

# **Mathematical Methods for Macroeconomics: Exam Solutions**

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## Question 1.

A. State variable:  $k(t)$ . Control variable:  $c(t)$ . Current-value Hamiltonian:

$$H^*(t) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma} + q(t) \cdot [k(t)^\alpha - c(t) - \delta \cdot k(t)],$$

where  $q(t)$  is the co-state variable. We have the following optimality conditions:

- First optimality condition:

$$\begin{aligned} \frac{\partial H^*(t)}{\partial c(t)} &= 0 \\ c(t)^{-\sigma} &= q(t). \end{aligned}$$

- Second optimality condition:

$$\begin{aligned} \frac{\partial H^*(t)}{\partial k(t)} &= \rho \cdot q(t) - \dot{q}(t) \\ q(t) \cdot [\alpha \cdot k(t)^{\alpha-1} - \delta - \rho] &= -\dot{q}(t) \\ [\alpha \cdot k(t)^{\alpha-1} - \delta - \rho] &= -\frac{\dot{q}(t)}{q(t)} \end{aligned}$$

- Third optimality condition: the transversality condition

$$\lim_{t \rightarrow +\infty} e^{-\rho \cdot t} \cdot q(t) \cdot k(t) = 0.$$

B. Take the log of the first optimality condition, differentiate with respect to time  $t$ , and plug the result from the second optimality condition:

$$\begin{aligned} -\sigma \cdot \ln[c(t)] &= \ln[q(t)] \\ \sigma \cdot \frac{\dot{c}(t)}{c(t)} &= -\frac{\dot{q}(t)}{q(t)} \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma} \cdot [\alpha \cdot k(t)^{\alpha-1} - \delta - \rho]. \end{aligned}$$

C. The optimal functions  $\{k(t), c(t)\}$  are described by a system of two equations: the Euler

equation and the law of motion of  $k(t)$ . If  $\alpha = 1$  and  $\sigma = 1$ , the system is

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} (1 - \rho - \delta) & 0 \\ -1 & (1 - \delta) \end{bmatrix} \begin{bmatrix} c(t) \\ k(t) \end{bmatrix}.$$

This is a linear, homogenous system of first-order differential equations.

We compute the eigenvalues  $\lambda$  of the system.  $\lambda$  solves

$$\det \begin{bmatrix} 1 - \rho - \delta - \lambda & 0 \\ -1 & 1 - \delta - \lambda \end{bmatrix} = 0.$$

Hence,  $\lambda$  solves

$$[(1 - \rho - \delta) - \lambda] \cdot [(1 - \delta) - \lambda] = 0.$$

Therefore the system admits two distinct positive eigenvalues:

$$\lambda_1 = 1 - (\rho + \delta) > 0$$

$$\lambda_2 = 1 - \delta > 0.$$

We are facing a linear homogenous system of first-order differential equations with two positive eigenvalues are positive: the system is unstable.

- D. The optimal functions  $\{k(t), c(t)\}$  are described by a system of two equations: the Euler equation and the law of motion of  $k(t)$ . If  $\alpha < 1$ , the system is

$$\begin{aligned} \dot{c}(t) &= \frac{1}{\sigma} \cdot \left[ \alpha \cdot k(t)^{\alpha-1} - \delta - \rho \right] \cdot c(t) \\ \dot{k}(t) &= k(t)^\alpha - c(t) - \delta \cdot k(t). \end{aligned}$$

This is a nonlinear system of first-order differential equations.

We draw the phase diagram with the state variable  $k$  on the x-axis and the control variable  $c$  on the y-axis. The locus  $\dot{c}(t) = 0$  and the locus  $\dot{k}(t) = 0$  satisfy

$$\begin{aligned} k(t) &= \left[ \frac{\delta + \rho}{\alpha} \right]^{-1/(1-\alpha)} \equiv k^* \\ c(t) &= k(t)^\alpha - \delta \cdot k(t). \end{aligned}$$

The locus  $\dot{c}(t) = 0$  is a vertical line. The locus  $\dot{k}(t) = 0$  is a concave curve that goes through the origin and that cuts the x-axis again at  $k^{**} = \delta^{-1/(1-\alpha)}$ . Since  $\rho > 0$  and  $\alpha < 1$ ,  $k^{**} > k^*$  and the concave curve crosses the vertical curve when it is positive. The steady state of the system is the intersection of the locus  $\dot{c}(t) = 0$  and the locus  $\dot{k}(t) = 0$ .

Look at the equation for  $\dot{c}(t)$  to determine the vertical arrows. Since  $\alpha \cdot k^{\alpha-1} - \delta - \rho$  decreases with  $k$ ,  $\dot{c}(t) > 0$  to the west of  $\dot{c}(t) = 0$  and  $\dot{c}(t) < 0$  to the east of  $\dot{c}(t) = 0$ . So the vertical arrows point northwards to the west of  $\dot{c}(t) = 0$  and southwards to the east of  $\dot{c}(t) = 0$ .

Look at the equation for  $\dot{k}(t)$  to determine the horizontal arrows. Clearly,  $\dot{k}(t) > 0$  to the south of  $\dot{k}(t) = 0$  and  $\dot{k}(t) < 0$  to the north of  $\dot{k}(t) = 0$ . So the horizontal arrows point eastwards to the north of  $\dot{k}(t) = 0$  and westwards to the south of  $\dot{k}(t) = 0$ .

Therefore, the steady state of the system is a saddle point. The saddle path goes through the south-west and north-east regions of the plane.

## Question 2.

A. The Lagrangian associated with the problem is

$$L = \sum_{t=0}^{+\infty} \beta^t \cdot \{\ln(c_t) - \lambda_t \cdot [k_{t+1} - (1+r) \cdot k_t + c_t]\},$$

where  $\{\lambda_t\}_{t=1}^{+\infty}$  is the sequences of Lagrange multipliers associated with the sequences of constraints.

B. The first-order conditions with respect to  $c_t$  and  $k_{t+1}$  are

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ \lambda_t &= \beta \cdot (1+r) \cdot \lambda_{t+1}. \end{aligned}$$

C. Combining the first-order conditions yields the Euler equation:

$$\frac{c_{t+1}}{c_t} = \beta \cdot (1+r).$$

D. State variable:  $k$ . Control variable:  $k'$  (the value of variable  $k$  next period). Bellman equation:

$$V(k) = \max_{k'} [\ln((1+r) \cdot k - k') + \beta \cdot V(k')].$$

E. The first-order condition with respect to  $k'$  in the Bellman equation is

$$\frac{1}{c} = \beta \cdot V'(k').$$

F. We apply the envelope theorem to the Bellman equation:

$$V'(k) = \frac{(1+r)}{c}.$$

G. The Benveniste-Scheinkman equation holds for any  $k$ . In particular,  $V'(k') = (1+r)/c'$ . Combining this equation with the first-order condition yields the Euler equation:

$$\frac{c'}{c} = \beta \cdot (1+r).$$

This Euler equation is the same as that obtained with the Lagrangian method. The two methods are equivalent.

H. We guess that optimal consumption  $c = h(k) = A \cdot (1 + r) \cdot k$ . A first implication is that

$$\frac{c'}{c} = \frac{A \cdot (1 + r) \cdot k'}{A \cdot (1 + r) \cdot k} = \frac{k'}{k}.$$

Using the Euler equation, we obtain

$$\frac{k'}{k} = \frac{c'}{c} = (1 + r) \cdot \beta.$$

The transition equation then implies

$$c = (1 + r) \cdot k - k' = (1 - \beta) \cdot (1 + r) \cdot k.$$

Therefore, it must be that

$$A = (1 - \beta).$$

I. The Bellman equation can be written in terms of the policy function:

$$V(k) = \ln(h(k)) + \beta \cdot V((1 + r) \cdot k - h(k)).$$

We plug our guess for the value function and the expression for the policy function into the Bellman equation:

$$B + D \cdot \ln(k) = \ln((1 - \beta) \cdot (1 + r) \cdot k) + \beta \cdot [B + D \cdot \ln(\beta \cdot (1 + r) \cdot k)].$$

Rearranging the terms on the right-hand side yields

$$B + D \cdot \ln(k) = [\ln((1 - \beta) \cdot (1 + r)) + \beta \cdot B + \beta \cdot D \cdot \ln(\beta \cdot (1 + r))] + [1 + \beta \cdot D] \cdot \ln(k).$$

This equation must hold for any  $k$  so it is necessary that

$$\begin{aligned} D &= 1 + \beta \cdot D \\ D &= \frac{1}{1 - \beta} \end{aligned}$$

and

$$B = \ln((1 - \beta) \cdot (1 + r)) + \beta \cdot B + \beta \cdot D \cdot \ln(\beta \cdot (1 + r))$$

$$B = \frac{(1 - \beta) \cdot \ln((1 - \beta) \cdot (1 + r)) + \beta \cdot \ln(\beta \cdot (1 + r))}{(1 - \beta)^2}$$

$$B = \frac{(1 - \beta) \cdot \ln(1 - \beta) + \beta \cdot \ln(\beta) + \ln(1 + r)}{(1 - \beta)^2}.$$