Game Theory I

COMP4418 Knowledge Representation and Reasoning

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These slides are based on lecture slides by Prof. Felix Brandt.

What is Game Theory?

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What is Game Theory?

- Mathematical study of strategic behavior in interactive environments, where the well-being of each agent depends on their own actions and those of other agents.
- "Interactive Decision Theory"
- Central questions:
  - How do we formalize rational decision-making?
  - How should decision makers act in interactive situations?
  - Can we compute solutions efficiently?

What is Game Theory?

An example: Rock-Paper-Scissors

			
	$\frac{1}{2}, \frac{1}{2}$	$0, 1$	$1, 0$
	$1, 0$	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
	$0, 1$	$1, 0$	$\frac{1}{2}, \frac{1}{2}$

Decision Making under Uncertainty

Rational Agents

- An agent is an autonomous entity that has the ability to interact with its environment.
  - e.g., humans, robots, software
- A prerequisite for rational decision making are preferences over the set of alternatives A.
  - typically modeled as binary preference relations on A
  - x is at least as good as y: $x \succsim y$
  - preference relations $\succsim$ can be decomposed into
    - a strict part $\succ$ ($x \succ y \iff x \succsim y$ and not $y \succsim x$)
    - an indifference part $\sim$ ($x \sim y \iff x \succsim y$ and $y \succsim x$)
  - E.g.,: I like apples more than bananas and am indifferent between bananas and cherries: apples $\succ$ bananas $\sim$ cherries

Rational Agents

- A preference relation is called rational if it is
  - complete: $\forall x, y \in A : x \succsim y \lor y \succsim x$
  - transitive: $\forall x, y, z \in A : x \succsim y \land y \succsim z \implies x \succsim z$
- Preference relations are agnostic to the intensity of preferences.
- An agent is called rational if he chooses the most desirable among all feasible alternatives according to some rational preference relation.
- Remark: transitivity is a strong and sometimes unrealistic assumption!
  - E.g.: I am always indifferent between adding one more grain of sugar to my coffee, but I strictly prefer coffee without sugar to coffee with a full spoon of sugar.

From Preferences to Utilities

- A utility function $u : A \to \mathbb{R}$ assigns a value to every alternative measuring its quality from an individual's perspective.
- A utility function u represents the preference relation $\succsim$ if $x \succsim y \iff u(x) \geq u(y)$ for all $x, y \in A$.
  - E.g.: $u(\text{apples}) = 2$, $u(\text{bananas}) = 1$, $u(\text{cherries}) = 1$
- Proposition: For a countable number of alternatives, a preference relation can be represented by a utility function if and only if it is rational.
- While utility functions are capable of representing intensities of preferences, it should be carefully considered whether it is sensible to interpret them in such a context.

Uncertainty

- Many decisions are based on stochastic consequences.
  - E.g.: Studying at UNSW results with a higher chance in a good job than studying elsewhere.
- Uncertainty is typically modeled via lotteries over the set of alternatives (or outcomes) A.
- The set of all lotteries is $\mathcal{L}(A)=\{p \in [0,1]^A : \sum_{x\in A} p(x)=1\}$.
  - E.g., if you attempt to pass an exam without studying, your lottery of passing may be $L = [0.05 : \text{pass}, 0.95 : \text{fail}]$
- $L = [\alpha : L_1, 1-\alpha : L_2]$ is the compound lottery that executes lottery L_1 with probability α and lottery L_2 with probability $1-\alpha$.
  - Compound lotteries can be turned into simple lotteries by multiplying probabilities: $L(a) = \alpha L_1(a) + (1-\alpha) L_2(a)$ for all $a \in A$

Preferences over Lotteries

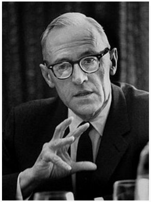
- We can still formalize preference relations over the (infinite) set of lotteries via preference relations!
  - e.g., $[1 : a] \succ [0.99 : a, 0.01 : b] \succ [0.98 : a, 0.02 : b] \succ \ldots$
  - e.g., $[1 : a] \succ [1 : b] \succ [0.99 : a, 0.01 : b] \succ \ldots$
- Preferences over lotteries typically follow some principles (e.g., comparing most likely outcomes, avoidance of randomization, expected utility).
- We will use the axiomatic method for explaining preferences over lotteries.

Continuity and Independence

- Continuity: For all lotteries L_1, L_2, L_3 with $L_1 \succ L_2 \succ L_3$, there is $\epsilon > 0$ such that $[1-\epsilon : L_1, \epsilon : L_3] \succ L_2 \succ [1-\epsilon : L_3, \epsilon : L_1]$
  - Assume you prefer going out with friends (L_1) to studying (L_2) to meeting your ex-partner (L_3). Continuity implies that, if the probability of meeting your ex-partner is sufficiently small, you still prefer going out with friends to studying.

- Independence: For all lotteries L_1, L_2, L_3 and all $p \in (0,1)$, it holds that
  $L_1 \succsim L_2 \iff [p : L_1, (1-p) : L_3] \succsim [p : L_2, (1-p) : L_3]$
  - Let L_1, L_2, L_3 as before. Independence implies that you prefer going out with your friends to studying when the chance of meeting your ex-partner is the same for both options.

vNM Utility Functions



Theorem (von Neumann and Morgenstern, 1947)
A preference relation $\succsim$ on $\mathcal{L}(A)$ is rational, continuous, and independent if and only if there is a utility function u on A such that

$$L_1 \succsim L_2 \iff \sum_{x \in A} L_1(x)u(x) \geq \sum_{x \in A} L_2(x)u(x)$$

for all lotteries $L_1, L_2 \in \mathcal{L}(A)$.

- This theorem justifies the use of expected utility.
- Utility functions are merely compact representations of preferences over lotteries.
- For every positive affine transformation $f(x) = \alpha x + \beta$ with $\alpha > 0$, $f(u(\cdot))$ is a new utility function representing the same preference relation.

The Basics of Game Theory

The Prisoner's Dilemma

- Two guilty suspects are interrogated separately as there is insufficient evidence.
- If both remain silent, they are only put in short-term pre-trial custody.
- If one testifies against the other, he will be immediately released and the silent accomplice receives a 10-year sentence
- If both betray each other, each suspect receives a 5-year sentence.

The Prisoner's Dilemma

	cooperate	defect
cooperate	2, 2	0, 3
defect	3, 0	1, 1

Verdict	Utility
10 years	0
5 years	1
1 week	2
Freedom	3

Normal-form Games

- Normal-form games formalize situations, where agents choose actions simultaneously, independent, and only once.
- A normal-form game is a tuple $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where
  - $N = \{1, \ldots, n\}$ is a set of players (or agents)
  - $A_i = \{a_i^1, \ldots, a_i^{k_i}\}$ is the set of actions available to player i,
  - $A = A_1 \times \cdots \times A_n$ is the set of all action profiles.
  - $A_{-i} = A_1 \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$ is the set of action profiles without player i.
  - $u_i : A \to \mathbb{R}$ is the utility function of player i.
  - For an action profile $a \in A$, $(u_1(a), \ldots, u_n(a))$ is called an outcome.

Normal-form Games

Example: Prisoner's dilemma

	cooperate	defect
cooperate	2, 2	0, 3
defect	3, 0	1, 1

- $N = \{1, 2\}$
- $A_1 = A_2 = \{\text{cooperate, defect}\}$
- $A = \{(\text{cooperate, cooperate}), (\text{cooperate, defect}),$
  $(\text{defect, cooperate}), (\text{defect, defect})\}$
- $u_1(\text{cooperate, cooperate}) = 2$, $u_1(\text{cooperate, defect}) = 0$
  $u_1(\text{defect, cooperate}) = 3$, $u_1(\text{defect, defect}) = 1$

Example: Diner's Dilemma (three-player version)

- Three friends go to a diner and agree to split the bill equally before ordering. There are two options, a cheap meal and an expensive one which is slightly better.
- We assume that each person prefers the expensive meal to the cheap one as long as he/she does not have to pay its full cost; if an agent gets the same dish in two situations, he/she prefers the cheaper situation.

Normal-form Games

Example: Diner's Dilemma (three-player version)

	cheap	expensive
cheap	(3, 3, 3)	(1, 5, 1)
expensive	(5, 1, 1)	(4, 4, 0)

cheap

	cheap	expensive
cheap	(1, 1, 5)	(0, 4, 4)
expensive	(4, 0, 4)	(2, 2, 2)

expensive

- $N = \{1, 2, 3\}$
- $A_1 = A_2 = A_3 = \{\text{cheap, expensive}\}$
- $A = \{(\text{cheap, cheap, cheap}), (\text{cheap, cheap, expensive}),$
  $(\text{cheap, expensive, cheap}), (\text{expensive, cheap, cheap}),$
  $(\text{cheap, expensive, expensive}), (\text{expensive, cheap, expensive}),$
  $(\text{expensive, expensive, cheap}), (\text{expensive, expensive, expensive})\}$
- $u_1(\text{cheap, cheap, cheap}) = 3,$
  $u_1(\text{cheap, cheap, expensive}) = 1, \ldots$

Dominance-based Solution Concepts

Pareto-optimality

- In both the Prisoner's dilemma and the Diner's dilemma, the likely outcomes seem undesirable for all agents.
- Outcome $(u_1(a), \ldots, u_n(a))$ Pareto-dominates outcome $(u_1(b), \ldots, u_n(b))$ if $u_i(a) \geq u_i(b)$ for all $i \in N$ and $u_i(a) > u_i(b)$ for at least one $i \in N$.
- An outcome is Pareto-optimal if it is not Pareto-dominated by any other outcome.
- In the Prisoner's dilemma, the outcome $(1, 1)$ is Pareto-dominated by $(2, 2)$. All other outcomes are Pareto-optimal.

2, 2	0, 3
3, 0	1, 1

Dominated and Dominant Actions

- Action a_i **dominates** action b_i if $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.
  - Action a_i dominates action b_i if, regardless of what the other agents do, a_i yields more utility than b_i.
- $b_i \in A_i$ is **dominated** if there is $a_i \in A_i$ such that a_i dominates b_i.
- $a_i \in A_i$ is **dominant** if it dominates all other strategies $b_i \in A_i \setminus \{a_i\}$.
- Example: To defect is a dominant action in the prisoner's dilemma.

2, 2	0, 3
3, 0	1, 1

An example

	x	y	z
a	1, 3	3, 0	1, 1
b	0, 0	2, 3	2, 1

- Which outcomes are Pareto-optimal?
- Which actions are dominated?
- Which actions are dominated when players may randomize and aim to maximize their expected utility?
- What if players assume that their opponents are rational?
- What if players assume that their opponents are rational and that there opponents know that they are rational?

Mixed Strategies and Mixed Dominance

- A mixed strategy $s_i \in S_i = \mathcal{L}(A_i)$ is a lottery over actions.
- The expected utility of a player in a given strategy profile $s \in S = S_1 \times \cdots \times S_n$ is $u_i(s) = \sum_{a \in A} u_i(a) \cdot \prod_{j=1}^{n} s_j(a_j)$.
- We assume vNM utility functions!
- Let $s_i, t_i \in S_i$. s_i dominates t_i if $u_i(s_i, s_{-i}) > u_i(t_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.
- Lemma: s_i dominates t_i if and only if $u_i(s_i, a_{-i}) > u_i(t_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

Iterated Dominance

- When agents are rational and the rationality of all agents is common knowledge, no agent will play a dominated strategy.
- By removing iterated strategies from consideration, more strategies may become dominated.
- A game can be solved via iterated (strict) dominance if only a single action profile survives the iterated elimination of dominated actions.
  - Actions may be dominated by mixed strategies!
- This process is independent of the order in which alternatives are removed.

Iterated Dominance

Can the below game be solved via iterated dominance?

	x	y	z
a	3, 1	0, 0	0, 0
b	1, 1	1, 2	5, 0
c	0, 1	4, 0	0, 0

Iterated Dominance

Can the below game be solved via iterated dominance?

	x	y
a	3, 1	0, 0
b	1, 1	1, 2
c	0, 1	4, 0

Iterated Dominance

Can the below game be solved via iterated dominance?

	x	y
a	3, 1	0, 0
c	0, 1	4, 0

Iterated Dominance

Can the below game be solved via iterated dominance?

	x
a	3, 1
c	0, 1

Iterated Dominance

Can the below game be solved via iterated dominance?

	x
a	3, 1

Iterated Weak Dominance

- Not every game is solvable with iterated dominance.
- A strategy s_i weakly dominates a strategy t_i if $u(s_i, s_{-i}) \geq u(t_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u(s_i, s_{-i}) > u(t_i, s_{-i})$ for at least one $s_{-i} \in S_{-i}$.
- Iterated weak dominance (IWD) is defined analogously to iterated strict dominance, but not every game is solvable by IWD.
- IWD is order-dependent and it is NP-hard to check whether a game can be solved by IWD (Conitzer and Sandholm, 2005).

2 1	0 0
0 0	1 2

Maximin and Security Level

Maximin

- In face of uncertainty, players may opt for maximizing their worst-case utility.
- The set of maximin strategies of player i is $\arg\max_{s_i} \min_{s_{-i}}(s_i, s_{-i})$.
- The security level of player i is $\max_{s_i} \min_{s_{-i}}(s_i, s_{-i})$.
- The security level is the minimal utility that a player can enforce, regardless of the behavior of other agents.
- Lemma: It holds for every game that $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \max_{s_i} \min_{a_{-i}} u_i(s_i, a_{-i})$.
- Maximin does not assume that agents are rational!

Maximin

Example: Battle of the Sexes

	boxing	ballet
boxing	2, 1	0, 0
ballet	0, 0	1, 2

The security level of both players is $\frac{2}{3}$. The row player achieves this by playing "boxing" with probability $\frac{1}{3}$ and "ballet" with probability $\frac{2}{3}$.

Further Reading

Reading

Social choice chapters of the following books:

- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic Game-Theoretic, and Logical Foundations. 2009. Section 3. `http://www.masfoundations.org/mas.pdf`
- N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Algorithmic Game Theory. Cambridge University Press, 2007. Section 1. `https://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf`
- M. Maschler, E. Solan, and S. Zamir. Game Theory. Cambridge University Press. 2015. Sections 2 to 5.

Image References

- Slide 2 (chess): https://assetsio.gnwcdn.com/chess-playing-hand.jpeg
- Slide 2 (poker): https://assets.editorial.aetnd.com/uploads/2013/04/gettyimages-200571196-004.jpg
- Slide 2 (auction): https://www.wine-searcher.com/images/news/40/73/christiesmainjpg-10004073.jpg
- Slide 2 (patrol): https://s.yimg.com/ny/api/res/1.2/kLhhEFRqM15nupDJRx402Q--/YXBwaWQ9aGlnaGxhbmRlcjt3PTI0MDA7aD0xMzUy/https://media.zenfs.com/en/nca_newswire_424/cff56ef79f5a0e3393aaa6c02cdf11ed
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