Homework 4

Total: 20 points
Due: Wed 12 Feb 2014 09:30 in class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **CHARACTERISTIC EQUATIONS WITH COMPLEX ROOTS**

   (a) Use Euler's Formula: $e^{ix} = \cos(x) + i\sin(x)$ to rewrite the following complex numbers in the form $a + ib$.

   i. e^{1+2i} Boyce 3.3 Q1

   ii. $e^{2-\frac{\pi}{2}i}$ Boyce 3.3 Q4

   iii. 2^{1-i} Boyce 3.3 Q5

   (b) Give the general solution to the following differential equations:

   i. $y'' - 2y' + 2y = 0$ Boyce 3.3 Q7

   ii. $y'' + 6y' + 13y = 0$ Boyce 3.3 Q11

   iii. $y'' + 4y' + 6.25y = 0$ Boyce 3.3 Q16

   (c) Find the solution to each of the following initial value problems. Sketch the graph of the solution and describe its behaviour for increasing t:

   i. $y'' + 4y = 0, \quad y(0) = 0, y'(0) = 1$ Boyce 3.3 Q17

   ii. $y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0, y'(\frac{\pi}{2}) = 2$ Boyce 3.3 Q19

   iii. $y'' + 2y' + 2y = 0, \quad y(\frac{\pi}{4}) = 2, y'(\frac{\pi}{4}) = -2$ Boyce 3.3 Q22

   (d) Boyce 3.3 Q34 & 42

   An equation of the form
   $$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \qquad t > 0,$$
   where α and β are real constants, is called an Euler equation.

   i. Let $x = \ln(t)$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   ii. Use the result of part (a) to transform the above differential equation to
   $$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$
   Observe that this equation has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to this equation, then $y_1(\ln(t))$ and $y_2(\ln(t))$ form a fundamental set of solutions to the original Euler equation.

   iii. Use parts (a) and (b) to find the general solution to
   $$t^2y'' + 7ty' + 10y = 0.$$

[More questions on the next page!]

2. **CHARACTERISTIC EQUATIONS WITH EQUAL ROOTS**

(a) Find the general solution to the following differential equations:

i. $9y'' + 6y' + y = 0$ — Boyce 3.4 Q2

ii. $y'' - 6y' + 9y = 0$ — Boyce 3.4 Q6

(b) Find the solution to each of the following initial value problems. Sketch the graph of the solution and describe its behaviour for increasing t:

i. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, y'(0) = -1$ — Boyce 3.4 Q11

ii. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, y'(-1) = 1$ — Boyce 3.4 Q14

(c) Boyce 3.4 Q16

Consider the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \qquad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b, and determine the critical value of b that separates solutions that grow positively from those that eventially grow negatively.

(d) Boyce 3.4 Q24

Consider the Euler equation

$$t^2y'' + 2ty' - 2y = 0, \qquad t > 0.$$

We may use the method detailed earlier in this homework to find the general solution to this equation. However, if we already know a specific solution we may instead use the method of reduction in order to find the DE's general solution.

Given that $y_1(t) = t$ is a solution to this differential equation, use the method of reduction of order to find the general solution to this DE.

3. **NONHOMOGENEOUS EQUATIONS**

(a) Find the general solution to the following differential equations:

i. $y'' - 2y' - 3y = 3e^{2t}$ — Boyce 3.5 Q1

ii. $y'' + 2y' + 5y = 3\sin(2t)$ — Boyce 3.5 Q2

iii. $y'' + 2y' + y = 2e^{-t}$ — Boyce 3.5 Q8

(b) Boyce 3.5 Q20

Solve the initial value problem

$$y'' + 2y' + 5y = 4e^{-t}\cos(2t), \qquad y(0) = 1, \quad y'(0) = 0.$$