Exam on Mathematical Methods for Macroeconomics: Solutions

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Solution to Problem 1.

A. State variable: $k(t)$. Control variable: $c(t)$. Current-value Hamiltonian:

$$H^*(t) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma} + q(t) \cdot \left[k(t)^{\alpha} - c(t) - \delta \cdot k(t)\right],$$

where $q(t)$ is the co-state variable. We have the following optimality conditions:

- First optimality condition:

$$\begin{aligned} \frac{\partial H^*(t)}{\partial c(t)} &= 0 \\ c(t)^{-\sigma} &= q(t). \end{aligned}$$

- Second optimality condition:

$$\begin{aligned} \frac{\partial H^*(t)}{\partial k(t)} &= \rho \cdot q(t) - \dot{q}(t) \\ q(t) \cdot \left[\alpha \cdot k(t)^{\alpha-1} - \delta - \rho\right] &= -\dot{q}(t) \\ \left[\alpha \cdot k(t)^{\alpha-1} - \delta - \rho\right] &= -\frac{\dot{q}(t)}{q(t)} \end{aligned}$$

- Third optimality condition: the transversality condition

$$\lim_{t \to +\infty} e^{-\rho \cdot t} \cdot q(t) \cdot k(t) = 0.$$

B. Take the log of the first optimality condition, differentiate with respect to time t, and plug the result from the second optimality condition:

$$\begin{aligned} -\sigma \cdot \ln[c(t)] &= \ln[q(t)] \\ \sigma \cdot \frac{\dot{c}(t)}{c(t)} &= -\frac{\dot{q}(t)}{q(t)} \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma} \cdot \left[\alpha \cdot k(t)^{\alpha-1} - \delta - \rho\right]. \end{aligned}$$

C. The optimal functions $\{k(t), c(t)\}$ are described by a system of two equations: the Euler equation and the law of motion of $k(t)$. If $\alpha = 1$ and $\sigma = 1$, the system is

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} (1-\rho-\delta) & 0 \\ -1 & (1-\delta) \end{bmatrix} \begin{bmatrix} c(t) \\ k(t) \end{bmatrix}.$$

This is a linear, homogenous system of first-order differential equations.

We compute the eigenvalues λ of the system. λ solves

$$\det \begin{bmatrix} 1-\rho-\delta-\lambda & 0 \\ -1 & 1-\delta-\lambda \end{bmatrix} = 0.$$

Hence, λ solves

$$[(1-\rho-\delta)-\lambda] \cdot [(1-\delta)-\lambda] = 0.$$

Therefore the system admits two distinct positive eigenvalues:

$$\lambda_1 = 1-(\rho+\delta) > 0$$
$$\lambda_2 = 1-\delta > 0.$$

We are facing a linear homogenous system of first-order differential equations with two positive eigenvalues are positive: the system is unstable.

D. The optimal functions $\{k(t), c(t)\}$ are described by a system of two equations: the Euler equation and the law of motion of $k(t)$. If $\alpha < 1$, the system is

$$\dot{c}(t) = \frac{1}{\sigma} \cdot \left[\alpha \cdot k(t)^{\alpha-1} - \delta - \rho\right] \cdot c(t)$$
$$\dot{k}(t) = k(t)^{\alpha} - c(t) - \delta \cdot k(t).$$

This is a nonlinear system of first-order differential equations.

We draw the phase diagram with the state variable k on the x-axis and the control variable c on the y-axis. The locus $\dot{c}(t) = 0$ and the locus $\dot{k}(t) = 0$ satisfy

$$k(t) = \left[\frac{\delta+\rho}{\alpha}\right]^{-1/(1-\alpha)} \equiv k^*$$
$$c(t) = k(t)^{\alpha} - \delta \cdot k(t).$$

The locus $\dot{c}(t) = 0$ is a vertical line. The locus $\dot{k}(t) = 0$ is a concave curve that goes through the origin and that cuts the x-axis again at $k^{**} = \delta^{-1/(1-\alpha)}$. Since $\rho > 0$ and $\alpha < 1$, $k^{**} > k^*$ and the concave curve crosses the vertical curve when it is positive. The steady state of the system is the intersection of the locus $\dot{c}(t) = 0$ and the locus $\dot{k}(t) = 0$.

Look at the equation for $\dot{c}(t)$ to determine the vertical arrows. Since $\alpha \cdot k^{\alpha-1} - \delta - \rho$ decreases with k, $\dot{c}(t) > 0$ to the west of $\dot{c}(t) = 0$ and $\dot{c}(t) < 0$ to the east of $\dot{c}(t) = 0$. So the vertical arrows point northwards to the west of $\dot{c}(t) = 0$ and southwards to the east of $\dot{c}(t) = 0$.

Look at the equation for $\dot{k}(t)$ to determine the horizontal arrows. Clearly, $\dot{k}(t) > 0$ to the south of $\dot{k}(t) = 0$ and $\dot{k}(t) < 0$ to the north of $\dot{k}(t) = 0$. So the horizontal arrows point eastwards to the north of $\dot{k}(t) = 0$ and westwards to the south of $\dot{c}(t) = 0$.

Therefore, the steady state of the system is a saddle point. The saddle path goes through the south-west and north-east regions of the plane.

Solution to Problem 2.

A. The Lagrangian associated with the problem is

$$L = \sum_{t=0}^{+\infty} \beta^t \cdot \{\ln(c_t) - \lambda_t \cdot [k_{t+1} - (1+r) \cdot k_t + c_t]\},$$

where $\{\lambda_t\}_{t=1}^{+\infty}$ is the sequences of Lagrange multipliers associated with the sequences of constraints.

B. The first-order conditions with respect to c_t and k_{t+1} are

$$\frac{1}{c_t} = \lambda_t$$
$$\lambda_t = \beta \cdot (1+r) \cdot \lambda_{t+1}.$$

C. Combining the first-order conditions yields the Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta \cdot (1+r).$$

D. State variable: k. Control variable: k' (the value of variable k next period). Bellman equation:

$$V(k) = \max_{k'}[\ln((1+r) \cdot k - k') + \beta \cdot V(k')].$$

E. The first-order condition with respect to k' in the Bellman equation is

$$\frac{1}{c} = \beta \cdot V'(k').$$

F. We apply the envelope theorem to the Bellman equation:

$$V'(k) = \frac{(1+r)}{c}.$$

G. The Benveniste-Scheinkman equation holds for any k. In particular, $V'(k') = (1+r)/c'$. Combining this equation with the first-order condition yields the Euler equation:

$$\frac{c'}{c} = \beta \cdot (1+r).$$

This Euler equation is the same as that obtained with the Lagrangian method. The two methods are equivalent.

H. We guess that optimal consumption $c = h(k) = A \cdot (1+r) \cdot k$. A first implication is that

$$\frac{c'}{c} = \frac{A \cdot (1+r) \cdot k'}{A \cdot (1+r) \cdot k} = \frac{k'}{k}.$$

Using the Euler equation, we obtain

$$\frac{k'}{k} = \frac{c'}{c} = (1+r) \cdot \beta.$$

The transition equation then implies

$$c = (1+r) \cdot k - k' = (1-\beta) \cdot (1+r) \cdot k.$$

Therefore, it must be that

$$A = (1-\beta).$$

I. The Bellman equation can be written in terms of the policy function:

$$V(k) = \ln(h(k)) + \beta \cdot V((1+r) \cdot k - h(k)).$$

We plug our guess for the value function and the expression for the policy function into the Bellman equation:

$$B + D \cdot \ln(k) = \ln((1-\beta) \cdot (1+r) \cdot k) + \beta \cdot [B + D \cdot \ln(\beta \cdot (1+r) \cdot k)].$$

Rearranging the terms on the right-hand side yields

$$B + D \cdot \ln(k) = [\ln((1-\beta) \cdot (1+r)) + \beta \cdot B + \beta \cdot D \cdot \ln(\beta \cdot (1+r))] + [1 + \beta \cdot D] \cdot \ln(k).$$

This equation must hold for any k so it is necessary that

$$D = 1 + \beta \cdot D$$
$$D = \frac{1}{1-\beta}$$

and

$$
\begin{aligned}
B &= \ln((1-\beta)\cdot(1+r)) + \beta\cdot B + \beta\cdot D\cdot\ln(\beta\cdot(1+r)) \\
B &= \frac{(1-\beta)\cdot\ln((1-\beta)\cdot(1+r)) + \beta\cdot\ln(\beta\cdot(1+r))}{(1-\beta)^2} \\
B &= \frac{(1-\beta)\cdot\ln(1-\beta) + \beta\cdot\ln(\beta) + \ln(1+r)}{(1-\beta)^2}.
\end{aligned}
$$