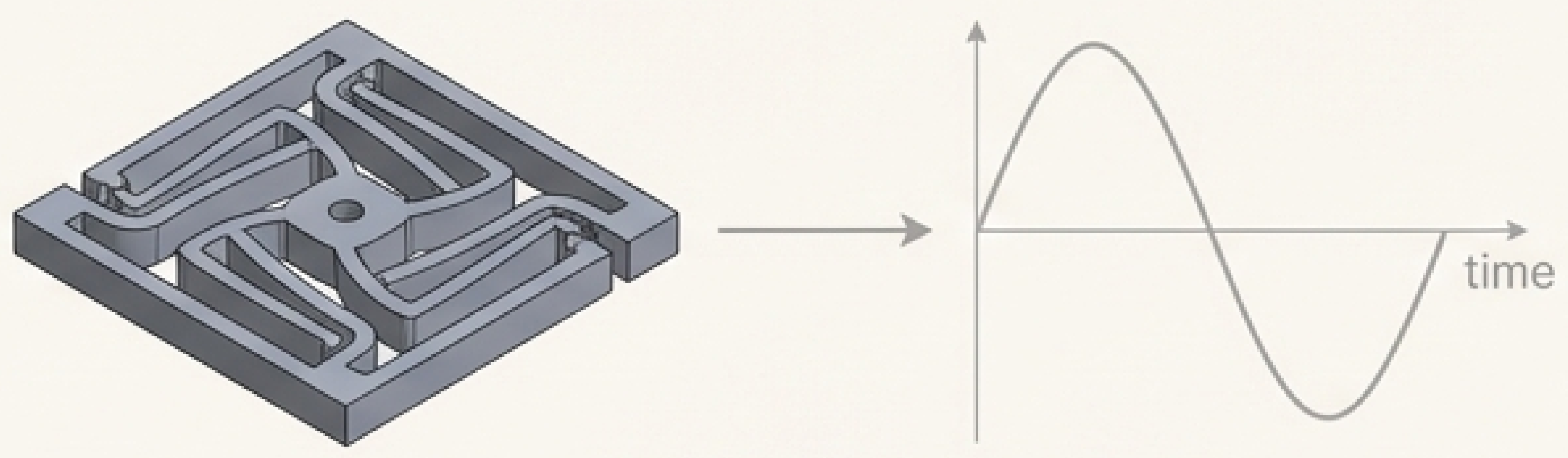
From Motion to Mechanism: The Inverse Problem of Synthesis

In the design of compliant mechanisms, we face two fundamental challenges:

- **Analysis (The Forward Problem):** Given a mechanism's structure (its pattern of constraints), what are its allowed motions?
  - *Constraint Pattern* $[\hat{W}]$ → *Freedom Space* $[\hat{T}]$
- **Synthesis (The Inverse Problem):** Given a desired motion (a specific freedom space), what structure or pattern of constraints will produce it?
  - *Freedom Space* $[\hat{T}]$ → *Constraint Pattern* $[\hat{W}]$

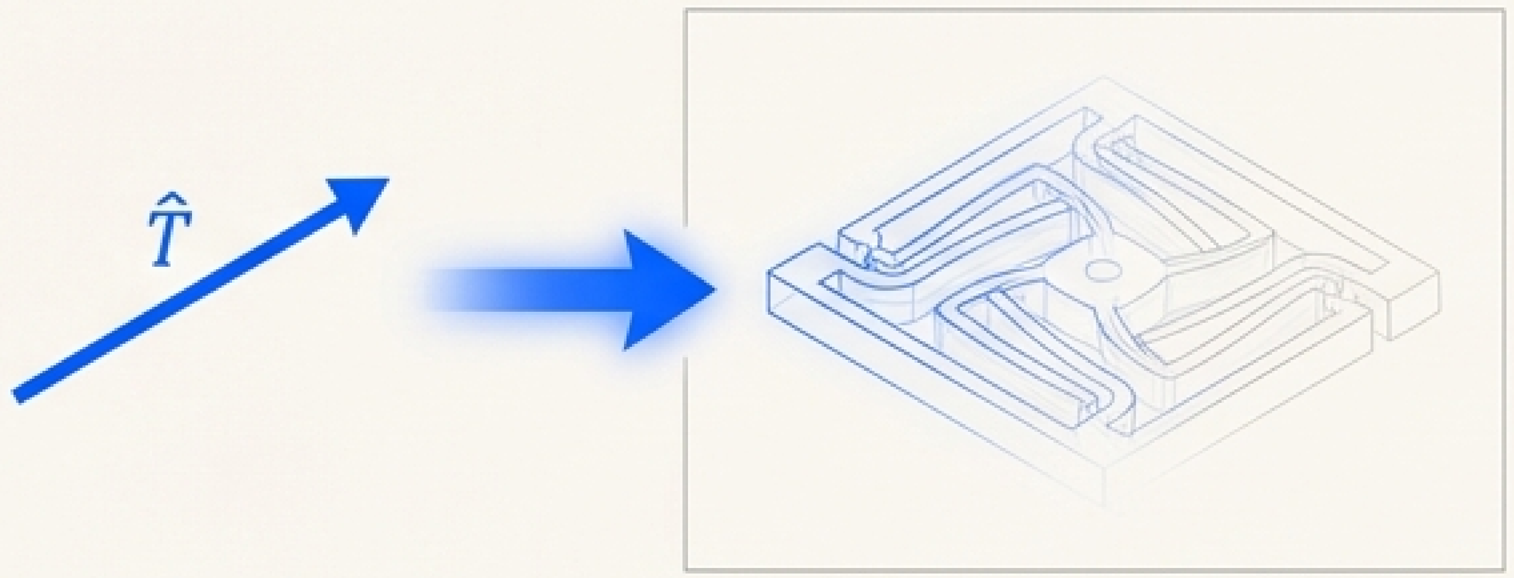
This lecture focuses on Synthesis. As defined by Su, et al. (2009): “Finding the wrench system for a given twist system is called flexure synthesis.” We are moving from being analysts of the given to architects of the desired.

Analysis



Constraint Pattern $[\hat{W}]$ → *Freedom Space* $[\hat{T}]$

Synthesis

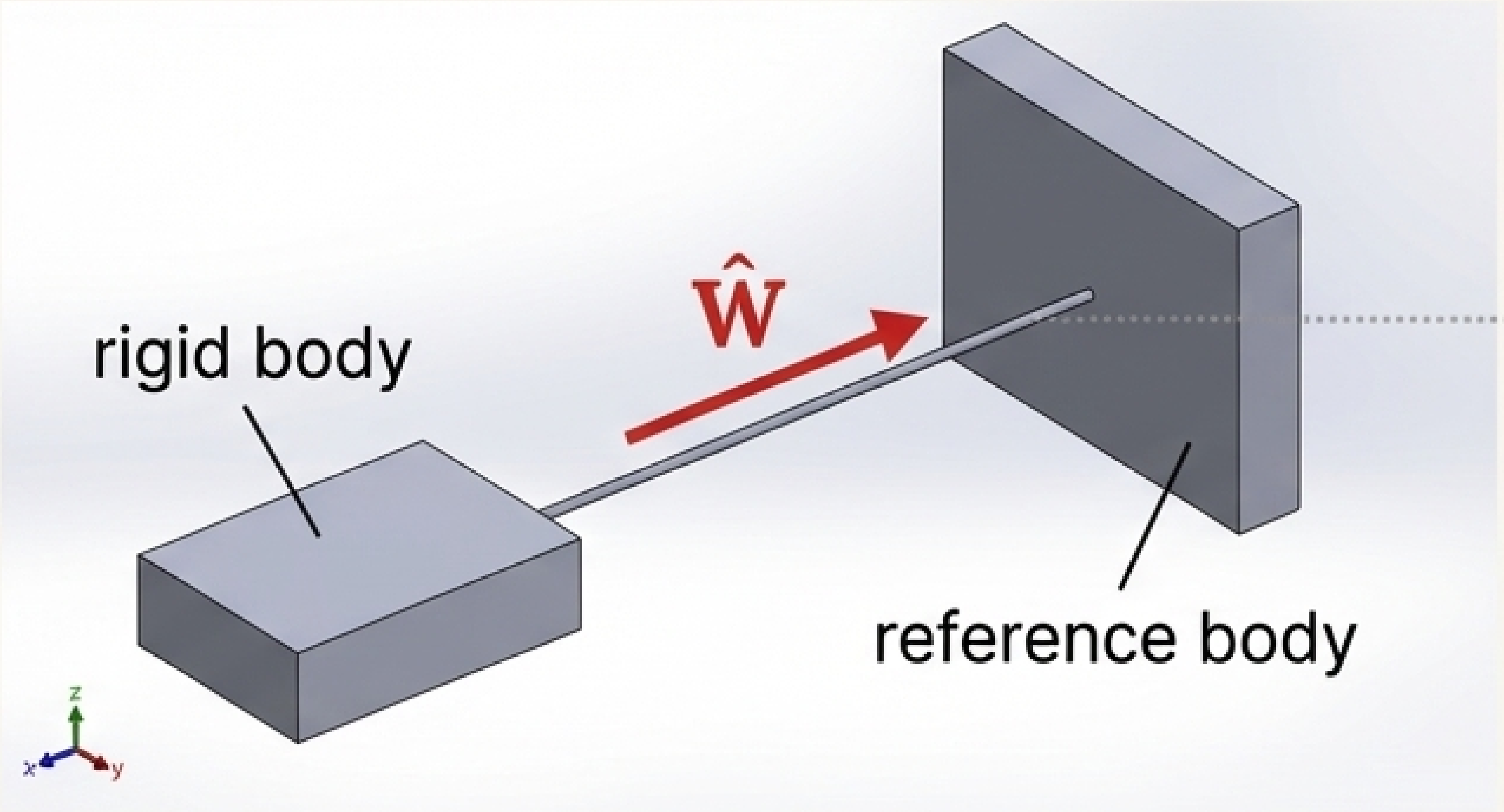


Freedom Space $[\hat{T}]$ → *Constraint Pattern* $[\hat{W}]$

Our Fundamental Building Block: **The Ideal Wire Flexure**

To synthesize mechanisms, we use simple, realizable constraints. The most fundamental is the ideal wire flexure.

- **Physical Definition**: "An ideal constraint is a slender structural member that is infinitly stiff along its axis but is infinitely compliant perpendicular to its axis." (Su, et al., 2009)
- **Kinematic Function**: It provides a pure translational constraint.
- **Screw Theory Representation**: It is modeled as a pure force wrench, which is a wrench with zero pitch (q=0).



A wrench representing a pure force is written as:

$$\hat{\mathbf{W}} = (\mathbf{f}\,\mathbf{u} \mid \mathbf{r} \times \mathbf{f}\,\mathbf{u})$$

This satisfies the general condition for any pure force wrench:

$$\boxed{\mathbf{F} \cdot \mathbf{M} = 0}$$

The Synthesis Algorithm: A Systematic Procedure

We can systematically derive a constraint pattern for any desired motion using a four-step process based on the **Rule of Complementary Patterns**. This rule states that a freedom space and its constraint space are reciprocal to each other.

1. **Define Desired Motion**
   Express the f desired degrees-of-freedom as a basis of twists $\{\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_f\}$.
2. **Find Reciprocal Constraints**
   Solve the system of linear equations $\hat{T}_j \circ \hat{W} = 0$ for all basis twists to find the general form of a wrench $\hat{W}$ in the complementary constraint space.
3. **Enforce Physical Realizability**
   To build with wire flexures, enforce the "ideal constraint" condition: the wrench $\hat{W}$ must be a pure force ($F \cdot M = 0$).
4. **Select a Basis**
   Choose $c = 6 - f$ independent wrenches that satisfy these conditions. This set of wrenches forms the basis for your constraint pattern.

Core Principle

$\hat{T}_j \circ \hat{W}_i = 0$ for $\hat{T}_j \circ \hat{W}_i = 0$, for $i = 1, \ldots, c$ and $j = 1, \ldots, f$, where $c + f = 6$

Case Study: Synthesizing a Compliant Slider (1-DOF)

Objective: To design a flexure mechanism that allows **only** pure translation along the x-axis.

Step 1: Define the Desired Motion

The desired freedom space has one degree-of-freedom ($f = 1$). This motion is a pure translation with infinite pitch, represented by the twist:

$$\hat{T} = (\Omega \,|\, V) = (0\,0\,0 \,|\, 1\,0\,0).$$

Therefore, we need to find a system of
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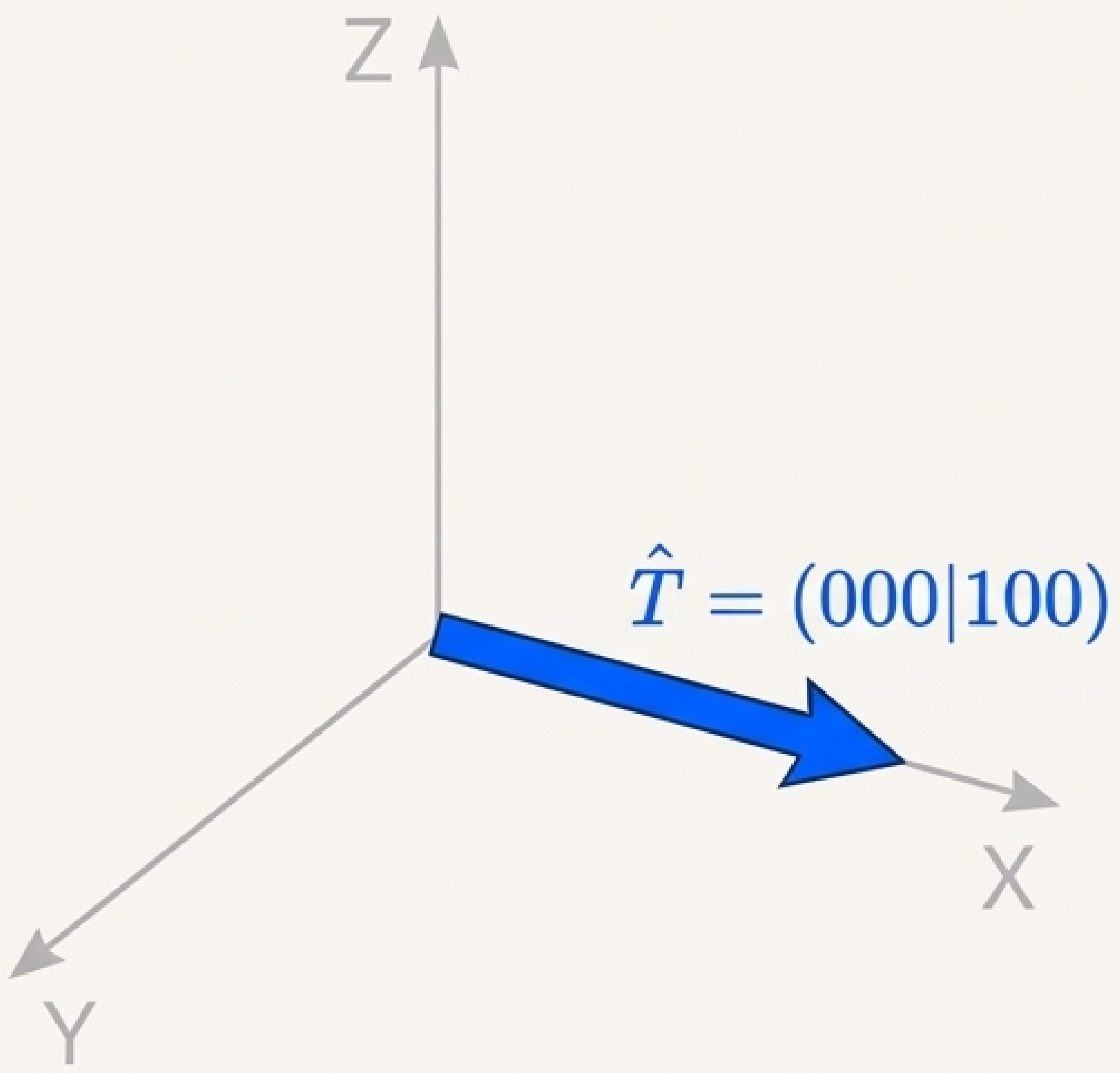
$c = 6 - 1 = 5$ independent constraints that are reciprocal to this twist.



Slider Synthesis: Finding the Reciprocal Constraints

Step 2: Find the General Reciprocal Wrench

Let $\hat{\mathbf{T}} = (0\ 0\ 0 \mid 1\ 0\ 0)$ and $\hat{\mathbf{W}} = (F_x\ F_y\ F_z \mid M_x\ M_y\ M_z)$.

The reciprocity condition is:

- $\hat{\mathbf{T}} \circ \hat{\mathbf{W}} = 0$
- $\hat{\mathbf{T}} \circ \hat{\mathbf{W}} = \mathbf{F}\cdot\mathbf{V} + \mathbf{M}\cdot\boldsymbol{\Omega}$

$$= F_x \cdot 1 + F_y \cdot 0 + F_z \cdot 0 + M_x \cdot 0 + M_y \cdot 0 + M_z \cdot 0$$
$$= F_x$$

For reciprocity, we must have: $F_x = 0$.

Step 3: Enforce Realizability with Wire Flexures

We want to use only ideal constraints (pure force wrenches), so we add the condition $\mathbf{F} \cdot \mathbf{M} = 0$:

$$F_x M_x + F_y M_y + F_z M_z = 0$$

Substituting $F_x = 0$ gives:

$$F_y M_y + F_z M_z = 0.$$

Final Conditions for Constraint Wrenches:

1. $F_x = 0$
2. $F_y M_y + F_z M_z = 0$

(Matches Eq. 36 and 37 from DETC2009-86684)

Slider Synthesis: Selecting a Basis of Constraints

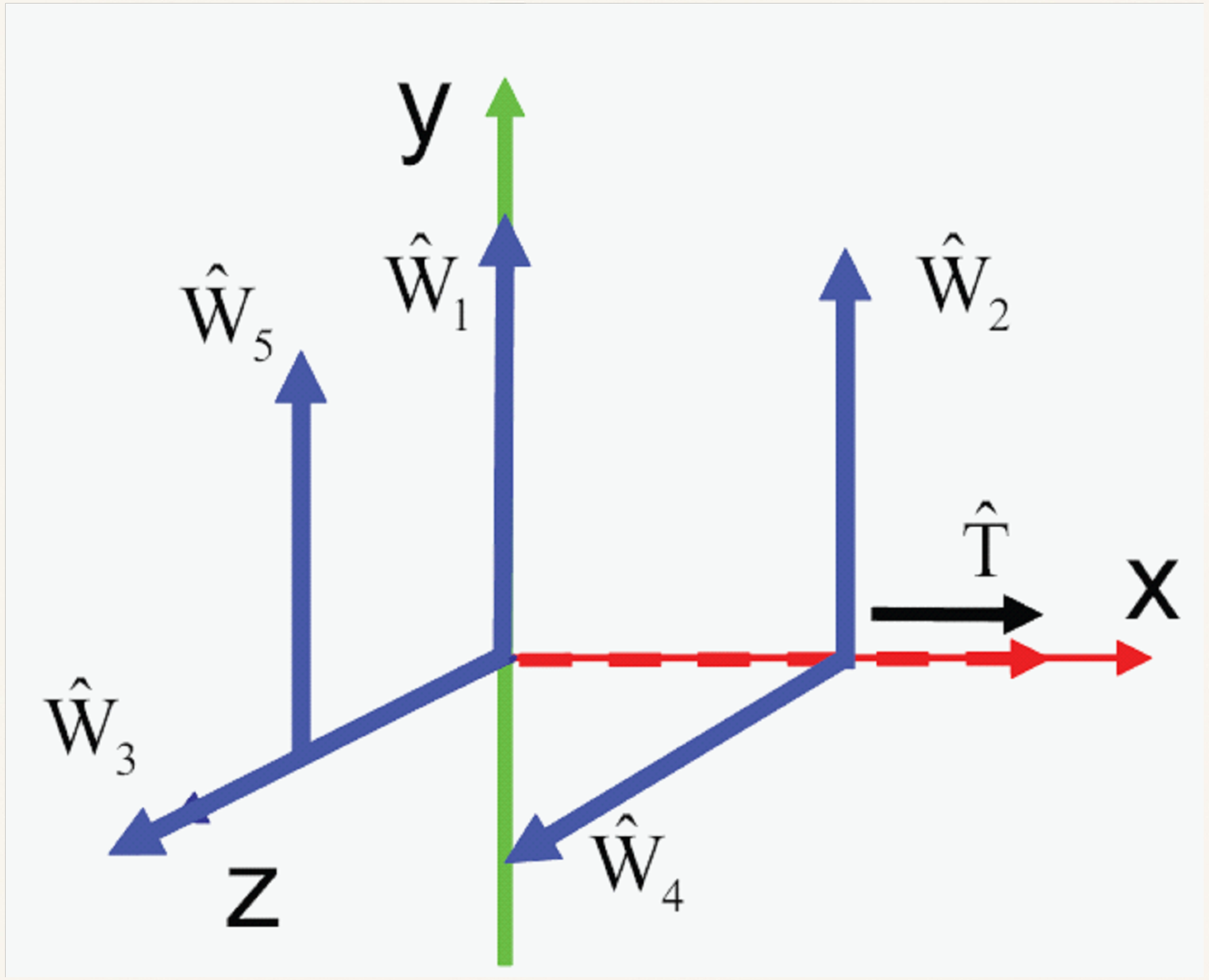
Step 4: Find a Basis for the Constraint Space

We need to find $c = 5$ independent pure force wrenches satisfying $F_x = 0$ and $F_y M_y + F_z M_z = 0$. For simplicity, we choose constraints parallel to the primary coordinate axes.

A possible basis set is:

- $\widehat{W}_1 = (0\ 1\ 0 \mid 0\ 0\ 0)$
- $\widehat{W}_2 = (0\ 1\ 0 \mid 0\ 0\ 1)$
- $\widehat{W}_3 = (0\ 0\ 1 \mid 0\ 0\ 0)$
- $\widehat{W}_4 = (0\ 0\ 1 \mid 0\ 1\ 0)$
- $\widehat{W}_5 = (0\ 1\ 0 \mid 1\ 0\ 0)$

These five wrenches form a valid constraint space that is recipirocal to the desired translation along the x-axis.



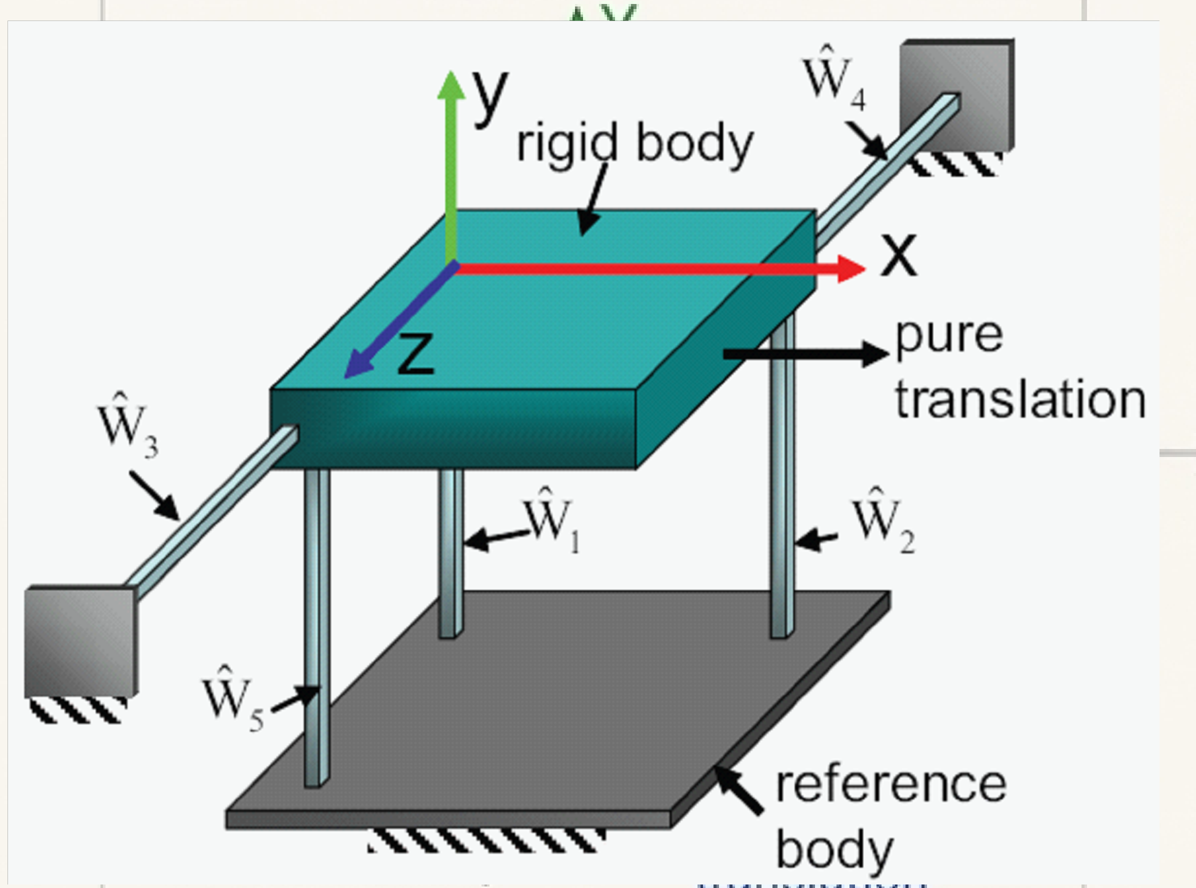
(Extracted from DETC2009-86684, Eq. 39 & 40)

From Abstract Vectors to Physical Reality

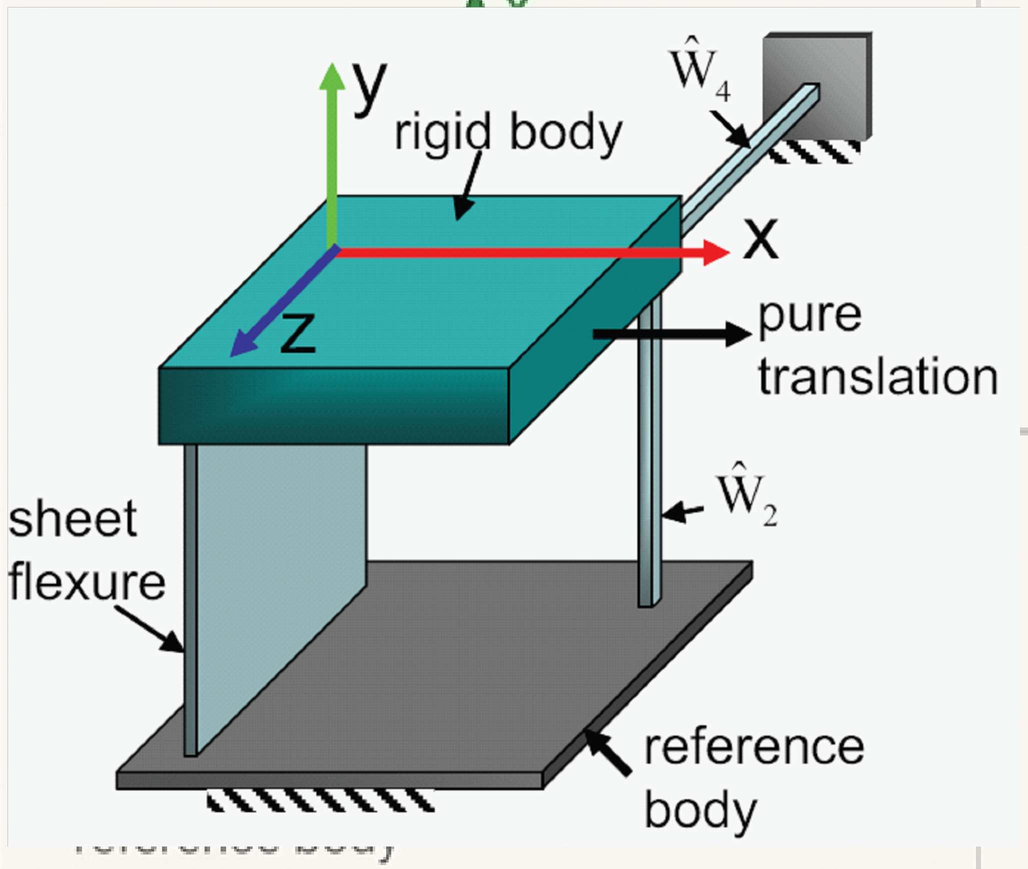
Step 5: Physical Arrangement

The five abstract wrench vectors can be physically realized in multiple ways. The choice of basis constraints is not unique, which allows for design flexibility.

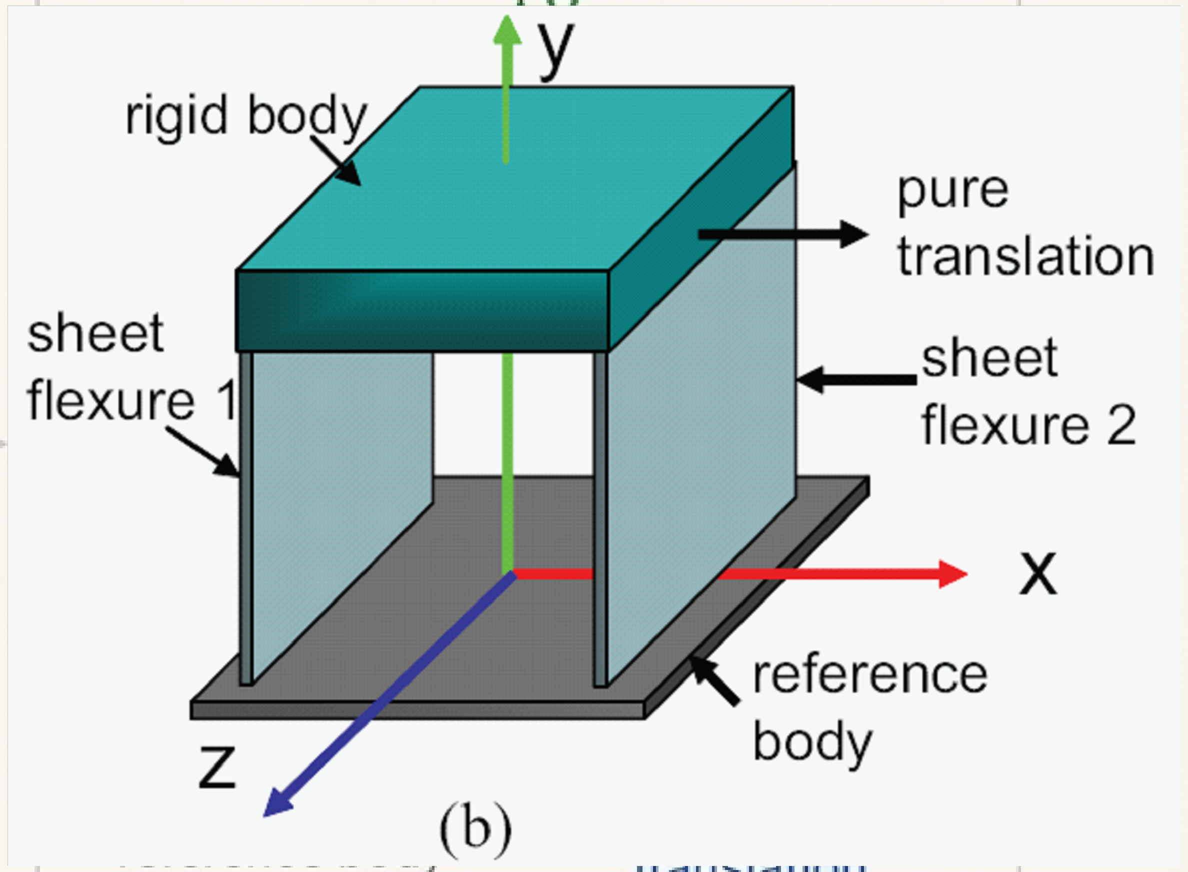
Design 1: Five Wire Flexures



Design 2: Sheet and Wire Hybrid



Design 3: Two Parallel Sheets

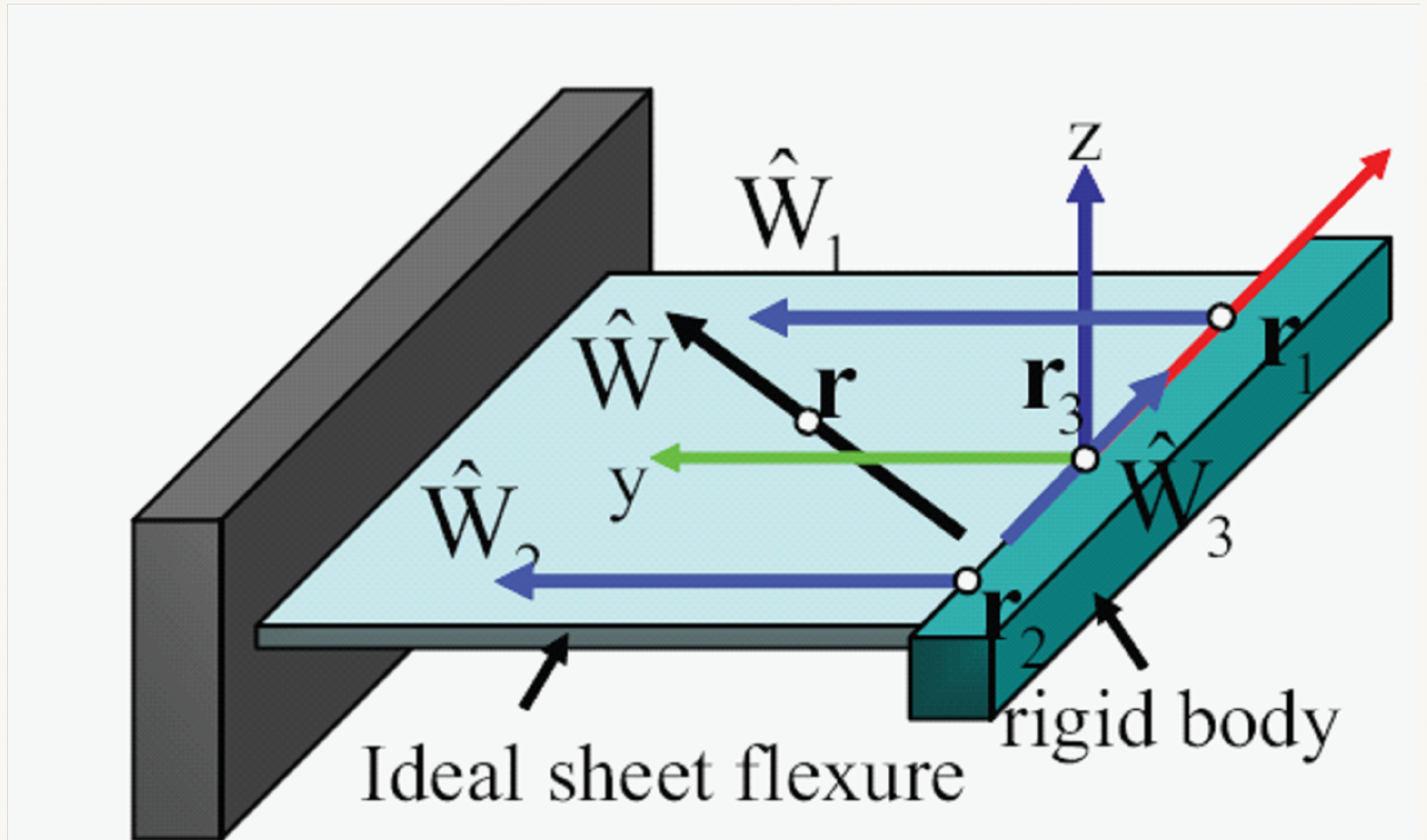


This shows how a rigorous mathematical derivation leads directly to an optimal and widely-used engineering solution.

The Duality of Freedom and Constraint: The Sheet Flexure

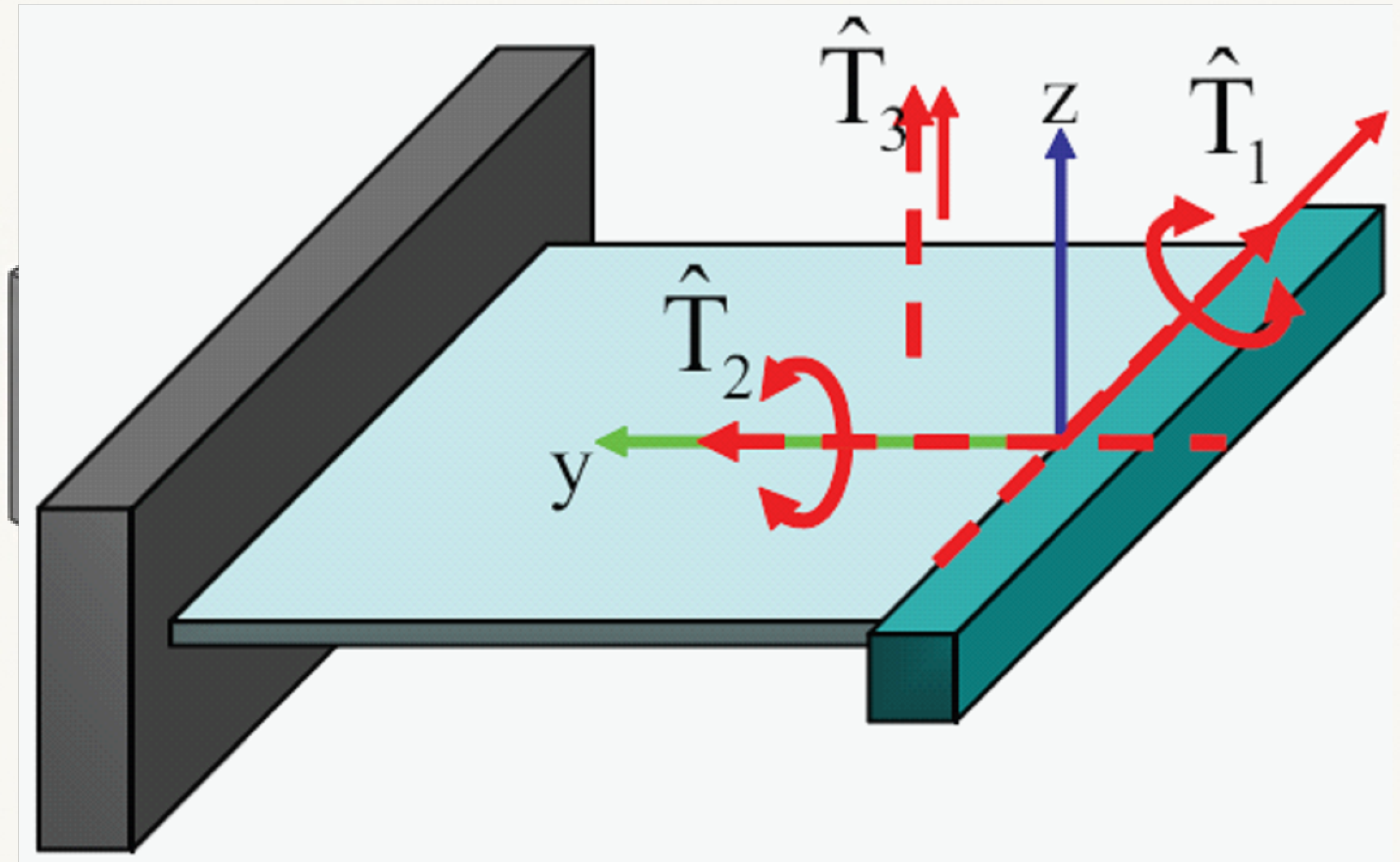
The **Rule of Complementary Patterns** states that freedom and constraint are reciprocal. An ideal sheet flexure provides a perfect example.

Constraint Space $(c = 3)$



(a) constraint space

Freedom Space $(f = 3)$



(b) freedom space

The mathematics confirms our intuition: constraining 3 DOFs frees the other 3.

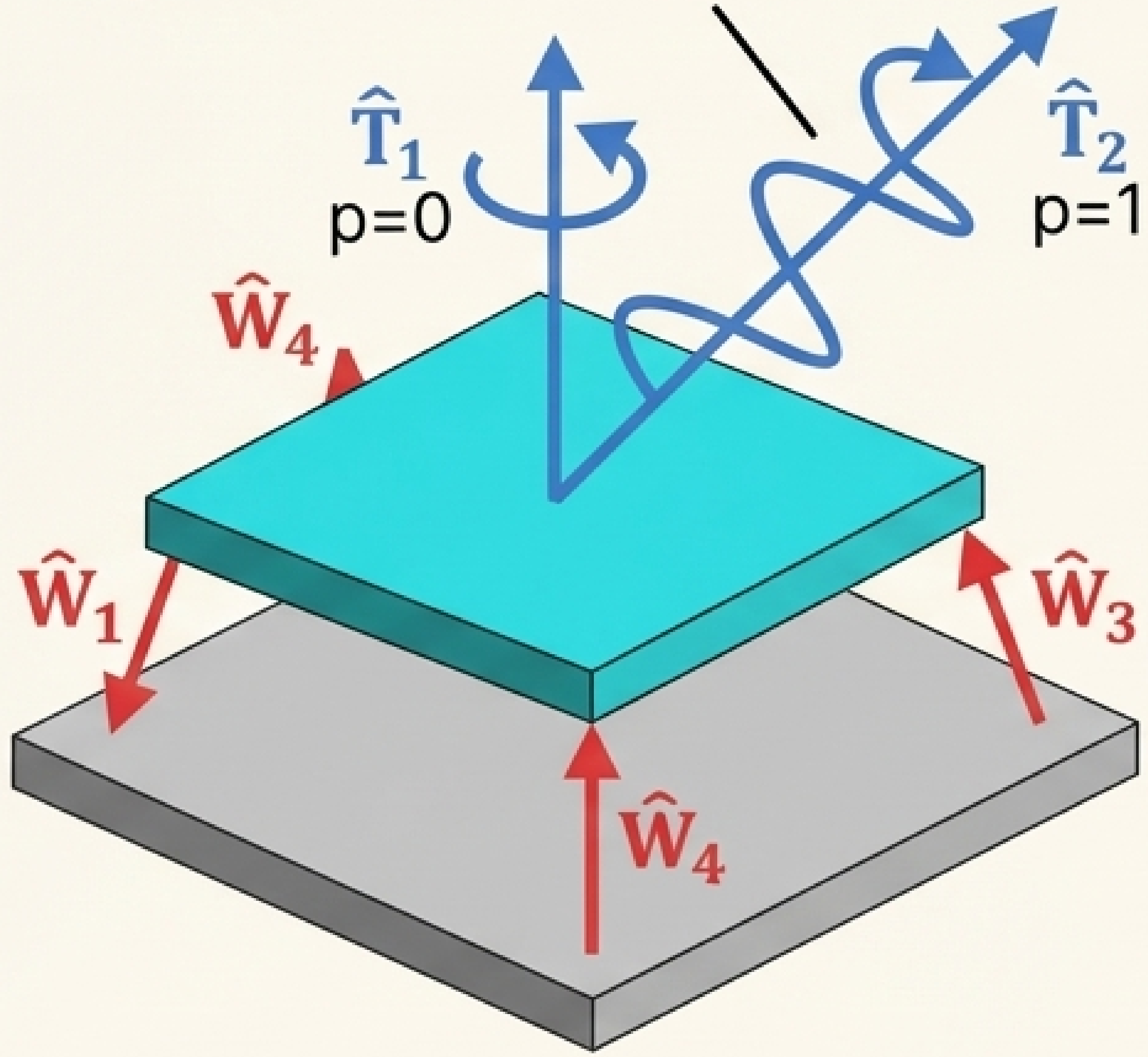
A Critical Question: The Realizability Problem

We have shown that we can find a mathematically valid constraint space for any desired motion. But can we always build it using our simple building block—the ideal wire flexure?

The Answer is NO.

* The synthesis procedure may result in a required constraint space that contains wrenches with **non-zero pitch (screw constraints)** or **infinite pitch (pure couples)**.
* If the required constraint space **cannot be spanned by only pure force wrenches**, the desired motion is **not realizable** with simple wire or sheet flexures.

Starting with **only pure forces** can still result in a required **screw motion.**



When the Math Works, but Wires Don't: An Unrealizable Motion

Let's attempt to synthesize a mechanism that allows for two independent translations.

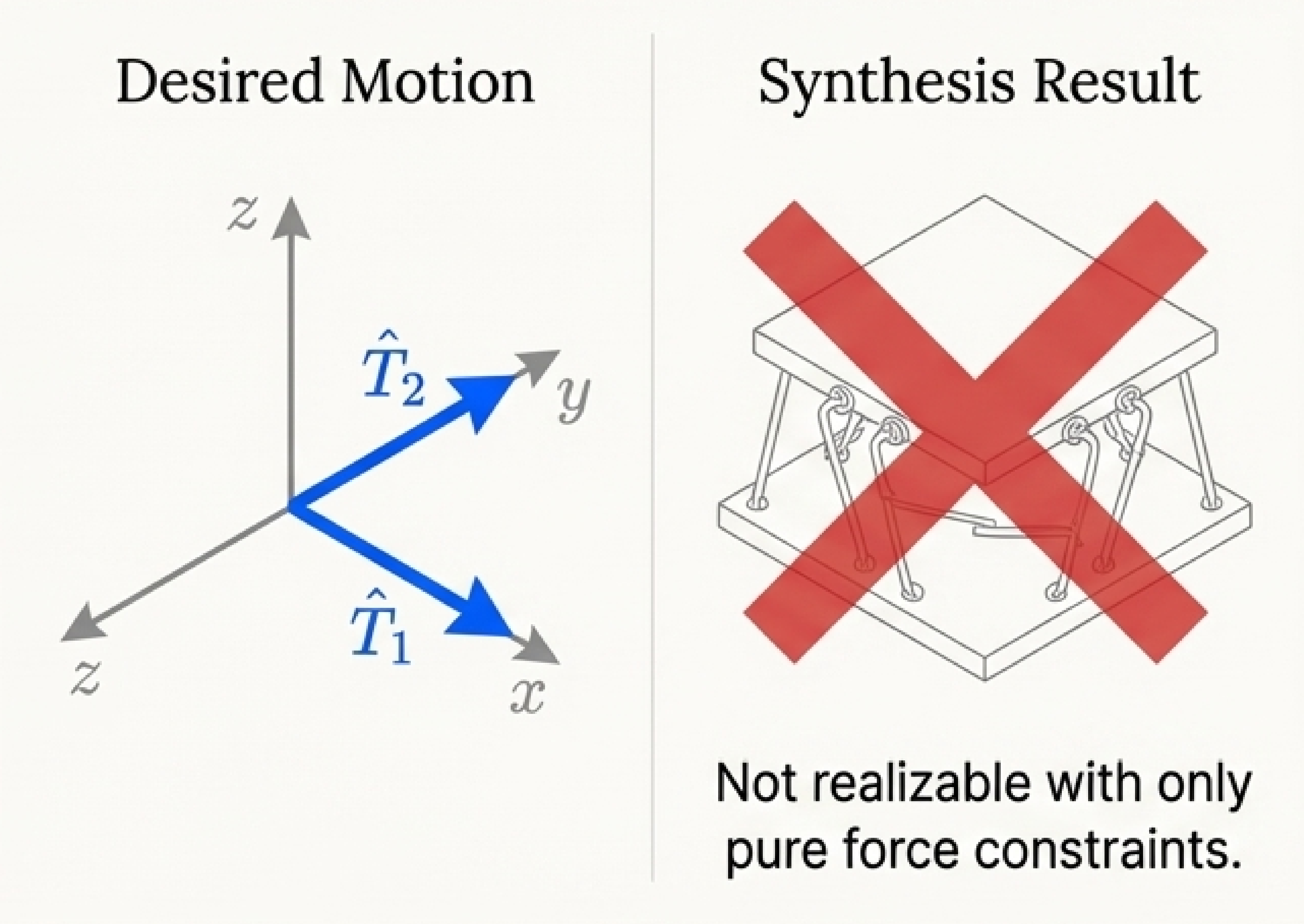
Desired Motion (f=2):

- $\hat{T}_1 = (0\ 0\ 0 \mid 1\ 0\ 0)$ (Translation along x-axis)
- $\hat{T}_2 = (0\ 0\ 0 \mid 0\ 1\ 0)$ (Translation along y-axis)

Synthesis Result:

"By going through the design steps, one can find that there do not exist c = 6 - 2 = 4 complementary ideal constraints." (Su, et al., 2009)

The reciprocal constraint space requires non-ideal constraints (screws or couples) to satisfy the reciprocity conditions for both twists simultaneously.



Case Study 2: A 2-DOF Spatial Joint

Objective: Synthesize a flexure that allows rotation around the z-axis AND translation along the vector (0, 1, 1).

Step 1: Define Desired Motion (f=2)

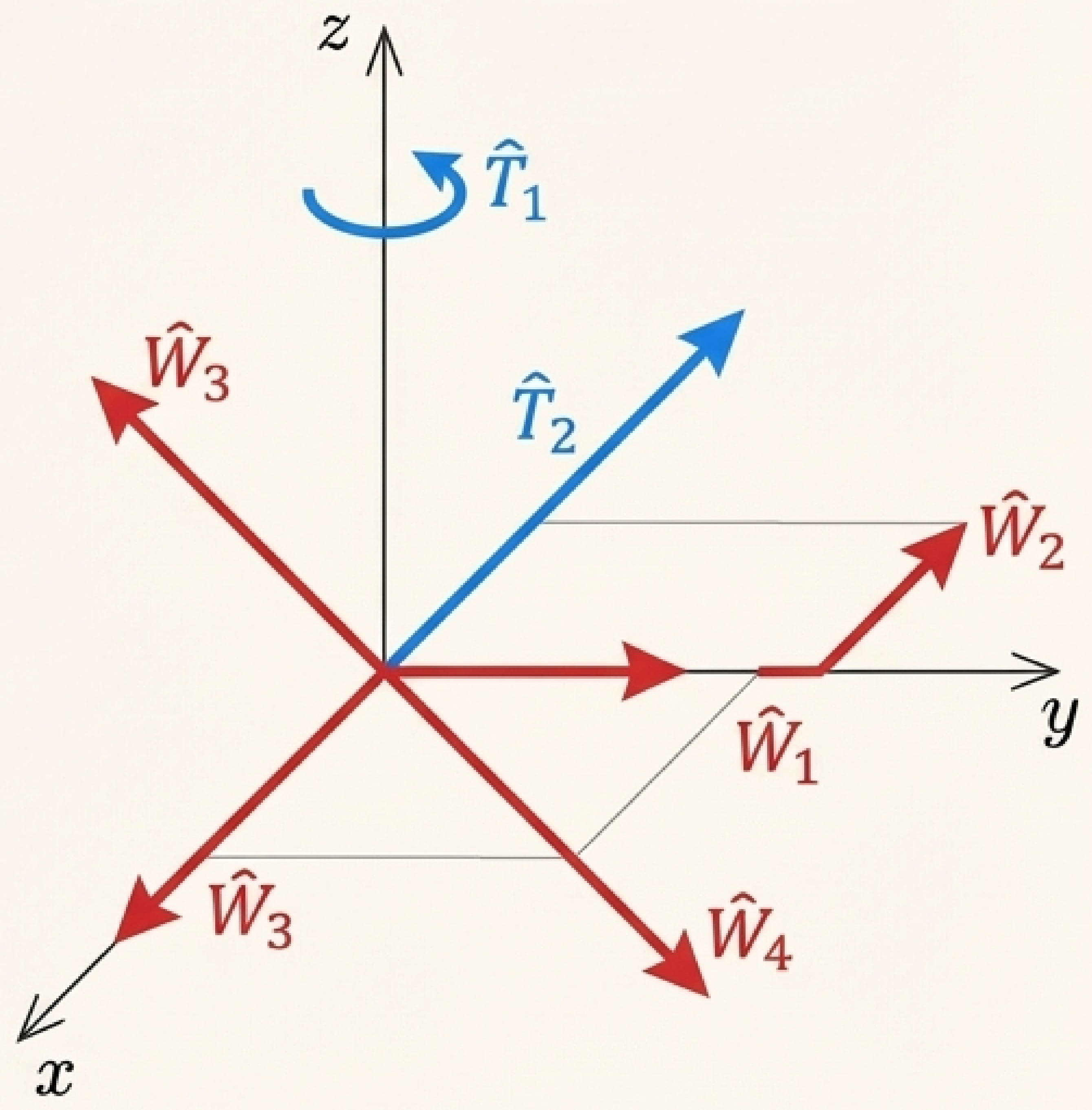
- $\hat{T}_1 = (0\ 0\ 1 \mid 0\ 0\ 0)$ (Pure rotation, p=0)
- $\hat{T}_2 = (0\ 0\ 0 \mid 0\ 1\ 1)$ (Pure translation, p=∞)

Steps 2-4: Find a Basis of Ideal Constraints (c=4)

Solving the reciprocity equations... yields a 4-dimensional constraint space. One possible basis is:

- $\hat{W}_1 = (1\ 0\ 0 \mid 0\ 0\ 0)$
- $\hat{W}_2 = (1\ 0\ 0 \mid 0\ 1\ 0)$
- $\hat{W}_3 = (0\ 1\ -1 \mid 0\ 0\ 0)$
- $\hat{W}_4 = (0\ 1\ -1 \mid 1\ 0\ 0)$

This specific combination of rotation and translation IS realizable with ideal wire flexures.



From Intuition to Algebra: The Power of Systematic Synthesis

Screw theory provides a complete mathematical framework for constraint-based design.

* Freedom (Motion) is represented by a Twist ($\hat{T}$).
* Constraint (Structure) is represented by a Wrench ($\hat{W}$).
* The link between them is Reciprocity ($\hat{T} \circ \hat{W} = 0$).

This framework transforms the creative challenge of “type synthesis” into a solvable linear algebra problem.
We can now move from a precise description of a desired motion to a guaranteed physical structure that achieves it.