Ph.D Dissertation

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1 Introduction

This is a check to see if we can successfully cite! [Bob11]

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2 Bounding the Regulator

To define the regulator of a rational elliptic curve, we must first define the naïve logarithmic height, Néron-Tate canonical height and the Néron-Tate pairing on points on E.

Let E be an elliptic curve over $\mathbb{Q}$ and $P \in E(\mathbb{Q})$ a rational point on E.

Definition 2.1. The *naïve logarithmic height* of P is a measure of the "complexity" of the coefficients of P. Specifically, any non-identity rational point P may be written as $P = (\frac{a}{d^2}, \frac{b}{d^3})$, with $a, b, d \in \mathbb{Z}$, $d > 0$ and $\gcd(a, b, d) = 1$; we then define the naïve height of P to be

$$h(P) := \ln(d). \tag{2.1}$$

Moreover, define $h(\mathcal{O}) = 0$.

If you compute the naïve heights of a number of points on an elliptic curve, you'll notice that the naïve height function is "almost a quadratic form" on E. That is $h(nP) \sim n^2 h(P)$ for integers n, up to some constant that doesn't depend on P. We can turn h into a true quadratic form as follows:

Definition 2.2. The *Néron-Tate height* height function $\hat{h} : E(\mathbb{Q}) \to \mathbb{R}$ is defined as

$$\hat{h}(P) := \lim_{n\to\infty} \frac{h(2^n P)}{(2^n)^2}, \tag{2.2}$$

where h is the naïve logarithmic height defined above.

Theorem 2.3 (Néron-Tate)**.** *Néron-Tate has defines a canonical quadratic form on $E(\mathbb{Q})$ modulo torsion. That is,*

1. *For all $P, Q \in E(\mathbb{Q})$,*
$$\hat{h}(P+Q) + \hat{h}(P-Q) = 2\left[\hat{h}(P) + \hat{h}(Q)\right] \tag{2.3}$$
i.e. $\hat{h}$ obeys the parallelogram law;

2. *For all $P \in E(\mathbb{Q})$ and $n \in \mathbb{Z}$,*
$$\hat{h}(nP) = n^2 \hat{h}(P) \tag{2.4}$$

3. *$\hat{h}$ is even, and the pairing $\langle\ ,\ \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$ by*

$$\langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q) \tag{2.5}$$

is bilinear;

4. *$\hat{h}(P) = 0$ iff P is torsion;*

5. *We may replace h with another height function on $E(\mathbb{Q})$ that is "almost quadratic" without changing $\hat{h}$.*

For a proof of this theorem and elaboration on the last point, see [Sil85, pp. 227-232].

Definition 2.4. The *Néron-Tate pairing* on $E/\mathbb{Q}$ is the bilinear form $\langle\ ,\ \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$ by

$$\langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q) \tag{2.6}$$

Note that this definition may be extended to all pairs of points over $\overline{\mathbb{Q}}$, but the definition above suffices for our purposes.

If $E(\mathbb{Q})$ has rank r, then $E(\mathbb{Q})/E_{\text{tor}}(\mathbb{Q}) \hookrightarrow \mathbb{R}^r$ under the quadratic form $\hat{h}$ as a (rank r) lattice.

Definition 2.5. The *regulator* Reg_E of $E/\mathbb{Q}$ is the covolume of the lattice that is the image of $E(\mathbb{Q})$ under $\hat{h}$. That is, if $\{P_1, \ldots, P_r\}$ generates $E(\mathbb{Q})$, then

$$\text{Reg}_E = \det\left(\langle P_i, P_j\rangle\right)_{1\le i,j\le r} \tag{2.7}$$

where $\left(\langle P_i, P_j\rangle\right)_{1\le i,j\le r}$ is the matrix whose (i,j)th entry is the value of the pairing $\langle P_i, P_j\rangle$. If $E/\mathbb{Q}$ has rank zero, then Reg_E is defined to be 1.

Loosely, the regulator measures the "density" of rational points on E: positive rank elliptic curves with small regulators have many points with small coordinates, while those with large regulators have few such points.

It turns out that one can construct elliptic curves with arbitrarily large regulators (for example, fix some x_0 and y_0 with large denominators, and then find A and B such that $P = (x, y)$ lies on $E : y^2 = x^2 + Ax + B$). However, the more interesting question to ask – and the one that is relevant to this thesis – is "how small can the regulator get?". Specifically, given $E/\mathbb{Q}$ with discriminant D_E, what is the smallest Reg_E can be as a function of D_E?

This is an open question. However, the following conjecture has been made by Lang [Lan97]:

Conjecture 2.6. *Let $E/\mathbb{Q}$ have minimal discriminant Δ_E. There exists an absolute constant $M_{\mathbb{Q}} > 0$ independent of E such that any non-torsion point $P \in E(\mathbb{Q})$ satisfies*

$$\hat{h}(P) >= M_{\mathbb{Q}} \log|\Delta_E|. \tag{2.8}$$

That is, the minimum height of a non-torsion point on E scales with the log of the absolute value of the curve's minimal discriminant. Hindry and Silverman in [HS88] show that the abc conjecture implies Lang's height conjecture; since we are already assuming strong abc, we have this result for free.

A corollary of the height conjecture is that the regulator is bounded from below in terms of the curve's conductor:

Corollary 2.7. *Let $E/\mathbb{Q}$ have conductor N_E, and let $M_{\mathbb{Q}}$ be the absolute constant in Lang's height conjecture. Then*

$$\text{Reg}_E \ge 3 \tag{2.9}$$

Proof. Let E have minimal discriminant Δ_E and algebraic rank r. Since $|\Delta_E| \ge N_E$, it follows that

$$\text{Reg}_E \ge (M_{\mathbb{Q}} \log N_E)^r \ge M_{\mathbb{Q}}^r$$

$\square$

References

[Bob11] Jonathan W. Bober, *Conditionally bounding analytic ranks of elliptic curves*, The arXiv (2011), 1–9.

[HS88] M. Hindry and J.H. Silverman, *The canonical height and integral points on elliptic curves*, Inventiones mathematicae **93** (1988), no. 2, 419–450 (English).

[Lan97] Serge Lang, *Survey of diophantine geometry*, corr. 2nd printing ed., Berlin: Springer, 1997 (English).

[Sil85] Joseph H. Silverman, *The arithmetic of elliptic curves (graduate texts in mathematics)*, Springer, Dec 1985.