Mathematical Foundations for Kinematic Analysis

A Preparatory Review of Trigonometric & Vector Algebra for the Vector Loop Method

Our Goal: To Systematically Solve Vector Loop Equations

The analysis of a closed-loop mechanism begins with a vector closure equation, which states that the sum of vectors around a loop is zero. For a standard four-bar linkage, this can be expressed as:

$$\vec{r}_2 + \vec{r}_3 = \vec{r}_1 + \vec{r}_4$$

$$\downarrow$$

This single vector equation expands into two non-linear scalar equations. Through a series of algebraic manipulations, the system is reduced to a standard trigonometric form containing a single unknown angle, θ_4:

$$\mathbf{A}\cos(\theta_4) + \mathbf{B}\sin(\theta_4) + \mathbf{C} = 0$$

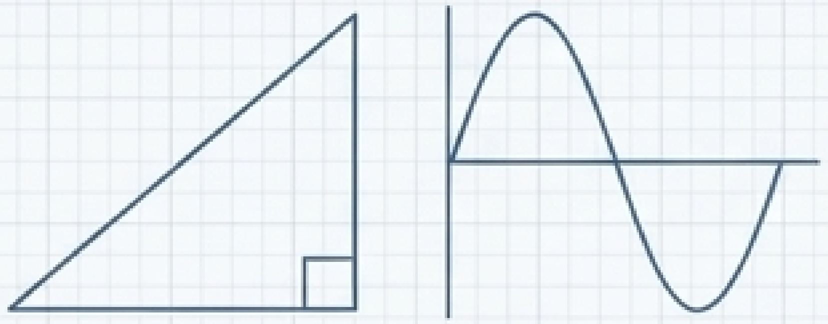
$$\downarrow$$

This equation is then transformed into a solvable polynomial using half-angle identities, where $\mathbf{t} = \tan(\theta_4/2)$:

$$(C\text{-}A)\mathbf{t}^2 + 2\mathbf{Bt} + (A+C) = 0$$

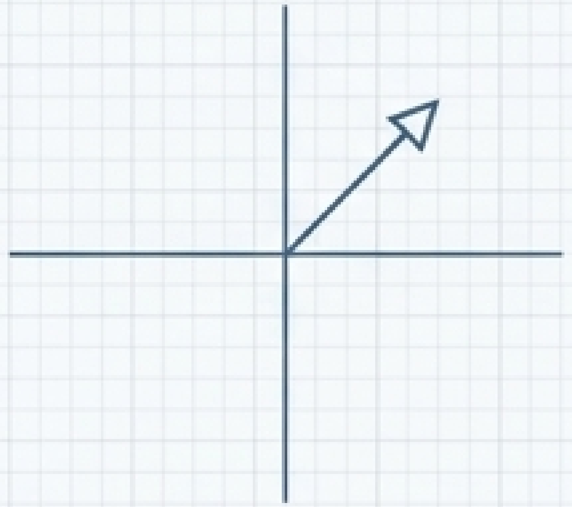
To confidently navigate this derivation, a firm grasp of the underlying trigonometry and vector algebra is essential. This review will sharpen those fundamental tools.

A Two-Part Review to Build Our Toolkit



Part 1: The Trigonometric Toolkit

- Essential Identities (Pythagorean, Half-Angle)
- The Law of Cosines
- Solving Quadratic Equations
- Resolving Ambiguity with `atan2`



Part 2: The Language of Vectors

- Defining 2D Vectors (Cartesian & Polar)
- Vector Operations (Addition & Subtraction)
- Key Calculations (Distance & Inversion)

Part 1: The Trigonometric Toolkit

Essential Identities for Simplification and Variable Elimination

Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

Purpose: This identity is the key to eliminating an unknown angle. By isolating the sin and cos terms of an angle on one side of two separate equations, we can square and add them, causing the angular variable to vanish and simplifying the system.

Half-Angle Identities

$$\boxed{\begin{aligned} \cos\theta &= \frac{1-t^2}{1+t^2} \\ \sin\theta &= \frac{2t}{1+t^2} \end{aligned}}$$

where $t = \tan(\theta/2)$

Purpose: This powerful substitution converts a transcendental equation containing $\sin(\theta)$ and $\cos(\theta)$ into a standard quadratic equation in terms of t. This transforms a difficult problem into a straightforward one that can be solved with the quadratic formula.

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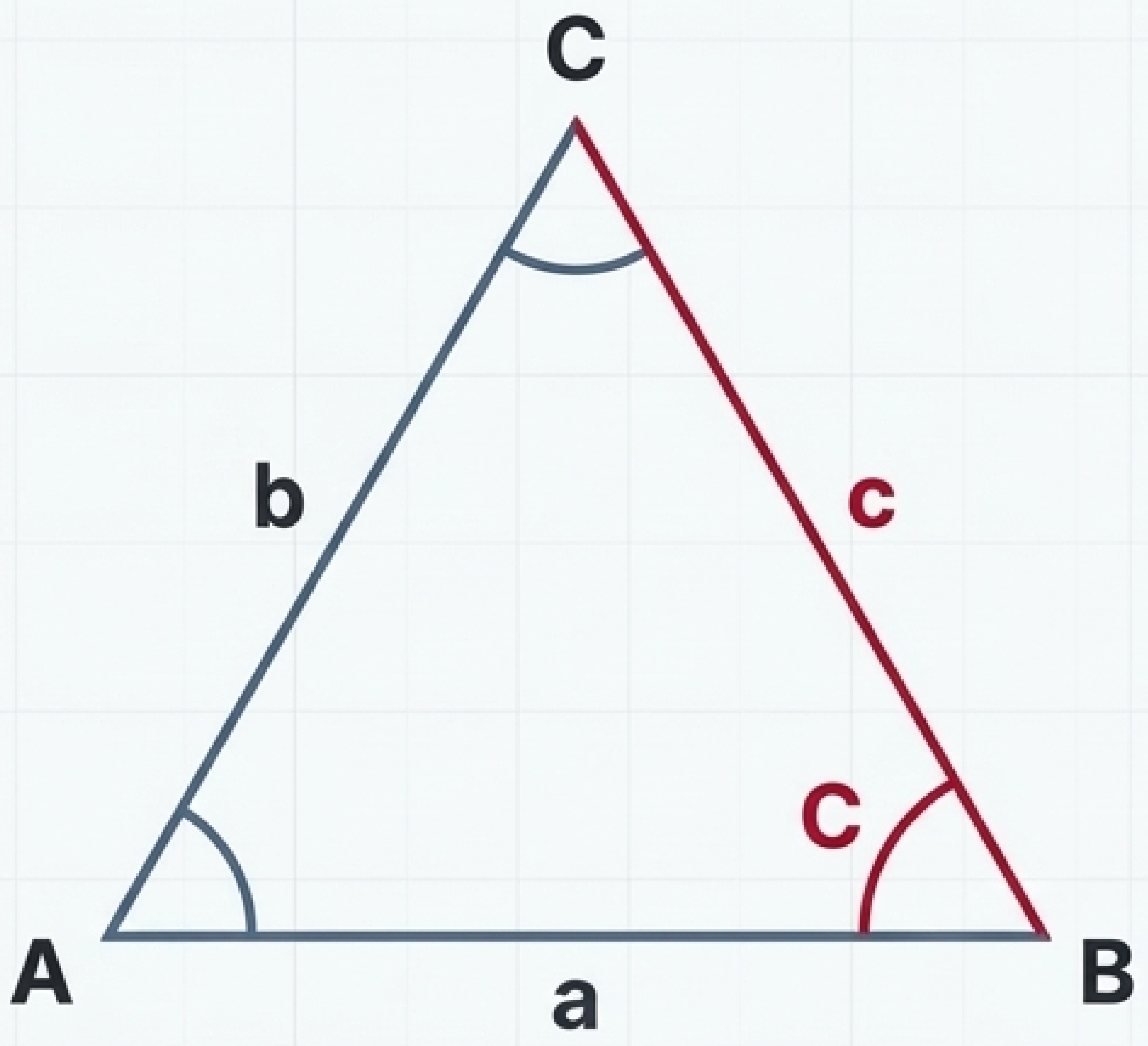
The Law of Cosines: Relating a Triangle's Sides and Angles



The Law of Cosines is a fundamental rule of geometry that relates the lengths of the sides of a triangle to the cosine of one of its angles. It is essential for solving geometric constraints.

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

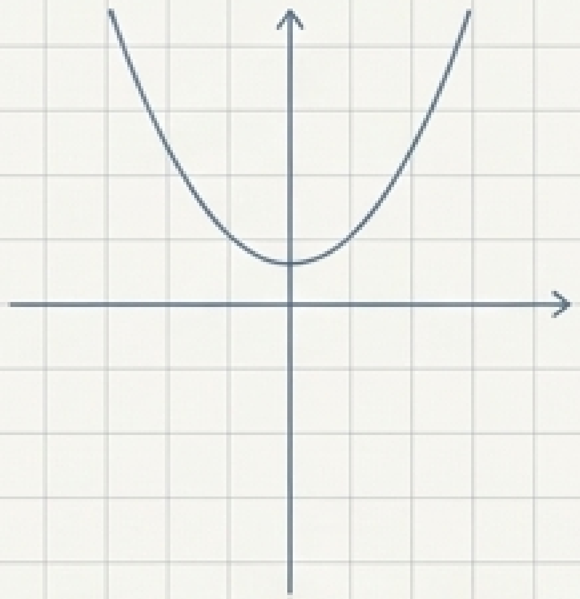
Solving Quadratic Equations and Interpreting the Roots

Standard Form: $ax^2 + bx + c = 0$

The Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

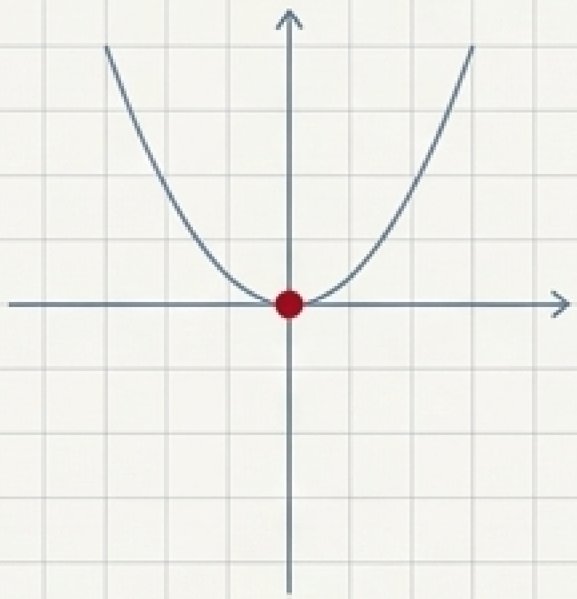
The Discriminant's Physical Meaning:
The term under the square root, $D = b^2 - 4ac$, determines the nature of the solution and has a direct physical correlation in kinematics.

$D < 0$ (No Real Roots)



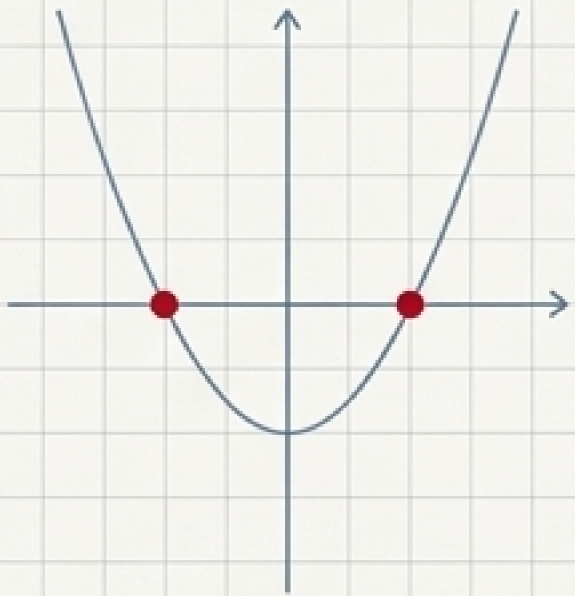
The linkage cannot be assembled. The specified link lengths and input position are physically impossible.

$D = 0$ (One Real Root)



Corresponds to an extreme position (a change-point or locking position) where the two assembly modes merge.

$D > 0$ (Two Real Roots)



Corresponds to the two possible assembly modes or branches of a linkage for a given input.

Resolving Quadrant Ambiguity: The Importance of `atan2`

The Problem with `atan(y/x)`

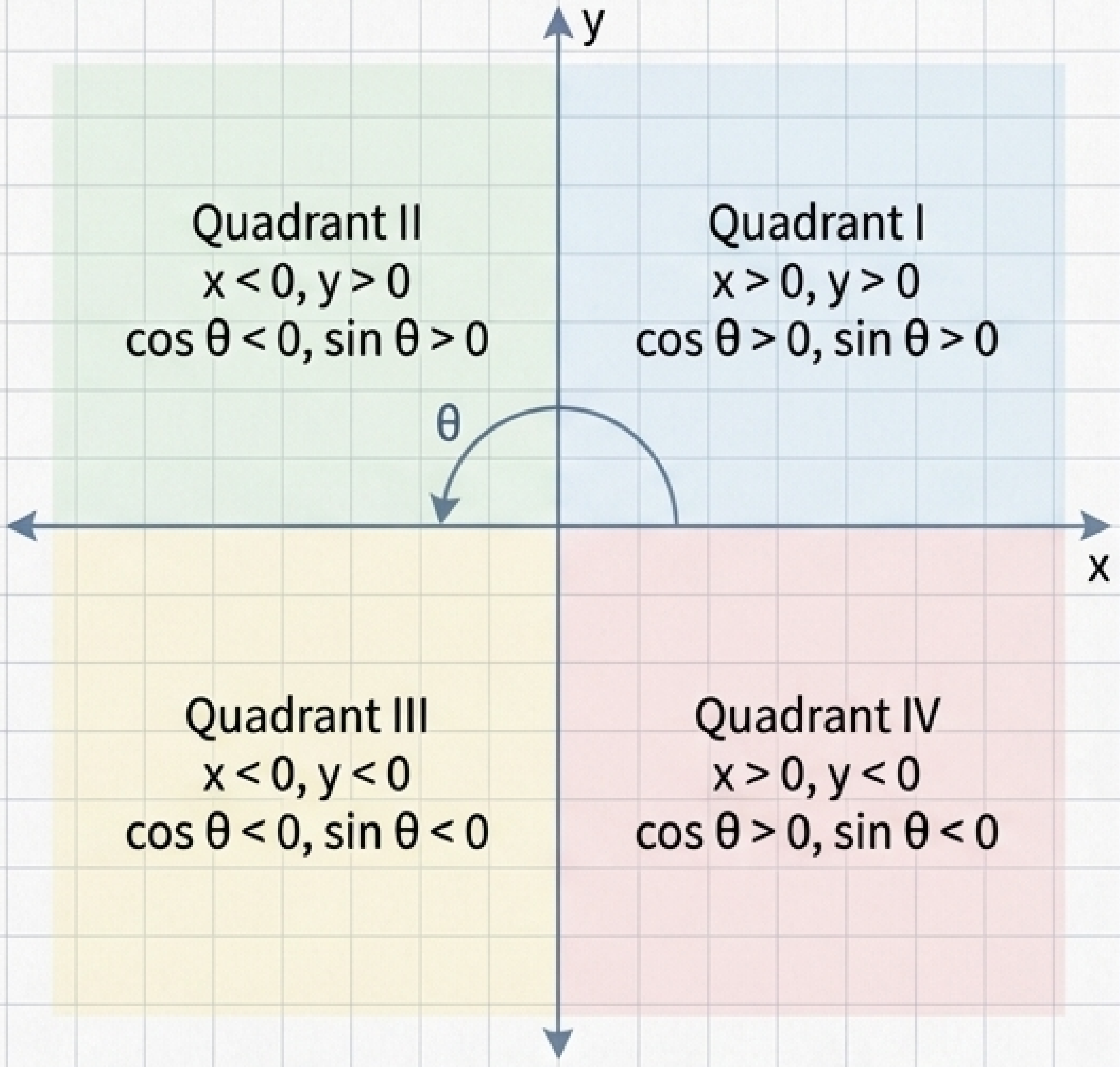
The standard arctangent function only returns angles between -90° and +90° (`−π/2` to `+π/2`). It loses quadrant information because the sign of the ratio `y/x` is the same in diagonally opposite quadrants (e.g., `1/1` is the same as `−1/-1`). This is insufficient for uniquely defining a vector's orientation.

The Solution: `atan2(y, x)`

This two-argument function uses the individual signs of both `y and `x to determine the correct quadrant. It returns a single, unambiguous angle over the full 360° range (−π to +π`).

Application

When solving for an unknown angle like θ_3, we calculate **$\tan(\theta_3) = N/D$**. Using **atan2(N, D)** is essential to find the correct solution.



`atan2` correctly identifies the angle's true location.

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Part 2: The Language of Vectors

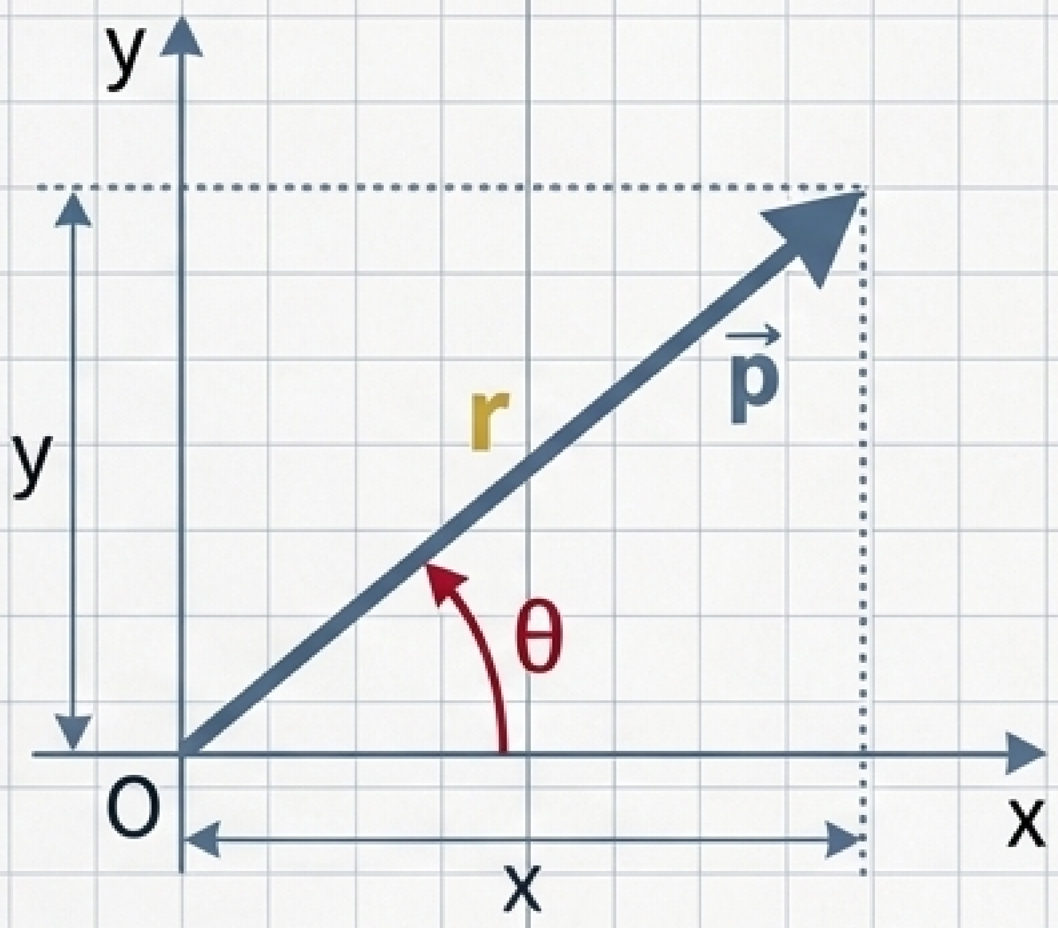
Defining a 2D Vector: Cartesian and Polar Coordinates

A 2D vector is defined by its magnitude (length, r) and direction (angle, θ). It can be represented in two equivalent coordinate systems.

Cartesian (x, y)

Describes a vector by its horizontal (x) and vertical (y) components.

$\vec{p} = (x, y)$



Polar (r, θ)

Describes a vector by its length (r) and its angle (θ) relative to the positive x-axis.

$\vec{p} = (r, \theta)$

Conversion Between Systems

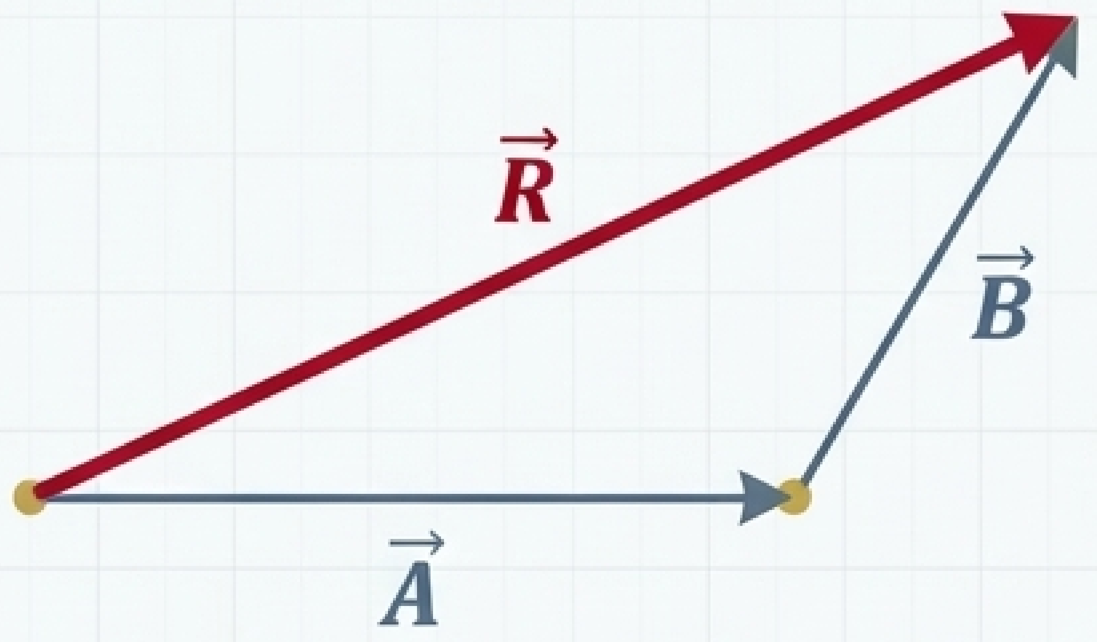
From Polar to Cartesian: $x = r\cos(\theta)$, $y = r\sin(\theta)$

From Cartesian to Polar: $r = \sqrt{(x^2 + y^2}$, $\theta = \mathrm{atan2}(y, x)$

Vector Operations: Addition and Subtraction

Graphical Method: The "Head-to-Tail" Rule

To add $\vec{A} + \vec{B}$, place the tail of $\vec{B}$ at the head of $\vec{A}$. The resultant vector $\vec{R}$ connects the tail of $\vec{A}$ to the head of $\vec{B}$. This visual representation is the foundation of a vector loop.



Computational Method: Component-wise Arithmetic

The most direct way to perform vector math is by operating on the Cartesian components.

Given $\vec{\boldsymbol{A}} = (A_x, A_y)$ and $\vec{\boldsymbol{B}} = (B_x, B_y)$:

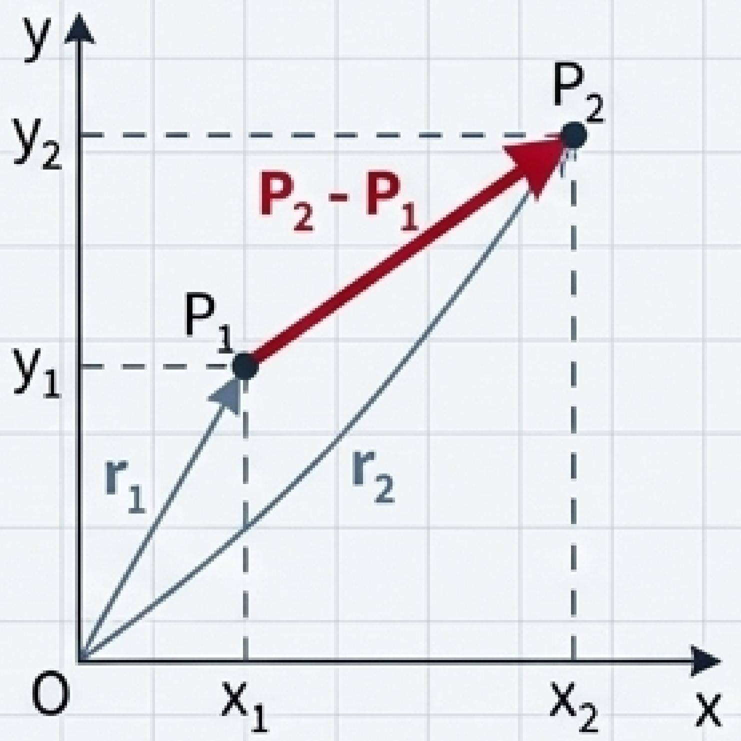
Addition: $\vec{\boldsymbol{R}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = (A_x + B_x, A_y + B_y)$

Subtraction: $\vec{\boldsymbol{S}} = \vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} = (A_x - B_x, A_y - B_y)$

Key Calculations: Distance and Inversion

1. Distance Between Two Points

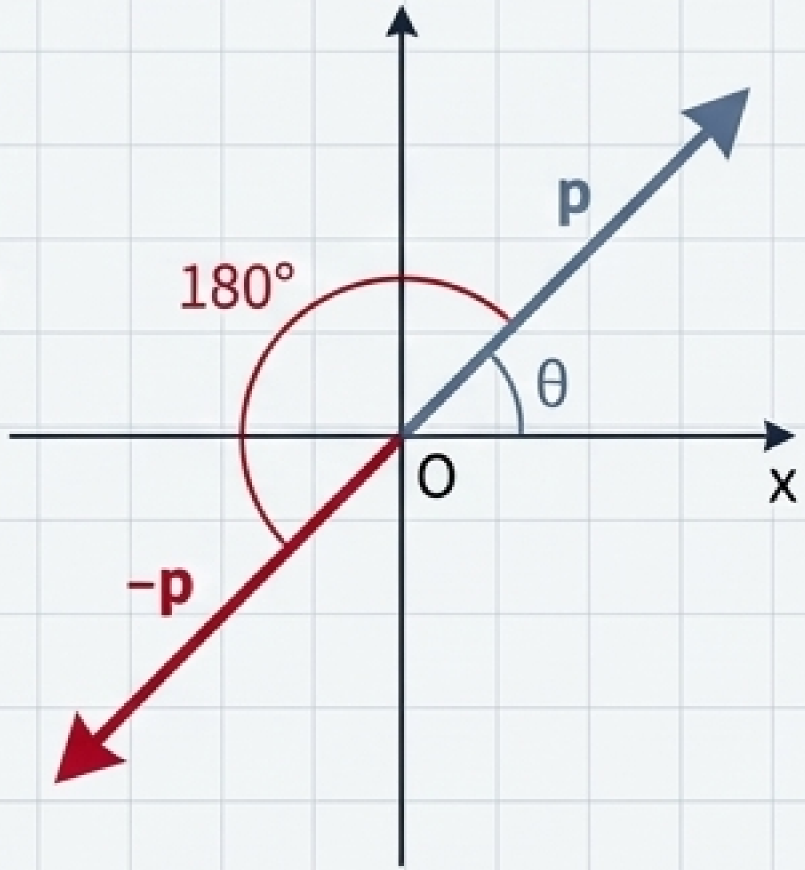
The distance between two points, P_1 and P_2, is the magnitude (length) of the vector connecting them. This vector is calculated as the difference between their position vectors: $\mathbf{d} = \mathbf{P_2} - \mathbf{P_1}$.



$$\text{Distance} = |P_2 - P_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Vector Inversion (Opposite Direction)

The vector $-\mathbf{p}$ has the same magnitude as $\mathbf{p}$ but points in the exact opposite direction (a 180° or π radian shift).



Cartesian:
If $\mathbf{p} = (x, y)$, then
$-\mathbf{p} = (-x, -y)$

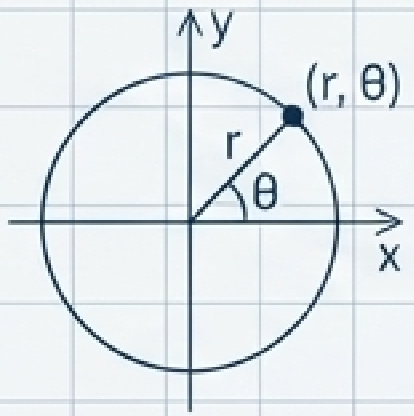
Polar:
If $\mathbf{p} = (r, \theta)$, then
$-\mathbf{p} = (r, \theta + \pi)$

Toolkit Assembled: Applying These Concepts with Interactive Tools

You now have the core trigonometric and vector algebra tools required to tackle the analytical position analysis of planar linkages using the vector loop method.

Interactive Learning Resources

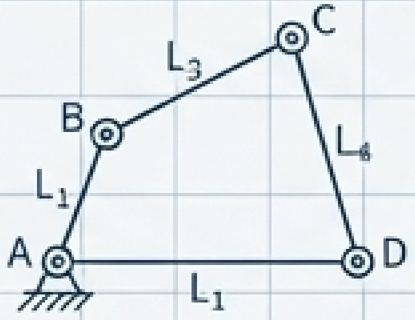
The following open-source tools provide a hands-on environment to deepen your understanding.



Tool 1: Polar & Cartesian Coordinates Visualizer

An interactive web app to explore the relationship between polar (r, θ) and Cartesian (x, y) coordinates. This is an excellent way to build intuition for vector representations and the `atan2` function.

https://github.com/haijunsu-osu/kinematics_ug/blob/main/polar_cartesian_visualizer.html



Tool 2: Interactive 4-Bar Linkage Kinematics Analysis

A repository containing Python code and an interactive GUI for analyzing a four-bar linkage. See how the equations you've reviewed are implemented to predict the mechanism's motion.

https://github.com/haijunsu-osu/kinematics_4bar

Q & A

Thank You