Problem Set on Differential Equations: Solutions

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Solution to Problem 1.

We multiply both sides of the differential equation by the integrating factor $\mu(t) = e^{-r \cdot t}$. We obtain

$$\dot{a}(t) \cdot e^{-r \cdot t} - r \cdot a(t) \cdot e^{-r \cdot t} = s \cdot e^{-r \cdot t}$$
$$\frac{d\left[a(t) \cdot e^{-r \cdot t}\right]}{dt} = s \cdot e^{-r \cdot t}$$

Integrating from time 0 to t,

$$\int_0^t d\left[a(t) \cdot e^{-r \cdot t}\right] = \int_0^t s \cdot e^{-r \cdot t} dt$$
$$a(t) \cdot e^{-r \cdot t} - a(0) = -\frac{s}{r} \cdot e^{-r \cdot t} + \frac{s}{r}.$$

Therefore, as $a(0) = a_0$, the solution to the initial value problem must satisfy

$$a(t) = a_0 \cdot e^{r \cdot t} + \frac{s}{r}\left(e^{r \cdot t} - 1\right).$$

Solution to Problem 2.

The integrating factor is now

$$\mu(t) = \exp\left(-\int_0^t r(w)dw\right).$$

Notice that the derivative of the integrating factor satisfies

$$\dot{\mu}(t) = -r(t) \cdot \mu(t)$$

(which is why we picked this specific integrating factor). We multiply both sides of the differential equation by the integrating factor. The differential equation becomes

$$\begin{aligned}
\dot{a}(t) \cdot \mu(t) - a(t) \cdot r(t) \cdot \mu(t) &= s(t) \cdot \mu(t) \\
\dot{a}(t) \cdot \mu(t) - a(t) \cdot \dot{\mu}(t) &= s(t) \cdot \mu(t) \\
\frac{d\left[a(t) \cdot \mu(t)\right]}{dt} &= s(t) \cdot \mu(t).
\end{aligned}$$

Integrating from time 0 to t,

$$\begin{aligned}
a(t) \cdot \mu(t) - a(0) \cdot \mu(0) &= \int_0^t s(z) \cdot \mu(z)dz \\
a(t) &= \frac{a_0}{\mu(t)} + \int_0^t s(z) \cdot \frac{\mu(z)}{\mu(t)}dz \\
a(t) &= a_0 \cdot \exp\left(\int_0^t r(z)dz\right) + \int_0^t s(z) \cdot \exp\left(\int_z^t r(w)dw\right) dz.
\end{aligned}$$

This equation reduces to the solution of Problem 6 when both r and s are constant.

Solution to Problem 3.

A. We are facing a linear, two-variable, homogenous system of first-order differential equations. To find the general solution of the system, we need the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

First, we determine the eigenvalues. The eigenvalues λ are the roots of the polynomial $\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I})$. So the eigenvalues λ solve

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

Hence, the eigenvalues λ are solutions to

$$(1-\lambda)^2 - 4 = 0$$

So there are two distinct eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = -1$.

Second, we determine the eigenvectors. The eigenvector $[\alpha, \beta]$ associated with the eigenvalue λ solves

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To determine the eigenvector associated with $\lambda_1 = 3$, we solve

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to the single equation

$$-2 \cdot \alpha + \beta = 0$$

thus $\beta = 2 \cdot \alpha$, and the eigenvector corresponding to $\lambda_1 = 3$ is

$$\boldsymbol{z}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Similarly, the eigenvector corresponding to $\lambda_2 = -1$ is

$$\boldsymbol{z}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Using the eigenvalues and eigenvectors that we have determined, we conclude that the general solution of the system is

$$\boldsymbol{x}(t) = c_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{3 \cdot t} + c_2 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot e^{-t},$$

where c_1 and c_2 are arbitrary constants.

B. To determine a specific solution, we would need two boundary conditions that would allow us to determine the two constants c_1 and c_2.

C. Since the linear, two-variable, homogenous system has two eigenvalues of opposite sign, the trajectories of the system have the origin as a saddle point. See the treatment of the two-variable linear system with two eigenvalues of opposite sign in the lecture notes.

Solution to Problem 4.

A. $f(k) = k^\alpha$ with $\alpha \in (0, 1)$ satisfies the Inada conditions.

B. Steady-state capital k^* is implicitly determined by

$$s \cdot f\left(k^*\right) = \delta \cdot k^*.$$

C. Plot k on the x-axis. Draw two curves $y = s \cdot f(k)$ and $y = \delta \cdot k$. The $y = s \cdot f(k)$ curve is the saving curve. It is increasing and concave . The $y = \delta \cdot k$ curve is the depreciation curve. It is an increasing straight line. The intersection of these two curves is the steady state. Starting from an initial k_0, $k(t)$ converge to k^*. This is because if $k(t)$ is to the left of k^*, $\dot{k} > 0$ so $k(t)$ increases to k^*; and if $k(t)$ is to the right of k^*, $\dot{k} < 0$, so $k(t)$ decreases to k^*.

Solution to Problem 5.

A. See lecture notes.

B. The Jacobian matrix at the steady state is

$$J^* = \begin{bmatrix} \rho & -1 \\ \alpha \cdot (\alpha - 1) \cdot A \cdot k^{\alpha-2} & 0 \end{bmatrix}$$

C. To show that the steady state is a saddle point locally, we must show that the eigenvalues of the Jacobian matrix evaluated at the steady state have opposite sign. The determinant of the Jacobian matrix is $\det(J^*) = \alpha \cdot (\alpha - 1) \cdot A \cdot k^{\alpha-2} < 0$. As explained in the lecture notes, the two eigenvalues have opposite sign and the steady state is a saddle point locally.

D. An unanticipated decrease in ρ at time t_0 means that the $\dot{c} = 0$ locus shifts to the right at time t_0. The new steady state is (k^{**}, c^{**}) with $k^{**} > k^*$ and $c^{**} > c^*$. There is a new saddle path for the new steady state. Given that k is predetermined, it must remain at its steady-state level at t_0: $k(t_0) = k^*$. Only consumption adjusts to bring the economy on the new saddle path. Thus at time t_0, the economy jumps to a point $(k^*, c(t_0))$ on the new saddle path. Then it moves along the saddle path to converge to the new steady state.

Solution to Problem 6.

A. We plot the phase diagram in a (k, q) plane. The $\dot{k}(t) = 0$ locus is horizontal. The $\dot{q}(t) = 0$ locus is described implicitly by

$$f''(k) \cdot \frac{\partial k}{\partial q} = r - \frac{q-1}{\chi}.$$

There is no clear sign for the slope of the $\dot{q}(t) = 0$ locus. However, if we are close to the steady state, q is close to 1. So the $\dot{q}(t) = 0$ locus must be downward sloping.

B. The two differential equations show that $k(t)$ increases if we are to the right of the $\dot{k}(t) = 0$ locus, and $q(t)$ increases if we are above the $\dot{q}(t) = 0$ locus. Again, we have a saddle point locally.

Solution to Problem 7.

A. By definition

$$\Delta k = f(k) - \delta \cdot k - c$$
$$\Delta c = \left[\beta \cdot \left(f'(k) + 1 - \delta\right) - 1\right] \cdot c.$$

Hence, the locus $\Delta k = 0$ is defined by

$$c = f(k) - \delta \cdot k,$$

and the locus $\Delta c = 0$ is defined by

$$f'(k) = \frac{1}{\beta} - 1 + \delta.$$

The intersection of these two curves is the steady state (k^*, c^*). The $\Delta k = 0$ locus is concave in the (k, c) plane while the $\Delta c = 0$ locus is a vertical line passing through k^*.

B. Follow the same procedure as that described in the lecture notes to analyze systems of nonlinear differential equations.

Solution to Problem 8.

A. $c(t)$ and $l(t)$ are the control variables. $k(t)$ and $h(t)$ are the state variables.

B. The present-value Hamiltonian is

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot \ln(c(t)) + \lambda^k(t) \cdot \left[y(t) - c(t) - \delta \cdot k(t)\right] + \lambda^h(t) B \cdot (1 - l(t)) \cdot h(t),$$

where $\lambda^h(t)$ and $\lambda^k(t)$ are the co-state variables associated with the law of motion of human capital $h(t)$ and physical capital $k(t)$.

C. The optimality conditions are

$$\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0$$
$$\frac{\partial \mathcal{H}(t)}{\partial l(t)} = 0$$
$$\frac{\partial \mathcal{H}(t)}{\partial k(t)} = -\dot{\lambda}^k(t)$$
$$\frac{\partial \mathcal{H}(t)\nu}{\partial h(t)} = -\dot{\lambda}^h(t).$$

These conditions simplify to

(1) $$e^{-\rho \cdot t} \cdot \frac{1}{c(t)} = \lambda^k(t)$$

(2) $$\lambda^k(t) \cdot \beta \cdot \frac{y(t)}{l(t)} = \lambda^h(t) \cdot B \cdot h(t)$$

(3) $$\lambda^k(t) \cdot \left[\alpha \cdot \frac{y(t)}{k(t)} - \delta\right] = -\dot{\lambda}^k(t)$$

(4) $$\lambda^k(t) \cdot \beta \cdot \frac{y(t)}{h(t)} + \lambda^h(t) \cdot B \cdot \left[1 - l(t)\right] = -\dot{\lambda}^h(t).$$

D. The growth rate of $c(t)$ follows from the combination of equations (1) and (3).

E. The equality of equation (2) holds for interior solution only, i.e. $0 < l < 1$. When $B = 0$, the optimal solution is $l = 1$.

F. The dynamic equations of the equilibrium are:

$$\dot{k} = A \cdot k^{\alpha} \cdot h_0^{\beta} - c - \delta \cdot k$$
$$\frac{\dot{c}}{c} = \alpha \cdot A \cdot k^{\alpha-1} \cdot h_0^{\beta} - (\delta + \rho)$$
$$\dot{h} = 0$$

Since h_0 is simply a constant, this system has a steady state (k^*, c^*) where $\dot{k} = \dot{c} = 0$. The steady state satisfies

$$\alpha \cdot A \cdot \left(k^*\right)^{\alpha-1} \cdot h_0^{\beta} = \delta + \rho.$$

To draw the phase diagram from here, see lecture notes.

G. To show that the steady state is a saddle point graphically, see lecture notes.

H. The Jacobian is given by

$$\boldsymbol{J}^* = \begin{bmatrix} \left.\frac{\partial \dot{k}}{\partial k}\right|_{(k^*,c^*)} & \left.\frac{\partial \dot{k}}{\partial c}\right|_{(k^*,c^*)} \\ \left.\frac{\partial \dot{c}}{\partial k}\right|_{(k^*,c^*)} & \left.\frac{\partial \dot{c}}{\partial c}\right|_{(k^*,c^*)} \end{bmatrix} = \begin{bmatrix} \rho & -1 \\ (\alpha - 1)\, \alpha \cdot A \cdot (k^*)^{\alpha-2}\, h_0^{\beta} & 0 \end{bmatrix}$$

I. It follows that the steady state is a saddle point locally because the determinant of the Jacobian matrix is negative:

$$\det\left(\boldsymbol{J}^*\right) = (\alpha - 1) \cdot \alpha \cdot A \left(k^*\right)^{\alpha-2} \cdot h_0^{\beta} < 0,$$

which implies that the two eigenvalues of the system have opposite sign.