Mathematical Methods for Macroeconomics: Exercise Solutions

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Solution to Exercise 1.

A. See lecture notes.

B. At the beginning of period t, one can choose c_t but not k_t. So the control variable is c_t and the state variable is k_t. But given k_t, c_t and k_{t+1} are tied via the resource constraint. We saw in lecture that choosing k_{t+1} simplifies the application of the Benveniste-Scheinkman equation. So we use k_{t+1} instead of c_t as a control variable. Below, k denotes capital in the current period (state variable) and k' denotes capital in the next period (control variable).

C. The Bellman equation is

$$V(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V(k') \right\}.$$

D. The first-order condition with respect to k' in the Bellman equation is

$$\frac{1}{c} = \beta \cdot \frac{dV}{dk}\left(k'\right) \tag{1}$$

and the Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \alpha \cdot A \cdot k^{\alpha-1} \cdot \frac{1}{c}$$

and by combining both equations we obtain the Euler equation

$$c' = \alpha \cdot \beta \cdot A \cdot \left(k'\right)^{\alpha-1} \cdot c.$$

E. Start with $V_0(k) = 0$. Plug $V_0(k)$ into the Bellman equation to calculate the value function

$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) + \beta \cdot V_0(k') \right\}$$
$$V_1(k) = \max_{k'} \left\{ \ln\left(A \cdot k^{\alpha} - k'\right) \right\}.$$

The policy function is $k' = 0$, which implies that $c = A \cdot k^{\alpha}$. Therefore, the value function after the first iteration is

$$V_1(k) = \ln\left(A \cdot k^{\alpha}\right)$$

Now substitute the value function $V_1(k)$ into the Bellman equation and calculate the value function

$$V_2(k) = \max_{k'} \left\{ \ln\left(A \cdot k^\alpha - k'\right) + \beta \cdot V_1(k') \right\}$$
$$V_2(k) = \max_{k'} \left\{ \ln\left(A \cdot k^\alpha - k'\right) + \beta \cdot \ln\left(A \cdot (k')^\alpha\right) \right\}.$$

The first-order condition with respect to k' is

$$\frac{-1}{A \cdot k^\alpha - k'} + \frac{\alpha \cdot \beta}{k'} = 0.$$

Thus, the policy function is

$$k' = \frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^\alpha$$

which also implies that

$$c = \frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^\alpha.$$

Therefore, the value function after the second iteration is

$$V_2(k) = \ln\left(\frac{1}{1 + \alpha \cdot \beta} \cdot A \cdot k^\alpha\right) + \beta \ln\left(A \cdot \left(\frac{\alpha \cdot \beta}{1 + \alpha \cdot \beta} \cdot A \cdot k^\alpha\right)^\alpha\right).$$

It is convenient to write

$$V_2(k) = \kappa_2 + (1 + \alpha \cdot \beta) \cdot \ln\left(k^\alpha\right)$$

where κ_2 is a constant.

F. Using (1), we infer that the policy function satisfies

$$k'(k) = \alpha \cdot \beta \cdot A \cdot k^\alpha$$

and equivalently

$$c(k) = (1 - \alpha \cdot \beta) \cdot A \cdot k^\alpha.$$

G. Dynamic programming sometimes allows us to find closed-form solution to optimization problems, which the Lagrangian method would not allow us to do. Even if it does not allow us to find closed-form solutions, dynamic programming sometimes allows us to find some theoretical properties of the solution. Last, dynamic programs can be (sometimes easily) solved with numerical methods.

Solution to Exercise 2.

A. The state variable are the amount of shares s_t and the dividend d_t. The control variables is consumption c_t. Since c_t and s_{t+1} are linked through the budget, we can also choose s_{t+1} as control variable. As usual, we pick s_{t+1} as control variable to simplify derivations.

B. The Bellman equation is

$$V(s,d) = \max_{s'}\left\{u\left((p+d)\cdot s - p\cdot s'\right) + \beta\cdot\mathbb{E}\left(V\left(s',d'\right) \mid d\right)\right\}$$

C. The first-order condition with respect to s' in the Bellman equation is

$$-p\cdot\frac{du}{dc}(c) + \beta\cdot\mathbb{E}\left(\frac{\partial V\left(s',d'\right)}{\partial s'} \mid d\right) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{\partial V(s,d)}{\partial s} = (p+d)\cdot\frac{du}{dc}(c).$$

Combining both equations we obtain the following Euler equation:

$$p\cdot\frac{du}{dc}(c) = \beta\cdot\mathbb{E}\left(\left(d'+p'\right)\cdot\frac{du}{dc}\left(c'\right) \mid d\right).$$

D. With $u(c) = c$, $du/dc = 1$ and the Euler equation becomes

$$p = \beta\cdot\mathbb{E}\left(\left(d'+p'\right) \mid d\right).$$

Let p_h be the price when today's dividend is high, and let p_l be the price when today's dividend is low.

$$p_h = \beta\cdot\left[\rho\cdot\left(d_h+p_h\right) + (1-\rho)\cdot\left(d_l+p_l\right)\right]$$
$$p_l = \beta\cdot\left[\rho\cdot\left(d_l+p_l\right) + (1-\rho)\cdot\left(d_h+p_h\right)\right]$$

which implies

$$p_h - p_l = \beta\cdot\frac{2\cdot\rho-1}{1-[\beta\cdot(2\cdot\rho-1)]}\cdot\left(d_h-d_l\right) > 0$$

because $0.5 < \rho < 1$. So the price is higher when the dividend is higher.

Solution to Exercise 3.

A. k is the state variable and (k', l) are the control variables.

B. The Bellman equation is

$$V(k) = \max_{k',l} \left\{ u\left[f(k,l) - k', l\right] + \beta \cdot V(k') \right\}$$

C. The first-order conditions with respect to k' and l in the Bellman equation are

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \frac{dV}{dk}(k') = 0$$
$$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation is

$$\frac{dV}{dk}(k) = \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l)$$

We combine these equations to get

$$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \frac{\partial u}{\partial c}(c', l') \cdot \frac{\partial f}{\partial k}(k', l') \tag{2}$$
$$\frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) = -\frac{\partial u}{\partial l}(c,l). \tag{3}$$

D. In steady state, we have $l = l^*$, $c = c^*$, and $k = k^*$. Using (2) and the functional form of f, we obtain

$$\alpha \cdot \beta \cdot \left(\frac{k^*}{l^*}\right)^{\alpha-1} = 1$$
$$\frac{k^*}{l^*} = (\alpha \cdot \beta)^{1/(1-\alpha)}.$$

Then use the law of motion of capital implies

$$\frac{c^*}{k^*} = \left(\frac{k^*}{l^*}\right)^{\alpha-1} - 1 = \frac{1}{\alpha \cdot \beta} - 1.$$

E. The Bellman equation is

$$V(A,k) = \max_{k',l} \left\{ u\left[A \cdot f(k,l) - k', l\right] + \beta \cdot \mathbb{E}\left(V\left(A',k'\right) \mid A\right) \right\}$$

where (A,k) are the state variables and $\left(k',l\right)$ are the control variables.

F. The first-order conditions with respect to k' and l become

$$-\frac{\partial u}{\partial c}(c,l) + \beta \cdot \mathbb{E}\left(\frac{\partial V}{\partial k'}\left(A',k'\right) \mid A\right) = 0$$
$$A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial l}(k,l) + \frac{\partial u}{\partial l}(c,l) = 0.$$

The Benveniste-Scheinkman equation becomes

$$\frac{\partial V}{\partial k}(A,k) = A \cdot \frac{\partial u}{\partial c}(c,l) \cdot \frac{\partial f}{\partial k}(k,l).$$

The Euler condition is

$$\frac{\partial u}{\partial c}(c,l) = \beta \cdot \mathbb{E}\left(A' \cdot \frac{\partial u}{\partial c}(c',l') \cdot \frac{\partial f}{\partial k}(k',l') \mid A\right).$$

Solution to Exercise 4.

A. The present-value Hamiltonian is

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot \ln(c(t)) + \lambda(t) \left[f\left(k(t)\right) - c(t) - \delta \cdot k(t) \right]$$

where $\lambda(t)$ is the co-state variable associated with the state variable $k(t)$.

B. The optimality conditions for the present-value Hamiltonian are

$$\begin{aligned} \frac{\partial \mathcal{H}(t)}{\partial c(t)} &= 0 \\ \frac{\partial \mathcal{H}(t)}{\partial k(t)} &= -\dot{\lambda}(t) \\ \lim_{t \to +\infty} \lambda(t) \cdot k(t) &= 0. \end{aligned}$$

The last condition is the transversality condition. The first two conditions imply that

$$e^{-\rho \cdot t} \cdot \frac{1}{c(t)} = \lambda(t) \tag{4}$$

$$\lambda(t) \cdot \left[f'\left(k(t)\right) - \delta \right] = -\dot{\lambda}(t). \tag{5}$$

We can eliminate $\lambda(t)$ by taking log and differentiating (4) with respect to time t. This procedure yields

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -\rho - \frac{\dot{c}(t)}{c(t)}$$

We can then substitute $\dot{\lambda}(t)/\lambda(t)$ into (5), which gives the following Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \alpha \cdot A \cdot k(t)^{\alpha - 1} - (\delta + \rho)\,.$$

C. The steady state is given by

$$\begin{aligned} k^* &= \left(\frac{\alpha \cdot A}{\delta + \rho} \right)^{1/(1-\alpha)} \\ c^* &= A^{1/(1-\alpha)} \left(\frac{\alpha}{\delta + \rho} \right)^{\alpha/(1-\alpha)} \cdot \left(\frac{\delta \cdot (1 - \alpha) + \rho}{\delta + \rho} \right). \end{aligned}$$

Solution to Exercise 5.

A. The current-value Hamiltonian is

$$\mathcal{H}^*(t) = f\,(k(t)) - i(t) - \frac{\chi}{2} \cdot \left(\frac{i(t)^2}{k(t)}\right) + q(t) \cdot i(t),$$

where $q(t)$ is the co-state variable associated with the state variable $k(t)$.

B. There are two optimality conditions for the current-value Hamiltonian. (We omit the transversality condition.) The first optimality condition is

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}^*(t)}{\partial i(t)} \\ 0 &= -1 - \chi \cdot \left[\frac{i(t)}{k(t)}\right] + q(t) \\ i(t) &= \left[\frac{q(t) - 1}{\chi}\right] \cdot k(t), \end{aligned}$$

which implies, using the law of motion of capital, that

$$\dot{k}(t) = \left[\frac{q(t) - 1}{\chi}\right] \cdot k(t).$$

The second optimality condition is

$$\begin{aligned} \frac{\partial \mathcal{H}(t)}{\partial k(t)} &= r \cdot q(t) - \dot{q}(t) \\ f'(k(t)) + \frac{\chi}{2} \cdot \left[\frac{i(t)}{k(t)}\right]^2 &= r \cdot q(t) - \dot{q}(t) \end{aligned}$$

The first optimality condition implies that $i(t)/k(t) = \dot{k}(t)/k(t) = (q(t) - 1)/\chi$. So this optimality condition becomes

$$\dot{q}(t) = r \cdot q(t) - f'(k(t)) - \frac{1}{2 \cdot \chi} \cdot (q(t) - 1)^2\,.$$

C. In steady state, $\dot{q}(t) = 0$ and $\dot{k}(t) = 0$, so $i^* = 0$. Notice that we can say that $\dot{q}(t) = 0$ only because $q(t)$ is the co-state variable used with a current-value Hamiltonian. The co-state variables used in a present-value Hamiltonian are not constant in steady state (which is a reason why we prefer to work with a current-value Hamiltonian). Since $\dot{k}(t) = 0$, the first optimality condition implies

$$q^* = 1.$$

Since $q^* = 1$ and $\dot{q}(t) = 0$, the second optimality condition implies

$$f'(k^*) = r.$$

Solution to Exercise 6.

We multiply both sides of the differential equation by the integrating factor $\mu(t) = e^{-r\cdot t}$. We obtain

$$\begin{aligned} \dot{a}(t) \cdot e^{-r\cdot t} - r \cdot a(t) \cdot e^{-r\cdot t} &= s \cdot e^{-r\cdot t} \\ \frac{d\left[a(t) \cdot e^{-r\cdot t}\right]}{dt} &= s \cdot e^{-r\cdot t} \end{aligned}$$

Integrating from time 0 to t,

$$\begin{aligned} \int_0^t d\left[a(t) \cdot e^{-r\cdot t}\right] &= \int_0^t s \cdot e^{-r\cdot t} dt \\ a(t) \cdot e^{-r\cdot t} - a(0) &= -\frac{s}{r} \cdot e^{-r\cdot t} + \frac{s}{r}. \end{aligned}$$

Therefore, as $a(0) = a_0$, the solution to the initial value problem must satisfy

$$a(t) = a_0 \cdot e^{r\cdot t} + \frac{s}{r}\left(e^{r\cdot t} - 1\right).$$

Solution to Exercise 7.

The integrating factor is now

$$\mu(t) = \exp\left(-\int_0^t r(w)dw\right).$$

Notice that the derivative of the integrating factor satisfies

$$\dot{\mu}(t) = -r(t) \cdot \mu(t)$$

(which is why we picked this specific integrating factor). We multiply both sides of the differential equation by the integrating factor. The differential equation becomes

$$\begin{aligned}
\dot{a}(t) \cdot \mu(t) - a(t) \cdot r(t) \cdot \mu(t) &= s(t) \cdot \mu(t) \\
\dot{a}(t) \cdot \mu(t) - a(t) \cdot \dot{\mu}(t) &= s(t) \cdot \mu(t) \\
\frac{d\left[a(t) \cdot \mu(t)\right]}{dt} &= s(t) \cdot \mu(t).
\end{aligned}$$

Integrating from time 0 to t,

$$\begin{aligned}
a(t) \cdot \mu(t) - a(0) \cdot \mu(0) &= \int_0^t s(z) \cdot \mu(z) dz \\
a(t) &= \frac{a_0}{\mu(t)} + \int_0^t s(z) \cdot \frac{\mu(z)}{\mu(t)} dz \\
a(t) &= a_0 \cdot \exp\left(\int_0^t r(z)dz\right) + \int_0^t s(z) \cdot \exp\left(\int_z^t r(w)dw\right) dz.
\end{aligned}$$

This equation reduces to the solution of exercise 6 when both r and s are constant.

Solution to Exercise 8.

A. We are facing a linear, two-variable, homogenous system of FODEs. To find the general solution of the system, we need the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

First, we determine the eigenvalues. The eigenvalues λ are the roots of the polynomial $\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I})$. So the eigenvalues λ solve

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

Hence, the eigenvalues λ are solutions to

$$(1-\lambda)^2 - 4 = 0$$

So there are two distinct eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = -1$.

Second, we determine the eigenvectors. The eigenvector $[\alpha, \beta]$ associated with the eigenvalue λ solves

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To determine the eigenvector associated with $\lambda_1 = 3$, we solve

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to the single equation

$$-2 \cdot \alpha + \beta = 0$$

thus $\beta = 2 \cdot \alpha$, and the eigenvector corresponding to $\lambda_1 = 3$ is

$$\boldsymbol{z}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Similarly, the eigenvector corresponding to $\lambda_2 = -1$ is

$$\boldsymbol{z}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Using the eigenvalues and eigenvectors that we have determined, we conclude that the general solution of the system is

$$\boldsymbol{x}(t) = c_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot e^{3 \cdot t} + c_2 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot e^{-t},$$

where c_1 and c_2 are arbitrary constants.

B. To determine a specific solution, we would need two boundary conditions that would allow us to determine the two constants c_1 and c_2.

C. Since the linear, two-variable, homogenous system has two eigenvalues of opposite sign, the trajectories of the system have the origin as a saddle point. See the treatment of the two-variable linear system with two eigenvalues of opposite sign in the lecture notes.

Solution to Exercise 9.

A. $f(k) = k^\alpha$ with $\alpha \in (0, 1)$ satisfies the Inada conditions.

B. Steady-state capital k^* is implicitly determined by

$$s \cdot f\left(k^*\right) = \delta \cdot k^*.$$

C. Plot k on the x-axis. Draw two curves $y = s \cdot f(k)$ and $y = \delta \cdot k$. The $y = s \cdot f(k)$ curve is the saving curve. It is increasing and concave . The $y = \delta \cdot k$ curve is the depreciation curve. It is an increasing straight line. The intersection of these two curves is the steady state. Starting from an initial k_0, $k(t)$ converge to k^*. This is because if $k(t)$ is to the left of k^*, $\dot{k} > 0$ so $k(t)$ increases to k^*; and if $k(t)$ is to the right of k^*, $\dot{k} < 0$, so $k(t)$ decreases to k^*.

Solution to Exercise 10.

A. See lecture notes.

B. The Jacobian matrix at the steady state is

$$\boldsymbol{J}^* = \begin{bmatrix} \rho & -1 \\ \alpha \cdot (\alpha - 1) \cdot A \cdot k^{\alpha-2} & 0 \end{bmatrix}$$

C. To show that the steady state is a saddle point locally, we must show that the eigenvalues of the Jacobian matrix evaluated at the steady state have opposite sign. The determinant of the Jacobian matrix is $\det(\boldsymbol{J}^*) = \alpha \cdot (\alpha - 1) \cdot A \cdot k^{\alpha-2} < 0$. As explained in the lecture notes, the two eigenvalues have opposite sign and the steady state is a saddle point locally.

D. An unanticipated decrease in ρ at time t_0 means that the $\dot{c} = 0$ locus shifts to the right at time t_0. The new steady state is (k^{**}, c^{**}) with $k^{**} > k^*$ and $c^{**} > c^*$. There is a new saddle path for the new steady state. Given that k is predetermined, it must remain at its steady-state level at t_0: $k(t_0) = k^*$. Only consumption adjusts to bring the economy on the new saddle path. Thus at time t_0, the economy jumps to a point $(k^*, c(t_0))$ on the new saddle path. Then it moves along the saddle path to converge to the new steady state.

Solution to Exercise 11.

A. We plot the phase diagram in a (k, q) plane. The $\dot{k}(t) = 0$ locus is horizontal. The $\dot{q}(t) = 0$ locus is described implicitly by

$$f''(k) \cdot \frac{\partial k}{\partial q} = r - \frac{q-1}{\chi}.$$

There is no clear sign for the slope of the $\dot{q}(t) = 0$ locus. However, if we are close to the steady state, q is close to 1. So the $\dot{q}(t) = 0$ locus must be downward sloping.

B. The two differential equations show that $k(t)$ increases if we are to the right of the $\dot{k}(t) = 0$ locus, and $q(t)$ increases if we are above the $\dot{q}(t) = 0$ locus. Again, we have a saddle point locally.

Solution to Exercise 12.

A. By definition

$$\begin{aligned} \Delta k &= f(k) - \delta \cdot k - c \\ \Delta c &= \left[\beta \cdot \left(f'(k) + 1 - \delta\right) - 1\right] \cdot c. \end{aligned}$$

Hence, the locus $\Delta k = 0$ is defined by

$$c = f(k) - \delta \cdot k,$$

and the locus $\Delta c = 0$ is defined by

$$f'(k) = \frac{1}{\beta} - 1 + \delta.$$

The intersection of these two curves is the steady state (k^*, c^*). The $\Delta k = 0$ locus is concave in the (k, c) plane while the $\Delta c = 0$ locus is a vertical line passing through k^*.

B. Follow the same procedure as that described in the lecture notes to analyze systems of nonlinear differential equations.

Solution to Exercise 13

A. $c(t)$ and $l(t)$ are the control variables. $k(t)$ and $h(t)$ are the state variables.

B. The present-value Hamiltonian is

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot \ln(c(t)) + \lambda^k(t) \cdot \left[y(t) - c(t) - \delta \cdot k(t)\right] + \lambda^h(t) B \cdot (1 - l(t)) \cdot h(t),$$

where $\lambda^h(t)$ and $\lambda^k(t)$ are the co-state variables associated with the law of motion of human capital $h(t)$ and physical capital $k(t)$.

C. The optimality conditions are

$$\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0$$
$$\frac{\partial \mathcal{H}(t)}{\partial l(t)} = 0$$
$$\frac{\partial \mathcal{H}(t)}{\partial k(t)} = -\dot{\lambda}^k(t)$$
$$\frac{\partial \mathcal{H}(t) \nu}{\partial h(t)} = -\dot{\lambda}^h(t).$$

These conditions simplify to

$$e^{-\rho \cdot t} \cdot \frac{1}{c(t)} = \lambda^k(t) \tag{6}$$

$$\lambda^k(t) \cdot \beta \cdot \frac{y(t)}{l(t)} = \lambda^h(t) \cdot B \cdot h(t) \tag{7}$$

$$\lambda^k(t) \cdot \left[\alpha \cdot \frac{y(t)}{k(t)} - \delta\right] = -\dot{\lambda}^k(t) \tag{8}$$

$$\lambda^k(t) \cdot \beta \cdot \frac{y(t)}{h(t)} + \lambda^h(t) \cdot B \cdot \left[1 - l(t)\right] = -\dot{\lambda}^h(t). \tag{9}$$

D. The growth rate of $c(t)$ follows from the combination of equations (6) and (8).

E. The equality of equation (7) holds for interior solution only, i.e. $0 < l < 1$. When $B = 0$, the optimal solution is $l = 1$.

F. The dynamic equations of the equilibrium are:

$$\dot{k} = A \cdot k^{\alpha} \cdot h_0^{\beta} - c - \delta \cdot k$$
$$\frac{\dot{c}}{c} = \alpha \cdot A \cdot k^{\alpha-1} \cdot h_0^{\beta} - (\delta + \rho)$$
$$\dot{h} = 0$$

Since h_0 is simply a constant, this system has a steady state (k^*, c^*) where $\dot{k} = \dot{c} = 0$. The steady state satisfies

$$\alpha \cdot A \cdot \left(k^*\right)^{\alpha-1} \cdot h_0^{\beta} = \delta + \rho.$$

To draw the phase diagram from here, see lecture notes.

G. To show that the steady state is a saddle point graphically, see lecture notes.

H. The Jacobian is given by

$$\boldsymbol{J}^* = \begin{bmatrix} \left.\frac{\partial \dot{k}}{\partial k}\right|_{(k^*,c^*)} & \left.\frac{\partial \dot{k}}{\partial c}\right|_{(k^*,c^*)} \\ \left.\frac{\partial \dot{c}}{\partial k}\right|_{(k^*,c^*)} & \left.\frac{\partial \dot{c}}{\partial c}\right|_{(k^*,c^*)} \end{bmatrix} = \begin{bmatrix} \rho & -1 \\ (\alpha - 1)\,\alpha \cdot A \cdot (k^*)^{\alpha-2}\, h_0^{\beta} & 0 \end{bmatrix}$$

I. It follows that the steady state is a saddle point locally because the determinant of the Jacobian matrix is negative:

$$\det\left(\boldsymbol{J}^*\right) = (\alpha - 1) \cdot \alpha \cdot A \left(k^*\right)^{\alpha-2} \cdot h_0^{\beta} < 0,$$

which implies that the two eigenvalues of the system have opposite sign.