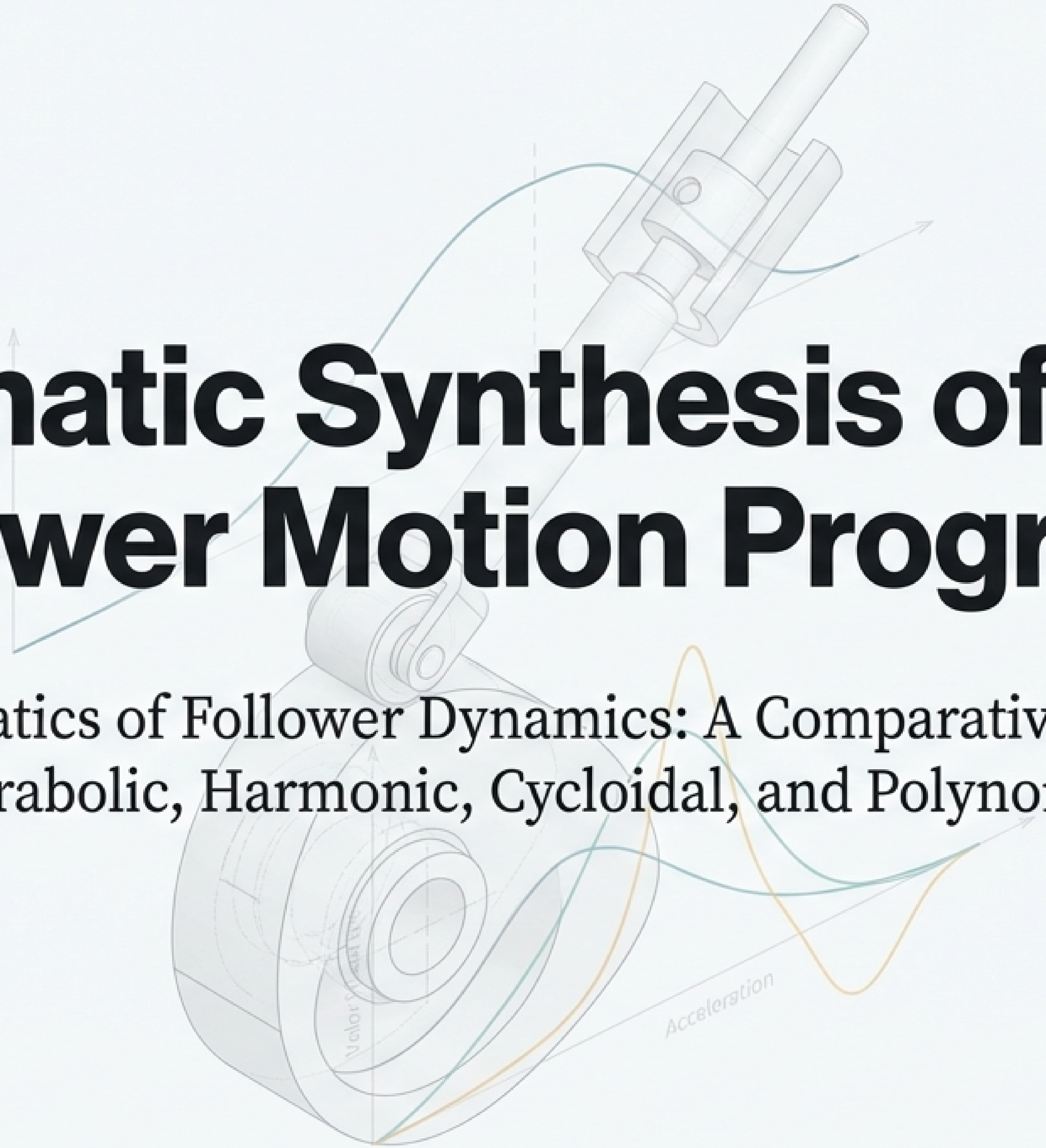
# Kinematic Synthesis of Cam Follower Motion Programs

The Mathematics of Follower Dynamics: A Comparative Analysis of Uniform, Parabolic, Harmonic, Cycloidal, and Polynomial Motions



# The Goal: Creating Smooth, Predictable Motion

## What is Motion Program Synthesis?

- In cam design, the overall cycle often includes required segments like dwells (zero motion). Motion synthesis is the process of defining the follower's displacement, velocity, and acceleration in the unspecified transition segments (the rises and returns) that connect these required portions.
- The choice of mathematical function for these transitions is critical for the dynamic performance of the system, dictating forces, vibrations, and operational speed.

## The Fundamental Law of Cam Design

For a cam to operate smoothly at high speeds without inducing shock or vibration, its motion profile must adhere to this core principle:

1. **The Law:** The cam displacement function must be continuous through its first (velocity) and second (acceleration) derivatives across the entire 360° interval.
2. **The Corollary:** The jerk function (the third derivative, $y'''$) must be finite across the entire interval.

Any discontinuities in these derivatives result in theoretical infinite forces, causing vibration, wear, and noise in real-world systems. This law is the standard for high-performance cam design.

# Motion Profile 1: Uniform Motion — Simplest Profile, Worst Dynamics

## Concept

Uniform (or linear) motion connects two points with a straight line, implying constant velocity. It can only match two boundary conditions (position at start and end).

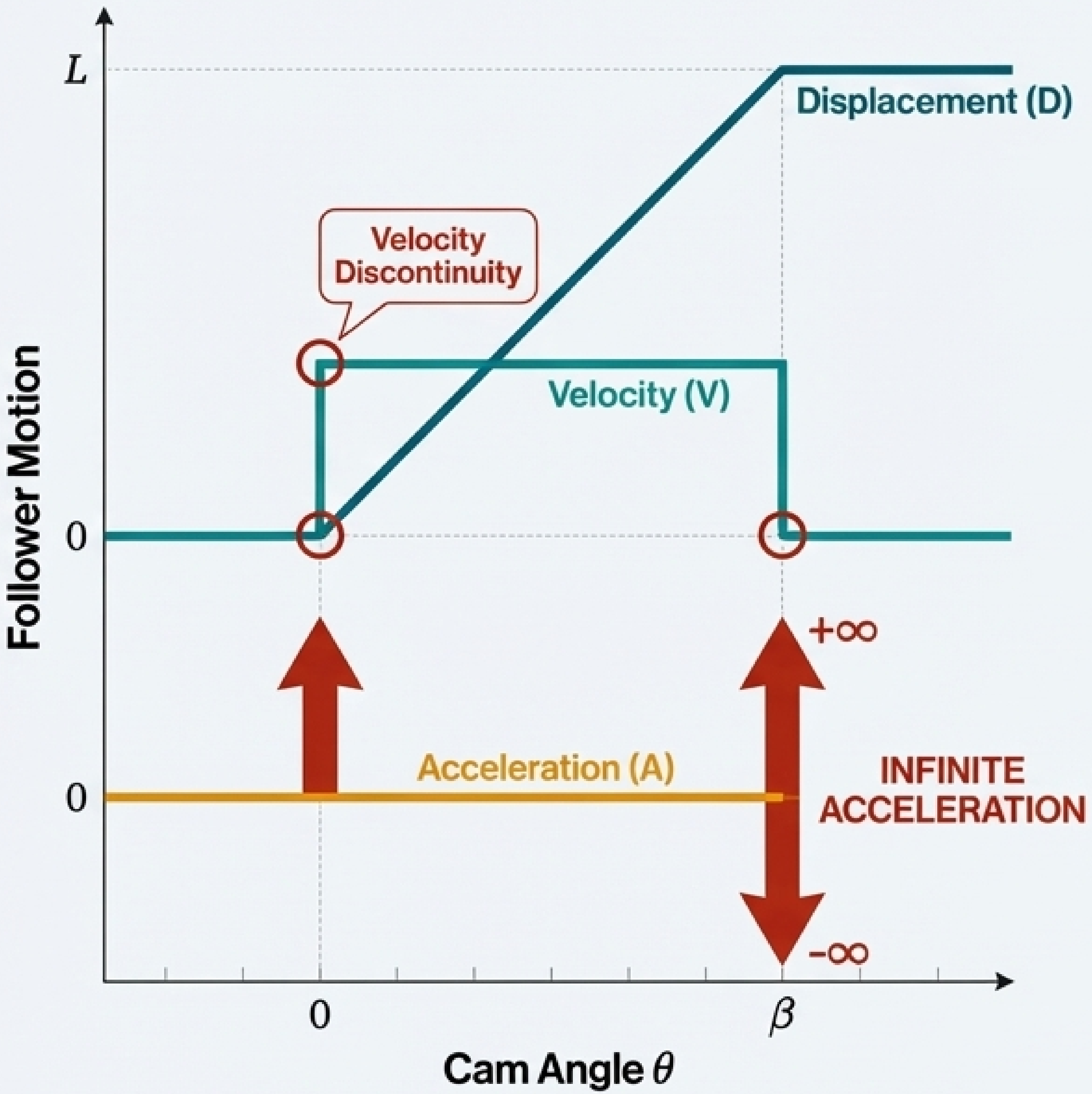
## Derivation

1. **General Equation:** $y(\theta) = C_0 + C_1\theta$
2. **Boundary Conditions (BCs):** For a rise of $L$ over an angle $\beta$:
   $y(0) = 0 \Rightarrow C_0 = 0$
   $y(\beta) = L \Rightarrow C_1\beta = L \Rightarrow C_1 = \frac{L}{\beta}$
3. **Resulting Equations:**
   **Displacement (y):** $y = \frac{L}{\beta}\theta$
   **Velocity (y'):** $y' = \frac{L}{\beta}$ (Constant)
   **Acceleration (y''):** $y'' = \frac{d^2y}{d\theta^2} = 0$

## Dynamic Analysis

**The Flaw:** The velocity is constant during the rise but zero during the dwell. This creates an instantaneous change—a step function—in velocity at the transitions. This velocity discontinuity means the theoretical acceleration ($y''$) must be infinite at the start and end of the motion.

**Conclusion:** This profile generates extreme shock forces and is almost never used in practice. It is a clear violation of the Fundamental Law.

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# Motion Profile 2: Parabolic Motion (Part 1 - Acceleration Derivation)

**Concept**

To eliminate infinite acceleration, Parabolic Motion uses two joined quadratic functions. The first half provides constant positive acceleration, and the second half provides constant negative acceleration (deceleration). Each parabola matches three boundary conditions, allowing for velocity control at the boundaries.

**Derivation: Parabola 1 (Acceleration Phase, $0 \leq \theta \leq \frac{\beta}{2}$)**

1. **General Equation:**

$$y(\theta) = C_0 + C_1\theta + C_2\theta^2$$
$$y'(\theta) = C_1 + 2C_2\theta$$

2. **Boundary Conditions (BCs):** We require the motion to start from rest and reach half the total lift (L/2) at the midpoint angle (β/2).
   - $y(0) = 0 \implies C_0 = 0$
   - $y'(0) = 0 \implies C_1 = 0$
   - $y\left(\frac{\beta}{2}\right) = \frac{L}{2} \implies C_2\left(\frac{\beta}{2}\right)^2 = \frac{L}{2} \implies C_2 = \frac{2L}{\beta^2}$
3. **Resulting Equations (Parabola 1):**
   - **Displacement ($y$):** $y = \frac{2L}{\beta^2}\theta^2$
   - **Velocity ($y'$):** $y' = \frac{4L}{\beta^2}\theta$
   - **Acceleration ($y''$):** $y'' = \frac{4L}{\beta^2}$ (Constant Positive)

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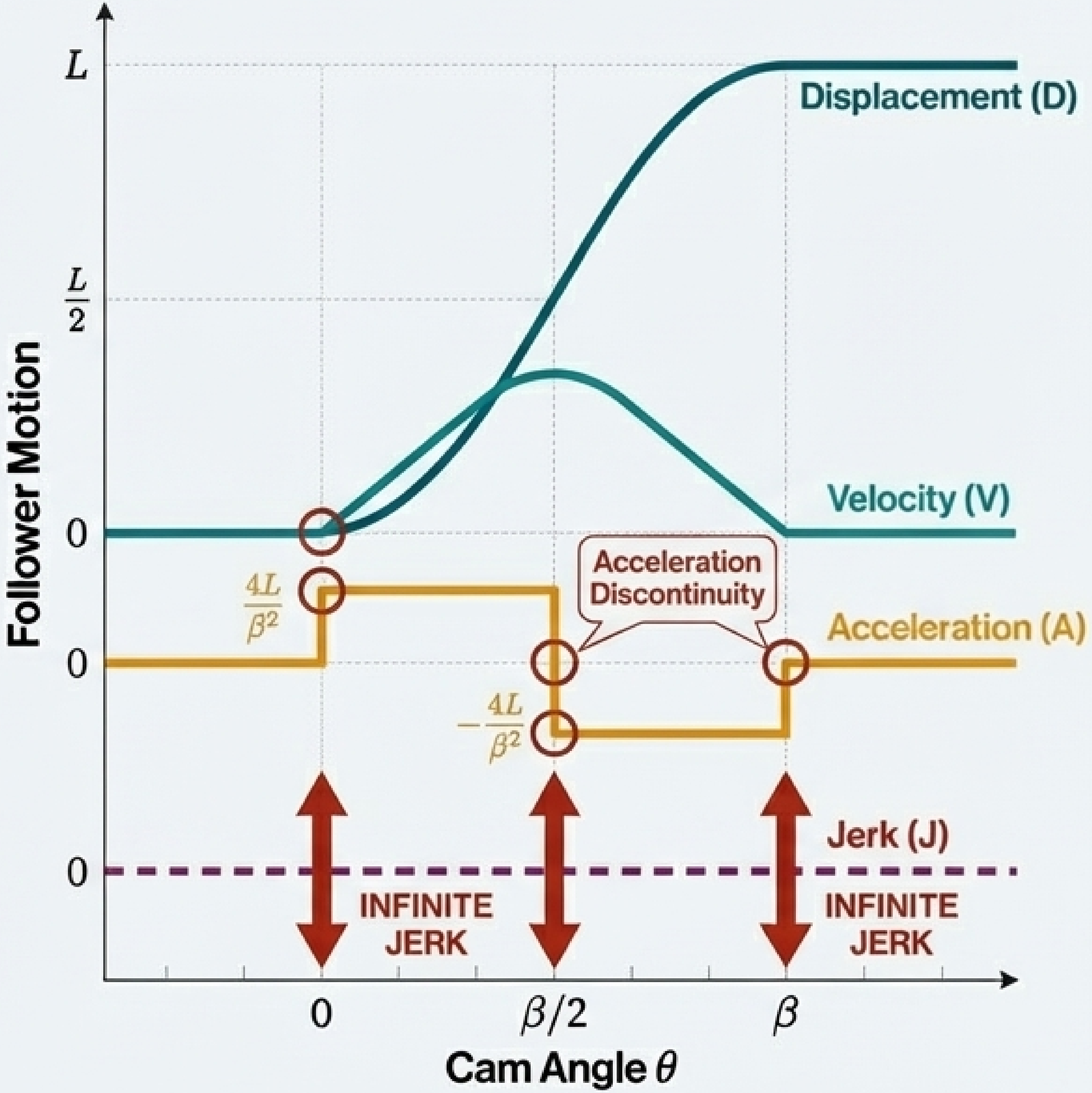
# Motion Profile 2: Parabolic Motion (Part 2 – Deceleration and Dynamics)

## Derivation: Parabola 2 (Deceleration Phase, $\frac{\beta}{2} \le \theta \le \beta$)

1. **Boundary Conditions (BCs):** The motion must reach the full lift ($L$) and come to rest ($y'=0$) at the end of the transition ($\beta$).
   - $y(\beta) = L$
   - $y'(\beta) = 0$
   - Match position at midpoint: $y(\beta/2) = L/2$
2. **Resulting Equations (Parabola 2):**
   - **Displacement (y):** $y = L\left[1 - 2\left(1 - \frac{\theta}{\beta}\right)^2\right]$
   - **Velocity (y′):** $y' = \frac{4L}{\beta}\left(1 - \frac{\theta}{\beta}\right)$
   - **Acceleration (y″):** $y'' = -\frac{4L}{\beta^2}$ (Constant Negative)

## Dynamic Analysis

- **The Improvement:** Velocity is now continuous across the entire motion, a major improvement over uniform motion.
- **The New Flaw:** The acceleration value **changes instantaneously** from 0 to a positive constant, then to a negative constant, and back to 0.
- This **acceleration discontinuity** means the jerk ($y'''$), or rate of change of acceleration, is **infinite** at these three points. This still induces shock and vibration.
- **Conclusion:** Better, but its high jerk makes it unsuitable for high-speed applications. It violates the corollary of the Fundamental Law.



# Motion Profile 3: Simple Harmonic Motion — A Smoother, But Still Flawed, Curve

## Concept

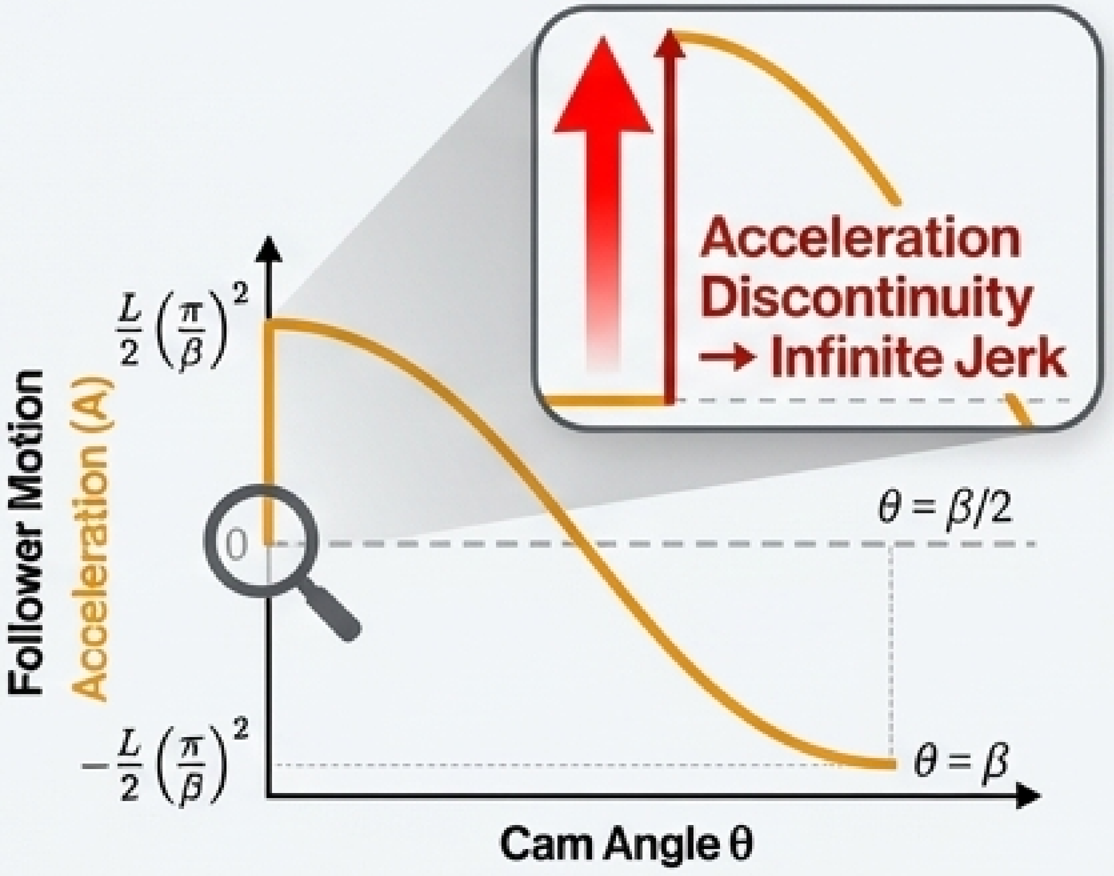
Simple Harmonic Motion (SHM) uses a single cosine-based function for the entire transition. This is inherently smoother than the piecewise approach of Parabolic motion and can be generated geometrically by an offset circular cam.

## Derivation

1. **General Equation:** $y(\theta) = C_0 + C_1 \cos(C_2\theta)$
2. **Boundary Conditions (BCs):** For a rise $L$ over angle $\beta$, starting and ending at zero velocity.
   - $y(0) = 0$ and $y(\beta) = L$.
   - Solving for the constants $C_0$, $C_1$, and $C_2$ to match these conditions over half a cosine wave period yields the final equation.
3. **Resulting Equations:**
   - **Displacement (y):** $y = \frac{L}{2}\left(1 - \cos\frac{\pi\theta}{\beta}\right)$
   - **Velocity (y'):** $y' = \frac{\pi L}{2\beta}\sin\frac{\pi\theta}{\beta}$
   - **Acceleration (y''):** $y'' = \frac{L}{2}\left(\frac{\pi}{\beta}\right)^2 \cos\frac{\pi\theta}{\beta}$

## Dynamic Analysis

- **The Flaw:** Like parabolic motion, velocity is continuous. However, the cosine acceleration curve starts and ends at its maximum and minimum values, not at zero.
- This creates an **acceleration discontinuity** at the transitions to the dwells ($\theta=0$ and $\theta=\beta$).
- An instantaneous change in acceleration means the jerk ($y'''$) is **infinite** at the boundaries.
- **Conclusion**: Despite its internal smoothness, the boundary discontinuities make it a poor choice for high-speed systems. It also violates the corollary of the Fundamental Law.



# Motion Profile 4: Cycloidal Motion — The First High-Speed Solution

## Concept

Cycloidal motion is a standard profile used specifically for its excellent dynamic properties. Its primary and defining characteristic is achieving **zero acceleration** at the beginning and end of the motion.

## Dynamic Analysis

**The Breakthrough:** Because acceleration starts and ends at zero, the acceleration curve is continuous when transitioning to and from a dwell. A continuous acceleration curve means the jerk ($y'''$) is **finite** throughout the entire cycle.

**Conclusion:** By adhering to the Fundamental Law and its corollary, Cycloidal motion prevents the dynamic shock and vibration common in the previous profiles. It is well-suited for high-speed applications.

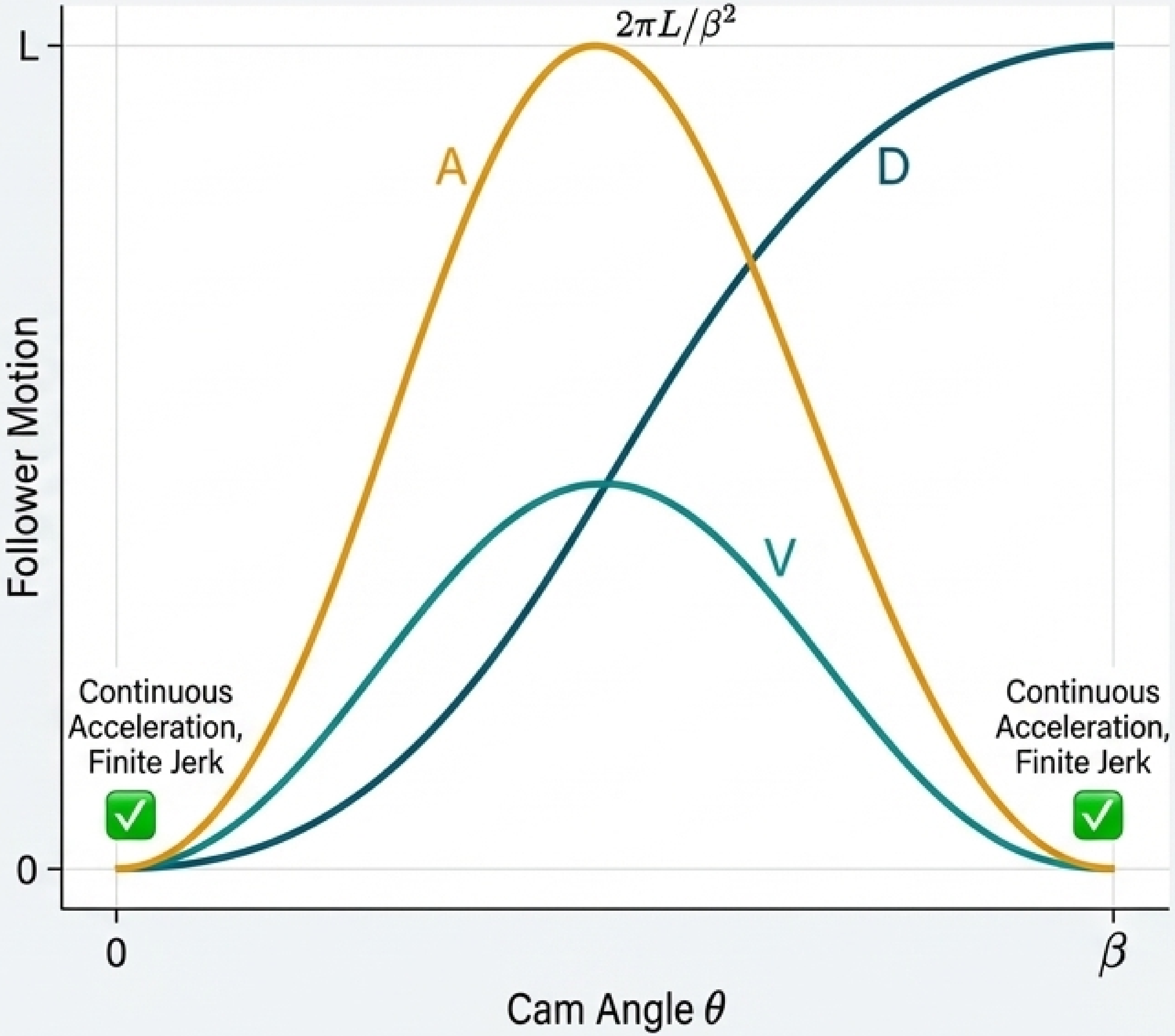
## Mathematical Representation (Rise)

The derivation is complex, but the resulting equations define a profile with a smooth, sinusoidal acceleration curve that starts and ends at zero.

**Displacement (y):** $y = L\left(\frac{\theta}{\beta} - \frac{1}{2\pi}\sin\frac{2\pi\theta}{\beta}\right)$

**Velocity (y′):** $y' = \frac{L}{\beta}\left(1 - \cos\frac{2\pi\theta}{\beta}\right)$

**Acceleration (y″):** $y'' = \frac{2\pi L}{\beta^2}\sin\frac{2\pi\theta}{\beta}$



# Motion Profile 5: 3-4-5 Polynomial — Ultimate Control via Boundary Conditions

## Concept

This 5th-order polynomial provides precise control by defining and satisfying six boundary conditions at the start and end of the transition. This ensures a perfectly smooth blend with adjacent dwell segments.

## Boundary Conditions for a Dwell-Rise-Dwell

To ensure continuity of position, velocity, AND acceleration, we impose six conditions for a rise `$L$` over angle `$\beta$`:

1. $y(0) = 0$ (Start Position)
2. $y'(0) = 0$ (Start Velocity)
3. $y''(0) = 0$ (Start Acceleration)
4. $y(\beta) = L$ (End Position)
5. $y'(\beta) = 0$ (End Velocity)
6. $y''(\beta) = 0$ (End Acceleration)

## Mathematical Representation & Dynamics

Solving a 5th-order polynomial ($y = \sum C_i \theta^i$) for these six conditions yields the "3-4-5" polynomial, as the coefficients for the 0th, 1st, and 2nd powers become zero.

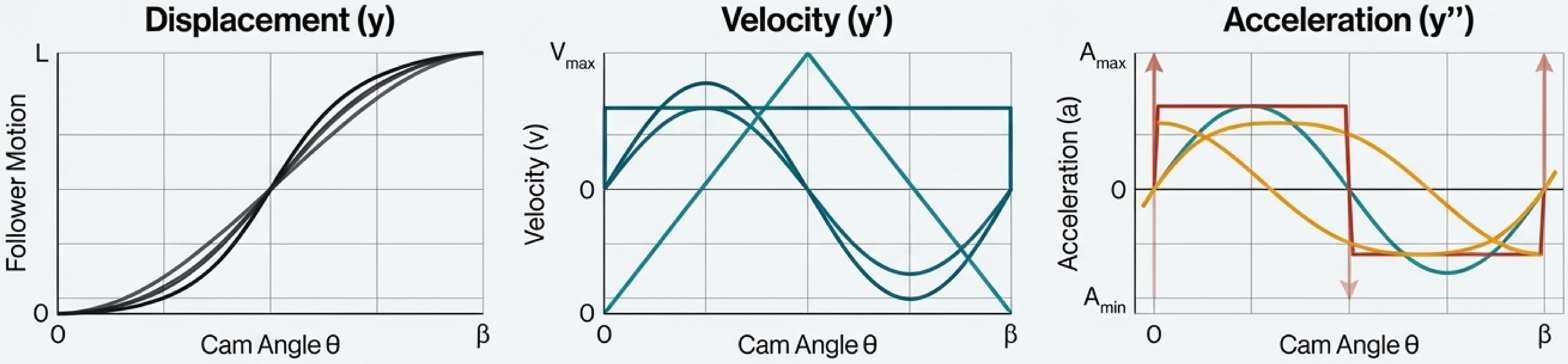
**Displacement (y):** $$y = L\left[10\left(\frac{\theta}{\beta}\right)^3 - 15\left(\frac{\theta}{\beta}\right)^4 + 6\left(\frac{\theta}{\beta}\right)^5\right]$$

## Conclusion

Like Cycloidal motion, this profile provides **zero acceleration** at the transitions, ensuring **finite jerk**. It is considered the practical maximum order for polynomials, as higher orders become overly sensitive to manufacturing imperfections.

# Dynamic Comparison: The Difference is in the Derivatives

**Key Insight** While the displacement curves for all five motions may appear visually similar, their derivatives reveal drastic differences in dynamic performance. A profile's suitability for high-sped applications depends entirely on its **adherence to the Fundamental Law.**



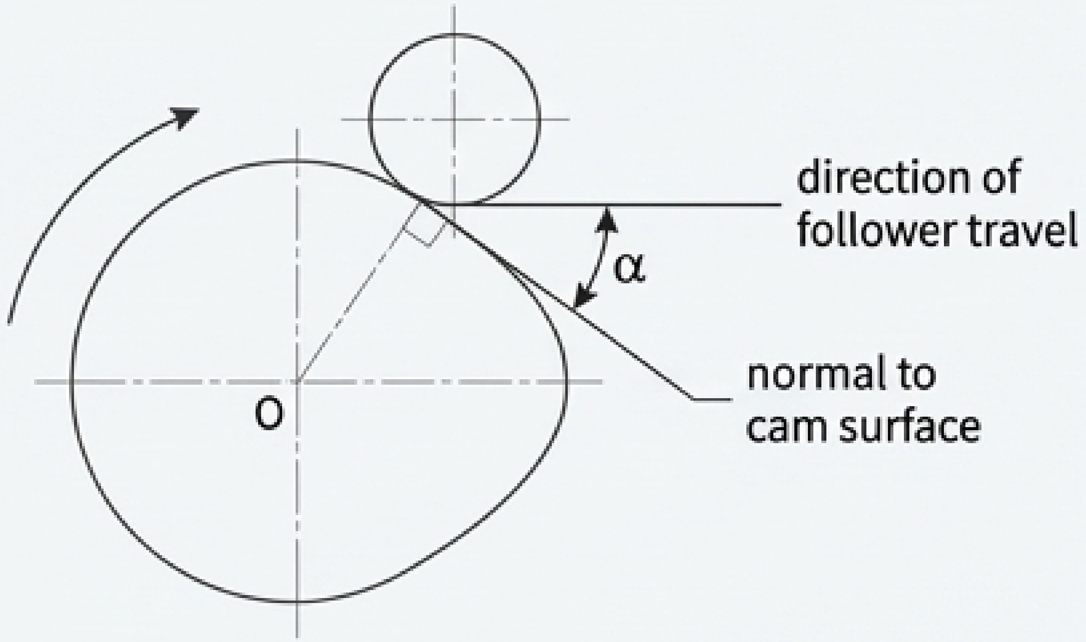

| Motion Profile | Continuous Velocity? | Continuous Acceleration? | Finite Jerk? | Obeys Fundamental Law? |
|---|---|---|---|---|
| **Uniform (Linear)** | ❌ No | ❌ No | ❌ No | ❌ **Violates** |
| **Parabolic** | ✅ Yes | ❌ No | ❌ No | ❌ **Violates** |
| **Simple Harmonic** | ✅ Yes | ❌ No | ❌ No | ❌ **Violates** |
| **Cycloidal** | ✅ Yes | ✅ Yes | ✅ Yes | ✅ **Obeys** |
| **3-4-5 Polynomial** | ✅ Yes | ✅ Yes | ✅ Yes | ✅ **Obeys** |

# From Motion Program to Physical Cam: Geometric Constraints

Synthesizing a smooth motion program (Stage 1) is not enough. The physical geometry of the cam must also satisfy critical constraints to function reliably.

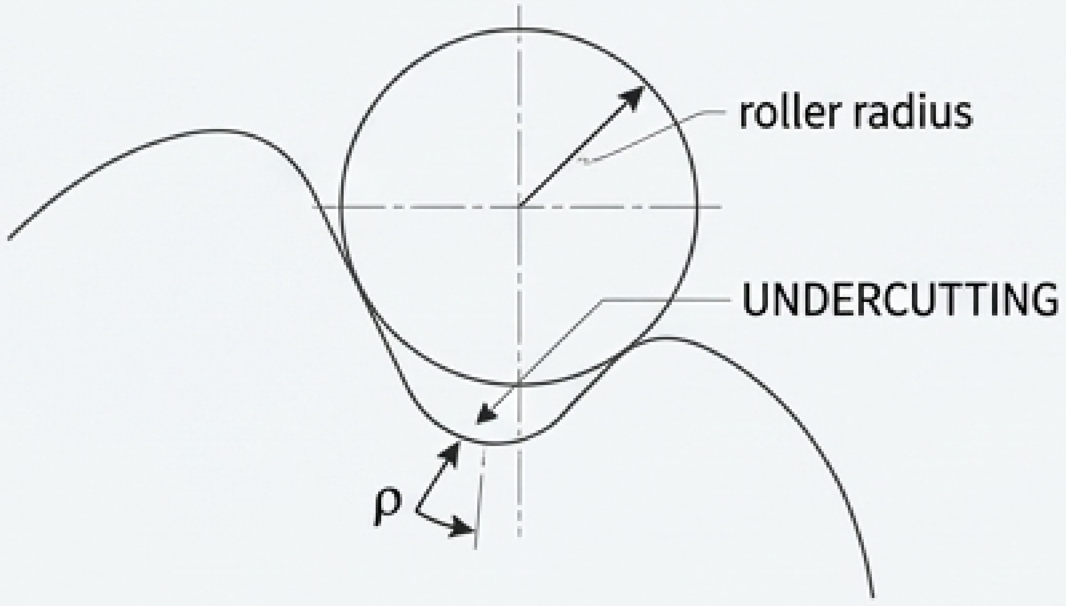
## 1. Pressure Angle (α)

- **Definition:** The angle between the direction of follower motion and the normal to the cam surface at the point of contact.
- **Why it Matters:** High pressure angles (>30°) increase the side-load on the follower, leading to high friction, wear, and potential for the follower to jam in its guide.
- **Design Consideration:** The largest pressure angle typically occurs at the inflection points of the displacement curve (where acceleration is zero). If the angle is too large, the cam's base circle radius must be increased.




## 2. Radius of Curvature (ρ)

- **Definition:** The radius of the cam's curve at a specific point.
- **Why it Matters:** The cam profile must not have a radius of curvature smaller than the roller follower's radius.
    - If $\rho <$ roller radius, the follower cannot trace the intended path, a phenomenon known as 'undercutting'.
    - A cusp ($\rho = 0$) or concave region ($\rho < 0$) creates manufacturing and contact stress problems.
- **Design Consideration:** Increase the cam's base circle radius to ensure the minimum radius of curvature ($\rho_{min}$) is sufficiently large.



# Generating the Cam Profile via Kinematic Inversion

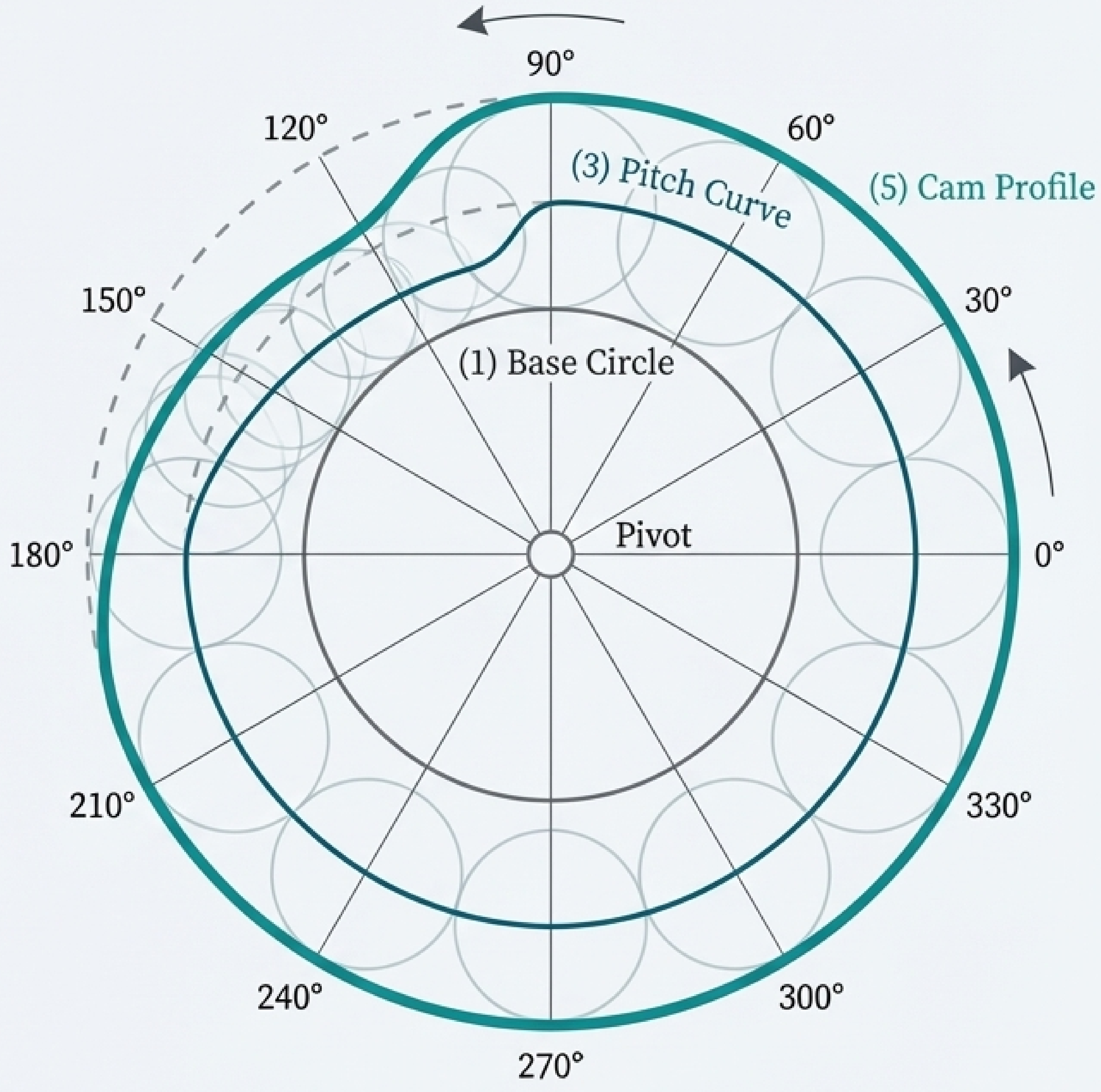
Stage 2 of cam design is generating the physical cam profile from the synthesized displacement data. This is achieved using the principle of of **kinematic inversion**.

## The Process

Instead of rotating the cam, we imagine the **cam is held stationary** and the follower assembly is rotated around it in the opposite direction. The final cam profile is the **envelope curve** that is tangent to all the inverted positions of the follower face.

## Graphical Generation Steps (for a roller follower):

1. **Draw Base & Prime Circles:** The base circle ($r_b$) is the smallest radius of the cam. The prime circle ($r_p = r_b + r_{roller}$) is the path of the roller's center at zero displacement.
2. **Lay off Angles:** Mark angular increments around the prime circle in the direction *opposite* to the cam's intended rotation.
3. **Plot Displacements:** At each angular station, measure the follower displacement $y(\theta)$ radially outward from the prime circle. This path is the "pitch curve".
4. **Draw Follower Circles:** At numerous points along the pitch curve, draw the roller follower's circle.
5. **Draw Envelope:** The final cam profile is the smooth curve drawn tangent to all the individual follower circles.



# Conclusion: From Theory to Reality

## Manufacturing Precision is Paramount

- The theoretical dynamic benefits of advanced profiles like Cycloidal or 3-4-5 Polynomial are only realized if the cam is manufactured with extreme accuracy.
- Small profile variations can re-introduce the shock and vibration the profile was designed to eliminate. This requires high-precision methods like CNC milling and grinding.

## Synthesis Summary

- The choice of a motion program is a trade-off between mathematical simplicity and dynamic performance.
- While simple profiles like Parabolic are easy to define, their inherent jerk makes them unsuitable for anything beyond low-speed applications.
- For high-performance machinery, profiles that obey the **Fundamental Law of Cam Design** are essential for smooth, reliable operation.

## Cam Systems: A Final Perspective

- **Advantages:** Can produce highly specific and complex motion programs; excellent at generating extended dwells, which is difficult for linkages.
- **Disadvantages:** High manufacturing cost for accurate profiles; high contact stresses can lead to wear; susceptible to noise and follower bounce at high speeds if not designed and manufactured correctly.