ME 3751: **Kinematics and mechanisms**

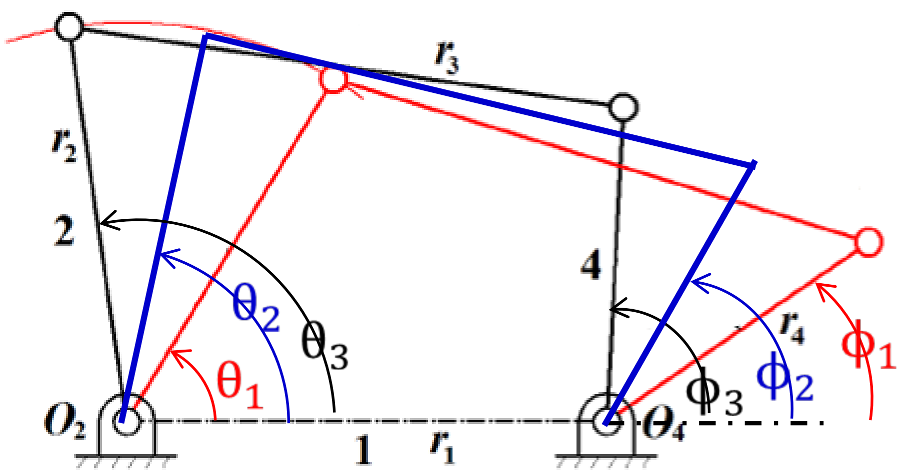
Lecture 12

Analytical synthesis of 4-bar with three rocker positions

Read 3.1-3.2, Kinzel

Ch 3.2 - Three Rocker Positions

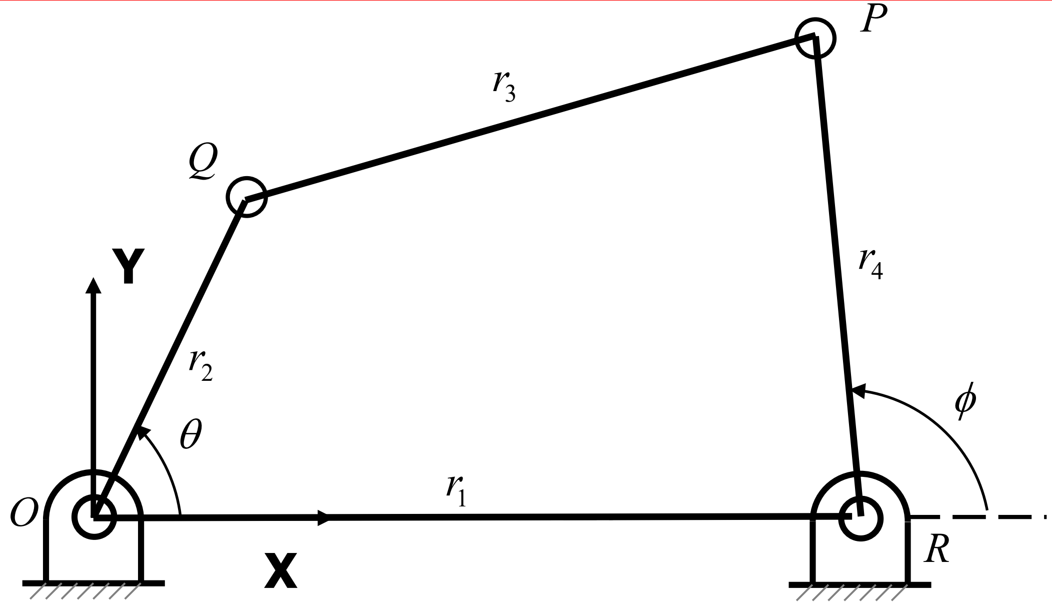
- Given:
  - Three positions of the input link $\theta_1, \theta_2, \theta_3$
  - And the corresponding two positions of the output link ϕ_1 , ϕ_2 , ϕ_3
- Find:
  - lengths of all 4 links



The Input-Output Equation of 4-Bar

- Input-Output function

$$\boxed{r_3^2 = r_1^2 + r_4^2 + r_2^2 + 2r_4 r_1 \cos\phi - 2r_2 r_1 \cos\theta - 2r_2 r_4 \cos(\theta - \phi)}$$



- The scaled design equation $\boxed{r_1 = 1}$

$$r_3^2 = 1 + r_4^2 + r_2^2 + 2r_4 \cos\phi - 2r_2 \cos\theta - 2r_2 r_4 \cos(\theta - \phi)$$

Simplify the design equation

- The design equation is nonlinear

$$r_3^2 = 1 + r_2^2 + r_4^2 - 2r_2\cos\theta + 2r_4\cos\phi - 2r_4r_2\cos(\phi-\theta)$$

Collect terms

$$2r_4r_2\cos(\phi-\theta) = 1 + r_2^2 + r_4^2 - r_3^2 - 2r_2\cos\theta + 2r_4\cos\phi$$

Simplify by dividing by $2r_4r_2$

$$\frac{2r_4r_2\cos(\phi-\theta)}{2r_4r_2} = \frac{1 + r_2^2 + r_4^2 - r_3^2}{2r_4r_2} - \frac{2r_2\cos\theta}{2r_4r_2} + \frac{2r_4\cos\phi}{2r_4r_2}$$

Which gives us:

$$\cos(\phi-\theta) = \frac{1 + r_2^2 + r_4^2 - r_3^2}{2r_4r_2} + \frac{\cos\phi}{r_2} - \frac{\cos\theta}{r_4}$$

Step 4, cont.

$$\cos(\phi-\theta)=\frac{1+r_2^2+r_4^2-r_3^2}{2r_4r_2}+\frac{\cos\phi}{r_2}-\frac{\cos\theta}{r_4}$$

Now make the following substitutions

$$z_1=\frac{1+r_2^2+r_4^2-r_3^2}{2r_4r_2}$$

$$z_2=\frac{1}{r_2}$$

$$z_3=\frac{1}{r_4}$$

Then

$$\cos(\phi-\theta)=z_1+z_2\cos\phi-z_3\cos\theta$$

The Final Design Equation

$$\cos(\phi - \theta) = z_1 + z_2 \cos\phi - z_3 \cos\theta$$

This is the basic design equation. Note that it is **linear** in z_1, z_2, z_3

Write the design equation three times, one for each precision point:

$$\cos(\phi_1 - \theta_1) = z_1 + z_2 \cos\phi_1 - z_3 \cos\theta_1$$
$$\cos(\phi_2 - \theta_2) = z_1 + z_2 \cos\phi_2 - z_3 \cos\theta_2$$
$$\cos(\phi_3 - \theta_3) = z_1 + z_2 \cos\phi_3 - z_3 \cos\theta_3$$

Rewrite in Matrix Form

$$\cos(\phi_1 - \theta_1) = z_1 + z_2 \cos\phi_1 - z_3 \cos\theta_1$$
$$\cos(\phi_2 - \theta_2) = z_1 + z_2 \cos\phi_2 - z_3 \cos\theta_2$$
$$\cos(\phi_3 - \theta_3) = z_1 + z_2 \cos\phi_3 - z_3 \cos\theta_3$$

Rewrite in matrix form

$$\begin{bmatrix} 1 & \cos\phi_1 & -\cos\theta_1 \\ 1 & \cos\phi_2 & -\cos\theta_2 \\ 1 & \cos\phi_3 & -\cos\theta_3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = \begin{Bmatrix} \cos(\phi_1 - \theta_1) \\ \cos(\phi_2 - \theta_2) \\ \cos(\phi_3 - \theta_3) \end{Bmatrix}$$

Solve for z's

Solve for z_1, z_2, z_3 using any matrix solution routine. For example, using inversion, we get

$$\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = \begin{bmatrix} 1 & \cos\phi_1 & -\cos\theta_1 \\ 1 & \cos\phi_2 & -\cos\theta_2 \\ 1 & \cos\phi_3 & -\cos\theta_3 \end{bmatrix}^{-1} \begin{Bmatrix} \cos(\phi_1 - \theta_1) \\ \cos(\phi_2 - \theta_2) \\ \cos(\phi_3 - \theta_3) \end{Bmatrix}$$

Solving with Matlab

$$\begin{bmatrix} 1 & \cos\phi_1 & -\cos\theta_1 \\ 1 & \cos\phi_2 & -\cos\theta_2 \\ 1 & \cos\phi_3 & -\cos\theta_3 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix} = \begin{Bmatrix} \cos(\phi_1 - \theta_1) \\ \cos(\phi_2 - \theta_2) \\ \cos(\phi_3 - \theta_3) \end{Bmatrix}$$

For MATLAB, let

$$A = \begin{bmatrix} 1 & \cos\phi_1 & -\cos\theta_1 \\ 1 & \cos\phi_2 & -\cos\theta_2 \\ 1 & \cos\phi_3 & -\cos\theta_3 \end{bmatrix} \qquad X = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad b = \begin{bmatrix} \cos(\phi_1 - \theta_1) \\ \cos(\phi_2 - \theta_2) \\ \cos(\phi_3 - \theta_3) \end{bmatrix}$$

Then

$$AX = b$$

Matlab, cont.

$$AX = b$$

To solve in MATLAB,

$$X = A \backslash b$$

Knowing z_1, z_2, z_3, we can find r_2, r_3, r_4

$$z_2 = \frac{1}{r_2} \qquad \Rightarrow r_2 = \frac{1}{z_2}$$

$$z_3 = \frac{1}{r_4} \qquad \Rightarrow r_4 = \frac{1}{z_3}$$

$$z_1 = \frac{1 + r_2^2 + r_4^2 - r_3^2}{2r_4 r_2} \qquad \Rightarrow r_3 = \sqrt{1 + r_2^2 + r_4^2 - 2r_4 r_2 z_1}$$

Interpretation

$$z_2 = \frac{1}{r_2} \qquad \Rightarrow r_2 = \frac{1}{z_2}$$

$$z_3 = \frac{1}{r_4} \qquad \Rightarrow r_4 = \frac{1}{z_3}$$

$$z_1 = \frac{1 + r_2^2 + r_4^2 - r_3^2}{2 r_4 r_2} \qquad \Rightarrow r_3 = \sqrt{1 + r_2^2 + r_4^2 - 2 r_4 r_2 z_1}$$

r_3 is the distance between the circle points, so it must be positive, in spite of the square root.

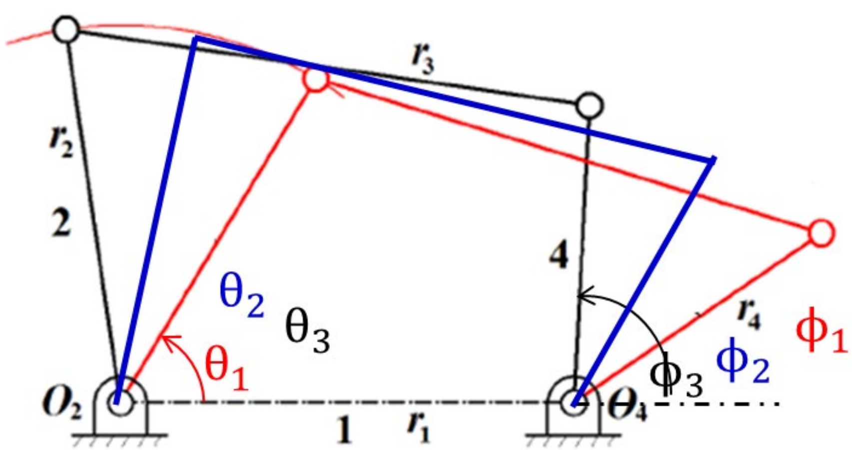
r_2 and r_4 can be either positive or negative in a vector sense.

Order and branch problems can still occur but less common than in rigid body guidance because the ranges are usually modest.

Lecture Problem: 3 Position Synthesis

- Design a four-bar linkage such that the input and the output links to pass the following three specified positions.
- $\theta_1 = 35.02^\circ, \theta_2 = 67.50^\circ, \theta_3 = 100.0^\circ,$
- $\phi_1 = 91.21^\circ, \phi_2 = 101.79^\circ, \phi_3 = 117.19^\circ$

The base link length must be r_1=4.5 inches. Please determine the lengths of the other three links.



Solution

Substitute values of $\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3$ into the solution matrix

$$\begin{bmatrix} 1 & \cos(\phi_1) & -\cos(\theta_1) \\ 1 & \cos(\phi_2) & -\cos(\theta_2) \\ 1 & \cos(\phi_3) & -\cos(\theta_3) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 - \phi_1) \\ \cos(\theta_2 - \phi_2) \\ \cos(\theta_3 - \phi_3) \end{bmatrix}$$

To obtain

$$\begin{bmatrix} 1 & -0.0210 & -0.8189 \\ 1 & -0.2043 & -0.3827 \\ 1 & -0.4569 & 0.1732 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0.5566 \\ 0.8262 \\ 0.9552 \end{bmatrix}$$

Solving yields the following.

$z_1 = 5.150,\ z_2 = 11.20,\ z_3 = 5.321,$

These values of z can now be used to find the link length proportions.

$r_2 = \frac{1}{z_2} = 0.0893,\ r_4 = \frac{1}{z_3} = 0.1879,\ r_3 = \sqrt{1 + r_2{}^2 + r_4{}^2 - 2r_2r_4z_1} = 0.9329$

Since the above equations assumes r_1=1. The un-scaled link lengths are shown below. $K = 4.5$

$R_1 = 4.5\ inches,$

$R_2 = (0.0893)4.5 = 0.4018\ inches,$

$R_3 = (0.9329)4.5 = 4.198\ inches,$

$R_4 = (0.1879)4.5 = 0.8456\ inches$

Demos

- Solidworks