Problem Set on Optimal Control: Solutions

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Solution to Problem 1

A) The present-value Hamiltonian is

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot \ln(c(t)) + \lambda(t)\left[f\left(k(t)\right) - c(t) - \delta \cdot k(t)\right]$$

where $\lambda(t)$ is the co-state variable associated with the state variable $k(t)$.

B) The optimality conditions for the present-value Hamiltonian are

$$\begin{aligned}
\frac{\partial \mathcal{H}(t)}{\partial c(t)} &= 0 \\
\frac{\partial \mathcal{H}(t)}{\partial k(t)} &= -\dot{\lambda}(t) \\
\lim_{t \to +\infty} \lambda(t) \cdot k(t) &= 0.
\end{aligned}$$

The last condition is the transversality condition. The first two conditions imply that

$$e^{-\rho \cdot t} \cdot \frac{1}{c(t)} = \lambda(t) \tag{1}$$

$$\lambda(t) \cdot \left[f'\left(k(t)\right) - \delta\right] = -\dot{\lambda}(t). \tag{2}$$

We can eliminate $\lambda(t)$ by taking log and differentiating (1) with respect to time t. This procedure yields

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -\rho - \frac{\dot{c}(t)}{c(t)}$$

We can then substitute $\dot{\lambda}(t)/\lambda(t)$ into (2), which gives the following Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \alpha \cdot A \cdot k(t)^{\alpha-1} - (\delta + \rho)\,.$$

C) The steady state is given by

$$\begin{aligned}
k^* &= \left(\frac{\alpha \cdot A}{\delta + \rho}\right)^{1/(1-\alpha)} \\
c^* &= A^{1/(1-\alpha)} \left(\frac{\alpha}{\delta + \rho}\right)^{\alpha/(1-\alpha)} \cdot \left(\frac{\delta \cdot (1-\alpha) + \rho}{\delta + \rho}\right).
\end{aligned}$$

Solution to Problem 2

A) The current-value Hamiltonian is

$$\mathcal{H}^*(t) = f(k(t)) - i(t) - \frac{\chi}{2} \cdot \left(\frac{i(t)^2}{k(t)} \right) + q(t) \cdot i(t),$$

where $q(t)$ is the co-state variable associated with the state variable $k(t)$.

B) There are two optimality conditions for the current-value Hamiltonian. (We omit the transversality condition.) The first optimality condition is

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}^*(t)}{\partial i(t)} \\ 0 &= -1 - \chi \cdot \left[\frac{i(t)}{k(t)} \right] + q(t) \\ i(t) &= \left[\frac{q(t) - 1}{\chi} \right] \cdot k(t), \end{aligned}$$

which implies, using the law of motion of capital, that

$$\dot{k}(t) = \left[\frac{q(t) - 1}{\chi} \right] \cdot k(t).$$

The second optimality condition is

$$\begin{aligned} \frac{\partial \mathcal{H}(t)}{\partial k(t)} &= r \cdot q(t) - \dot{q}(t) \\ f'(k(t)) + \frac{\chi}{2} \cdot \left[\frac{i(t)}{k(t)} \right]^2 &= r \cdot q(t) - \dot{q}(t). \end{aligned}$$

The first optimality condition implies that $i(t)/k(t) = \dot{k}(t)/k(t) = (q(t) - 1)/\chi$. So this optimality condition becomes

$$\dot{q}(t) = r \cdot q(t) - f'(k(t)) - \frac{1}{2 \cdot \chi} \cdot (q(t) - 1)^2 .$$

C) In steady state, $\dot{q}(t) = 0$ and $\dot{k}(t) = 0$, so $i^* = 0$. Notice that we can say that $\dot{q}(t) = 0$ only because $q(t)$ is the co-state variable used with a current-value Hamiltonian. The co-state variables used in a present-value Hamiltonian are not constant in steady state (which is a reason why we prefer to work with a current-value Hamiltonian). Since $\dot{k}(t) = 0$, the first optimality condition implies

$$q^* = 1.$$

Since $q^* = 1$ and $\dot{q}(t) = 0$, the second optimality condition implies

$$f'(k^*) = r.$$