Mathematical Methods for Macroeconomics: Exercises

Pascal Michaillat

Dynamic Programming

Exercise 1.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose consumption $\{c_t\}_{t=0}^{+\infty}$ to maximize utility

$$\sum_{t=0}^{\infty} \beta^t \cdot \ln(c_t)$$

subject to the resource constraint

$$k_{t+1} = A \cdot k_t^{\alpha} - c_t.$$

The parameters satisfy $0 < \beta < 1,\ A > 0,\ 0 < \alpha < 1$.

A. Derive the optimal law of motion of consumption c_t using a Lagrangian.

B. Identify the state variable and the control variable.

C. Write down the Bellman equation.

D. Derive the following Euler equation:

$$c_{t+1} = \beta \cdot \alpha \cdot A \cdot k_{t+1}^{\alpha-1} \cdot c_t.$$

E. Derive the first two value functions, $V_1(k)$ and $V_2(k)$, obtained by iteration on the Bellman equation starting with the value function $V_0(k) \equiv 0$.

F. The process of determining the value function by iterations using the Bellman equation is commonly used to solve dynamic programs numerically. The algorithm is called *value function iteration*. For this optimal growth problem, one can show show using value function iteration that the value function is

$$V(k) = \kappa + \frac{\ln(k^{\alpha})}{1 - \alpha \cdot \beta},$$

where κ is a constant. Using the Bellman equation, determine the policy function $k'(k)$ associated with this value function.

G. In light of these results, for which reasons would you prefer to use the dynamic-programming approach instead of the Lagrangian approach to solve the optimal growth problem? And for which reasons would you prefer to use the Lagrangian approach instead of the dynamic-programming approach?

Exercise 2.

Consider the problem of choosing consumption $\{c_t\}_{t=0}^{+\infty}$ to maximize expected utility

$$\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \cdot u\left(c_t\right)$$

subject to the budget constraint

$$c_t + p_t \cdot s_{t+1} = \left(d_t + p_t\right) \cdot s_t.$$

d_t is the dividend paid out for one share of the asset, p_t is the price of one share of the asset, and s_t is the number of shares of the asset held at the beginning of period t. In equilibrium, the price p_t of one share is solely a function of dividends d_t. Dividends can only take two values d_l and d_h, with $0 < d_l < d_h$. Dividends follow a Markov process with transition probabilities

$$\mathbb{P}\left(d_{t+1} = d_l \mid d_t = d_l\right) = \mathbb{P}\left(d_{t+1} = d_h \mid d_t = d_h\right) = \rho$$

with $1 > \rho > 0.5$.

A. Identify state and control variables.

B. Write down the Bellman equation.

C. Derive the following Euler equation:

$$p_t \cdot u'\left(c_t\right) = \beta \cdot \mathbb{E}\left(\left(d_{t+1} + p_{t+1}\right) \cdot u'\left(c_{t+1}\right) \mid d_t\right).$$

D. Suppose that $u(c) = c$. Show that the asset price is higher when the current dividend is high.

Exercise 3.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose consumption and labor $\{c_t, l_t\}_{t=0}^{+\infty}$ to maximize utility

$$\sum_{t=0}^{+\infty} \beta^t \cdot u(c_t, l_t)$$

subject to the law of motion of capital

$$k_{t+1} = A_t \cdot f(k_t, l_t) - c_t.$$

In addition, we impose $0 \leq l_t \leq 1$. The discount factor $\beta \in (0, 1)$. The function f is increasing and concave in both arguments. The function u is increasing and concave in c, decreasing and convex in l.

Deterministic case. First, suppose $A_t = 1$ for all t.

A. What are the state and control variables?

B. Write down the Bellman equation.

C. Derive the following optimality conditions:

$$\frac{\partial u(c_t, l_t)}{\partial c_t} = \beta \cdot \frac{\partial u(c_{t+1}, l_{t+1})}{\partial c_{t+1}} \cdot \frac{\partial f(k_{t+1}, l_{t+1})}{\partial k_{t+1}}$$

$$\frac{\partial u(c_t, l_t)}{\partial c_t} \cdot \frac{\partial f(k_t, l_t)}{\partial l_t} = -\frac{\partial u(c_t, l_t)}{\partial l_t}.$$

D. Suppose that the production function $f(k, l) = k^\alpha \cdot l^{1-\alpha}$. Determine the ratios c/k and l/k in steady state.

Stochastic case. Now, suppose A_t is a stochastic process that takes values A_1 and A_2 with the following probability:

$$\mathbb{P}(A_{t+1} = A_1 \mid A_t = A_1) = \mathbb{P}(A_{t+1} = A_2 \mid A_t = A_2) = \rho.$$

E. Write down the Bellman equation.

F. Derive the optimality conditions.

Optimal Control

Exercise 4.

Consider the following optimal growth problem: Given initial capital $k_0 > 0$, choose a consumption path $\{c_t\}_{t \geq 0}$ to maximize utility

$$\int_0^{\infty} e^{-\rho \cdot t} \cdot \ln(c_t) dt$$

subject to the law of motion of capital

$$\dot{k}_t = f(k_t) - c_t - \delta \cdot k_t.$$

The discount factor $\rho > 0$, and the production function f satisfies

$$f(k) = A \cdot k^{\alpha},$$

where $\alpha \in (0, 1)$ and $A > 0$.

A. Write down the present-value Hamiltonian.

B. Show that the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho).$$

C. Solve for the steady state of the system.

Exercise 5.

Consider the following investment problem: Given initial capital k_0, choose the investment path $\{i_t\}_{t \geq 0}$ to maximize profits

$$\int_0^{\infty} e^{-r \cdot t} \left[f(k_t) - i_t - \frac{\chi}{2} \cdot \left(\frac{i_t^2}{k_t} \right) \right] dt$$

subject to the law of motion of capital (we assume no capital depreciation)

$$\dot{k}_t = i_t.$$

The interest rate $r > 0$, the capital adjustment cost $\chi > 0$, and the production function f satisfies $f' > 0$ and $f'' < 0$.

A. Write down the current-value Hamiltonian.

B. Use the optimality conditions for the current-value Hamiltonian to derive the following differential equations:

$$\dot{k}_t = \left(\frac{q_t - 1}{\chi}\right) \cdot k_t$$
$$\dot{q}_t = r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi}(q_t - 1)^2$$

C. Solve for the steady state.

Differential Equations

Exercise 6.

Find the solution of the initial value problem

$$\begin{aligned} \dot{a}(t) &= r \cdot a(t) + s \\ a(0) &= a_0 \end{aligned}$$

where both r and s are known constant.

Exercise 7.

Find the solution of the initial value problem

$$\begin{aligned} \dot{a}(t) &= r(t) \cdot a(t) + s(t) \\ a(0) &= a_0 \end{aligned}$$

where both $r(t)$ and $s(t)$ are known functions of t.

Exercise 8.

Consider the linear system of FODEs given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}(t).$$

A. Find the general solution of the system.

B. What would you need to find a specific solution of the system?

C. Draw the trajectories of the system.

Exercise 9.

Consider the initial value problem

$$\begin{aligned} \dot{k}(t) &= s \cdot f(k(t)) - \delta \cdot k(t) \\ k(0) &= k_0 \end{aligned}$$

where the saving rate $s \in (0,1)$, the capital depreciation rate $\delta \in (0,1)$, and the production function f satisfies the *Inada conditions*. That is, f is continuously differentiable and

$$\begin{aligned} f(0) &= 0 \\ f'(x) &> 0 \\ f''(x) &< 0 \\ \lim_{x \to 0} f'(x) &= +\infty \\ \lim_{x \to +\infty} f'(x) &= 0. \end{aligned}$$

A. Give a production function f that satisfies the Inada conditions.

B. Find the steady state of the system.

C. Draw the dynamic path of $k(t)$ and show that it converges to the steady state.

Exercise 10.

The solution of the problem studied in Exercise 4 is characterized by a system of two nonlinear first-order differential equations:

$$\begin{aligned} \dot{k}_t &= f(k_t) - c_t - \delta \cdot k_t \\ \frac{\dot{c}_t}{c_t} &= \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho). \end{aligned}$$

The first FODE is the law of motion of capital. The second FODE is the Euler equation, which describes the optimal path of consumption over time.

A. Draw the phase diagram of the system.

B. Linearize the system around its steady state.

C. Show that the steady state is a saddle point locally.

D. Suppose the economy is in steady state at time t_0 and there is an unanticipated decrease in the discount factor ρ. Show on your phase diagram the transition dynamics of the model.

Exercise 11.

The solution of the investment problem studied in Exercise 5 is characterized by a system of two nonlinear first-order differential equations:

$$\dot{k}_t = \left(\frac{q_t - 1}{\chi}\right) \cdot k_t$$
$$\dot{q}_t = r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi}(q_t - 1)^2 .$$

The first FODE is the law of motion of capital k_t. The second FODE is the law of motion of the co-state variable q_t.

A. Draw the phase diagram.

B. Show that the steady state is a saddle point locally.

Exercise 12.

Consider a discrete time version of the typical growth model:

$$k(t+1) = f(k(t)) - c(t) + (1 - \delta) \cdot k(t)$$
$$c(t+1) = \beta \cdot \left[1 + f'(k(t)) - \delta\right] \cdot c(t).$$

The discount factor $\beta \in (0, 1)$, the rate of depreciation of capital $\delta \in (0, 1)$, initial capital k_0 is given, and the production function f satisfies the Inada conditions. These two equations are a system of first-order difference equations. Whereas a system of first-order differential equations relates $\dot{\boldsymbol{x}}(t)$ to $\boldsymbol{x}(t)$, a system of first-order difference equations relate $\boldsymbol{x}(t+1)$ to $\boldsymbol{x}(t)$.

In this exercise, we will see that we can study a system of first-order difference equations with the tools that we used to study systems of first-order differential equations. In particular, we can use phase diagrams to understand the dynamics of the system.

A. Construct a phase diagram for the system. First, define

$$\Delta k \equiv k(t+1) - k(t),$$
$$\Delta c \equiv c(t+1) - c(t).$$

Second, draw the $\Delta k = 0$ locus and the $\Delta c = 0$ locus on the (k, c) plane. Finally, find the steady state as the intersection of the $\Delta k = 0$ locus and the $\Delta c = 0$ locus.

B. Show that the steady state is a saddle point in the phase diagram.

Exercise 13.

We consider the following optimal growth problem. Given initial human capital h_0 and initial physical capital k_0, choose consumption $c(t)$ and labor $l(t)$ to maximize utility

$$\int_0^{\infty} e^{-\rho \cdot t} \cdot \ln(c) dt$$

subject to

$$\dot{k}_t = y_t - c_t - \delta \cdot k_t$$
$$\dot{h}_t = B \cdot (1 - l_t) \cdot h_t.$$

Output y_t is defined by

$$y_t \equiv A \cdot k_t^{\alpha} \cdot (l_t \cdot h_t)^{\beta} .$$

We also impose that $0 \leq l_t \leq 1$. The discount factor $\rho > 0$, the rate of depreciation of physical capital $\delta > 0$, the constants $A > 0$ and $B > 0$, and the production function parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$.

A. Give state and control variables.

B. Write down the present-value Hamiltonian for this problem.

C. Derive the optimality conditions.

D. Show that the growth rate of consumption $c(t)$ is

$$\frac{\dot{c}}{c} = \frac{\alpha \cdot y}{k} - (\delta + \rho) .$$

E. From now on, we assume that $B = 0$. Show that $l = 1$.

F. Draw the phase diagram in the (k, c) plane.

G. Show on the diagram that the steady state of the system is a saddle point.

H. Derive the Jacobian of the system.

I. Show that the steady state of the system is a saddle point.