2</sub>
$a_1$	(2, 3, 2)	(0, 5, 2)	$a_1$	(4, 5, 1)	(1, 0, 1)
a <sub>3</sub>	(1, 1, 1)	<b>(5, 4,</b> 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
	C	ì		C	2

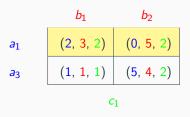
	$b_1$	$b_2$		$b_1$	<i>b</i> <sub>2</sub>
$a_1$	(2, 3, 2)	(0, <b>5</b> , 2)			(1, 0, 1)
a <sub>3</sub>	(1, 1, 1)	<b>(5, 4,</b> 2)	a <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
	C	ì		C	2

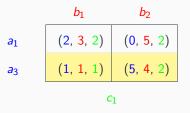


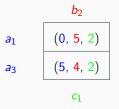
	$b_1$	$b_2$		$b_1$	<i>b</i> <sub>2</sub>
	(2, 3, 2)				(1, 0, 1)
<i>a</i> <sub>3</sub>	(1, 1, 1)	<b>(5, 4,</b> 2)	<i>a</i> <sub>3</sub>	(1, 2, 0)	(2, 2, 1)
	C	ì		c	2



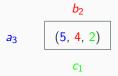












The game can be solved via iterated strict dominance!

a) Consider the following formulation of rock-paper-scissors. What are the maximin strategies and the security level of both players?

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
Р	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
Р	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
Р	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

$$u_{\min}(s) = \min(u_1(s, R), u_1(s, P), u_i(s, S))$$

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
P	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

$$u_{\min}(s) = \min(u_1(s, R), u_1(s, P), u_i(s, S))$$
  
=  $\min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S),$ 

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
P	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

$$u_{\min}(s) = \min(u_1(s, R), u_1(s, P), u_i(s, S))$$
  
=  $\min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S),$   
 $-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S),$ 

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
Р	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

$$\begin{aligned} u_{\min}(s) &= \min(u_1(s, R), u_1(s, P), u_i(s, S)) \\ &= \min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S), \\ &-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S), \\ &1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S)) \end{aligned}$$

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \ge u$$
 (1)

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \ge u \tag{3}$$

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \ge u$$
 (1)

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \ge u$$
 (3)  
 $s(R) + s(P) + s(S) = 1$ 

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \ge u$$
 (1)

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \ge u \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

$$(0-1+1)s(R)+(1+0-1)s(P)+(-1+1+0)s(S)\geq 3u$$

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \ge u$$
 (1)

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \ge u$$

$$s(R) + s(P) + s(S) = 1$$
(3)

$$s(R) \ge 0, s(P) \ge 0, s(S) \ge 0$$

$$(0-1+1)s(R) + (1+0-1)s(P) + (-1+1+0)s(S) \ge 3u$$
  
 $\iff 0 \ge u$ 

$$s(P) - s(S) \ge 0 \tag{1}$$

$$-s(R) + s(S) \ge 0 \tag{2}$$

$$s(R) - s(P) \ge 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

$$s(P) - s(S) \ge 0 \tag{1}$$

$$-s(R) + s(S) \ge 0 \tag{2}$$

$$s(R) - s(P) \ge 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \ge 0, s(P) \ge 0, s(S) \ge 0$$

$$s(P) \ge s(S) \ge s(R) \ge s(P)$$

$$s(P) - s(S) \ge 0 \tag{1}$$

$$-s(R) + s(S) \ge 0 \tag{2}$$

$$s(R) - s(P) \ge 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

$$s(P) \ge s(S) \ge s(R) \ge s(P)$$
  
 $\implies s(P) = s(S) = s(R)$ 

$$s(P) - s(S) \ge 0 \tag{1}$$

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$$s(R) - s(P) \ge 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

$$s(P) \geq s(S) \geq s(R) \geq s(P)$$

$$\implies s(P) = s(S) = s(R)$$

$$\Longrightarrow s(P) = s(S) = s(R) = \frac{1}{3}$$

	F	?	F	)	5	5
R	0	0	-1	1	1	-1
Р	1	-1	0	0	-1	1
S	-1	1	1	-1	0	0

The maximin strategy of player 1 is given by  $s(R) = s(P) = s(S) = \frac{1}{3}$ . His security level is 0.

A symmetric analysis shows that player 2 has the same maximin strategy and security level.

	R		F	)	S		
R	0	0	-1	1	1	-1	
Р	1	-1	0	0	-1	1	
S	-1	1	1	-1	0	0	

	R		Р		S		W
R	0	0	-1	1	1	-1	
Р	1	-1	0	0	-1	1	
S	-1	1	1	-1	0	0	
W							

	R		Р		S		W	
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1		
S	-1	1	1	-1	0	0		
W								

	R		Р		S		W	
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1		
S	-1	1	1	-1	0	0		
W	1	-1						

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	F	?	F	)		5	V	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0		
W	1	-1	-1	1				

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	F	3	F	)		5	V	<b>/</b>
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1		

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	F	3	F	)		5	V	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

	F	2	F	)	9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

	F	?	F	Р		S		V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

max u

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u$$
 (1)

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u$$
 (2)

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \qquad (3)$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \qquad (4)$$
  
$$s \in \mathcal{L}(A_1)$$

	F	?	F	Р		S		V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u \tag{1}$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

	F	?	F	Р		S		V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u \tag{1}$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

It is always weakly better for player 1 to put probability on W rather than on R.

	F	?	F		9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

	F	?	F	Р		S		V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

	F	?	F	)	9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \ge u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

Inequality (4) makes than Inequality (1) redundant.

	F	?	F		9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

	F	?	F		9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

These are the same conditions we had before!

	F	?	F	)	9	5	٧	V
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	1	-1
S	-1	1	1	-1	0	0	-1	1
W	1	-1	-1	1	1	-1	0	0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \ge u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \ge u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \ge u \tag{4}$$

These are the same conditions we had before! The maximin strategy of player 1 is given by  $s(P) = s(S) = s(W) = \frac{1}{3}$  and his security level is 0.

c) Assume there is lava as a fourth option. Lava beats all other option, but if both players play lava, they both experience a super lose with a utility of -100. What are the maximin strategies and the security levels of both players?

c) Assume there is lava as a fourth option. Lava beats all other option, but if both players play lava, they both experience a super lose with a utility of -100. What are the maximin strategies and the security levels of both players?

	R	Р	S	L	
R	0 0	-1 1	1 -1	-1	
Р	1 -1	0 0	-1 1	-1	
S	-1	1 -1	0 0	-1	
L	1 -1	1 -1	1 -1	-100	

	R		F	)	5	5	L	
R	0	0	-1	1	1	-1	-1	1
Р	1	-1	0	0	-1	1	-1	1
S	-1	1	1	-1	0	0	-1	1
L	1	-1	1	-1	1	-1	-100	-100

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \ge u$$

$$s \in \mathcal{L}(A_1)$$

	R	Р	S	L
R	0 0	-1	1 -1	-1
Р	1 -1	0 0	-1	-1
S	-1 1	1 -1	0 0	-1
L	1 -1	1 -1	1 -1	-100

subject to 
$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(L) \ge u$$

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \ge u$$

$$s \in \mathcal{L}(A_1)$$

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \ge u$$

The security level of player 1 is at most -1.

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \ge u$$

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The security level of player 1 can only be -1 if he never plays lava!

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \ge u$$

The security level of player 1 is at most -1.

The security level of player 1 can only be -1 if he never plays lava!

$$\begin{aligned} \max u \\ \text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) &\geq u \\ -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) &\geq u \\ 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) &\geq u \\ -1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) &\geq u \\ s &\in \mathcal{L}(A_1) \end{aligned}$$

	R	Р	S	L
R	0 0	-1	1 -1	-1
Р	1 -1	0 0	-1	-1
S	-1	1 -1	0 0	-1
L	1 -1	1 -1	1 -1	-100

Every strategy s with s(L) = 0 is a maximin strategy! The security level of both players is -1.

Assume that  $A=\{a,b,c\}$  and let  $\succsim$  denote a rational and independent preference relation on  $\mathcal{L}(A)$  such that  $[1:a]\succ [1:b]$  and  $[\frac{1}{2}:b,\frac{1}{2}:c]\sim [\frac{2}{3}:a,\frac{1}{3}:c]$ .

a) Show that  $[1:c] \succ [1:a]$ .

A preference relation  $\succsim$  on  $\mathcal{L}(A)$  is

• rational if

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- rational if
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  - and transitive:  $L_1 \succsim L_2$  and  $L_2 \succsim L_3$  implies  $L_1 \succsim L_3$  for all  $L_1, L_2, L_3 \in \mathcal{L}(A)$ .

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  - and transitive:  $L_1 \succsim L_2$  and  $L_2 \succsim L_3$  implies  $L_1 \succsim L_3$  for all  $L_1, L_2, L_3 \in \mathcal{L}(A)$ .
- continuous if, for all  $L_1, L_2, L_3 \in \mathcal{L}(A)$  with  $L_1 \succ L_2 \succ L_3$ , there is  $\epsilon > 0$  such that

$$[1-\epsilon:L_1,\epsilon:L_3]\succ L_2\succ [1-\epsilon:L_3,\epsilon:L_1].$$

A preference relation  $\succeq$  on  $\mathcal{L}(A)$  is

- rational if
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  - and transitive:  $L_1 \succsim L_2$  and  $L_2 \succsim L_3$  implies  $L_1 \succsim L_3$  for all  $L_1, L_2, L_3 \in \mathcal{L}(A)$ .
- continuous if, for all  $L_1, L_2, L_3 \in \mathcal{L}(A)$  with  $L_1 \succ L_2 \succ L_3$ , there is  $\epsilon > 0$  such that

$$[1-\epsilon:L_1,\epsilon:L_3] \succ L_2 \succ [1-\epsilon:L_3,\epsilon:L_1].$$

• independent if, for all lotteries  $L_1, L_2, L_3$  and all  $p \in (0, 1)$ , it holds that

$$L_1 \succsim L_2 \iff [p:L_1,(1-p):L_3] \succsim [p:L_2,(1-p):L_3].$$

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- Let  $L_x = [1:x]$  for  $x \in \{a, b, c\}$ ,  $L_1 = [\frac{2}{3}:a, \frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b, \frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .

- Let  $L_x = [1:x]$  for  $x \in \{a, b, c\}$ ,  $L_1 = [\frac{2}{3}:a, \frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b, \frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .

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- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .

- Let  $L_x = [1:x]$  for  $x \in \{a,b,c\}$ ,  $L_1 = [\frac{2}{3}:a,\frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b,\frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .
- Now, assume that  $L_b \gtrsim L_c$ .

- Let  $L_x = [1:x]$  for  $x \in \{a,b,c\}$ ,  $L_1 = [\frac{2}{3}:a,\frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b,\frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = \left[\frac{2}{3} : L_a, \frac{1}{3} : L_c\right]$ .
- Now, assume that  $L_b \succsim L_c$ .
  - By independence,  $[\frac{3}{4}:L_b,\frac{1}{4}:L_b] \succsim [\frac{3}{4}:L_b,\frac{1}{4}:L_c]$

- Let  $L_x = [1:x]$  for  $x \in \{a,b,c\}$ ,  $L_1 = [\frac{2}{3}:a,\frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b,\frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .
- Now, assume that  $L_b \succsim L_c$ .
  - By independence,  $\left[\frac{3}{4}:L_b,\frac{1}{4}:L_b\right] \succsim \left[\frac{3}{4}:L_b,\frac{1}{4}:L_c\right]$
  - This shows that  $L_a \succ L_b \succsim L_3$ , contradiction.

- Let  $L_x = [1:x]$  for  $x \in \{a,b,c\}$ ,  $L_1 = [\frac{2}{3}:a,\frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b,\frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .
- Now, assume that  $L_a \succsim L_c \succ L_b$ .

- Let  $L_x = [1:x]$  for  $x \in \{a, b, c\}$ ,  $L_1 = [\frac{2}{3}:a, \frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b, \frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .
- Now, assume that  $L_a \succsim L_c \succ L_b$ .
  - By independence,  $\left[\frac{3}{4}:L_c,\frac{1}{4}:L_c\right]\succ \left[\frac{3}{4}:L_b,\frac{1}{4}:L_c\right]$

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- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$ .
- Now, assume that  $L_a \succsim L_c \succ L_b$ .
  - By independence,  $[\frac{3}{4}:L_c,\frac{1}{4}:L_c] \succ [\frac{3}{4}:L_b,\frac{1}{4}:L_c]$
  - This shows that  $L_a \succ L_c \succsim L_3$ , contradiction.

- Let  $L_x = [1:x]$  for  $x \in \{a, b, c\}$ ,  $L_1 = [\frac{2}{3}:a, \frac{1}{3}:c]$ , and  $L_2 = [\frac{1}{2}:b, \frac{1}{2}:c]$
- By assumption,  $L_2 \sim L_1$ .
- Let  $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$ . It holds that  $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$ .
- By independence,  $L_3 \sim L_a$  since  $L_1 = \left[\frac{2}{3} : L_a, \frac{1}{3} : L_c\right]$ .
- Now, assume that  $L_a \succsim L_c \succ L_b$ .
  - By independence,  $[\frac{3}{4}:L_c,\frac{1}{4}:L_c] \succ [\frac{3}{4}:L_b,\frac{1}{4}:L_c]$
  - This shows that  $L_a \succ L_c \succsim L_3$ , contradiction.
- Hence, the only possibility is that  $L_c \succ L_a \succ L_b$ .

Assume that  $A = \{a, b, c\}$  and let  $\succsim$  denote a rational and independent preference relation on  $\mathcal{L}(A)$  such that  $[1:a] \succ [1:b]$  and  $[\frac{1}{2}:b,\frac{1}{2}:c] \sim [\frac{2}{3}:a,\frac{1}{3}:c]$ .

b) Show that, if  $\succsim$  is additionally continuous, then it can be represented by the vNM utility function u given by u(c)=1,  $u(a)=\frac{1}{4},\ u(b)=0$ .

8

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- vNM utility functions are invariant under addition. Hence, define u' by u'(x) = u(x) u(b) for all  $x \in \{a, b, c\}$
- vNM utility functions are invariant under multiplication with a positive scalar. Hence, define v(x) = u'(x)/u'(c) for all  $x \in \{a, b, c\}$ .

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- Finally,  $\left[\frac{1}{2}:b,\frac{1}{2}:c\right]\sim\left[\frac{2}{3}:a,\frac{1}{3}:c\right]$  implies that

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• Hence,  $\succeq$  is represented by the utility function v with v(c)=1,  $v(a)=\frac{1}{4}$ , and v(b)=0.

Let  $\succsim$  denote the rational preference relation over a set

$$A = \{x_1, \dots, x_m\}$$
 given by  $x_1 \succ x_2 \succ \dots \succ x_m$ .

Is the following relation a rational preference relation on  $\mathcal{L}(A)$ ? Is it continuous and independent? Prove your answers!

The relation  $\succsim_1$  is defined by  $L_1 \succsim_1 L_2$  if and only if  $x \succsim y$  for all  $x, y \in A$  with  $L_1(x) > 0$  and  $L_2(y) > 0$ .

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  - It holds that  $x_1 \succ x_2$  and  $L_1(x_2) > 0$  and  $L_2(x_1) > 0$ , so  $L_1 \not\succsim_1 L_2$ .

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  - Similarly,  $L(x_1) > 0$  and  $L_2(x_2) > 0$ , so  $L_2 \not \succsim_1 L_1$ .

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  - It holds that
    - $x \succeq y$  for all  $x, y \in A$  with  $L_1(x) > 0$  and  $L_2(y) > 0$  and
    - $y \gtrsim z$  for all  $y, z \in A$  with  $L_2(y) > 0$  and  $L_3(z) > 0$ .

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  - This means that  $L_1 \succsim_1 L_3$ .

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    - However, for every  $\epsilon > 0$ ,  $[1 \epsilon : L_1, \epsilon : L_3] \not\gtrsim L_2$  because  $L(x_3) = \epsilon > 0$  for  $L = [1 \epsilon : L_1, \epsilon : L_3]$  and  $L_2(x_2) > 0$ .

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  - $\succsim_1$  fails independence:
    - Let  $L_4 = [\frac{1}{2} : L_1, \frac{1}{2} : L_3]$  and  $L_5 = [\frac{1}{2} : L_2, \frac{1}{2} : L_3]$ .

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    - While  $L_1 \succ_1 L_2$ , we have  $L_4 \not\succ_1 L_5$ .

Let  $\succeq$  denote the rational preference relation over a set  $A = \{x_1, \dots, x_m\}$  given by  $x_1 \succ x_2 \succ \dots \succ x_m$ .

Is the following relation a rational preference relation on  $\mathcal{L}(A)$ ? Is it continuous and independent? Prove your answers!

b) We define  $\max(\succsim, X)$  as the most preferred alternative in X and  $\Delta(L_1, L_2) = \max(\succsim, \{x \in A \colon L_1(x) \neq L_2(x)\})$ . The relation  $\succsim_2$  is defined by  $L_1 \succsim_2 L_2$  if and only if  $L_1 = L_2$  or  $L_1(\Delta(L_1, L_2)) \geq L_2(\Delta(L_1, L_2))$ .

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"Lexicographic preferences"

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  - If  $L1 \neq L_2$ , then  $\Delta(L_1, L_2)$  is well-defined, so either  $L_1 \succsim_2 L_2$  or  $L_2 \succsim_2 L_2$ .

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  - Assume  $L_1 \neq L_2$  and  $L_2 \neq L_3$ . Let  $x_1 = \Delta(L_1, L_2)$ ,  $x_2 = \Delta(L_2, L_3)$ , and  $x^* = \max(\succsim, \{x_1, x_2\})$ .

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  - By definition, we have that  $L_1(x) = L_2(x) = L_3(x)$  for all x with  $x \succ x^*$ .
  - If  $x^* = x_1$ , then  $L_1(x^*) > L_2(x^*) \ge L_3(x^*)$ . Hence,  $\Delta(L_1, L_3) = x^*$  and  $L_1 \succsim_2 L_3$ .

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  - If  $L_1 = L_2$  or  $L_2 = L_3$ , it trivially holds that  $L_1 \succsim_2 L_3$ .
  - Assume  $L_1 \neq L_2$  and  $L_2 \neq L_3$ . Let  $x_1 = \Delta(L_1, L_2)$ ,  $x_2 = \Delta(L_2, L_3)$ , and  $x^* = \max(\succsim, \{x_1, x_2\})$ .
  - By definition, we have that L<sub>1</sub>(x) = L<sub>2</sub>(x) = L<sub>3</sub>(x) for all x with x ≻ x\*.
  - If  $x^* = x_1$ , then  $L_1(x^*) > L_2(x^*) \ge L_3(x^*)$ . Hence,  $\Delta(L_1, L_3) = x^*$  and  $L_1 \succsim_2 L_3$ .
  - If  $x^* = x_2$ , then  $L_1(x^*) \ge L_2(x^*) > L_3(x^*)$ . Hence,  $\Delta(L_1, L_3) = x^*$  and  $L_1 \succsim_2 L_3$ .

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- Let  $L_1, L_2, L_3 \in \Delta(A)$ . Moreover, let  $\lambda \in (0, 1)$  and  $L_4 = [\lambda : L_1, 1 \lambda : L_3]$  and  $L_5 = [\lambda : L_2, 1 \lambda : L_3]$ .

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$$L_4(x) - L_5(x) = \lambda L_1(x) + (1 - \lambda)L_3(x) - \lambda L_2(x) + (1 - \lambda)L_3(x)$$
  
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  - However, for every  $\epsilon > 0$ , we have that  $\Delta([\epsilon : L_1, 1 \epsilon : L_3], L_2) = x_1$ .
  - Hence, it holds for every  $\epsilon > 0$  that  $[\epsilon : L_1, 1 \epsilon : L_3] \succ_2 L_2$ .

Let  $\succeq$  denote the rational preference relation over a set

 $A = \{x_1, \dots, x_m\}$  given by  $x_1 \succ x_2 \succ \dots \succ x_m$ .

Is following relation a rational preference relation on  $\mathcal{L}(A)$ ? Is it continuous and independent? Prove your answers!

The relation  $\succeq_3$  is defined by  $L_1 \succeq_3 L_2$  if and only if  $L_1(x_1) \geq L_2(x_1)$ .

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    - Hence,  $L_1(x_1) \ge L_2(x_1) \ge L_3(x_1)$  and thus also  $L_1 \succsim_3 L_3$

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  - Hence,  $L_1(x_1) \ge L_2(x)$  if and only if  $L_4(x_1) \ge L_5(x)$ .

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  - There is  $\epsilon \in (0,1)$  such that

$$(1-\epsilon)L_1(x_1) + \epsilon L_3(x_1) > L_2(x_1) > (1-\epsilon)L_3(x_1) + \epsilon L_1(x).$$

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• Hence,  $[1 - \epsilon : L_1, \epsilon : L_3] \succ_3 L_2 \succ_3 [1 - \epsilon : L_3, \epsilon : L_1]$ .