

Mathematical Methods for Macroeconomics: One-Hour Exam

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Question 1. (50 pts)

Let $\alpha \in (0, 1)$, $\delta \in (0, 1)$, $\rho \in (0, 1)$, and $\sigma > 0$. Impose that $\rho + \delta < 1$. Given $k(0)$, we want to find the function $c(t)$ to maximize

$$\int_0^{+\infty} e^{-\rho \cdot t} \cdot \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt,$$

subject to the law of motion

$$\dot{k}(t) = k(t)^\alpha - c(t) - \delta \cdot k(t).$$

- A. (20 pts) Which variable do you choose as a state variable? Which variable do you choose as a control variable? Write down the current-value Hamiltonian and derive the optimality conditions.
- B. (5 pts) The Euler equation is the first-order differential equation that characterizes the optimal function $c(t)$. Determine the Euler equation.
- C. (10 pts) Suppose $\alpha = 1$ and $\sigma = 1$. Show that the system describing the optimal functions $\{k(t), c(t)\}$ reduces to a linear, homogenous system of first-order differential equations. Show that the system is unstable by computing the eigenvalues.
- D. (15 pts) Suppose $\alpha < 1$ and $\sigma > 0$. Show that the system describing the optimal functions $\{k(t), c(t)\}$ reduces to a nonlinear system of first-order differential equations. Use a phase-diagram to show that the steady state of the system is a saddle point. Explain how you draw the phase diagram.

Question 2. (50 pts)

Let $\beta \in (0, 1)$ and $r > 0$. Given $k_0 > 0$, we want to find a collection of sequences $\{c_t, k_{t+1}\}_{t=0}^{+\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \cdot \ln(c_t),$$

subject to the constraints

$$k_{t+1} = (1 + r) \cdot k_t - c_t$$

for all $t \geq 0$.

Lagrangian. We first solve the maximization problem using the Lagrangian method.

- A. (5 pts) Write down the Lagrangian of the problem.
- B. (5 pts) Derive the first-order condition(s) of the maximization problem.
- C. (5 pts) Derive the Euler equation.

Dynamic Programming. Next we solve the maximization problem using the dynamic programming method.

- D. (5 pts) Which variable do you choose as a state variable? Which variable do you choose as a control variable? Write down the Bellman equation.
- E. (5 pts) Derive the first-order condition associated with the Bellman equation.
- F. (5 pts) Derive the Benveniste-Scheinkman equation.
- G. (5 pts) Derive the Euler equation. Compare it with the Euler equation obtained with the Lagrangian method and discuss.
- H. (5 pts) Suppose that the policy function takes the form $h(k) = A \cdot (1 + r) \cdot k$ where $A \in (0, 1)$. Derive A .
- I. (10 pts) Suppose that the value function takes the form $V(k) = B + D \cdot \ln(k)$, where B and D are constants. Using the expression for the policy function that you derived in the previous question, derive B and D .