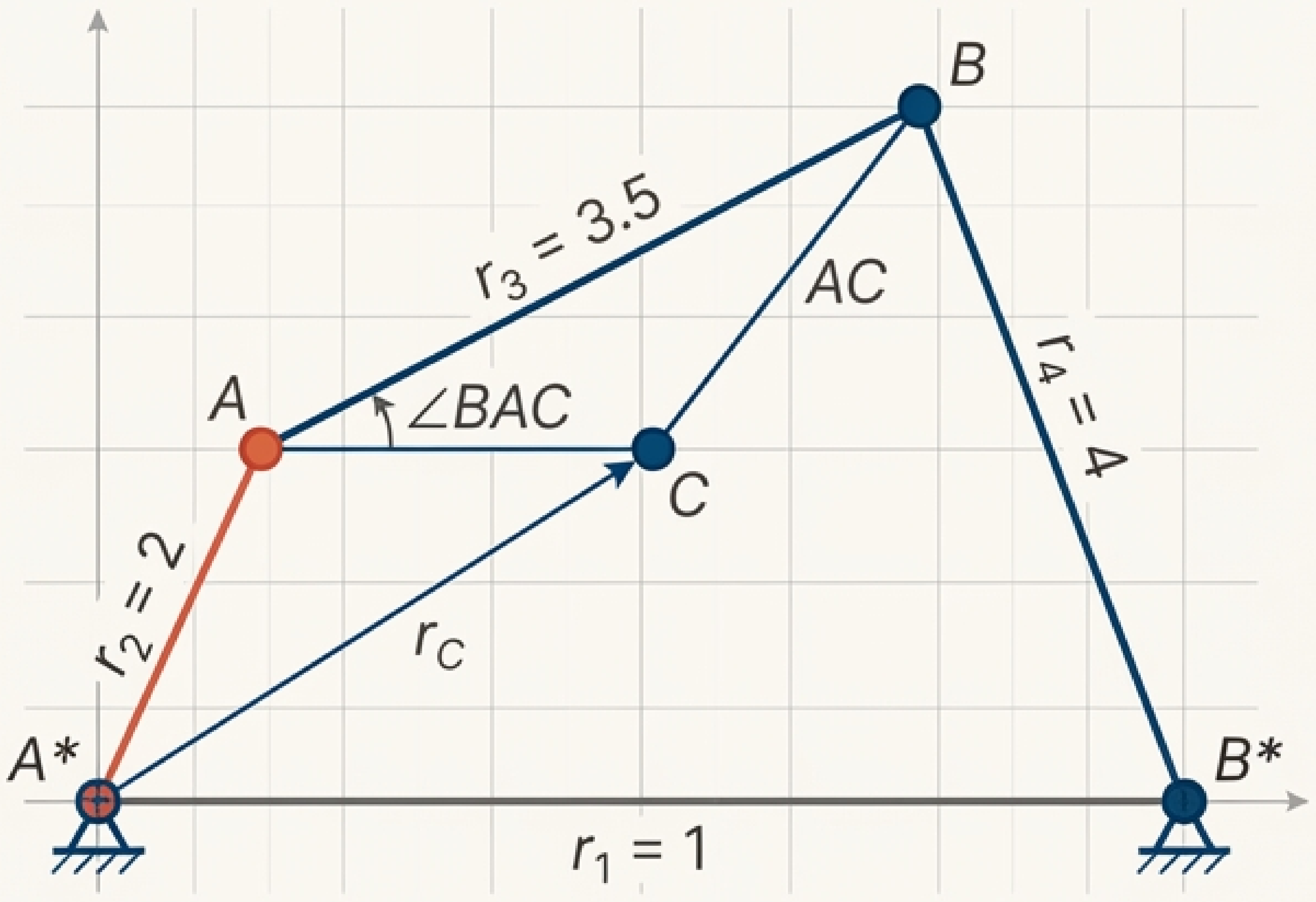
# Kinematics of Machines: Positional Analysis of the Planar 4-Bar Linkage

## An Analytical Approach using Vector Loop Closure



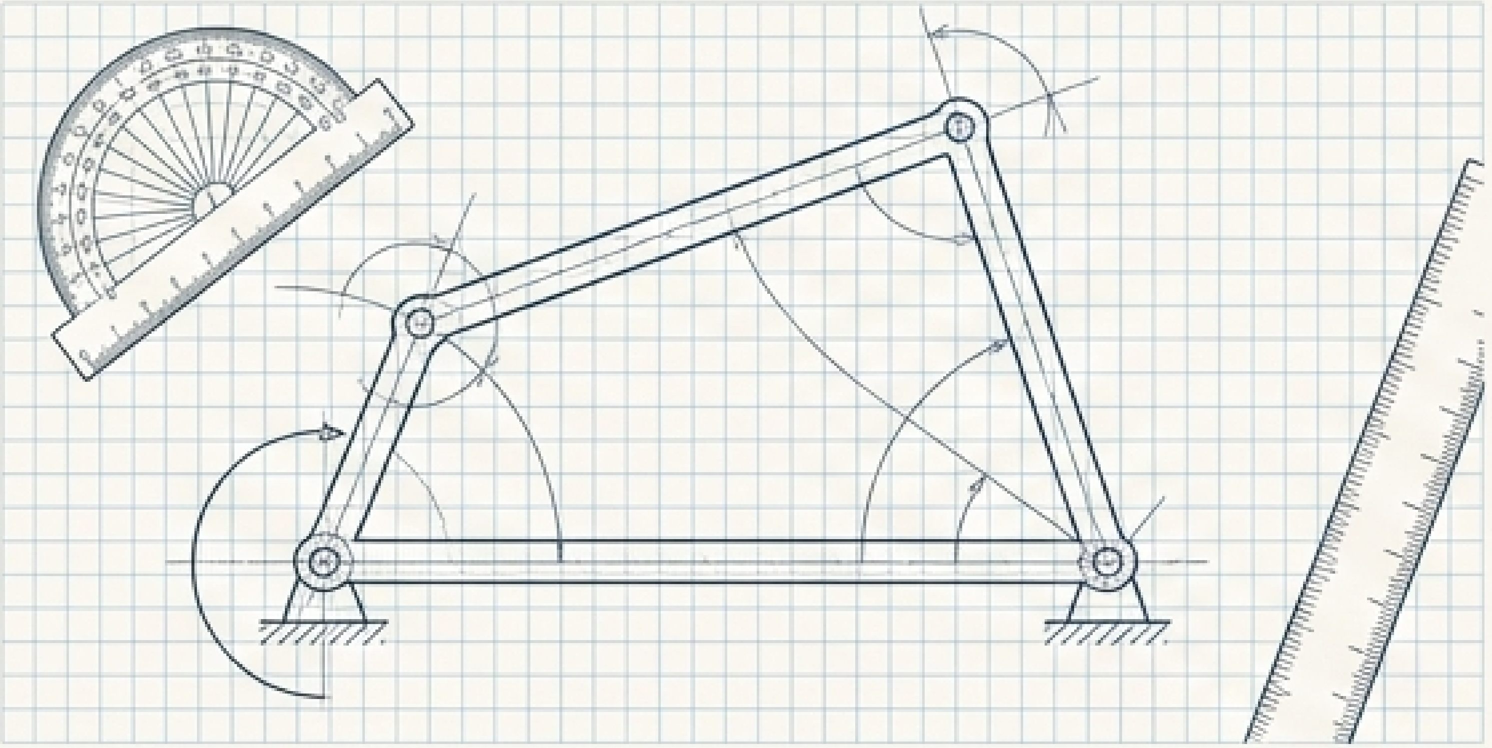


Course: ME 3751: Kinematics and Mechanism Design
Professor: Haijun Su
University: The Ohio State University

# From Sketch to Simulation: The Need for Precision

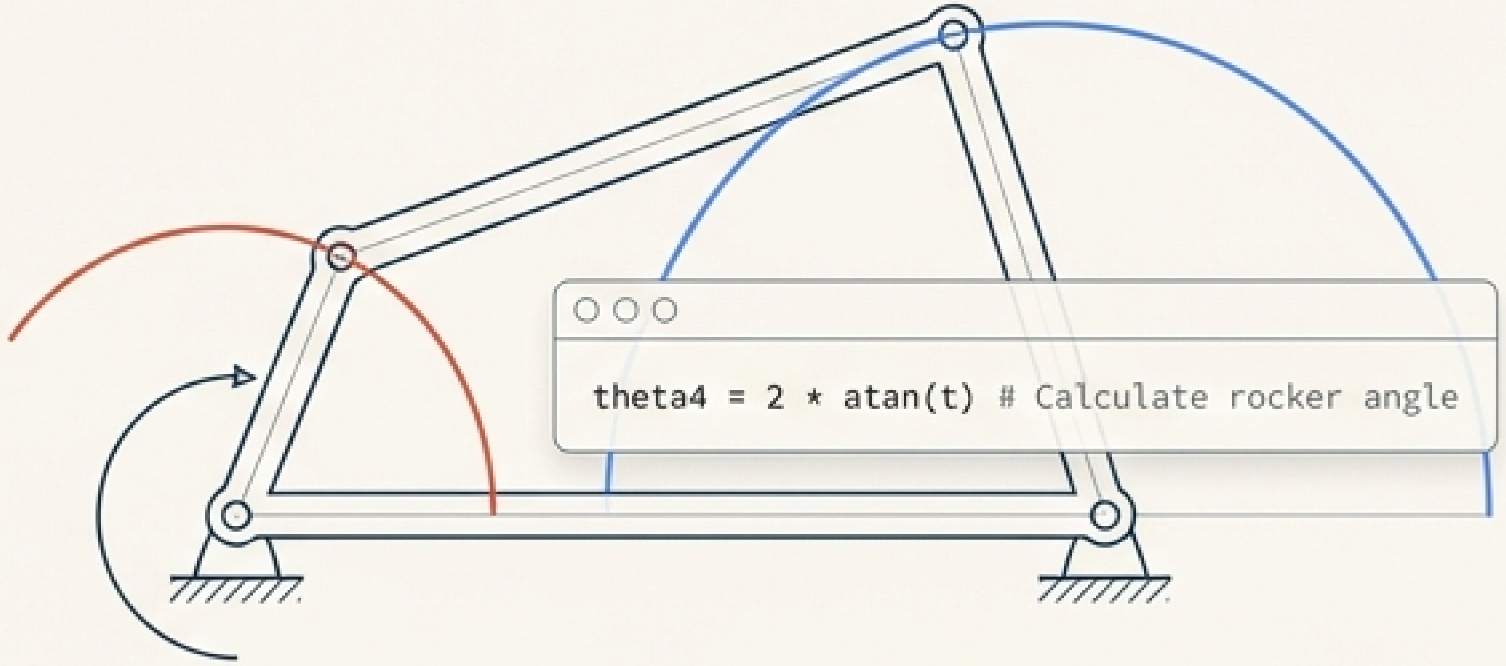
While graphical methods are great for initial concepts, modern engineering demands the precision, speed, and automation that only analytical methods can provide.

## Graphical Methods



- Intuitive and excellent for visualizing a mechanism's position at a single point.
- Limitations: Suffer from imprecision due to drawing inaccuracies. Slow and tedious for analyzing a full range of motion. Not suitable for computer implementation.

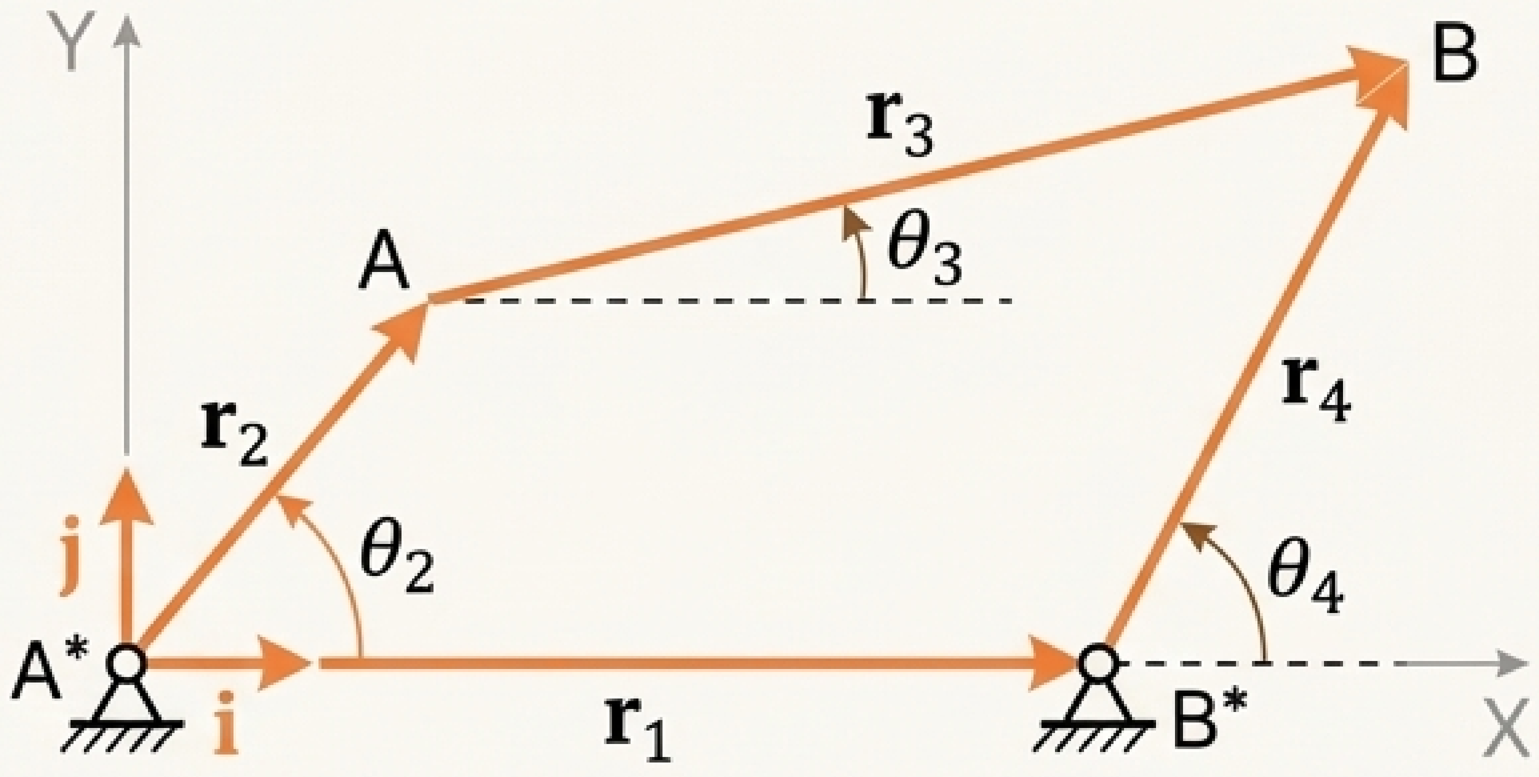
## Analytical Methods




- Provide exact, closed-form solutions for any input position.
- Advantages: The equations can be easily programmed for rapid analysis, dynamic simulation, and design optimization. This is the foundation of modern computer-aided engineering (CAE).

# Translating Mechanics into Mathematics

We can represent each link of the four-bar mechanism as a vector, defined by its length ($r_k$) and angle ($\theta_k$). Because the links form a closed chain, the vectors must also form a closed loop. This geometric fact is our starting point.




## Vector Loop Closure Equation:

$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4 \quad \text{(Eq. 7.24)}$$

This equation states that the path from pivot A* to pivot B through the crank and coupler is always identical to the path through the ground link and rocker. It holds true for any position of the linkage.

# The Puzzle: Two Equations, Two Unknowns

The single vector equation contains two independent pieces of information, one for the x-direction and one for the y-direction. This gives us a system of two equations.

x-component: $r_2\cos\theta_2 + r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4$ (Eq. 7.26)

y-component: $r_2\sin\theta_2 + r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4$ (Eq. 7.27)

**Knowns:** All link lengths ($r_1$, $r_2$, $r_3$, $r_4$), the ground link angle ($\theta_1$), and the input crank angle ($\theta_2$).

**Unknowns:** The coupler angle ($\theta_3$) and the rocker angle ($\theta_4$).

These are coupled, non-linear trigonometric equations. We cannot solve for one unknown without information about the other, which requires a strategic approach.

# The Strategy, Step 1: Isolate and Eliminate

Our first goal is to find $\theta_4$. To do this, we must first eliminate $\theta_3$ from the system of equations.

## Step 1: Isolate

Isolate the $r_3$ terms on one side of the scalar equations.

$r_3\cos(\theta_3) = r_1\cos(\theta_1) + r_4\cos(\theta_4) - r_2\cos(\theta_2)$ (Eq. 7.28)

$r_3\sin(\theta_3) = r_1\sin(\theta_1) + r_4\sin(\theta_4) - r_2\sin(\theta_2)$ (Eq. 7.29)

## Step 2: Eliminate (The "Aha!" Moment)

- Square both sides of both equations.
- Add the two new equations together.
- The left side becomes $r_3^2(\cos^2(\theta_3) + \sin^2(\theta_3))$.

$$\cos^2(\theta_3) + \sin^2(\theta_3) \longrightarrow 1$$

Using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$, the left side simplifies to just $r_3^2$, and $\theta_3$ is eliminated entirely.

# The Strategy, Step 2: Revealing the Underlying Structure

After eliminating $\theta_3$ and expanding the terms, we get a single, complex equation. By rearranging and grouping coefficients, we can express this equation in a much cleaner, standard trigonometric form.

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 + 2r_1r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4)$$
$$- 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$- 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) \text{ (Eq. 7.30)}$$

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$$A\cos\theta_4 + B\sin\theta_4 + C = 0 \text{ (Eq. 7.31)}$$

(**Note:** For a given input angle $\theta_2$, these are just constants.)

$$A = 2r_1r_4\cos\theta_1 - 2r_2r_4\cos\theta_2$$
$$B = 2r_1r_4\sin\theta_1 - 2r_2r_4\sin\theta_2$$
$$C = r_1^2 + r_2^2 + r_4^2 - r_3^2 - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

# The Strategy, Step 3: The Final Transformation

**The Challenge:** We need to solve the equation $A\cos\theta_4 + B\sin\theta_4 + C = 0$ for the unknown angle $\theta_4$.

**The Tool:** We use the tangent half-angle identities, a standard technique for solving this type of equation.

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### The Substitution

Let $t = \tan(\theta_4/2)$. Then we can express $\sin\theta_4$ and $\cos\theta_4$ in terms of $t$.

$$\cos\theta_4 = \frac{1 - t^2}{1 + t^2} \quad \text{(Eq. 7.34)}$$

$$\sin\theta_4 = \frac{2t}{1 + t^2} \quad \text{(Eq. 7.33)}$$

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### The Result

Substituting these into our simplified equation and multiplying by $(1 + t^2)$ converts the trigonometric problem into a standard quadratic equation in the variable $t$.

$$(C - A)t^2 + 2Bt + (A + C) = 0$$

**Key Insight:** We have successfully transformed a difficult trigonometric problem into a solvable quadratic equation.

# The Closed-Form Solution

We now apply the quadratic formula to solve for $t$. This yields two possible values, which in turn give us two possible solutions for $\theta_4$. Once $\theta_4$ is known, we can directly calculate the corresponding $\theta_3$.

**Solving for $\theta_4$:**

- **Step 1:** Solve the quadratic equation for $t$, using $\sigma = \pm 1$ to represent the two roots:

$$t = \frac{-B + \sigma\sqrt{B^2 - C^2 + A^2}}{C - A} \text{ (Eq. 7.35)}$$

- **Step 2:** Invert the tangent half-angle substitution to find $\theta_4$:

$$\theta_4 = 2\tan^{-1}(t) \text{ (Eq. 7.36)}$$

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**Solving for $\theta_3$:**

- **Step 1:** With $\theta_4$ known, we can solve for $\theta_3$ by dividing Eq. 7.29 by Eq. 7.28.
- **Step 2:** **Crucial Note:** We must use a two-argument arctangent function (**atan2**) to ensure the angle is in the correct quadrant.

$$\theta_3 = \text{atan2}(N, D) \text{ where:}$$

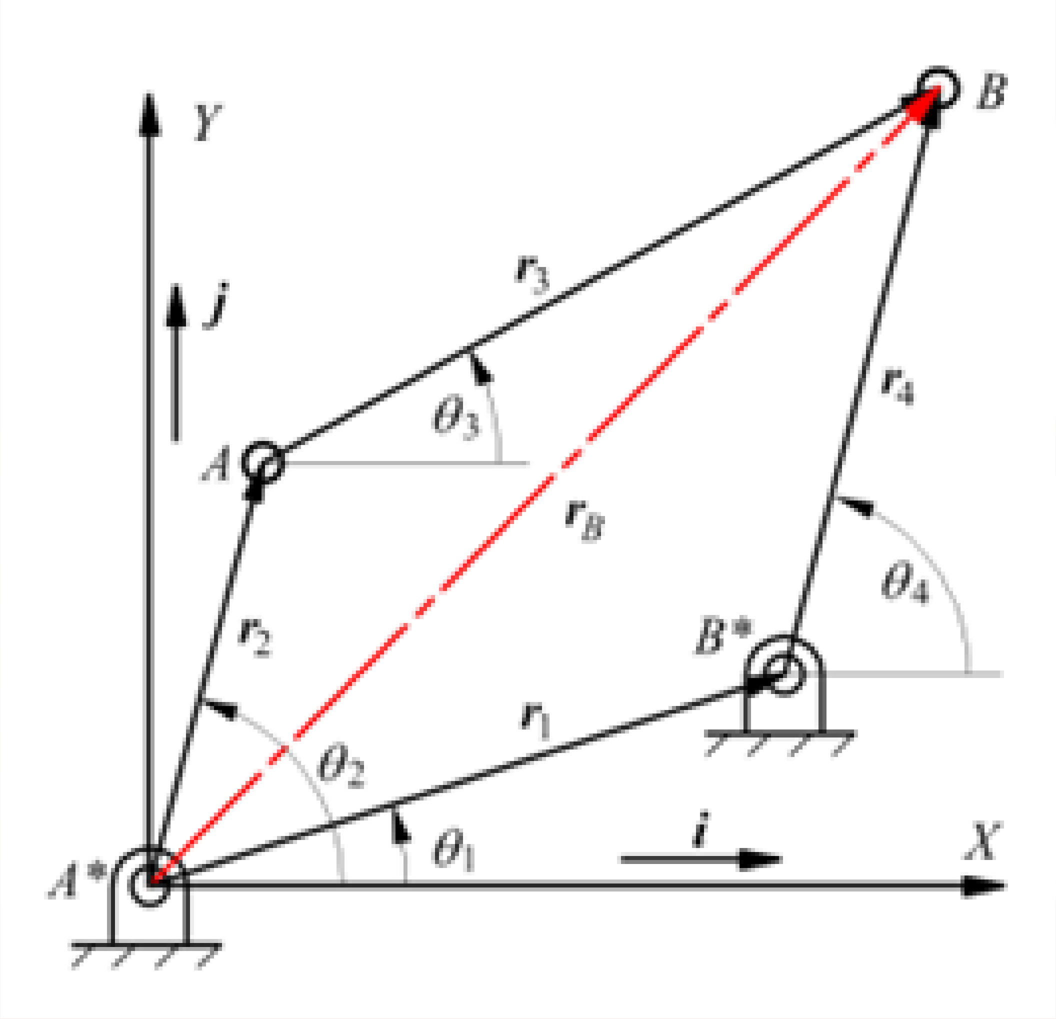
$$N = r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2$$

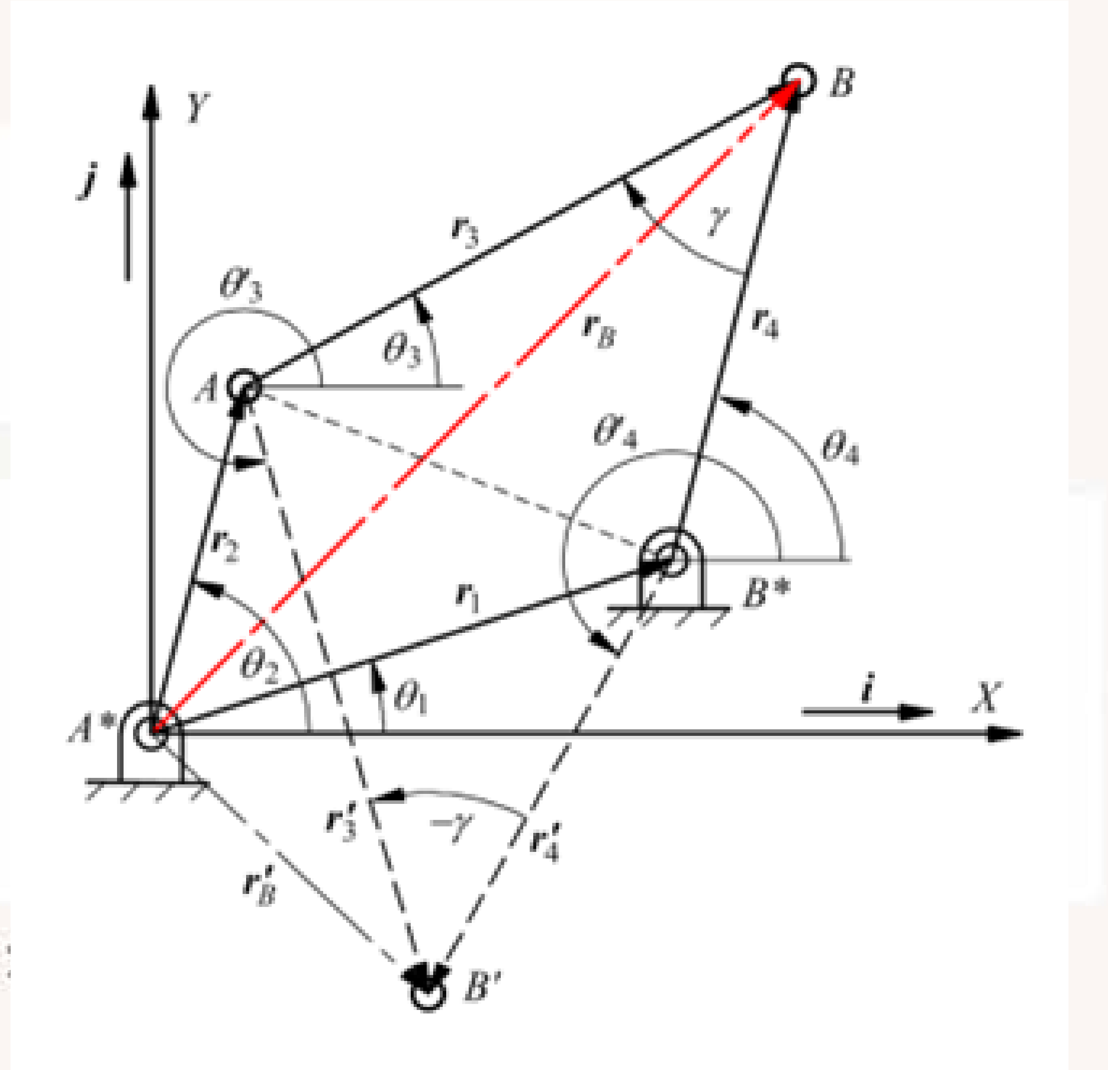
$$D = r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2$$

(derived from Eq. 7.37)

# Two Solutions, Two Physical Configurations

The $\sigma = \pm 1$ in our solution is not just a mathematical artifact; it represents the two physically distinct ways a four-bar linkage can be assembled for a given input angle. A real linkage must exist in one of these “assembly modes.”



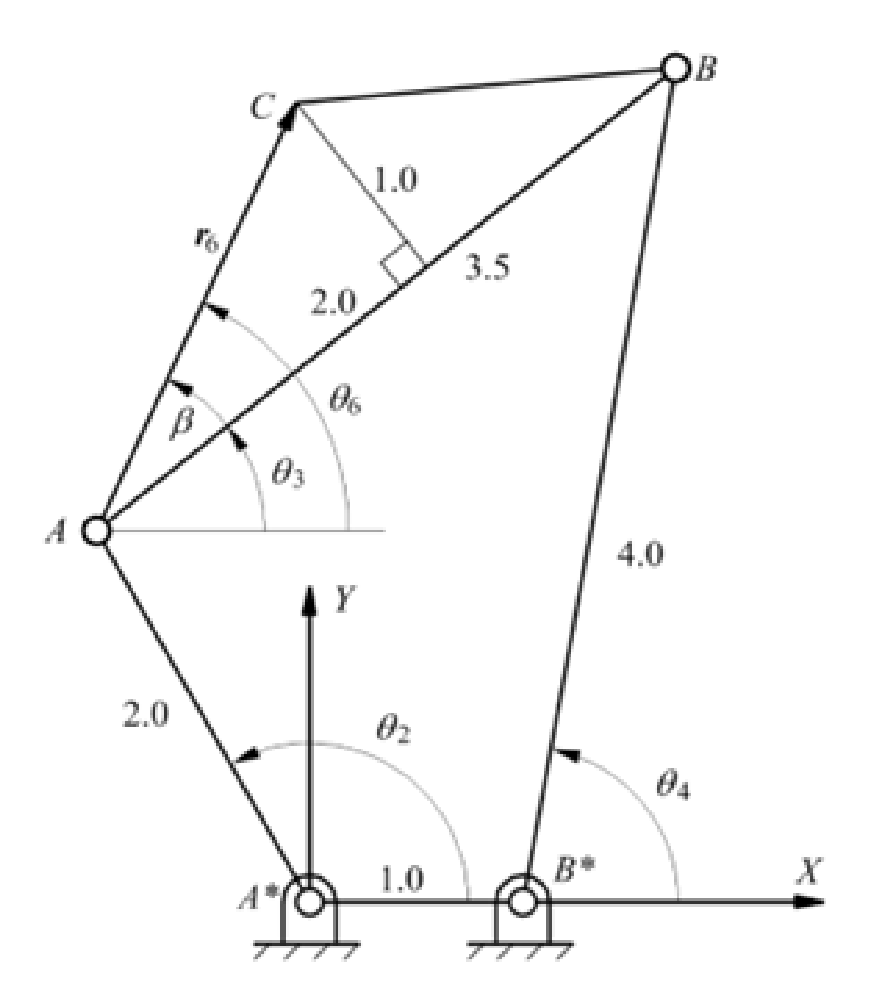

The two positions of point B (B and B’) are symmetric about the link AB*. The two resulting configurations for links 3 and 4 are mirror images about the line connecting the moving pivots A and B*.

# Putting Theory into Practice: Example 7.1

For a linkage with the given dimensions, find the output angles $\theta_3$ and $\theta_4$ when the input crank is at $\theta_2 = 0°$.

## Given Parameters

Link Lengths: $r_1 = 1$, $r_2 = 2$, $r_3 = 3.5$, $r_4 = 4$
Input: $\theta_1 = 0°$, $\theta_2 = 0°$




## Step-by-Step Calculation (for σ = +1)

1. **Calculate Coefficients:**
   $A = 2(1)(4)\cos(0) - 2(2)(4)\cos(0) = 8 - 16 = \mathbf{-8}$
   $B = 2(1)(4)\sin(0) - 2(2)(4)\sin(0) = 0$
   $C = 1^2 + 2^2 + 4^2 - 3.5^2 - 2(1)(2)(\cos(0)\cos(0) + \sin(0)\sin(0))$
   $= 1+4+16-12.25-4 = \mathbf{4.75}$
2. Solve for $\theta_4$:
   $$t = \frac{-0 + 1\sqrt{(0^2 - 4.75^2 + (-8)^2}}{4.75 - (-8)} = \frac{\sqrt{41.4375}}{12.75} \approx \mathbf{0.5049}$$
   $\theta_4 = 2\tan^{-1}(0.5049) = \mathbf{53.58°}$
3. Solve for $\theta_3$:
   $N = 1\sin(0) + 4\sin(53.58°) - 2\sin(0) \approx \mathbf{3.2187}$
   $D = 1\cos(0) + 4\cos(53.58°) - 2\cos(0) \approx 1 + 2.3748 - 2 = \mathbf{1.3748}$
   $\theta_3 = \text{atan2}(3.2187, 1.3748) = \mathbf{66.87°}$

**TABLE 7.2 Summary of results for Example 7.1**

| $\theta_2$ | $\sigma$ | $A$ | $B$ | $C$ | $\theta_4$ | $\theta_3$ |
|---|---|---|---|---|---|---|
| 0 | 1 | -8 | 0 | 4.75 | 53.58° | 66.87° |
| | –1 | | | | -53.58° | -66.87° |
| π/2 | 1 | 8 | -16 | 8.75 | 177.28° | -143.85° |
| | -1 | | | | 55.85° | 21.98° |
| π | 1 | 24 | 0 | 12.75 | -122.09° | -75.52° |
| | -1 | | | | 122.09° | 75.52° |
| -π/2 | 1 | 8 | 16 | 8.75 | -55.85° | -21.98° |
| | -1 | | | | -177.28° | 148.85° |

# From Equations to Executable Code

The true power of an analytical solution is that it can be directly implemented in code. A simple function can solve the position for any input angle, for either assembly mode.

## Summary of Formulas

$$A = 2r_1r_4\cos\theta_1 - 2r_2r_4\cos\theta_2$$

$$B = 2r_1r_4\sin\theta_1 - 2r_2r_4\sin\theta_2$$

$$C = r_1^2 + r_2^2 + r_4^2 - r_3^2 - 2r_1r_2(\cos(\theta_1 - \theta_2))$$

$$t = \left[\frac{-B + \sigma\sqrt{A^2 + B^2 - C^2}}{C - A}\right]$$

$$\theta_4 = 2 * \mathrm{atan}(t)$$

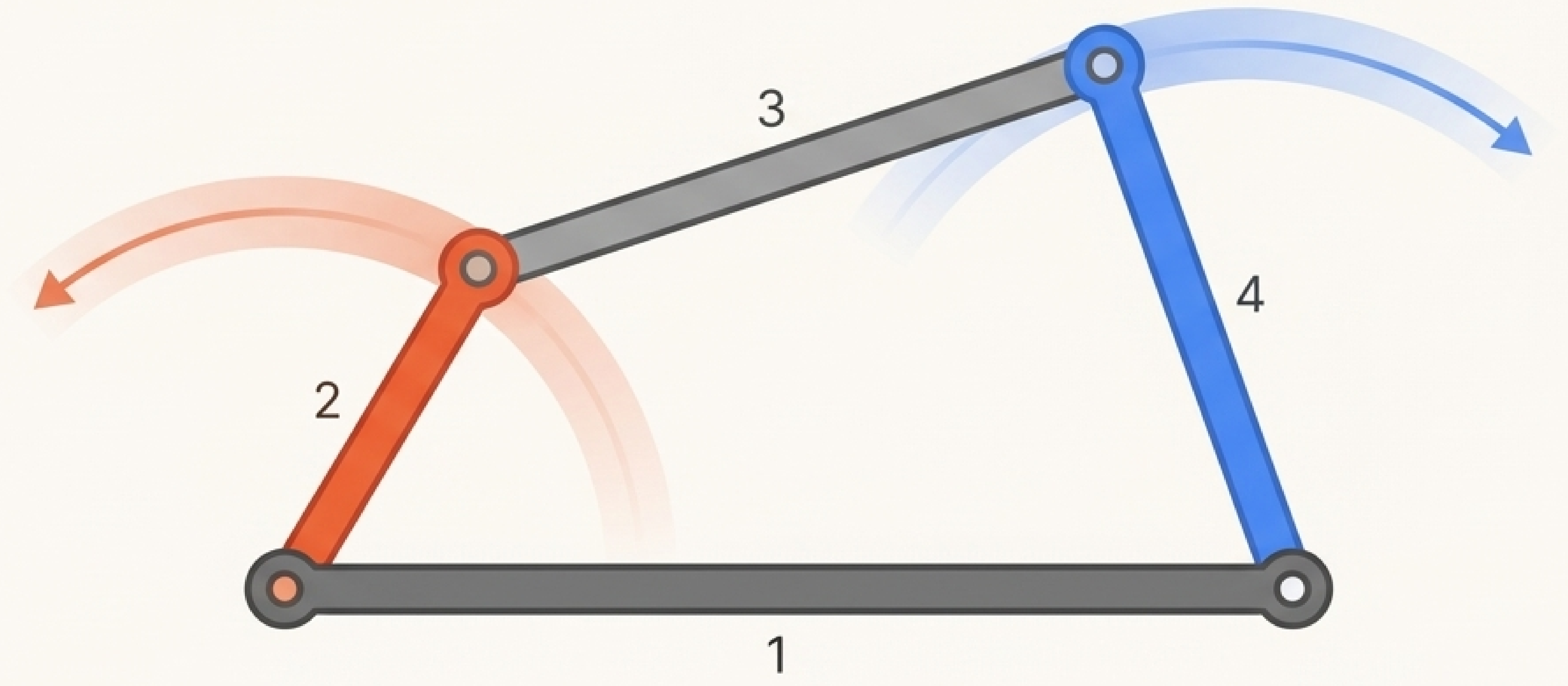
$$\theta_3 = \mathrm{atan2}(N, D)$$

## Python Code Implementation

```
import numpy as np

def solve_position(r1, r2, r3, r4, theta1, theta2, sigma):
    """Solves for the position of a 4-bar linkage."""

    # Calculate intermediate coefficients A, B, C (Eq. 7.32)
    A = 2*r1*r4*np.cos(theta1) - 2*r2*r4*np.cos(theta2)
    B = 2*r1*r4*np.sin(theta1) - 2*r2*r4*np.sin(theta2)
    C = r1**2 + r2**2 + r4**2 - r3**2 - 2*r1*r2*np.cos(theta1 - theta2)

    # Check for valid assembly (discriminant must be non-negative)
    discriminant = A**2 + B**2 - C**2
    if discriminant < 0:
        return None, None  # No real solution

    # Solve for t using quadratic formula (Eq. 7.35)
    t = (-B + sigma * np.sqrt(discriminant)) / (C - A)

    # Solve for theta4 (Eq. 7.36)
    theta4 = 2 * np.arctan(t)

    # Solve for theta3 using atan2 (Eq. 7.37)
    N = r1*np.sin(theta1) + r4*np.sin(theta4) - r2*np.sin(theta2)
    D = r1*np.cos(theta1) + r4*np.cos(theta4) - r2*np.cos(theta2)
    theta3 = np.arctan2(N, D)

    return theta3, theta4
```

# Bringing the Kinematics to Life: Assembly Mode 1

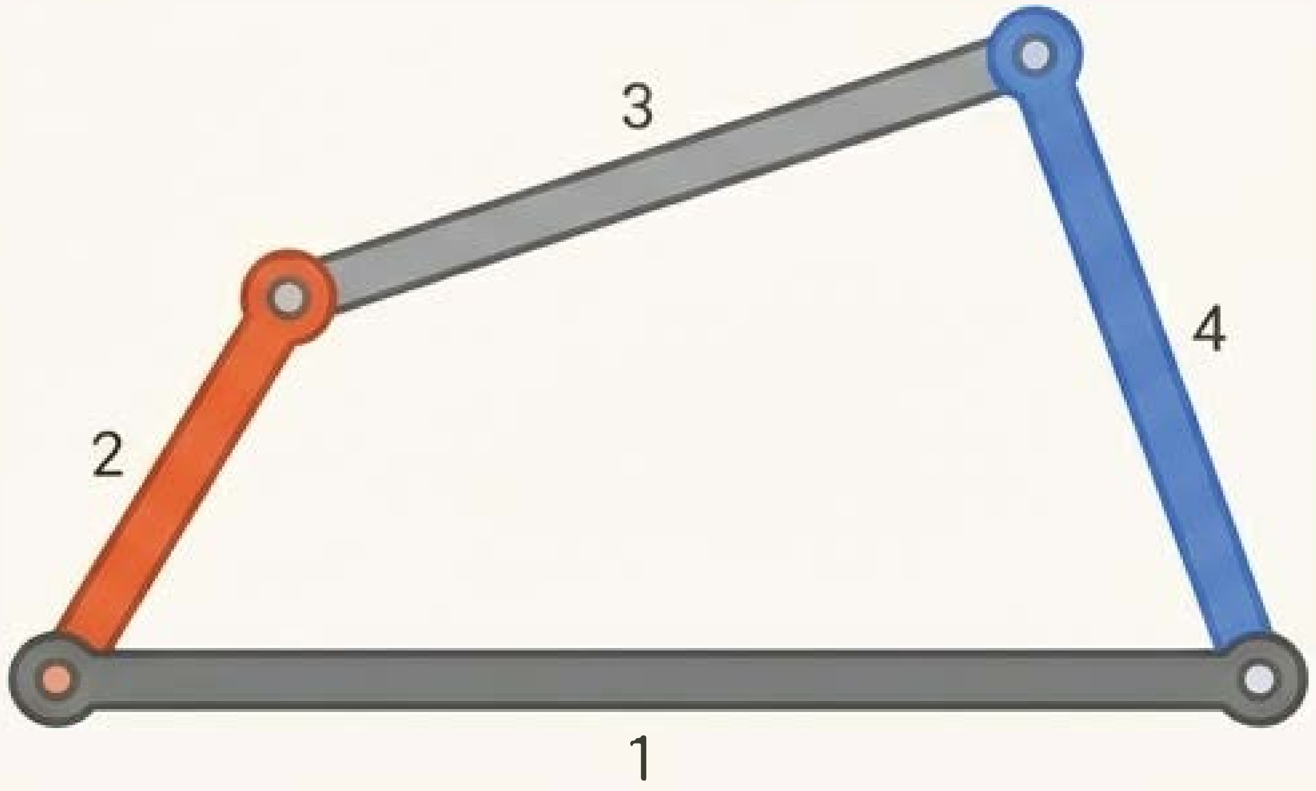
We use our Python function in a loop, calculating $\theta_3$ and $\theta_4$ for a full 360° rotation of the input crank ($\theta_2$). By plotting the joint coordinates at each step, we generate a smooth animation of the linkage's motion.



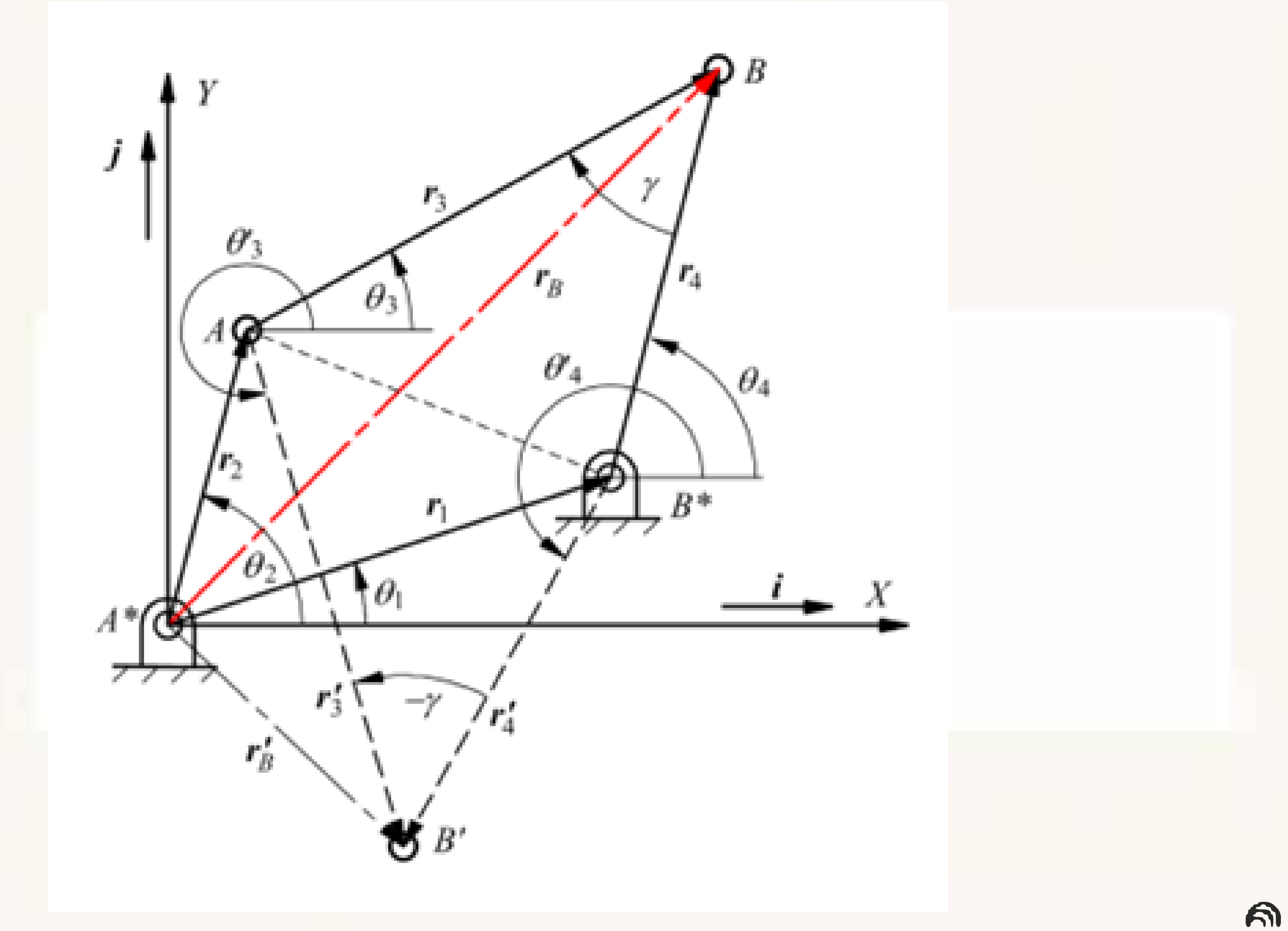
# Visualizing the Second Solution: Assembly Mode 2

To generate the motion for the second assembly mode, we simply change the sigma parameter in our Python function from +1 to -1 and re-run the simulation. The same equations generate a completely different, yet equally valid, motion.

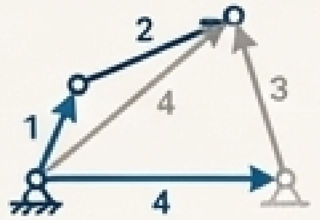
Assembly Mode: $\sigma = +1$




Assembly Mode: $\sigma = -1$



# Recap: The 5-Step Process for Positional Analysis




### 1. Model

Represent the linkage with vectors and write the vector loop closure equation.

$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4$$


### 2. Isolate

Decompose into scalar equations and algebraically rearrange to isolate the terms of the first unknown to be eliminated (e.g., $\theta_3$).

$$r_3\cos\theta_3 = \ldots \text{ and } r_3\sin\theta_3 = \ldots$$

$\sin^2\theta + \cos^2\theta \to 1$

### 3. Eliminate

Square and add the rearranged equations. Use the identity $\sin^2\theta + \cos^2\theta = 1$ to eliminate one variable. A single equation with one unknown angle (e.g., $\theta_4$) remains.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

### 4. Solve

$$(C - A)t^2 + 2Bt + (A + C) = 0$$

Simplify the equation into the form $A\cos\theta + B\sin\theta + C = 0$. Use the tangent half-angle substitution to convert it to a quadratic equation and solve for the two roots ($\sigma = \pm 1$), which represent the two assembly modes.


### 5. Back-Substitute

With one angle known, use the isolated equations from Step 2 to solve for the other unknown angle. Use the atan2(y, x) function for quadrant correctness.

# The Foundation is Set

## Summary of Accomplishments

- We have successfully derived a complete, closed-form analytical solution for the position of a four-bar linkage.
- This method provides precise results and is the basis for computational kinematic analysis and simulation.
- We have seen how the mathematics directly corresponds to the physical behavior of the mechanism, including its dual assembly modes.

## Looking Ahead: Next Steps

- This position analysis is the essential first step.
- **Next Lecture:** We will build directly on this foundation by differentiating our position equations with respect to time to perform **Velocity and Acceleration Analysis**.
- Full **dynamic and force analysis**, which builds upon velocity and acceleration, is a key topic in advanced dynamics courses such as **ME 5751**.