Problem Set on Dynamic Programming: Solutions

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Solution to Problem 1

A) See lecture notes.

B) At the beginning of period t, one can choose c_t but not k_t. So the control variable is c_t and the state variable is k_t. But given k_t, c_t and k_{t+1} are tied via the resource constraint. We saw in lecture that choosing k_{t+1} simplifies the application of the Benveniste-Scheinkman equation. So we use k_{t+1} instead of c_t as a control variable. Below, k denotes capital in the current period (state variable) and k' denotes capital in the next period (control variable).

C) The Bellman equation is

$$V(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V(k') \right\}.$$

D) The first-order condition with respect to k' in the Bellman equation is

$$\frac{1}{c} = \beta \cdot \frac{dV}{dk}\left(k'\right) \tag{1}$$

and the Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \alpha \cdot A \cdot k^{\alpha-1} \cdot \frac{1}{c}$$

and by combining both equations we obtain the Euler equation

$$c' = \alpha \cdot \beta \cdot A \cdot \left(k'\right)^{\alpha-1} \cdot c.$$

E) Start with $V_0(k) = 0$. Plug $V_0(k)$ into the Bellman equation to calculate the value function

$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V_0(k') \right\}$$
$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) \right\}.$$

The policy function is $k' = 0$, which implies that $c = A \cdot k^{\alpha}$. Therefore, the value function after the first iteration is

$$V_1(k) = \ln\left(A \cdot k^{\alpha}\right)$$

Now substitute the value function $V_1(k)$ into the Bellman equation and calculate the value function

$$V_2(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V_1\left(k'\right) \right\}$$
$$V_2(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot \ln\left(A \cdot (k')^{\alpha}\right) \right\}.$$

The first-order condition with respect to k' is

$$\frac{-1}{A \cdot k^{\alpha} - k'} + \frac{\alpha \cdot \beta}{k'} = 0.$$

Thus, the policy function is

$$k' = \frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}$$

which also implies that

$$c = \frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}.$$

Therefore, the value function after the second iteration is

$$V_2(k) = \ln\left(\frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}\right) + \beta \ln\left(A \cdot \left(\frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^{\alpha}\right)^{\alpha}\right).$$

It is convenient to write

$$V_2(k) = \kappa_2 + (1 + \alpha \cdot \beta) \cdot \ln\left(k^{\alpha}\right)$$

where κ_2 is a constant.

F) Using (1), we infer that the policy function satisfies

$$k'(k) = \alpha \cdot \beta \cdot A \cdot k^{\alpha}$$

and equivalently

$$c(k) = (1 - \alpha \cdot \beta) \cdot A \cdot k^{\alpha}.$$

G) Dynamic programming sometimes allows us to find closed-form solution to optimization problems, which the Lagrangian method would not allow us to do. Even if it does not allow us to find closed-form solutions, dynamic programming sometimes allows us to find some theoretical properties of the solution. Last, dynamic programs can be (sometimes easily) solved with numerical methods.

Solution to Problem 2

A) The state variable are the amount of shares s_t and the dividend d_t. The control variables is consumption c_t. Since c_t and s_{t+1} are linked through the budget, we can also choose s_{t+1} as control variable. As usual, we pick s_{t+1} as control variable to simplify derivations.

B) The Bellman equation is

$$V(s, d) = \max_{s'}\left\{u\left((p+d) \cdot s - p \cdot s'\right) + \beta \cdot \mathbb{E}\left(V\left(s', d'\right) \mid d\right)\right\}$$

C) The first-order condition with respect to s' in the Bellman equation is

$$-p \cdot \frac{du}{dc}(c) + \beta \cdot \mathbb{E}\left(\frac{\partial V\left(s', d'\right)}{\partial s'} \mid d\right) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{\partial V(s, d)}{\partial s} = (p+d) \cdot \frac{du}{dc}(c).$$

Combining both equations we obtain the following Euler equation:

$$p \cdot \frac{du}{dc}(c) = \beta \cdot \mathbb{E}\left(\left(d' + p'\right) \cdot \frac{du}{dc}\left(c'\right) \mid d\right).$$

D) With $u(c) = c$, $du/dc = 1$ and the Euler equation becomes

$$p = \beta \cdot \mathbb{E}\left(\left(d' + p'\right) \mid d\right).$$

Let p_h be the price when today's dividend is high, and let p_l be the price when today's dividend is low.

$$\begin{aligned} p_h &= \beta \cdot \left[\rho \cdot \left(d_h + p_h\right) + (1-\rho) \cdot \left(d_l + p_l\right)\right] \\ p_l &= \beta \cdot \left[\rho \cdot \left(d_l + p_l\right) + (1-\rho) \cdot \left(d_h + p_h\right)\right] \end{aligned}$$

which implies

$$p_h - p_l = \beta \cdot \frac{2 \cdot \rho - 1}{1 - [\beta \cdot (2 \cdot \rho - 1)]} \cdot \left(d_h - d_l\right) > 0$$

because $0.5 < \rho < 1$. So the price is higher when the dividend is higher.

Solution to Problem 3

A) k is the state variable and (k', l) are the control variables.

B) The Bellman equation is

$$V(k) = \max_{k',l} \left\{ u\left[f(k,l) - k', l\right] + \beta \cdot V(k') \right\}$$

C) The first-order conditions with respect to k' and l in the Bellman equation are

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \frac{dV}{dk}(k') = 0$$
$$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l)$$

We combine these equations to get

(2) $$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \frac{\partial u}{\partial c}(c', l') \cdot \frac{\partial f}{\partial k}(k', l')$$

(3) $$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) = -\frac{\partial u}{\partial l}(c,l).$$

D) In steady state, we have $l = l^*$, $c = c^*$, and $k = k^*$. Using (2) and the functional form of f, we obtain

$$\alpha \cdot \beta \cdot \left(\frac{k^*}{l^*}\right)^{\alpha-1} = 1$$
$$\frac{k^*}{l^*} = (\alpha \cdot \beta)^{1/(1-\alpha)}.$$

Then use the law of motion of capital implies

$$\frac{c^*}{k^*} = \left(\frac{k^*}{l^*}\right)^{\alpha-1} - 1 = \frac{1}{\alpha \cdot \beta} - 1.$$

E) The Bellman equation is

$$V(A,k) = \max_{k',l} \left\{ u\left[A \cdot f(k,l) - k', l\right] + \beta \cdot \mathbb{E}\left(V\left(A',k'\right) \mid A\right) \right\}$$

where (A,k) are the state variables and $\left(k',l\right)$ are the control variables.

F) The first-order conditions with respect to k' and l become

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \mathbb{E}\left(\frac{\partial V}{\partial k'}\left(A',k'\right) \mid A\right) = 0$$
$$A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation becomes

$$\frac{\partial V}{\partial k}(A,k) = A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l).$$

The Euler condition is

$$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \mathbb{E}\left(A' \cdot \frac{\partial u}{\partial c}(c',l') \cdot \frac{\partial f}{\partial k}(k',l') \mid A\right).$$