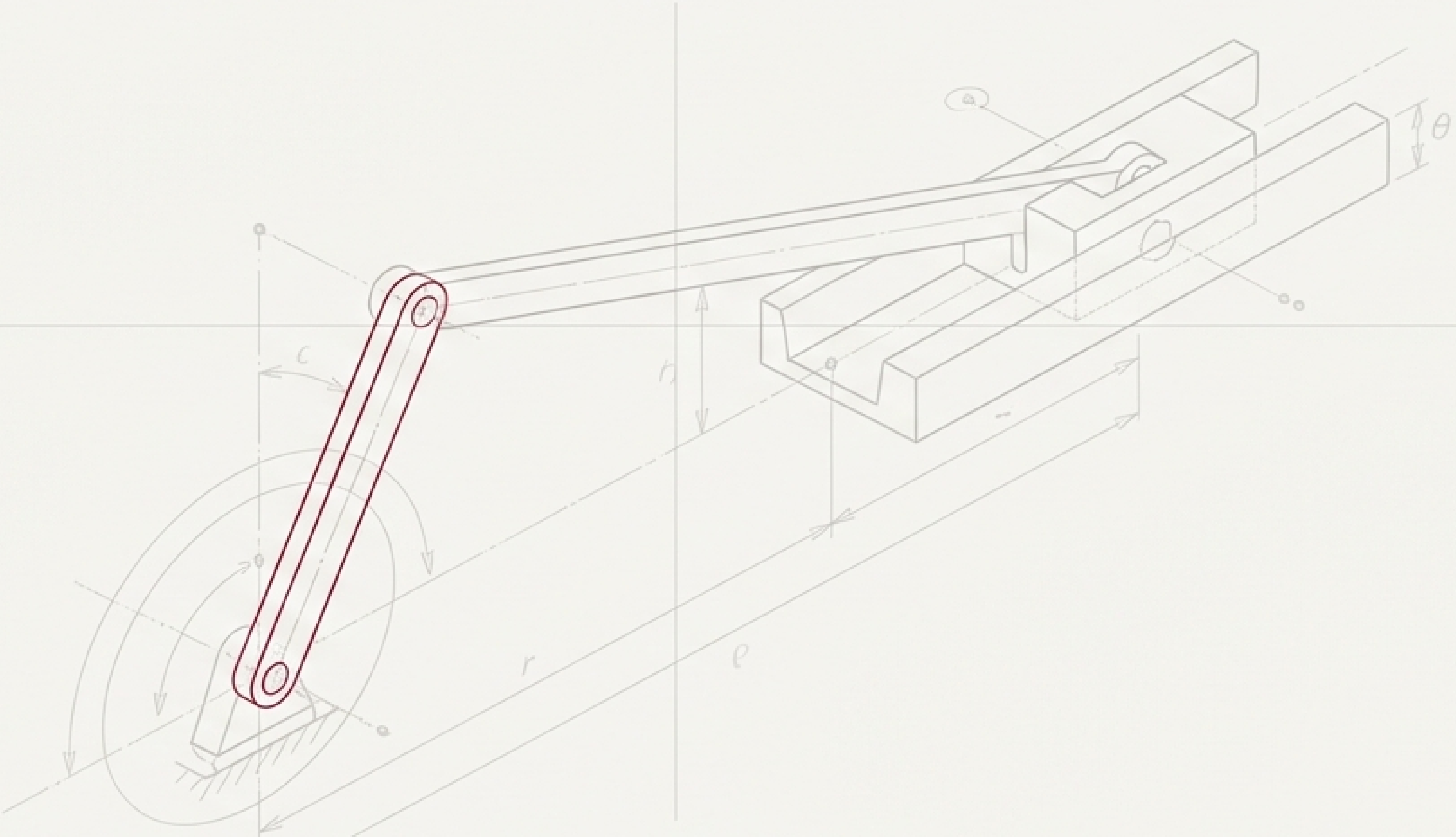
ME 3751: Kinematics and Mechanism Design

Analytical Positional Analysis of the Planar Slider-Crank



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Today's Agenda & Learning Objectives

Lecture Agenda

1. **The Challenge**: Why analytical methods are essential for mechanism analysis.
2. **The Derivation**: A step-by-step mathematical formulation for the slider-crank's position.
   - Step 1: The Vector Loop
   - Step 2: Scalar Equations
   - Step 3: Solving for Slider Position (r_1)
   - Step 4: Solving for Coupler Angle (θ_3)
3. **The Application**: Applying the derived equations to a practical example (Example 7.4).
4. **The Implementation**: Translating the analysis into a computational script.

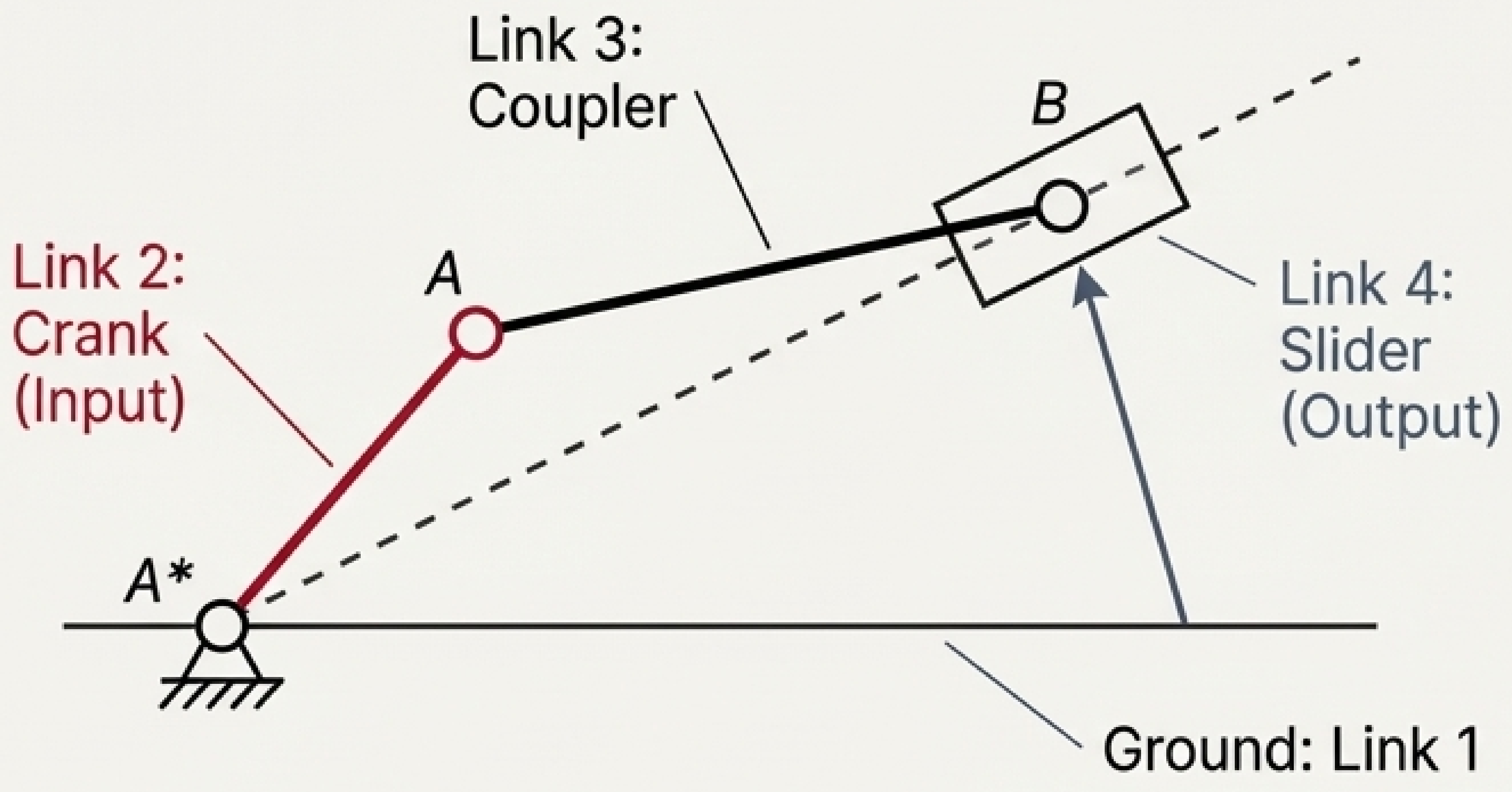
Learning Objectives

By the end of this lecture, you will be able to:

- ✓ Formulate the **vector loop closure equation** for a slider-crank linkage.
- ✓ Derive the **analytical solution** for the **slider position** (r_1) and **coupler angle** (θ_3).
- ✓ Explain the physical meaning of **assembly modes**.
- ✓ Apply the **derived equations** to solve for the position of a given slider-crank mechanism.

The Slider-Crank Mechanism & The Analytical Challenge

The Mechanism



The slider-crank (an RRRP mechanism) is a fundamental component in countless machines, from internal combustion engines to pumps and manufacturing equipment.

The Challenge

While graphical analysis is intuitive, it is insufficient for the repetitive or extensive analyses required in modern engineering design. We need a robust, analytical solution that can be programmed into a computer.

Knowns

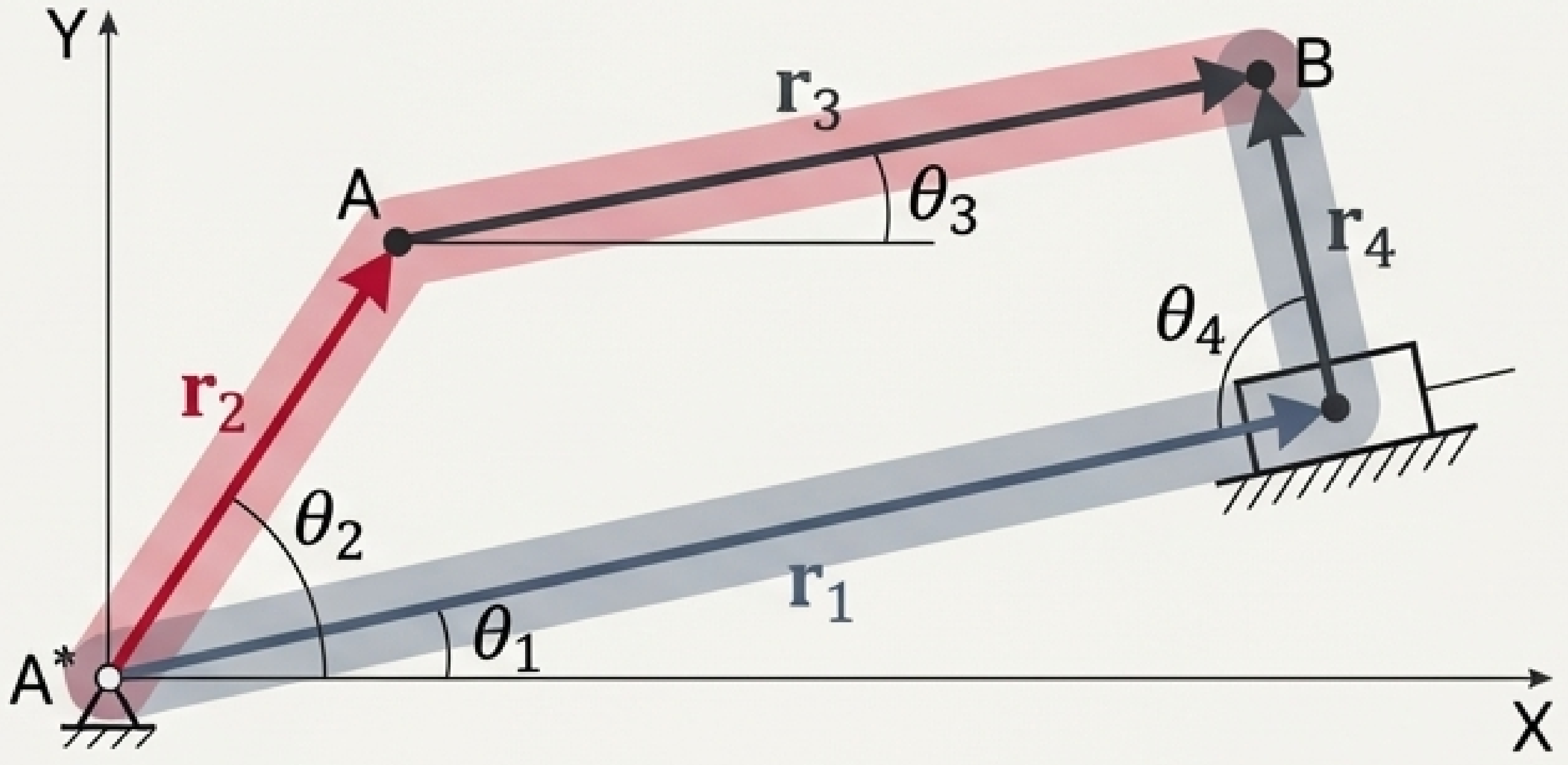
- Link lengths: r_2 (crank), r_3 (coupler) (r_2 is scarlet)
- Slider offset: r_4
- Slider path angle: θ_1
- Input crank angle: θ_2 (θ_2 is scarlet)

Unknowns to Solve For

- Slider position: r_1 (r_1 is slate blue)
- Coupler angle: θ_3 (θ_3 is slate blue)

Step 1: Formulating the Vector Loop Closure Equation

The geometric constraints of the closed-loop mechanism can be expressed by stating that the path from A* to B is the same, whether you trace the crank and coupler or the ground and slider offset.



$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4 \qquad \text{(Eq. 7.60)}$$

Vector Definitions

$\mathbf{r}_2$: Vector representing the crank (length r_2, angle θ_2)

$\mathbf{r}_3$: Vector representing the coupler (length r_3, angle θ_3)

$\mathbf{r}_1$: Vector along the slider path (variable length r_1, constant angle θ_1)

$\mathbf{r}_4$: Vector for the slider offset (constant length r_4, constant angle θ_4)

Key Relationship

For this mechanism, the offset vector $\mathbf{r}_4$ is always perpendicular to the slider path $\mathbf{r}_1$. Therefore:

$$\theta_4 = \theta_1 + \pi/2 \qquad \text{(Eq. 7.62)}$$

Step 2: From a Single Vector to Two Scalar Equations

To solve the vector equation, we decompose it into its scalar components along the fixed X and Y axes. Each vector $\mathbf{r}_k$ is expressed as $\mathbf{r}_k(\cos\theta_k\,\mathbf{i} + \sin\theta_k\,\mathbf{j})$.

$$\mathbf{r}_2(\cos\theta_2\,\mathbf{i} + \sin\theta_2\,\mathbf{j}) + \mathrm{r}_3(\cos\theta_3\,\mathbf{i} + \sin\theta_3\,\mathbf{j}) = \mathbf{r}_1(\cos\theta_1\,\mathbf{i} + \sin\theta_1\,\mathbf{j}) + \mathrm{r}_4(\cos\theta_4\,\mathbf{i} + \sin\theta_4\,\mathbf{j}) \quad \text{(Eq. 7.61)}$$

Decomposition yields two scalar equations with our two unknowns, $\mathbf{r}_1$ and θ_3:

X Component

$$\mathbf{r}_2\cos\theta_2 + \mathrm{r}_3\cos\theta_3 = \mathbf{r}_1\cos\theta_1 + \mathrm{r}_4\cos\theta_4 \quad \text{(Eq. 7.63)}$$

Y Component

$$\mathbf{r}_2\sin\theta_2 + \mathrm{r}_3\sin\theta_3 = \mathbf{r}_1\sin\theta_1 + \mathrm{r}_4\sin\theta_4 \quad \text{(Eq. 7.64)}$$

We now have a system of two nonlinear, coupled equations. Our next task is to manipulate them algebraically to solve for one unknown at a time.

Step 3 (Part A): The Strategy — Isolate to Eliminate

Our goal is to find r_1. The variable θ_3 is currently in both equations, making them difficult to solve directly. The strategy is to first eliminate θ_3.

The Process

1. **Isolate**: Rearrange both scalar equations to isolate the terms containing θ_3 on one side.
2. **Eliminate**: Use a trigonometric identity to combine the two equations into a single equation that only contains the unknown r_1.

[Two Scalar Equations] → Isolate θ_3 Terms → [Two Rearranged Equations] → **Eliminate θ_3** → [Single Equation for r_1]

Isolating the θ_3 terms:

From the X-component equation: $r_3 \cos\theta_3 = r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2$ (Eq. 7.65)

From the Y-component equation: $r_3 \sin\theta_3 = r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_2 \sin\theta_2$ (Eq. 7.66)

Step 3 (Part B): The Key Move — Eliminating θ_3

We now employ a powerful technique: square both rearranged equations and then add them together.

$$(r_3\cos\theta_3)^2 = (r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2)^2$$
$$(r_3\sin\theta_3)^2 = (r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2)^2$$

$$+$$

$$r_3^2(\cos^2\theta_3 + \sin^2\theta_3) = (r_1\cos\theta_1 + \dots)^2 + (r_1\sin\theta_1 + \dots)^2$$

$$\cos^2\theta_3 + \sin^2\theta_3 = 1$$

This identity causes θ_3 to vanish from the left side of the equation, leaving us with a single equation containing only the unknown r_1.

$$r_3^2 = r_1^2 + r_2^2 + r_4^2 + 2r_1r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4)$$
$$- 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4) \quad \text{(Eq. 7.67)}$$

Step 3 (Part C): The Quadratic Solution for Slider Position (r_1)

Equation 7.67 appears complex, but it is simply a quadratic equation in terms of r_1. By collecting terms, we can rearrange it into the standard form:

$$r_1^2 + Ar_1 + B = 0 \qquad \text{(Eq. 7.68)}$$

Where the coefficients A and B are constants based on known geometry and the input angle θ_2:

$A = 2r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4) - 2r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$	$B = r_2^2 + r_4^2 - r_3^2 - 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4)$

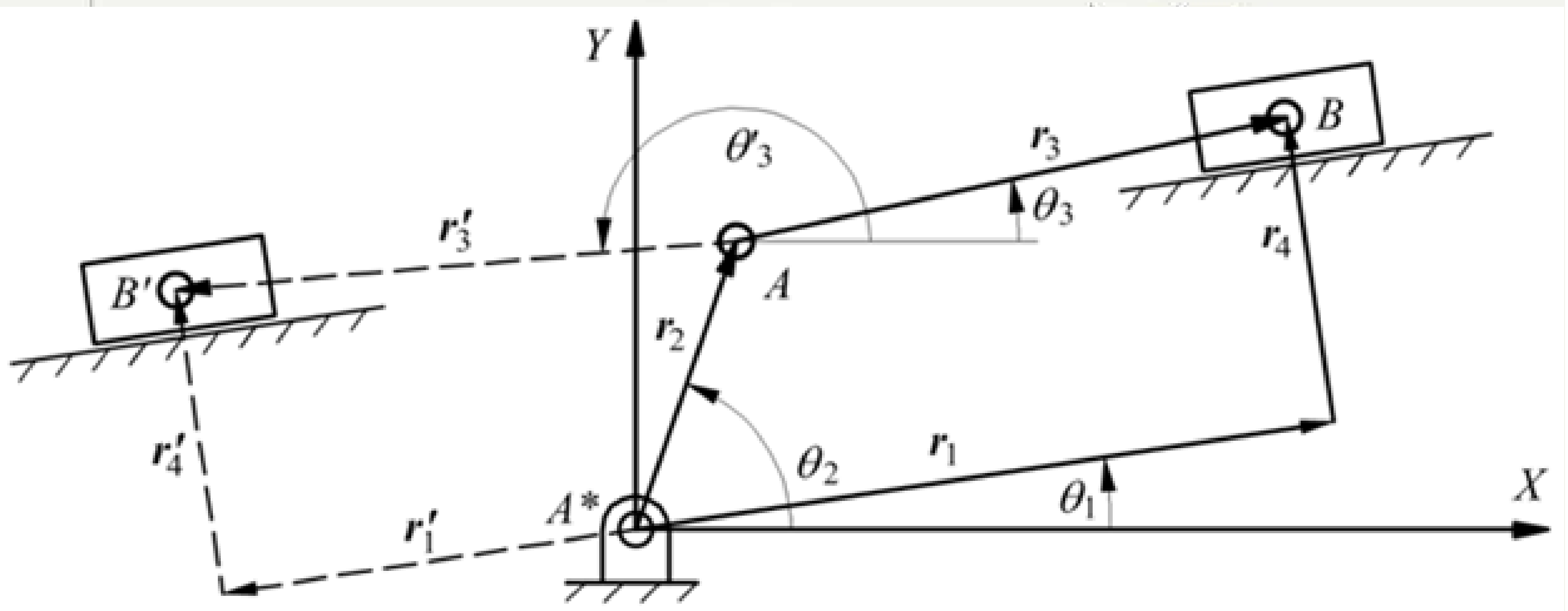
Note: The source text uses 'C' in one equation (7.69) but 'B' in the quadratic solution (7.70). We will use 'B' for consistency.

Solving for r_1 using the quadratic formula gives our final solution:

$$r_1 = \frac{-A \pm \sqrt{A^2 - 4B}}{2} \qquad \text{(Eq. 7.70)}$$

Understanding the Solution: Assembly Modes

The quadratic formula provides two possible mathematical solutions for r_1. These correspond to the two physically distinct ways the linkage can be assembled for a single input angle θ_2.



The Assembly Mode Indicator (σ)

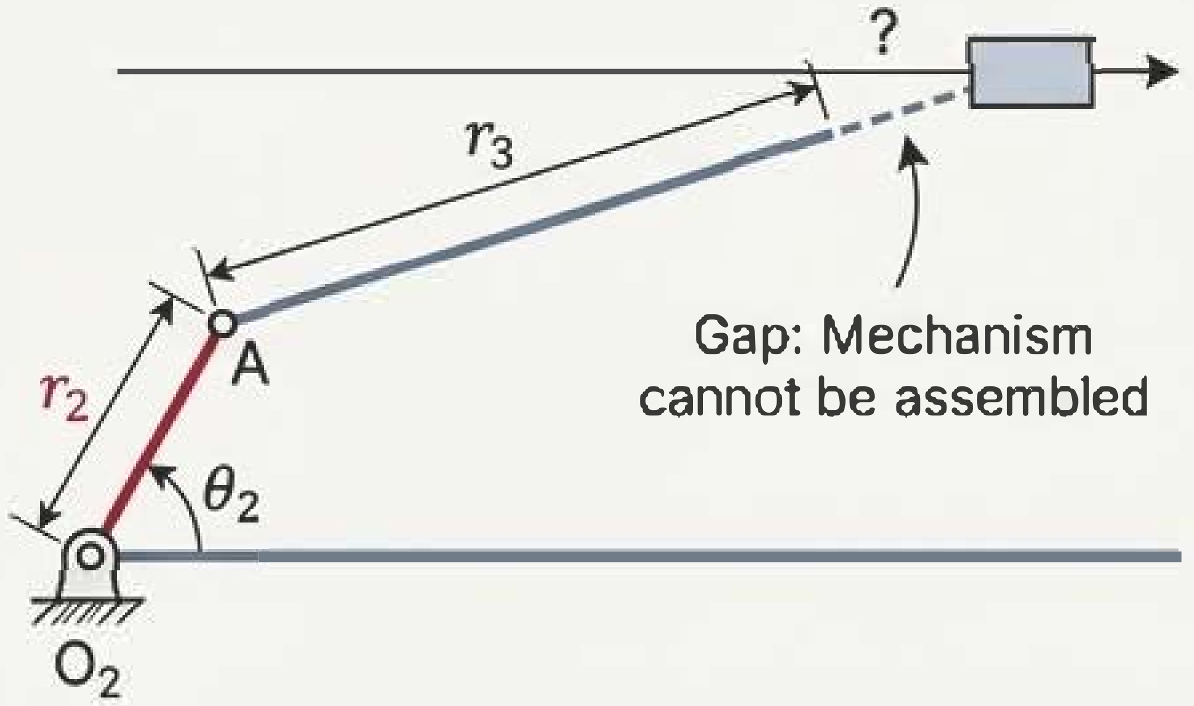
We introduce a sign variable, $\sigma = \pm 1$, to select the desired assembly mode.

$$r_1 = \frac{-A + \boldsymbol{\sigma}\sqrt{A^2 - 4B}}{2}$$

- $\sigma = +1$ selects one assembly mode.
- $\sigma = -1$ selects the other.
- For a continuous motion, σ remains constant.

Condition for Assemblability

If the discriminant ($A^2 - 4B$) is negative, the square root is imaginary. This means there is no real solution, and the mechanism cannot be physically assembled for the given link lengths and input angle θ_2.



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Step 4: Solving for the Coupler Angle (θ_3)

Now that we have a value for r_1 (for a chosen assembly mode σ), we can substitute it back into our isolated scalar equations (Eqs. 7.65 & 7.66) to solve for θ_3.

$r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2$	$r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2$

By dividing the sine equation by the cosine equation, we get the tangent of θ_3:

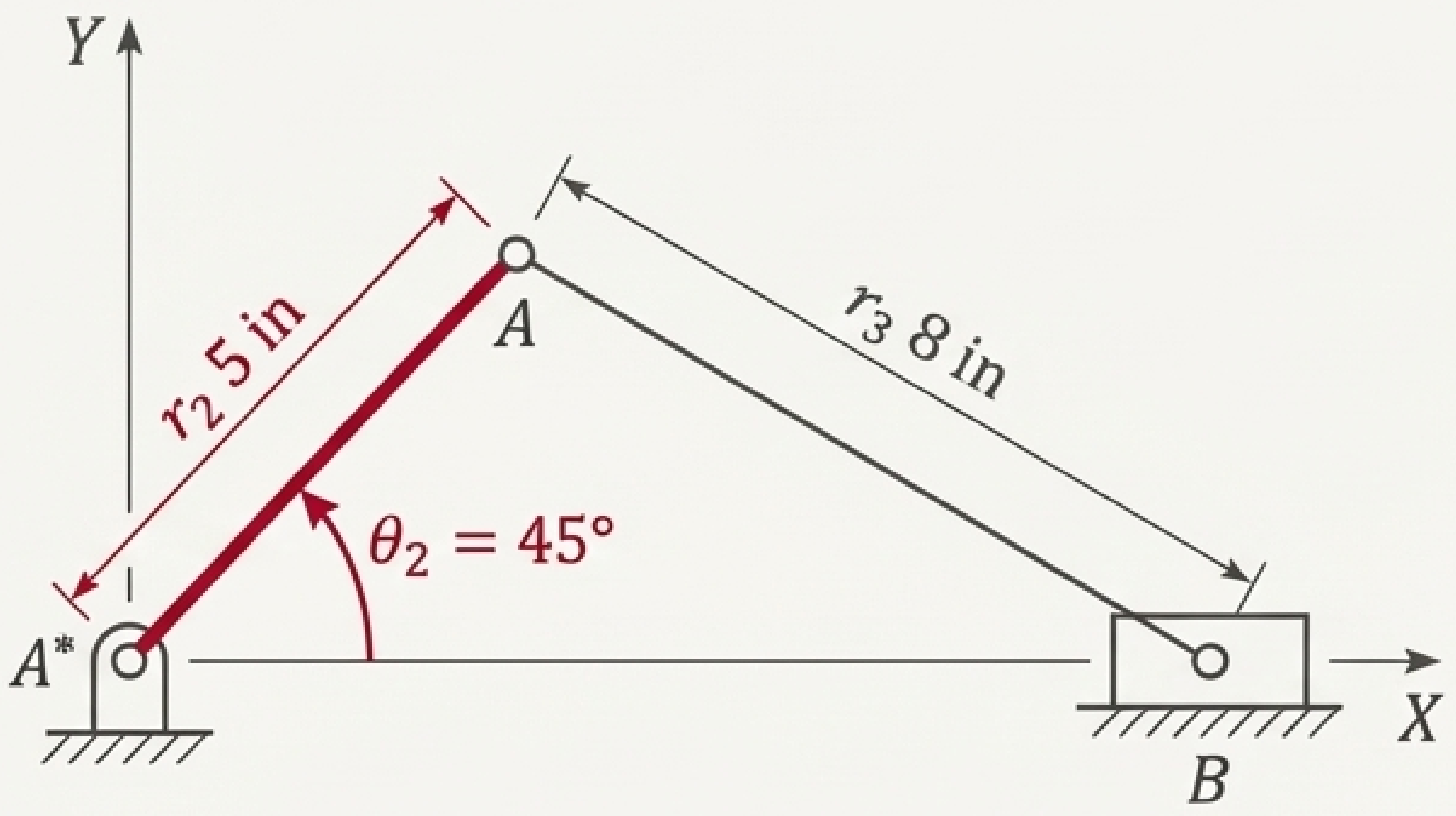
$$\tan(\theta_3) = \frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}$$

$$\theta_3 = \tan^{-1}\left[\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}\right] \text{(Eq. 7.71)}$$

To ensure θ_3 is in the correct quadrant, always use the two-argument arctangent function (ATAN2 or arctan2) available in most programming languages. This function uses the signs of the numerator and the denominator to resolve the ambiguity.

$$\theta_3 = \text{ATAN2}((r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2), (r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2))$$

Application: Example 7.4 — Problem Setup



Given Parameters

Crank length, $r_2 = 5$ in

Coupler length, $r_3 = 8$ in

Slider offset, $r_4 = 0$ in (In-line slider-crank)

Slider path angle, $\theta_1 = 0°$

Input crank angle, $\theta_2 = 45°$

Desired Assembly Mode: $\sigma = +1$

Goal

Find the slider position (r_1) and the coupler angle (θ_3).

Since $r_4 = 0$ and $\theta_1 = 0°$, our general equations for A and B will simplify significantly.

Example 7.4 — Step-by-Step Calculation

1. Calculate Coefficients A and B

With $r_4 = 0$ and $\theta_1 = 0$, the general equations simplify:

$$A = -2r_2\cos\theta_2 = -2(5)\cos(45°)$$
$$= -7.707$$

$$B = r_2^2 - r_3^2 = 5^2 - 8^2 = 25 - 64 = -39$$

2. Calculate Slider Position (r_1)

Using $r_1 = \frac{-A + \sigma\sqrt{A^2 - 4B}}{2}$ with $\sigma = +1$:

$$r_1 = \frac{-(-7.707) + 1\sqrt{(-7.707)^2 - 4(-39)}}{2}$$

$$r_1 = \frac{7.707 + \sqrt{59.39 + 156}}{2}$$

$$r_1 = \frac{7.707 + \sqrt{215.39}}{2} = \frac{7.707 + 14.676}{2}$$

$$\boxed{r_1 = 10.712 \text{ in}}$$

Values presented follow the source textbook's calculation.

3. Calculate Coupler Angle (θ_3)

Using the simplified equation $\theta_3 = \mathrm{ATAN2}(-r_2\sin\theta_2, r_1 - r_2\cos\theta_2)$:

$$-5 * \sin(45°) = -3.5355$$

$$10.712 - 5 * \cos(45°)$$
$$= 10.712 - 3.5355 = 7.1765$$

$$\theta_3 = \mathrm{ATAN2}(-3.5355, 7.1765)$$

$$\boxed{\theta_3 = -26.228°}$$

Solution from textbook

Start with the position analysis, and first compute the constants A and B from Eq. (7.69):

$$A = 2r_4(\cos\theta_1 \cos\theta_4 + \sin\theta_1 \sin\theta_4) - 2r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$
$$= -2(5)(\cos 0^\circ \cos 45^\circ + \sin 0^\circ \sin 45^\circ) = 7.707$$
$$B = r_2^2 + r_4^2 - r_3^2 - 2r_2 r_4(\cos\theta_2 \cos\theta_4 + \sin\theta_2 \sin\theta_4)$$
$$= 5^2 - 8^2 = -39$$

The desired configuration of the linkage corresponds to the position of the slider with the larger x coordinate. Therefore, $\sigma = +1$. Then,

$$r_1 = \frac{-A + \sigma\sqrt{A^2 - 4B}}{2} = \frac{-(7.707) + \sqrt{7.707^2 - 4(-39)}}{2} = 10.712 \text{ in}$$

Then θ_3 is given by

$$\theta_3 = \tan^{-1}\left[\frac{r_1 \sin\theta_1 + r_4 \sin\theta_4 - r_2 \sin\theta_2}{r_1 \cos\theta_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2}\right] = \tan^{-1}\left[\frac{-(5)\sin 45^\circ}{10.712 - (5)\cos 45^\circ}\right] = -26.228^\circ$$

Computational Implementation: Python Script

The true power of the analytical approach is its suitability for computer implementation. Below is a Python script that performs the calculations from Example 7.4.

```
import numpy as np

# --- Given Parameters (Example 7.4) ---
r2 = 5.0            # Crank length (in)
r3 = 8.0            # Coupler length (in)
r4 = 0.0            # Slider offset (in)
theta1_deg = 0.0
theta2_deg = 45.0
sigma = 1           # Assembly mode (+1 or -1)

# --- Convert angles to radians ---
theta1 = np.deg2rad(theta1_deg)
theta2 = np.deg2rad(theta2_deg)
theta4 = theta1 + np.pi/2

# --- Step 1: Calculate Coefficients A and B ---
# Using simplified equations for r4=0, theta1=0
A = -2 * r2 * np.cos(theta2)
B = r2**2 - r3**2

# --- Step 2: Solve for slider position r1 ---
discriminant = A**2 - 4 * B
if discriminant < 0:
    print("Error: Mechanism cannot be assembled.")
else:
    r1 = (-A + sigma * np.sqrt(discriminant)) / 2

    # --- Step 3: Solve for coupler angle theta3 ---
    num = r1*np.sin(theta1) + r4*np.sin(theta4) - r2*np.sin(theta2)
    den = r1*np.cos(theta1) + r4*np.cos(theta4) - r2*np.cos(theta2)
    theta3 = np.arctan2(num, den)
    theta3_deg = np.rad2deg(theta3)

    # --- Print Results ---
    print(f"Slider Position (r1): {r1:.3f} in")
    print(f"Coupler Angle (theta3): {theta3_deg:.3f} degrees")
```

Script Output

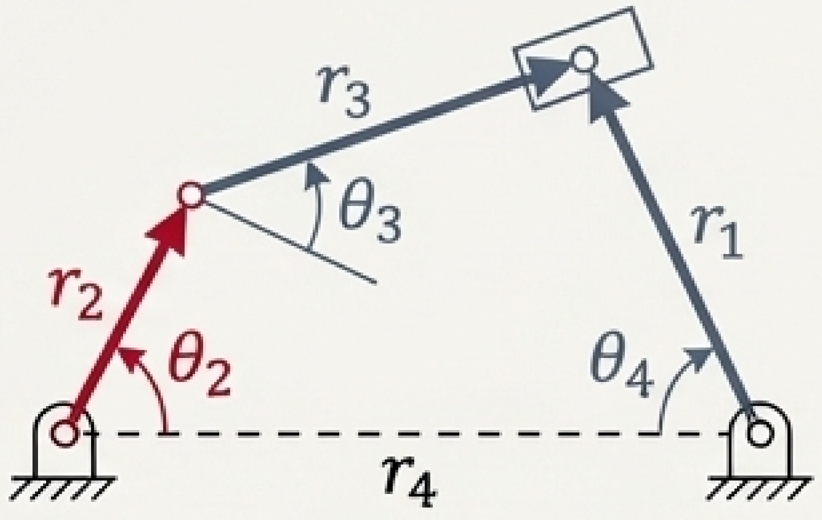
```
Slider Position (r1): 11.192 in
Coupler Angle (theta3): -24.878 degrees
```

A Note on Precision

Notice the slight difference between these computationally precise results and the values from the textbook. This highlights the importance of using robust computational tools to verify hand calculations and avoid propagating small rounding errors.

Summary of Key Position Equations

This slide summarizes the final equations for the positional analysis of a planar slider-crank, with the crank (θ_2) as the input.



Section 1: To Find Slider Position, r_1

1. Calculate coefficients A and B:

$$A = 2r_4(\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4) - 2r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$
$$B = r_2^2 + r_4^2 - r_3^2 - 2r_2r_4(\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4)$$

2. Solve the quadratic equation:

$$r_1 = \frac{-A + \sigma\sqrt{A^2 - 4B}}{2}$$

where $\sigma = \pm 1$ defines the assembly mode.

Section 2: To Find Coupler Angle, θ_3

Use the ATAN2 function with the now-known value of r_1:

$$\theta_3 = \mathrm{ATAN2}\left(\frac{r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2}{r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2}\right)$$

Conclusion & Next Steps in Kinematic Analysis

The Process We Followed

Physical Mechanism → Vector Loop Equation → Decompose to Scalar Equations → Isolate & Eliminate → Solve for Unknowns (r_1, θ_3)

Recap

We have successfully translated the geometric constraints of a physical slider–crank linkage into a set of analytical equations. This mathematical model allows for precise, repeatable, and programmable position analysis—the essential first step in understanding any mechanism.

Next Steps

Position is only the beginning. The real power of this analytical approach is that these position equations can be differentiated with respect to time to find velocity and acceleration. In our next lecture, we will perform this differentiation to complete the full kinematic analysis of the slider-crank.