

## Chapter 12

# Spatial Kinematics

In this chapter we develop the geometry of spatial displacements defined by coordinate transformations consisting of spatial rotations and translations. We consider the invariants of these transformations and find that there are no invariant points. Instead there is an invariant line, called the screw axis. Thus, the geometry of lines becomes important to our study of spatial kinematics. We find that a configuration of three lines, called a spatial triangle, generalizes our results for planar and spherical triangles to three-dimensional space.

A convenient set of coordinates for lines, known as Plücker coordinates, are introduced, then generalized to yield screws. Dual vector algebra manipulates these coordinates using the same rules as the usual vector algebra. This yields a screw form of Rodrigues's formula that defines the screw axis of a composite displacement in terms of the screw axes of the two factor displacements.

### 12.1 Spatial Displacements

A spatial displacement is the composition of a spatial rotation followed by a spatial translation. This transformation takes the coordinates  $\mathbf{x} = (x, y, z)^T$  of a point in the moving frame  $M$  and computes its coordinates  $\mathbf{X} = (X, Y, Z)^T$  in the fixed frame  $F$ , by the formula

$$\mathbf{X} = T(\mathbf{x}) = [A]\mathbf{x} + \mathbf{d}, \quad (12.1)$$

where  $[A]$  is a  $3 \times 3$  *rotation matrix* and  $\mathbf{d}$  is a  $3 \times 1$  *translation vector*. A spatial displacement preserves the distance between points measured in both  $M$  and  $F$ .

### 12.1.1 Homogeneous Transforms

The transformation that defines a spatial displacement is not a linear operation. To see this compute  $T(\mathbf{x} + \mathbf{y})$ . The result does not equal to  $T(\mathbf{x}) + T(\mathbf{y})$ . This can be attributed to the inhomogeneous translation term in (12.1). A standard strategy to adjust for this inhomogeneity is to add a fourth component to our position vectors that will always equal 1. Then we have the  $4 \times 4$  *homogeneous transform*

$$\begin{Bmatrix} \mathbf{X} \\ 1 \end{Bmatrix} = \begin{bmatrix} A & \mathbf{d} \\ 000 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ 1 \end{Bmatrix}, \quad (12.2)$$

which we write as

$$\mathbf{X} = [T]\mathbf{x}. \quad (12.3)$$

Notice that we have not distinguished between the point coordinates that have a 1 as their fourth component. In general, these vectors will have three components. Please assume the addition of the fourth component, when it is appropriate for the use of these  $4 \times 4$  transforms. We use  $[T] = [A, \mathbf{d}]$  to denote the  $4 \times 4$  homogeneous transform with rotation matrix  $[A]$  and translation vector  $\mathbf{d}$ .

### 12.1.2 Composition of Displacements

The set of matrices that have the structure shown in (12.2) form a matrix group, denoted by SE(3), with matrix multiplication as its operation. The matrix product of  $[T_1] = [A_1, \mathbf{d}_1]$  and  $[T_2] = [A_2, \mathbf{d}_2]$  yields

$$[T_3] = [A_1, \mathbf{d}_1][A_2, \mathbf{d}_2] = [A_1 A_2, \mathbf{d}_1 + A_1 \mathbf{d}_2]. \quad (12.4)$$

It is easy to see that the  $4 \times 4$  transform  $[T_3]$  has the same structure as (12.2) with  $A_3 = A_1 A_2$  as its rotation matrix and  $\mathbf{d}_3 = \mathbf{d}_1 + [A_1]\mathbf{d}_2$  as its translation vector.

The composition of the displacements  $[T_1] = [A_1, \mathbf{d}_1]$  and  $[T_2] = [A_2, \mathbf{d}_2]$  can be interpreted as follows. Let  $[T_1]$  define the position of a frame  $M'$  relative to  $F$  such that  $\mathbf{X} = [A_1]\mathbf{y} + \mathbf{d}_1$ . Then the position of  $M$  relative to  $M'$  is defined by  $[T_2]$  such that  $\mathbf{y} = [A_2]\mathbf{x} + \mathbf{d}_2$ . Thus, the position of  $M$  relative to  $F$  is given by

$$\mathbf{X} = [A_1 A_2]\mathbf{x} + \mathbf{d}_1 + A_1 \mathbf{d}_2. \quad (12.5)$$

Compare this equation to (12.4) to see that the product of two homogeneous transforms defines this composition of displacements.

Similarly, the matrix inverse  $[T]^{-1} = [A, \mathbf{d}]^{-1}$  defines the inverse displacement

$$[T^{-1}] = [A, \mathbf{d}]^{-1} = [A^T, -A^T \mathbf{d}]. \quad (12.6)$$

It is easy to use (12.4) to check that  $[A, \mathbf{d}][A^T, -A^T \mathbf{d}] = [I]$ .

### 12.1.2.1 Changing Coordinates of a Displacement

Consider the displacement  $\mathbf{X} = [T]\mathbf{x}$  that defines the position of  $M$  relative to  $F$ . We now consider the transformation  $[T']$  between the frames  $M'$  and  $F'$  that are displaced by the same amount from both  $M$  and  $F$ . In particular, let  $[R] = [B, \mathbf{c}]$  be the displacement that transforms the coordinates between the primed and unprimed frames, that is,  $\mathbf{Y} = [R]\mathbf{X}$  and  $\mathbf{y} = [R]\mathbf{x}$  are the coordinates in  $F'$  and  $M'$ , respectively. Then, from  $\mathbf{X} = [T]\mathbf{x}$  we can compute

$$\mathbf{Y} = [R][T][R^{-1}]\mathbf{y}. \quad (12.7)$$

Thus, the original matrix  $[T]$  is transformed by the change of coordinates into  $[T'] = [R][T][R^{-1}]$ .

### 12.1.3 Relative Displacements

For a set of displacements  $[T_i] = [A_i, \mathbf{d}_i]$ ,  $i = 1, \dots, n$ , the relative displacement between any two is given by

$$[T_{ij}] = [T_j][T_i^{-1}]. \quad (12.8)$$

If  $\mathbf{X}^i = [T_i]\mathbf{x}$  denotes the coordinates in  $F$  for points in position  $M_i$ , then we have

$$\mathbf{X}^j = [T_{ij}]\mathbf{X}^i. \quad (12.9)$$

Notice that both  $\mathbf{X}^i$  and  $\mathbf{X}^j$  are measured in the fixed reference frame  $F$ ; they are the coordinates of corresponding points of  $M$  in positions  $M_i$  and  $M_j$ .

#### 12.1.3.1 Relative Inverse Displacements

The relative inverse displacement  $[T_{ik}^\dagger]$  between two inverse positions  $F_i$  and  $F_k$  is given by

$$[T_{ik}^\dagger] = [T_k^{-1}][T_i]. \quad (12.10)$$

Notice that this is not the inverse of the relative displacement  $[T_{ik}]$ , which would be  $[T_{ik}^{-1}] = [T_i][T_k^{-1}]$ .

The relative inverse displacement  $[T_{ik}^\dagger]$  is defined from the point of view of the moving frame  $M$ . However, we can choose a specific position  $M_j$  and transform this displacement by  $[T_j]$ , to obtain

$$[T_{ik}^j] = [T_j][T_{ik}^\dagger][T_j^{-1}]. \quad (12.11)$$

This is known as the *image* of the relative inverse transformation for position  $M_j$  in  $F$ .

Notice that if  $M_j$  is one of the frames used in computing the relative inverse displacement, for example  $j = i$ , then we have

$$[T_{ik}^i] = [T_i]([T_k^{-1}][T_i])[T_i^{-1}] = [T_i][T_k^{-1}] = [T_{ik}^{-1}]. \quad (12.12)$$

This same result is obtained when  $j = k$ . Thus, for  $j = i$  or  $j = k$  the image of the relative inverse displacement  $[T_{ik}^j]$  is the inverse of the relative displacement.

### 12.1.4 Screw Displacements

We now consider the invariants of spatial displacements. If a point  $\mathbf{C}$  has the same coordinates before and after a spatial displacement  $[T]$ , then it satisfies the equation

$$\mathbf{C} = [T]\mathbf{C}, \quad \text{or} \quad [I - T]\mathbf{C} = \mathbf{0}, \quad (12.13)$$

which simplifies to

$$[I - A]\mathbf{C} = \mathbf{d}. \quad (12.14)$$

Recall that all spatial rotations have 1 as an eigenvalue. Therefore, the  $3 \times 3$  matrix  $[I - A]$  is singular. Thus, a spatial displacement has no fixed points.

While there are no fixed points, there is a line, called the *screw axis*, that remains fixed during a spatial displacement. To determine this line, we decompose the translation component of the displacement  $[T] = [A, \mathbf{d}]$  into vectors parallel and perpendicular to the rotation axis  $\mathbf{S}$  of  $[A]$ , that is,

$$\mathbf{d} = \mathbf{d}^* + k\mathbf{S}, \quad \text{where} \quad k = \mathbf{d} \cdot \mathbf{S}. \quad (12.15)$$

The displacement  $[T]$  can now be written as the composition of the rotational displacement  $[R] = [A, \mathbf{d}^*]$  and the translation  $[S] = [I, d\mathbf{S}]$ ,

$$[T] = [S][R] = [I, k\mathbf{S}][A, \mathbf{d}^*] = [A, \mathbf{d}^* + k\mathbf{S}] = [A, \mathbf{d}]. \quad (12.16)$$

Notice that all spatial displacements can be decomposed in this way.

We have already seen in (8.81) that a rotational displacement  $[R] = [A, \mathbf{d}^*]$  has a fixed point is given by

$$\mathbf{C} = \frac{\mathbf{b} \times (\mathbf{d}^* - \mathbf{b} \times \mathbf{d}^*)}{2\mathbf{b} \cdot \mathbf{b}}, \quad (12.17)$$

where  $\mathbf{b} = \tan(\phi/2)\mathbf{S}$  is Rodrigues's vector of the rotation  $[A]$ . Now consider the line  $S$  through this point in the direction of the rotation axis of  $[A]$ , defined by

$$\mathbf{S} : \mathbf{P}(t) = \mathbf{C} + t\mathbf{S}. \quad (12.18)$$

Points on this line remain fixed during the rotational displacement  $[R] = [A, \mathbf{d}^*]$ . Furthermore, the translation  $[S] = [I, k\mathbf{S}]$  slides points along this line the distance  $k$ . This line remains fixed during the displacement. Thus, a general spatial displace-

ment consists of a rotation by  $\phi$  about this line and the sliding distance  $k$  along it. This is called a *screw displacement* and the line  $S$  is called the *screw axis*.

### 12.1.5 The Screw Matrix

It is often convenient to define a spatial displacement in terms of its screw axis  $S$  and the angle  $\phi$  and slide  $k$  around and along it. We have already determined a formula for a rotation matrix  $[A(\phi, S)]$  in terms of its rotation axis and angle. From (12.16) we see that the translation vector is given by

$$\mathbf{d} = [I - A]\mathbf{C} + k\mathbf{S}. \quad (12.19)$$

Now use the notation  $\hat{\phi} = (\phi, k)$  for the rotation and slide of the screw displacement and define the *screw matrix*

$$[T(\hat{\phi}, S)] = [A(\phi, S), [I - A]\mathbf{C} + k\mathbf{S}]. \quad (12.20)$$

This is the  $4 \times 4$  homogeneous transform with elements defined in terms of the screw parameters of the displacement.

This form of a spatial displacement allows us to write the transformation of  $\mathbf{x}$  in  $M$  to  $\mathbf{X}$  in  $F$  as

$$\mathbf{X} - \mathbf{C} = [A](\mathbf{x} - \mathbf{C}) + k\mathbf{S}, \quad (12.21)$$

which shows directly that the displacement consists of a rotation about  $\mathbf{C}$  followed by a translation along the screw axis  $S$ .

## 12.2 Lines and Screws

The geometry of the screw axis of a spatial displacement is best studied using Plücker's coordinates that define the line directly. Plücker coordinates for a line are six-vectors assembled from the direction of the line and its moment about the origin of the reference frame. The generalization of these coordinates, called a *screw*, is familiar from the study of elementary statics and dynamics where it appears as the pair formed by the resultant force and moment on a body.

### 12.2.1 Plücker Coordinates of a Line

Consider the line  $S$  through two points  $\mathbf{C}$  and  $\mathbf{Q}$  in space, given by the parameterized equation

$$S : \mathbf{P}(t) = \mathbf{C} + t\mathbf{S}, \quad (12.22)$$

where  $\mathbf{C}$  has been selected as a reference point on the line, and  $\mathbf{S}$  is the unit vector along  $\mathbf{Q} - \mathbf{C}$ . To eliminate the free parameter  $t$  in the definition of  $S$ , we introduce the *Plücker coordinates* of the line

$$S = \left\{ \begin{array}{c} \mathbf{S} \\ \mathbf{C} \times \mathbf{S} \end{array} \right\}. \quad (12.23)$$

The vector  $\mathbf{C} \times \mathbf{S}$  is the moment of the line about the origin of the reference frame. Notice that these coordinates do not depend on the choice of the reference point  $\mathbf{C}$ , because any other point  $\mathbf{C}' = \mathbf{C} + k\mathbf{S}$  yields the same moment  $\mathbf{C}' \times \mathbf{S} = \mathbf{C} \times \mathbf{S}$ .

A general pair of vectors  $W = (\mathbf{W}, \mathbf{V})^T$  can be the Plücker coordinates of a line only if  $\mathbf{W} \cdot \mathbf{V} = 0$ . This is equivalent to saying that there must be a vector  $\mathbf{C}$  such that

$$\mathbf{C} \times \mathbf{W} = \mathbf{V}. \quad (12.24)$$

Solve this equation by computing the vector product of both sides by  $\mathbf{W}$  to obtain

$$\mathbf{C} = \frac{\mathbf{W} \times \mathbf{V}}{\mathbf{W} \cdot \mathbf{W}}. \quad (12.25)$$

This formula defines the coordinates for the reference point directly in terms of the Plücker coordinates of the line.

Plücker coordinates are homogeneous, which means that  $\mathbf{W} = w\mathbf{S}$  defines the same line as the unit vector  $\mathbf{S}$ . For convenience, we normalize our the Plücker coordinates so  $S = (\mathbf{S}, \mathbf{C} \times \mathbf{S})$ , where  $|\mathbf{S}| = 1$ .

### 12.2.2 Screws

A general pair of vectors  $W = (\mathbf{W}, \mathbf{V})^T$  for which  $\mathbf{W} \cdot \mathbf{V} \neq 0$  and  $|\mathbf{W}| = w \neq 1$  is called a *screw*. We can associate with any screw  $W$  a line  $S$ , called the *axis* of the screw. To do this, decompose the second vector  $\mathbf{V}$  into components parallel and perpendicular to  $\mathbf{W}$ , so we have  $\mathbf{V} = p_w \mathbf{W} + \mathbf{V}^*$ . Since  $\mathbf{W} \cdot \mathbf{V}^* = 0$ , we can determine a point  $\mathbf{C}$  such that  $\mathbf{C} \times \mathbf{W} = \mathbf{V}^*$ . This is given by

$$\mathbf{C} = \frac{\mathbf{W} \times \mathbf{V}^*}{\mathbf{W} \cdot \mathbf{W}} = \frac{\mathbf{W} \times \mathbf{V}}{\mathbf{W} \cdot \mathbf{W}}. \quad (12.26)$$

Notice that the vector product with  $\mathbf{W}$  automatically eliminates the component of  $\mathbf{V}$  in the direction  $\mathbf{W}$ .

The line  $S = (\mathbf{W}, \mathbf{C} \times \mathbf{W})^T$  is the axis of the screw  $W$ . Let  $\mathbf{W} = w\mathbf{S}$ . Then the components of this screw can be written in the form

$$W = \left\{ \begin{array}{c} w\mathbf{S} \\ w\mathbf{C} \times \mathbf{S} + wp_w\mathbf{S} \end{array} \right\}. \quad (12.27)$$

The parameter  $w = |\mathbf{W}|$  is called the *magnitude* of the screw, and

$$p_w = \frac{\mathbf{W} \cdot \mathbf{V}}{\mathbf{W} \cdot \mathbf{W}} \quad (12.28)$$

is its *pitch*. Lines are often called *zero-pitch screws*.

### 12.2.3 Dual Vector Algebra

We now introduce *dual vector algebra*, which allows us to manipulate the pairs of vectors that define lines and screws using the same operations as vector algebra.

#### 12.2.3.1 The Dual Magnitude of a Screw

A multiplication operation can be defined so that a general screw  $W$ , given by (12.27), can be obtained as the product of the pair of scalars  $\hat{w} = (w, wp_w)$  with the pair of vectors  $S = (\mathbf{S}, \mathbf{C} \times \mathbf{S})$ . This operation is formulated by introducing the dual unit  $\varepsilon$  that has all the properties of a real scalar with the additional feature that  $\varepsilon^2 = 0$ . Using this symbol, we define the dual number

$$\hat{w} = (w, wp_w) = w + \varepsilon wp_w. \quad (12.29)$$

Notice that we do not distinguish symbolically between the dual number written as a pair of numbers or written using the dual unit  $\varepsilon$ . Similarly, we can define the dual vector

$$\mathbf{S} = (\mathbf{S}, \mathbf{C} \times \mathbf{S})^T = \mathbf{S} + \varepsilon \mathbf{C} \times \mathbf{S}. \quad (12.30)$$

Again, we do not distinguish between the screw written as a pair of vectors or a dual vector.

Now multiply the dual scalar  $\hat{w}$  and the components of the dual vector  $\mathbf{S}$  and impose the rule  $\varepsilon^2 = 0$  to obtain

$$\hat{w}\mathbf{S} = (w + \varepsilon wp_w)(\mathbf{S} + \varepsilon \mathbf{C} \times \mathbf{S}) = w\mathbf{S} + \varepsilon(w\mathbf{C} \times \mathbf{S} + wp_w\mathbf{S}). \quad (12.31)$$

Compare this to (12.27) to see that this equation defines a general screw  $W$ . The dual number  $\hat{w} = w + \varepsilon wp_w$  is the *dual magnitude* of the screw  $W$ .

#### 12.2.3.2 Dual Numbers

The set of dual numbers  $\hat{a} = a + \varepsilon a^\circ$ , where  $a$  and  $a^\circ$  are real numbers and  $\varepsilon^2 = 0$ , has all the properties of complex numbers. Addition and subtraction are obtained componentwise, and multiplication is performed as though these numbers were polynomials in  $\varepsilon$ . Division is defined so that for  $\hat{a} = a + \varepsilon a^\circ$  and  $\hat{b} = b + \varepsilon b^\circ$ , we have

$$\frac{\hat{b}}{\hat{a}} = \frac{(b + \varepsilon b^\circ)(a - \varepsilon a^\circ)}{(a + \varepsilon a^\circ)(a - \varepsilon a^\circ)} = \frac{b}{a} + \varepsilon \frac{b^\circ a - ba^\circ}{a^2}. \quad (12.32)$$

Notice that if the dual number has a zero real part then this division operation is undefined. Such numbers are known as *pure* dual numbers. See Appendix D for a summary of the properties of dual numbers.

### 12.2.3.3 The Dual Scalar Product

The linearity of the scalar product of vectors allows us to define the dual scalar product as

$$\mathbf{W} \cdot \mathbf{V} = (\mathbf{W} + \varepsilon \mathbf{W}^\circ) \cdot (\mathbf{V} + \varepsilon \mathbf{V}^\circ) = \mathbf{W} \cdot \mathbf{V} + \varepsilon (\mathbf{W} \cdot \mathbf{V}^\circ + \mathbf{W}^\circ \cdot \mathbf{V}). \quad (12.33)$$

This equation can be written in matrix form by introducing the  $6 \times 6$  matrix  $[\Pi]$  defined by

$$[\Pi]\mathbf{V} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{V} \\ \mathbf{V}^\circ \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}^\circ \\ \mathbf{V} \end{Bmatrix}. \quad (12.34)$$

The second component of the dual scalar product (12.33) is  $\mathbf{W}^T [\Pi]\mathbf{V}$ , and we have

$$\mathbf{W} \cdot \mathbf{V} = (\mathbf{W}^T \mathbf{V}, \mathbf{W}^T [\Pi]\mathbf{V}) \quad (12.35)$$

as the matrix form of the dual scalar product.

Notice that if a screw is given by  $\mathbf{W} = \hat{w}\mathbf{S}$ , where  $\mathbf{S}$  is the Plücker coordinates of its axis, then we can compute

$$\mathbf{W} \cdot \mathbf{W} = \hat{w}^2 (\mathbf{S} \cdot \mathbf{S}) = \hat{w}^2. \quad (12.36)$$

This is because  $\mathbf{S} \cdot \mathbf{S} = 1$  for normalized Plücker coordinates. For this reason Plücker vectors are often called *unit screws*.

The dual magnitude of a screw  $\mathbf{W} = (\mathbf{W}, \mathbf{W}^\circ)^T$  can be computed using the dual scalar product to obtain

$$|\mathbf{W}| = (\mathbf{W} \cdot \mathbf{W})^{1/2}. \quad (12.37)$$

Furthermore, the axis of a screw  $\mathbf{W}$  can be found by dividing by its dual magnitude, that is,

$$\mathbf{S} = \frac{\mathbf{W}}{|\mathbf{W}|} = \frac{\mathbf{W} + \varepsilon \mathbf{W}^\circ}{w + \varepsilon w p_w} = \frac{1}{w} \mathbf{W} + \varepsilon \frac{1}{w} (\mathbf{W}^\circ - p_w \mathbf{W}). \quad (12.38)$$

This yields the same screw axis as was defined above.

### 12.2.3.4 The Dual Vector Product

The linearity of the vector vector product allows its extension to dual vectors as well. Consider the two screws  $\mathbf{W} = (\mathbf{W}, \mathbf{W}^\circ)^T$  and  $\mathbf{V} = (\mathbf{V}, \mathbf{V}^\circ)^T$ , and compute

$$\mathbf{W} \times \mathbf{V} = (\mathbf{W} + \varepsilon \mathbf{W}^\circ) \times (\mathbf{V} + \varepsilon \mathbf{V}^\circ) = \mathbf{W} \times \mathbf{V} + \varepsilon (\mathbf{W} \times \mathbf{V}^\circ + \mathbf{W}^\circ \times \mathbf{V}). \quad (12.39)$$

In what follows we show that this screw has as its axis the common normal to the axes of the screws  $\mathbf{W}$  and  $\mathbf{V}$ .

### 12.2.4 Orthogonal Components of a Line

Let the Plücker coordinates of the  $x$ ,  $y$ , and  $z$  axes of the fixed frame  $F$  be  $\mathbf{l} = (\vec{i}, \mathbf{o} \times \vec{i})^T$ ,  $\mathbf{j} = (\vec{j}, \mathbf{o} \times \vec{j})^T$ , and  $\mathbf{k} = (\vec{k}, \mathbf{o} \times \vec{k})^T$ , where  $\mathbf{o}$  is the origin of  $F$ . We now determine the orthogonal components of a general line measured against these coordinate lines.

Consider the line  $\mathbf{S}$  that intersects the  $z$ -axis  $\mathbf{K}$  in a right angle at a distance  $d$  from  $\mathbf{o}$ , such that it lies at an angle  $\theta$  measured from the  $x$ -axis  $\mathbf{l}$ . The direction of  $\mathbf{S}$  is  $\mathbf{S} = \cos \theta \vec{i} + \sin \theta \vec{j}$ , and its moment term is  $(\mathbf{o} + d\vec{k}) \times \mathbf{w}$ . Therefore, we have

$$\begin{aligned} \mathbf{S} &= \left\{ \begin{array}{c} \cos \theta \vec{i} + \sin \theta \vec{j} \\ (\mathbf{o} + d\vec{k}) \times (\cos \theta \vec{i} + \sin \theta \vec{j}) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \cos \theta \vec{i} \\ \cos \theta \mathbf{o} \times \vec{i} - d \sin \theta \vec{i} \end{array} \right\} + \left\{ \begin{array}{c} \sin \theta \vec{j} \\ \sin \theta \mathbf{o} \times \vec{j} + d \cos \theta \vec{j} \end{array} \right\}. \end{aligned} \quad (12.40)$$

Thus, the coordinates of  $\mathbf{S}$  can be written as the sum of two screws. The first screw has the dual magnitude  $(\cos \theta, -d \sin \theta)$  and the line  $\mathbf{l}$  as its axis. The second screw has the dual magnitude  $(\sin \theta, d \cos \theta)$  and  $\mathbf{j}$  as its axis.

#### 12.2.4.1 The Dual Angle

We now introduce the *dual angle*  $\hat{\theta} = \theta + \varepsilon d$ , which measures the angle  $\theta$  and distance  $d$  around and along the axis  $\mathbf{K}$  from the  $x$ -axis  $\mathbf{l}$  to the line  $\mathbf{S}$ , [Figure 12.1](#). The cosine and sine functions of this dual angle are defined such that

$$\cos \hat{\theta} = \cos \theta - \varepsilon d \sin \theta, \quad \sin \hat{\theta} = \sin \theta + \varepsilon d \cos \theta. \quad (12.41)$$

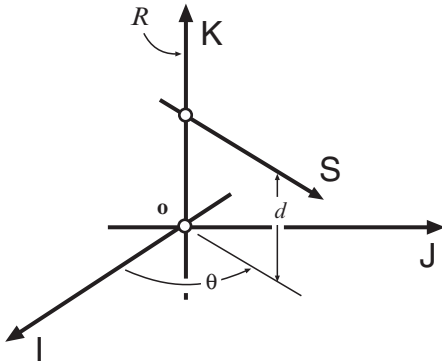
Equation (12.40) can now be written in the form

$$\mathbf{S} = \cos \hat{\theta} \mathbf{l} + \sin \hat{\theta} \mathbf{j}, \quad (12.42)$$

where the screws  $\cos \hat{\theta} \mathbf{l}$  and  $\sin \hat{\theta} \mathbf{j}$  are the *orthogonal components* of  $\mathbf{S}$  in the frame  $F$ .

The dual scalar product can be used to compute the component of  $\mathbf{S}$  along the line  $\mathbf{l}$ ,

$$\mathbf{S} \cdot \mathbf{l} = \mathbf{S} \cdot \vec{i} + \varepsilon (\mathbf{S} \cdot \mathbf{o} \times \vec{i} + \mathbf{C} \times \mathbf{S} \cdot \vec{i}) = \cos \hat{\theta}. \quad (12.43)$$



**Fig. 12.1** The dual angle  $\hat{\theta}$  to a line  $S$  in the reference frame  $R$  formed by the mutually orthogonal lines  $I$ ,  $J$ , and  $K$ .

This calculation uses the fact that  $\mathbf{S} \times \vec{r} = -\sin \theta \vec{k}$  and that the component of  $\mathbf{C} - \mathbf{o}$  along  $\vec{k}$  is  $d$ . Thus, the dual scalar product allows us to calculate the dual angle between any two lines about their common normal. In fact, for general screws  $W$  and  $V$ , we have

$$W \cdot V = |W||V| \cos \hat{\theta}, \quad (12.44)$$

where  $|W|$  and  $|V|$  are the dual magnitudes of these screws and  $\hat{\theta}$  is the dual angle between their axes.

#### 12.2.4.2 The Intersection of Two Lines

We now consider the meaning of a zero value for the dual scalar product between two screws, that is,  $W \cdot V = 0$ . Let the dual magnitudes of these screws be  $|W| = w(1 + \varepsilon p_w)$  and  $|V| = v(1 + \varepsilon p_v)$ , so we have

$$W \cdot V = wv(1 + \varepsilon p_w)(1 + \varepsilon p_v)(\cos \theta - \varepsilon d \sin \theta). \quad (12.45)$$

This shows that  $W \cdot V = 0$  implies that  $\cos \hat{\theta} = 0$ , which means that  $\hat{\theta} = (\pi/2, 0)$ . Thus, the axes of the two screws must intersect in right angles.

Two screws that satisfy the weaker condition  $W^T [II] V = 0$  are said to be *reciprocal*. Notice that this is equivalent to the requirement that  $W \cdot V = k$ , where  $k$  is a real constant. Expand this relation to obtain

$$W^T [II] V = \mathbf{w} \cdot \mathbf{v}^\circ + \mathbf{w}^\circ \cdot \mathbf{v} = wv((p_w + p_v) \cos \theta - d \sin \theta) = 0. \quad (12.46)$$

Thus, the condition that two screws are reciprocal is

$$d \tan \theta = p_w + p_v. \quad (12.47)$$

Applying this condition to two lines for which  $p_w = p_v = 0$ , we see that to be reciprocal the lines must be parallel ( $\theta = 0$ ), or they must intersect ( $d = 0$ ).

### 12.2.4.3 The Common Normal

The dual vector product between two lines defines a screw that has the common normal between the lines as its axis. To see this, we first consider the dual vector product between the coordinate axes  $I$  and  $J$ . By direct computation we obtain

$$\begin{aligned} I \times J &= \vec{i} \times \vec{j} + \varepsilon(\vec{i} \times (\mathbf{o} \times \vec{j}) + (\mathbf{o} \times \vec{i}) \times \vec{j}) \\ &= \vec{i} \times \vec{j} + \varepsilon \mathbf{o} \times (\vec{i} \times \vec{j}) = K. \end{aligned} \quad (12.48)$$

The simplification of the dual component in this equation uses the identity for triple vector products

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0, \quad (12.49)$$

which in our case yields the relation

$$-\mathbf{o} \times (\vec{j} \times \vec{i}) = \vec{i} \times (\mathbf{o} \times \vec{j}) + \vec{j} \times (\vec{i} \times \mathbf{o}). \quad (12.50)$$

Similar calculations show that  $J \times K = I$  and  $K \times I = J$ .

Consider a general pair of lines  $S = (\mathbf{S}, \mathbf{C} \times \mathbf{S})^T$  and  $L = (\mathbf{W}, \mathbf{Q} \times \mathbf{W})^T$ . Let  $N$  be the common normal between these lines, and let  $\mathbf{p}$  and  $\mathbf{r}$  be its points of intersection with  $S$  and  $L$ , respectively, [Figure 12.2](#). Now use these two points to define the moment terms in the Plücker coordinates for these lines, so we have  $S = (\mathbf{S}, \mathbf{p} \times \mathbf{S})^T$  and  $L = (\mathbf{W}, \mathbf{r} \times \mathbf{W})^T$ . We can now compute the vector product

$$\begin{aligned} S \times L &= (\mathbf{S} + \varepsilon \mathbf{p} \times \mathbf{S}) \times (\mathbf{W} + \varepsilon \mathbf{r} \times \mathbf{W}), \\ &= \mathbf{S} \times \mathbf{W} + \varepsilon(\mathbf{S} \times (\mathbf{r} \times \mathbf{W}) + (\mathbf{p} \times \mathbf{S}) \times \mathbf{W}). \end{aligned} \quad (12.51)$$

Notice that  $\mathbf{r} = \mathbf{p} + d\mathbf{N}$ , where  $\mathbf{N}$  is the direction of the common normal. Substitute this into the equation above, and use the vector identity (12.49) to obtain

$$S \times L = \sin \theta \mathbf{N} + \varepsilon(\cos \theta \mathbf{N} + \sin \theta \mathbf{o} \times \mathbf{N}) = \sin \hat{\theta} \mathbf{N}. \quad (12.52)$$

Thus, the dual vector product is an operation that computes the common normal to two given lines. Furthermore, for general screws  $W$  and  $V$  we have

$$W \times V = |W||V| \sin \hat{\theta} \mathbf{N}, \quad (12.53)$$

where  $|W|$  and  $|V|$  are the dual magnitudes of  $W$  and  $V$ .

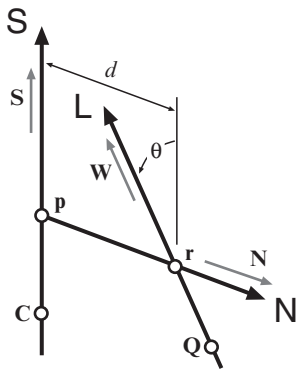


Fig. 12.2 Two general lines  $S$  and  $L$  define a common normal line  $N$ .

### 12.2.5 The Spatial Displacement of Screws

A spatial displacement  $[T] = [A, \mathbf{d}]$  transforms the coordinates of points that form a line. By applying this transformation to two points on the line we obtain a  $6 \times 6$  transformation  $[\hat{T}]$  for Plücker coordinates. This transformation applies to general screws as well.

Consider the line  $\mathbf{x} = (\mathbf{x}, \mathbf{p} \times \mathbf{x})^T$  for which every point is displaced by the  $4 \times 4$  homogeneous transform  $[T] = [A, \mathbf{d}]$  to define a new line  $\mathbf{X} = (\mathbf{X}, \mathbf{P} \times \mathbf{X})^T$ . We now determine the associated transformation  $[\hat{T}]$  that acts directly on Plücker coordinates such that

$$\mathbf{X} = [\hat{T}] \mathbf{x}. \quad (12.54)$$

Let  $\mathbf{q}$  be a point on the line  $\mathbf{x}$  a unit distance from  $\mathbf{p}$ , so  $\mathbf{x} = \mathbf{q} - \mathbf{p}$ . Then, we can compute the new coordinates  $\mathbf{P}$  and  $\mathbf{Q}$  to define the line  $\mathbf{X}$

$$\left\{ \begin{array}{c} \mathbf{X} \\ \mathbf{P} \times \mathbf{X} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{Q} - \mathbf{P} \\ \mathbf{P} \times (\mathbf{Q} - \mathbf{P}) \end{array} \right\} = \left\{ \begin{array}{c} [A] \mathbf{x} \\ [D][A] \mathbf{x} + [A](\mathbf{p} \times \mathbf{x}) \end{array} \right\}. \quad (12.55)$$

This calculation uses the skew-symmetric matrix  $[D]$  defined by  $[D]\mathbf{y} = \mathbf{d} \times \mathbf{y}$  for any  $\mathbf{y}$ . Thus, we obtain  $[\hat{T}]$  as the  $6 \times 6$  matrix

$$[\hat{T}] = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix}. \quad (12.56)$$

The inverse of this transformation is easily obtained as

$$[\hat{T}^{-1}] = \begin{bmatrix} A^T & 0 \\ A^T D^T & A^T \end{bmatrix}. \quad (12.57)$$

Note that because  $[D]$  is skew-symmetric, we have  $[D + D^T] = 0$ .

### 12.2.5.1 The Transformation of Screws

The transformation  $[\hat{T}]$  defined by (12.56) applies to general screws as well. To see this, consider the screw  $\mathbf{w} = (\mathbf{w}, \mathbf{v})^T$  and compute  $[\hat{T}]\mathbf{w}$  to obtain

$$\begin{Bmatrix} \mathbf{W} \\ \mathbf{V} \end{Bmatrix} = \begin{Bmatrix} [A]\mathbf{w} \\ [D][A]\mathbf{w} + [A]\mathbf{v} \end{Bmatrix}. \quad (12.58)$$

Clearly, the transformation of the direction  $\mathbf{w} = k\mathbf{s}$  of the screw is the same as for lines, that is,

$$\mathbf{W} = w\mathbf{S} = w[A]\mathbf{s} = [A]\mathbf{w}. \quad (12.59)$$

Therefore, we focus attention on the term  $\mathbf{V} = [D][A]\mathbf{w} + [A]\mathbf{v}$ .

Let  $\mathbf{w}$  be written in terms of its axis  $\mathbf{s} = (\mathbf{s}, \mathbf{p} \times \mathbf{s})^T$ , so we have  $\mathbf{w} = (w\mathbf{s}, w\mathbf{p} \times \mathbf{s} + wp_w\mathbf{s})^T$ , where  $w$  is the magnitude and  $p_w$  the pitch of  $\mathbf{w}$ . Clearly, we have

$$[D][A](w\mathbf{s}) = w\mathbf{d} \times \mathbf{S}. \quad (12.60)$$

We can now compute

$$[A]\mathbf{v} = [A](w\mathbf{p} \times \mathbf{s} + wp_w\mathbf{s}) = w([A]\mathbf{p}) \times \mathbf{S} + wp_w\mathbf{S}. \quad (12.61)$$

Combining these results we have

$$\mathbf{V} = [D][A]\mathbf{w} + [A]\mathbf{v} = \mathbf{P} \times \mathbf{W} + p_w\mathbf{W}, \quad (12.62)$$

where  $\mathbf{P} = [A]\mathbf{p} + \mathbf{d}$ . Thus, for a general screw  $\mathbf{w}$  in  $M$  we obtain

$$\mathbf{W} = [\hat{T}]\mathbf{w}. \quad (12.63)$$

The transformation  $[\hat{T}]$  preserves the magnitude and pitch of screws.

The matrix form of the dual scalar product makes it easy to show that the transformation  $[\hat{T}]$  preserves the dual magnitude of a screw. In particular, we show that  $\mathbf{W} \cdot \mathbf{W} = \mathbf{w} \cdot \mathbf{w}$  by the calculation

$$\begin{aligned} \mathbf{W} \cdot \mathbf{W} &= ([\hat{T}]\mathbf{w}) \cdot ([\hat{T}]\mathbf{w}) = (\mathbf{w}^T [A]^T [A] \mathbf{w}, \mathbf{w}^T [\hat{T}]^T [\Pi] [\hat{T}] \mathbf{w}) \\ &= (\mathbf{w}^T \mathbf{w}, \mathbf{w}^T [\Pi] \mathbf{w}) = \mathbf{w} \cdot \mathbf{w}. \end{aligned} \quad (12.64)$$

This computation uses the identities  $[A]^T [A] = [I]$  and  $[\hat{T}]^T [\Pi] [\hat{T}] = [\Pi]$ .

## 12.3 The Geometry of Screw Axes

### 12.3.1 The Screw Axis of a Displacement

We have seen that for every spatial displacement there is a fixed line, called its screw axis. Here we show that the Plücker coordinates  $S = (\mathbf{S}, \mathbf{V})$  of this screw axis satisfy the condition

$$S = [\hat{T}]S. \quad (12.65)$$

This shows that the screw axis is an invariant of the  $6 \times 6$  transformation matrix  $[\hat{T}]$ .

We rewrite (12.65) as the equation

$$[I - \hat{T}]S = 0, \quad (12.66)$$

and seek solutions other than  $S = 0$ . This is easily done if we separate it into the pair of  $3 \times 1$  vector equations

$$[I - A]S = 0 \quad \text{and} \quad [I - A]V - [DA]S = 0. \quad (12.67)$$

We already know how to determine the vector  $S = (s_x, s_y, s_z)^T$ , which is the rotation axis of the rotation matrix  $[A]$ .

Notice that because  $[A]S = S$ , the second equation of (12.67) can be written as

$$[I - A]V = [D]S. \quad (12.68)$$

Now  $[D]S = \mathbf{d} \times S$  must be orthogonal to  $S$ . Therefore, it does not have a component in the direction of the null space of  $[I - A]$ , which is  $S$ . This means that we can solve this equation for  $V$ .

Substitute Cayley's formula for  $[A]$ , and simplify to obtain

$$[B]V = \frac{1}{2}[I - B][S]\mathbf{d}; \quad (12.69)$$

for convenience we have introduced  $[D]S = -[S]\mathbf{d}$ . Using the fact that  $[B] = \tan(\phi/2)[S]$ , we can write this equation as

$$[S]V = \frac{1}{2 \tan \frac{\phi}{2}}[S][I - B]\mathbf{d} = \mathbf{C}. \quad (12.70)$$

Finally, multiply both sides by  $[S]$  and simplify to obtain

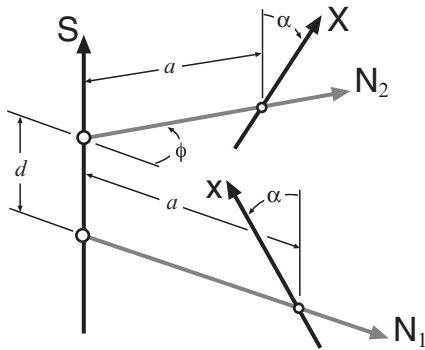
$$V = \frac{-1}{2 \tan \frac{\phi}{2}}[S^2][I - B]\mathbf{d}. \quad (12.71)$$

Because  $S \cdot V = 0$ , we see that  $S = (S, V)^T$  are the Plücker coordinates of a line.

The reference point  $\mathbf{C}$  for  $S$  is determined by  $\mathbf{C} = \mathbf{S} \times \mathbf{V}$ , which is given in (12.70) above. The line  $S = (\mathbf{S}, \mathbf{C} \times \mathbf{S})^T$  is exactly the screw axis that was formulated earlier for the a spatial displacement  $[T] = [A, \mathbf{d}]$ .

### 12.3.2 Perpendicular Bisectors and the Screw Axis

A spatial displacement preserves the distances and angles between all points in the moving body. Therefore, it preserves the dual angles between lines in the body. In particular, the dual angle  $\hat{\alpha}$  between the screw axis  $S$  of a displacement and the axis of a general screw must be the same before and after the displacement, [Figure 12.3](#).



**Fig. 12.3** Corresponding positions of lines  $x$  and  $X$  lie at the same dual angle  $\hat{\alpha}$  relative to the screw axis  $S$  of a spatial displacement.

This is seen by letting  $x$  be the coordinates of a screw in the initial position, so we have  $X = [\hat{T}]x$  as its coordinates after the displacement. From the fact that  $S = [\hat{T}]S$ , we can compute

$$S \cdot X = ([\hat{T}]S) \cdot ([\hat{T}]x) = S \cdot x. \quad (12.72)$$

Thus, the screw axis  $S$  forms the same dual angle  $\hat{\alpha}$  with the axes of both  $x$  and its corresponding screw  $X$ .

This allows us to compute

$$S \cdot (X - x) = 0, \quad (12.73)$$

which shows that the axis of the difference of any two corresponding screws  $X - x$  must intersect  $S$  in a right angle.

### 12.3.2.1 The Screw Perpendicular Bisector

We now examine equation (12.73) in detail. To do this focus on the pair of corresponding screws  $\mathbf{x} = \mathbf{p}$  and  $\mathbf{X} = \mathbf{P}$  and consider all the screws  $\mathbf{Y}$  that satisfy the equation

$$\mathbf{Y} \cdot (\mathbf{P} - \mathbf{p}) = 0. \quad (12.74)$$

For example, the screw  $(\mathbf{P} + \mathbf{p})/2$  is a member of this set, as can be seen from the calculation

$$\frac{\mathbf{P} + \mathbf{p}}{2} \cdot (\mathbf{P} - \mathbf{p}) = \frac{\mathbf{P} \cdot \mathbf{P} - \mathbf{p} \cdot \mathbf{p}}{2} = 0. \quad (12.75)$$

Recall from (12.64) that  $|\mathbf{P}| = |\mathbf{p}|$ .

Let  $D$  be the common normal to the lines  $\mathbf{p}$  and  $\mathbf{P}$  with points of intersection  $\mathbf{c}_1$  on  $\mathbf{p}$  and  $\mathbf{c}_2$  on  $\mathbf{P}$ . Introduce the line  $V$  that passes through the midpoint  $\mathbf{c}$  of the segment  $\mathbf{c}_2 - \mathbf{c}_1$  and is directed along the bisector of the directions  $\mathbf{p}$  and  $\mathbf{P}$ . Finally, let  $\mathbf{N} = D \times V$ , so  $D$ ,  $V$ , and  $\mathbf{N}$  are the coordinate axes of a reference frame  $R$  located at  $\mathbf{c}$ . See Figure 12.4.

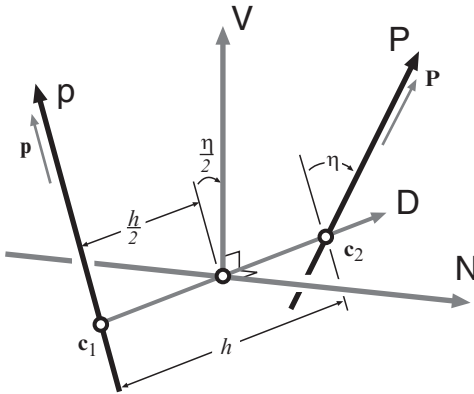


Fig. 12.4 The screws  $\mathbf{P} + \mathbf{p}$  and  $\mathbf{P} - \mathbf{p}$  have  $V$  and  $N$  as their respective axes.

If we denote the dual angle between  $\mathbf{p}$  and  $\mathbf{P}$  by  $\hat{\eta} = (\eta, h)$ , then we can write the components of these screws as

$$\mathbf{p} = |\mathbf{p}| \left( \cos \frac{\hat{\eta}}{2} \mathbf{V} - \sin \frac{\hat{\eta}}{2} \mathbf{N} \right) \quad \text{and} \quad \mathbf{P} = |\mathbf{p}| \left( \cos \frac{\hat{\eta}}{2} \mathbf{V} + \sin \frac{\hat{\eta}}{2} \mathbf{N} \right). \quad (12.76)$$

This allows us to compute

$$\mathbf{P} + \mathbf{p} = 2|\mathbf{p}| \cos \frac{\hat{\eta}}{2} \mathbf{V}. \quad (12.77)$$

The line  $V$  is the axis of the midpoint screw. Furthermore, from

$$P - p = 2|p| \sin \frac{\hat{\eta}}{2} N \quad (12.78)$$

we see that  $N$  is the axis of the screw difference  $P - p$ .

Thus, the set of screws  $Y$  that satisfy (12.74) must have axes that intersect  $N$  in right angles. This defines a two parameter family of screws that we call a *screw perpendicular bisector*.

### 12.3.2.2 Constructing the Screw Axis

The equation (12.73) shows that the screw axis  $S$  must lie on the screw perpendicular bisector for all segments  $X - x$  in the moving body. This provides a convenient way to construct the screw axis of a displacement. Consider two specific segments  $P - p$  and  $Q - q$  formed by the two screws  $p, q$  and their corresponding screws  $P = [\hat{T}]p$  and  $Q = [\hat{T}]q$ . This defines two screw perpendicular bisectors

$$Y \cdot (P - p) = 0 \quad \text{and} \quad Y \cdot (Q - q) = 0. \quad (12.79)$$

Let  $N_1$  be the axis of  $P - p$ , and let  $N_2$  be the axis of  $Q - q$ . Then the screw axis of the displacement  $S$  must intersect both of these axes in right angles. Thus,  $S$  must be the common normal to the axes  $N_1$  and  $N_2$ .

The algebra of dual vectors allows us to compute  $S$  from the dual vector product of the screws  $P - p$  and  $Q - q$ , that is,

$$S = \frac{(P - p) \times (Q - q)}{|(P - p) \times (Q - q)|}. \quad (12.80)$$

This provides a direct way to compute the screw axis of a spatial displacement from data that define the positions of two screws.

### 12.3.2.3 The Dual Displacement Angle

We can determine the dual angle  $\hat{\phi}$  of a spatial displacement using any screw  $p$  and its corresponding displaced screw  $P = [\hat{T}]p$ . This is done by computing the dual scalar and vector products

$$\sin \hat{\phi} = \frac{(S \times p) \times (S \times P) \cdot S}{|(S \times p) \cdot (S \times P)|}, \quad \cos \hat{\phi} = \frac{(S \times p) \cdot (S \times P)}{|(S \times p) \cdot (S \times P)|}. \quad (12.81)$$

Thus, we have

$$\tan \hat{\phi} = \frac{(S \times p) \cdot P}{(S \times p) \cdot (S \times P)}. \quad (12.82)$$

The simplification in the numerator is obtained using dual vector identities that are identical to those of vector algebra.

### 12.3.3 Rodrigues's Equation for Screws

We now examine in more detail the geometric relationship between the screw axis  $S$  of a displacement  $[\hat{T}]$  and the initial and final positions of a general screw. Because  $X = [\hat{T}]x$ , we have  $X \cdot X - x \cdot x = 0$ , which can also be written as

$$(X - x) \cdot (X + x) = 0. \quad (12.83)$$

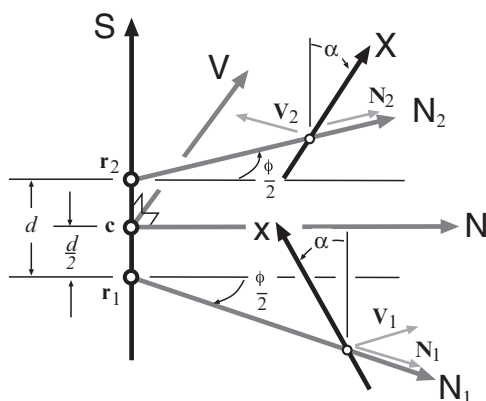
This can be interpreted as stating that the axes of the diagonals of a screw rhombus must intersect at right angles. In what follows, we will determine the components of these diagonals and obtain a screw version of Rodrigues's equation.

Let the common normals between the screw axis  $S$  and the axes of the corresponding screws  $x$  and  $X$  be the lines  $N_1 = (N_1, r_1 \times N_1)^T$  and  $N_2 = (N_2, r_2 \times N_2)^T$ , where  $r_i$  are the respective points of intersection with  $S$ , [Figure 12.5](#). Also introduce the lines  $V_1$  and  $V_2$ , given by  $V_i = S \times N_i$ . Then  $x$  and  $X$  can be expanded into components

$$x = |x|(\cos \hat{\alpha} S - \sin \hat{\alpha} V_1), \quad X = |x|(\cos \hat{\alpha} S - \sin \hat{\alpha} V_2). \quad (12.84)$$

From these equations we obtain the screws  $X - x$  and  $X + x$  as

$$\begin{aligned} X - x &= -|x| \sin \hat{\alpha} (V_2 - V_1), \\ X + x &= |x| (2 \cos \hat{\alpha} S - \sin \hat{\alpha} (V_2 + V_1)). \end{aligned} \quad (12.85)$$



**Fig. 12.5** The components of the screws  $X + x$  and  $X - x$  can be determined in the  $S, N, V$  frame.

To simplify (12.85), we introduce the line  $N$  through the midpoint  $c$  of the segment  $r_2 - r_1$  along  $S$ . Choose the direction of  $N$  so that it bisects the rotation angle  $\phi$ , that is, so  $N$  is aligned with the vector  $(N_1 + N_2)/2$ . We complete the frame at  $c$  by introducing  $V$ , given by  $V = S \times N$ . In the  $S, N, V$  frame the lines  $N_1$  and  $N_2$

become

$$N_1 = \cos \frac{\hat{\phi}}{2} N - \sin \frac{\hat{\phi}}{2} V, \quad N_2 = \cos \frac{\hat{\phi}}{2} N + \sin \frac{\hat{\phi}}{2} V. \quad (12.86)$$

Notice that  $V_2 - V_1 = S \times (N_2 - N_1) = 2 \sin(\hat{\phi}/2) S \times V$ . Therefore,

$$X - x = 2|x| \sin \hat{\alpha} \sin \frac{\hat{\phi}}{2} N. \quad (12.87)$$

From the fact that  $V_1 + V_2 = S \times (N_1 + N_2) = 2 \cos(\hat{\phi}/2) V$ , we have

$$X + x = 2|x| \left( \cos \hat{\alpha} S - \sin \hat{\alpha} \cos \frac{\hat{\phi}}{2} V \right). \quad (12.88)$$

From equations (12.87) and (12.88) we obtain the screw form of Rodrigues's equation as

$$X - x = \tan \frac{\hat{\phi}}{2} S \times (X + x). \quad (12.89)$$

The screw  $B = \tan(\hat{\phi}/2) S$  is known as *Rodrigues's screw*.

## 12.4 The Spatial Screw Triangle

### 12.4.1 The Screw Axis of a Composite Displacement

Rodrigues's equation can be used to derive a formula that defines the screw axis of a composite displacement in terms of the screw axes of the two individual displacements.

Let  $[T(\hat{\alpha}, A)]$  be the displacement with screw axis  $A$  and rotation angle and slide distance  $\hat{\alpha} = (\alpha, a)$ . Given another displacement  $[T(\hat{\beta}, B)]$ , we can compute the composite displacement by matrix multiplication

$$[T(\hat{\gamma}, C)] = [T(\hat{\beta}, B)][T(\hat{\alpha}, A)]. \quad (12.90)$$

Our goal is to obtain a formula for the screw axis  $C$  and the dual angle  $\hat{\gamma}$  in terms of  $\hat{\beta}$ ,  $B$  and  $\hat{\alpha}$ ,  $A$ .

The displacement  $[T(\hat{\alpha}, A)]$  has the associated  $6 \times 6$  transformation  $[\hat{T}(\hat{\alpha}, A)]$  that transforms screws  $x$  in  $M$  to  $y$  in  $M'$ . The displacement  $[T(\hat{\beta}, B)]$  also has an associated  $6 \times 6$  transformation  $[\hat{T}(\hat{\beta}, B)]$  that transforms screws  $y$  in  $M'$  to  $X$  in  $F$ . This sequence of displacements can be written using Rodrigues's equation (12.89) to yield

$$\begin{aligned} y - x &= \tan \frac{\alpha}{2} A \times (y + x), \\ X - y &= \tan \frac{\beta}{2} B \times (X + y). \end{aligned} \quad (12.91)$$

We eliminate the screw  $y$  in these equations in order to obtain a formula for the screw axis  $C$  and dual angle  $\hat{\gamma}$ .

The following calculations use dual vector algebra and follow exactly the derivation for the spherical version of Rodrigues's formula. The first step is to introduce  $X$  in the first equation and  $x$  in the second, so we have

$$\begin{aligned} y - x &= \tan \frac{\hat{\alpha}}{2} A \times (X + x - (X - y)), \\ X - y &= \tan \frac{\hat{\beta}}{2} B \times (X + x + (y - x)). \end{aligned} \quad (12.92)$$

Add these equations and use (12.91) to obtain

$$\begin{aligned} X - x &= (\tan \frac{\hat{\beta}}{2} B + \tan \frac{\hat{\alpha}}{2} A) \times (X + x) - \tan \frac{\hat{\alpha}}{2} A \times (\tan \frac{\hat{\beta}}{2} B \times (X + y)) \\ &\quad + \tan \frac{\hat{\beta}}{2} B (\tan \frac{\hat{\alpha}}{2} A \times (y + x)). \end{aligned} \quad (12.93)$$

Triple product identities for dual vectors that are identical to those for vectors simplify this equation to yield

$$X - x = \tan \frac{\hat{\gamma}}{2} C \times (X + x), \quad (12.94)$$

where

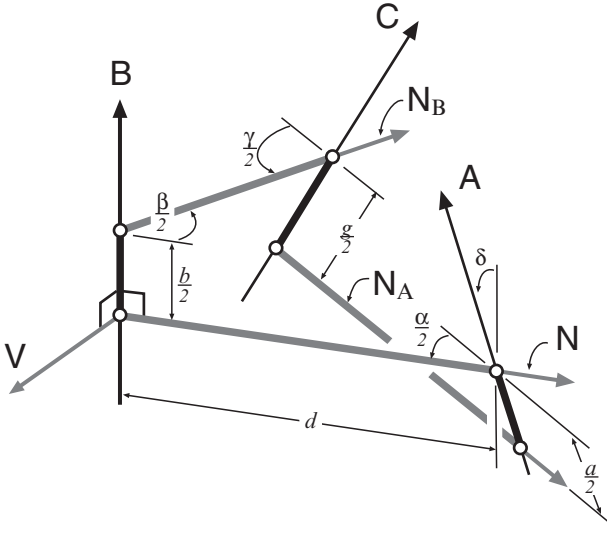
$$\tan \frac{\hat{\gamma}}{2} C = \frac{\tan \frac{\hat{\beta}}{2} B + \tan \frac{\hat{\alpha}}{2} A + \tan \frac{\hat{\beta}}{2} \tan \frac{\hat{\alpha}}{2} B \times A}{1 - \tan \frac{\hat{\beta}}{2} \tan \frac{\hat{\alpha}}{2} B \cdot A}. \quad (12.95)$$

This is *Rodrigues's formula* for screws.

### 12.4.2 The Spatial Triangle

We now show that (12.95) is the equation of an assembly of three lines  $A$ ,  $B$ , and  $C$  known as a *spatial triangle*. Introduce the common normal  $N$  directed from the line  $B$  to  $A$ . Now introduce the common normals  $N_A$  and  $N_B$  directed from  $A$  and  $B$  to  $C$ , respectively. The lines  $A$ ,  $B$ , and  $C$  form the *vertices* of the spatial triangle, and the common normals  $N$ ,  $N_A$ , and  $N_B$  form its *sides*, [Figure 12.6](#).

Let the interior dual angle between the sides  $N_A$  and  $N$  be  $\hat{\alpha}/2$ . Similarly, let the interior dual angle between the sides  $N$  and  $N_B$  be  $\hat{\beta}/2$ . We now show that for



**Fig. 12.6** The spatial triangle formed from the lines A, B, and C and their common normals  $N$ ,  $N_A$ , and  $N_B$ .

this configuration the line C and exterior dual angle  $\hat{\gamma}/2$  are defined by Rodrigues's formula (12.95).

From these definitions we see that

$$N_A \times N_B = \sin \frac{\hat{\gamma}}{2} C \quad \text{and} \quad N_A \cdot N_B = \cos \frac{\hat{\gamma}}{2}. \quad (12.96)$$

A formula for C is easily obtained by determining  $N_A$  and  $N_B$  explicitly in terms of A and B.

Let  $V = N \times B$  complete the reference frame formed by N, B, and V. The line  $N_B$  intersects B and lies parallel to the NV plane at the angle  $\hat{\beta}/2$  relative to N. Therefore,

$$N_B = \cos \frac{\hat{\beta}}{2} N - \sin \frac{\hat{\beta}}{2} V. \quad (12.97)$$

Note that N and V are computed from the given lines A and B.

Introduce the line  $T = N \times A$ . Then a computation similar to (12.97) yields the coordinates of T as

$$T = -\sin \hat{\delta} B + \cos \hat{\delta} V. \quad (12.98)$$

The line  $N_A$  lies parallel to the TN plane such that the dual angle measured from  $N_A$  to N is  $\hat{\alpha}/2$ . Thus,  $N_A$  is given by

$$N_A = \sin \frac{\hat{\alpha}}{2} T + \cos \frac{\hat{\alpha}}{2} N. \quad (12.99)$$

### 12.4.2.1 The Equation of the Spatial Triangle

We now compute the scalar and vector products in (12.96) to obtain

$$\begin{aligned}\sin \frac{\hat{\gamma}}{2} \mathbf{C} &= \cos \frac{\hat{\alpha}}{2} \sin \frac{\hat{\beta}}{2} \mathbf{B} + \sin \frac{\hat{\alpha}}{2} \cos \frac{\hat{\beta}}{2} \mathbf{A} + \sin \frac{\hat{\alpha}}{2} \sin \frac{\hat{\beta}}{2} \mathbf{B} \times \mathbf{A}, \\ \cos \frac{\hat{\gamma}}{2} &= \cos \frac{\hat{\alpha}}{2} \cos \frac{\hat{\beta}}{2} - \sin \frac{\hat{\alpha}}{2} \sin \frac{\hat{\beta}}{2} \mathbf{B} \cdot \mathbf{A}.\end{aligned}\quad (12.100)$$

Notice that  $\mathbf{B} \cdot \mathbf{A} = \mathbf{T} \cdot \mathbf{V} = \cos \hat{\delta}$ . Dividing these two equations, we obtain

$$\tan \frac{\hat{\gamma}}{2} \mathbf{C} = \frac{\tan \frac{\hat{\beta}}{2} \mathbf{B} + \tan \frac{\hat{\alpha}}{2} \mathbf{A} + \tan \frac{\hat{\beta}}{2} \tan \frac{\hat{\alpha}}{2} \mathbf{B} \times \mathbf{A}}{1 - \tan \frac{\hat{\beta}}{2} \tan \frac{\hat{\alpha}}{2} \mathbf{B} \cdot \mathbf{A}}. \quad (12.101)$$

Equation (12.101) defines the coordinates of the line  $\mathbf{C}$  in terms of those of  $\mathbf{A}$  and  $\mathbf{B}$  and their interior dual angles  $\hat{\beta}/2$  and  $\hat{\alpha}/2$ , respectively. Comparing this to (12.95) we see immediately that Rodrigues's formula is the equation of the spatial triangle formed by the three screw axes  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ .

### 12.4.3 The Composite Screw Axis Theorem

The screw form of Rodrigues's equation is separated into two parts by the dual unit  $\varepsilon$ . The real part is simply Rodrigues's equation for the composition of rotations. We have seen that this defines a spherical triangle  $\triangle \mathbf{ABC}$ , which we call the *spherical image* of the spatial triangle  $\triangle \mathbf{ABC}$ .

We have already seen that there are two forms of the spherical image  $\triangle \mathbf{ABC}$  depending on the magnitude of the rotation angles  $\alpha$ ,  $\beta$ , and  $\gamma$ .

1. If  $\sin(\gamma/2) > 0$ , that is,  $\gamma < 2\pi$ , then the vertex  $\mathbf{C}$  has a positive component along  $\mathbf{B} \times \mathbf{A}$ . In this case  $\alpha/2$  and  $\beta/2$  are the interior angles of  $\triangle \mathbf{ABC}$  at the vertices  $\mathbf{A}$  and  $\mathbf{B}$ . The angle  $\gamma/2$  is the exterior angle at  $\mathbf{C}$ .
2. If  $\sin(\gamma/2) < 0$ , that is,  $\gamma > 2\pi$ , then the vertex  $\mathbf{C}$  is directed opposite to the vector  $\mathbf{B} \times \mathbf{A}$ . The angles  $\alpha/2$  and  $\beta/2$  are the exterior angles of  $\triangle \mathbf{ABC}$  at  $\mathbf{A}$  and  $\mathbf{B}$ , respectively. If the angle  $\kappa$  is the interior angle at  $\mathbf{C}$ , then  $\gamma/2 = \kappa + \pi$ .

The dual part of Rodrigues's formula (12.95) is linear in the slide parameters of the dual angles. This means that the spatial configuration of lines can be adjusted by changing the slide parameters  $a$  and  $b$  without changing the directions of any of the lines  $\mathbf{A}$ ,  $\mathbf{B}$ , or  $\mathbf{C}$ . Thus, we have the following theorem:

**Theorem 14 (The Composite Screw Axis Theorem).** *The axis  $\mathbf{C}$  of a composite displacement  $[T_C] = [T_B][T_A]$  forms a spatial triangle with the screw axes  $\mathbf{B}$  and  $\mathbf{A}$  of the displacements  $[T_B]$  and  $[T_A]$ . If  $\sin(\gamma/2) > 0$ , then the interior dual angles of this triangle at  $\mathbf{A}$  and  $\mathbf{B}$  are  $\hat{\alpha}/2$  and  $\hat{\beta}/2$ , respectively. If  $\sin(\gamma/2) < 0$ , then  $\hat{\alpha}/2$  and*

$\hat{\beta}/2$  are the exterior dual angles at these vertices. In this case, if  $\hat{\kappa}$  is the interior dual angle at  $C$ , then  $\hat{\gamma}/2 = \hat{\kappa} + \pi$ .

#### 12.4.4 Dual Quaternions and the Spatial Triangle

W. K. Clifford [13] generalized Hamilton's quaternions to obtain hypercomplex numbers known as *dual quaternions* (Yang and Freudenstein [154]). A dual quaternion  $\hat{P}$  is the formal sum of a dual number  $\hat{p}_0 = (p, p^\circ)$  and a screw  $P = (\mathbf{p}, \mathbf{a})$ , written as  $\hat{P} = \hat{p}_0 + P$ . Dual quaternions can be added together componentwise, and multiplied by a scalar like eight-dimensional vectors. They can also be multiplied by dual scalars like four-dimensional vectors of dual numbers.

Furthermore, Clifford extended Hamilton's product for quaternions to a product for dual quaternions, given by the formula

$$\begin{aligned}\hat{R} = \hat{P}\hat{Q} &= (\hat{p}_0 + P)(\hat{q}_0 + Q) \\ &= (\hat{p}_0\hat{q}_0 - P \cdot Q) + (\hat{q}_0P + \hat{p}_0Q + P \times Q).\end{aligned}\quad (12.102)$$

The scalar and vector products are operations between dual vectors.

The *conjugate* of a dual quaternion  $\hat{Q} = \hat{q}_0 + Q$  is  $\hat{Q}^* = \hat{q}_0 - Q$ , and the product  $\hat{Q}\hat{Q}^*$  is the dual number

$$\hat{Q}\hat{Q}^* = (\hat{q}_0 + Q)(\hat{q}_0 - Q) = \hat{q}_0^2 + Q \cdot Q = |\hat{Q}|^2. \quad (12.103)$$

The dual number  $|\hat{Q}|$  is called the *norm* of the dual quaternion.

We are interested in dual quaternions  $\hat{Q}$  of unit norm, which means that  $|\hat{Q}| = 1$ . These unit dual quaternions can be written in the form

$$\hat{Q} = \cos \frac{\hat{\phi}}{2} + \sin \frac{\hat{\phi}}{2} S, \quad (12.104)$$

where  $S = (\mathbf{S}, \mathbf{C} \times \mathbf{S})^T$  is the Plücker coordinate vector of a line.

Now consider the product of the two unit dual quaternions  $\hat{A} = \cos(\hat{\alpha}/2) + \sin(\hat{\alpha}/2)A$  and  $\hat{B} = \cos(\hat{\beta}/2) + \sin(\hat{\beta}/2)B$ , that is,

$$\hat{C} = \cos \frac{\hat{\gamma}}{2} + \sin \frac{\hat{\gamma}}{2} C = \left( \cos \frac{\hat{\beta}}{2} + \sin \frac{\hat{\beta}}{2} B \right) \left( \cos \frac{\hat{\alpha}}{2} + \sin \frac{\hat{\alpha}}{2} A \right). \quad (12.105)$$

Expanding this product, we obtain

$$\begin{aligned}\cos \hat{\gamma} + \sin \hat{\gamma} C &= \left( \cos \frac{\hat{\beta}}{2} \cos \frac{\hat{\alpha}}{2} - \sin \frac{\hat{\beta}}{2} \sin \frac{\hat{\alpha}}{2} B \cdot A \right) \\ &\quad + \left( \sin \frac{\hat{\beta}}{2} \cos \frac{\hat{\alpha}}{2} B + \sin \frac{\hat{\alpha}}{2} \cos \frac{\hat{\beta}}{2} A + \sin \frac{\hat{\beta}}{2} \sin \frac{\hat{\alpha}}{2} B \times A \right).\end{aligned}\quad (12.106)$$

Compare this to (12.100) to see that dual quaternion multiplication computes the Plücker coordinates of one vertex of a spatial triangle and its associated exterior dual angle from the Plücker coordinates of the other two vertices and their interior dual angles. Thus, the algebra of dual quaternions provides a useful tool for exploiting the geometry of the spatial triangle.

### 12.4.5 The Triangle of Relative Screw Axes

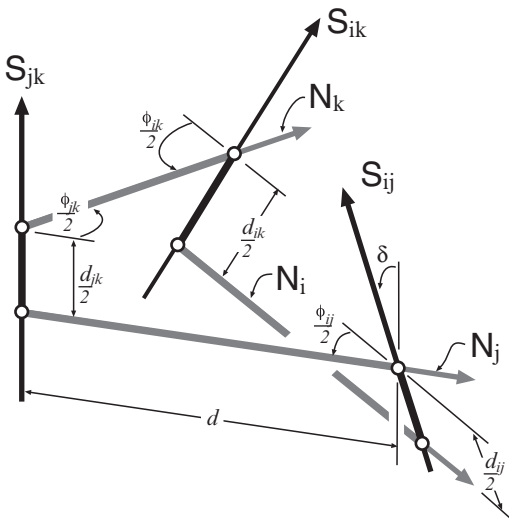
Given three positions  $M_i$ ,  $M_j$ , and  $M_k$  for a moving body, we have the three relative transformations  $[\hat{T}_{ij}] = [\hat{T}_j][\hat{T}_i^{-1}]$ ,  $[\hat{T}_{jk}] = [\hat{T}_k][\hat{T}_j^{-1}]$ , and  $[\hat{T}_{ik}] = [\hat{T}_k][\hat{T}_i^{-1}]$ . From the fact that

$$[\hat{T}_{ik}] = [\hat{T}_{jk}][\hat{T}_{ij}], \quad (12.107)$$

we can use Rodrigues's formula for screws to obtain

$$\tan \frac{\hat{\phi}_{ik}}{2} S_{ik} = \frac{\tan \frac{\hat{\phi}_{jk}}{2} S_{jk} + \tan \frac{\hat{\phi}_{ij}}{2} S_{ij} + \tan \frac{\hat{\phi}_{jk}}{2} \tan \frac{\hat{\phi}_{ij}}{2} S_{jk} \times S_{ij}}{1 - \tan \frac{\hat{\phi}_{jk}}{2} \tan \frac{\hat{\phi}_{ij}}{2} S_{jk} \cdot S_{ij}}. \quad (12.108)$$

The spatial triangle formed by the three relative screw axes  $S_{ij}$ ,  $S_{ik}$ , and  $S_{jk}$  is analogous to the pole triangle for three planar displacements and is called the *screw triangle*, [Figure 12.7](#).



**Fig. 12.7** The screw triangle formed by the screw axes  $S_{ij}$ ,  $S_{jk}$ , and  $S_{ik}$ .

## 12.5 Summary

This chapter has developed the basic geometric properties of spatial rigid displacements. The central role played by the screw axis of a displacement lead to the introduction of line geometry, screws, and dual vector algebra. We obtain the screw form of Rodrigues's formula and see that it is the equation of a spatial triangle formed by three lines and their common normals. Its properties generalize results for the planar and spherical pole triangles.

## 12.6 References

The kinematics of spatial displacements is developed in detail in Roth [104, 105] and Bottema and Roth [5]. Dimentberg [25] introduced the algebra of dual vectors and Yang [155] applied it to the analysis of spatial linkages. Also see Woo and Freudenstein [152]. Yang and Freudenstein [154] formulated dual quaternion algebra for use in spatial kinematic theory. Pennock and Yang [93] use dual-number matrices to solve the inverse kinematics of robots. Fischer [34] presents the kinematic, static, and dynamic analysis of spatial linkages using dual vectors and matrices.

## Exercises

1. A spatial displacement has as its axis the line through the origin in the direction  $\mathbf{S} = (0, \cos(45^\circ), \sin(45^\circ))^T$ . Let the rotation and slide around and along this axis be  $(45^\circ, \sqrt{2})$ . Determine the  $4 \times 4$  homogeneous transform.
2. Determine the spatial displacement  $[D] = [T_1][T_2][T_2]$  defined by a sequence of transformations: (i)  $[T_1]$ , a translation by  $(5, 4, 1)^T$ ; (ii)  $[T_2]$ , a rotation by  $30^\circ$  about the  $x$ -axis; and (iii)  $[T_3]$ , a rotation by  $60^\circ$  about the unit vector through the point  $(2, 0, 2)^T$  (Crane and Duffy [16]).
3. Determine the  $4 \times 4$  homogeneous transform  $[T_{12}]$  from the initial and final positions of four points by constructing the matrix equation using homogeneous coordinates,  $[\mathbf{A}^2, \mathbf{B}^2, \mathbf{C}^2, \mathbf{D}^2] = [T_{12}][\mathbf{A}^1, \mathbf{B}^1, \mathbf{C}^1, \mathbf{D}^1]$ . Solve this equation for  $[T_{12}]$  using the coordinates in Table 12.1 (Sandor and Erdman [112]).

**Table 12.1** Point coordinates defining two spatial positions

Point	$M_1$	$M_2$
$\mathbf{A}^i$	$(0, 3, 7)^T$	$(1.90, 11.23, 7.19)^T$
$\mathbf{B}^i$	$(2, 7, 10)^T$	$(3.29, 14.44, 11.29)^T$
$\mathbf{C}^i$	$(0, 5, 10)^T$	$(4.26, 13.41, 8.84)^T$
$\mathbf{D}^i$	$(-2, 5, 7)^T$	$(3.92, 9.83, 8.59)^T$

4. Use the coordinates in Table 12.1 to determine the screw axis  $S$  and rotation and slide around and along this axis for the displacement  $[T_{12}]$ .
5. Given the screw axis  $S$  and rotation and slide  $\phi$  and  $d$  let  $N_1$  and  $N_2$  be two lines that intersect  $S$  at right angles that are separated by the dual angle  $\hat{\phi}/2 = (\phi/2, d/2)$ . Show that this displacement is equivalent to the sequence of reflections through  $N_1$  and then  $N_2$  (Bottema and Roth [5]).
6. Show that Rodrigues's screw  $B$  can be computed from the initial and final positions of two lines  $p, P$  and  $q, Q$  to obtain  $B = (P - p) \times (Q - q) / (P - p) \cdot (Q + q)$ .
7. Show that three positions  $\mathbf{P}^i, i = 1, 2, 3$ , of a point can be obtained from the reflection of a cardinal point  $\mathbf{P}^*$  through the three sides of the screw triangle defined by three specified spatial positions.
8. Obtain three positions  $L^i, i = 1, 2, 3$ , of a line by the reflection of a cardinal line  $L^*$  through the three sides of the screw triangle.