Homework 3

Total: 20 points
Due: Wed 5 Feb 2014 09:30 in class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **AUTONOMOUS EQUATIONS**
   (a) In the following two autonomous equations $\frac{dy}{dt} = f(y)$, sketch the graph of $f(y)$ versus y, determin the critical (equilibrium) points, and classify each equilibrium solution as asymptotically stable, unstable or semistable. Then sketch a graph of several solutions on the ty-plane, including the equilibrium solutions and a few other solutions to indicate asymptotic behaviour.
      i. $\frac{dy}{dt} = y(y-1)(y-2)$ Boyce 2.5 Q3
      ii. $\frac{dy}{dt} = e^{-y} - 1$ Boyce 2.5 Q5

   (b) Boyce 2.5 Q18
   A pond forms as water collects in a conical depression of radius a and depth h. Suppose that the water flows in at a constant rate k, and is lost through evaporation at a rate proportional to the pond's surface area.
      i. Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation
      $$\frac{dV}{dt} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}},$$
      where α is the coefficient of evaporation.
      ii. Find the equilibrium depth of the water in the pond. Is the equilibrium asymptotically stable or unstable?
      iii. Find a condition relating k and α that must be satisfied if the pond is not to overflow.

2. **EULER'S METHOD** Consider the initial value problem
   $$\frac{dy}{dt} = (t-1)(y+1), \quad y(1) = 1.$$
   (a) Let $y = \phi(t)$ be the unique solution to this IVP. Estimate the value of $\phi(2)$ using Euler's method with a step size of $h = 1$. Then do the same for step sizes of $h = 0.5$ and $h = 0.2$.
   (b) Solve the IVP and state the true value of $\phi(2)$. Do your estimates underpredict or overpredict $\phi(2)$? Do they get more accurate as h decreases?

[More questions overleaf!]

3. **2ND ORDER LINEAR DIFFERENTIAL EQUATIONS**

(a) In each of the following problems, find the general solution to the given differential equation

i. $y'' + 2y' - 3y = 0$ Boyce 3.1 Q1

ii. $y'' + 3y' + 2y = 0$ Boyce 3.1 Q2

iii. $y'' + 5y' = 0$ Boyce 3.1 Q5

iv. $y'' - 2y' - 2y = 0$ Boyce 3.1 Q8

(b) In each of the following problems, find the solution to the given initial value problem, and sketch a graph of th solution, indicating the behaviour as t increases.

i. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$ Boyce 3.1 Q9

ii. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$ Boyce 3.1 Q10

iii. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$ Boyce 3.1 Q12

(c) Boyce 3.1 Q23

Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0,$$

where α is a given constant. Determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.