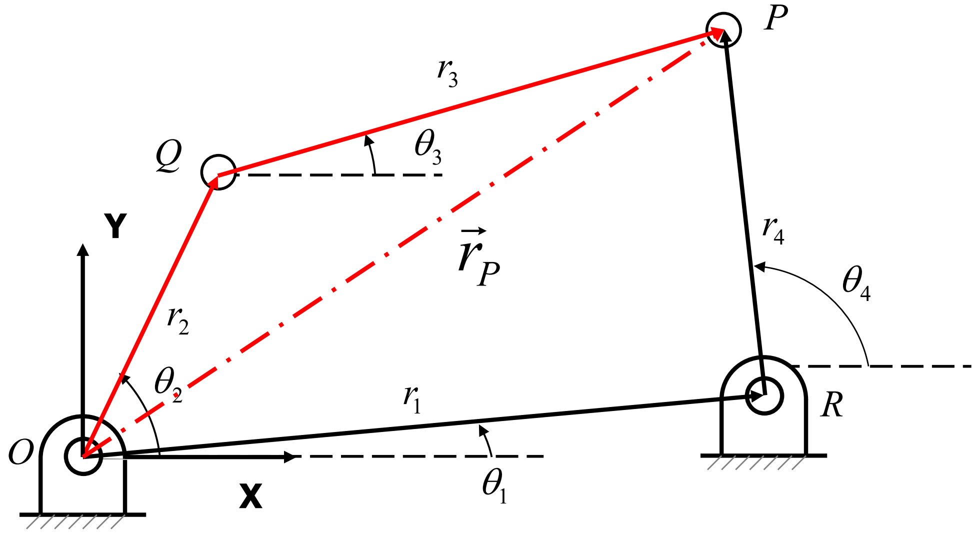
§ Section 5.3 Analytical Analysis of 4-Bar Linkages

A general 4-bar linkage is shown below.



Review on Four-Bar Definitions

- Two turning links (crank/rocker) joining to the ground
- Coupler link – Any link that has no joints with the ground link
- Coupler point – Any point on a coupler
    - Coupler radius: distance from the coupler point to the crank point
    - Coupler angle: angle between coupler radius and the crank
- Input link also called driver link is the turning link whose motion including position/velocity/acceleration of input link is given.
- Output link called driven link is the other turning link whose motion is determined by the motion of the input link
- Grashof conditions: s+l<p+q
    - Yes: Grashof type 1 linkage. Subtypes include
        - Crank-rocker: shortest link is a turning link
        - Double rocker: shortest link is the coupler link
        - Double crank: shortest link is the ground link
    - No: Grashof type 2 linkage: Always double rocker.
- Open chain vs. closed chain
- RR-Dyad: two links joined sequentially to the ground by two pin-joints

The goal of analysis:

- Known: all link dimensions, i.e. r_1, r_2, r_3, r_4 (the lengths of four links), θ_1 the angle of the frame link
- **Driver parameter:** θ_2
- Find driven (unknown) parameter: θ_3, θ_4

Derivation steps:

1. Use vectors $\vec{r}_1 = \vec{r}_{OR}, \vec{r}_2 = \vec{r}_{OQ}, \vec{r}_3 = \vec{r}_{QP}, \vec{r}_4 = \vec{r}_{RP}$ to represents the vectors of four links.

2. The position vector of the pin-joint P can be written in two ways

   $$\vec{r}_P = \vec{r}_2 + \vec{r}_3 = \vec{r}_1 + \vec{r}_4$$

3. This is called "loop closure equation" which can be written in x ($\hat{i}$) and y($\hat{j}$) components as

   $$\hat{i}: r_2 \cos\theta_2 + r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 \qquad (5.26)$$
   $$\hat{j}: r_2 \sin\theta_2 + r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 \qquad (5.27)$$

4. Identify known: $r_1, r_2, r_3, r_4, \theta_1$ and driver parameter θ_2. Unknowns: θ_3, θ_4.

5. Use the following steps to solve equations (5.26) and (5.27) for θ_3, θ_4. Rearrange eqs. (5.26) and (5.27)to obtain

   $$\hat{i}: r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2 \qquad (5.28)$$
   $$\hat{j}: r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_2 \sin\theta_2 \qquad (5.29)$$

   Rearranging strategy: move the unknown to be eliminated to one hand side.

6. Square both sides of eqs. (1) and (2), add and simplify the result using trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. Or requiring distance $|QP| = r_3$ gives

   $$r_3^2 = r_1^2 + r_2^2 + r_4^2 + 2r_1r_4(\cos\theta_1 \cos\theta_4 + \sin\theta_1 \sin\theta_4)$$
   $$-2r_1r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) - 2r_2r_4(\cos\theta_2 \cos\theta_4 + \sin\theta_2 \sin\theta_4) \qquad (5.30)$$

7. Rearrange the above equation into

$$A\cos\theta_4 + B\sin\theta_4 + C = 0 \qquad (5.31)$$

where

$$A = 2r_1r_4\cos\theta_1 - 2r_2r_4\cos\theta_2$$

$$B = 2r_1r_4\sin\theta_1 - 2r_2r_4\sin\theta_2$$

$$\begin{aligned} C &= r_1^2 + r_2^2 + r_4^2 - r_3^2 - 2r_1r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \\ &= r_1^2 + r_2^2 + r_4^2 - r_3^2 - 2r_1r_2\cos(\theta_2 - \theta_1) \end{aligned}$$

8. Calculate numerical values of A, B and C by substituting known parameters.

9. To obtain solutions of θ_4 from (3), we use the standard trigonometric identities for half angles

$$\sin\theta_4 = \frac{2\tan\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} = \frac{2t}{1+t^2} \qquad \cos\theta_4 = \frac{1-\tan^2\left(\frac{\theta_4}{2}\right)}{1+\tan^2\left(\frac{\theta_4}{2}\right)} = \frac{1-t^2}{1+t^2} \qquad \text{where } t = \tan\left(\frac{\theta_4}{2}\right)$$

10. Substitute the above formula into (5.31) to obtain

$$(C-A)t^2 + 2Bt + (A+C) = 0$$

11. If $(C-A) \neq 0$ Solving the above quadratic equation for t gives

$$t_{1,2} = \frac{-2B \pm \sqrt{4B^2 - 4(C-A)(C+A)}}{2(C-A)} = \frac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C-A}$$

where $\sigma = \pm 1$ is a sign varaible identifying the assembly mode

For each solution of t, calculate angle θ_4 from t as

$$\theta_4 = 2\tan^{-1} t, \qquad -\pi < \theta_4 < \pi$$

12. If $(C-A) = 0$, the quadratic equation degenerates into a linear equation $2Bt + (A+C) = 0$, which we can solve for $t_1 = -(A+C)/(2B)$.

The other solution will be $\theta_4 = \pi$. The derivation is as the following.
The equation (5.31) becomes

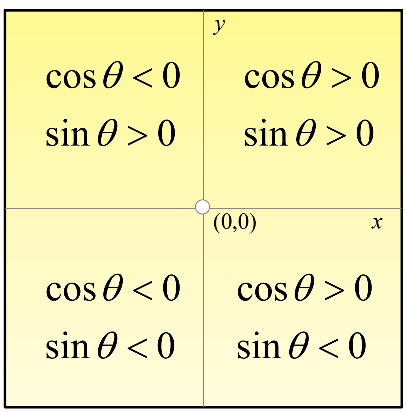
$$A(\cos\theta_4 + 1) + B\sin\theta_4 = 0\,,$$

which is satisfied for any value of A and B if $\cos\theta_4 = -1$ and $\sin\theta_4 = 0$. This gives $\theta_4 = \pi$

13. Now back substitute θ_4 into (1) and (2) to solve for θ_3.

$$\theta_3 = \tan^{-1}\left[\frac{\sin\theta_3}{\cos\theta_3}\right] = \tan^{-1}\left[\frac{r_3\sin\theta_3}{r_3\cos\theta_3}\right] = \tan^{-1}\left[\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}\right]$$

Use the sign of $\sin\theta_3, \cos\theta_3$ to determine the quadrant in which the angle θ_3 lies



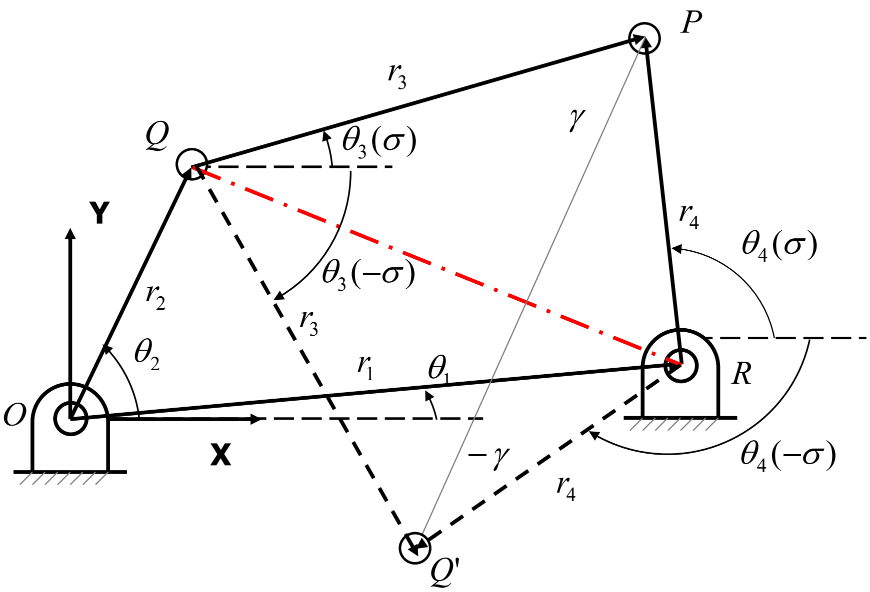
Or use Matlab function

$$\begin{aligned}\theta_3 &= ATAN2(\sin\theta_3, \cos\theta_3)\\ &= ATAN2(r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2, r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2)\end{aligned}$$

14. Calculate the position of points P and Q as

$$\vec{r}_Q = \vec{r}_2 = r_2\left(\cos\theta_2\hat{i} + \sin\theta_2\hat{j}\right)$$
$$\vec{r}_P = \vec{r}_1 + \vec{r}_4 = r_1\left(\cos\theta_1\hat{i} + \sin\theta_1\hat{j}\right) + r_4\left(\cos\theta_4\hat{i} + \sin\theta_4\hat{j}\right)$$

- Geometric interpretation of two analytical solutions:
  The two solutions represent two assembled modes or branches. P and P' are symmetric about the line QR.



Could also use the sign of transmission angle $\gamma = \theta_4 - \theta_3$ for distinguish two assembly modes.

- If the variable t is complex, i.e.

  $t = \dfrac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C - A}$ is complex when $B^2 - C^2 + A^2 < 0$

  Mechanism can't be assembled in the position specified, i.e. the given driver angle θ_2
  The linkage has to be dissembled at this position. This also represents the transition between assembly modes.

  If $B^2 - C^2 + A^2 \geq 0$ for all values of θ_2 , θ_2 is a crank. Otherwise it is a rocker.

- If the coupler link is the driver θ_3
    - **Driver parameter:** θ_3
    - Find driven (unknown) parameter: θ_2, θ_4

Rearrange (5.26) and (5.27),

$$\hat{i} : r_2 \cos\theta_2 + r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 \qquad (5.26)$$
$$\hat{j} : r_2 \sin\theta_2 + r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 \qquad (5.27)$$

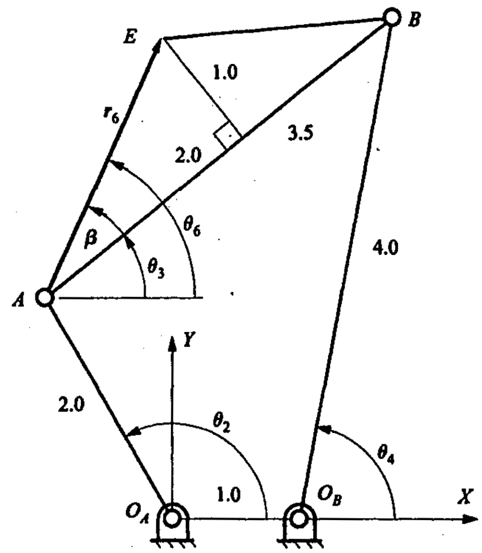
to obtain

$$\hat{i} : r_2 \cos\theta_2 = r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_3 \cos\theta_3 \qquad (5.42)$$
$$\hat{j} : r_2 \sin\theta_2 = r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_3 \sin\theta_3 \qquad (5.43)$$

Identify known: $r_1, r_2, r_3, r_4, \theta_1$ and driver parameter θ_3. Unknowns: θ_2, θ_4.
Square both sides of eqs. (5.42) and (5.43), add and simplify the result using trigonometric identity $\cos^2\theta + \sin^2\theta = 1$.

Follow similar steps (5-13) to solve equations (5.42) and (5.43) for θ_2, θ_4.

Example 5.1: For a given 4-bar linkage, r_1=1", r_2=2", r_3=3.5", r_4=4" and θ_1=0°.



- What is the linkage type (Grashof Type 1 or 2)? What is the linkage subtype? (drag link, double rocker, etc.)
- When the driver angle θ_2= 0°, compute the values of θ_4, θ_3 and coordinates of point A and B.
- Roughly draw these two configurations. (Build a cardboard model)
- What is the assembly mode σ gives the solution show in the Figure above?
- Full positional analysis using Matlab.

§ Section 5.3.3 Velocity Equations for 4-bar Linkages

For velocity analysis, the position of the linkage must be known. So calculate the position of the linkage first before the velocity analysis.

1. The vector loop equation

   $$\hat{i} : r_2 \cos\theta_2 + r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 \qquad (5.26)$$

   $$\hat{j} : r_2 \sin\theta_2 + r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 \qquad (5.27)$$

2. Take derivative of the above equations w.r.t. time and obtain two linear equations:

   $$\hat{i} : r_2\dot{\theta}_2 \sin\theta_2 + r_3\dot{\theta}_3 \sin\theta_3 = r_4\dot{\theta}_4 \sin\theta_4 \qquad (5.45)$$

   $$\hat{j} : r_2\dot{\theta}_2 \cos\theta_2 + r_3\dot{\theta}_3 \cos\theta_3 = r_4\dot{\theta}_4 \cos\theta_4 \qquad (5.46)$$

   Note all link lengths r_1, r_2, r_3, r_4 and frame angle θ_1 are constants.

3. If link 2 is the driver, i.e. $\dot{\theta}_2$ is known. $\dot{\theta}_3, \dot{\theta}_4$ are unknowns. Solve two equations (5.45) and (5.46) for $\dot{\theta}_3, \dot{\theta}_4$

   $$\begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ -r_3 \cos\theta_3 & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_2\dot{\theta}_2 \sin\theta_2 \\ r_2\dot{\theta}_2 \cos\theta_2 \end{bmatrix}$$

4. If link 3 is the driver, i.e. $\dot{\theta}_3$ is known. $\dot{\theta}_2, \dot{\theta}_4$ are unknowns. Solve two equations (5.45) and (5.46) for $\dot{\theta}_2, \dot{\theta}_4$

   $$\begin{bmatrix} -r_2 \sin\theta_2 & r_4 \sin\theta_4 \\ -r_2 \cos\theta_2 & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_3\dot{\theta}_3 \sin\theta_3 \\ r_3\dot{\theta}_3 \cos\theta_3 \end{bmatrix}$$

5. Once the angular velocities are known, it is simple to calculate linear velocity of any point on the vector loop.

   $$\vec{r}_Q = \vec{r}_2 = r_2\left(\cos\theta_2 \hat{i} + \sin\theta_2 \hat{j}\right) \longrightarrow \dot{\vec{r}}_Q = \dot{\vec{r}}_2 = r_2\dot{\theta}_2\left(-\sin\theta_2 \hat{i} + \cos\theta_2 \hat{j}\right)$$

   $$\vec{r}_P = \vec{r}_2 + \vec{r}_3 = \vec{r}_1 + \vec{r}_4 \longrightarrow \dot{\vec{r}}_p = \dot{\vec{r}}_2 + \dot{\vec{r}}_3 = \dot{\vec{r}}_1 + \dot{\vec{r}}_4$$

§ Section 5.3.3 Acceleration Equations for 4-bar Linkages

For acceleration analysis, the position and velocity of the linkage must be known. So calculate the position and velocity of the linkage first before the acceleration velocity analysis.

1. The velocity equation

$$\hat{i} : r_2\dot{\theta}_2 \sin\theta_2 + r_3\dot{\theta}_3 \sin\theta_3 = r_4\dot{\theta}_4 \sin\theta_4 \qquad (5.45)$$

$$\hat{j} : r_2\dot{\theta}_2 \cos\theta_2 + r_3\dot{\theta}_3 \cos\theta_3 = r_4\dot{\theta}_4 \cos\theta_4 \qquad (5.46)$$

   Find all velocities $\dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4$ by velocity analysis.

2. Take derivative of the above velocity equations w.r.t. time and obtain:

$$r_2\ddot{\theta}_2 \sin\theta_2 + r_2\dot{\theta}_2^2 \cos\theta_2 + r_3\ddot{\theta}_3 \sin\theta_3 + r_3\dot{\theta}_3^2 \cos\theta_3 = r_4\ddot{\theta}_4 \sin\theta_4 + r_4\dot{\theta}_4^2 \cos\theta_4 \qquad (5.51)$$

$$r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2 + r_3\ddot{\theta}_3 \cos\theta_3 - r_3\dot{\theta}_3^2 \sin\theta_3 = r_4\ddot{\theta}_4 \cos\theta_4 - r_4\dot{\theta}_4^2 \sin\theta_4 \qquad (5.52)$$

3. If link 2 is the driver, i.e. $\ddot{\theta}_2$ is known. $\ddot{\theta}_3, \ddot{\theta}_4$ are unknowns. Solve two equations (5.51) and (5.52) for $\ddot{\theta}_3, \ddot{\theta}_4$

$$\begin{bmatrix} -r_3 \sin\theta_3 & r_4 \sin\theta_4 \\ -r_3 \cos\theta_3 & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_2\ddot{\theta}_2 \sin\theta_2 + r_2\dot{\theta}_2^2 \cos\theta_2 + r_3\dot{\theta}_3^2 \cos\theta_3 - r_4\dot{\theta}_4^2 \cos\theta_4 \\ r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2 - r_3\dot{\theta}_3^2 \sin\theta_3 + r_4\dot{\theta}_4^2 \sin\theta_4 \end{bmatrix}$$

4. If link 3 is the driver, i.e. $\ddot{\theta}_3$ is known. $\ddot{\theta}_2, \ddot{\theta}_4$ are unknowns. Solve two equations (5.51) and (5.52) for $\ddot{\theta}_2, \ddot{\theta}_4$

$$\begin{bmatrix} -r_2 \sin\theta_2 & r_4 \sin\theta_4 \\ -r_2 \cos\theta_2 & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_3\ddot{\theta}_3 \sin\theta_3 + r_3\dot{\theta}_3^2 \cos\theta_3 + r_2\dot{\theta}_2^2 \cos\theta_2 - r_4\dot{\theta}_4^2 \cos\theta_4 \\ r_3\ddot{\theta}_3 \cos\theta_3 - r_3\dot{\theta}_3^2 \sin\theta_3 - r_2\dot{\theta}_2^2 \sin\theta_2 + r_4\dot{\theta}_4^2 \sin\theta_4 \end{bmatrix}$$

5. Once the angular accelerations are known, it is simple to calculate linear acceleration of any point on the vector loop.

$$\dot{\vec{r}}_Q = \dot{\vec{r}}_2 = r_2\dot{\theta}_2\left(-\sin\theta_2\hat{i} + \cos\theta_2\hat{j}\right) \longrightarrow \ddot{\vec{r}}_Q = \ddot{\vec{r}}_2 = \left[-r_2\dot{\theta}_2^2 \cos\theta_2 - r_2\ddot{\theta}_2 \sin\theta_2\right]\hat{i} + \left[r_2\ddot{\theta}_2 \cos\theta_2 - r_2\dot{\theta}_2^2 \sin\theta_2\right]\hat{j}$$

$$\dot{\vec{r}}_p = \dot{\vec{r}}_2 + \dot{\vec{r}}_3 = \dot{\vec{r}}_1 + \dot{\vec{r}}_4 \longrightarrow \ddot{\vec{r}}_P = \ddot{\vec{r}}_2 + \ddot{\vec{r}}_3 = \ddot{\vec{r}}_1 + \ddot{\vec{r}}_4$$

❖ Table 5.1 Summarize position, velocity and acceleration equations for 4-bar linkages.
The position, velocity and acceleration of the driver link $\theta_M, \dot{\theta}_M, \ddot{\theta}_M$ are given.

- Use M=2, J=3 if link 2 is the input/driver link
- Use M=3, J=2 if link 3 is the input/driver link

Position

$$A = 2r_1 r_4 \cos\theta_1 - 2r_M r_4 \cos\theta_M$$
$$B = 2r_1 r_4 \sin\theta_1 - 2r_M r_4 \sin\theta_M$$
$$C = r_1^2 + r_M^2 + r_4^2 - r_J^2 - 2r_1 r_M \cos(\theta_M - \theta_1)$$

$$\theta_4 = 2\tan^{-1}\left[\frac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C - A}\right]; \sigma = \pm 1 \quad -\pi < \theta_4 < \pi$$

$$\theta_J = \tan^{-1}\left[\frac{r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_M \sin\theta_M}{r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_M \cos\theta_M}\right]$$

Use the sign of $\sin\theta_3, \cos\theta_3$ to determine the quadrant in which the angle θ_3 lies

$$\vec{r}_Q = \vec{r}_2 = r_2\left(\cos\theta_2 \hat{i} + \sin\theta_2 \hat{j}\right)$$
$$\vec{r}_P = \vec{r}_2 + \vec{r}_3 = r_2\left(\cos\theta_2 \hat{i} + \sin\theta_2 \hat{j}\right) + r_3\left(\cos\theta_3 \hat{i} + \sin\theta_3 \hat{j}\right)$$
$$= \vec{r}_1 + \vec{r}_4 = r_1\left(\cos\theta_1 \hat{i} + \sin\theta_1 \hat{j}\right) + r_4\left(\cos\theta_4 \hat{i} + \sin\theta_4 \hat{j}\right)$$

Velocity

$$\begin{bmatrix} -r_J \sin\theta_J & r_4 \sin\theta_4 \\ -r_J \cos\theta_J & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_J \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_M \dot{\theta}_M \sin\theta_M \\ r_M \dot{\theta}_M \cos\theta_M \end{bmatrix}$$

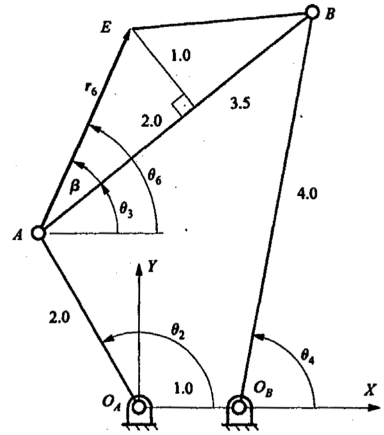
$$\dot{\vec{r}}_Q = \dot{\vec{r}}_2 = r_2 \dot{\theta}_2\left(-\sin\theta_2 \hat{i} + \cos\theta_2 \hat{j}\right)$$
$$\dot{\vec{r}}_p = \dot{\vec{r}}_1 + \dot{\vec{r}}_4 = \dot{\vec{r}}_4 = r_4 \dot{\theta}_4\left(-\sin\theta_4 \hat{i} + \cos\theta_4 \hat{j}\right)$$

Acceleration

$$\begin{bmatrix} -r_J \sin\theta_J & r_4 \sin\theta_4 \\ -r_J \cos\theta_J & r_4 \cos\theta_4 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_J \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_M \ddot{\theta}_M \sin\theta_M + r_M \dot{\theta}_M^2 \cos\theta_M + r_J \dot{\theta}_J^2 \cos\theta_J - r_4 \dot{\theta}_4^2 \cos\theta_4 \\ r_M \ddot{\theta}_M \cos\theta_M - r_M \dot{\theta}_M^2 \sin\theta_M - r_J \dot{\theta}_J^2 \sin\theta_J + r_4 \dot{\theta}_4^2 \sin\theta_4 \end{bmatrix}$$

$$\ddot{\vec{r}}_Q = \ddot{\vec{r}}_2 = -r_2\left[\dot{\theta}_2^2 \cos\theta_2 + \ddot{\theta}_2 \sin\theta_2\right]\hat{i} + r_2\left[\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right]\hat{j}$$
$$\ddot{\vec{r}}_P = \ddot{\vec{r}}_1 + \ddot{\vec{r}}_4 = \ddot{\vec{r}}_4 = -r_4\left[\dot{\theta}_4^2 \cos\theta_4 + r_4 \ddot{\theta}_4 \sin\theta_4\right]\hat{i} + r_4\left[\ddot{\theta}_4 \cos\theta_4 - \dot{\theta}_4^2 \sin\theta_4\right]\hat{j}$$

Example 5.2: For the linkage and its position (θ_2= 0°) given in Example 5.1, if the velocity of the driver link $\dot{\theta}_2 = 10rad/s, \ddot{\theta}_2 = 0$, compute $\dot{\theta}_3, \dot{\theta}_4,\ \ddot{\theta}_3, \ddot{\theta}_4$

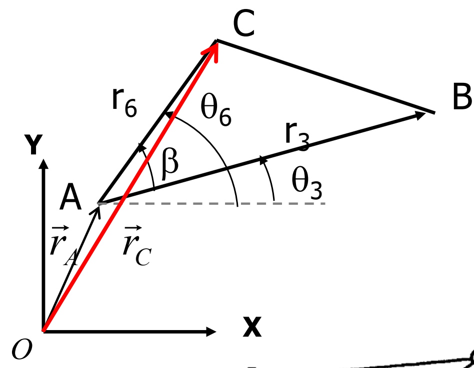


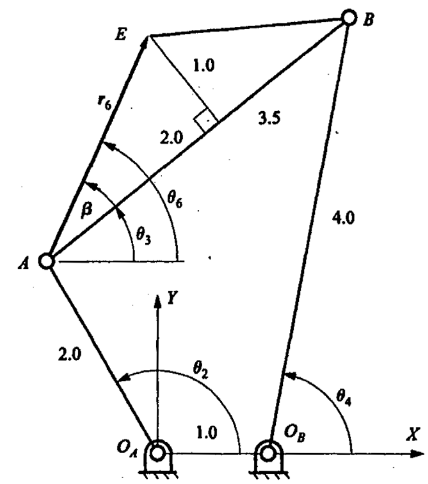
§ Section 5.4 Analytical Equations for a Rigid Body after The kinematic properties of two points are known

Suppose the position and orientation of a link AB (r_3 and θ_3) is known, the position of any other point C on this link can be obtained as

$$\vec{r}_C = \vec{r}_A + \vec{r}_{AC} = \vec{r}_A + r_6(\cos\theta_6 i + \sin\theta_6 j)$$

where $\theta_6 = \theta_3 + \beta$



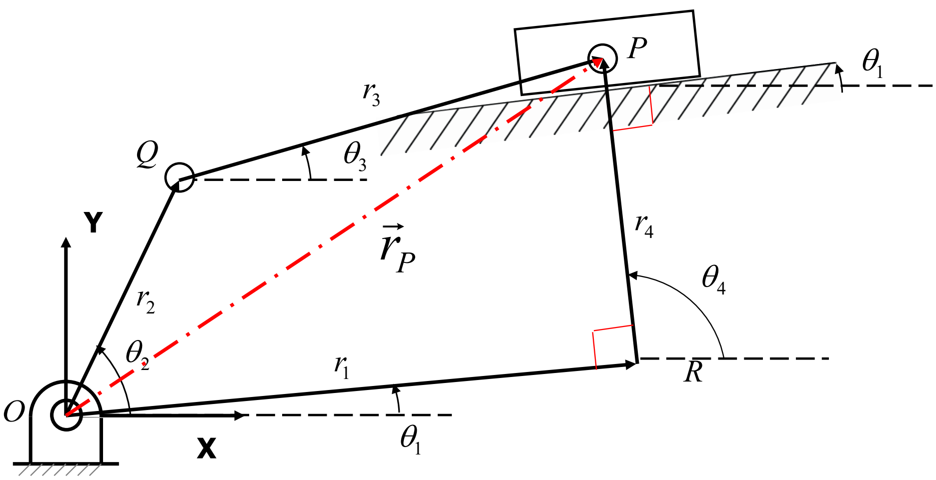


Example 5.3:

- For a given 4-bar linkage given in Example 5.1 and the driver angle θ_2= 0°
- Compute coordinates of the coupler point E $\vec{r}_E$
- Compute the velocity and acceleration of the coupler point E $\dot{\vec{r}}_E, \ddot{\vec{r}}_E$

§ Section 5.5 Analytical Equations for Slider-Crank Mechanisms

A general slider-crank linkage is shown below.



The goal of analysis:

- Known: all link dimensions, i.e. θ_1, r_2, r_3, r_4
- $\theta_4 = \theta_1 + \pi / 2$
- **Driver parameter:** θ_2
- Find driven (unknown) parameter: r_1, θ_3

Derivation steps:

1. Write the loop closure equation

$$\vec{r}_P = \vec{r}_2 + \vec{r}_3 = \vec{r}_1 + \vec{r}_4 \qquad (5.24)$$

2. which can be written in x ($\hat{i}$) and y($\hat{j}$) components as

$$\hat{i} : r_2 \cos\theta_2 + r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 \qquad (5.26)$$
$$\hat{j} : r_2 \sin\theta_2 + r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 \qquad (5.27)$$

3. Identify known: $\theta_1, \theta_4 (= \theta_1 + \pi / 2), r_2, r_3, r_4$ and driver parameter θ_2. Unknowns: r_1, θ_3.

4. Rearrange eqs. (5.26) and (5.27) to obtain

$$\hat{i} : r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2 \qquad (5.65)$$
$$\hat{j} : r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_2 \sin\theta_2 \qquad (5.66)$$

5. Square both sides of above equations, add and simplify the result using trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. This gives

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 + 2r_1r_4\left(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4\right)$$
$$-2r_1r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) - 2r_2r_4\left(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4\right) \qquad (5.67)$$

6. Arrange the above equation into

$$r_1^2 + Ar_1 + B = 0 \qquad (5.68)$$

where

$$A = 2r_4\left(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4\right) - 2r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)$$
$$= 2r_4\cos(\theta_1 - \theta_4) - 2r_2\cos(\theta_1 - \theta_2) = 2r_4\cos(\pi/2) - 2r_2\cos(\theta_1 - \theta_2)$$
$$= -2r_2\cos(\theta_1 - \theta_2)$$

$$B = r_2^2 + r_4^2 - r_3^2 - 2r_2r_4\cos\left(\theta_2 - \theta_4\right)$$

7. Solving (5.68) for r_1 gives two solutions of r_1

$$r_1 = \frac{-A + \sigma\sqrt{A^2 - 4B}}{2}$$

where $\sigma = \pm 1$ is a sign varaible identifying the assembly mode

8. For each solution of r_1, back substitute r_1 into (5.65) and (5.66) and solve for θ_3.

$$\theta_3 = \tan^{-1}\left[\frac{\sin\theta_3}{\cos\theta_3}\right] = \tan^{-1}\left[\frac{r_3\sin\theta_3}{r_3\cos\theta_3}\right] = \tan^{-1}\left[\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}\right] \qquad (5.71)$$

Similar to the 4-bar, use the sign of $\sin\theta_3, \cos\theta_3$ to determine the quadrant in which the angle θ_3 lies or use Matlab function ATAN2.

9. Calculate the position of points P and Q as

$$\vec{r}_Q = \vec{r}_2 = r_2\left(\cos\theta_2\hat{i} + \sin\theta_2\hat{j}\right) \qquad (5.72)$$
$$\vec{r}_P = \vec{r}_1 + \vec{r}_4 = r_1\left(\cos\theta_1\hat{i} + \sin\theta_1\hat{j}\right) + r_4\left(\cos\theta_4\hat{i} + \sin\theta_4\hat{j}\right) \qquad (5.73)$$

- If the slider link is the driver r_1
  - **Driver parameter:** r_1
  - Find driven (unknown) parameter: θ_2, θ_3

1. Start from Eq.(5.67),

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 + 2r_1r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4)$$
$$-2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) \qquad (5.67)$$

2. Rearrange the above equation into

$$A\cos\theta_2 + B\sin\theta_2 + C = 0 \qquad (5.74)$$

where

$$A = -2r_1r_2\cos\theta_1 - 2r_2r_4\cos\theta_4$$
$$B = -2r_1r_2\sin\theta_1 - 2r_2r_4\sin\theta_4$$
$$\begin{aligned} C &= r_1^2 + r_2^2 + r_4^2 - r_3^2 + 2r_1r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4) \\ &= r_1^2 + r_2^2 + r_4^2 - r_3^2 + 2r_1r_4\cos(\theta_4 - \theta_1) \\ &= r_1^2 + r_2^2 + r_4^2 - r_3^2 + 2r_1r_4\cos(\pi/2) \\ &= r_1^2 + r_2^2 + r_4^2 - r_3^2 \end{aligned}$$

3. Substitute $t = \tan\left(\frac{\theta_2}{2}\right)$ into (5.74) to obtain

$$(C-A)t^2 + 2Bt + (A+C) = 0$$

4. Solving the quadratic equation for t gives

$$t = \frac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C - A}$$

where $\sigma = \pm 1$ is a sign varaible identifying the assembly mode

5. For each solution of t, calculate angle θ_2 from t as

$$\theta_2 = 2\tan^{-1} t, \qquad -\pi < \theta_2 < \pi$$

6. Use (5.71) to calculate angle θ_3 and use (5.72) and (5.73) for calculating the position of points P and Q.

❖ Geometric interpretation of two analytical solutions (slider-crank):

➢ When θ_2 is the driver, the two solutions P and P' are symmetric about the point Q. See figure below

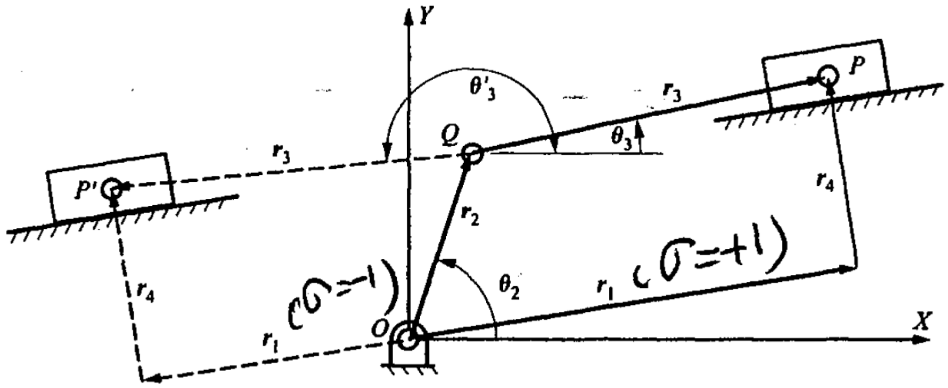


FIGURE 5.15 The two possible positions (P and P') of the point P for a given value of θ_2 in a slider-crank mechanism.

➢ When r_1 is the driver, the two solutions Q and Q' are symmetric about the line OP. See figure below

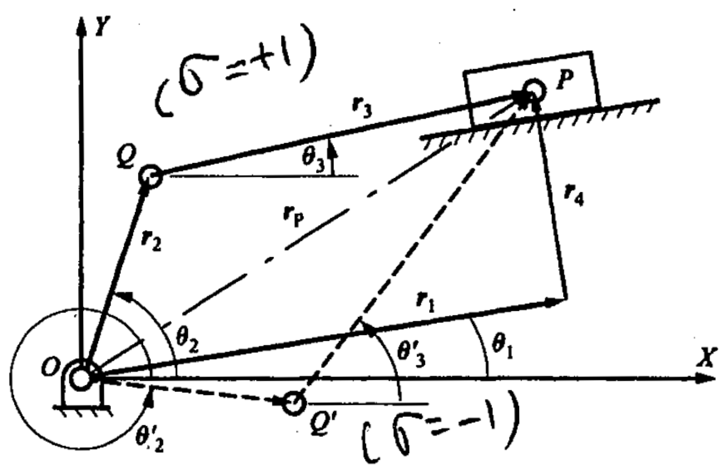


FIGURE 5.17 Two possible assembly modes when the position r_1 of the slider is given as an input.

§ Section 5.5.4 Velocity Equations for Slider-Crank Mechanism

For velocity analysis, the position of the linkage must be known. So calculate the position of the linkage first before the velocity analysis.

Derivation of velocity equations:

1. The vector loop equation

   $$\hat{i}: r_2\cos\theta_2 + r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4 \qquad (5.63)$$
   $$\hat{j}: r_2\sin\theta_2 + r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4 \qquad (5.64)$$

2. Take derivative of the above equations w.r.t. time and obtain two linear equations:

   $$\hat{i}: -r_2\dot{\theta}_2\sin\theta_2 - r_3\dot{\theta}_3\sin\theta_3 = \dot{r}_1\cos\theta_1 \qquad (5.81)$$
   $$\hat{j}: r_2\dot{\theta}_2\cos\theta_2 + r_3\dot{\theta}_3\cos\theta_3 = \dot{r}_1\sin\theta_1 \qquad (5.82)$$

   Note link lengths r_2, r_3, offset r_4, sliding angle θ_1 and $\theta_4=\theta_1+\pi/2$ are constants (not changing w.r.t time).

   Equations (5.81) and (5.82) are linear in terms of three velocity variables $\dot{r}_1, \dot{\theta}_2, \dot{\theta}_3$

3. Pick one of crank/coupler/slider as the driver, i.e. the velocity of the driver is known, solve solve (5.81) and (5.82) for the other two variables.

§ Section 5.5.5 Acceleration Equations for Slider-Crank Mechanism

For acceleration analysis, the position and velocity of the linkage must be known. So calculate the position and velocity of the linkage first before the acceleration analysis.

Derivation of acceleration equations:

1. Take derivative of the velocity equations (5.81) and (5.82) w.r.t. time to obtain two linear equations:

   $$-r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3 = \ddot{r}_1\cos\theta_1 \qquad (5.87)$$
   $$r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 + r_3\ddot{\theta}_3\cos\theta_3 - r_3\dot{\theta}_3^2\sin\theta_3 = \ddot{r}_1\sin\theta_1 \qquad (5.88)$$

   Equations (5.87) and (5.88) are linear in terms of three acceleration variables $\ddot{\theta}_2, \ddot{\theta}_3, \ddot{r}_1$

2. Pick one joint as the driver, solve equations (5.87) and (5.88) for the other two acceleration variables.

See Table 5.4 and 5.5 for summary of position, velocity and acceleration equations for a slider-crank mechanism when crank, coupler or slider is the input (driver). A slightly modified version of these two tables are in below.

TABLE 5.4 Summary of Position, Velocity, and Acceleration Equations for a Slider-Crank Mechanism When Either the Crank or the Coupler Is the Input. Link 2 Is the Input Link When $M = 2$ and $J = 3$. Link 3 Is the Input Link When $M = 3$ and $J = 2$. The Link Numbers and Points Are Defined in Fig. 5.14

Position

$$A = 2r_4\left(\cos\theta_1\cos\theta_4 + \overset{0}{\underline{\sin\theta_1\sin\theta_4}}\right) - 2r_M\underbrace{\left(\cos\theta_1\cos\theta_M + \sin\theta_1\sin\theta_M\right)}_{\cos(\theta_1-\theta_M)}$$

$$B = r_M^2 + r_4^2 - r_J^2 - 2r_M r_4\underbrace{\left(\cos\theta_M\cos\theta_4 + \sin\theta_M\sin\theta_4\right)}_{\cos(\theta_M-\theta_4)}$$

$$r_1 = \frac{-A + \sigma\sqrt{A^2 - 4B}}{2}; \quad \sigma = \pm 1$$

$$\theta_J = \tan^{-1}\left[\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_M\sin\theta_M}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_M\cos\theta_M}\right]$$

$$\boldsymbol{r}_Q = \boldsymbol{r}_2 = r_2\left(\cos\theta_2\boldsymbol{i} + \sin\theta_2\boldsymbol{j}\right)$$

$$\boldsymbol{r}_P = \boldsymbol{r}_2 + \boldsymbol{r}_3 = r_2\left(\cos\theta_2\boldsymbol{i} + \sin\theta_2\boldsymbol{j}\right) + r_3\left(\cos\theta_3\boldsymbol{i} + \sin\theta_3\boldsymbol{j}\right)$$

Velocity

$$\begin{bmatrix}\cos\theta_1 & r_J\sin\theta_J\\ \sin\theta_1 & -r_J\cos\theta_J\end{bmatrix}\begin{Bmatrix}\dot r_1\\ \dot\theta_J\end{Bmatrix} = \begin{Bmatrix}-r_M\dot\theta_M\sin\theta_M\\ r_M\dot\theta_M\cos\theta_M\end{Bmatrix}$$

$$\dot{\boldsymbol{r}}_Q = r_2\dot\theta_2\left(-\sin\theta_2\boldsymbol{i} + \cos\theta_2\boldsymbol{j}\right)$$

$$\dot{\boldsymbol{r}}_P = \left(-r_2\dot\theta_2\sin\theta_2 + r_3\dot\theta_3\sin\theta_3\right)\boldsymbol{i} + \left(r_2\dot\theta_2\cos\theta_2 + r_3\dot\theta_3\cos\theta_3\right)\boldsymbol{j}$$

Acceleration

$$\begin{bmatrix}\cos\theta_1 & r_J\sin\theta_J\\ \sin\theta_1 & -r_J\cos\theta_J\end{bmatrix}\begin{Bmatrix}\ddot r_1\\ \ddot\theta_J\end{Bmatrix} = \begin{Bmatrix}-r_M\ddot\theta_M\sin\theta_M - r_M\dot\theta_M^2\cos\theta_M - r_J\dot\theta_J^2\cos\theta_J\\ r_M\ddot\theta_M\cos\theta_M - r_M\dot\theta_M^2\sin\theta_M - r_J\dot\theta_J^2\sin\theta_J\end{Bmatrix}$$

$$\ddot{\boldsymbol{r}}_Q = \left(r_2\ddot\theta_2\sin\theta_2 - r_2\dot\theta_2^2\cos\theta_2\right)\boldsymbol{i} + \left(r_2\ddot\theta_2\cos\theta_2 - r_2\dot\theta_2^2\sin\theta_2\right)\boldsymbol{j}$$

$$\begin{aligned}\ddot{\boldsymbol{r}}_P = &-\left(r_2\ddot\theta_2\sin\theta_2 + r_2\dot\theta_2^2\cos\theta_2 + r_3\ddot\theta_3\sin\theta_3 + r_3\dot\theta_3^2\cos\theta_3\right)\boldsymbol{i}\\ &+\left(r_2\ddot\theta_2\cos\theta_2 - r_2\dot\theta_2^2\sin\theta_2 + r_3\ddot\theta_3\cos\theta_3 - r_3\dot\theta_3^2\sin\theta_3\right)\boldsymbol{j}\end{aligned}$$

TABLE 5.5 Summary of Position, Velocity, and Acceleration Equations for a Slider-Crank Mechanism When the Slider (Link 4) is the Input Link. The Link Numbers and Points Are Defined in Fig. 5.14

Position

$$A = 2r_1r_2\cos\theta_1 - 2r_2r_4\cos\theta_4$$

$$B = 2r_1r_2\sin\theta_1 - 2r_2r_4\sin\theta_4$$

$$C = r_1^2 + r_2^2 + r_4^2 - r_3^2 - 2r_1r_4\left(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4\right)$$

$$0 = \cos(\theta_4 - \theta_1) = \cos\frac{\pi}{2}$$

$$\theta_2 = 2\tan^{-1}\left[\frac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C - A}\right]; \quad \sigma = \pm 1$$

$$\theta_3 = \tan^{-1}\left[\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}\right]$$

$$\boldsymbol{r}_Q = \boldsymbol{r}_2 = r_2\left(\cos\theta_2\boldsymbol{i} + \sin\theta_2\boldsymbol{j}\right)$$

$$\boldsymbol{r}_P = \boldsymbol{r}_2 + \boldsymbol{r}_3 = r_2\left(\cos\theta_2\boldsymbol{i} + \sin\theta_2\boldsymbol{j}\right) + r_3\left(\cos\theta_3\boldsymbol{i} + \sin\theta_3\boldsymbol{j}\right)$$

Velocity

$$\begin{bmatrix} -r_2\sin\theta_2 & -r_3\sin\theta_3 \\ r_2\cos\theta_2 & r_3\cos\theta_3 \end{bmatrix}\begin{Bmatrix} \dot\theta_2 \\ \dot\theta_3 \end{Bmatrix} = \begin{Bmatrix} \dot r_1\cos\theta_1 \\ \dot r_1\sin\theta_1 \end{Bmatrix}$$

$$\dot{\boldsymbol{r}}_Q = r_2\dot\theta_2\left(-\sin\theta_2\boldsymbol{i} + \cos\theta_2\boldsymbol{j}\right)$$

$$\dot{\boldsymbol{r}}_P = \left(-r_2\dot\theta_2\sin\theta_2 - r_3\dot\theta_3\sin\theta_3\right)\boldsymbol{i} + \left(r_2\dot\theta_2\cos\theta_2 + r_3\dot\theta_3\cos\theta_3\right)\boldsymbol{j}$$

Acceleration

$$\begin{bmatrix} -r_2\sin\theta_2 & -r_3\sin\theta_3 \\ r_2\cos\theta_2 & r_3\cos\theta_3 \end{bmatrix}\begin{Bmatrix} \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} = \begin{Bmatrix} r_2\dot\theta_2^2\cos\theta_2 + r_3\dot\theta_3^2\cos\theta_3 + \ddot r_1\cos\theta_1 \\ r_2\dot\theta_2^2\sin\theta_2 + r_3\dot\theta_3^2\sin\theta_3 + \ddot r_1\sin\theta_1 \end{Bmatrix}$$

$$\ddot{\boldsymbol{r}}_Q = \left(-r_2\ddot\theta_2\sin\theta_2 - r_2\dot\theta_2^2\cos\theta_2\right)\boldsymbol{i} + \left(r_2\ddot\theta_2\cos\theta_2 - r_2\dot\theta_2^2\sin\theta_2\right)\boldsymbol{j}$$

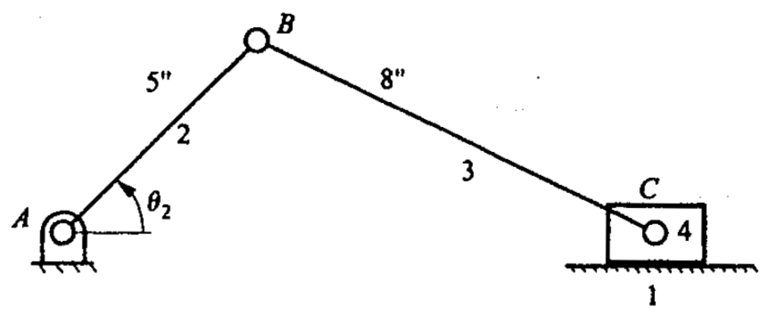
$$\ddot{\boldsymbol{r}}_P = -\left(r_2\ddot\theta_2\sin\theta_2 + r_2\dot\theta_2^2\cos\theta_2 + r_3\ddot\theta_3\sin\theta_3 + r_3\dot\theta_3^2\cos\theta_3\right)\boldsymbol{i}$$
$$+\left(r_2\ddot\theta_2\cos\theta_2 - r_2\dot\theta_2^2\sin\theta_2 + r_3\ddot\theta_3\cos\theta_3 - r_3\dot\theta_3^2\sin\theta_3\right)\boldsymbol{j}$$

Example 5.4: For a slider-crank, the crank is the input. The known input information is

$$\theta_1 = 0°, \qquad \theta_2 = 45°, \qquad \dot{\theta}_2 = 10 \text{ rad/s}, \qquad \ddot{\theta}_2 = 0$$

$$r_2 = 5 \text{ in}, \qquad r_3 = 8 \text{ in}, \qquad r_4 = 0 \text{ in}$$

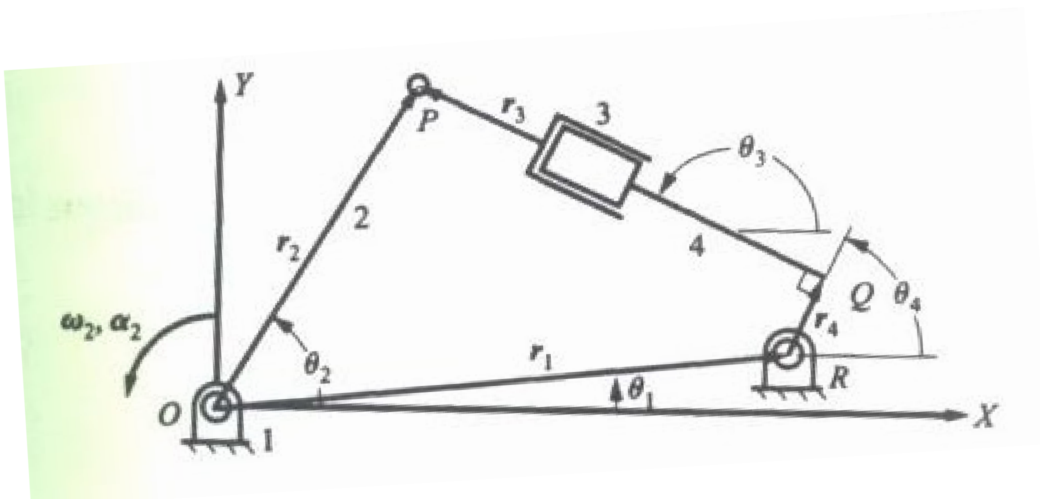
- Derive the positional analysis equations
- Position analysis: roughly draw these two configurations. what is the assemble mode for the solution shown in the Figure?
- Derive the velocity analysis equations
- Velocity analysis: at what position of the linkage, the velocity of the slider is zero.



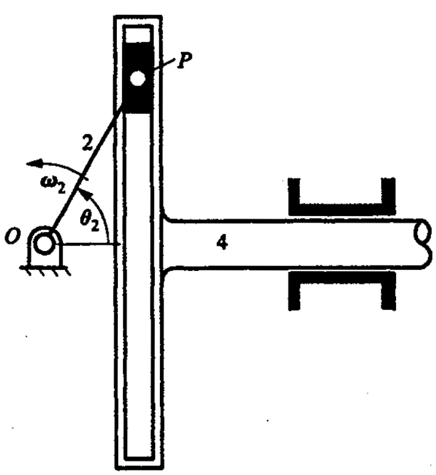
§ Section 5.6 Analytical Equations for The Slider-Crank Inversion

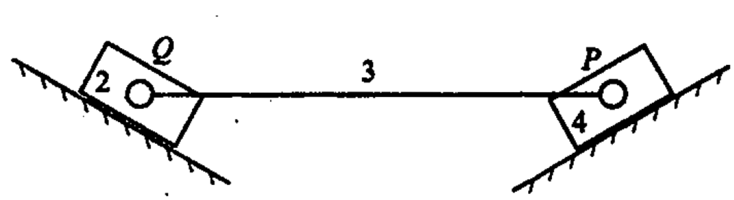
The figure below shows an inversion of the slider-crank mechanism. Suppose the crank link r_2 is the driver. In order to do the positional analysis,

- Please derive the loop closure equations of the linkage.
- Identify constant parameters (does not change over time) and variable parameters.
- Among the variable parameters, identify known parameters and unknown parameters.



How about other linkages like scotch yoke mechanism (Fig. 5.37), Elliptical trammel (Fig. 5.39)



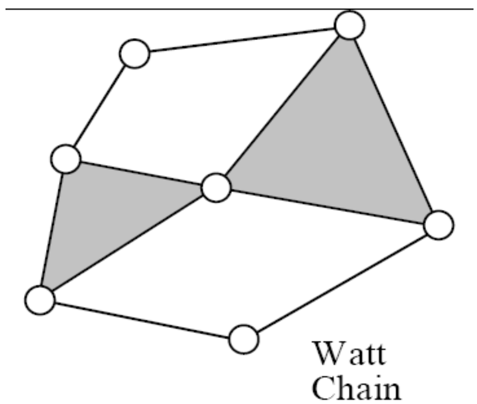


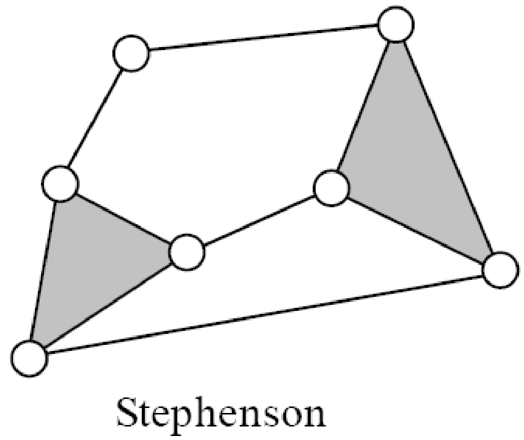
scotch yoke mechanism

Elliptical trammel

§ Kinematic Analysis of Linkages Beyond Four-Bar.

How could you analyze the following two six-bar linkages?
Strategy: breaking the linkage into four-bar/crank-slider/double slider and dyads etc.





How about this mechanism?