Problem Set on Dynamic Programming: Solutions

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Solution to Problem 1.

A. See lecture notes.

B. At the beginning of period t, one can choose c_t but not k_t. So the control variable is c_t and the state variable is k_t. But given k_t, c_t and k_{t+1} are tied via the resource constraint. We saw in lecture that choosing k_{t+1} simplifies the application of the Benveniste-Scheinkman equation. So we use k_{t+1} instead of c_t as a control variable. Below, k denotes capital in the current period (state variable) and k' denotes capital in the next period (control variable).

C. The Bellman equation is

$$V(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V(k') \right\}.$$

D. The first-order condition with respect to k' in the Bellman equation is

$$\frac{1}{c} = \beta \cdot \frac{dV}{dk}\left(k'\right) \tag{1}$$

and the Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \alpha \cdot A \cdot k^{\alpha-1} \cdot \frac{1}{c}$$

and by combining both equations we obtain the Euler equation

$$c' = \alpha \cdot \beta \cdot A \cdot \left(k'\right)^{\alpha-1} \cdot c.$$

E. Start with $V_0(k) = 0$. Plug $V_0(k)$ into the Bellman equation to calculate the value function

$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V_0(k') \right\}$$
$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) \right\}.$$

The policy function is $k' = 0$, which implies that $c = A \cdot k^{\alpha}$. Therefore, the value function after the first iteration is

$$V_1(k) = \ln\left(A \cdot k^{\alpha}\right)$$

Now substitute the value function $V_1(k)$ into the Bellman equation and calculate the value function

$$V_2(k) = \max_{k'}\left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V_1\left(k'\right)\right\}$$
$$V_2(k) = \max_{k'}\left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot \ln\left(A \cdot (k')^{\alpha}\right)\right\}.$$

The first-order condition with respect to k' is

$$\frac{-1}{A \cdot k^{\alpha} - k'} + \frac{\alpha \cdot \beta}{k'} = 0.$$

Thus, the policy function is

$$k' = \frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}$$

which also implies that

$$c = \frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}.$$

Therefore, the value function after the second iteration is

$$V_2(k) = \ln\left(\frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}\right) + \beta \ln\left(A \cdot \left(\frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}\right)^{\alpha}\right).$$

It is convenient to write

$$V_2(k) = \kappa_2 + (1 + \alpha \cdot \beta) \cdot \ln\left(k^{\alpha}\right)$$

where κ_2 is a constant.

F. Using (1), we infer that the policy function satisfies

$$k'(k) = \alpha \cdot \beta \cdot A \cdot k^{\alpha}$$

and equivalently

$$c(k) = (1 - \alpha \cdot \beta) \cdot A \cdot k^{\alpha}.$$

G. Dynamic programming sometimes allows us to find closed-form solution to optimization problems, which the Lagrangian method would not allow us to do. Even if it does not allow us to find closed-form solutions, dynamic programming sometimes allows us to find some theoretical properties of the solution. Last, dynamic programs can be (sometimes easily) solved with numerical methods.

Solution to Problem 2.

A. The state variable are the amount of shares s_t and the dividend d_t. The control variables is consumption c_t. Since c_t and s_{t+1} are linked through the budget, we can also choose s_{t+1} as control variable. As usual, we pick s_{t+1} as control variable to simplify derivations.

B. The Bellman equation is

$$V(s,d) = \max_{s'}\left\{u\left((p+d)\cdot s - p\cdot s'\right) + \beta\cdot\mathbb{E}\left(V\left(s',d'\right) \mid d\right)\right\}$$

C. The first-order condition with respect to s' in the Bellman equation is

$$-p\cdot\frac{du}{dc}(c) + \beta\cdot\mathbb{E}\left(\frac{\partial V\left(s',d'\right)}{\partial s'} \mid d\right) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{\partial V(s,d)}{\partial s} = (p+d)\cdot\frac{du}{dc}(c)\,.$$

Combining both equations we obtain the following Euler equation:

$$p\cdot\frac{du}{dc}(c) = \beta\cdot\mathbb{E}\left(\left(d'+p'\right)\cdot\frac{du}{dc}\left(c'\right) \mid d\right).$$

D. With $u(c) = c$, $du/dc = 1$ and the Euler equation becomes

$$p = \beta\cdot\mathbb{E}\left(\left(d'+p'\right) \mid d\right).$$

Let p_h be the price when today's dividend is high, and let p_l be the price when today's dividend is low.

$$\begin{aligned}
p_h &= \beta\cdot\left[\rho\cdot\left(d_h+p_h\right) + (1-\rho)\cdot\left(d_l+p_l\right)\right] \\
p_l &= \beta\cdot\left[\rho\cdot\left(d_l+p_l\right) + (1-\rho)\cdot\left(d_h+p_h\right)\right]
\end{aligned}$$

which implies

$$p_h - p_l = \beta\cdot\frac{2\cdot\rho-1}{1-[\beta\cdot(2\cdot\rho-1)]}\cdot\left(d_h-d_l\right) > 0$$

because $0.5 < \rho < 1$. So the price is higher when the dividend is higher.

Solution to Problem 3.

A. k is the state variable and (k', l) are the control variables.

B. The Bellman equation is

$$V(k) = \max_{k',l} \left\{ u\left[f(k,l) - k', l\right] + \beta \cdot V(k') \right\}$$

C. The first-order conditions with respect to k' and l in the Bellman equation are

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \frac{dV}{dk}(k') = 0$$
$$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l)$$

We combine these equations to get

$$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \frac{\partial u}{\partial c}(c', l') \cdot \frac{\partial f}{\partial k}(k', l') \tag{2}$$
$$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) = -\frac{\partial u}{\partial l}(c,l). \tag{3}$$

D. In steady state, we have $l = l^*$, $c = c^*$, and $k = k^*$. Using (2) and the functional form of f, we obtain

$$\alpha \cdot \beta \cdot \left(\frac{k^*}{l^*}\right)^{\alpha-1} = 1$$
$$\frac{k^*}{l^*} = (\alpha \cdot \beta)^{1/(1-\alpha)}.$$

Then use the law of motion of capital implies

$$\frac{c^*}{k^*} = \left(\frac{k^*}{l^*}\right)^{\alpha-1} - 1 = \frac{1}{\alpha \cdot \beta} - 1.$$

E. The Bellman equation is

$$V(A,k) = \max_{k',l} \left\{ u\left[A \cdot f(k,l) - k', l\right] + \beta \cdot \mathbb{E}\left(V\left(A',k'\right) \mid A\right) \right\}$$

where (A,k) are the state variables and $\left(k',l\right)$ are the control variables.

F. The first-order conditions with respect to k' and l become

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \mathbb{E}\left(\frac{\partial V}{\partial k'}\left(A',k'\right) \mid A\right) = 0$$
$$A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation becomes

$$\frac{\partial V}{\partial k}(A,k) = A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l).$$

The Euler condition is

$$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \mathbb{E}\left(A' \cdot \frac{\partial u}{\partial c}(c',l') \cdot \frac{\partial f}{\partial k}(k',l') \mid A\right).$$