Your Name

Student ID #

- Cellphones off please!
- Please box all of your answers.
- You are allowed one two-sided handwritten notesheet for this midterm. You may use a scientific calculator; graphing calculators and all other course-related materials may not be used.
- In order to receive credit, you must **show all of your work** unless explicitly stated otherwise by the question. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $e^{-5\sqrt{3}}$) unless explicity stated otherwise by the question.
- If you need more room, use the backs of the pages, and indicate on the front of the page that you have done so.
- Raise your hand if you have a question.
- This exam has 5 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 50 minutes to complete the exam.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total | 50 | |

1. (10 total points)

   (a) (4 points) Find the expicit general solution to the following first-order differential equation. Your answer should be a function in the form $y = g(x,C)$, where $C$ is an integration constant parameterizing the family of solutions to the DE.

$$\frac{dy}{dx} - 2xy - x = 0$$

   (b) (6 points) Solve the following initial value problem (your answer should be in the form $y = g(t)$, where there is no undetermined constant in $g$). State for what $t$-interval the solution is defined.

$$\frac{dy}{dt} = e^{2t-3y}, \quad y(1) = 2.$$

2. (10 total points) Consider the following initial value problem:

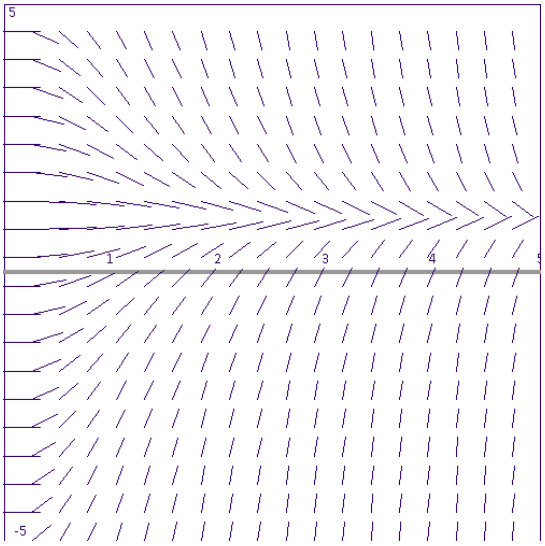
$$\frac{dy}{dx} = \sqrt{\frac{y}{x}}, \qquad y(x_0) = y_0$$

where (in order to avoid having to make sense of square roots of negative numbers or infinity) we may assume that $y_0 \geq 0$ and $x_0 > 0$.

(a) (2 points) Using the existence and uniqueness theorem for nonlinear differential equations, state for which values of $y_0$ the DE is **not** guaranteed to have a unique solution.

(b) (8 points) There is more than one distinct solution to the above DE for the initial condition $y(1) = 0$. One such solution is $y = 0$. Solve the differential equation to find a second different solution.

3. (10 total points)
   The slope field to the differential equation $\frac{dy}{dx} = f(x,y)$ is plotted below for $0 \le x \le 5, -5 \le y \le 5$:




   (a) (4 points) Circle the differential equation that corresponds to the above slope field (you do not need to show your working to receive full grade for this part of the question).

$$\frac{dy}{dx} = x(y+1) \qquad \frac{dy}{dx} = -x(y+1) \qquad \frac{dy}{dx} = x(y-1) \qquad \frac{dy}{dx} = -x(y-1)$$

   (b) (6 points) Let $y = \phi(x)$ be the solution to the differential equation you circled above that satisfies the initial condition $y(0) = 0$. Use Euler's method with a step size of $h = 0.5$ to estimate the value of the solution at $x = 1.5$. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

4. (10 total points) Consider the autonomous differential equation

$$\frac{dy}{dt} = sin^2(y) - K,$$

where $K$ is a constant such that $y = \frac{\pi}{3}$ is an equilibrium solution.

(a) (5 points) Find $K$, and state whether $y = \frac{\pi}{3}$ is a stable, unstable or semistable equilibrium solution. Be sure to justify your answer.

(b) (5 points) Suppose $y = \phi(t)$ is the unique solution to the differential equation satisfying the initial condition $y(0) = 0$. Find $\lim_{t\to\infty} \phi(t)$.

5. (10 total points) A reservoir on a farm initially contains 10000 liters of water, in which 200kg nitrate fertilizer is dissolved. The owner of the reservoir decides the amount of dissolved nitrate needs to be increased, so starts pumping in a 1kg nitrate:1 liter water solution at a rate of 100 l/min. However, unbeknownst to the farmer the reservoir simultaneously develops a leak, and starts draining at a rate of 200 l/min.

   (a) (7 points) Assuming the solution remains perfectly mixed at all times, find the mass of nitrate in the reservoir at time $t$.

   (b) (3 points) What is the maximum amount of nitrate in the reservoir, and when does it occur?