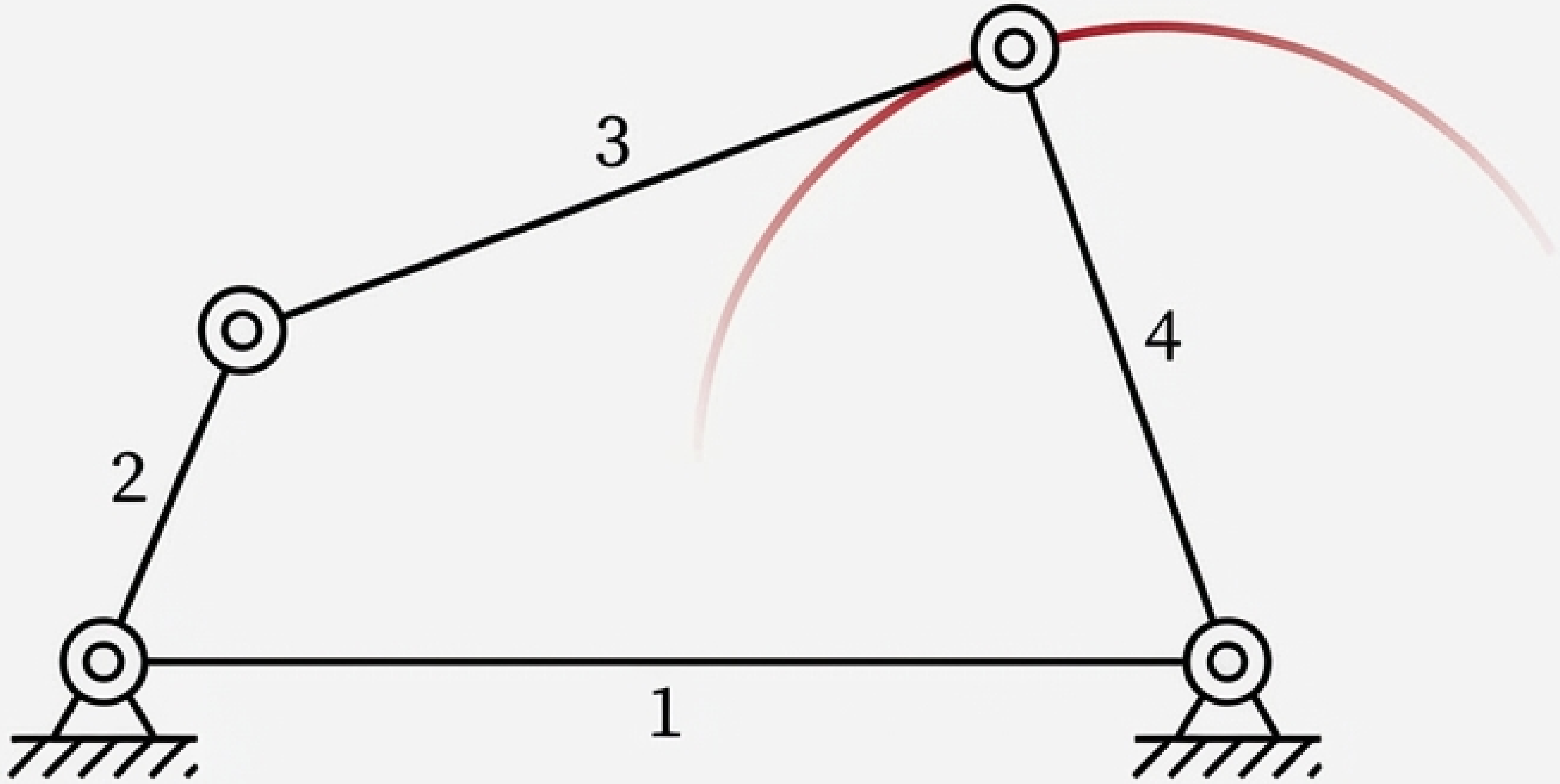
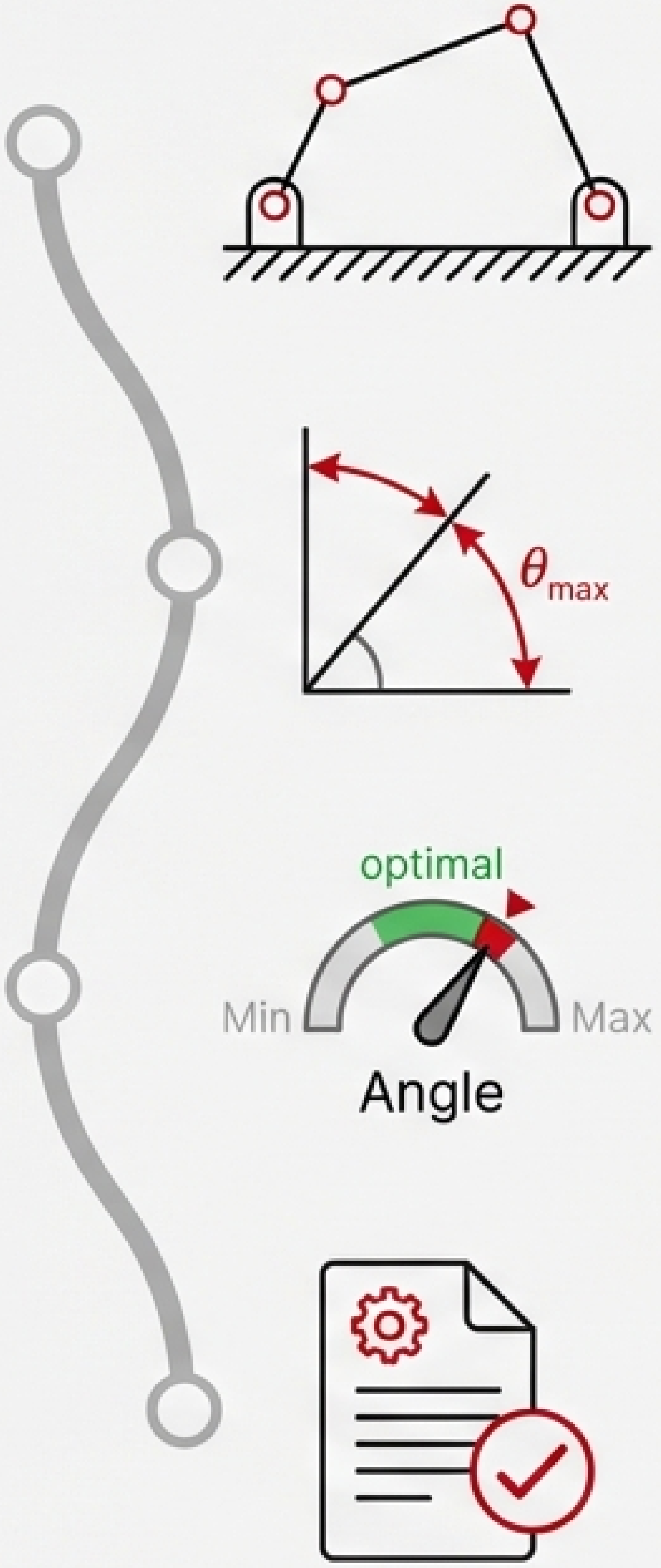
ME 3751: Limit Analysis of Crank-Rocker Linkages

Understanding a Mechanism's Motion and Performance



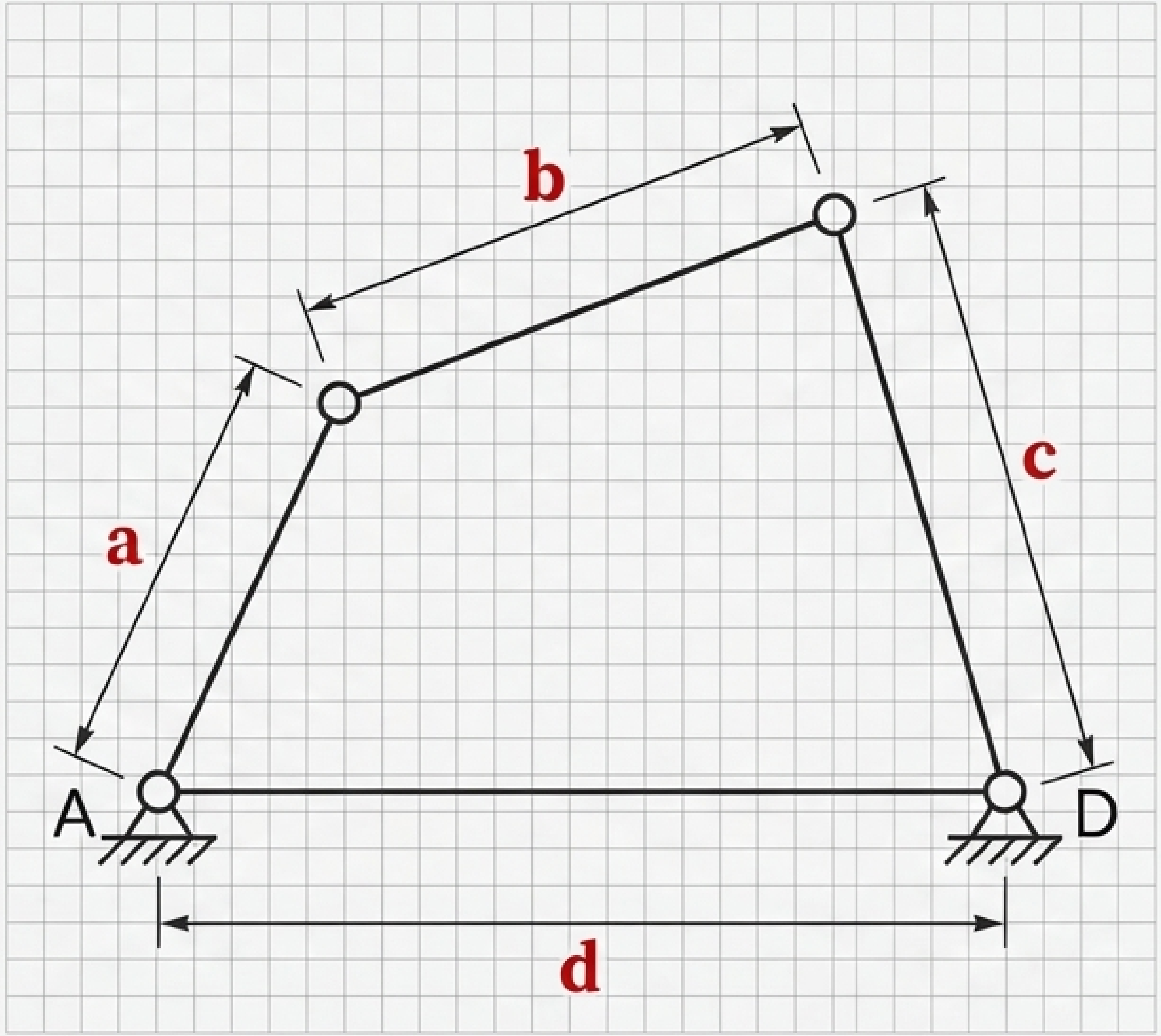
Our Roadmap: From Motion Limits to Design Quality

1. **Re-Orient**: What are the key components of a 4-bar linkage?
2. **Limit Analysis**: How far can the output link rock back and forth? (The geometry and the math).
3. **Transmission Angle**: How effectively is motion transmitted through the linkage?
   (The definition and its importance).
4. **Synthesis**: How do we apply these concepts to solve a practical problem?



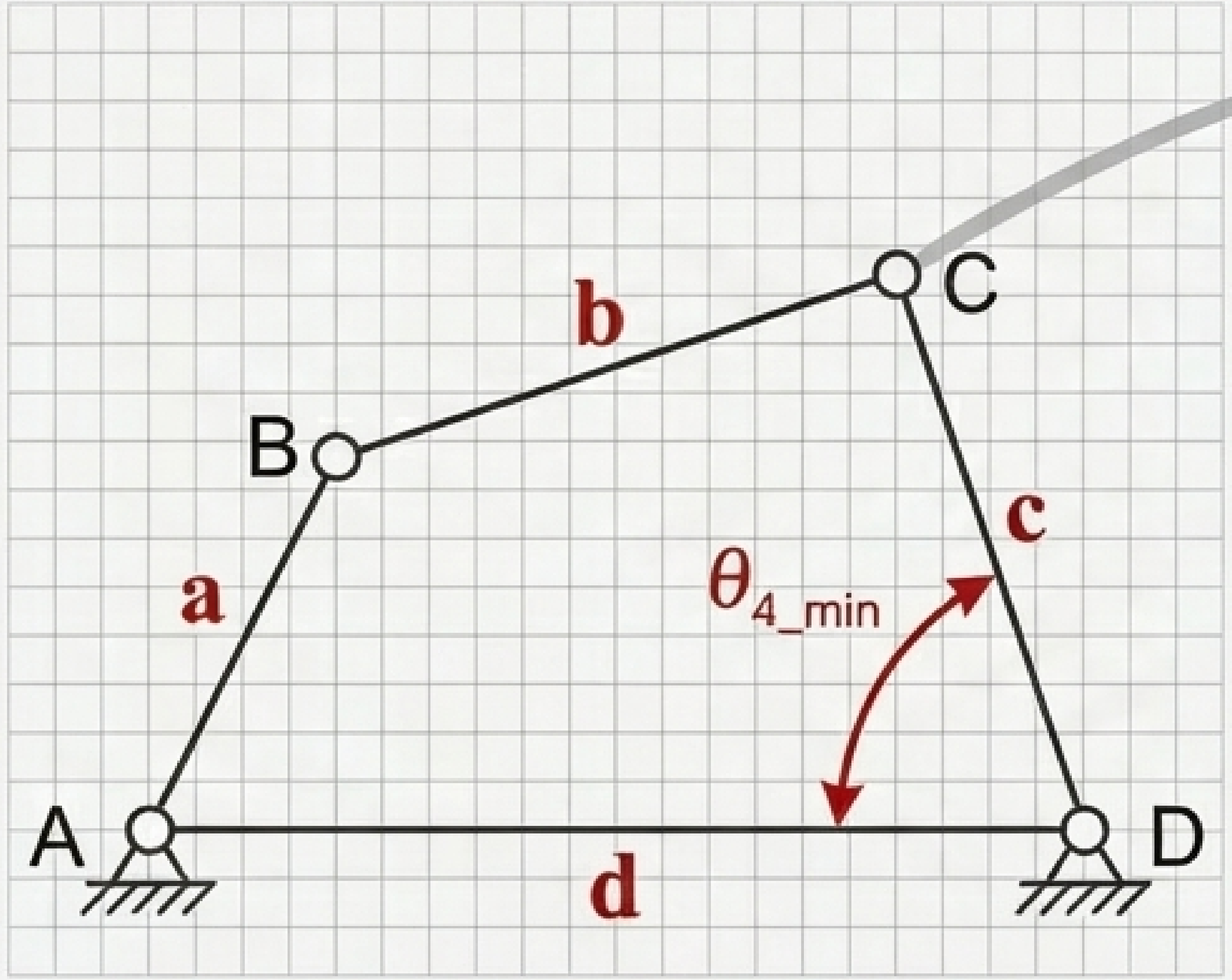
The Anatomy of a Four-Bar Linkage

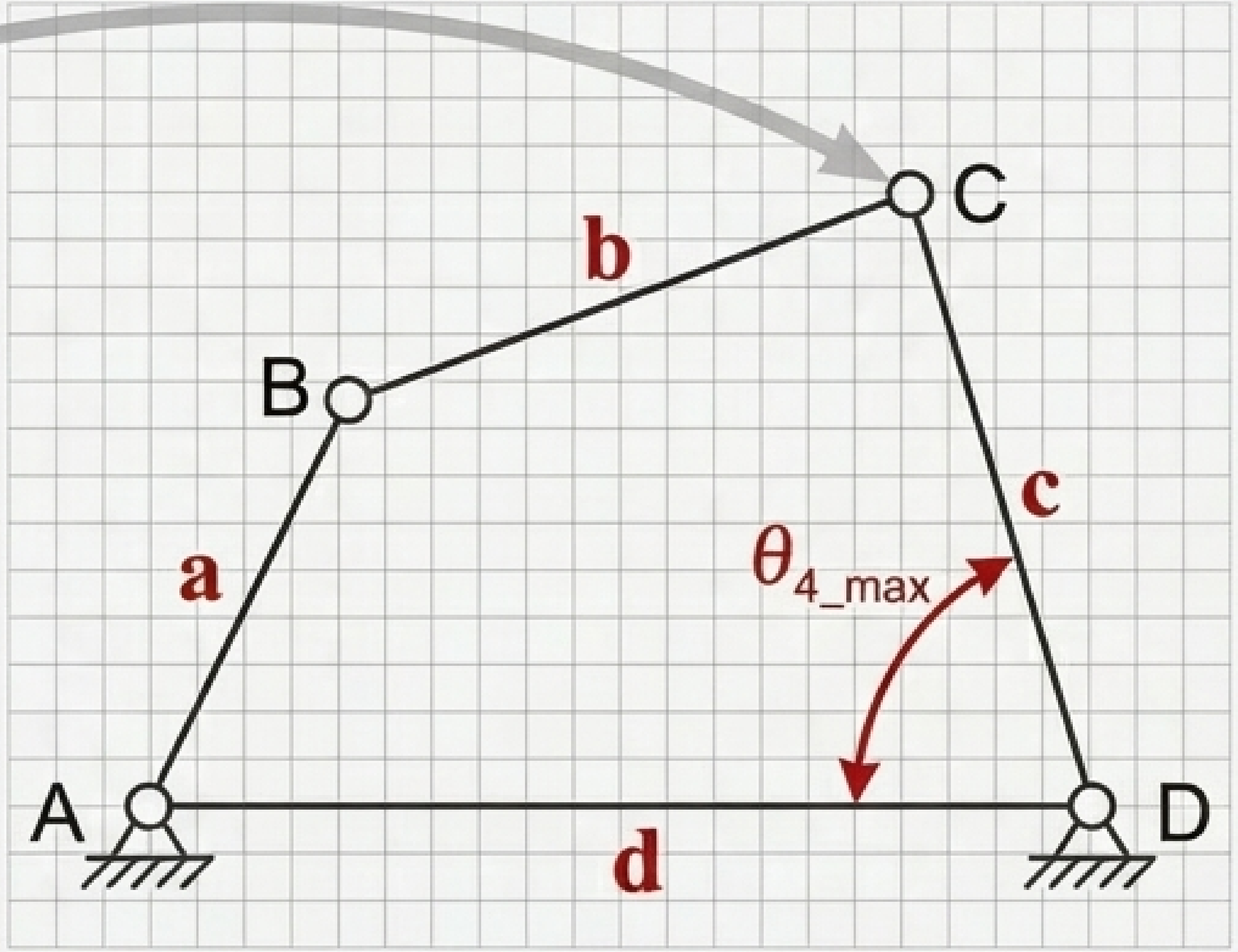
- **Ground (d):** The fixed frame of reference.
- **Crank (a):** The input link, capable of 360° rotation.
- **Coupler (b):** Connects the crank and the rocker.
- **Rocker (c):** The output link, which oscillates. (Also called the Follower).



What Defines the Rocker's Range of Motion?

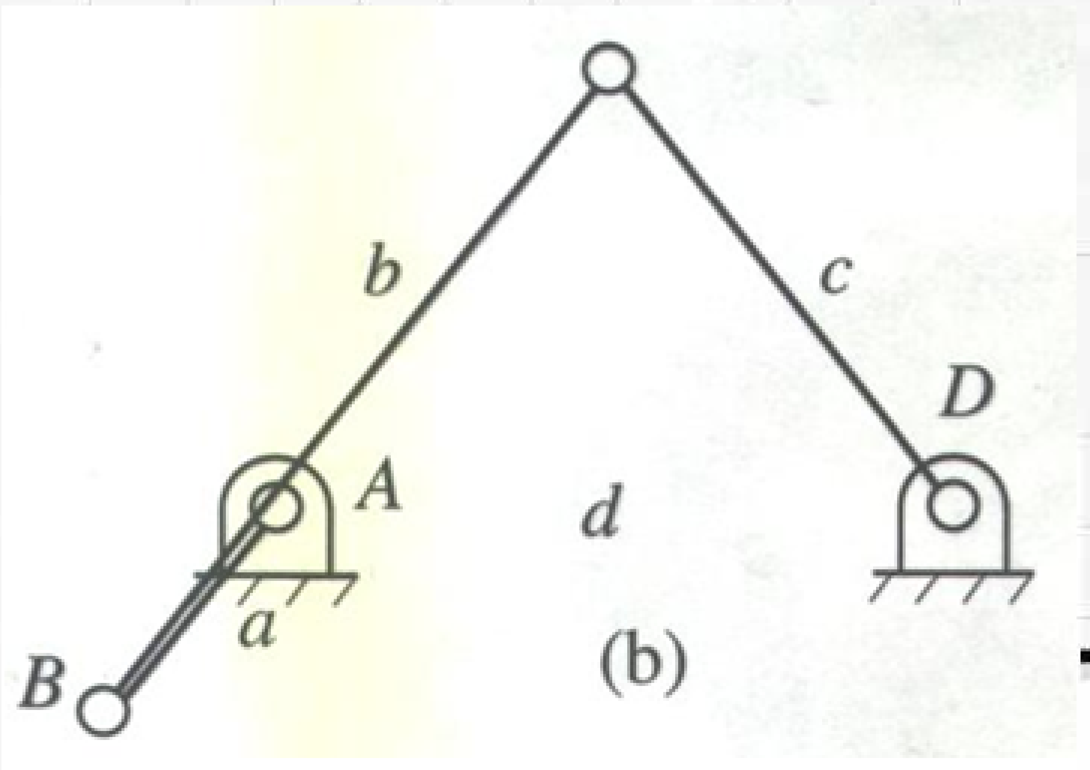
The rocker arm oscillates between two extreme positions. These are called the **limit positions**. The total angle between them defines the mechanism's range of motion.





The Limit Positions Occur When Two Links Align

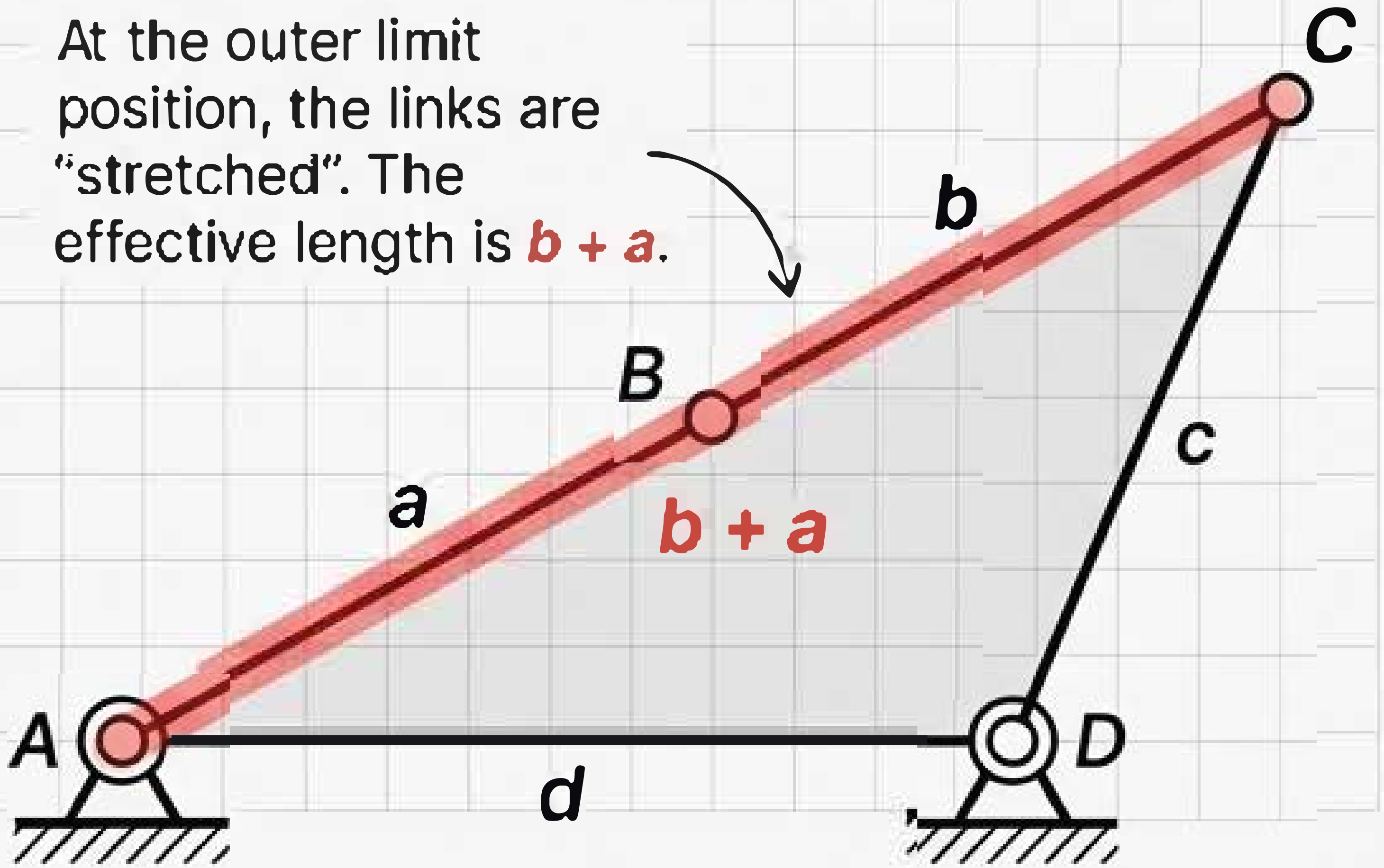
The limits of motion are reached when the crank (*a*) and the coupler (*b*) become **collinear**.



At the inner limit position, the links are "folded". The effective length is *b - a*.

Folded

At the outer limit position, the links are "stretched". The effective length is *b + a*.



Stretched

From Geometry to Algebra: The Vector Loop

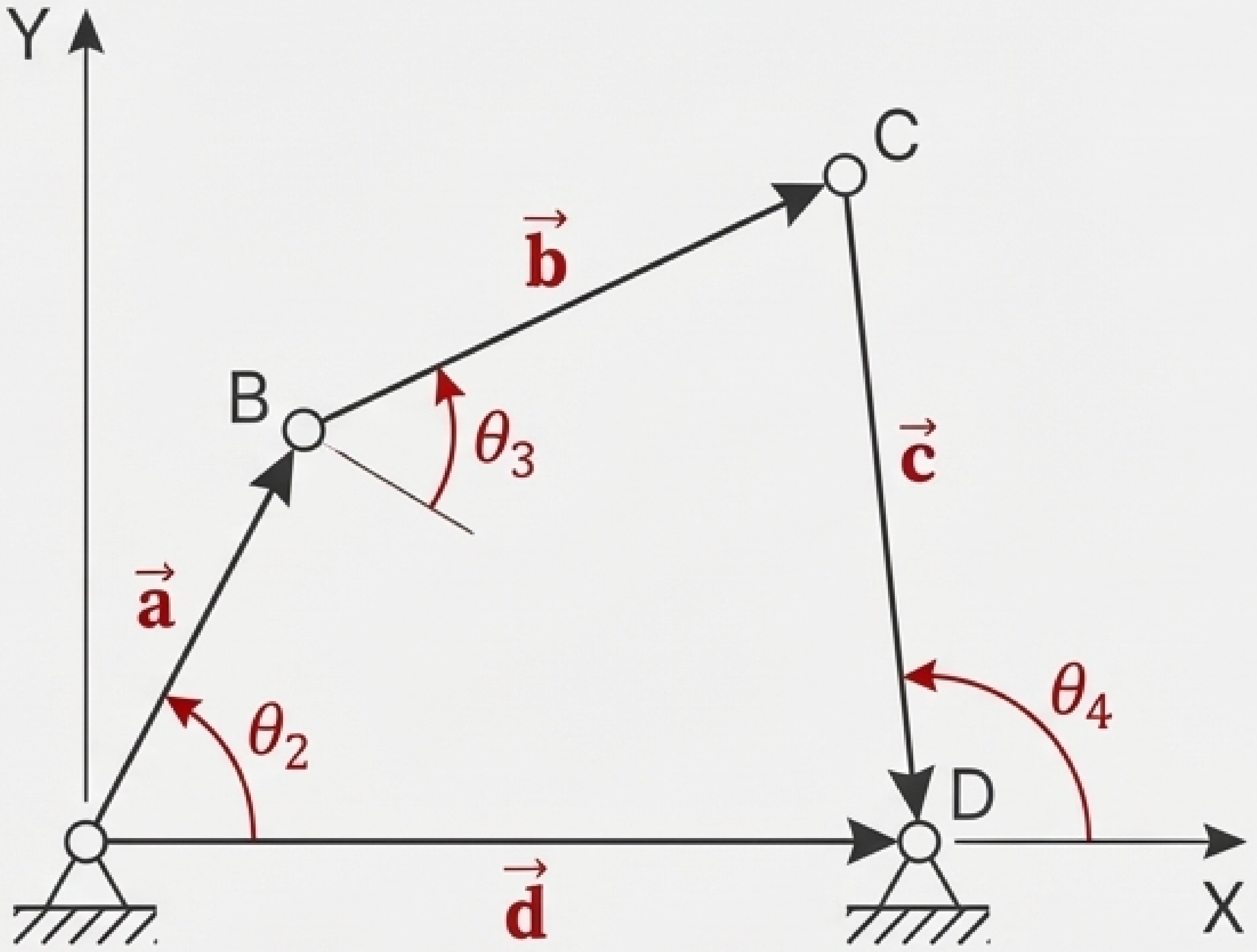
1. We can represent each link as a vector: $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$.
2. The linkage forms a closed loop, which gives us the fundamental vector loop closure equation:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{d}} + \vec{\mathbf{c}}$$

3. This vector equation can be resolved into two scalar equations for the X and Y components:

X-component: $a\cos(\theta_2) + b\cos(\theta_3) = d + c\cos(\theta_4)$

Y-component: $a\sin(\theta_2) + b\sin(\theta_3) = c\sin(\theta_4)$



Finding the Limit Condition Analytically

To find a direct relationship between the input angle (θ_2) and the output angle (θ_4), we must eliminate the unknown coupler angle (θ_3) from the two scalar equations. The result of this algebraic manipulation gives us the mathematical condition for the limits of motion.

X & Y Component Equations $\xrightarrow{\text{Algebraic Elimination of } \theta_3}$

The Limit Condition

The input crank reaches its limit positions (corresponding to the rocker's limits) when:

$$B^2 - C^2 + A^2 = 0$$

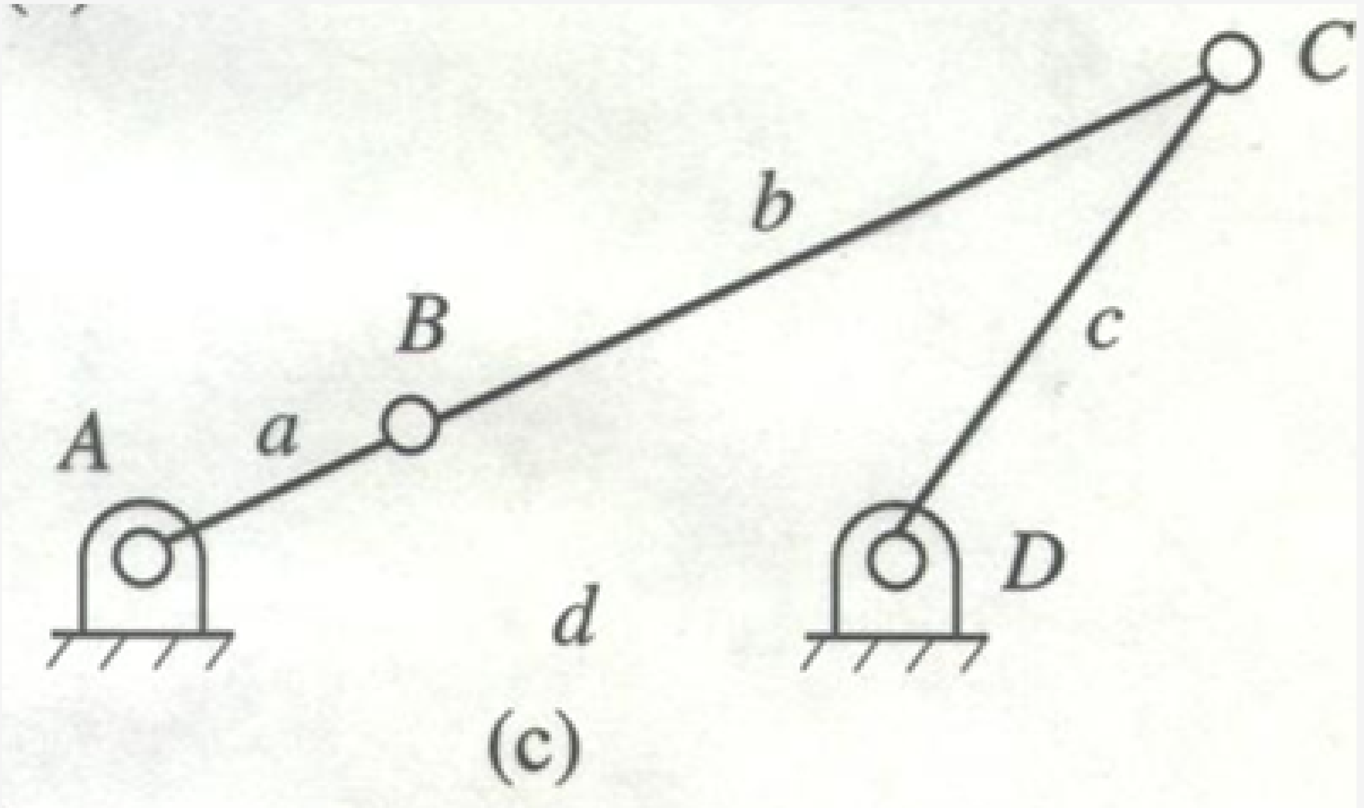
where A, B, and C are functions of the link lengths and θ_4.

The Meaning: This equation is the mathematical signature for the geometric condition we saw earlier—when the crank and coupler are collinear.

Calculating the Minimum and Maximum Rocker Angles

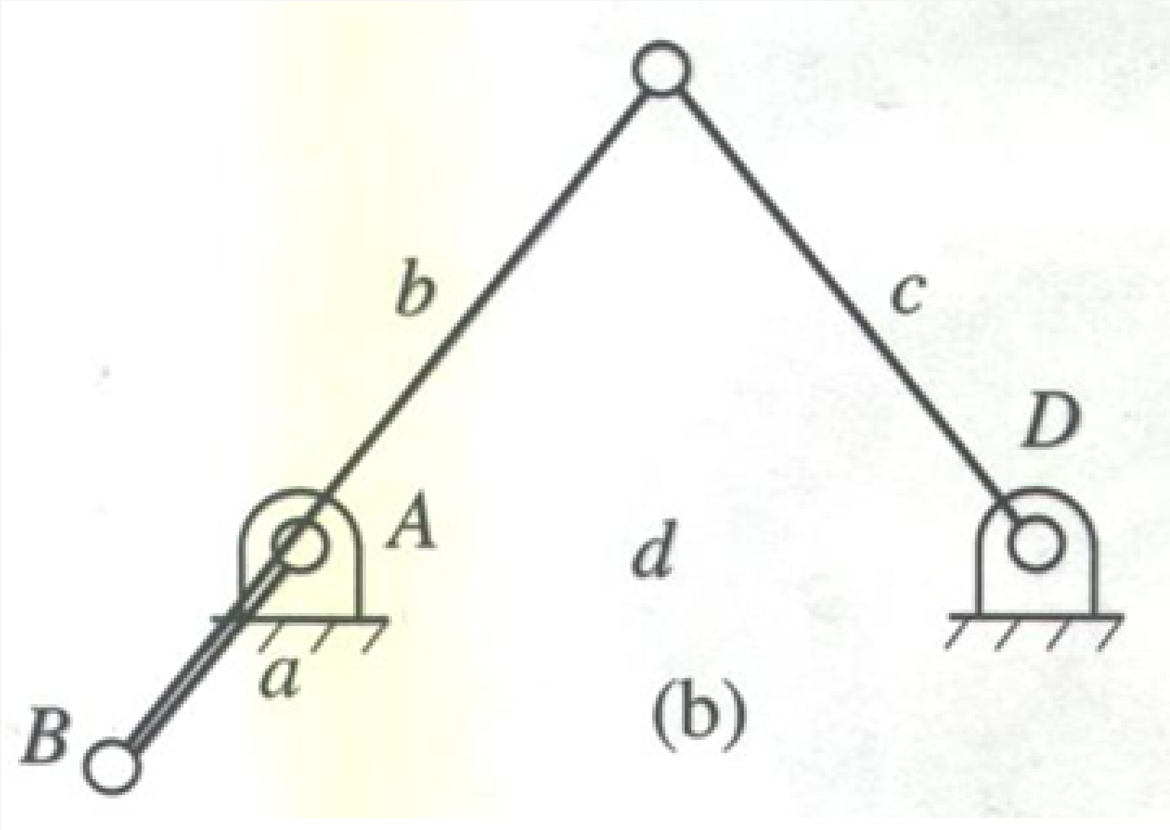
The final formulas for the rocker limit angles are derived by applying the Law of Cosines to the two triangles we identified in Slide 5.

Maximum Angle (Stretched position)



$$\theta_{4\_max} = \pi - \cos^{-1}\left(\frac{c^2 + d^2 - (b+a)^2}{2cd}\right)$$

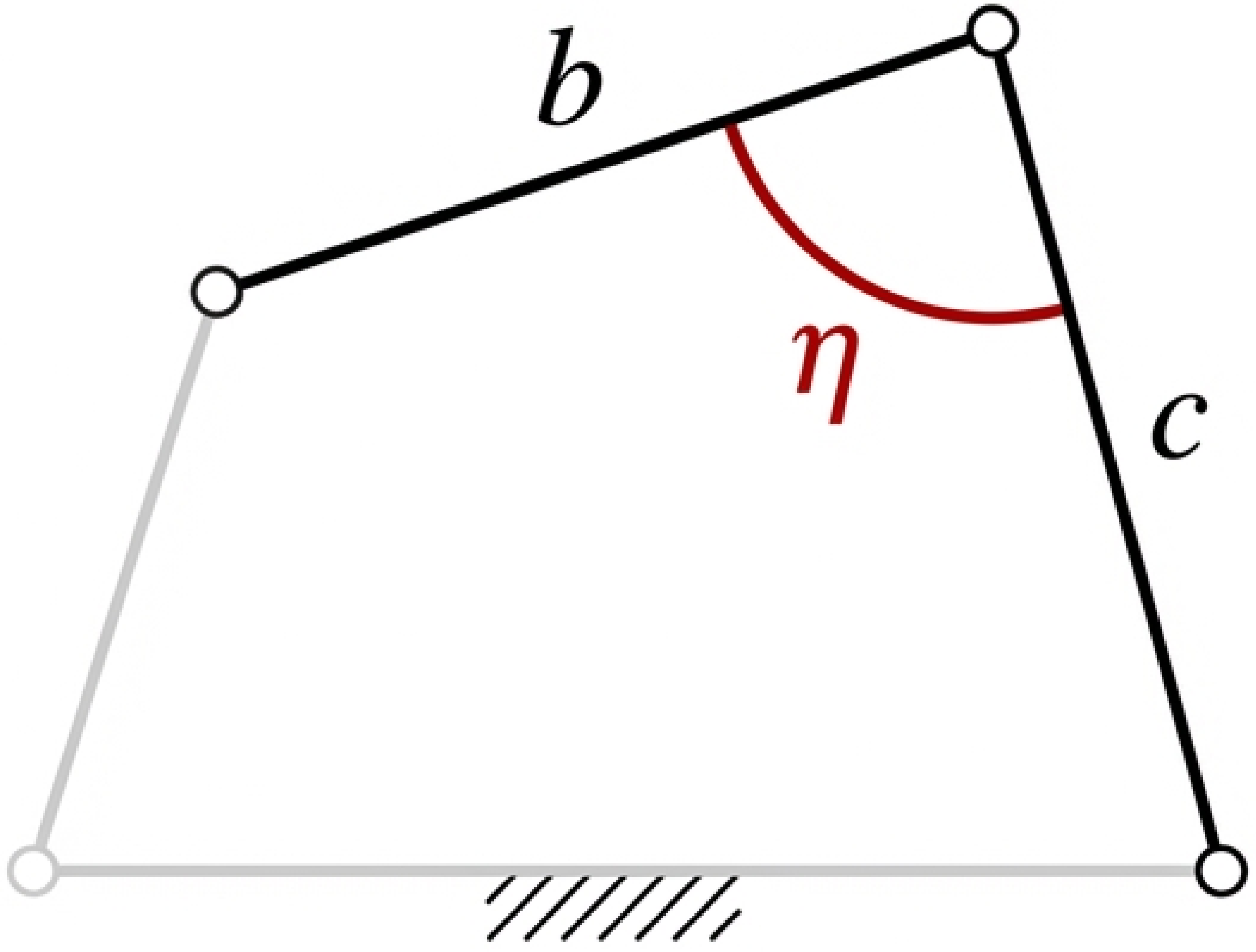
Minimum Angle (Folded position)



$$\theta_{4\_min} = \pi - \cos^{-1}\left(\frac{c^2 + d^2 - (b-a)^2}{2cd}\right)$$

A New Question: What is the Transmission Angle (η)?

- **Definition:** The transmission angle (η) is the angle between the coupler link (‘b) and the output link/rocker (‘c’).
- **Significance:** It is a critical metric for measuring the quality of force and motion transmission from the coupler to the output. A good transmission angle ensures smooth operation, while a poor one can lead to high joint forces and binding.

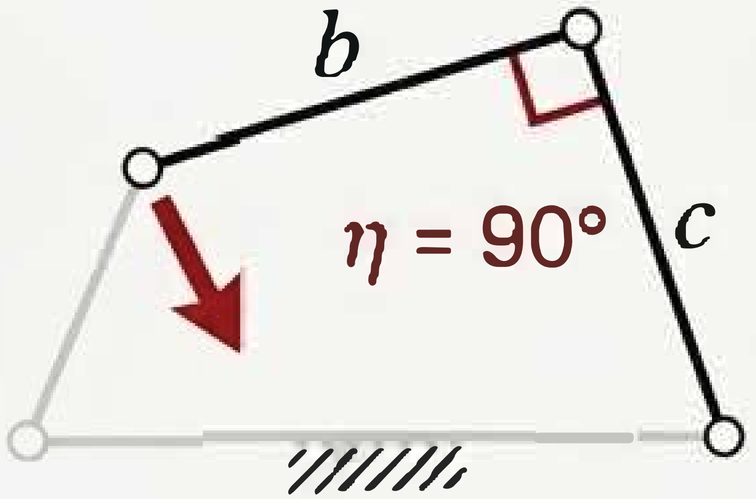


Not All Angles Are Created Equal

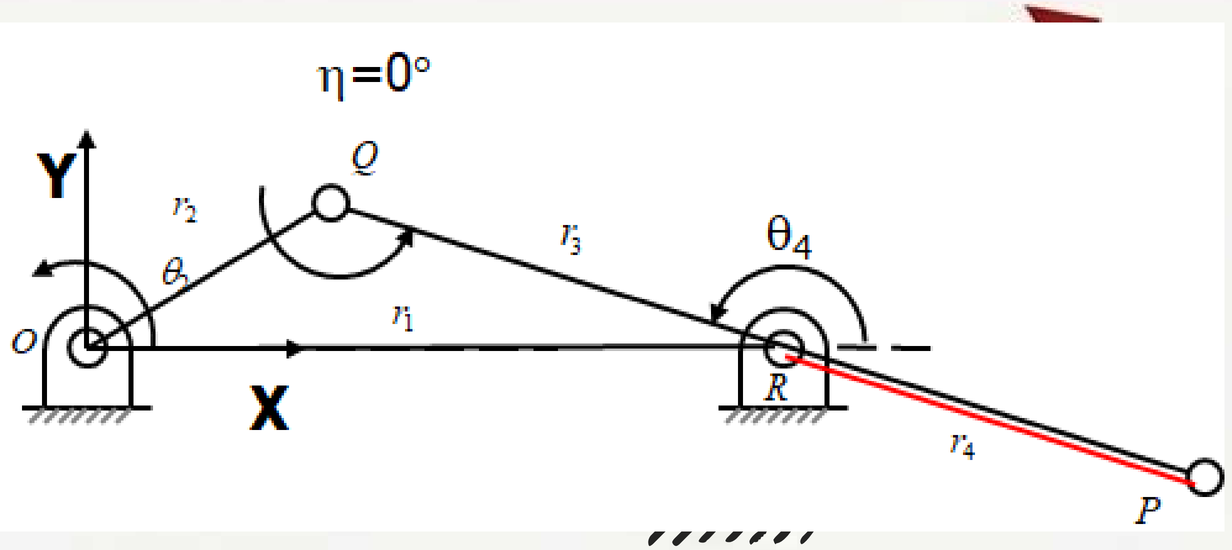
Best Case (η = 90°): All force from the coupler is acting to rotate the output link. Transmission is most effective.

Worst Case (η = 0° or 180°): This is a 'dead point' or 'toggle position.' The mechanism can lock up, as no amount of force on the coupler can rotate the output. This should be avoided.

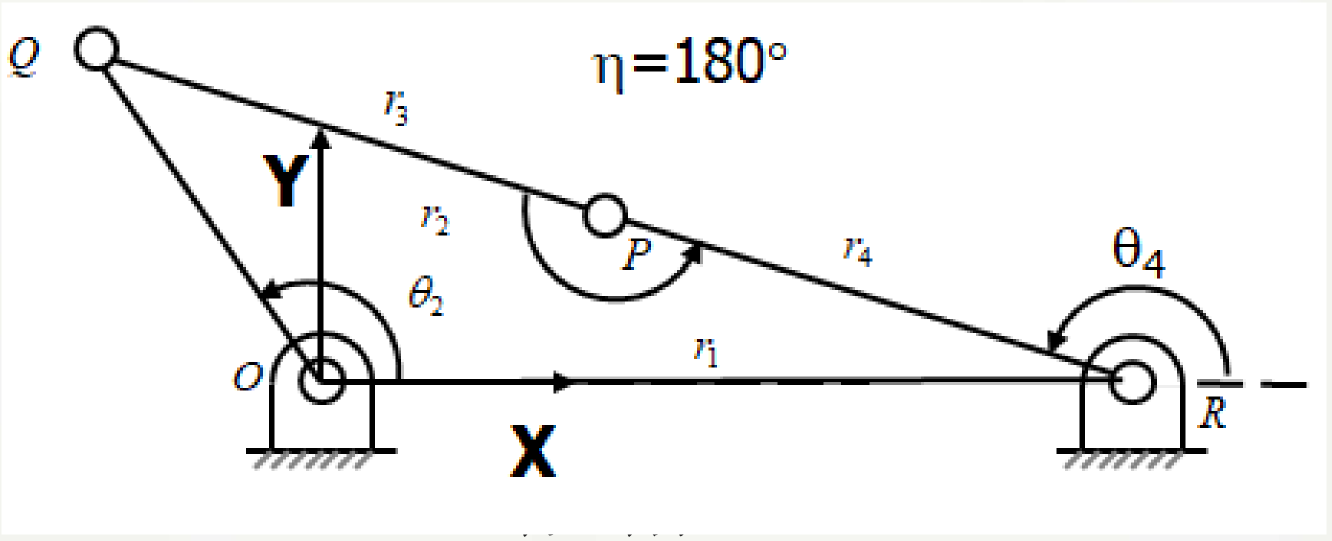
Design Rule of Thumb: For smooth operation and good force transmission, designers ideally aim for **30° < η < 150°**.



Best Case



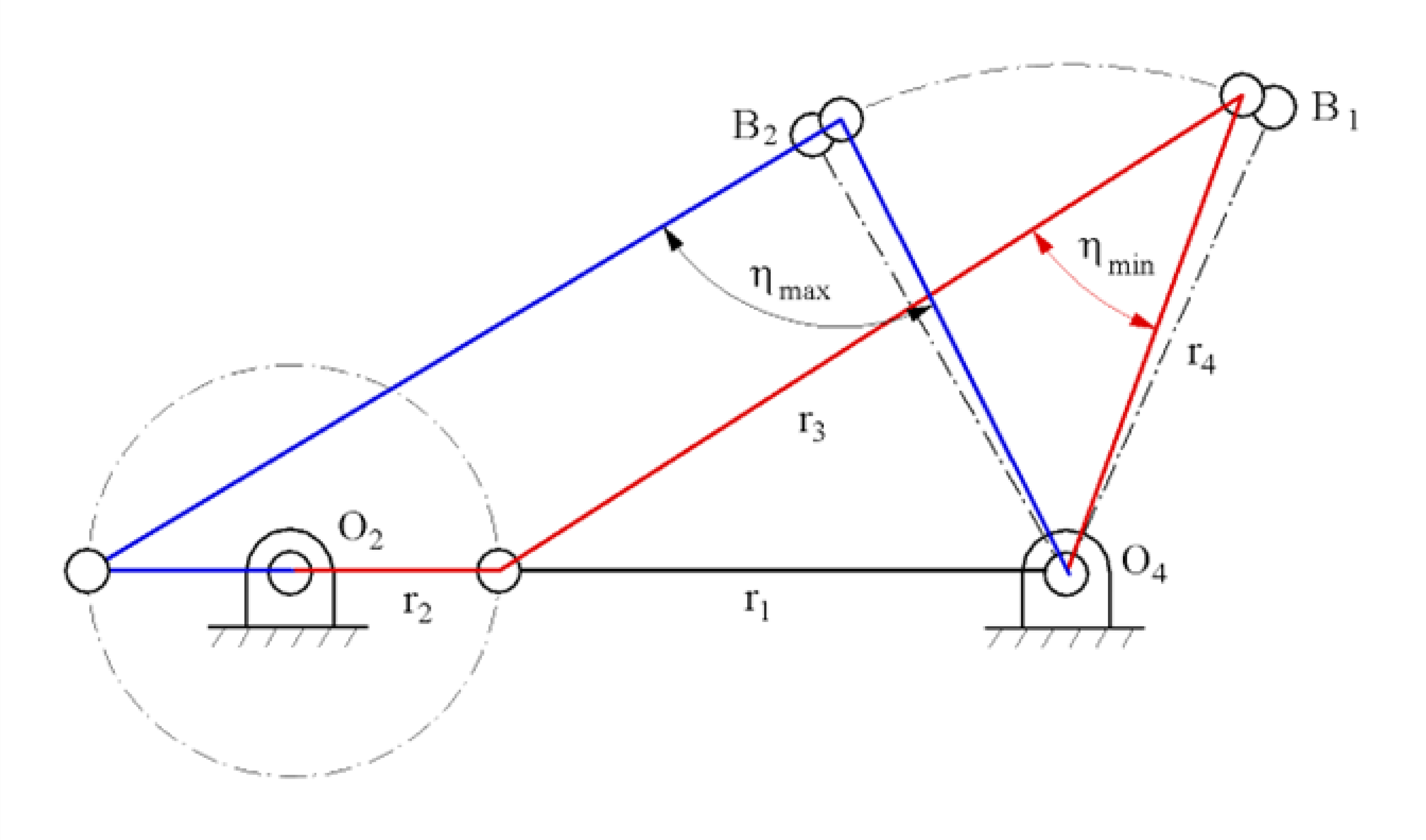
Worst Case (Dead Point)



Worst Case (Dead Point)

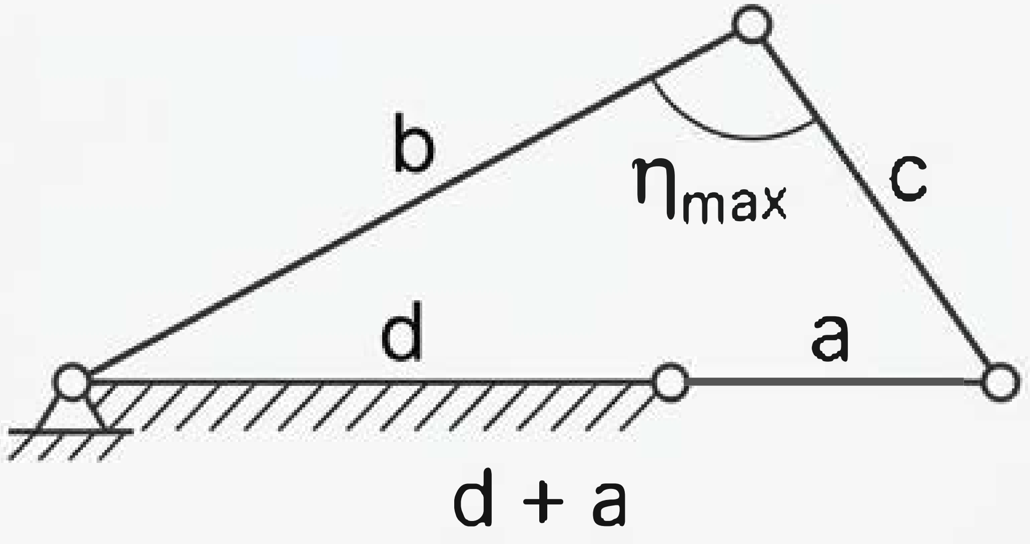
When are Transmission Angles Minimum and Maximum?

The geometric condition for extreme transmission angles in a crank-rocker is different from the rocker limits. They occur when the **input crank** (*a*) **becomes collinear with the ground link** (*d*).



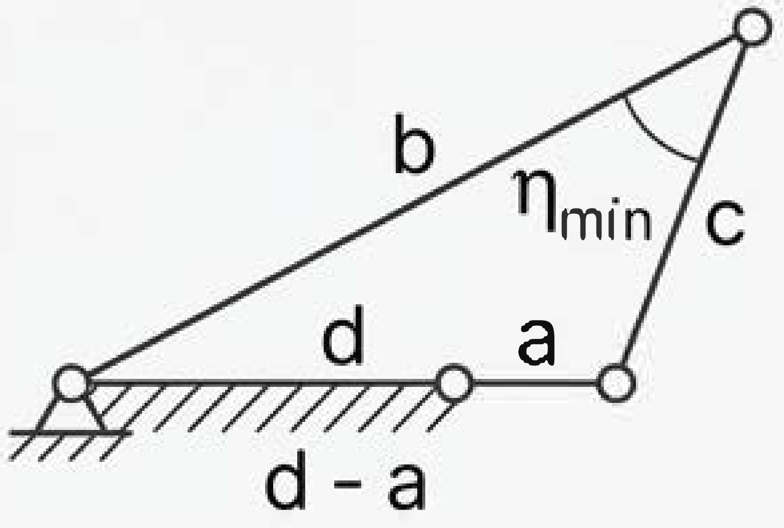
Calculating η_{min} and η_{max}

The formulas for the minimum and maximum transmission angle are found by applying the Law of Cosines to the triangles formed when the crank and ground links are collinear.



Maximum Transmission Angle

$$\eta_{max} = \cos^{-1}\left(\frac{b^2 + c^2 - (d+a)^2}{2bc}\right)$$



Minimum Transmission Angle

$$\eta_{min} = \cos^{-1}\left(\frac{b^2 + c^2 - (d-a)^2}{2bc}\right)$$

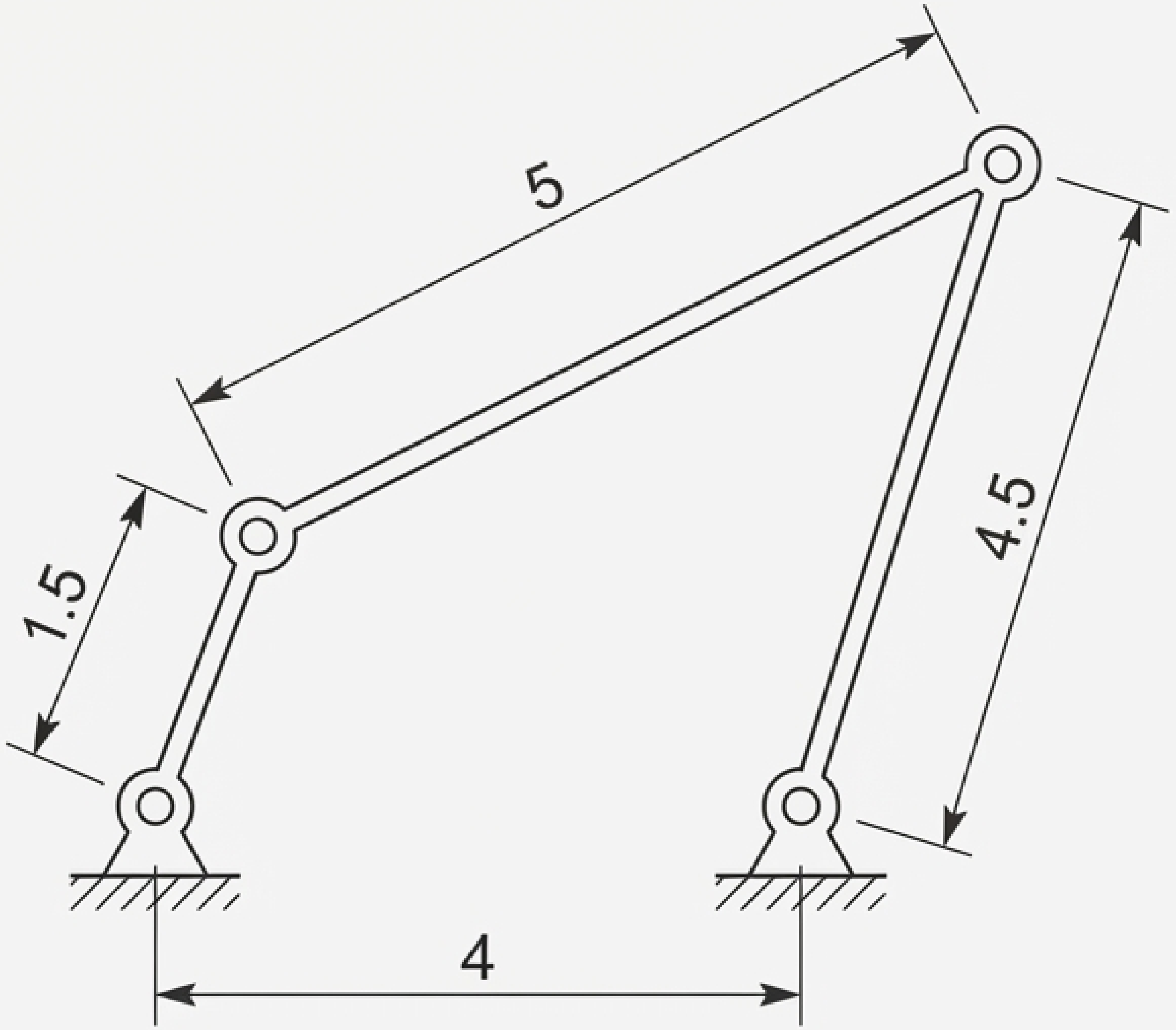
Let’s Put Theory Into Practice

Given:

- Crank (a) = 1.5 in
- Coupler (b) = 5 in
- Rocker (c) = 4.5 in
- Ground (d) = 4 in

Find:

1. The motion limits of the rocker (θ_{4_min} and θ_{4_max}).
2. The minimum and maximum transmission angle (η_{min} and η_{max}).



Applying Our Formulas: Step-by-Step Solution

Part 1: Rocker Limits

$$\theta_{4\_max} = \pi - \cos^{-1}\left(\frac{4.5^2 + 4^2 - (5 - 1.5)^2}{2 * 4.5 * 4}\right)$$

2.3 rad = 131.81°

$$\theta_{4\_min} = \pi - \cos^{-1}\left(\frac{4.5^2 + 4^2 - (5 + 1.5)^2}{2 * 4.5 * 4}\right)$$

1.404 rad = 80.40°

Part 2: Transmission Angle Extremes

$$\eta_{max} = \cos^{-1}\left(\frac{5^2 + 4.5^2 - (4 + 1.5)^2}{2 * 5 * 4.5}\right)$$

1.231 rad = 70.53°

$$\eta_{min} = \cos^{-1}\left(\frac{5^2 + 4.5^2 - (4 - 1.5)^2}{2 * 5 * 4.5}\right)$$

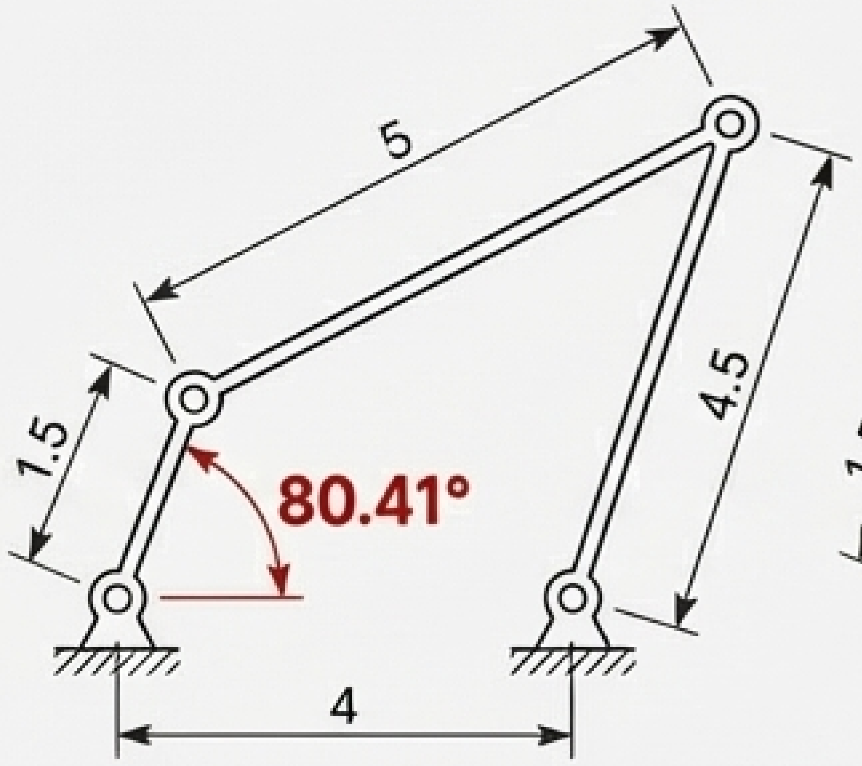
0.522 rad = 29.93°

Summary: Geometry, Analysis, and Design

Key Takeaways

- **Rocker Limits** are defined by the **collinearity of the crank and coupler**.
- **Transmission Angle Extremes** are defined by the **collinearity of the crank and ground link**.
- **Design Implication**: Understanding both analyses is crucial for designing functional, reliable mechanisms that operate smoothly and efficiently within their intended workspace.

Calculated Rocker Limits



Calculated Transmission Angle Extremes