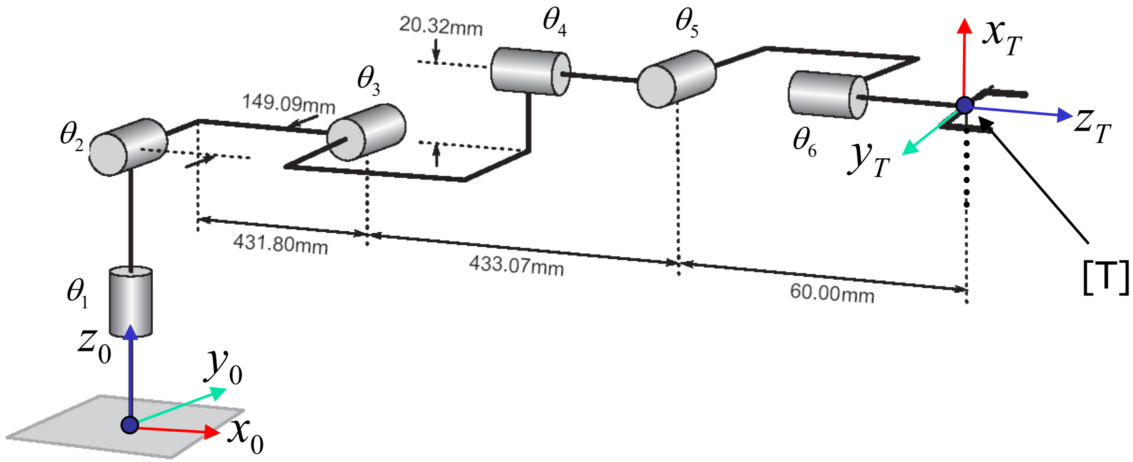
Forward Kinematics of PUMA560 Robot



- D-H Table of PUMA 560

i	$\alpha_{i-1,i}$	$a_{i-1,i}$	d_i	θ_i
1	–	–	0	θ_1
2	$-90°$	0	0	θ_2
3	0	0.43180	0.14909	θ_3
4	$-90°$	0.02032	0.43307	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

$$[G]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0.5\\0&0&0&1\end{bmatrix},\quad [H]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0.06\\0&0&0&1\end{bmatrix}$$

- Compute individual link and joint transformations

$$[Z(d_i,\theta_i)]\text{ and }[X(a_{i-1,i},\alpha_{i-1,i})]$$

- Compute the end-effector frame [T] relative to the world frame using matrix multiplications

$$[T]=[G][Z(d_1,\theta_1)][X(a_{12},\alpha_{12})][Z(d_2,\theta_2)][X(a_{23},\alpha_{23})][Z(d_3,\theta_3)][X(a_{34},\alpha_{34})]$$
$$\boxed{[Z(d_4,\theta_4)]}[X(a_{45},\alpha_{45})][Z(d_5,\theta_5)][X(a_{56},\alpha_{56})][Z(d_6,\theta_6)][H]$$

$$[T]=[G][Z(d_1,\theta_1)][X(a_{12},\alpha_{12})][Z(d_2,\theta_2)][X(a_{23},\alpha_{23})][Z(d_3,\theta_3)][X(a_{34},\alpha_{34})]\boxed{[Z(d_4,0)]}$$
$$\underbrace{\boxed{[Z(0,\theta_4)]}[X(a_{45},\alpha_{45})][Z(d_5,\theta_5)][X(a_{56},\alpha_{56})][Z(d_6,\theta_6)][H]}_{[W]\text{(Spherical Wrist Transformation)}}$$

- Geometric Meaning

$$[T'] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} r_{11}(\theta_1,\cdots,\theta_6) & r_{12}(\theta_1,\cdots,\theta_6) & r_{13}(\theta_1,\cdots,\theta_6) & p_x(\theta_1,\theta_2,\theta_3) \\ r_{21}(\theta_1,\cdots,\theta_6) & r_{22}(\theta_1,\cdots,\theta_6) & r_{23}(\theta_1,\cdots,\theta_6) & p_y(\theta_1,\theta_2,\theta_3) \\ r_{31}(\theta_1,\cdots,\theta_6) & r_{32}(\theta_1,\cdots,\theta_6) & r_{33}(\theta_1,\cdots,\theta_6) & p_z(\theta_1,\theta_2,\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} p_x = c_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) - d_3 s_1 \\ p_y = s_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) + d_3 c_1, \\ p_z = -a_{23}s_2 - a_{34}s_{23} - d_4 c_{23} \end{cases}$$

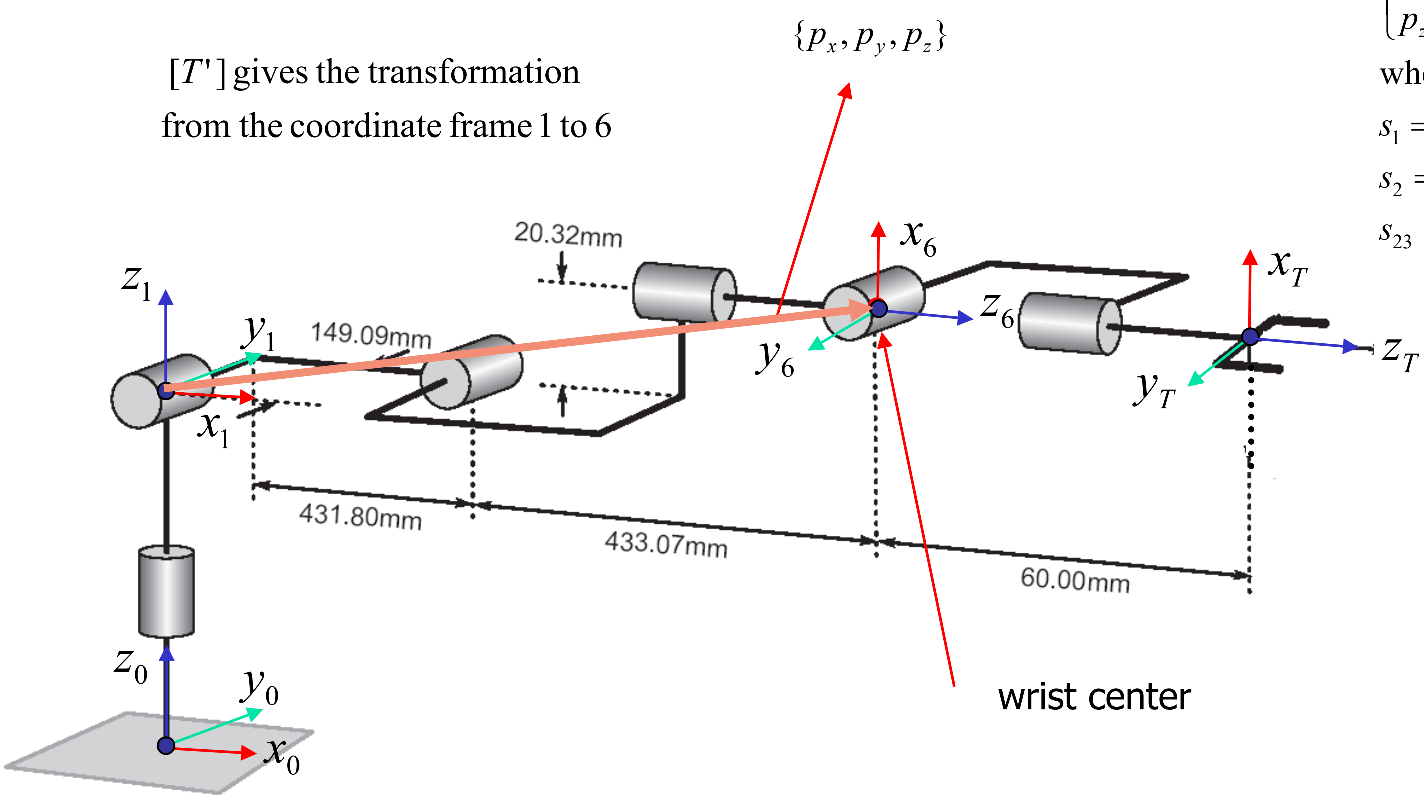
where

$s_1 = \sin(\theta_1), c_1 = \cos(\theta_1),$

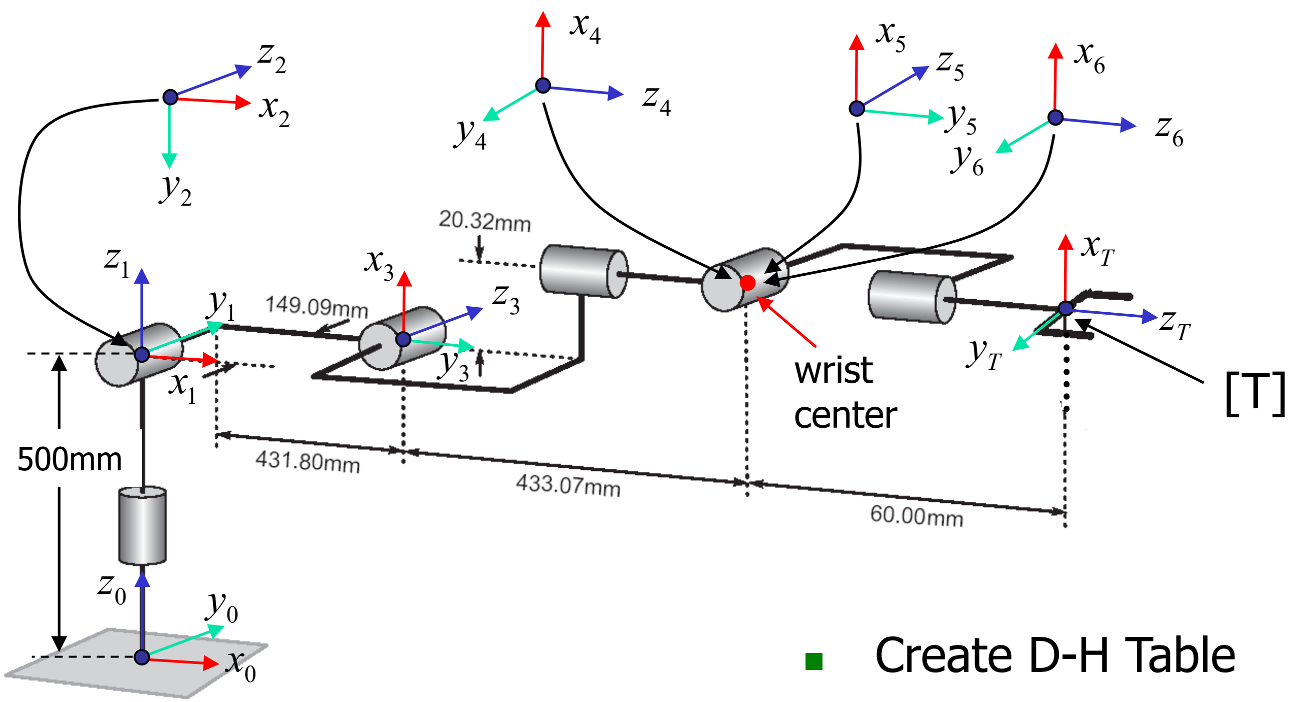
$s_2 = \sin(\theta_2), c_2 = \cos(\theta_2),$

$s_{23} = \sin(\theta_2 + \theta_3), c_{23} = \cos(\theta_2 + \theta_3)$

$[T']$ gives the transformation from the coordinate frame 1 to 6



D-H Table of PUMA560 Robot



- [G]: frame 0 to frame 1
- [H]: frame 6 to frame [T]

- Create D-H Table unit: meter

i	$\alpha_{i-1,i}$	$a_{i-1,i}$	d_i	θ_i
1	–	–	0	θ_1
2	$-90°$	0	0	θ_2
3	0	0.43180	0.14909	θ_3
4	$-90°$	0.02032	0.43307	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

spherical wrist center

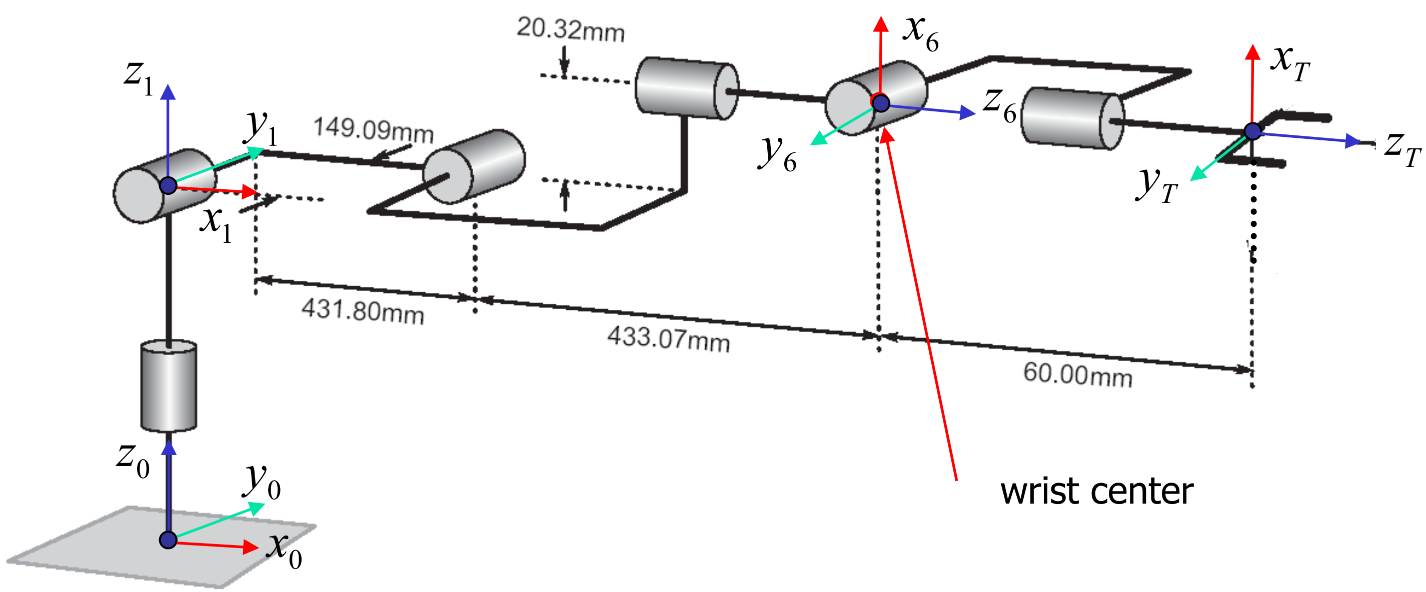
Numerical Example

- Ignore [G] and [H]
- Forward Kinematics

$$[T] =$$

- D-H Table of PUMA 560

i	$\alpha_{i-1\,i}$	$a_{i-1\,i}$	d_i	θ_i
1	–	–	0	0
2	$-90°$	0	0	0
3	0	0.43180	0.14909	-90
4	$-90°$	0.02032	0.43307	0
5	$90°$	0	0	0
6	$-90°$	0	0	0



- Transform the equation

$$[T] = \overbrace{[G][Z(d_1,\theta_1)][X(a_{12},\alpha_{12})][Z(d_2,\theta_2)][X(a_{23},\alpha_{23})][Z(d_3,\theta_3)][X(a_{34},\alpha_{34})][Z(d_4,0)]}^{[R(\theta_1,\theta_2,\theta_3)]}$$
$$\underbrace{[Z(0,\theta_4)][X(a_{45},\alpha_{45})][Z(d_5,\theta_5)][X(a_{56},\alpha_{56})][Z(d_6,\theta_6)]}_{[W(\theta_4,\theta_5,\theta_6)]}[H]$$

$$[T] = [G][R(\theta_1,\theta_2,\theta_3)][W(\theta_4,\theta_5,\theta_6)][H]$$

i	$\alpha_{i-1,i}$	$a_{i-1,i}$	d_i	θ_i
1	–	–	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_{23}	d_3	θ_3
4	$-90°$	a_{34}	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

$a_{23} = 0.43180$
$a_{34} = 0.02032$
$d_3 = 0.14909$
$d_4 = 0.43307$
$b = 0.5$
$l = 0.06$

$$\text{Define}\,[T'] = [G]^{-1}[T][H]^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[G] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [H] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}[T'] &= [R(\theta_1,\theta_2,\theta_3)][W(\theta_4,\theta_5,\theta_6)] \\ &= [Z(0,\theta_1)][X(0,-90°)][Z(0,\theta_2)][X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)][W] \\ &= \begin{bmatrix} f_{11}(\theta_1,\cdots,\theta_6) & f_{12}(\theta_1,\cdots,\theta_6) & f_{13}(\theta_1,\cdots,\theta_6) & p_x(\theta_1,\theta_2,\theta_3) \\ f_{21}(\theta_1,\cdots,\theta_6) & f_{22}(\theta_1,\cdots,\theta_6) & f_{23}(\theta_1,\cdots,\theta_6) & p_y(\theta_1,\theta_2,\theta_3) \\ f_{31}(\theta_1,\cdots,\theta_6) & f_{32}(\theta_1,\cdots,\theta_6) & f_{33}(\theta_1,\cdots,\theta_6) & p_z(\theta_1,\theta_2,\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Inverse Kinematics of PUMA 560 (Position Part)

$$\begin{cases} p_x = c_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) - d_3 s_1 \\ p_y = s_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) + d_3 c_1, \\ p_z = -a_{23}s_2 - a_{34}s_{23} - d_4 c_{23} \end{cases} \longrightarrow \text{Goal: find } \theta_1, \theta_2, \theta_3$$

- Numerical Iteration Method
  - Mathematica solver: NSolve[]
  - Matlab command: fsolve(f,x,x0)
  - Can only find one solution
  - Need an initial guess to the solution
- Symbolic Solver
  - Mathematica solver: Solve[]
  - Matlab command: solve(f,x)
- Analytical Method (closed form solution)

Inverse Kinematics of PUMA 560 (Analytical Method 1)

$$[T'] = [Z(0,\theta_1)][X(0,-90°)][Z(0,\theta_2)][X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)][W]$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & c_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) - d_3 s_1 \\ * & * & * & s_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) + d_3 c_1 \\ * & * & * & -a_{23}s_2 - a_{34}s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[Z(0,-\theta_1)][T'] = [X(0,-90°)][Z(0,\theta_2)][X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)][W]$$

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * & c_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) - d_3 s_1 \\ * & * & * & s_1(a_{23}c_2 + a_{34}c_{23} - d_4 s_{23}) + d_3 c_1 \\ * & * & * & -a_{23}s_2 - a_{34}s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & * & * & c_1 p_x + s_1 p_y \\ * & * & * & \boxed{-s_1 p_x + c_1 p_y} \\ * & * & * & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & a_{23}c_2 + a_{34}c_{23} - d_4 s_{23} \\ * & * & * & \boxed{d_3} \\ * & * & * & -a_{23}s_2 - a_{34}s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \theta_1 \text{ (2 solutions)}$$

square and add the three equations

$$p_x^2 + p_y^2 + p_z^2 = a_{23}^2 + a_{34}^2 + d_3^2 + d_4^2 + 2a_{23}(a_{34}c_3 - d_4 s_3) \Rightarrow \theta_3 \text{ (2 solutions for each } \theta_1)$$

$$A\cos\psi + B\sin\psi = C$$

$$\psi^{\pm} = \delta \pm \arccos\left(\frac{C}{\sqrt{A^2 + B^2}}\right)$$

where $\delta = \mathrm{ArcTan}[A, B]$

Inverse Kinematics of PUMA 560 (Analytical Method 2)

$$\begin{bmatrix} * & * & * & c_1 p_x + s_1 p_y \\ * & * & * & -s_1 p_x + c_1 p_y \\ * & * & * & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & a_{23} c_2 + a_{34} c_{23} - d_4 s_{23} \\ * & * & * & d_3 \\ * & * & * & -a_{23} s_2 - a_{34} s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[Z(0,-\theta_2)][X(0,+90°)][Z(0,-\theta_1)][T'] = [X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)][W]$$

$$\begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ -s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * & c_1 p_x + s_1 p_y \\ * & * & * & -s_1 p_x + c_1 p_y \\ * & * & * & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ -s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * & a_{23} c_2 + a_{34} c_{23} - d_4 s_{23} \\ * & * & * & d_3 \\ * & * & * & -a_{23} s_2 - a_{34} s_{23} - d_4 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use the two identities:

$$c_2 c_{23} + s_2 s_{23} = c_3$$
$$c_2 s_{23} - s_2 c_{23} = s_3$$

$$\begin{bmatrix} * & * & * & \boxed{\begin{matrix} c_2(c_1 p_x + s_1 p_y) - s_2 p_z \\ -s_2(c_1 p_x + s_1 p_y) - c_2 p_z \end{matrix}} \\ * & * & * & -s_1 p_x + c_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & \boxed{\begin{matrix} a_{23} + a_{34} c_3 - d_4 s_3 \\ a_{34} s_3 + d_4 c_3 \end{matrix}} \\ * & * & * & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ θ_2 (1 solutions)

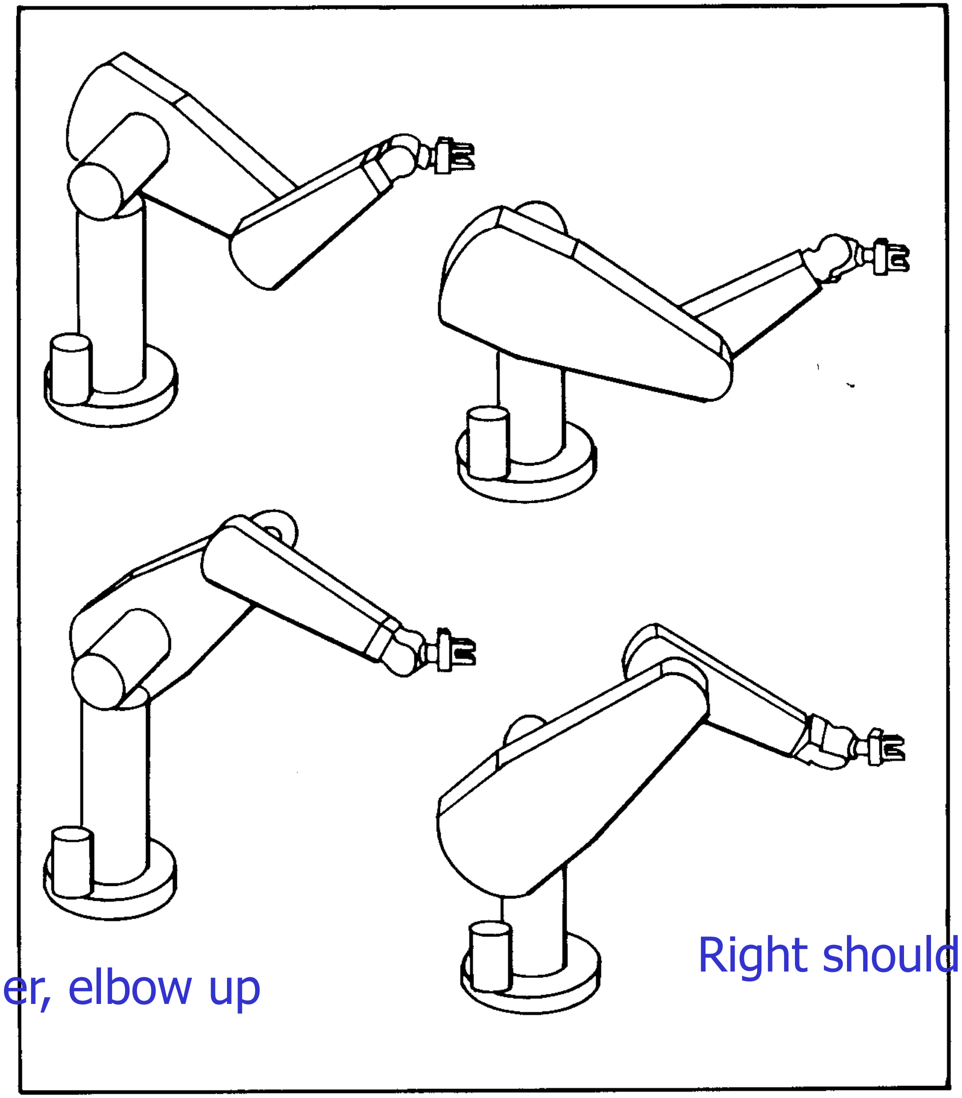
Substitute θ_1, and θ_3

Conclusion: there are 2*2=4 set of solutions (θ_1, θ_2, θ_3)
for the inverse kinematics of PUMA 560 (position part)

Four Solutions of the Position Part of PUMA 560

Left shoulder, elbow down

Right shoulder, elbow down



Left shoulder, elbow up

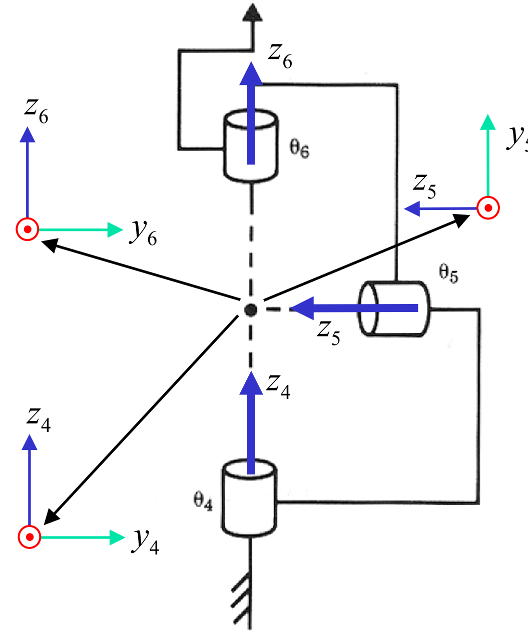
Right shoulder, elbow up

FIGURE 4.4 Four solutions of the PUMA 560.

Forward Kinematics of Spherical Wrist

- D-H Table

i	$\alpha_{i-1,i}$	$a_{i-1,i}$	d_i	θ_i
4	–	–	–	θ_1
5	$90°$	0	0	θ_2
6	$-90°$	0	0	θ_3



⊗ : pointing inside
⊙ : pointing outside

- Compute the end-effector frame [W] relative to the world frame using matrix multiplications

$$[W] = [Z(0,\theta_4)][X(0,90°)][Z(0,\theta_5)][X(0,-90°)][Z(0,\theta_6)] =$$

$$\begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6 - \sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6 + \cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics of Spherical Wrist

- Objective: given the position and orientation of the end-effector [W], find joint angles θ_4,θ_5 ,θ_6.

$$[W]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6-\sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6+\cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6+\cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Step 1: Solve θ_5 from a_{33} to obtain two solutions $\Longrightarrow$ $\theta_5 = \pm \mathrm{ArcCos}[a_{33}]$
- Step 2: Back substitute each solution of of θ_5 to solve θ_4 from a_{13} and a_{23}

$$\theta_4 = \mathrm{ArcTan}[\frac{a_{13}}{-\sin\theta_5}, \frac{a_{23}}{-\sin\theta_5}]$$

In Mathematica, use ArcTan[x,y] to obtain the $\boxed{\text{unique}}$ solution of θ_4 for each θ_5 .
In Matlab, use atan2[y,x] instead

- Step 3: Back substitute each solution of θ_5 to solve θ_6 from a_{31} and a_{32}

$$\theta_6 = \mathrm{ArcTan}[\frac{a_{31}}{\sin\theta_5}, \frac{a_{32}}{-\sin\theta_5}]$$

Conclusion: there are two set of solutions (θ_4, θ_5, θ_6) for the inverse kinematics of spherical wrist.

Singular Cases

- What if $\sin\theta_5 = 0, \text{that is } \theta_5 = 0, \pi \quad \cos\theta_5 = \pm 1$

If $\theta_5 = 0, \cos\theta_5 = 1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_4\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\sin\theta_6 - \sin\theta_4\cos\theta_6 & 0 & 0 \\ \sin\theta_4\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\sin\theta_6 + \cos\theta_4\cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_4+\theta_6) & -\sin(\theta_4+\theta_6) & 0 & 0 \\ \sin(\theta_4+\theta_6) & \cos(\theta_4+\theta_6) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\theta_5 = 0, \cos\theta_5 = 1$

$$\theta_4 + \theta_6 = \mathrm{ArcTan}[a_{11}, a_{21}]$$

If $\theta_5 = \pi, \cos\theta_5 = -1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta_4\cos\theta_6 - \sin\theta_4\sin\theta_6 & +\cos\theta_4\sin\theta_6 - \sin\theta_4\cos\theta_6 & 0 & 0 \\ -\sin\theta_4\cos\theta_6 + \cos\theta_4\sin\theta_6 & +\sin\theta_4\sin\theta_6 + \cos\theta_4\cos\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos(\theta_4-\theta_6) & -\sin(\theta_4-\theta_6) & 0 & 0 \\ -\sin(\theta_4-\theta_6) & \cos(\theta_4-\theta_6) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 - \theta_6 = \mathrm{ArcTan}[-a_{21}, -a_{11}]$$

  - Solutions are not isolated but in higher dimension
  - Robots at singular configurations lose at least one degree of freedom

Numerical Example for Kinematics of Spherical Wrist

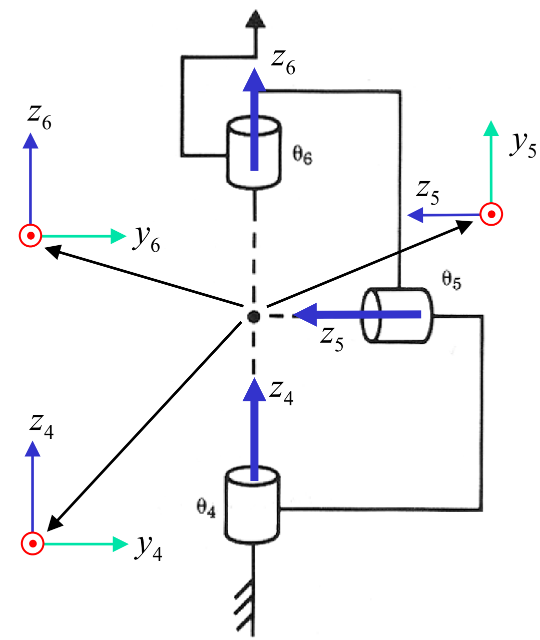
- Forward Kinematics

$$[W]=[Z(0,\theta_4)][X(0,90°)][Z(0,\theta_5)][X(0,-90°)][Z(0,\theta_6)]=$$

$$\begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6-\sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6+\cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6+\cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

D-H Table

i	$\alpha_{i-1,i}$	$a_{i-1,i}$	d_i	θ_i
4	–	–	–	30°
5	90°	0	0	–45°
6	–90°	0	0	90°



⊗ : pointing inside
⊙ : pointing outside

- Inverse Kinematics (Mathematica)

$$\theta_5=\pm\text{AcrCos}[a_{33}]$$

$$\theta_4=\text{ArcTan}\left[\frac{a_{13}}{-\sin\theta_5},\frac{a_{23}}{-\sin\theta_5}\right]$$

$$\theta_6=\text{ArcTan}\left[\frac{a_{31}}{+\sin\theta_5},\frac{a_{32}}{-\sin\theta_5}\right]$$

Inverse Kinematics of PUMA 560 (Wrist Part)

$$[T'] = [R(\theta_1,\theta_2,\theta_3)][W(\theta_4,\theta_5,\theta_6)]$$
$$[R(\theta_1,\theta_2,\theta_3)]^{-1}[T'] = [W(\theta_4,\theta_5,\theta_6)]$$

Inverse Kinematics of Spherical Wrist

$$[T'] = [R(\theta_1,\theta_2,\theta_3)][W(\theta_4,\theta_5,\theta_6)]$$
$$[Z(0,-\theta_2)][X(0,+90°)][Z(0,-\theta_1)][T'] = [X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)][W(\theta_4,\theta_5,\theta_6)]$$
$$[Z(-d_4,0)][X(-a_{34},+90°)][Z(-d_3,-\theta_3)][X(-a_{23},0)][Z(0,-\theta_2)][X(0,+90°)][Z(0,-\theta_1)][T'] = [W(\theta_4,\theta_5,\theta_6)]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6 - \sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6 + \cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_5 = \pm \mathrm{AcrCos}\left[a_{33}\right]$$

$$\theta_4 = \mathrm{ArcTan}\left[\frac{a_{13}}{-\sin\theta_5}, \frac{a_{23}}{-\sin\theta_5}\right]$$

$$\theta_6 = \mathrm{ArcTan}\left[\frac{a_{31}}{+\sin\theta_5}, \frac{a_{32}}{-\sin\theta_5}\right]$$

There are 2*2=4 set of solutions (θ_1, θ_2, θ_3) for the inverse kinematics of PUMA 560 (wrist position part)

For each set of solutions (θ_1, θ_2, θ_3), we can obtain two solutions of (θ_4, θ_5, θ_6)

Conclusion: there are 2*4=8 set of solutions (θ_1, θ_2, θ_3 θ_4, θ_5, θ_6) for the inverse kinematics of PUMA 560.

Summary of IK of PUMA 560 (the first three joints)

1. Goal: given the target pose matrix [T],
2. Compute $$[T'] = [G]^{-1}[T][H]^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
3. Solving the following equation for $\theta_1^{\pm}$ by using the solution to the equation $A cos\varphi + B sin\varphi = C$ (see the bottom)
$$-s_1 p_x + c_1 p_y = d_3$$
4. Solve the following equation for $\theta_3^{\pm}$ by using the solution to $A cos\varphi + B sin\varphi = C$
$$p_x^2 + p_y^2 + p_z^2 = a_{23}^2 + a_{34}^2 + d_3^2 + d_4^2 + 2a_{23}(a_{34}c_3 - d_4 s_3)$$
5. There are 4 combinations of $\theta_1^{\pm}$ and $\theta_3^{\pm}$: (θ_1^+, θ_3^+), (θ_1^+, θ_3^-), (θ_1^-, θ_3^+), (θ_1^-, θ_3^-)
6. Solve the following two linear equations for s_2 and c_2 (consider them as independent unknowns
$$\boxed{\begin{gathered} c_2(c_1 p_x + s_1 p_y) - s_2 p_z = a_{23} + a_{34}c_3 - d_4 s_3 \\ -s_2(c_1 p_x + s_1 p_y) - c_2 p_z = a_{34}s_3 + d_4 c_3 \end{gathered}}$$
7. Substitute each of the 4 solutions θ_1 , θ_3 into $\theta_2 = atan2(s_2, c_2)$
8. There are 4 sets of solutions $(\theta_1^+, \theta_{11}, \theta_3^+)$, $(\theta_1^+, \theta_{12}, \theta_3^-)$, $(\theta_1^-, \theta_{13}, \theta_3^+)$, $(\theta_1^-, \theta_{14}, \theta_3^-)$

The solutions to equation $A cos\varphi + B sin\varphi = C$ are
$$\varphi^{\pm} = atan2(B, A) \pm \operatorname{acos}\left(\frac{C}{\sqrt{A^2+B^2}}\right)$$

Summary of IK of PUMA 560 (the last three joints)

1. Substitute each of 4 solutions of $(\theta_1\ ,\theta_2\ ,\ \theta_3\ )$ to compute the reach matrix [R]= $[Z(0,\theta_1)][X(0,-90°)][Z(0,\theta_2)][X(a_{23},0)][Z(d_3,\theta_3)][X(a_{34},-90°)][Z(d_4,0)]$

2. Compute the numerical form of the wrist matrix [W]=[R]$^{-1}$[T′]= $\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. Calculate two solutions of $\theta_5^{\pm} = \pm\mathrm{acos}(a_{33})$

4. Each solution of $\theta_5^{\pm}$, calculate $\theta_4^{\pm} = atan2\left(\frac{a_{23}}{-sin\theta_5}, \frac{a_{13}}{-sin\theta_5}\right)$

5. Each solution of $\theta_5^{\pm}$, calculate $\theta_6^{\pm} = atan2\left(\frac{a_{32}}{-sin\theta_5}, \frac{a_{31}}{sin\theta_5}\right)$

6. There should be total of 8 sets of solutions.

Simulation of PUMA 560 Robot in RoboDK

RoboDK DH Parameters

	Joint Min (deg)	Joint Max (deg)	α (deg)	a (mm)	θ (deg)	d (mm)
J1	-160.000	160.000	0.000	0.000	0.000	0.000
J2	-180.000	70.000	-90.000	0.000	0.000	139.700
J3	-45.000	220.000	0.000	431.800	-180.000	0.000
J4	-150.000	150.000	-90.000	0.000	0.000	433.700
J5	-100.000	100.000	90.000	0.000	0.000	0.000
J6	-266.000	266.000	-90.000	0.000	180.000	55.880