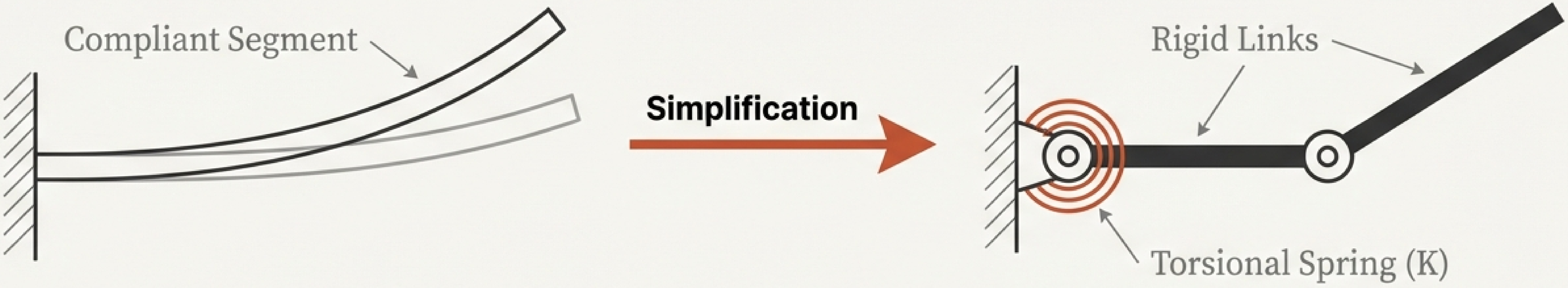
Kinetostatic Analysis of Compliant Mechanisms

Applying the Principle of Virtual Work to Pseudo-Rigid-Body Models

Visual metaphor demonstrating the scalability of compliant mechanism principles from macro-scale devices (left, parallel motion brakes) to microscopic technology (right, MEMS device). Key elements and actuation paths are highlighted in rust-orange.

From Kinematic Models to Force Analysis



Recap: In the last lecture, we learned to simplify complex flexible segments into Pseudo-Rigid-Body Models (PRBMs), consisting of rigid links and torsional springs. This is a powerful tool for kinematic analysis.

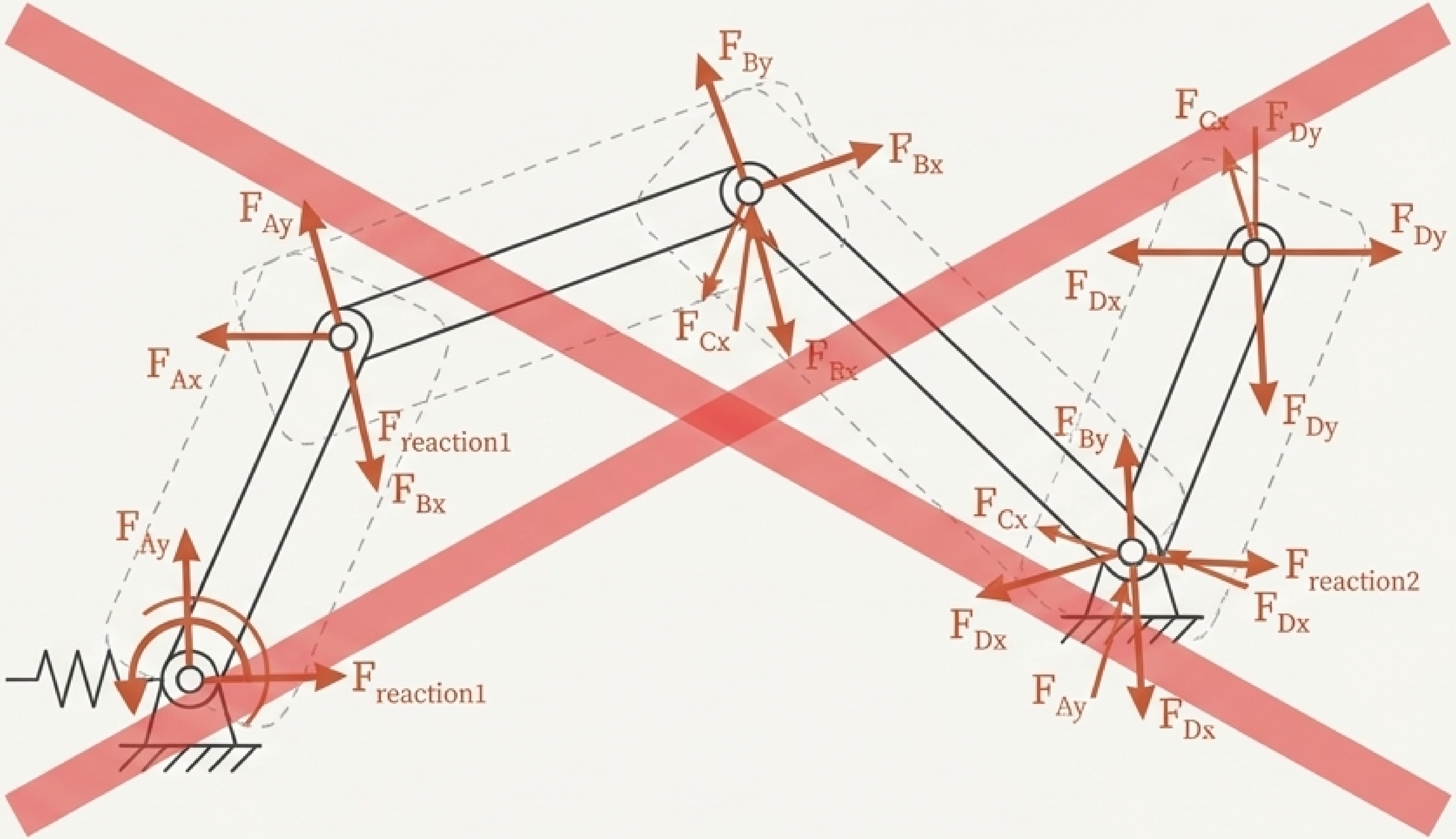
The Challenge: How do we use this simplified model to determine the relationship between applied forces and the resulting deflections? This is the core question of *kinetostatic analysis*.

Why it's complex: Unlike purely rigid mechanisms, compliant mechanisms store potential energy in their flexible members. This energy storage, combined with large, nonlinear deflections, makes force analysis non-trivial.

The Challenge with Traditional Statics

Consider a compliant mechanism modeled as a multi-link PRBM. A traditional approach using free-body diagrams for each link would require:

1. Isolating each rigid link.
2. Drawing all external forces, spring torques, and internal reaction forces at each pin joint.
3. Writing multiple equilibrium equations ($\Sigma F=0$, $\Sigma M=0$) for each link.
4. Solving a large system of simultaneous equations.



Key Insight: This process is cumbersome and forces us to solve for many internal reaction forces that we often don't need. We need a more elegant, energy-based approach.

A More Powerful Tool: The Principle of Virtual Work

"For a system in static equilibrium, the total virtual work done by all active forces and moments for any virtual displacement is zero."

$$\delta W = \sum (\mathbf{F}_i \cdot \delta \mathbf{r}_i) + \sum (\mathbf{T}_i \cdot \delta \theta_i) = 0$$

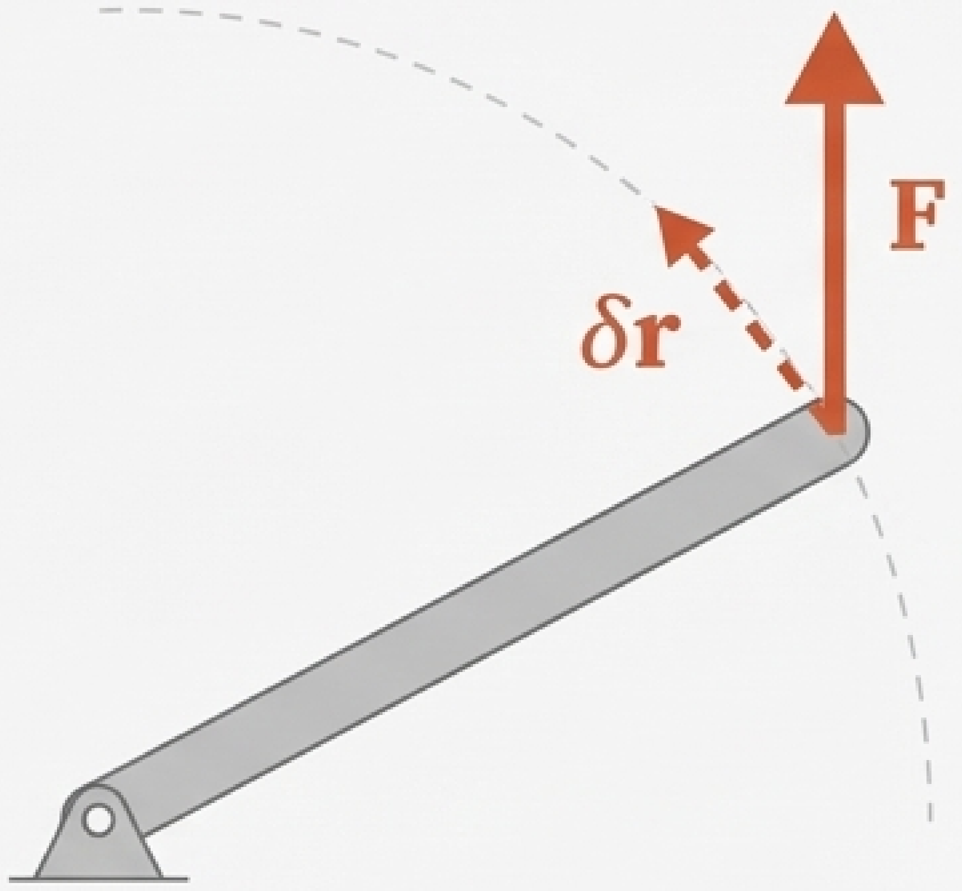
δW is the total virtual work.

$\mathbf{F}_i$ is an applied external force vector.

$\delta \mathbf{r}_i$ is the virtual displacement of the point of application of $\mathbf{F}_i$.

$\mathbf{T}_i$ is an applied external torque.

$\delta \theta_i$ is the virtual angular displacement.



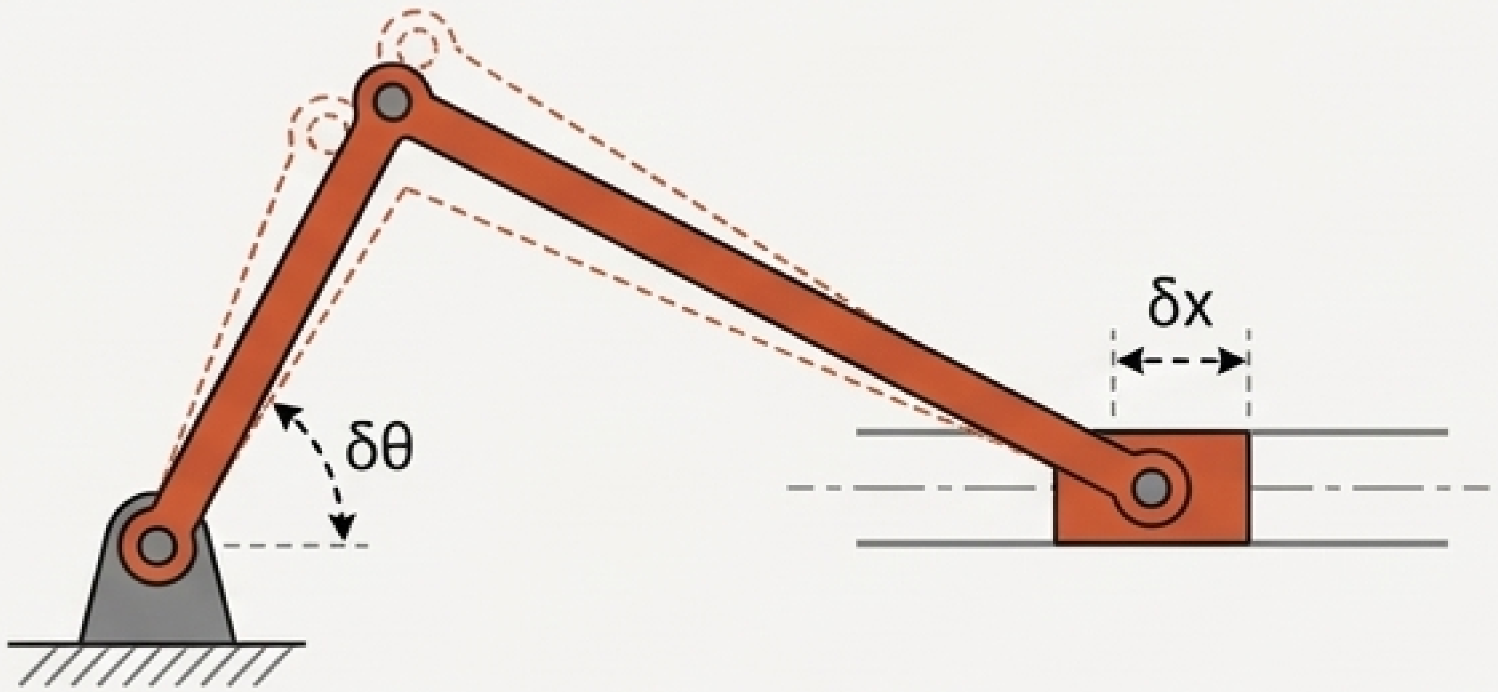
Demystifying the Key Concepts

Concept 1: Virtual Displacement (δr, $\delta\theta$)

A virtual displacement is a small, imaginary, instantaneous displacement of a point or body.

It is not a real displacement that occurs over time.

Crucially, it must be *consistent with the constraints* of the mechanism (e.g., a link pinned to ground can only rotate).

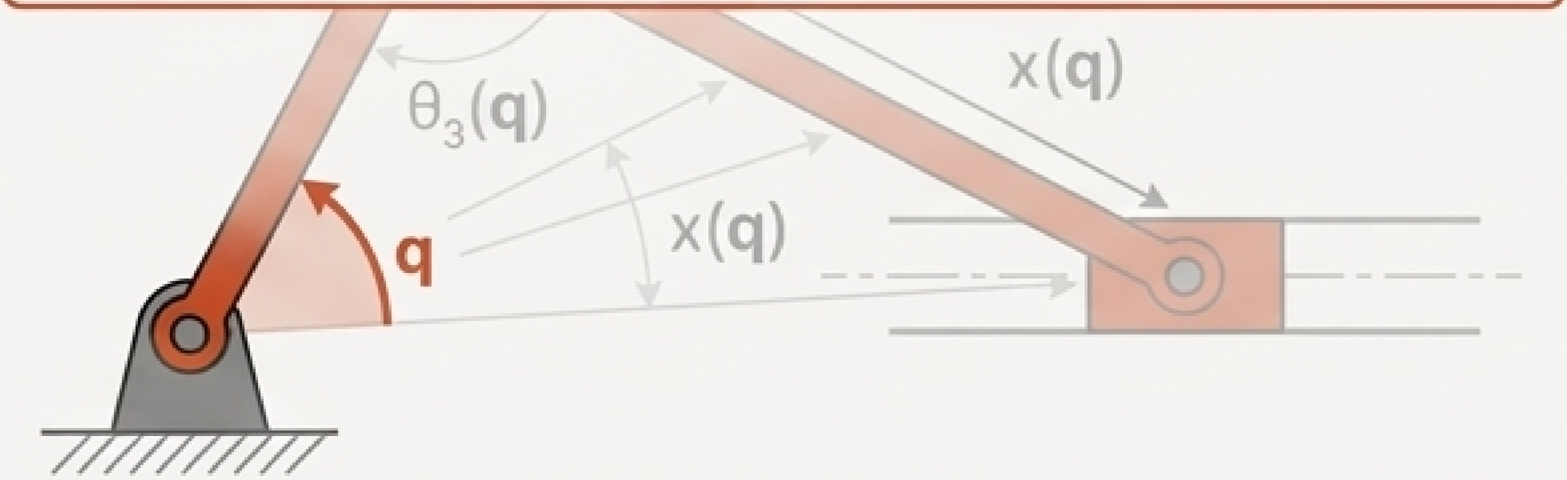


Concept 2: Generalized Coordinate (q)

For a single-degree-of-freedom mechanism, the position of every link can be described by a single independent variable. This is the generalized coordinate.

Examples: The input angle of a crank (θ_2), the displacement of a slider.

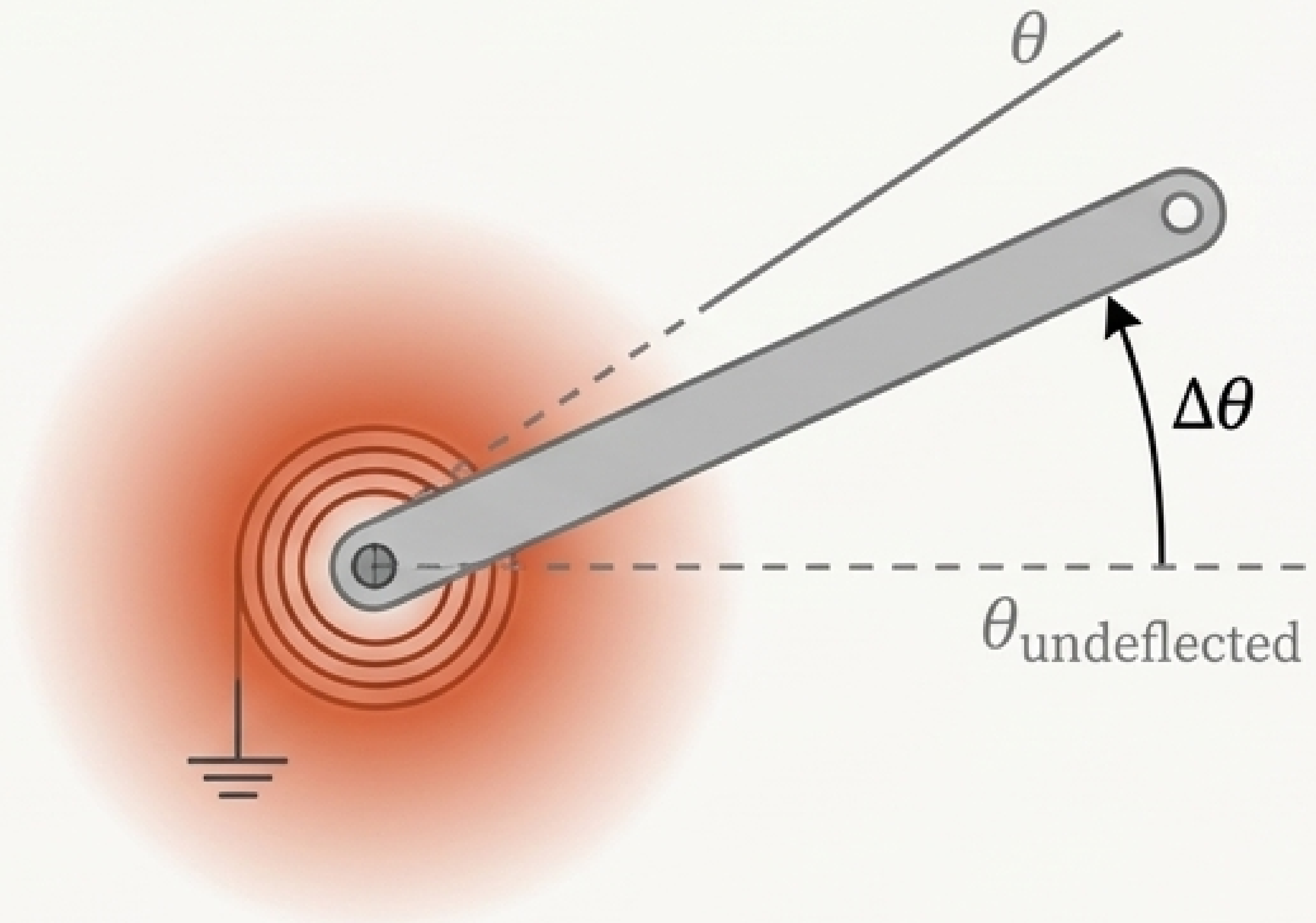
The Power of 'q': By expressing all forces and displacements in terms of `q` and its virtual change `δq`, we can reduce a complex system of equations to a single equation with a single variable.



The Energy Perspective: Work and Potential

An alternative and often simpler formulation of the PVW states that the virtual work done by ***external non-conservative forces*** (δW_{ext}) must equal the change in the system's stored ***potential energy*** (δU).

$$\delta W_{\text{ext}} = \delta U$$



Where does the potential energy come from? In our PRBMs, potential energy (U) is stored in the deflected torsional springs.

The potential energy stored in a single torsional spring is:

$$U_{\text{spring}} = \frac{1}{2} K \, (\Delta\theta)^2$$

Where K is the torsional spring constant and $\Delta\theta$ is the angular deflection from its undeflected position.

Total Potential Energy: The total potential energy of the system is the sum of the potential energy stored in all springs:

$$U_{\text{total}} = \sum \frac{1}{2} K_i \, (\Delta\theta_i)^2$$

The Kinetostatic Analysis Method: A Step-by-Step Guide

1. **Model the System**
   Draw the compliant mechanism and its corresponding Pseudo-Rigid-Body Model (PRBM).

2. **Select a Generalized Coordinate (q)**
   Choose a single independent variable (like an input angle θ_2) that defines the entire mechanism's configuration.

3. **Identify Energies and Work**
   Write an expression for the total potential energy (U) stored in all torsional springs as a function of q.
   Write expressions for the position vectors ($\vec{\mathbf{r}}_i$) of all externally applied forces ($\vec{\mathbf{F}}_i$).

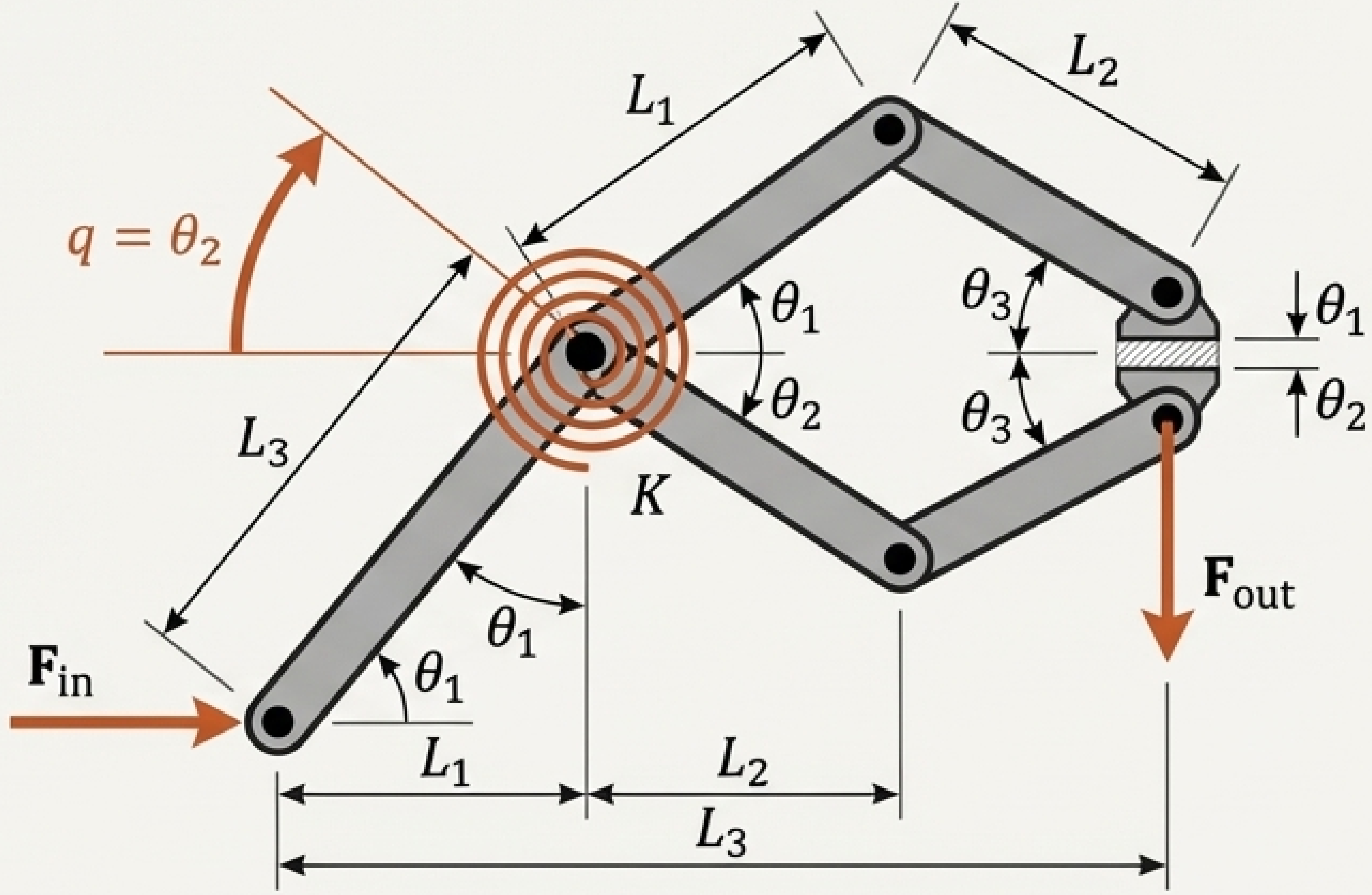
4. **Apply the PVW (Energy Method)**
   Calculate the change in potential energy, δU, by taking the derivative of U with respect to q: $\delta U = (dU/dq)\delta q$.
   Calculate the external virtual work, $\delta W_{\text{ext}} = \sum \vec{\mathbf{F}}_i \cdot \delta\vec{r}_i$. Note that $\delta\vec{r}_i = (d\vec{r}_i/dq)\delta q$.

5. **Solve**
   Set $\delta W_{\text{ext}} = \delta U$ and solve the resulting equation for the unknown force, torque, or displacement.

Example 1: A Compliant Gripper (Part 1 - Setup)

Determine the relationship between the **input force** (F_{in}), the **gripping force** (F_{out}), and the mechanism's position.

- Step 1: The Model
  - The compliant hand tool is simplified to its corresponding PRBM, highlighting the rigid links, pin joints, and torsional spring (K).
- Step 2: Generalized Coordinate
  - We select the input angle, θ_2, as the generalized coordinate, so $q = \theta_2$.
- Forces:
  - Input Force: $\mathbf{F}_{in}$ (applied horizontally)
  - Output Force: $\mathbf{F}_{out}$ (gripping force, vertical)



Example 1: Compliant Gripper (Part 2 - Analysis)

Step 3: Position Vectors

- Define an origin at the grounded pin.
- Vector to input force: $\vec{\mathbf{r}}_{\text{in}} = L_1 \cos(\theta_2)\,\vec{\mathbf{i}} + L_1 \sin(\theta_2)\,\vec{\mathbf{j}}$
- Vector to output force: $\vec{\mathbf{r}}_{\text{out}} = L_2 \sin(\theta_3)\,\vec{\mathbf{i}} - L_2 \cos(\theta_3)\,\vec{\mathbf{j}}$
- (Note: θ_3 is geometrically related to θ_2)

Step 4: Apply PVW

- **External Work:** $\delta W_{\text{ext}} = \mathbf{F}_{\text{in}} \cdot \delta\vec{\mathbf{r}}_{\text{in}} + \mathbf{F}_{\text{out}} \cdot \delta\vec{\mathbf{r}}_{\text{out}}$
- **Potential Energy**: $U = \frac{1}{2}K\,(\Delta\theta)^2$ (assuming one main spring for simplicity, where $\Delta\theta$ depends on the mechanism's angles)
- **Differentiate:** $\delta U = K(\Delta\theta)\,\delta(\Delta\theta)$
- Relate all virtual displacements to $\delta q = \delta\theta_2$ using the chain rule.

Step 5: Solve

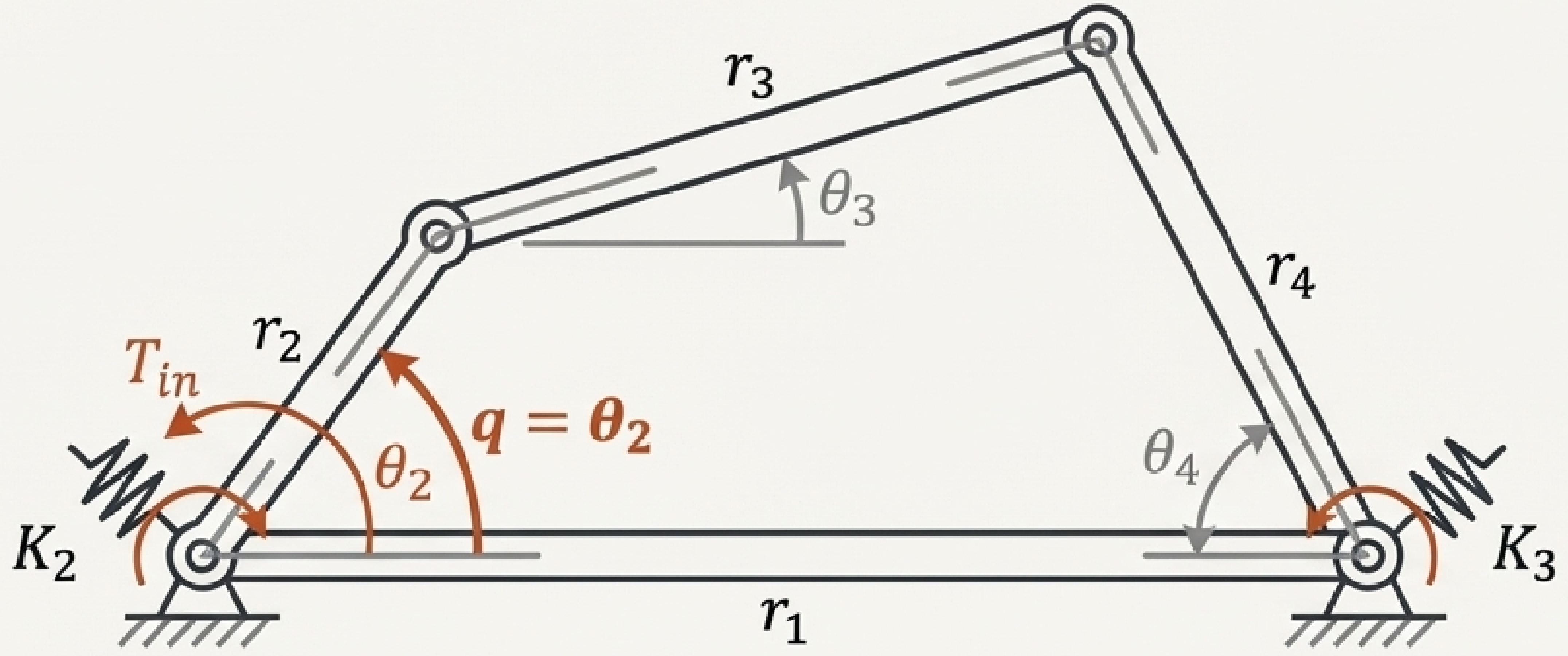
- Set $\delta W_{\text{ext}} = \delta U$.
- The $\delta\theta_2$ terms on both sides cancel out, leaving a single static equilibrium equation.
- **Result:**

$$\boxed{\boldsymbol{F}_{\text{in}} = f(F_{\text{out}}, \theta_2, K, \text{geometry})}$$

- This final equation provides the complete force-deflection relationship for the gripper.

Example 2: The Compliant Four-Bar Mechanism (Part 1 - Setup)

Determine the input torque (T_{in}) required to hold the mechanism in equilibrium for a given configuration (θ_2).



- **The Model:** A compliant four-bar mechanism is represented by its PRBM, with four rigid links (r_1, r_2, r_3, r_4) and two torsional springs at the grounded pivots (K_2 and K_3, corresponding to deflections Ψ_2 and Ψ_3).
- **Generalized Coordinate:** The input crank angle, $q = \theta_2$.
- **Goal:** Find the relationship: $T_{\text{in}} = f(\theta_2, K_2, K_3, \text{geometry})$

Example 2: The Compliant Four-Bar (Part 2 - Kinematics)

The Core Challenge

To calculate the total potential energy $U(\theta_2)$, we must express the deflection of both springs in terms of the single generalized coordinate, θ_2.

Relative Spring Deflections

The potential energy depends on the relative deflections, Ψ_2 and Ψ_3, where $\Psi_i = \theta_i - \theta_{i,\text{undeflected}}$.

Kinematic Coefficients

We need the derivatives that relate the angles of the other links to the input angle. These are the kinematic coefficients:

$$\frac{d\theta_3}{d\theta_2} = \frac{r_2 \sin(\theta_4 - \theta_2)}{r_3 \sin(\theta_3 - \theta_4)} \qquad \frac{d\theta_4}{d\theta_2} = \frac{r_2 \sin(\theta_3 - \theta_2)}{r_4 \sin(\theta_3 - \theta_4)}$$

****Significance**:**

These coefficients are the key. They allow us to use the chain rule when differentiating the total potential energy with respect to our single generalized coordinate, θ_2.

Example 2: The Compliant Four-Bar (Part 3 - PVW Analysis)

Step 3 & 4: Potential Energy & PVW

- **Total Potential Energy**: $U = \frac{1}{2}K_2\Psi_2^2 + \frac{1}{2}K_3\Psi_3^2$
- **Virtual Work by Input Torque**: $\delta W_{\text{ext}} = T_{\text{in}} \cdot \delta\theta_2$
- **Apply Energy Balance**: $\delta W_{\text{ext}} = \delta U$
- $T_{\text{in}} \cdot \delta\theta_2 = \frac{dU}{d\theta_2} \cdot \delta\theta_2$

Step 5: Differentiate and Solve

- Canceling $\delta\theta_2$ gives: $T_{\text{in}} = \frac{dU}{d\theta_2} = \frac{d}{d\theta_2}\left(\frac{1}{2}K_2\Psi_2^2 + \frac{1}{2}K_3\Psi_3^2\right)$
- Applying the chain rule:
- $T_{\text{in}} = K_2\Psi_2\frac{d\Psi_2}{d\theta_2} + K_3\Psi_3\frac{d\Psi_3}{d\theta_2}$
- Since $\Psi_i = \theta_i$ - constant, we know $\frac{d\Psi_i}{d\theta_2} = \frac{d\theta_i}{d\theta_2}$. We can now substitute the kinematic coefficients from the previous slide.

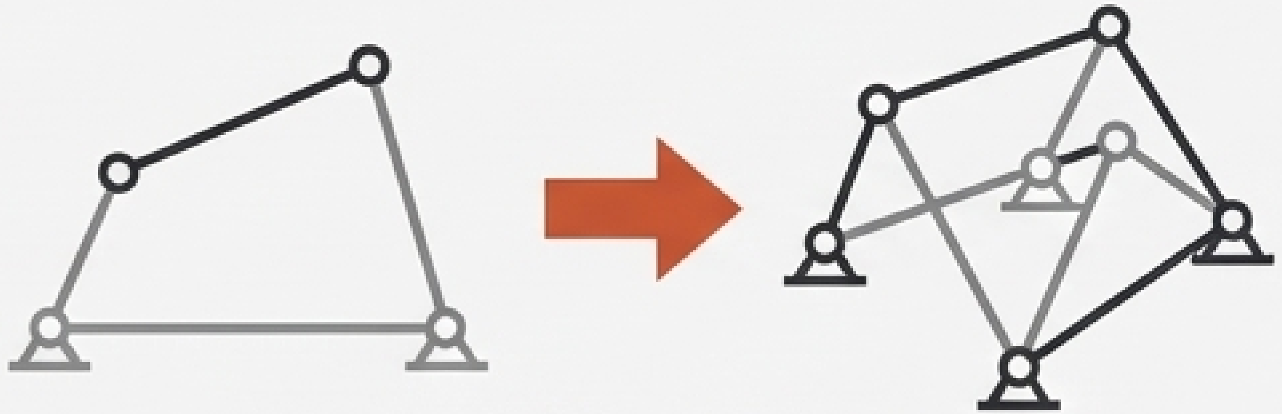
$$\boxed{T_{\text{in}} = K_2\left(\theta_2 - \theta_{2,\text{undeflected}}\right) + K_3\left(\theta_3 - \theta_{3,\text{undeflected}}\right)\left[\frac{r_2\sin(\theta_4 - \theta_2)}{r_3\sin(\theta_3 - \theta_4)}\right]}$$

This is the complete kinetostatic model, relating input torque to the mechanism's position, stiffness, and geometry.

The Power of the Virtual Work Method

Bypasses Internal Forces

It allows us to relate external forces, torques, and stored energy directly, without ever needing to calculate the complex reaction forces at the internal joints.

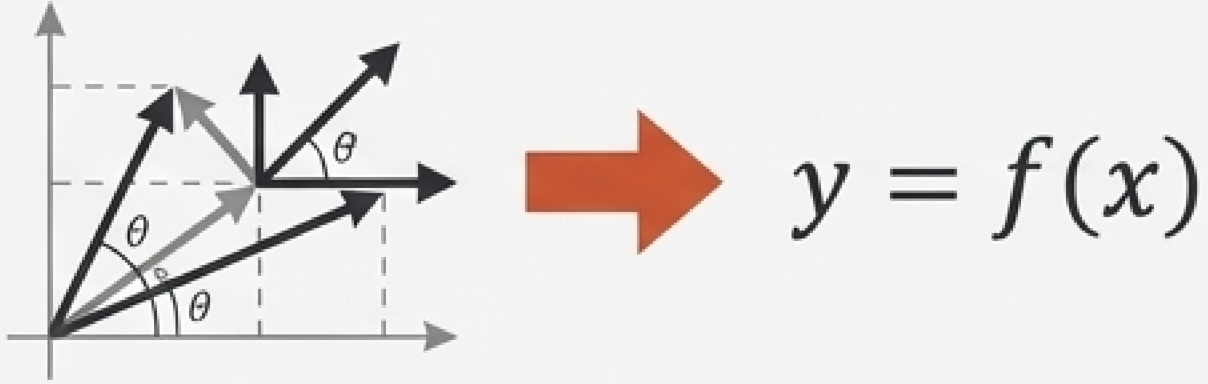


Systematic & Scalable

Provides a clear, step-by-step procedure that works for both simple and complex mechanisms.

Energy-Centric

Perfectly suited for compliant mechanisms, where stored potential energy is a fundamental part of the system's behavior.



Simplifies the Problem

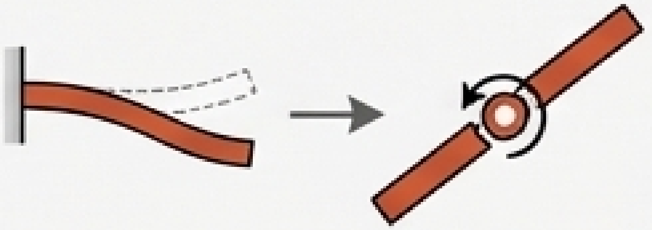
It transforms a complex vector-based statics problem into a more straightforward scalar differentiation problem based on a single generalized coordinate.

NotebookLM

Key Takeaways

We can accurately analyze the force-deflection behavior of complex compliant mechanisms by following a three-step process:

1. Simplify with PRBMs



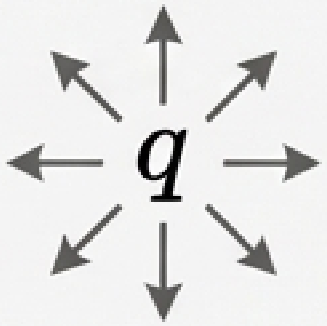
Model flexible segments as rigid links with torsional springs to make the kinematics tractable.

2. Apply the Principle of Virtual Work

$$\delta W_{\text{ext}} = \delta U$$

Use the energy balance equation as the governing principle for static equilibrium.

3. Use a Generalized Coordinate



Express the entire system's energy and work in terms of a single independent variable (q) to reduce the problem to a single, solvable equation.

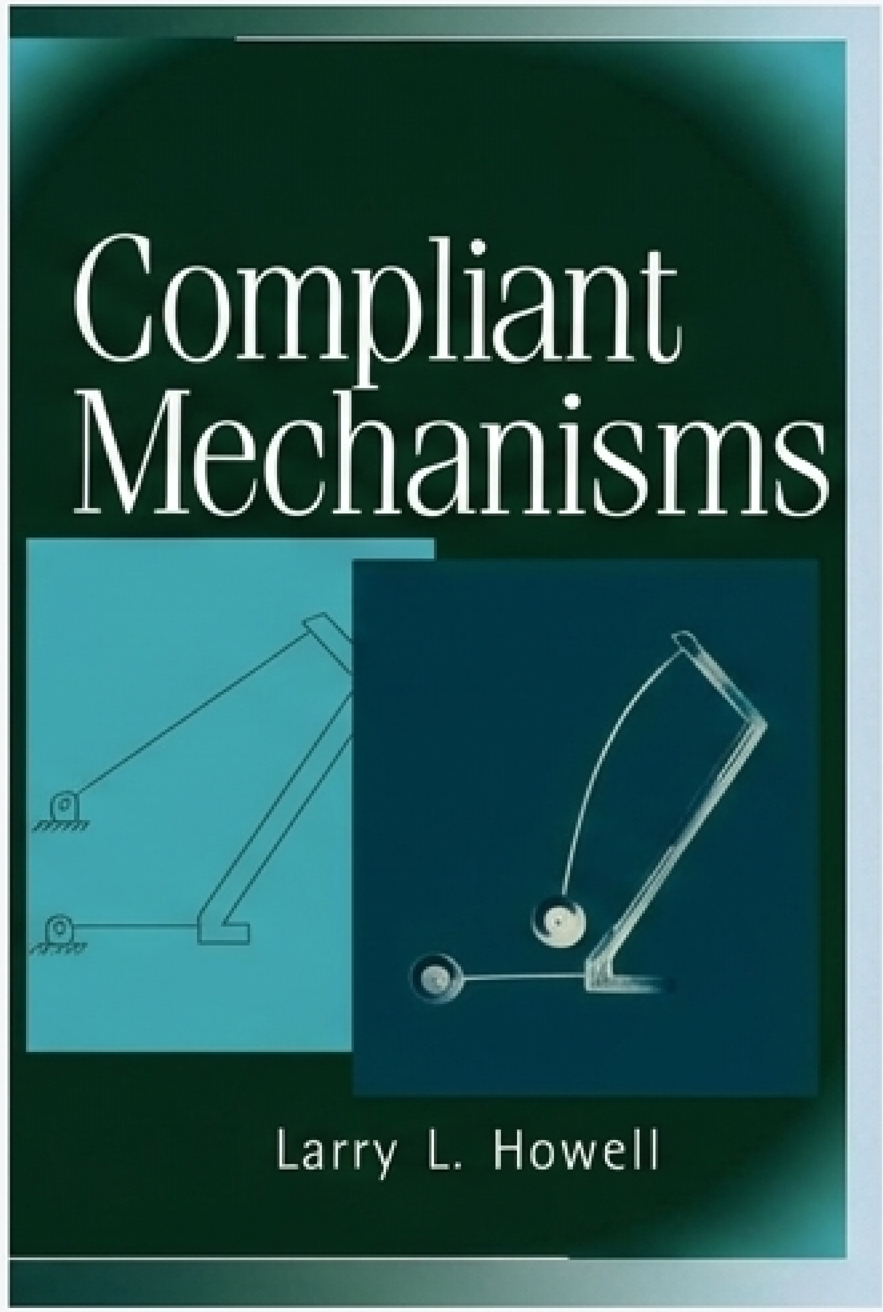
Result

Force-Deflection Relationship

Further Study & Next Steps

Primary Reference

The methods and examples discussed are detailed in Larry L. Howell's foundational text, *Compliant Mechanisms*. Chapter 6 (Force-Deflection Analysis) and Chapter 11 (Bistable Mechanisms) are particularly relevant.



Looking Ahead

In our next lecture, we will explore methods for the *synthesis* of compliant mechanisms—how to design new mechanisms from scratch to achieve desired motion and force characteristics.

Questions?