Chapter 8
Spherical Kinematics

In this chapter we consider spatial displacements that are pure rotations in three-dimensional space. These are transformations that have the property that one point of the moving body M has the same coordinates in F before and after the displacement. Because the distance between this fixed point and points in M are constant, each point in the moving body moves on a sphere about this point. If the origins for both the fixed and moving frames are located at this fixed point, then the spatial displacement is defined by a 3×3 rotation matrix. The study of spherical kinematics benefits from both the properties of linear transformations and the geometry of a sphere.

8.1 Isometry

A *spatial displacement* preserves the distance between every pair of points in the moving body and is an isometry of three-dimensional space. As in the plane, this displacement is the composition of a translation and a rotation.

Let $\mathbf{P} = (P_x, P_y, P_z)^T$ and $\mathbf{Q} = (Q_x, Q_y, Q_z)^T$ be the coordinate vectors of two points in three-dimensional space. The distance between these points is the magnitude of their relative position vector $\mathbf{Q}-\mathbf{P}$,

$$|\mathbf{Q}-\mathbf{P}| = \sqrt{(Q_x-P_x)^2+(Q_y-P_y)^2+(Q_z-P_z)^2}, \tag{8.1}$$

which is also called the *Euclidean metric*. Using vector notation, this formula takes the same form as that used for planar kinematics, that is,

$$|\mathbf{Q}-\mathbf{P}|^2 = (\mathbf{Q}-\mathbf{P})\cdot(\mathbf{Q}-\mathbf{P}) = (\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{P}). \tag{8.2}$$

The second equality is the matrix form of the vector scalar product.

J.M. McCarthy and G.S. Soh, *Geometric Design of Linkages*, Interdisciplinary Applied Mathematics 11, DOI 10.1007/978-1-4419-7892-9_8,

© Springer Science+Business Media, LLC 2011

8.1.1 Spatial Translations

As we saw in the plane, the addition of a vector $\mathbf{d} = (d_x, d_y, d_z)^T$ to the coordinates of all the points in a body, such that $\mathbf{X} = \mathbf{x} + \mathbf{d}$, is called a *translation*. Let the points $\mathbf{p}$ and $\mathbf{q}$ be translated so $\mathbf{P} = \mathbf{p} + \mathbf{d}$ and $\mathbf{Q} = \mathbf{q} + \mathbf{d}$, then we can compute

$$|\mathbf{Q} - \mathbf{P}| = |(\mathbf{q} + \mathbf{d}) - (\mathbf{p} + \mathbf{d})| = |\mathbf{q} - \mathbf{p}|. \tag{8.3}$$

Thus, translations preserve the distance between points.

8.1.2 Spatial Rotations

A spatial rotation has the same basic properties as a planar rotation, though now applied to three-dimensional vectors. A rotation takes M from a position initially aligned with F and reorients it, while keeping the origins of the two frames located at the same point $\mathbf{c}$. Let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the unit vectors along the coordinate axes of F. The rotation changes the direction of each of these vectors. Let these new directions be given by the orthogonal unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, such that $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$. This last condition ensures that $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ form a right-handed frame like $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. A point with coordinates $\mathbf{x} = (x, y, z)^T = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ before the rotation will have coordinates $\mathbf{X}$ after the rotation, given by

$$\mathbf{X} = X\vec{\imath} + Y\vec{\jmath} + Z\vec{k} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z. \tag{8.4}$$

Let the components of $\mathbf{e}_x$ be $(e_{x,1}, e_{x,2}, e_{x,3})^T$. A similar definition for $\mathbf{e}_y$ and $\mathbf{e}_z$ allows us to form the matrix equation

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} e_{x,1} & e_{y,1} & e_{z,1} \\ e_{x,2} & e_{y,2} & e_{z,2} \\ e_{x,3} & e_{y,3} & e_{z,3} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}, \tag{8.5}$$

or

$$\mathbf{X} = \left[\mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z\right] \mathbf{x} = [A]\mathbf{x}. \tag{8.6}$$

All spatial rotations are represented by matrices constructed in this way, which are known as *rotation matrices*.

8.1.2.1 Distances

We now show that rotations preserve distances between points. Let $\mathbf{p}$ and $\mathbf{q}$ be the coordinates of two points before the rotation, and let $\mathbf{P} = [A]\mathbf{p}$ and $\mathbf{Q} = [A]\mathbf{q}$ be their coordinates after the rotation. We compute

$$|\mathbf{Q}-\mathbf{P}|^2 = (\mathbf{p}-\mathbf{q})^T[A^T][A](\mathbf{p}-\mathbf{q})^T. \tag{8.7}$$

Notice that this equality is satisfied only if $[A^T][A] = [I]$.

This condition is always true for matrices constructed from orthogonal unit vectors as in (8.6), which can be seen from the computation

$$[A^T][A] = \begin{bmatrix} \mathbf{e}_x^T \\ \mathbf{e}_y^T \\ \mathbf{e}_z^T \end{bmatrix} \begin{bmatrix} \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \mathbf{e}_x^T\mathbf{e}_x & \mathbf{e}_x^T\mathbf{e}_y & \mathbf{e}_x^T\mathbf{e}_z \\ \mathbf{e}_y^T\mathbf{e}_x & \mathbf{e}_y^T\mathbf{e}_y & \mathbf{e}_y^T\mathbf{e}_z \\ \mathbf{e}_z^T\mathbf{e}_x & \mathbf{e}_z^T\mathbf{e}_y & \mathbf{e}_z^T\mathbf{e}_z \end{bmatrix} = [I]. \tag{8.8}$$

Important examples are the coordinate rotations $[X(\cdot)]$, $[Y(\cdot)]$, and $[Z(\cdot)]$ presented in (7.3) and (7.1) in the previous chapter.

In general, a spatial rotation is a linear transformation that preserves the distances between points and the orientation of the reference frames. Matrices $[A]$ that satisfy the condition (8.8) are termed *orthogonal*. However, in order to preserve the orientation of the coordinate frame we must add the requirement that the determinant $|A|$ be positive. From the calculation

$$\det([A^T][A]) = |A|^2 = 1 \tag{8.9}$$

we see that an orthogonal matrix can have a determinant of either $+1$ or -1. Those with $|A| = +1$ are *rotations*. Those with $|A| = -1$ are *reflections*. An example of a reflection is the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{8.10}$$

The columns of this matrix are orthogonal unit vectors, and its transpose is its inverse. However, it changes the orientation of the frame by reversing the direction of the z-axis relative to the xy-plane.

8.1.2.2 Angles

A spatial rotation preserves the relative distances between three points $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. Therefore, it preserves the angle $\kappa = \angle\mathbf{QRP}$. In order to show this, it is useful to recall that the sine and cosine of the angle about $\mathbf{R}$ from $\mathbf{P}$ to $\mathbf{Q}$ can be computed from the relative vectors $\mathbf{P}-\mathbf{R}$ and $\mathbf{Q}-\mathbf{R}$ by the formulas

$$\sin\kappa = \frac{(\mathbf{P}-\mathbf{R})\times(\mathbf{Q}-\mathbf{R})\cdot\mathbf{N}}{|\mathbf{P}-\mathbf{R}||\mathbf{Q}-\mathbf{R}|}, \quad \cos\kappa = \frac{(\mathbf{P}-\mathbf{R})\cdot(\mathbf{Q}-\mathbf{R})}{|\mathbf{P}-\mathbf{R}||\mathbf{Q}-\mathbf{R}|}, \tag{8.11}$$

where $\mathbf{N}$ is the unit vector in the direction of $(\mathbf{P}-\mathbf{R})\times(\mathbf{Q}-\mathbf{R})$.

Now let the triangle $\triangle\mathbf{QRP}$ be the result of a rotation by $[A]$, so we can make the substitutions $\mathbf{P}-\mathbf{R} = [A](\mathbf{p}-\mathbf{r})$, $\mathbf{Q}-\mathbf{R} = [A](\mathbf{q}-\mathbf{r})$, and $\mathbf{N} = [A]\mathbf{n}$ in (8.11). Note that for rotation matrices only, we have the identity

$$([A]\mathbf{x}) \times ([A]\mathbf{y}) = [A](\mathbf{x} \times \mathbf{y}). \tag{8.12}$$

This allows us to factor $[A]$ from the vector product in $\sin\kappa$ and obtain

$$(\mathbf{P}-\mathbf{R}) \times (\mathbf{Q}-\mathbf{R}) \cdot \mathbf{N} = (\mathbf{p}-\mathbf{r}) \times (\mathbf{q}-\mathbf{r}) \cdot \mathbf{n}, \tag{8.13}$$

where $[A]$ cancels in the scalar product because $[A^T][A] = [I]$. For the same reason $[A]$ cancels in the expression for $\cos\kappa$. The conclusion is that these formulas apply without change to coordinates in M and in F, and therefore the angle κ is the same before and after the rotation.

If the point $\mathbf{R} = (0,0,0)^T = \mathbf{c}$ is the origin of F, then (8.11) simplifies to define the angle between the vectors $\mathbf{P}$ and $\mathbf{Q}$ as

$$\sin\kappa = \frac{\mathbf{P} \times \mathbf{Q} \cdot \mathbf{N}}{|\mathbf{P}||\mathbf{Q}|}, \quad \cos\kappa = \frac{\mathbf{P} \cdot \mathbf{Q}}{|\mathbf{P}||\mathbf{Q}|}. \tag{8.14}$$

And we have that $\angle\mathbf{PcQ}=\angle\mathbf{pcq}$ for any two points $\mathbf{p}$ and $\mathbf{q}$ in the moving body.

8.1.3 Spatial Displacements

A spatial displacement consists of a spatial rotation $[A]$ of the moving frame M from its initial position to a new orientation M' followed by a translation $\mathbf{d}$ to M''. The initial position of M is aligned with F, and its final position is aligned with M''. As we did in the plane, we let F and M be the initial and final positions, and define the *spatial displacement* of M relative to F by the transformation $[T] = [A, \mathbf{d}]$.

Clearly, if $\mathbf{d} = 0$, then the displacement is a spatial rotation about the origin of F. Therefore, the *orientation* of M relative to F is defined by the rotation matrix $[A]$. Below we consider spatial displacements that are equivalent to pure rotations about other points in F. The properties of these rotational displacements are the same as for rotations about the origin of F, which are our focus of study in what follows.

8.1.4 Composition of Rotations

Consider two rotation matrices $[A_1]$ and $[A_2]$. Their product $[A_1][A_2]$ is an orthogonal matrix, as can be determined from the computation

$$([A_1][A_2])^T([A_1][A_2]) = [A_2^T][A_1^T][A_1][A_2] = [I]. \tag{8.15}$$

This is a rotation matrix because its determinant is the product of the determinants of $[A_1]$ and $[A_2]$.

The orientation of M defined by this product results from the orientation of a frame M' relative to F defined by the equation $\mathbf{X} = [A_1]\mathbf{y}$, combined with the orientation of M relative to M' given by $\mathbf{y} = [A_2]\mathbf{x}$. Compose these two rotations by direct substitution for $\mathbf{y}$, and the result is the orientation of M relative to F, given by

$$\mathbf{X} = [A_3]\mathbf{x} = [A_1][A_2]\mathbf{x}. \tag{8.16}$$

The composite rotation is obtained from the matrix product.

The *inverse* $[A^{-1}]$ of a rotation $[A]$ is the rotation defined such that the composition $[A^{-1}][A]$ is the identity. Therefore, $[A^T] = [A^{-1}]$ is the inverse rotation.

8.1.4.1 Changing Coordinates of a Rotation Matrix

Consider the rotation $\mathbf{X} = [A]\mathbf{x}$ of M relative to F. We now determine the rotation matrix $[A']$ between a pair of fixed and moving frames F' and M' that are rotated by the same matrix $[R]$ relative to the original frames. Coordinates $\mathbf{Y}$ and $\mathbf{y}$ in the new frames are related to the original coordinates by $\mathbf{Y} = [R]\mathbf{X}$ and $\mathbf{y} = [R]\mathbf{x}$. Therefore, we have

$$\mathbf{X} = [A]\mathbf{x} = [R^T]\mathbf{Y} = [A][R^T]\mathbf{y},$$

or

$$\mathbf{Y} = [R][A][R^T]\mathbf{y}. \tag{8.17}$$

Thus, the original matrix $[A]$ is transformed by the change of coordinates into $[A'] = [R][A][R^T]$.

8.1.5 Relative Rotations

Consider two orientations M_1 and M_2 of a body relative to F defined by the rotation matrices $[A_1]$ and $[A_2]$. Let $\mathbf{X}$ be the coordinates in F of a point $\mathbf{x}$ in M when in the orientation M_1. Similarly, let $\mathbf{Y}$ be the coordinates of the same point when M is in orientation M_2. Then we have $\mathbf{X} = [A_1]\mathbf{x}$ and $\mathbf{Y} = [A_2]\mathbf{x}$, respectively. The *relative rotation* matrix $[A_{12}]$ that transforms the coordinates $\mathbf{X}$ into $\mathbf{Y}$ is defined by

$$\mathbf{Y} = [A_{12}]\mathbf{X}. \tag{8.18}$$

Substitute for $\mathbf{X}$ and $\mathbf{Y}$ in order to obtain

$$[A_2]\mathbf{x} = [A_{12}][A_1]\mathbf{x}. \tag{8.19}$$

Equating the matrices on both sides of this equation, we see that $[A_{12}]$ is given by

$$[A_{12}] = [A_2][A_1^T]. \tag{8.20}$$

This defines the orientation of M_2 relative to M_1 measured in the frame F.

Relative rotations are easy to compute for coordinate rotations $[X(\cdot)]$, $[Y(\cdot)]$, and $[Z(\cdot)]$. Consider, for example, two orientations of M defined by the z-rotations $[Z(\theta_1)]$ and $[Z(\theta_2)]$. The relative rotation is given by

$$[Z(\theta_2)][Z(\theta_1)^T] = [Z(\theta_2)][Z(-\theta_1)] = [Z(\theta_2 - \theta_1)] = [Z(\theta_{12})], \tag{8.21}$$

where $\theta_{12} = \theta_2 - \theta_1$. This calculation uses the fact that the inverse of a coordinate rotation is just the rotation by the negative value of the rotation angle.

In general, given a set of orientations M_i, $i = 1,\dots,n$, we have the relative rotations defined by

$$[A_{ij}] = [A_j][A_i^T]. \tag{8.22}$$

8.1.6 Relative Inverse Rotations

For two orientations M_1 and M_2 of a body, we can determine the inverse orientations F_1 and F_2 of the fixed frame F as viewed from M. These are defined by the inverse rotations $[A_1^{-1}]$ and $[A_2^{-1}]$. Let $\mathbf{X}$ be a point in F that corresponds to a point $\mathbf{x}$ when M is in orientation M_1, or equivalently, when F coincides with F_1. Let this $\mathbf{X}$ correspond to $\mathbf{y}$ in M when in orientation M_2, which is the same as when F aligns with F_2 as viewed from M. These coordinates are related by the equations $\mathbf{x} = [A_1^{-1}]\mathbf{X}$ and $\mathbf{y} = [A_2^{-1}]\mathbf{X}$. The *relative inverse rotation* $[A_{12}^{\dagger}]$ transforms the coordinates $\mathbf{x}$ into $\mathbf{y}$ by

$$\mathbf{y} = [A_{12}^{\dagger}]\mathbf{x},$$

or

$$[A_2^{-1}]\mathbf{X} = [A_{12}^{\dagger}][A_1^{-1}]\mathbf{X}. \tag{8.23}$$

Thus, we have

$$[A_{12}^{\dagger}] = [A_2^T][A_1]. \tag{8.24}$$

This rotation defines the rotation of F from F_1 into F_2 as viewed from the moving frame M. Notice that this is not the inverse of the relative rotation matrix $[A_{12}]$, which is $[A_{12}^{-1}] = [A_1][A_2^T]$.

For a general set of orientations M_i, $i = 1,\dots,n$, we have the relative inverse rotations

$$[A_{ij}^{\dagger}] = [A_j^T][A_i]. \tag{8.25}$$

The relative inverse rotation is defined from the point of view of the moving frame M. We can choose a specific orientation M_j and transform its coordinates to the fixed frame. In particular, transform the relative inverse rotation $[A_{ik}^{\dagger}]$ to M_j in F by the rotation $[A_j]$, to obtain

$$[A^j_{ik}] = [A_j][A^\dagger_{ik}][A^T_j]. \tag{8.26}$$

This is known as the image of the relative inverse rotation for position M_j in F.

Notice that if M_j is one of the orientations of the relative inverse rotation, say $j = i$, then

$$[A^i_{ik}] = [A_i][A^\dagger_{ik}][A^T_i] = [A_i][A^T_k][A_i][A^T_i] = [A_i][A^T_k] = [A^T_{ik}]. \tag{8.27}$$

This result is also obtained for $j = k$. Thus, in these cases the image of the relative inverse rotation is the inverse of the relative rotation.

8.2 The Geometry of Rotation Axes

Every rotation has an axis, which is the set of points that are invariant under the transformation. The geometric properties of these axes are fundamental tools in the synthesis of spherical RR chains.

8.2.1 The Rotation Axis

The points that remain fixed during a rotation $[A]$ form its *rotation axis*. To find these points we consider the transformation equation

$$\mathbf{X} = [A]\mathbf{X}. \tag{8.28}$$

This shows that a fixed point $\mathbf{X}$ is the solution to

$$[I - A]\mathbf{X} = 0. \tag{8.29}$$

This equation has the solution $\mathbf{X} = 0$, which tells us that origin is a fixed point, as expected. For there to be other fixed points, the determinant of the coefficient matrix must be zero, that is, $|I - A| = 0$.

It happens that this condition is satisfied for all spatial rotation matrices. Another way of saying this is that these matrices always have $\lambda = 1$ as an eigenvalue. Notice that if $\mathbf{S}$ is a nonzero solution, then every point $\mathbf{P} = t\mathbf{S}$ on the line through the origin and $\mathbf{S}$ is also a solution. This line of points is the rotation axis.

8.2.1.1 Cayley's Formula

In order to obtain an explicit equation for the rotation axis we first derive *Cayley's formula* for a spatial rotation matrix. Consider the points $\mathbf{x}$ and $\mathbf{X}$ in F that represent the initial and final positions obtained from the rotation $\mathbf{X} = [A]\mathbf{x}$. Using the fact that

$|\mathbf{x}| = |\mathbf{X}|$, we compute

$$\mathbf{X}\cdot\mathbf{X} - \mathbf{x}\cdot\mathbf{x} = (\mathbf{X}-\mathbf{x})\cdot(\mathbf{X}+\mathbf{x}) = 0. \tag{8.30}$$

This equation states that the diagonals $\mathbf{X}-\mathbf{x}$ and $\mathbf{X}+\mathbf{x}$ of the rhombus formed by the vertices $\mathbf{O}$, $\mathbf{x}$, $\mathbf{X}$, and $\mathbf{x}+\mathbf{X}$ must be perpendicular.

The diagonals $\mathbf{X}-\mathbf{x}$ and $\mathbf{X}+\mathbf{x}$ are also given by the equations

$$\mathbf{X}-\mathbf{x} = [A-I]\mathbf{x} \quad \text{and} \quad \mathbf{X}+\mathbf{x} = [A+I]\mathbf{x}. \tag{8.31}$$

Substitute for $\mathbf{x}$ on the right side of these two equations in order obtain

$$\mathbf{X}-\mathbf{x} = [A-I][A+I]^{-1}(\mathbf{X}+\mathbf{x}) = [B](\mathbf{X}+\mathbf{x}). \tag{8.32}$$

The matrix $[B]$ operates on the diagonal $\mathbf{X}+\mathbf{x}$ to rotate it 90° and change its length. The result is the other diagonal $\mathbf{X}-\mathbf{x}$. From the fact that

$$(\mathbf{X}+\mathbf{x})^T[B](\mathbf{X}+\mathbf{x}) = 0, \tag{8.33}$$

we can see that $[B]$ must have the form

$$[B] = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix}. \tag{8.34}$$

This matrix is *skew-symmetric*, which means that $[B^T] = -[B]$.

The elements of $[B]$ can be assembled into the vector $\mathbf{b} = (b_x, b_y, b_z)^T$ that has the property that for any vector $\mathbf{y}$,

$$[B]\mathbf{y} = \mathbf{b}\times\mathbf{y}, \tag{8.35}$$

where $\times$ is the usual vector product. The vector $\mathbf{b}$ is called *Rodrigues's vector* and (8.32) is often written as

$$\mathbf{X}-\mathbf{x} = \mathbf{b}\times(\mathbf{X}+\mathbf{x}). \tag{8.36}$$

The equation $[B] = [A-I][A+I]^{-1}$ can be solved to obtain Cayley's formula for rotation matrices

$$[A] = [I-B]^{-1}[I+B]. \tag{8.37}$$

This shows that the nine elements of a 3×3 rotation matrix depend on three independent parameters. Another way to say this is that the set of rotation matrices SO(3) is three-dimensional.

We now solve (8.29) explicitly to determine a nonzero point $\mathbf{S}$ on the rotation axis. Substitute Cayley's formula for $[A]$ into this equation to obtain

$$\left[I - [I-B]^{-1}[I+B]\right]\mathbf{X} = 0, \tag{8.38}$$

which simplifies to

$$[B]\mathbf{X} = 0. \tag{8.39}$$

Since $[B]\mathbf{X} = \mathbf{b} \times \mathbf{X}$, it is clear that $\mathbf{X} = \mathbf{b}$ is a solution. Thus, Rodrigues's vector defines the rotation axis.

We denote by $\mathbf{S}$ the unit vector in the direction of Rodrigues's vector $\mathbf{b}$, and use it to identify the rotation axis of $[A]$.

8.2.2 Perpendicular Bisectors and the Rotation Axis

The angle between the rotation axis $\mathbf{S}$ and vectors through the origin $\mathbf{c}$ to any point $\mathbf{x}$ in the moving body is preserved by the rotation. This means that $\angle\mathbf{xcS}$ equals $\angle\mathbf{XcS}$ and we have

$$\mathbf{S} \cdot \mathbf{X} = \mathbf{S} \cdot \mathbf{x}. \tag{8.40}$$

This equation states that the component of any point $\mathbf{x}$ in the direction of the rotation axis $\mathbf{S}$ is unchanged by the rotation $[A]$. This can be made explicit by computing

$$\mathbf{S} \cdot \mathbf{X} - \mathbf{S} \cdot \mathbf{x} = \mathbf{S} \cdot (\mathbf{X} - \mathbf{x}) = 0. \tag{8.41}$$

Thus, the vector joining the initial position of a point $\mathbf{x}$ to its final position $\mathbf{X}$ is perpendicular to the direction of the rotation axis.

In order to examine this relation (8.41) in more detail, choose the pair of corresponding points $\mathbf{p}$ and $\mathbf{P}$ and consider the set of points $\mathbf{Y}$ that satisfy the equation

$$\mathbf{Y} \cdot (\mathbf{P} - \mathbf{p}) = 0. \tag{8.42}$$

This defines a plane through the origin, because $\mathbf{Y} = 0$ satisfies this equation. Furthermore, the midpoint $\mathbf{V}$ of the segment $\mathbf{P} - \mathbf{p}$ lies on this plane because

$$\mathbf{V} \cdot (\mathbf{P} - \mathbf{p}) = \frac{\mathbf{P} + \mathbf{p}}{2} \cdot (\mathbf{P} - \mathbf{p}) = \frac{\mathbf{P} \cdot \mathbf{P} - \mathbf{p} \cdot \mathbf{p}}{2} = 0. \tag{8.43}$$

The last equality simply restates that $|\mathbf{P}| = |\mathbf{p}|$.

This shows that the plane defined by (8.42) is perpendicular to the segment $\mathbf{pP}$ and passes through its mid-point $\mathbf{V}$. So it is the *perpendicular bisector* of the vector $\mathbf{P} - \mathbf{p}$. Thus, we see that the rotation axis lies on the perpendicular bisector of all vectors $\mathbf{X} - \mathbf{x}$ for every point $\mathbf{x}$ in the moving body.

8.2.2.1 Constructing the Rotation Axis

This result provides a convenient way to determine the rotation axis. Choose two points $\mathbf{p}$ and $\mathbf{q}$ and determine their transformed positions $\mathbf{P} = [A]\mathbf{p}$ and $\mathbf{Q} = [A]\mathbf{q}$. The perpendicular bisectors of the segments $\mathbf{P} - \mathbf{p}$ and $\mathbf{Q} - \mathbf{q}$ are

$$\begin{aligned}\mathbf{Y}\cdot(\mathbf{P}-\mathbf{p}) &= 0,\\ \mathbf{Y}\cdot(\mathbf{Q}-\mathbf{q}) &= 0.\end{aligned} \tag{8.44}$$

The rotation axis is the line of intersection of these two planes, and we have the solution $\mathbf{Y}=\mathbf{S}$ given by

$$\mathbf{S}=\frac{(\mathbf{P}-\mathbf{p})\times(\mathbf{Q}-\mathbf{q})}{|(\mathbf{P}-\mathbf{p})\times(\mathbf{Q}-\mathbf{q})|}. \tag{8.45}$$

8.2.3 The Rotation Angle

The plane defined by $\triangle\mathbf{xcS}$ rotates about $\mathbf{S}$ in order to reach its final position containing $\triangle\mathbf{XcS}$, Figure 8.1. The dihedral angle ϕ between these two planes is called the *rotation angle* of $[A]$. It can be determined from the two vectors $\mathbf{S}\times\mathbf{x}$ and $\mathbf{S}\times\mathbf{X}$ that are perpendicular to these respective planes, as well as perpendicular to $\mathbf{S}$. Compute the sine and cosine of the angle between these vectors using (8.14),

$$\sin\phi=\frac{(\mathbf{S}\times\mathbf{x})\times(\mathbf{S}\times\mathbf{X})\cdot\mathbf{S}}{|\mathbf{S}\times\mathbf{x}||\mathbf{S}\times\mathbf{X}|},\quad \cos\phi=\frac{(\mathbf{S}\times\mathbf{x})\cdot(\mathbf{S}\times\mathbf{X})}{|\mathbf{S}\times\mathbf{x}||\mathbf{S}\times\mathbf{X}|}. \tag{8.46}$$

The numerator of $\sin\phi$ simplifies to $(\mathbf{S}\times\mathbf{x})\cdot\mathbf{X}$, so we have

$$\phi=\arctan\left(\frac{(\mathbf{S}\times\mathbf{x})\cdot\mathbf{X}}{(\mathbf{S}\times\mathbf{x})\cdot(\mathbf{S}\times\mathbf{X})}\right). \tag{8.47}$$

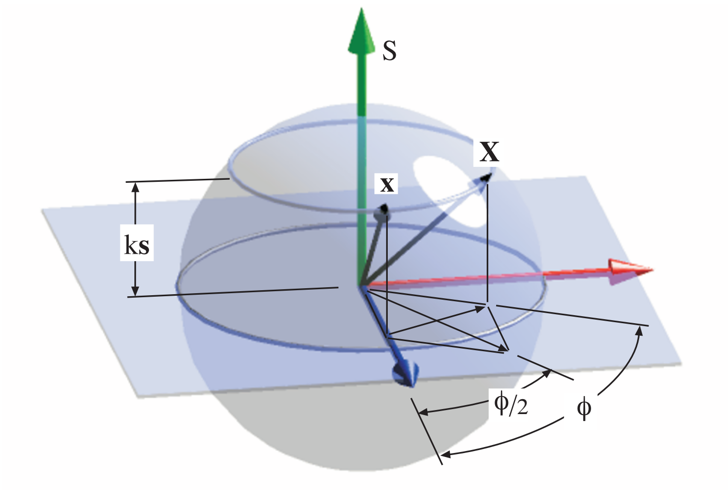

Fig. 8.1 The rotation axis and two positions of a general point.

8.2.3.1 Rodrigues's Equation

Consider the projections of the points $\mathbf{x}$ and $\mathbf{X}$ onto the plane perpendicular to the rotation axis $\mathbf{S}$ through $\mathbf{c}$, which we denote by $\mathbf{x}^*$ and $\mathbf{X}^*$, respectively. The isosceles triangle $\triangle\mathbf{x}^*\mathbf{c}\mathbf{X}^*$ has the rotation angle ϕ as its vertex angle. The altitude $\mathbf{V}^*$ of this triangle is the projection of the midpoint of the segment $\mathbf{X}-\mathbf{x}$. Therefore, we have

$$\tan\frac{\phi}{2} = \frac{|\mathbf{X}^* - \mathbf{V}^*|}{|\mathbf{V}^*|}. \tag{8.48}$$

Notice that the vectors $\mathbf{X}^* - \mathbf{V}^*$ and $\mathbf{V}^*$ are related by the equation

$$\mathbf{X}^* - \mathbf{V}^* = \tan\frac{\phi}{2}\mathbf{S}\times\mathbf{V}^*, \tag{8.49}$$

where the vector product by $\mathbf{S}$ rotates $\mathbf{V}^*$ by 90°.

We expand this relation to obtain Rodrigues's equation. First, notice that

$$\mathbf{S}\times\mathbf{V}^* = \mathbf{S}\times\frac{\mathbf{X}^*+\mathbf{x}^*}{2} = \mathbf{S}\times\frac{\mathbf{X}+\mathbf{x}}{2}. \tag{8.50}$$

The components of $\mathbf{x}$ and $\mathbf{X}$ in the direction $\mathbf{S}$ are canceled by the vector product with $\mathbf{S}$. Next, we have

$$\mathbf{X}^* - \mathbf{V}^* = \mathbf{X}^* - \frac{\mathbf{X}^*+\mathbf{x}^*}{2} = \frac{\mathbf{X}^*-\mathbf{x}^*}{2} = \frac{\mathbf{X}-\mathbf{x}}{2}, \tag{8.51}$$

because the components of $\mathbf{x}$ and $\mathbf{X}$ along $\mathbf{S}$ cancel. Combining these results, we obtain

$$\mathbf{X}-\mathbf{x} = \tan\frac{\phi}{2}\mathbf{S}\times(\mathbf{X}+\mathbf{x}). \tag{8.52}$$

This is another derivation of *Rodrigues's equation*. However, in this case, we see that the magnitude of Rodrigues's vector is $\tan(\phi/2)$, that is, $\mathbf{b} = \tan(\phi/2)\mathbf{S}$.

8.2.4 The Rotation Defined by ϕ and S

A rotation matrix $[A]$ is characterized by its rotation axis $\mathbf{S}$ and rotation angle ϕ. Cayley's formula combines with the definition of Rodrigues's vector to yield an explicit formula for $[A(\phi,\mathbf{S})]$. Because $\mathbf{b} = \tan(\phi/2)\mathbf{S}$, we have that the matrix $[B]$ is given by

$$[B] = \tan\frac{\phi}{2}\begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}, \tag{8.53}$$

where $\mathbf{S} = (s_x, s_y, s_z)^T$. Thus, Cayley's formula yields

$$[A(\phi,\mathbf{S})] = [I - \tan\frac{\phi}{2}[S]]^{-1}[I + \tan\frac{\phi}{2}[S]], \tag{8.54}$$

which can be expanded to obtain

$$[A(\phi,\mathbf{S})] = [I] + \sin\phi[S] + (1-\cos\phi)[S^2]. \tag{8.55}$$

This equation defines the rotation matrix in terms of its rotation axis and the angle of rotation about this axis.

Important examples of (8.55) are the coordinate rotations $[X(\theta)]$, $[Y(\theta)]$, and $[Z(\theta)]$, which are rotations by the angle θ about the axes $\vec{\imath} = (1,0,0)^T$, $\vec{\jmath} = (0,1,0)^T$, and $\vec{k} = (0,0,1)^T$, respectively.

8.2.4.1 Inverse Rotations

The rotation matrix $[A]$ and its inverse $[A^T]$ have the same rotation axis $\mathbf{S}$. This is easily seen by multiplying $[A]\mathbf{S} = \mathbf{S}$ by $[A^T]$ to obtain $\mathbf{S} = [A^T]\mathbf{S}$. We now compute $[A(\phi,\mathbf{S})^T]$ using (8.55). Notice that $[S^T] = -[S]$ and $[S^2]^T = [S^2]$, so we have

$$[A^T] = [I] - \sin\phi[S] + (1-\cos\phi)[S^2], \tag{8.56}$$

where ϕ is the rotation angle of $[A]$. Let ϕ' be the rotation angle of $[A^T]$. Then we see from $\sin\phi' = -\sin\phi$ and $\cos\phi' = \cos\phi$ that $\phi' = -\phi$. Thus, the inverse rotation is simply the rotation by the negative angle around the same axis.

8.2.4.2 A Change of Coordinates

Equation (8.55) provides a convenient way to understand the change of coordinates of a rotation matrix. Consider the transformation $[A'] = [R][A(\phi,\mathbf{S})][R^T]$, where $[A]$ is defined in terms of its rotation angle and axis. Then we have

$$\begin{aligned}[A'] =& [R]\big([I] + \sin\phi[S] + (1-\cos\phi)[S^2]\big)[R^T]\\ =& [I] + \sin\phi([R][S][R^T]) + (1-\cos\phi)([R][S^2][R^T]).\end{aligned} \tag{8.57}$$

It is easy to show that $[S'] = [R][S][R^T]$ is the skew symmetric matrix associated with the vector $\mathbf{S}' = [R]\mathbf{S}$, and we have

$$[A'] = [I] + \sin\phi[S'] + (1-\cos\phi)[S'^2]. \tag{8.58}$$

Thus, a change of coordinates $[R][A(\phi,\mathbf{S})][R^T]$ leaves the rotation angle unchanged, and transforms the rotation axis by $[R]$, so $[A'] = [A(\phi,[R]\mathbf{S})]$.

8.2.5 Eigenvalues of a Rotation Matrix

In this section we consider the matrix equation (8.29) in more detail. The matrix $[I-A]$ can be considered to be $[\lambda I-A]$, where $\lambda = 1$. This leads us to consider the eigenvalue equation

$$[A]\mathbf{X} = \lambda\mathbf{X}, \tag{8.59}$$

which has a solution only if the determinant $|\lambda I-A|$ is zero. This yields the characteristic polynomial

$$\lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + (M_{11}+M_{22}+M_{33})\lambda - 1 = 0, \tag{8.60}$$

where M_{ij} is the minor of the submatrix of $[A]$ obtained by removing row i and column j.

Rotation matrices have the property that each element is equal to its associated minor, that is,

$$M_{ij} = a_{ij}. \tag{8.61}$$

This follows directly from the fact that the inverse of a rotation matrix is its transpose.

We use this to simplify (8.60) to obtain

$$\lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + (a_{11}+a_{22}+a_{33})\lambda - 1 = 0. \tag{8.62}$$

It is now easy to check that $\lambda = 1$ is a root of this polynomial for all rotation matrices. This means that $|I-A| = 0$, so (8.29) always has solutions other than $\mathbf{X} = 0$.

To obtain the other two roots of (8.62), divide by $(\lambda - 1)$ to obtain

$$\lambda^2 - (a_{11}+a_{22}+a_{33}-1)\lambda + 1 = 0. \tag{8.63}$$

The roots of this equation are $\lambda = e^{i\phi}$ and $\lambda = e^{-i\phi}$, where the angle ϕ is given by

$$\phi = \arccos\left(\frac{a_{11}+a_{22}+a_{33}-1}{2}\right). \tag{8.64}$$

8.2.6 Rotation Axis of a Relative Rotation

For two orientations defined by the rotations $[A_1]$ and $[A_2]$ we have the relative rotation matrix $[A_{12}] = [A_2][A_1^T]$. This matrix transforms the coordinates $\mathbf{X}^1$ of point $\mathbf{x}$ in M in orientation M_1 into the coordinates $\mathbf{X}^2$ when the body is in M_2, that is,

$$\mathbf{X}^2 = [A_{12}]\mathbf{X}^1. \tag{8.65}$$

The axis of rotation $\mathbf{S}_{12}$ satisfies the condition (8.40), which in this case becomes

$$\mathbf{S}_{12} \cdot \mathbf{X}^2 = \mathbf{S}_{12} \cdot \mathbf{X}^1, \tag{8.66}$$

or

$$\mathbf{S}_{12} \cdot (\mathbf{X}^2 - \mathbf{X}^1) = 0. \tag{8.67}$$

Thus, the relative rotation axis $\mathbf{S}_{12}$ lies on the perpendicular bisector of all segments joining corresponding points in orientations M_1 and M_2.

Let $\mathbf{P}^1$ and $\mathbf{P}^2$ and $\mathbf{Q}^1$ and $\mathbf{Q}^2$ be a pair of corresponding points in orientations M_1 and M_2. Then $\mathbf{S}_{12}$ is the solution to the pair of equations

$$\begin{aligned} \mathbf{S}_{12} \cdot (\mathbf{P}^2 - \mathbf{P}^1) &= 0, \\ \mathbf{S}_{12} \cdot (\mathbf{Q}^2 - \mathbf{Q}^1) &= 0, \end{aligned} \tag{8.68}$$

given by

$$\mathbf{S}_{12} = \frac{(\mathbf{P}^2 - \mathbf{P}^1) \times (\mathbf{Q}^2 - \mathbf{Q}^1)}{|(\mathbf{P}^2 - \mathbf{P}^1) \times (\mathbf{Q}^2 - \mathbf{Q}^1)|}. \tag{8.69}$$

The relative rotation angle ϕ_{12} about $\mathbf{S}_{12}$ is the dihedral angle between the planes containing $\triangle \mathbf{X}^1\mathbf{O}\mathbf{S}_{12}$ and $\triangle \mathbf{X}^1\mathbf{O}\mathbf{S}_{12}$, which by (8.47) is

$$\phi_{12} = \arctan\left(\frac{(\mathbf{S}_{12} \times \mathbf{X}^1) \cdot \mathbf{X}^2}{(\mathbf{S} \times \mathbf{X}^1) \cdot (\mathbf{S} \times \mathbf{X}^2)}\right). \tag{8.70}$$

Finally, we see from (8.55) that the relative rotation $[A_{12}]$ is defined in terms of its rotation angle ϕ_{12} and axis $\mathbf{S}_{12}$ by the formula

$$[A(\phi_{12}, \mathbf{S}_{12})] = [I] + \sin\phi_{12}[S_{12}] + (1 - \cos\phi_{12})[S_{12}^2]. \tag{8.71}$$

8.2.7 Rotation Axis of a Relative Inverse Rotation

Given the two orientations M_1 and M_2, we can compute the inverse rotations $[A_1^T]$ and $[A_2^T]$ that define the orientations F_1 and F_2 of the fixed frame relative to M. The *relative inverse rotation* is given by $[A_{12}^\dagger] = [A_2^T][A_1]$. The rotation axis $\mathbf{s}_{12}$ of $[A_{12}^\dagger]$ is computed using the formulas above, but it now lies in M. Let $\phi_{12}^\dagger$ be the relative rotation angle.

In general, we can transform the relative inverse rotation $[A_{ik}^\dagger]$ to the fixed frame F when M is in orientation M_j by the computation

$$[A_{ik}^j] = [A_j][A_{ik}^\dagger][A_j^T]. \tag{8.72}$$

This transforms the coordinates of $\mathbf{s}_{ik}$ to

$$\mathbf{S}_{ik}^j = [A_j]\mathbf{s}_{ik}, \tag{8.73}$$

which is the *image* of the relative inverse rotation axis.

For the two cases $j = 1$ and $j = 2$ we have

$$[A^1_{12}] = [A^2_{12}] = [A^T_{12}]. \tag{8.74}$$

Thus, the relative inverse rotation angle is $\phi^{\dagger}_{12} = -\phi_{12}$. The rotation axis $\mathbf{s}_{12}$ is transformed to the fixed frame such that

$$\mathbf{S}_{12} = [A_1]\mathbf{s}_{12} = [A_2]\mathbf{s}_{12}, \tag{8.75}$$

which is what we expect for a relative rotation axis.

8.2.8 Rotational Displacements

We now consider spatial displacements that are rotations but around points other than the origin of F. In particular, we determine the condition under which the displacement $[T] = [A, \mathbf{d}]$ has a nonzero fixed point $\mathbf{c}$. We seek the points $\mathbf{X}$ such that

$$\mathbf{X} = [A]\mathbf{X} + \mathbf{d}, \tag{8.76}$$

or

$$[I - A]\mathbf{X} = \mathbf{d}. \tag{8.77}$$

We have already seen that $|I - A| = 0$. Therefore, this equation does not have a solution, in general, and there are no fixed points. However, we can determine a condition that the translation vector $\mathbf{d}$ must satisfy in order for a solution to exist.

Substitute Cayley's formula for $[A]$, as was done in (8.39). The result can be simplified to the form

$$\mathbf{b} \times \mathbf{X} = \frac{1}{2}(\mathbf{b} \times \mathbf{d} - \mathbf{d}). \tag{8.78}$$

The left side of this equation is orthogonal to the Rodrigues's vector $\mathbf{b}$. This means that the right side must be orthogonal to $\mathbf{b}$ as well. Thus, we obtain the condition

$$\mathbf{b} \cdot (\mathbf{b} \times \mathbf{d} - \mathbf{d}) = 0, \quad \text{or} \quad \mathbf{b} \cdot \mathbf{d} = 0. \tag{8.79}$$

This is clearly satisfied when $\mathbf{d} = 0$. However, we now see that a spatial displacement has a fixed point when the translation vector is orthogonal to the rotation axis.

In this case, we can solve (8.78) by computing

$$\mathbf{b} \times (\mathbf{b} \times \mathbf{X}) = \mathbf{b} \times \frac{1}{2}(\mathbf{b} \times \mathbf{d} - \mathbf{d}), \tag{8.80}$$

which simplifies to yield

$$\mathbf{X} = \frac{\mathbf{b} \times (\mathbf{d} - \mathbf{b} \times \mathbf{d})}{2\mathbf{b} \cdot \mathbf{b}}. \tag{8.81}$$

Not only is the point $\mathbf{X} = \mathbf{c}$ fixed under this displacement, but every point on the line $\mathsf{L} : \mathbf{Y} = \mathbf{c} + t\mathbf{S}$ is fixed as well. Therefore, the displacement is a pure rotation about the axis L, and we call it a *rotational displacement*.

8.3 The Spherical Pole Triangle

8.3.1 The Axis of Composite Rotation

There is an important geometric relationship between the axes of two rotations $[A]$ and $[B]$ and the axis of their product $[C] = [B][A]$. This is easily derived by using Rodrigues's equation (8.52) to represent each of the rotations. Let $[C]$ have the rotation axis $\mathbf{C}$ and rotation angle γ, so we have Rodrigues's vector $\tan(\gamma/2)\mathbf{C}$. Similarly, let Rodrigues's vectors for $[A]$ and $[B]$ be $\tan(\alpha/2)\mathbf{A}$ and $\tan(\beta/2)\mathbf{B}$, respectively.

We now consider the composite rotation $[B][A]$ as the transformation $\mathbf{y} = [A]\mathbf{x}$, followed by the transformation $\mathbf{X} = [B]\mathbf{y}$. Thus, for $[A]$ we have

$$\mathbf{y} - \mathbf{x} = \tan\frac{\alpha}{2}\mathbf{A} \times (\mathbf{y} + \mathbf{x}). \tag{8.82}$$

And for the rotation $[B]$ we have

$$\mathbf{X} - \mathbf{y} = \tan\frac{\beta}{2}\mathbf{B} \times (\mathbf{X} + \mathbf{y}). \tag{8.83}$$

The vector $\mathbf{y}$ can be eliminated between these two equations to yield

$$\mathbf{X} - \mathbf{x} = \tan\frac{\gamma}{2}\mathbf{C} \times (\mathbf{X} + \mathbf{x}),$$

where

$$\tan\frac{\gamma}{2}\mathbf{C} = \frac{\tan\frac{\beta}{2}\mathbf{B} + \tan\frac{\alpha}{2}\mathbf{A} + \tan\frac{\beta}{2}\tan\frac{\alpha}{2}\mathbf{B} \times \mathbf{A}}{1 - \tan\frac{\beta}{2}\tan\frac{\alpha}{2}\mathbf{B} \cdot \mathbf{A}}. \tag{8.84}$$

This result is known as *Rodrigues's formula* for the composition of rotations.

8.3.1.1 A Spherical Triangle

We now show that (8.84) is the equation of the spherical triangle formed by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, with interior angles $\alpha/2$ and $\beta/2$ at $\mathbf{A}$ and $\mathbf{B}$, and the exterior angle $\gamma/2$ at $\mathbf{C}$, Figure 8.2.

Introduce the planes E_{A} and E_{B} through the center of the sphere, which define the sides $\mathbf{AC}$ and $\mathbf{BC}$ of the spherical triangle. These planes intersect along the vector $\mathbf{C}$ and lie at the dihedral angle $\gamma/2$ relative to each other. Let $\mathbf{n}_{\mathrm{A}}$ be the unit vector in the direction $\mathbf{C} \times \mathbf{A}$ normal to E_{A}, and let $\mathbf{n}_{\mathrm{B}}$ be the unit vector along $\mathbf{B} \times \mathbf{C}$ normal

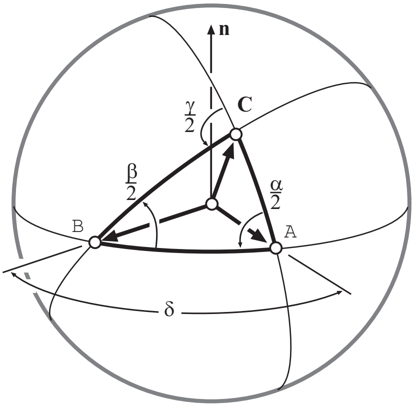

Fig. 8.2 The spherical triangle $\triangle\mathbf{ABC}$ with interior angles $\alpha/2$ and $\beta/2$ at $\mathbf{A}$ and $\mathbf{B}$, and $\gamma/2$ as its exterior angle at $\mathbf{C}$.

to E_{B}. Using these conventions we have

$$\mathbf{n}_{\mathrm{A}} \times \mathbf{n}_{\mathrm{B}} = \sin\frac{\gamma}{2}\mathbf{C} \quad \text{and} \quad \mathbf{n}_{\mathrm{B}} \cdot \mathbf{n}_{\mathrm{A}} = \cos\frac{\gamma}{2}, \tag{8.85}$$

where $\gamma/2$ is the exterior angle at the vertex $\mathbf{C}$. We now compute $\mathbf{C}$ in terms of the vertices $\mathbf{A}$ and $\mathbf{B}$ and their interior angles.

We can expand $\mathbf{n}_{\mathrm{A}}$ and $\mathbf{n}_{\mathrm{B}}$ in terms of the unit vectors $\mathbf{B}$, $\mathbf{v}$, and $\mathbf{n}$, where $\mathbf{n}$ is the unit vector in the direction $\mathbf{B}\times\mathbf{A}$ and $\mathbf{v} = \mathbf{n}\times\mathbf{B}$. If δ is the angle measured from $\mathbf{B}$ to $\mathbf{A}$ in the $\mathbf{AB}$ plane, that is, $\cos\delta = \mathbf{B}\cdot\mathbf{A}$, then we have

$$\begin{aligned} \mathbf{n}_{\mathrm{A}} &= \sin\frac{\alpha}{2}(\cos\delta\mathbf{v} - \sin\delta\mathbf{B}) + \cos\frac{\alpha}{2}\mathbf{n}, \\ \mathbf{n}_{\mathrm{B}} &= -\sin\frac{\beta}{2}\mathbf{v} + \cos\frac{\beta}{2}\mathbf{n}. \end{aligned} \tag{8.86}$$

Computing the scalar and vector products in (8.85) we obtain

$$\begin{aligned} \sin\frac{\gamma}{2}\mathbf{C} &= \sin\frac{\beta}{2}\cos\frac{\alpha}{2}\mathbf{B} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathbf{A} + \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\mathbf{B}\times\mathbf{A}, \\ \cos\frac{\gamma}{2} &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{B}\cdot\mathbf{A}. \end{aligned} \tag{8.87}$$

Divide these two equations to obtain

$$\tan\frac{\gamma}{2}\mathbf{C} = \frac{\tan\frac{\beta}{2}\mathbf{B} + \tan\frac{\alpha}{2}\mathbf{A} + \tan\frac{\beta}{2}\tan\frac{\alpha}{2}\mathbf{B}\times\mathbf{A}}{1 - \tan\frac{\beta}{2}\tan\frac{\alpha}{2}\mathbf{B}\cdot\mathbf{A}}. \tag{8.88}$$

Compare this equation with (8.84) to see that Rodrigues's formula is the equation of the triangle formed by the rotation axes of $[A]$, $[B]$, and $[C]$ with interior angles $\alpha/2$ and $\beta/2$ at the vertices $\mathbf{A}$ and $\mathbf{B}$, and the exterior angle $\gamma/2$ at $\mathbf{C}$.

8.3.1.2 The Composite Axis Theorem

The rotation axes of the composition of rotations $[C] = [B][A]$ form a spherical triangle $\triangle\mathbf{ABC}$ with vertex angles directly related to the rotation angles α, β, and γ. We examine this triangle using equation (8.87).

Notice that $\alpha/2$ and $\beta/2$ take values between zero and π, therefore the sine of these angles are always positive. Thus, the vector part of (8.87) has a positive component along $\mathbf{B}\times\mathbf{A}$. The component along $\mathbf{B}$ is positive for $\beta < \pi$ and negative for $\beta > \pi$. We now introduce the convention that $\mathbf{C}$ is directed so it always has a positive component along $\mathbf{B}$. This allows $\sin(\gamma/2)$ to take positive and negative values. Notice that if $\sin(\gamma/2)$ is negative, then $\gamma/2 > \pi$.

Because $\mathbf{A} = \cos\delta\mathbf{B} + \sin\delta\mathbf{v}$, we have

$$\sin\frac{\gamma}{2}\mathbf{C} = \Big(\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\delta\Big)\mathbf{B} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\delta\mathbf{v} \\ + \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\mathbf{B}\times\mathbf{A}. \tag{8.89}$$

Introduce the angle τ so that $\cos\tau = \mathbf{B}\cdot\mathbf{C}$. Then we have

$$\sin\frac{\gamma}{2}\cos\tau = \sin\frac{\beta}{2}\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\delta. \tag{8.90}$$

Our convention for the direction of $\mathbf{C}$ ensures that $\cos\tau$ is always positive, so we have the two cases:

Case 1. $\sin(\gamma/2) > 0$, that is, $\gamma < 2\pi$.

In this case the vertex $\mathbf{C}$ has a positive component along $\mathbf{B}\times\mathbf{A}$. The derivation above shows that $\alpha/2$ and $\beta/2$ are the interior angles of $\triangle\mathbf{ABC}$ at the vertices $\mathbf{A}$ and $\mathbf{B}$. The angle $\gamma/2$ is the exterior angle at $\mathbf{C}$.

Case 2. $\sin(\gamma/2) < 0$, that is, $\gamma > 2\pi$.

In this case the vertex $\mathbf{C}$ lies below the $\mathbf{AB}$ plane and has a component directed opposite to $\mathbf{B}\times\mathbf{A}$. The angles $\alpha/2$ and $\beta/2$ are the exterior angles of $\triangle\mathbf{ABC}$ at $\mathbf{A}$ and $\mathbf{B}$, respectively. If the angle κ is the interior angle at $\mathbf{C}$, then $\gamma/2 = \kappa + \pi$.

We collect these results in the following theorem:

Theorem 9 (The Composite Axis Theorem). *The axis* $\mathbf{C}$ *of a composite rotation* $[C] = [B][A]$ *forms a triangle with the axes* $\mathbf{B}$ *and* $\mathbf{A}$ *of the rotations* $[B]$ *and* $[A]$*, respectively. If* $\sin(\gamma/2) > 0$*, then the interior angles of this triangle at* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $\alpha/2$ *and* $\beta/2$*, respectively. If* $\sin(\gamma/2) < 0$*, then* $\alpha/2$ *and* $\beta/2$ *are the exterior angles at these vertices. In this case, if* κ *is the interior angle at* $\mathbf{C}$*, then* $\gamma/2 = \kappa + \pi$*.*

8.3.1.3 Quaternions and the Spherical Triangle

W. R. Hamilton [43] introduced quaternions to generalize to three dimensions the geometric properties of complex numbers. A quaternion is the formal sum of a scalar q_0 and a vector $\mathbf{q} = (q_x, q_y, q_z)^T$, written as $Q = q_0 + \mathbf{q}$.

Quaternions can be added together, and multiplied by a scalar, componentwise like four-dimensional vectors. A new operation, invented by Hamilton, defines the product of two quaternions $P = p_0 + \mathbf{p}$ and $Q = q_0 + \mathbf{q}$ by the rule

$$R = PQ = (p_0 + \mathbf{p})(q_0 + \mathbf{q}) = (p_0 q_0 - \mathbf{p}\cdot\mathbf{q}) + (q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{p}\times\mathbf{q}), \tag{8.91}$$

where the dot and cross denote the usual vector operations.

The conjugate of the quaternion $Q = q_0 + \mathbf{q}$ is $Q^* = q_0 - \mathbf{q}$, and the product QQ^* is the positive real number

$$QQ^\star = (q_0 + \mathbf{q})(q_0 - \mathbf{q}) = q_0^2 + \mathbf{q}\cdot\mathbf{q} = |Q|^2. \tag{8.92}$$

The scalar $|Q|$ is called the *norm* of the quaternion.

We are interested in quaternions of norm equal to 1. These so-called unit quaternions can be written in the form

$$Q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{S}, \tag{8.93}$$

where $\mathbf{S} = (s_x, s_y, s_z)^T$ is a unit vector. The quaternion product of $A(\alpha/2) = \cos(\alpha/2) + \sin(\alpha/2)\mathbf{A}$ and $B(\beta/2) = \cos(\beta/2) + \sin(\beta/2)\mathbf{B}$ yields the unit quaternion $C(\gamma/2) = B(\beta/2)A(\alpha/2)$, given by

$$\begin{aligned}\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\mathbf{C} = &\Big(\cos\frac{\beta}{2}\cos\frac{\alpha}{2} - \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\mathbf{B}\cdot\mathbf{A}\Big)\\ &+ \Big(\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\mathbf{B} + \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathbf{A} + \sin\frac{\beta}{2}\sin\frac{\alpha}{2}\mathbf{B}\times\mathbf{A}\Big).\end{aligned} \tag{8.94}$$

Compare this equation to (8.87) to see that quaternion multiplication yields one vertex of a spherical triangle from the other two. We conclude that each rotation $[A(\phi, \mathbf{S})]$ can be identified with a quaternion $S(\phi/2) = \cos(\phi/2) + \sin(\phi/2)\mathbf{S}$.

8.3.2 The Triangle of Relative Rotation Axes

For three orientations M_i, M_j, and M_k of a moving body, we can construct the relative rotations $[A_{ij}]$, $[A_{jk}]$, and $[A_{ik}]$. Notice that the relative rotation $[A_{ik}]$ is given by the product $[A_{jk}][A_{ij}]$, as is seen from

$$[A_{ik}] = [A_k][A_i^T] = \left([A_k][A_j^T]\right)\left([A_j][A_i^T]\right) = [A_{jk}][A_{ij}]. \tag{8.95}$$

Rodrigues's formula for this composition of rotations yields

$$\tan\frac{\phi_{ik}}{2}\mathbf{S}_{ik} = \frac{\tan\frac{\phi_{jk}}{2}\mathbf{S}_{jk} + \tan\frac{\phi_{ij}}{2}\mathbf{S}_{ij} + \tan\frac{\phi_{jk}}{2}\tan\frac{\phi_{ij}}{2}\mathbf{S}_{jk}\times\mathbf{S}_{ij}}{1-\tan\frac{\phi_{jk}}{2}\tan\frac{\phi_{ij}}{2}\mathbf{S}_{jk}\cdot\mathbf{S}_{ij}}. \tag{8.96}$$

This is the equation of the spherical triangle formed by the relative rotation axes $\triangle\mathbf{S}_{ij}\mathbf{S}_{jk}\mathbf{S}_{ik}$. The composite-axis theorem defines the relationship between the vertex angles of this triangle and the relative rotation angles $\phi_{ij}/2$, $\phi_{jk}/2$, and $\phi_{ik}/2$. For example, if $\mathbf{S}_{ik}$ lies above the plane through $\mathbf{S}_{ij}\mathbf{S}_{jk}$, then the interior angles at $\mathbf{S}_{ij}$ and $\mathbf{S}_{jk}$ are $\phi_{ij}/2$ and $\phi_{jk}/2$, respectively, and the exterior angle at $\mathbf{S}_{ik}$ is $\phi_{ik}/2$, Figure 8.3. This triangle is analogous to the planar pole triangle and is called the *spherical pole triangle*.

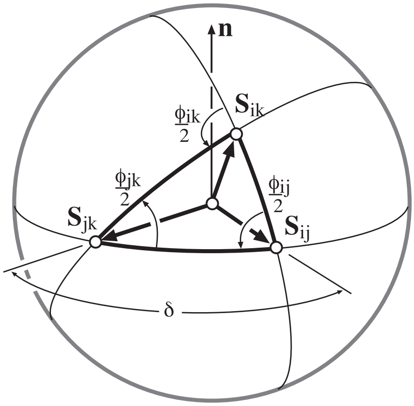

Fig. 8.3 The spherical pole triangle.

8.3.3 The Spherical Image Pole Triangle

We now consider the inverse rotations associated with the orientations M_i, M_j, M_k, given by $[A_i^T]$, $[A_j^T]$, and $[A_k^T]$. The relative inverse rotation $[A_{ik}^\dagger]$ is the composition of the relative inverse rotations $[A^{\dagger}_{jk}][A^{\dagger}_{ij}]$, as can be seen from

$$[A^{\dagger}_{ik}] = [A^T_k][A_i] = \big([A^T_k][A_j]\big)\big([A^T_j][A_i]\big) = [A^{\dagger}_{jk}][A^{\dagger}_{ij}]. \tag{8.97}$$

We can transform each relative inverse rotation to the fixed frame F for M aligned with M_m, that is, we compute

$$[A^m_{ik}] = [A_m][A^{\dagger}_{ik}][A^T_m]. \tag{8.98}$$

We obtain the composition of the relative inverse rotations as seen from F,

$$[A^m_{ik}] = [A^m_{jk}][A^m_{ij}]. \tag{8.99}$$

Rodrigues's formula for this composition defines the triangle of image relative rotation axes, which is known as the *spherical image pole triangle*.

Let $m = i$, for example, and notice that $\mathbf{S}^i_{ij} = \mathbf{S}_{ij}$ and $\mathbf{S}^i_{ik} = \mathbf{S}_{ik}$, and we have the image pole triangle $\triangle\mathbf{S}_{ij}\mathbf{S}^i_{jk}\mathbf{S}_{ik}$. The relative inverse rotation angles are the negatives of the relative rotation angles. Therefore, $\phi^{\dagger}_{ik} = -\phi_{ik}$. Thus, if the spherical pole triangle has $\phi_{ij}/2$ as its interior angle at $\mathbf{S}_{ij}$, then the image pole triangle $\triangle\mathbf{S}_{ij}\mathbf{S}^i_{jk}\mathbf{S}_{ik}$ has $-\phi_{12}/2$ as its associated interior angle. The result is that the axis $\mathbf{S}^i_{jk}$ is the reflection of $\mathbf{S}_{jk}$ through the plane defined by $\mathbf{S}_{ij}\mathbf{S}_{ik}$, Figure 8.4.

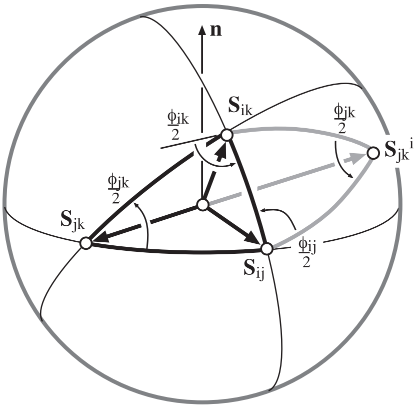

Fig. 8.4 The image pole $\mathbf{S}^i_{jk}$ is the reflection of $\mathbf{S}_{jk}$ through the side $\mathbf{S}_{ij}\mathbf{S}_{ik}$ of the spherical pole triangle.

8.4 Summary

This chapter has presented the geometric theory of spatial rotations. Of fundamental importance is the spherical pole triangle, which is the analog of the planar pole triangle. Notice that Hamilton's quaternions can be viewed as the generalization of complex vectors, and that they provide a convenient tool for computations using the spherical triangle. The similar form of the planar and spherical results provides an avenue for visualizing three-dimensional geometry using intuition drawn from plane geometry.

8.5 References

The kinematic theory of spatial rotations can be found in Bottema and Roth [5]. Crane and Duffy [16] present a detailed development of the trigonometric formulas for spherical triangles. Cheng and Gupta [11] discuss the history of the various representations of the rotation matrix. The interesting history surrounding Hamilton's quaternions and Rodrigues's formula is described by Altmann [1].

Exercises

1. Determine the rotation axis $\mathbf{S}$ and angle ϕ for the rotation $[A] = [X(30°)][Y(30°)][Z(30°)]$ (Sandor and Erdman [112]).
2. Let the axis of a rotation be along the vector $\mathbf{q} = (2,2,2\sqrt{2})^T$ and let the rotation angle be $\phi = 30°$. Determine the rotation matrix $[A(\phi,\mathbf{S})]$ (Sandor and Erdman [112]).
3. Table 8.1 gives four locations of a pair of points $\mathbf{P}$ and $\mathbf{Q}$. Determine the three relative rotation matrices $[A_{12}]$, $[A_{13}]$, and $[A_{14}]$ (Suh and Radcliffe [134]).

Table 8.1 Point coordinates defining four orientations

M_i	$\mathbf{P}^i$	$\mathbf{Q}^i$
1	$(0.105040, 0.482820, 0.869397)^T$	$(-0.464640, -0.676760, 0.571057)^T$
2	$(0.090725, 0.541283, 0.835931)^T$	$(-0.133748, -0.751642, 0.645868)^T$
3	$(0.104155, 0.620000, 0.777658)^T$	$(0.161113, -0.702067, 0.693646)^T$
4	$(0.096772, 0.725698, 0.681173)^T$	$(0.400762, -0.564306, 0.721769)^T$

4. Prove that each element of a rotation matrix is equal to its associated minor.
5. Show that the change of coordinates $[R][A][R^T]$ of a rotation matrix $[A]$ has the rotation axis $[R]\mathbf{S}$, where $\mathbf{S}$ is the rotation axis of $[A]$.

6. Let $c_1 = s_x \sin(\phi/2)$, $c_2 = s_y \sin(\phi/2)$, $c_3 = s_z \sin(\phi/2)$ and $c_4 = \cos(\phi/2)$ denote the components of a unit quaternion. Use (8.54) to obtain a formula for $[A(\phi, \mathbf{S})]$ with each element quadratic in c_i.
7. Derive Rodrigues's formula for the composition of rotations (8.84).
8. Consider a spherical pole triangle $\triangle \mathbf{S}_{12}\mathbf{S}_{23}\mathbf{S}_{13}$. Show that the first position of a point $\mathbf{Q}^1$ reflects through the side $N_1 : \mathbf{S}_{12}\mathbf{S}_{13}$ to the cardinal point $\mathbf{Q}^*$ and that the corresponding points $\mathbf{Q}^i$ reflect through the sides N_i to the same point.