Exercise Session: Game Theory I

COMP4418 Knowledge Representation and Reasoning

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These slides are based on lecture slides by Prof. Felix Brandt.

Exercise I: Iterated Dominance

a) Consider the following game. Which outcomes are Pareto-optimal? Can the game be solved by iterated strict dominance?

	w	x	y	z
a	0, 0	3, 2	2, 1	1, 0
b	3, 1	0, 1	5, 1	3, 2
c	3, 2	2, 1	0, 5	0, 1
d	5, 1	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

Which outcomes are Pareto-optimal?

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b	3, 1	0, 1	5, 1	3, 2
c	3, 2	2, 1	0, 5	0, 1
d	5, 1	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

Can the game be solved by iterated strict dominance?

	w	x	y	z
a	0, 0	3, 2	2, 1	1, 0
b	3, 1	0, 1	5, 1	3, 2
c	3, 2	2, 1	0, 5	0, 1
d	5, 1	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

	w	x	y	z	$\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z$
a	0, 0	3, 2	2, 1	1, 0	2, 1
b	3, 1	0, 1	5, 1	3, 2	$\frac{8}{3}$, $\frac{4}{3}$
c	3, 2	2, 1	0, 5	0, 1	$\frac{2}{3}$, $\frac{7}{3}$
d	5, 1	1, 4	4, 0	0, 0	$\frac{2}{3}$, $\frac{4}{3}$

Exercise I: Iterated Dominance

	w	x	y	z	$\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z$
a	0, 0	3, 2	2, 1	1, 0	2, 1
b	3, 1	0, 1	5, 1	3, 2	$\frac{8}{3}$, $\frac{4}{3}$
c	3, 2	2, 1	0, 5	0, 1	$\frac{2}{3}$, $\frac{7}{3}$
d	5, 1	1, 4	4, 0	0, 0	$\frac{2}{3}$, $\frac{4}{3}$

Exercise I: Iterated Dominance

	x	y	z
a	3, 2	2, 1	1, 0
b	0, 1	5, 1	3, 2
c	2, 1	0, 5	0, 1
d	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

	x	y	z
a	3, 2	2, 1	1, 0
b	0, 1	5, 1	3, 2
c	2, 1	0, 5	0, 1
d	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

	x	y	z
a	3, 2	2, 1	1, 0
b	0, 1	5, 1	3, 2
d	1, 4	4, 0	0, 0

Exercise I: Iterated Dominance

	x	y	z	$\frac{2}{3}x + \frac{1}{3}y$
a	3, 2	2, 1	1, 0	$\frac{7}{3}$, $\frac{4}{3}$
b	0, 1	5, 1	3, 2	1, $\frac{4}{3}$
d	1, 4	4, 0	0, 0	$\frac{2}{3}$, $\frac{8}{3}$

Exercise I: Iterated Dominance

	x	y	z	$\frac{2}{3}x + \frac{1}{3}z$
a	3, 2	2, 1	1, 0	$\frac{7}{3}$, $\frac{4}{3}$
b	0, 1	5, 1	3, 2	1, $\frac{4}{3}$
d	1, 4	4, 0	0, 0	$\frac{2}{3}$, $\frac{8}{3}$

Exercise I: Iterated Dominance

	x	z
a	3, 2	1, 0
b	0, 1	3, 2
d	1, 4	0, 0

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a	3, 2	1, 0
b	0, 1	3, 2
d	1, 4	0, 0

Exercise I: Iterated Dominance

	x	z
a	3, 2	1, 0
b	0, 1	3, 2

Exercise I: Iterated Dominance

	x	z
a	3, 2	1, 0
b	0, 1	3, 2

None of the remaining actions is dominated.

Exercise I: Iterated Dominance

b) Consider the following game. Which outcomes are Pareto-optimal? Can the game be solved by iterated strict dominance?

c_1:

	b_1	b_2
a_1	(2, 3, 2)	(0, 5, 2)
a_2	(1, 4, 1)	(2, 1, 1)
a_3	(1, 1, 1)	(5, 4, 2)

c_2:

	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
a_3	(1, 2, 0)	(2, 2, 1)

Exercise I: Iterated Dominance

Which outcomes are Pareto-optimal?

	b_1	b_2
a_1	(2, 3, 2)	(0, 5, 2)
a_2	(1, 4, 1)	(2, 1, 1)
a_3	(1, 1, 1)	(5, 4, 2)

c_1

	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
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c_2

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Which outcomes are Pareto-optimal?

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a_1	(2, 3, 2)	(0, 5, 2)
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a_3	(1, 1, 1)	(5, 4, 2)

c_2:

	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
a_3	(1, 2, 0)	(2, 2, 1)

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Which outcomes are Pareto-optimal?

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	b_1	b_2
a_1	(2, 3, 2)	(0, 5, 2)
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	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
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Which outcomes are Pareto-optimal?

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	b_1	b_2
a_1	$(4, 5, 1)$	$(1, 0, 1)$
a_2	$(2, 0, 3)$	$(1, 5, 3)$
a_3	$(1, 2, 0)$	$(2, 2, 1)$

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Which outcomes are Pareto-optimal?

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a_1	(2, 3, 2)	(0, 5, 2)
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	b_1	b_2
a_1	$(4, 5, 1)$	$(1, 0, 1)$
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a_1	(2, 3, 2)	(0, 5, 2)
a_2	(1, 4, 1)	(2, 1, 1)
a_3	(1, 1, 1)	(5, 4, 2)

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	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
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c_2

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Which outcomes are Pareto-optimal?

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	b_1	b_2
a_1	(2, 3, 2)	(0, 5, 2)
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a_3	(1, 2, 0)	(2, 2, 1)

c_2

Exercise I: Iterated Dominance

Can the game be solved by iterated strict dominance?

c_1:

	b_1	b_2
a_1	(2, 3, 2)	(0, 5, 2)
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c_2:

	b_1	b_2
a_1	(4, 5, 1)	(1, 0, 1)
a_2	(2, 0, 3)	(1, 5, 3)
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Exercise I: Iterated Dominance

	b_1	b_2
a_1	$(2, 3, 2)$	$(0, 5, 2)$
a_2	$(1, 4, 1)$	$(1, 1, 1)$
a_3	$(1, 1, 1)$	$(5, 4, 2)$
$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{3}{2}, 2, \frac{3}{2})$	$(\frac{5}{2}, \frac{9}{2}, 2)$

c_1

	b_1	b_2
a_1	$(4, 5, 1)$	$(1, 0, 1)$
a_2	$(2, 0, 3)$	$(1, 5, 3)$
a_3	$(1, 2, 0)$	$(2, 2, 1)$
$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{5}{2}, \frac{7}{2}, \frac{1}{2})$	$(\frac{3}{2}, 1, 1)$

c_2

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	b_1	b_2
a_1	$(2, 3, 2)$	$(0, 5, 2)$
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$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{3}{2}, 2, \frac{3}{2})$	$(\frac{5}{2}, \frac{9}{2}, 2)$

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a_2	$(2, 0, 3)$	$(1, 5, 3)$
a_3	$(1, 2, 0)$	$(2, 2, 1)$
$\frac{1}{2}a_1 + \frac{1}{2}a_3$	$(\frac{5}{2}, \frac{7}{2}, \frac{1}{2})$	$(\frac{3}{2}, 1, 1)$

c_2

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Exercise I: Iterated Dominance

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c_1

Exercise I: Iterated Dominance

	b_2
a_1	$(0, 5, 2)$
a_3	$(5, 4, 2)$
	c_1

Exercise I: Iterated Dominance

	b_2
a_3	$(5, 4, 2)$

c_1

Exercise I: Iterated Dominance

	b_2
a_3	$(5, 4, 2)$
	c_1

The game can be solved via iterated strict dominance!

Exercise II: Maximin Strategies and Security Levels

a) Consider the following formulation of rock-paper-scissors. What are the maximin strategies and the security level of both players?

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Let s denote a mixed strategy of the row player.The minimal expected utility $u_{\min}(s)$ guaranteed by s to the row player is

$$u_{\min}(s) = \min(u_1(s, R), u_1(s, P), u_i(s, S))$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Let s denote a mixed strategy of the row player.The minimal expected utility $u_{\min}(s)$ guaranteed by s to the row player is

$$\begin{aligned} u_{\min}(s) &= \min(u_1(s, R), u_1(s, P), u_i(s, S)) \\ &= \min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S), \end{aligned}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Let s denote a mixed strategy of the row player.The minimal expected utility $u_{\min}(s)$ guaranteed by s to the row player is

$$\begin{aligned} u_{\min}(s) &= \min(u_1(s,R), u_1(s,P), u_i(s,S)) \\ &= \min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S), \\ &\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S), \end{aligned}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Let s denote a mixed strategy of the row player.The minimal expected utility $u_{\min}(s)$ guaranteed by s to the row player is

$$\begin{aligned} u_{\min}(s) &= \min(u_1(s,R), u_1(s,P), u_i(s,S)) \\ &= \min(0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S), \\ &\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S), \\ &\qquad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S)) \end{aligned}$$

Exercise II: Maximin Strategies and Security Levels

Exercise II: Maximin Strategies and Security Levels

$$\max u$$

Exercise II: Maximin Strategies and Security Levels

$$\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \geq u && (1)\\
&\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \geq u && (2)\\
&\qquad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \geq u && (3)
\end{aligned}$$

Exercise II: Maximin Strategies and Security Levels

$$
\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \geq u && (1)\\
&-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \geq u && (2)\\
&1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \geq u && (3)\\
&s(R) + s(P) + s(S) = 1 \\
&s(R) \geq 0, s(P) \geq 0, s(S) \geq 0
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

$$
\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \geq u && (1)\\
&\quad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \geq u && (2)\\
&\quad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \geq u && (3)\\
&\quad s(R) + s(P) + s(S) = 1 \\
&\quad s(R) \geq 0, s(P) \geq 0, s(S) \geq 0
\end{aligned}
$$

$$(0 - 1 + 1)s(R) + (1 + 0 - 1)s(P) + (-1 + 1 + 0)s(S) \geq 3u$$

Exercise II: Maximin Strategies and Security Levels

$$
\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \geq u && (1) \\
&\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \geq u && (2) \\
&\qquad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \geq u && (3) \\
&\qquad s(R) + s(P) + s(S) = 1 \\
&\qquad s(R) \geq 0, s(P) \geq 0, s(S) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
(0 - 1 + 1)s(R) + (1 + 0 - 1)s(P) + (-1 + 1 + 0)s(S) &\geq 3u \\
\iff 0 &\geq u
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

Suppose $u = 0$.

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$$s(P) - s(S) \geq 0 \tag{1}$$

$$-s(R) + s(S) \geq 0 \tag{2}$$

$$s(R) - s(P) \geq 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

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Suppose $u = 0$.

$$s(P) - s(S) \geq 0 \tag{1}$$

$$-s(R) + s(S) \geq 0 \tag{2}$$

$$s(R) - s(P) \geq 0 \tag{3}$$

$$s(R) + s(P) + s(S) = 1$$

$$s(R) \geq 0, s(P) \geq 0, s(S) \geq 0$$

$$s(P) \geq s(S) \geq s(R) \geq s(P)$$

Exercise II: Maximin Strategies and Security Levels

Suppose $u = 0$.

$$
\begin{aligned}
s(P) - s(S) &\geq 0 && (1)\\
-s(R) + s(S) &\geq 0 && (2)\\
s(R) - s(P) &\geq 0 && (3)\\
s(R) + s(P) + s(S) &= 1\\
s(R) \geq 0, s(P) \geq 0, s(S) &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
&s(P) \geq s(S) \geq s(R) \geq s(P)\\
\implies &s(P) = s(S) = s(R)
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

Suppose $u = 0$.

$$
\begin{aligned}
s(P) - s(S) &\geq 0 && (1)\\
-s(R) + s(S) &\geq 0 && (2)\\
s(R) - s(P) &\geq 0 && (3)\\
s(R) + s(P) + s(S) &= 1\\
s(R) \geq 0, s(P) \geq 0, s(S) &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
& s(P) \geq s(S) \geq s(R) \geq s(P)\\
\Longrightarrow\ & s(P) = s(S) = s(R)\\
\Longrightarrow\ & s(P) = s(S) = s(R) = \frac{1}{3}
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

The maximin strategy of player 1 is given by $s(R) = s(P) = s(S) = \frac{1}{3}$. His security level is 0.

A symmetric analysis shows that player 2 has the same maximin strategy and security level.

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	
P	1, -1	0, 0	-1, 1	
S	-1, 1	1, -1	0, 0	
W				

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	
S	-1, 1	1, -1	0, 0	
W				

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	
S	-1, 1	1, -1	0, 0	
W	1, -1			

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	
W	1, -1	-1, 1		

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	

Exercise II: Maximin Strategies and Security Levels

b) Model the situation with well as a fourth option that beats rock and scissor but loses again paper. What are the maximin strategies and the security levels of both players?

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$\max u$$

$$\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \qquad (1)$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \qquad (2)$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \qquad (3)$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \qquad (4)$$

$$s \in \mathcal{L}(A_1)$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \tag{1}$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \tag{1}$$

$$-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$-1 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

It is always weakly better for player 1 to put probability on W rather than on R.

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(W) \geq u \tag{1}$$

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

Inequality (4) makes than Inequality (1) redundant.

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

These are the same conditions we had before!

Exercise II: Maximin Strategies and Security Levels

	R	P	S	W
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	1, -1
S	-1, 1	1, -1	0, 0	-1, 1
W	1, -1	-1, 1	1, -1	0, 0

$$0 \cdot s(P) + 1 \cdot s(S) - 1 \cdot s(W) \geq u \tag{2}$$

$$-1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(W) \geq u \tag{3}$$

$$1 \cdot s(P) - 1 \cdot s(S) + 0 \cdot s(W) \geq u \tag{4}$$

These are the same conditions we had before!

The maximin strategy of player 1 is given by

$s(P) = s(S) = s(W) = \frac{1}{3}$ and his security level is 0.

Exercise II: Maximin Strategies and Security Levels

c) Assume there is lava as a fourth option. Lava beats all other option, but if both players play lava, they both experience a super lose with a utility of -100. What are the maximin strategies and the security levels of both players?

Exercise II: Maximin Strategies and Security Levels

c) Assume there is lava as a fourth option. Lava beats all other option, but if both players play lava, they both experience a super lose with a utility of -100. What are the maximin strategies and the security levels of both players?

	R	P	S	L
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	-1, 1
S	-1, 1	1, -1	0, 0	-1, 1
L	1, -1	1, -1	1, -1	-100, -100

Exercise II: Maximin Strategies and Security Levels

	R	P	S	L
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	-1, 1
S	-1, 1	1, -1	0, 0	-1, 1
L	1, -1	1, -1	1, -1	-100, -100

$$
\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(L) \geq u \\
&\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) + 1 \cdot s(L) \geq u \\
&\qquad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(L) \geq u \\
&\qquad -1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \geq u \\
&\qquad s \in \mathcal{L}(A_1)
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	L
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	-1, 1
S	-1, 1	1, -1	0, 0	-1, 1
L	1, -1	1, -1	1, -1	-100, -100

$$
\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) + 1 \cdot s(L) \geq u \\
&-1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) + 1 \cdot s(L) \geq u \\
&1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) + 1 \cdot s(L) \geq u \\
&-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \geq u \\
&s \in \mathcal{L}(A_1)
\end{aligned}
$$

Exercise II: Maximin Strategies and Security Levels

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \geq u$$

The security level of player 1 is at most -1.

Exercise II: Maximin Strategies and Security Levels

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \geq u$$

The security level of player 1 is at most -1.

The security level of player 1 can only be -1 if he never plays lava!

Exercise II: Maximin Strategies and Security Levels

$$-1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) - 100 \cdot s(L) \geq u$$

The security level of player 1 is at most -1.

The security level of player 1 can only be -1 if he never plays lava!

$$\begin{aligned}
&\max u \\
&\text{subject to } 0 \cdot s(R) + 1 \cdot s(P) - 1 \cdot s(S) \geq u \\
&\qquad -1 \cdot s(R) + 0 \cdot s(P) + 1 \cdot s(S) \geq u \\
&\qquad 1 \cdot s(R) - 1 \cdot s(P) + 0 \cdot s(S) \geq u \\
&\qquad -1 \cdot s(R) - 1 \cdot s(P) - 1 \cdot s(S) \geq u \\
&\qquad s \in \mathcal{L}(A_1)
\end{aligned}$$

Exercise II: Maximin Strategies and Security Levels

	R	P	S	L
R	0, 0	-1, 1	1, -1	-1, 1
P	1, -1	0, 0	-1, 1	-1, 1
S	-1, 1	1, -1	0, 0	-1, 1
L	1, -1	1, -1	1, -1	-100, -100

Every strategy s with $s(L) = 0$ is a maximin strategy!
The security level of both players is -1.

Exercise III: Independence

Assume that $A = \{a, b, c\}$ and let $\succsim$ denote a rational and independent preference relation on $\mathcal{L}(A)$ such that $[1 : a] \succ [1 : b]$ and $[\frac{1}{2} : b, \frac{1}{2} : c] \sim [\frac{2}{3} : a, \frac{1}{3} : c]$.

a) Show that $[1 : c] \succ [1 : a]$.

Exercise III: Independence

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if

Exercise III: Independence

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if
  - it is complete: $L_1 \succsim L_2$ or $L_2 \succsim L_1$ for all $L_1, L_2 \in \mathcal{L}(A)$

Exercise III: Independence

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if
  - it is complete: $L_1 \succsim L_2$ or $L_2 \succsim L_1$ for all $L_1, L_2 \in \mathcal{L}(A)$
  - and transitive: $L_1 \succsim L_2$ and $L_2 \succsim L_3$ implies $L_1 \succsim L_3$ for all $L_1, L_2, L_3 \in \mathcal{L}(A)$.

Exercise III: Independence

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if
  - it is complete: $L_1 \succsim L_2$ or $L_2 \succsim L_1$ for all $L_1, L_2 \in \mathcal{L}(A)$
  - and transitive: $L_1 \succsim L_2$ and $L_2 \succsim L_3$ implies $L_1 \succsim L_3$ for all $L_1, L_2, L_3 \in \mathcal{L}(A)$.
- continuous if, for all $L_1, L_2, L_3 \in \mathcal{L}(A)$ with $L_1 \succ L_2 \succ L_3$, there is $\epsilon > 0$ such that

$$[1-\epsilon : L_1, \epsilon : L_3] \succ L_2 \succ [1-\epsilon : L_3, \epsilon : L_1].$$

Exercise III: Independence

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if
  - it is complete: $L_1 \succsim L_2$ or $L_2 \succsim L_1$ for all $L_1, L_2 \in \mathcal{L}(A)$
  - and transitive: $L_1 \succsim L_2$ and $L_2 \succsim L_3$ implies $L_1 \succsim L_3$ for all $L_1, L_2, L_3 \in \mathcal{L}(A)$.
- continuous if, for all $L_1, L_2, L_3 \in \mathcal{L}(A)$ with $L_1 \succ L_2 \succ L_3$, there is $\epsilon > 0$ such that

$$[1-\epsilon : L_1, \epsilon : L_3] \succ L_2 \succ [1-\epsilon : L_3, \epsilon : L_1].$$

- independent if, for all lotteries L_1, L_2, L_3 and all $p \in (0,1)$, it holds that

$$L_1 \succsim L_2 \iff [p : L_1, (1-p) : L_3] \succsim [p : L_2, (1-p) : L_3].$$

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_b \succsim L_c$.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_b \succsim L_c$.
    - By independence, $[\frac{3}{4} : L_b, \frac{1}{4} : L_b] \succsim [\frac{3}{4} : L_b, \frac{1}{4} : L_c]$

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_b \succsim L_c$.
    - By independence, $[\frac{3}{4} : L_b, \frac{1}{4} : L_b] \succsim [\frac{3}{4} : L_b, \frac{1}{4} : L_c]$
    - This shows that $L_a \succ L_b \succsim L_3$, contradiction.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_a \succsim L_c \succ L_b$.

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_a \succsim L_c \succ L_b$.
    - By independence, $[\frac{3}{4} : L_c, \frac{1}{4} : L_c] \succ [\frac{3}{4} : L_b, \frac{1}{4} : L_c]$

Exercise III: Independence

- Let $L_x = [1 : x]$ for $x \in \{a, b, c\}$, $L_1 = [\frac{2}{3} : a, \frac{1}{3} : c]$, and $L_2 = [\frac{1}{2} : b, \frac{1}{2} : c]$
- By assumption, $L_2 \sim L_1$.
- Let $L_3 = [\frac{3}{4} : b, \frac{1}{4} : c]$. It holds that $L_2 = [\frac{2}{3} : L_3, \frac{1}{3} : L_c]$.
- By independence, $L_3 \sim L_a$ since $L_1 = [\frac{2}{3} : L_a, \frac{1}{3} : L_c]$.
- Now, assume that $L_a \succsim L_c \succ L_b$.
    - By independence, $[\frac{3}{4} : L_c, \frac{1}{4} : L_c] \succ [\frac{3}{4} : L_b, \frac{1}{4} : L_c]$
    - This shows that $L_a \succ L_c \succsim L_3$, contradiction.

Exercise III: Independence

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- Now, assume that $L_a \succsim L_c \succ L_b$.
  - By independence, $[\frac{3}{4} : L_c, \frac{1}{4} : L_c] \succ [\frac{3}{4} : L_b, \frac{1}{4} : L_c]$
  - This shows that $L_a \succ L_c \succsim L_3$, contradiction.
- Hence, the only possibility is that $L_c \succ L_a \succ L_b$.

Exercise III: Independence

Assume that $A = \{a, b, c\}$ and let $\succsim$ denote a rational and independent preference relation on $\mathcal{L}(A)$ such that $[1 : a] \succ [1 : b]$ and $[\frac{1}{2} : b, \frac{1}{2} : c] \sim [\frac{2}{3} : a, \frac{1}{3} : c]$.

b) Show that, if $\succsim$ is additionally continuous, then it can be represented by the vNM utility function u given by $u(c) = 1$, $u(a) = \frac{1}{4}$, $u(b) = 0$.

Exercise III: Independence

- By the von-Neuman-Morgenstern Theorem: If $\succsim$ is rational, continuous, and independent, it can be represented by a vNM utility function.

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- Since $L_c \succ L_a \succ L_b$, it must be that $u(c) > u(a) > u(b)$.
- vNM utility functions are invariant under addition. Hence, define u' by $u'(x) = u(x) - u(b)$ for all $x \in \{a, b, c\}$

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- vNM utility functions are invariant under addition. Hence, define u' by $u'(x) = u(x) - u(b)$ for all $x \in \{a, b, c\}$
- vNM utility functions are invariant under multiplication with a positive scalar. Hence, define $v(x) = u'(x)/u'(c)$ for all $x \in \{a, b, c\}$.

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v(a) &= \frac{1}{4}
\end{aligned}$$

- Hence, $\succsim$ is represented by the utility function v with $v(c) = 1$, $v(a) = \frac{1}{4}$, and $v(b) = 0$.

Exercise IV: Preferences over Lotteries

Let $\succsim$ denote the rational preference relation over a set $A = \{x_1, \ldots, x_m\}$ given by $x_1 \succ x_2 \succ \cdots \succ x_m$.

Is the following relation a rational preference relation on $\mathcal{L}(A)$? Is it continuous and independent? Prove your answers!

a) The relation $\succsim_1$ is defined by $L_1 \succsim_1 L_2$ if and only if $x \succsim y$ for all $x, y \in A$ with $L_1(x) > 0$ and $L_2(y) > 0$.

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- $\succsim_1$ is not rational:
  - $\succsim_1$ is not complete.
  - Let $L_1 = [\frac{1}{3} : x_1, \frac{2}{3} : x_2]$ and $L_2 = [x_1 : \frac{1}{2}, x_2 : \frac{1}{2}]$.

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  - It holds that $x_1 \succ x_2$ and $L_1(x_2) > 0$ and $L_2(x_1) > 0$, so $L_1 \not\succsim_1 L_2$.

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  - It holds that $x_1 \succ x_2$ and $L_1(x_2) > 0$ and $L_2(x_1) > 0$, so $L_1 \not\succsim_1 L_2$.
  - Similarly, $L(x_1) > 0$ and $L_2(x_2) > 0$, so $L_2 \not\succsim_1 L_1$.

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- $\succsim_1$ is not rational:
  - $\succsim_1$ is transitive.
  - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succsim_1 L_2$ and $L_2 \succsim_1 L_3$.

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    - $y \succsim z$ for all $y, z \in A$ with $L_2(y) > 0$ and $L_3(z) > 0$.
  - By the transitivity of $\succsim$, it follows that $x \succsim z$ for all $x, z \in A$ with $L_1(x) > 0$ and $L_3(z) > 0$.

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        - $y \succsim z$ for all $y, z \in A$ with $L_2(y) > 0$ and $L_3(z) > 0$.
    - By the transitivity of $\succsim$, it follows that $x \succsim z$ for all $x, z \in A$ with $L_1(x) > 0$ and $L_3(z) > 0$.
    - This means that $L_1 \succsim_1 L_3$.

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- $\succsim_1$ fails continuity:

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    - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$ and $L_3 = [1 : x_3]$.

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    - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$ and $L_3 = [1 : x_3]$.
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- $\succsim_1$ fails continuity:
    - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$ and $L_3 = [1 : x_3]$.
        - It holds that $L_1 \succ_1 L_2 \succ_1 L_3$.
        - However, for every $\epsilon > 0$, $[1 - \epsilon : L_1, \epsilon : L_3] \not\succsim L_2$ because $L(x_3) = \epsilon > 0$ for $L = [1 - \epsilon : L_1, \epsilon : L_3]$ and $L_2(x_2) > 0$.

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    - $\succsim_1$ fails independence:

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- $\succsim_1$ fails continuity:
  - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$ and $L_3 = [1 : x_3]$.
    - It holds that $L_1 \succ_1 L_2 \succ_1 L_3$.
    - However, for every $\epsilon > 0$, $[1 - \epsilon : L_1, \epsilon : L_3] \not\succsim L_2$ because $L(x_3) = \epsilon > 0$ for $L = [1 - \epsilon : L_1, \epsilon : L_3]$ and $L_2(x_2) > 0$.
  - $\succsim_1$ fails independence:
    - Let $L_4 = [\frac{1}{2} : L_1, \frac{1}{2} : L_3]$ and $L_5 = [\frac{1}{2} : L_2, \frac{1}{2} : L_3]$.

Exercise IV: Preferences over Lotteries

The relation $\succsim_1$ is defined by $L_1 \succsim_1 L_2$ if and only if $x \succsim y$ for all $x, y \in A$ with $L_1(x) > 0$ and $L_2(y) > 0$.

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  - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$ and $L_3 = [1 : x_3]$.
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    - However, for every $\epsilon > 0$, $[1 - \epsilon : L_1, \epsilon : L_3] \not\succsim L_2$ because $L(x_3) = \epsilon > 0$ for $L = [1 - \epsilon : L_1, \epsilon : L_3]$ and $L_2(x_2) > 0$.
  - $\succsim_1$ fails independence:
    - Let $L_4 = [\frac{1}{2} : L_1, \frac{1}{2} : L_3]$ and $L_5 = [\frac{1}{2} : L_2, \frac{1}{2} : L_3]$.
    - While $L_1 \succ_1 L_2$, we have $L_4 \not\succ_1 L_5$.

Exercise IV: Preferences over Lotteries

Let $\succsim$ denote the rational preference relation over a set $A = \{x_1, \ldots, x_m\}$ given by $x_1 \succ x_2 \succ \cdots \succ x_m$.

Is the following relation a rational preference relation on $\mathcal{L}(A)$? Is it continuous and independent? Prove your answers!

b) We define $\max(\succsim, X)$ as the most preferred alternative in X and $\Delta(L_1, L_2) = \max(\succsim, \{x \in A\colon L_1(x) \neq L_2(x)\})$.
The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) \geq L_2(\Delta(L_1, L2))$.

Exercise IV: Preferences over Lotteries

We define $\max(\succsim, X)$ as the most preferred alternative in X with respect to $\succsim$ and $\Delta(L_1, L_2) = \max(\succsim, \{x \in A\colon L_1(x) \neq L_2(x)\})$. The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) \geq L_2(\Delta(L_1, L2))$.

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- "Lexicographic preferences"

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- "Lexicographic preferences"
- $\succsim_2$ is rational:

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- "Lexicographic preferences"
- $\succsim_2$ is rational:
    - $\succsim_2$ is complete:

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We define $\max(\succsim, X)$ as the most preferred alternative in X with respect to $\succsim$ and $\Delta(L_1, L_2) = \max(\succsim, \{x \in A\colon L_1(x) \neq L_2(x)\})$. The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) \geq L_2(\Delta(L_1, L2))$.

- "Lexicographic preferences"
- $\succsim_2$ is rational:
    - $\succsim_2$ is complete:
    - If $L_1 = L_2$, then $L_1 \sim_2 L_2$ by definition.

Exercise IV: Preferences over Lotteries

We define $\max(\succsim, X)$ as the most preferred alternative in X with respect to $\succsim$ and $\Delta(L_1, L_2) = \max(\succsim, \{x \in A \colon L_1(x) \neq L_2(x)\})$. The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) \geq L_2(\Delta(L_1, L2))$.

- "Lexicographic preferences"
- $\succsim_2$ is rational:
  - $\succsim_2$ is complete:
  - If $L_1 = L_2$, then $L_1 \sim_2 L_2$ by definition.
  - If $L1 \neq L_2$, then $\Delta(L_1, L_2)$ is well-defined, so either $L_1 \succsim_2 L_2$ or $L_2 \succsim_2 L_2$.

Exercise IV: Preferences over Lotteries

The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) > L_2(\Delta(L_1, L2))$.

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- $\succsim_2$ is rational:
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    - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succsim_2 L_2$ and $L_2 \succsim_2 L_3$.

Exercise IV: Preferences over Lotteries

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  - If $L_1 = L_2$ or $L_2 = L_3$, it trivially holds that $L_1 \succsim_2 L_3$.

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    - Assume $L_1 \neq L_2$ and $L_2 \neq L_3$. Let $x_1 = \Delta(L_1, L_2)$, $x_2 = \Delta(L_2, L_3)$, and $x^* = \max(\succsim, \{x_1, x_2\})$.

Exercise IV: Preferences over Lotteries

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  - By definition, we have that $L_1(x) = L_2(x) = L_3(x)$ for all x with $x \succ x^*$.

Exercise IV: Preferences over Lotteries

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  - Assume $L_1 \neq L_2$ and $L_2 \neq L_3$. Let $x_1 = \Delta(L_1, L_2)$, $x_2 = \Delta(L_2, L_3)$, and $x^* = \max(\succsim, \{x_1, x_2\})$.
  - By definition, we have that $L_1(x) = L_2(x) = L_3(x)$ for all x with $x \succ x^*$.
  - If $x^* = x_1$, then $L_1(x^*) > L_2(x^*) \geq L_3(x^*)$. Hence, $\Delta(L_1, L_3) = x^*$ and $L_1 \succsim_2 L_3$.

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  - By definition, we have that $L_1(x) = L_2(x) = L_3(x)$ for all x with $x \succ x^*$.
  - If $x^* = x_1$, then $L_1(x^*) > L_2(x^*) \geq L_3(x^*)$. Hence, $\Delta(L_1, L_3) = x^*$ and $L_1 \succsim_2 L_3$.
  - If $x^* = x_2$, then $L_1(x^*) \geq L_2(x^*) > L_3(x^*)$. Hence, $\Delta(L_1, L_3) = x^*$ and $L_1 \succsim_2 L_3$.

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- $\succsim_2$ is independent:
- Let $L_1, L_2, L_3 \in \Delta(A)$. Moreover, let $\lambda \in (0, 1)$ and $L_4 = [\lambda : L_1, 1 - \lambda : L_3]$ and $L_5 = [\lambda : L_2, 1 - \lambda : L_3]$.

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- Let $L_1, L_2, L_3 \in \Delta(A)$. Moreover, let $\lambda \in (0, 1)$ and $L_4 = [\lambda : L_1, 1 - \lambda : L_3]$ and $L_5 = [\lambda : L_2, 1 - \lambda : L_3]$.
- It holds that $\Delta(L_1, L_2) = \Delta(L_4, L_5)$ and that $L_1(\Delta(L_1, L_2)) > L_2(\Delta(L_1, L_2))$ if and only if $L_4(\Delta(L_4, L_5)) > L_5(\Delta(L_4, L_5))$ because for all $x \in A$

$$\begin{aligned} L_4(x) - L_5(x) &= \lambda L_1(x) + (1 - \lambda)L_3(x) - \lambda L_2(x) + (1 - \lambda)L_3(x) \\ &= \lambda(L_1(x) - L_2(x)). \end{aligned}$$

Exercise IV: Preferences over Lotteries

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- $\succsim_2$ is independent:
- Let $L_1, L_2, L_3 \in \Delta(A)$. Moreover, let $\lambda \in (0, 1)$ and $L_4 = [\lambda : L_1, 1 - \lambda : L_3]$ and $L_5 = [\lambda : L_2, 1 - \lambda : L_3]$.
- It holds that $\Delta(L_1, L_2) = \Delta(L_4, L_5)$ and that $L_1(\Delta(L_1, L_2)) > L_2(\Delta(L_1, L_2))$ if and only if $L_4(\Delta(L_4, L_5)) > L_5(\Delta(L_4, L_5))$ because for all $x \in A$

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- This shows that $L_1 \succsim_2 L_2$ if and only if $L_4 \succsim_2 L_5$.

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The relation $\succsim_2$ is defined by $L_1 \succsim_2 L_2$ if and only if $L_1 = L_2$ or $L_1(\Delta(L_1, L_2)) > L_2(\Delta(L_1, L2))$.

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- $\succsim_2$ fails continuity:

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- $\succsim_2$ fails continuity:
  - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$, and $L_3 = [1 : x_3]$.

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    - Let $L_1 = [1 : x_1]$, $L_2 = [1 : x_2]$, and $L_3 = [1 : x_3]$.
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  - It holds that $L_1 \succ_2 L_2 \succ_2 L_3$.
  - However, for every $\epsilon > 0$, we have that $\Delta([\epsilon : L_1, 1 - \epsilon : L_3], L_2) = x_1$.

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- $\succsim_2$ fails continuity:
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  - It holds that $L_1 \succ_2 L_2 \succ_2 L_3$.
  - However, for every $\epsilon > 0$, we have that $\Delta([\epsilon : L_1, 1 - \epsilon : L_3], L_2) = x_1$.
  - Hence, it holds for every $\epsilon > 0$ that $[\epsilon : L_1, 1 - \epsilon : L_3] \succ_2 L_2$.

Exercise IV: Preferences over Lotteries

Let $\succsim$ denote the rational preference relation over a set $A = \{x_1, \ldots, x_m\}$ given by $x_1 \succ x_2 \succ \cdots \succ x_m$.

Is following relation a rational preference relation on $\mathcal{L}(A)$? Is it continuous and independent? Prove your answers!

c) The relation $\succsim_3$ is defined by $L_1 \succsim_3 L_2$ if and only if $L_1(x_1) \geq L_2(x_1)$.

Exercise IV: Preferences over Lotteries

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The relation $\succsim_3$ is defined by $L_1 \succsim_3 L_2$ if and only if $L_1(x_1) \geq L_2(x_1)$.

- "We compare lotteries only based on the probability of x_1."

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- $\succsim_3$ is rational:

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- $\succsim_3$ is rational:
  - $\succsim_3$ is complete.

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- "We compare lotteries only based on the probability of x_1."
- $\succsim_3$ is rational:
  - $\succsim_3$ is complete.
    - For all lotteries L_1, L_2 it holds that either $L_1(x_1) \geq L_2(x_1)$ or $L_1(x_1) \leq L_1(x_3)$.

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  - $\succsim_3$ is transitive.

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  - $\succsim_3$ is transitive.
    - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succsim_3 L_2 \succsim_3 L_3$.

Exercise IV: Preferences over Lotteries

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- $\succsim_3$ is rational:
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    - $\succsim_3$ is transitive.
        - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succsim_3 L_2 \succsim_3 L_3$.
        - Hence, $L_1(x_1) \geq L_2(x_1) \geq L_3(x_1)$ and thus also $L_1 \succsim_3 L_3$

Exercise IV: Preferences over Lotteries

The relation $\succsim_3$ is defined by $L_1 \succsim_3 L_2$ if and only if $L_1(x_1) \geq L_2(x_1)$.

- $\succsim_3$ is independent:

Exercise IV: Preferences over Lotteries

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- $\succsim_3$ is independent:
  - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$. Let $\lambda \in (0, 1)$, $L_4 = [\lambda : L_1, (1-\lambda) : L_3]$, and $L_5 = [\lambda L_2, (1-\lambda) : L_3]$

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  - It holds that $L_4(x_1) = \lambda L_1(x_1) + (1 - \lambda) L_3(x_1)$ and $L_5(x_1) = \lambda L_2(x_1) + (1 - \lambda) L_3(x_1)$.

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    - It holds that $L_4(x_1) = \lambda L_1(x_1) + (1-\lambda) L_3(x_1)$ and $L_5(x_1) = \lambda L_2(x_1) + (1-\lambda) L_3(x_1)$.
    - Hence, $L_1(x_1) \geq L_2(x)$ if and only if $L_4(x_1) \geq L_5(x)$.

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    - It holds that $L_4(x_1) = \lambda L_1(x_1) + (1-\lambda)L_3(x_1)$ and $L_5(x_1) = \lambda L_2(x_1) + (1-\lambda)L_3(x_1)$.
    - Hence, $L_1(x_1) \geq L_2(x)$ if and only if $L_4(x_1) \geq L_5(x)$.
    - This shows that $L_1 \succsim_3 L_2$ if and only if $L_4 \succsim_3 L_5$.

Exercise IV: Preferences over Lotteries

The relation $\succsim_3$ is defined by $L_1 \succsim_3 L_2$ if and only if $L_1(x_1) \geq L_2(x_1)$.

- $\succsim_3$ is continuous:

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  - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succ_3 L_2 \succ_3 L_3$.

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- $\succsim_3$ is continuous:
    - Let $L_1, L_2, L_3 \in \mathcal{L}(A)$ such that $L_1 \succ_3 L_2 \succ_3 L_3$.
    - This means that $L_1(x_1) > L_2(x_1) > L_3(x_1)$.

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  - This means that $L_1(x_1) > L_2(x_1) > L_3(x_1)$.
  - There is $\epsilon \in (0, 1)$ such that

$$(1-\epsilon)L_1(x_1) + \epsilon L_3(x_1) > L_2(x_1) > (1-\epsilon)L_3(x_1) + \epsilon L_1(x).$$

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  - This means that $L_1(x_1) > L_2(x_1) > L_3(x_1)$.
  - There is $\epsilon \in (0, 1)$ such that

$$(1-\epsilon)L_1(x_1) + \epsilon L_3(x_1) > L_2(x_1) > (1-\epsilon)L_3(x_1) + \epsilon L_1(x).$$

  - Hence, $[1-\epsilon : L_1, \epsilon : L_3] \succ_3 L_2 \succ_3 [1-\epsilon : L_3, \epsilon : L_1]$.