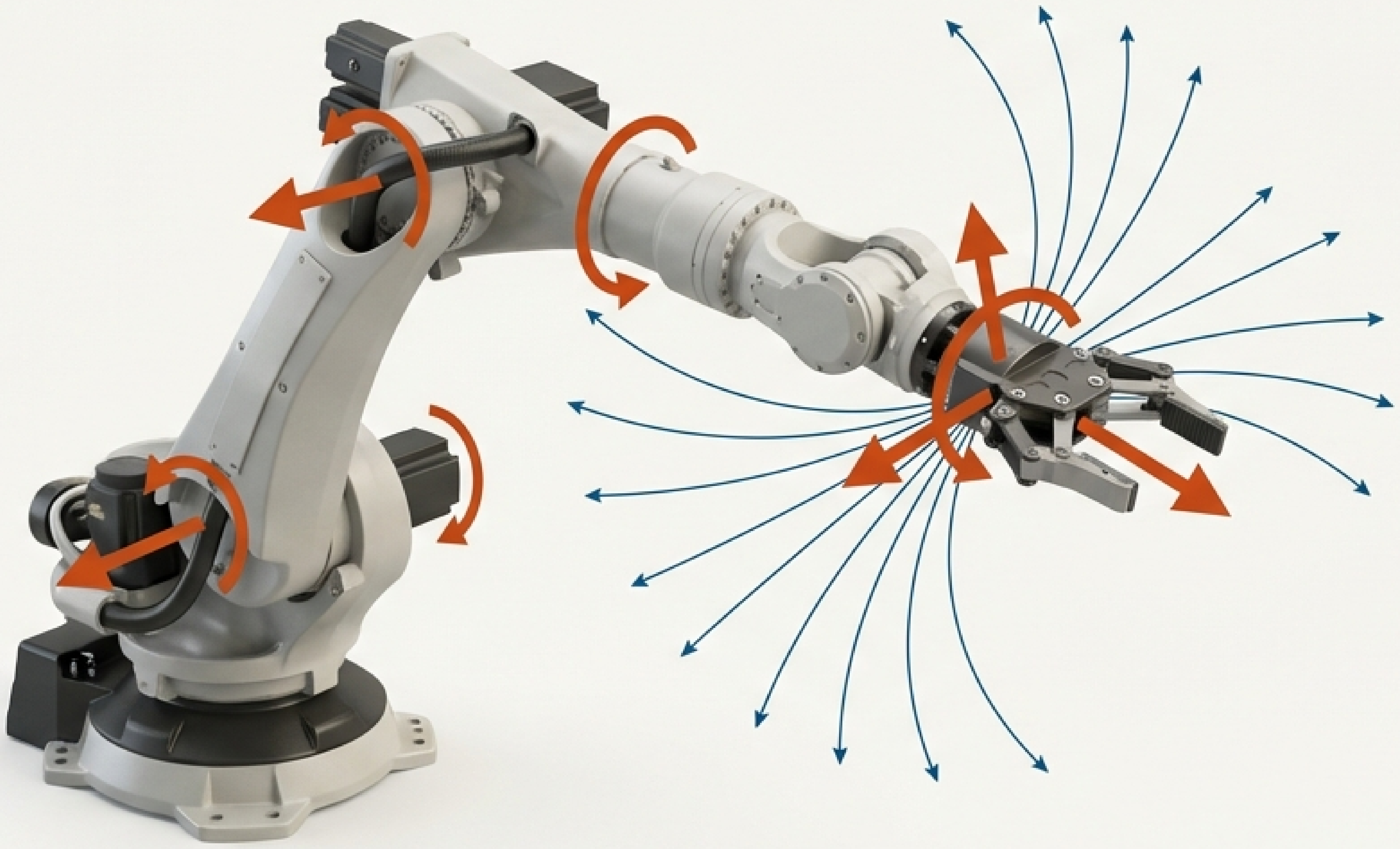
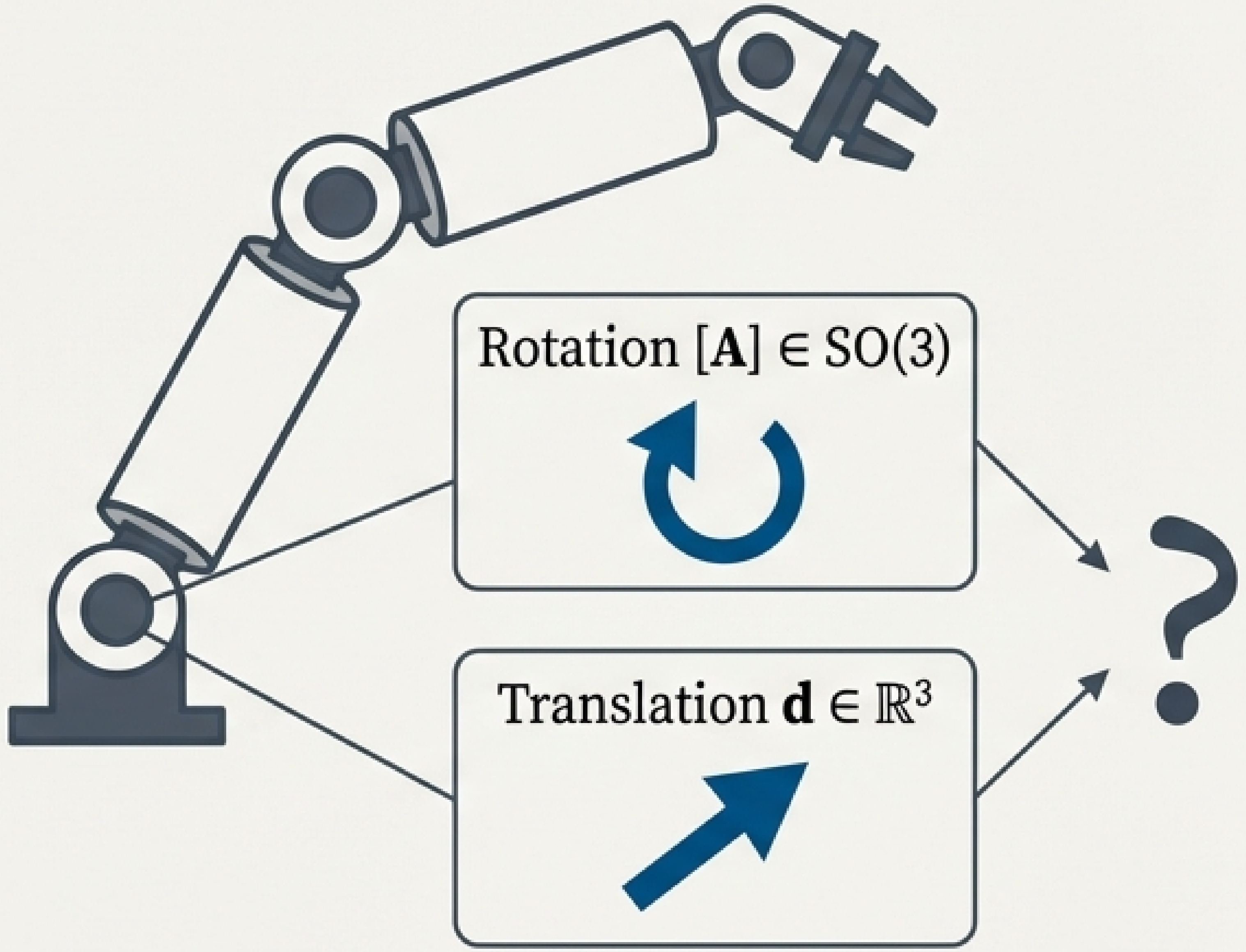
The Unified Language of Motion and Force

A Review of Screw Theory for Robotic Analysis



The Challenge: Describing 3D Motion and Forces

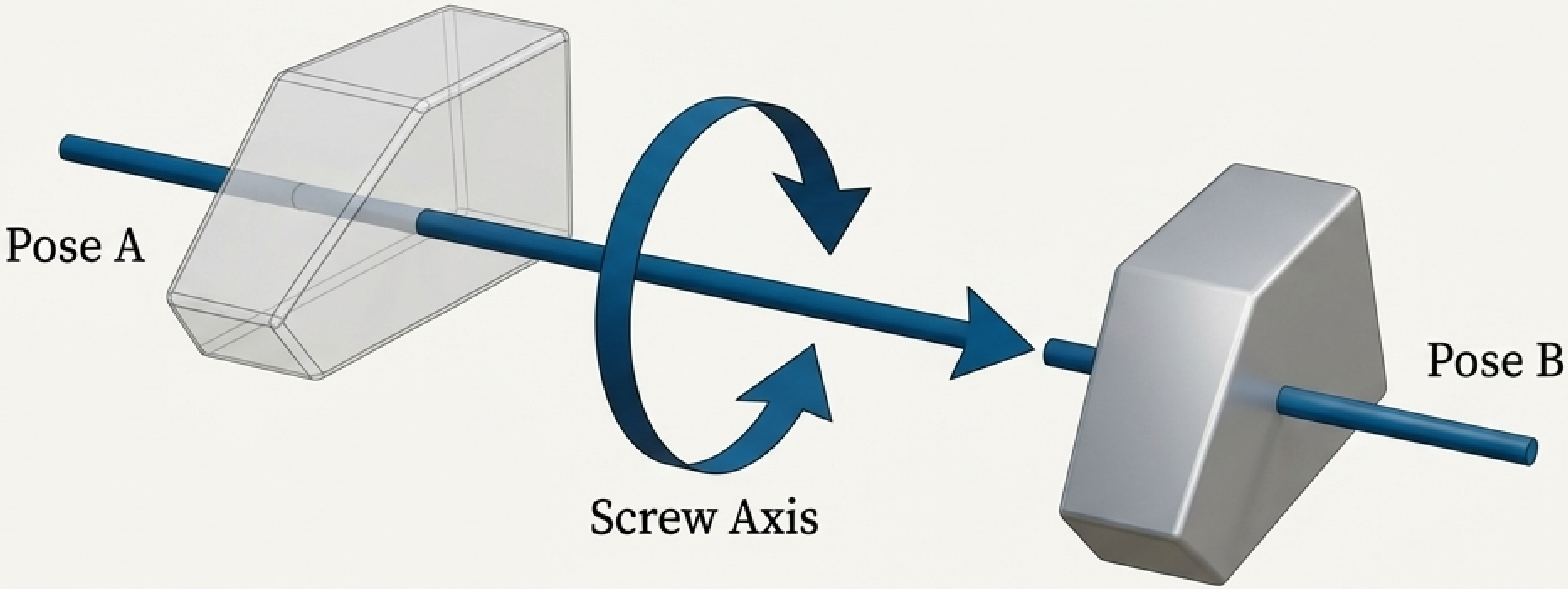
- Analyzing robotic systems requires describing the position, velocity, and forces of each link.
- Traditional methods often treat rotation and translation as separate mathematical entities (e.g., rotation matrices in SO(3) and translation vectors in $\mathbb{R}^3$).
- For complex spatial mechanisms, tracking these separate components through a kinematic chain becomes cumbersome and non-intuitive.



Can we find a single mathematical object that elegantly combines rotation and translation, motion and force?

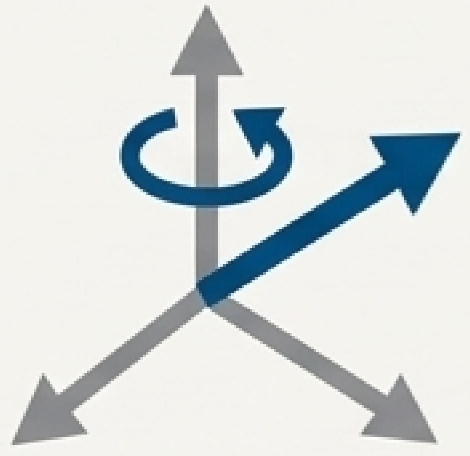
The Unifying Insight: Chasles' Theorem

Any rigid body displacement can be described as a rotation about a unique axis combined with a translation along that same axis.



This combined motion is called a screw motion. It reduces any complex 3D transformation between two poses into a single, intuitive action defined by an axis, a rotation angle, and a slide distance.

The Screw: A Unified 6D Vector for Motion and Force

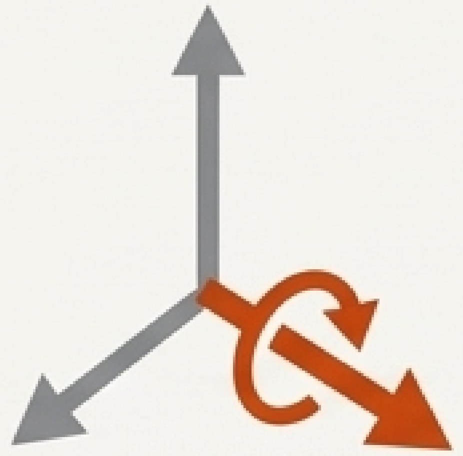


The Twist (Instantaneous Velocity)

Represents the motion of a rigid body.

A 6D vector $\hat{\mathbf{T}} = (\mathbf{\Omega} \mid \mathbf{V})$

- $\mathbf{\Omega}$: The 3D angular velocity vector.
- $\mathbf{V}$: The 3D linear velocity vector of a point at the origin.



The Wrench (Force System)

Represents the forces and moments acting on a body.

A 6D vector $\hat{\mathbf{W}} = (\mathbf{F} \mid \mathbf{M})$

- $\mathbf{F}$: The 3D resultant force vector.
- $\mathbf{M}$: The 3D resultant moment vector about the origin.

Two sides of the same coin: one describes how a body *can move*, the other describes how it is *constrained*.

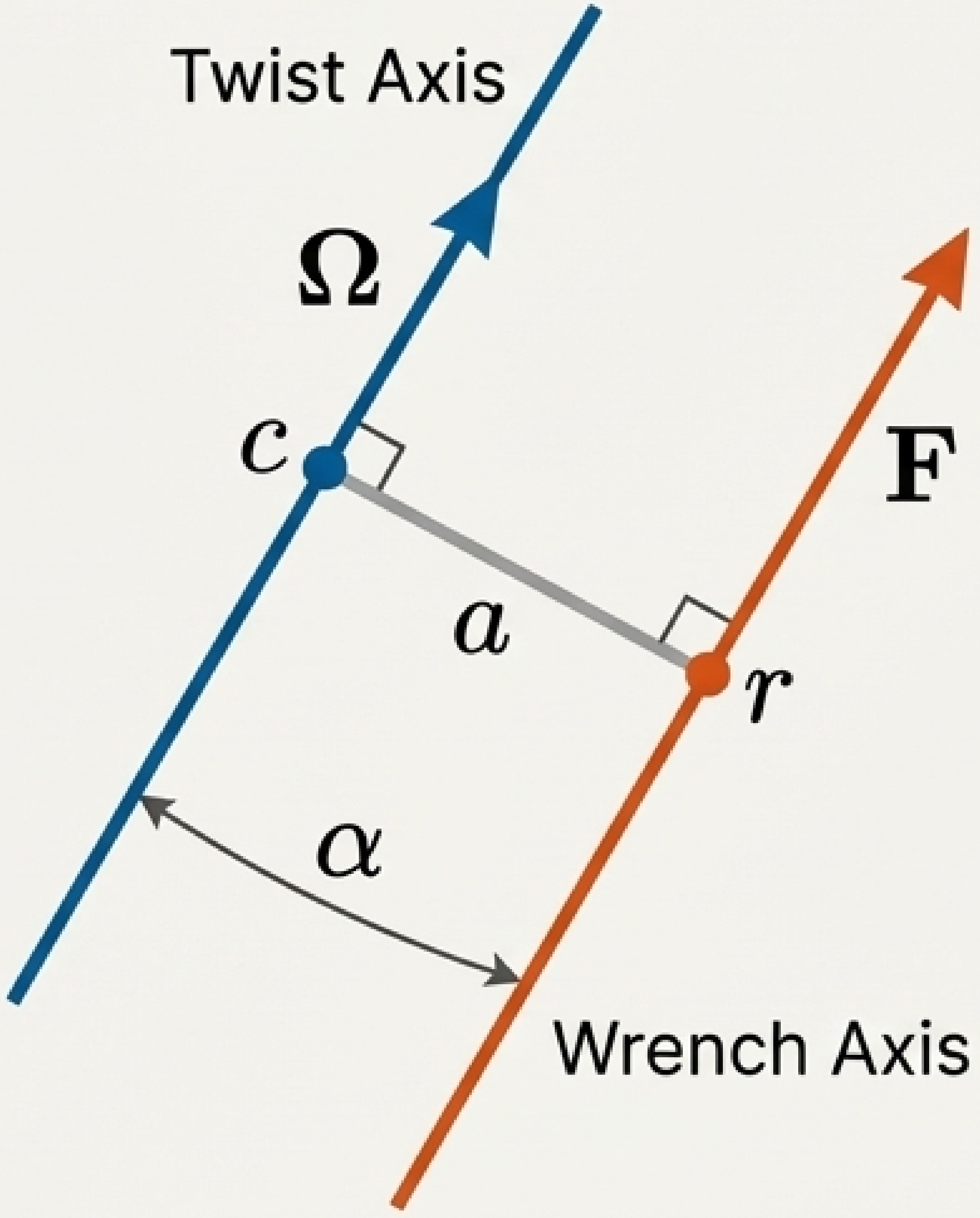
Deconstructing a Screw: Axis and Pitch

A screw is defined by a unique line in space (its axis) and a scalar value (its pitch) that couples the linear and angular components.

Twist:

$$\hat{\mathbf{T}} = (\omega_{\mathbf{s}} \mid \mathbf{c} \times \omega_{\mathbf{s}} + \mathbf{v}_{\mathbf{s}})$$

- **s**: Direction of the twist axis.
- **c**: A point on the twist axis.
- Pitch $p = \frac{v}{\omega}$: The ratio of linear to angular velocity along the axis.

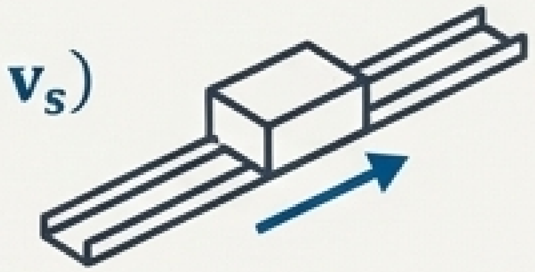
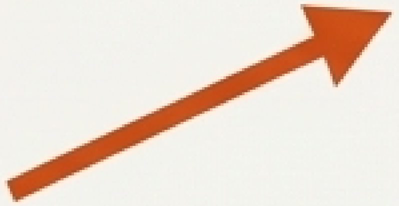
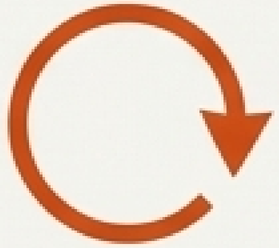


Wrench:

$$\hat{\mathbf{W}} = (\mathbf{f}_{\mathbf{u}} \mid \mathbf{r} \times \mathbf{f}_{\mathbf{u}} + \mathbf{m}_{\mathbf{u}})$$

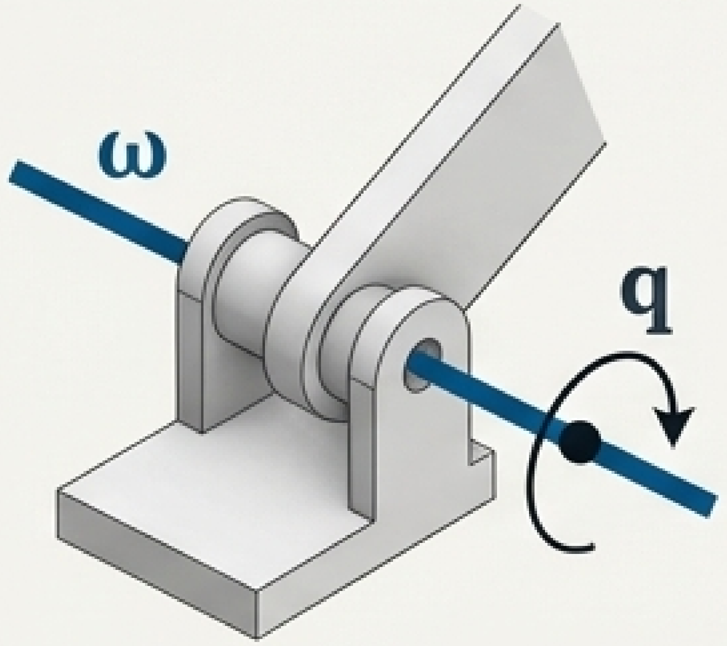
- **u**: Direction of the wrench axis.
- **r**: A point on the wrench axis.
- Pitch $q = \frac{m}{f}$: The ratio of moment to force along the axis.

The Spectrum of Screws Defined by Pitch

	Zero Pitch	Finite Pitch	Infinite Pitch
Twist (Motion)	**Zero Pitch ($p = 0$): Pure Rotation** The motion of a revolute joint. $\hat{T} = (\omega_s \mid c \times \omega_s)$ 	**Finite Pitch ($0 < p < \infty$): General Screw Motion** A helical or “corkscrew” motion. 	**Infinite Pitch ($p = \infty$): Pure Translation** The motion of a prismatic joint. $\hat{T} = (0 \mid \mathbf{v}_s)$ 
Wrench (Force/Constraint)	**Zero Pitch ($q = 0$): Pure Force** A force acting along a line. 	**Finite Pitch ($0 < q < \infty$): General Wrench** A force combined with a coupled moment. 	**Infinite Pitch ($q = \infty$): Pure Couple** A pure moment, with no net force. 

Representing Robot Joints as Twists

Revolute Joint (Rotation)

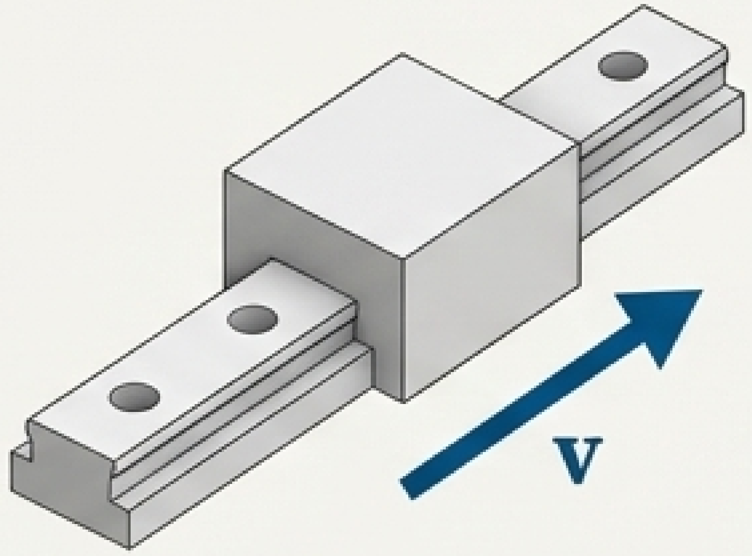


Allows pure rotation about an axis. It has zero pitch.

If the axis of rotation has direction $\boldsymbol{\omega}$ and passes through point $\mathbf{q}$, the joint twist ξ is:

$$\xi = (\boldsymbol{\omega} \mid \mathbf{q} \times \boldsymbol{\omega})$$

Prismatic Joint (Translation)



Allows pure translation along an axis. It has infinite pitch.

If the direction of translation is given by the vector $\mathbf{v}$, the joint twist ξ is:

$$\xi = (\mathbf{0} \mid \mathbf{v})$$

The Algebra of Interaction: Work and Reciprocity

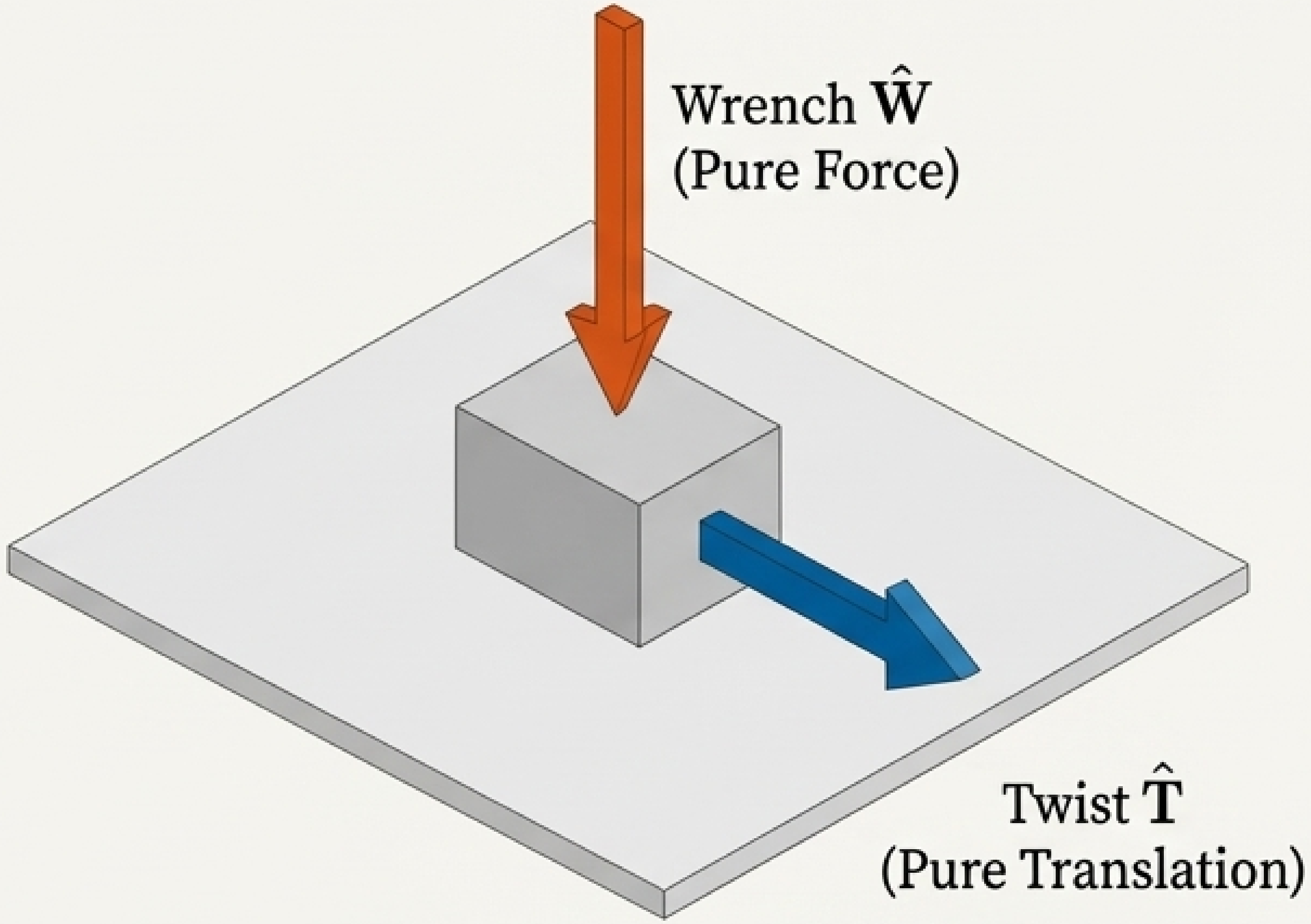
The interaction between a wrench and a twist is quantified by their reciprocal product, which represents virtual power.

$$\hat{\mathbf{T}} \circ \hat{\mathbf{W}} = \mathbf{F} \cdot \mathbf{V} + \mathbf{M} \cdot \boldsymbol{\Omega}$$

The Fundamental Principle of Reciprocity

When the reciprocal product is zero ($\hat{\mathbf{T}} \circ \hat{\mathbf{W}} = 0$), the wrench and twist are **reciprocal.**

- **Physical Interpretation:** This means the wrench (constraint) does **no work** on the body undergoing the motion defined by the twist.
- **In short:** A twist is reciprocal to a wrench if the wrench represents a constraint that allows the motion. This elegantly defines the relationship between a system's freedoms and its constraints.



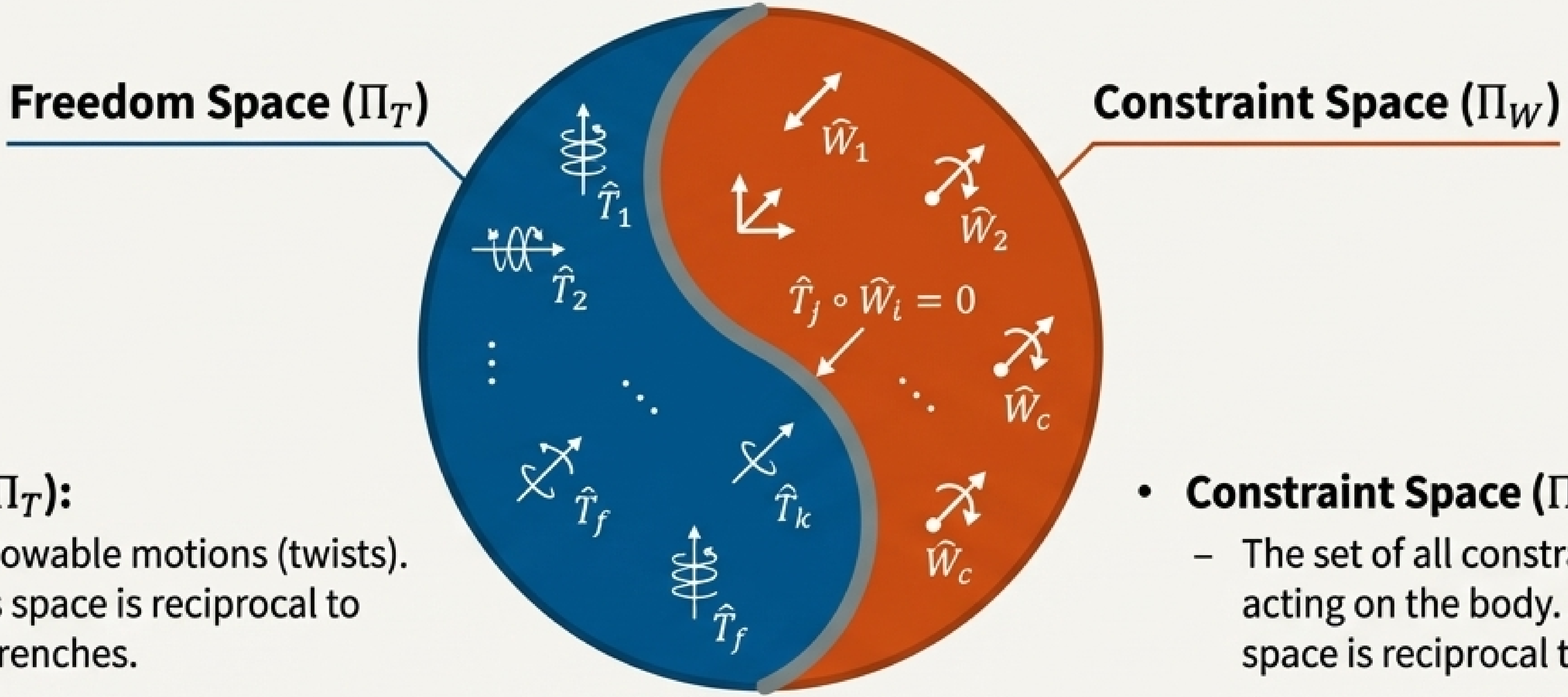
$$\hat{\mathbf{T}} \circ \hat{\mathbf{W}} = 0$$

The force is perpendicular to the motion, therefore no work is done.

Freedom and Constraint: The Rule of Complementary Patterns

For a non-redundant system in 3D space, the number of degrees-of-freedom (f) plus the number of degrees-of-constraint (c) is always six.

$$f + c = 6$$



- **Freedom Space (Π_T):**
  - The set of all allowable motions (twists). Any twist in this space is reciprocal to *all* constraint wrenches.

- **Constraint Space (Π_W):**
  - The set of all constraints (wrenches) acting on the body. Any wrench in this space is reciprocal to *all* freedom twists.

A system's kinematics (its freedom to move) and its statics (the constraints upon it) are mathematically complementary spaces.

Application I: Forward Velocity Kinematics

The Goal

Determine the velocity of the robot's end-effector given the speeds of its joints.

The Screw Theory Approach

1. Represent each joint's motion as a twist, ξ_i.
2. Express all joint twists in a common frame (e.g., the base frame).
3. The end-effector twist (T_{ee}) is a linear combination of the joint twists, where the coefficients are the joint velocities ($\dot{q}_i$).

The Result: The Geometric Jacobian

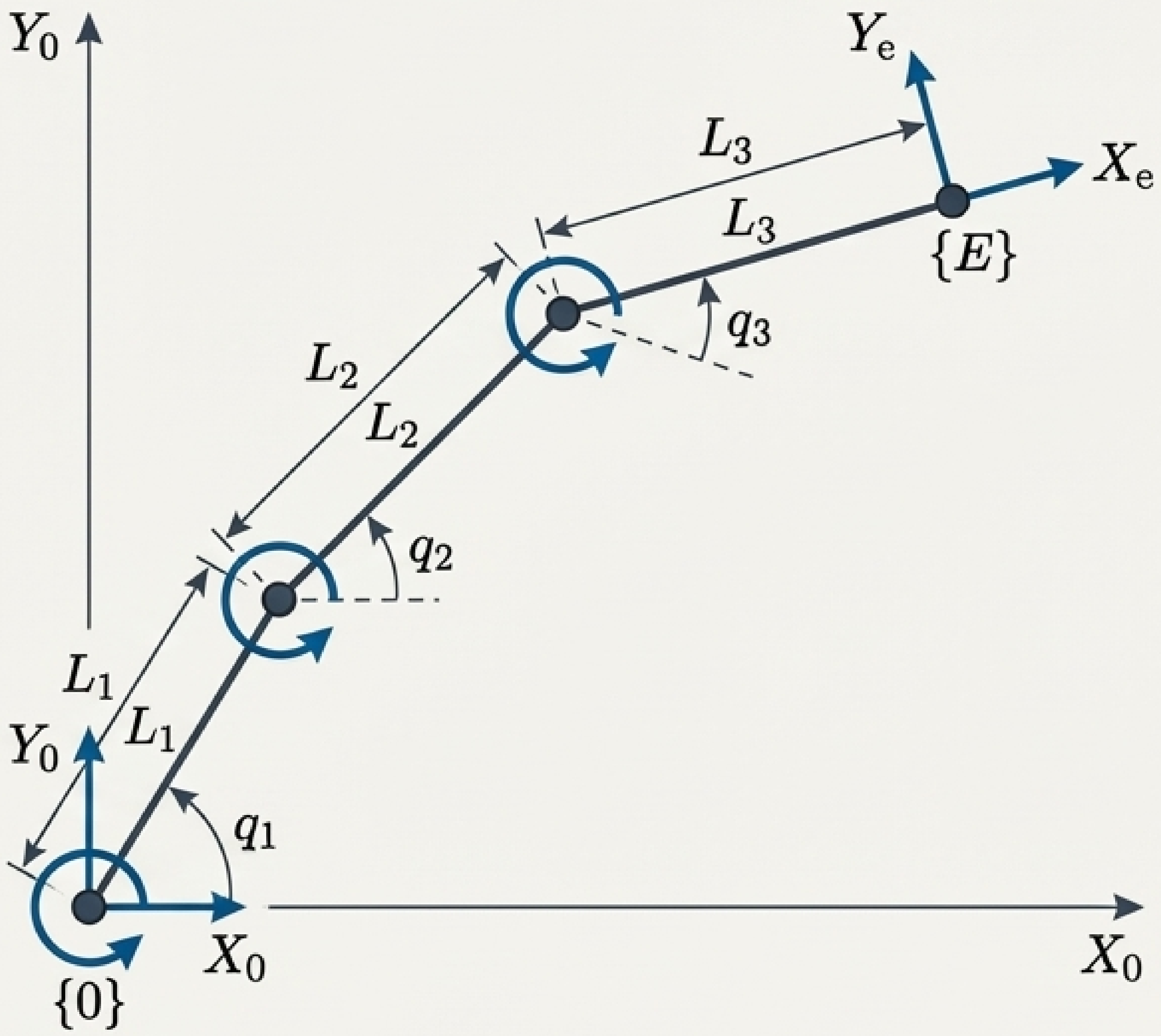
$$T_{\text{ee}} = J(\mathbf{q}) * \dot{\mathbf{q}}$$

$$J \left[\begin{array}{c|c|c|c} \xi_1 & \xi_2 & & \xi_n \\ \vdots & \vdots & \cdots & \vdots \\ \xi_1 & \xi_2 & & \xi_n \end{array} \right]$$

$$J = [\, \xi_1 \mid \xi_2 \mid \ldots \mid \xi_n \,]$$

This provides a powerful geometric interpretation of the Jacobian, which is often derived through more complex algebraic methods like using Denavit-Hartenberg parameters.

Example: Velocity Analysis of a Planar 3R Robot



1. Define Joint Twists

Each joint i is a revolute joint rotating about the Z-axis. Its twist is $\xi_i = (\omega \mid q_i \times \omega)$, where $\omega = [0\ 0\ 1]^T$.

2. Transform to Base Frame

Use coordinate transformations to express each ξ_i in the base frame $\{0\}$. The position vector q_i for each joint will depend on the preceding joint angles.

3. Assemble Jacobian

The 3x3 Jacobian for this planar case is constructed column by column:

$J = [\ \xi_1|_0 \mid \xi_2|_0 \mid \xi_3|_0\ ]$.

4. Calculate Velocity

The end-effector's linear and angular velocity in the plane is given by

$V_{ee} = J * [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T$.

Tool Spotlight: Transforming Screws Between Frames

The Need

To build the Jacobian, we must express all joint twists in a single common frame.

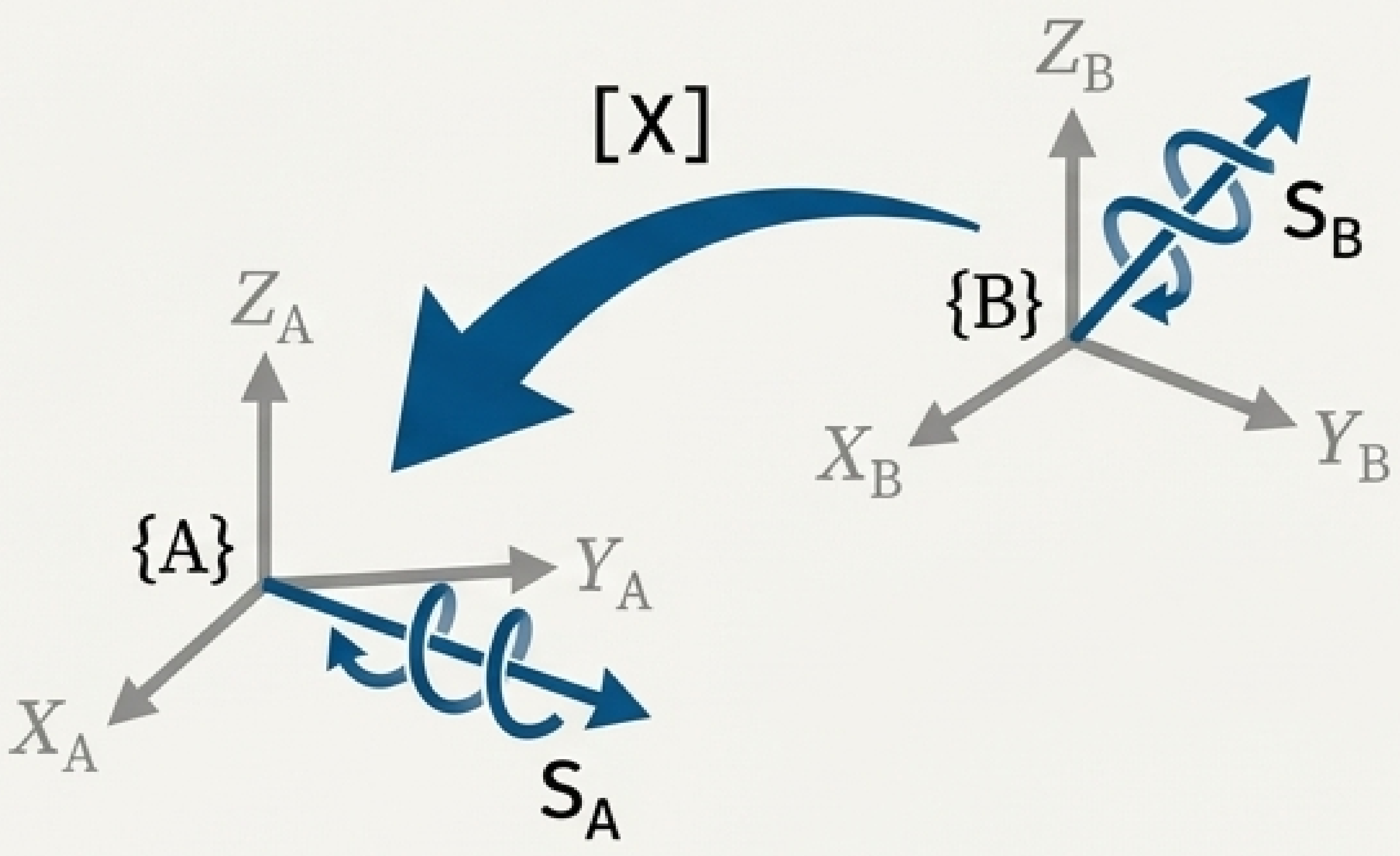
The Method

A spatial displacement from frame {A} to {B}, defined by rotation **[A]** and translation **d**, corresponds to a 6x6 transformation matrix **[X]**.

Transformation Matrix

$$[X] = \begin{bmatrix} [A] & 0 \\ [D][A] & [A] \end{bmatrix}$$

Where [D] is the 3x3 skew-symmetric matrix corresponding to the translation vector **d**.



Application

A screw S_B in frame {B} is transformed to S_A in frame {A} by the matrix multiplication:

$$S_A = [X] * S_B$$

Scaling Up: The Spatial 6R Robot

The Principle is Identical

The forward velocity kinematics for a general 6-DOF spatial robot follows the exact same logic.

The Geometric Jacobian

- The end-effector twist is still the sum of the joint twists:

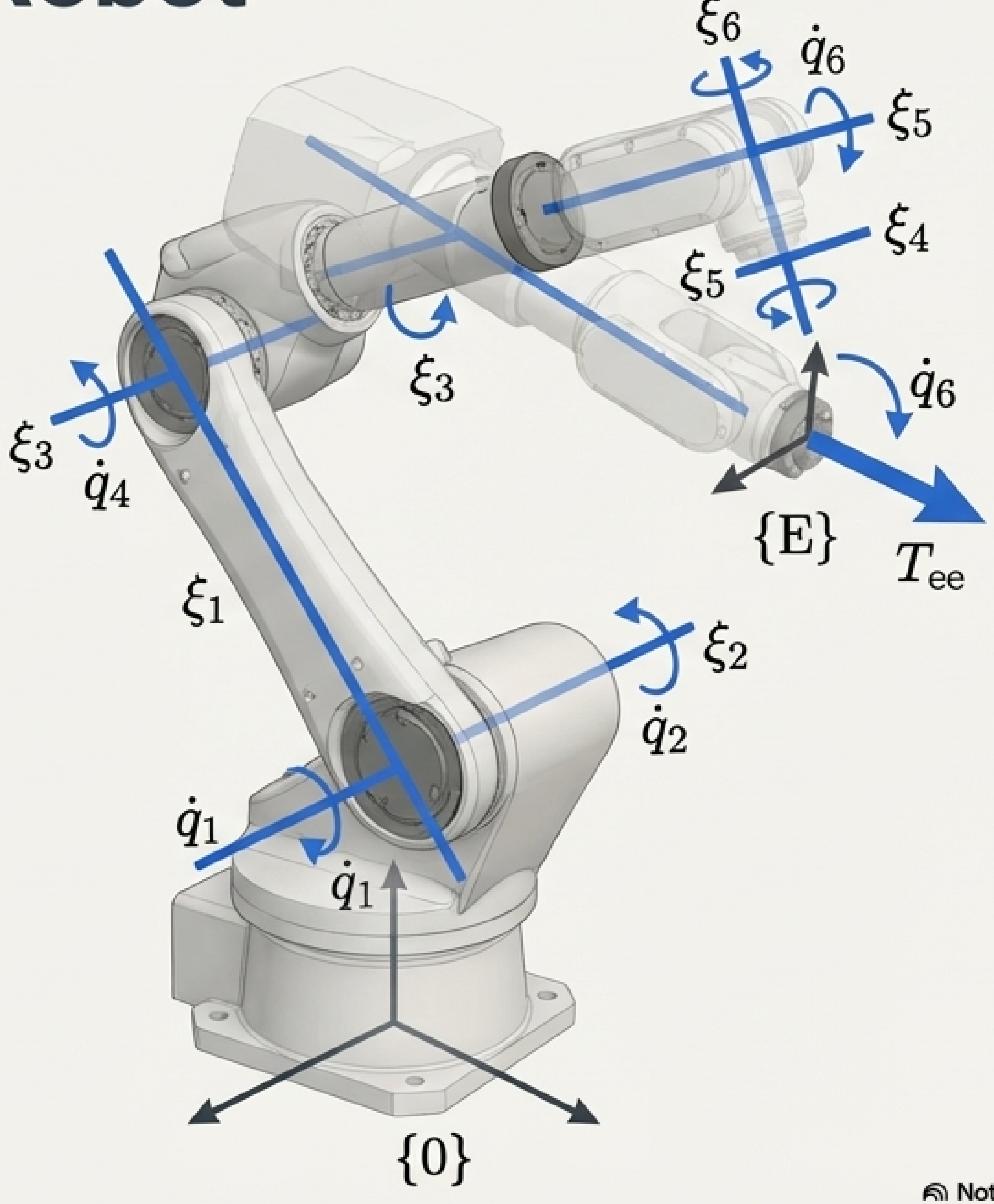
$$T_{\text{ee}} = J \cdot \dot{q}$$

- The 6x6 Jacobian matrix is still formed by the twists of the six joint axes, expressed in the base frame:

$$J = [\xi_1 \mid \xi_2 \mid \xi_3 \mid \xi_4 \mid \xi_5 \mid \xi_6]$$

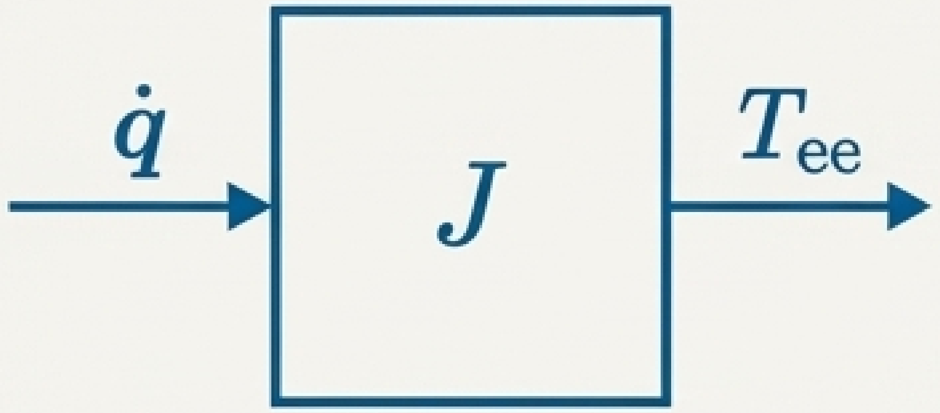
The Advantage

This method naturally handles any combination of revolute and prismatic joints and any arbitrary spatial orientation of their axes, bypassing many of the edge cases and complexities found in other formalisms like Denavit-Hartenberg.



The Duality Payoff: Static Force Analysis

Kinematics



Joint Velocities to
End-Effector Twist

The Goal

Determine the joint torques ($\boldsymbol{\tau}$) needed to balance an external wrench ($\mathbf{W}_{\text{ee}}$) applied at the end-effector.

The Link

The relationship is derived from the principle of virtual work, which states that the power generated by the joint motors must equal the power exerted by the external wrench.

Power Equations

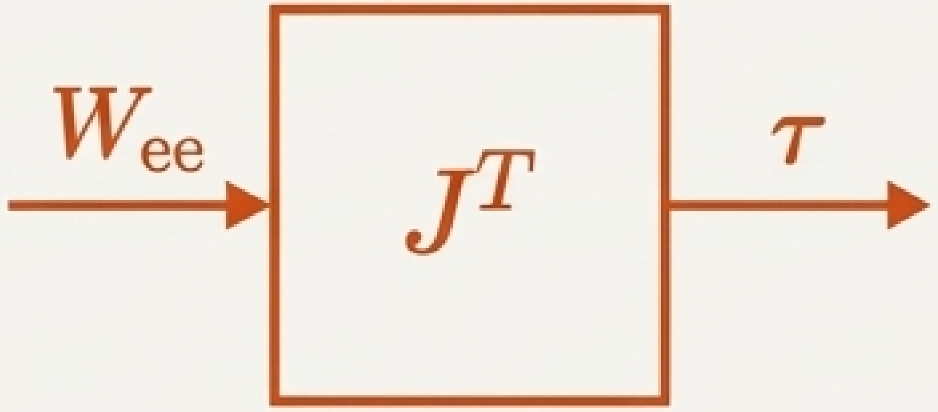
$$\text{Power}_{\text{in}} = \boldsymbol{\tau}^T * \dot{\boldsymbol{q}}$$

$$\text{Power}_{\text{out}} = {W_{\text{ee}}}^T * T_{\text{ee}} = {W_{\text{ee}}}^T * (J * \dot{\boldsymbol{q}})$$

By equating $\text{Power}_{\text{in}} = \text{Power}_{\text{out}}$, we arrive at the fundamental equation for statics:

$$\boldsymbol{\tau} = J^T * W_{\text{ee}}$$

Statics



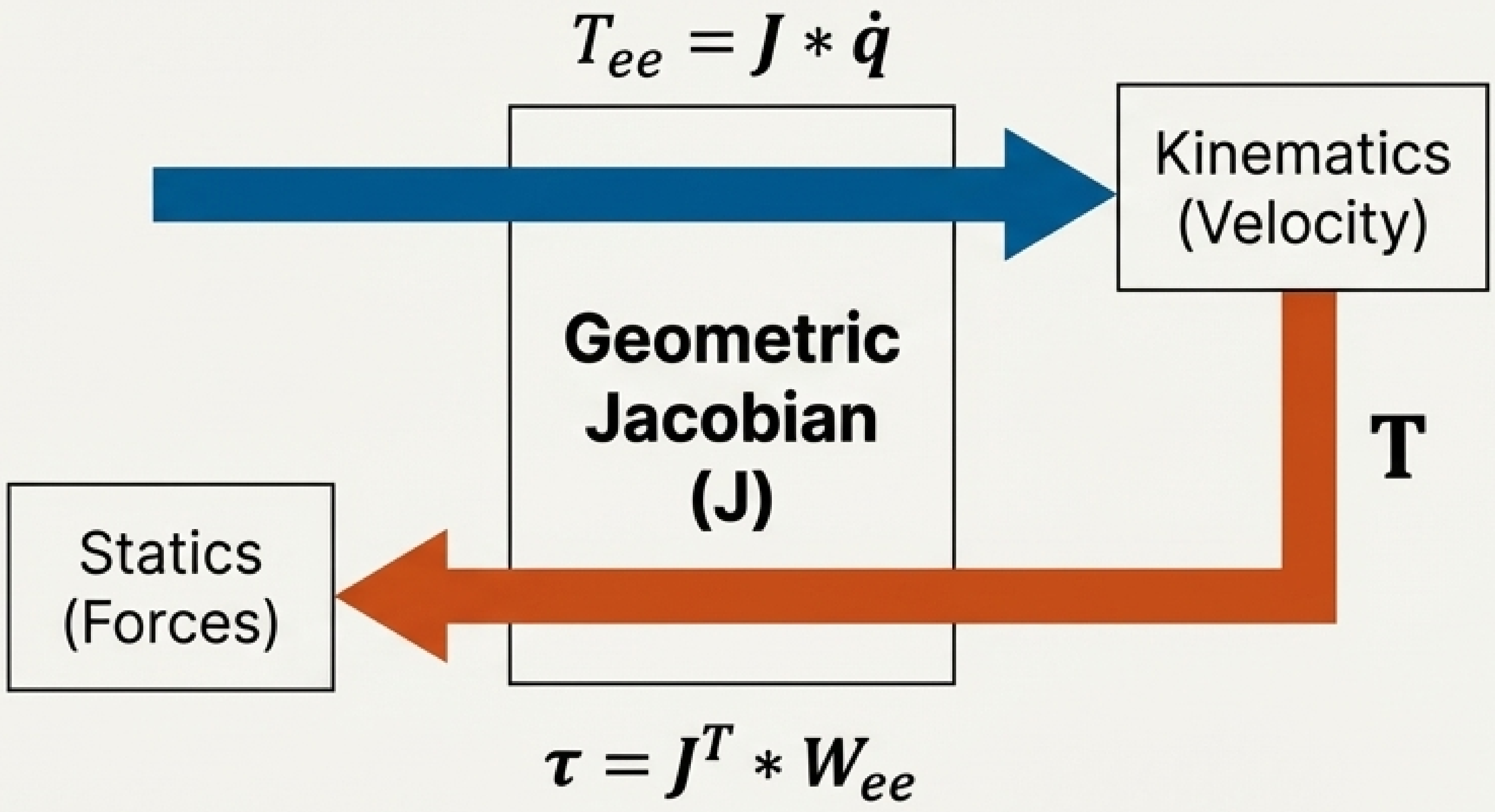
End-Effector Wrench
to Joint Torques

The **transpose of the geometric Jacobian** directly maps end-effector wrenches to the required joint torques. The same matrix governs both velocity and force relationships.

Screw Theory: The Unified Framework

Summary advantages:

- **A Single Language**
  Provides one consistent mathematical framework for both instantaneous kinematics (velocity) and statics (forces).
- **Powerful Geometric Intuition**
  Treats robot joints and end-effector motion as clear geometric entities (screws) in space, rather than abstract algebraic quantities.
- **Elegant Duality**
  The deep connection between motion and force is made explicit through the Jacobian and its transpose.



NotebookLM