Wed 19 Feb MATH307 LECTURE 16 BOYCE 3.8

§2.6 - FORCED VIBRATIONS

We now consider ~~vibrationing~~ vibrating systems where some known external force is applied. This corresponds to the DE $my'' + \gamma y' + ky = g(t)$, where the function $g(t)$ is no longer 0.

The most important case to consider is where $g(t)$ is periodic in nature, eg. $A\cos(\omega t)$ etc. as this is what we most often encounter in real-world situations.

2.6.1 Forced Vibrations Without Damping

In some situations friction is negligible over short-enough time periods. We investigate such systems when $g(t)$ is sinusoidal, i.e. we consider the equation

$$my'' + ky = F_0\cos(\omega t) \quad (1)$$

for known constants m, k, F_0 & ω.

Let ω_0 be the natural frequency of the system i.e. $\omega_0 = \sqrt{\frac{k}{m}}$. The ~~nature~~ behaviour of a solution to (1) depends on whether $\omega = \omega_0$ or $\omega \neq \omega_0$

2.6.2 Case 1: $\omega \neq \omega_0$:

We know that a general solution to the homogeneous solution looks like $y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$.

Claim: $Y = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$ is a particular solution to the nonhomogeneous ~~soluti~~ DE.

Proof: $Y'' = \frac{-\omega^2 F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)$.

So $mY'' + kY = \left(\frac{-\omega^2 F_0}{\omega_0^2-\omega^2} + \frac{kF_0}{m(\omega_0^2-\omega^2)}\right)\cos(\omega t) = \frac{F_0}{(\omega_0^2-\omega^2)}\left(-\omega^2 + \frac{k}{m}\right)\cos(\omega t)$

$= F_0\cos(\omega t)$ ✓.

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Hence $y = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$ is a full general solution to the DE.

So any solution to the DE will be the weighted sum of 2 sinusoids with different frequency.

Of particular interest is when ω is close to ω_0, i.e. $|\omega - \omega_0|$ is small: in that case we observe <u>interference</u>.

For simplicity let $y(0) = 0$, $y'(0) = 0$ i.e. the vibrating object starts from rest.

$\implies C_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}$, $C_2 = 0$,

So the solution to the IVP is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[\cos(\omega t) - \cos(\omega_0 t) \right]$$

Cool trick: Let $A = \frac{1}{2}(\omega_0 + \omega)t$ & $B = \frac{1}{2}(\omega_0 - \omega)t$

Then $\omega = A - B$ & $\omega_0 = A + B$.

$$\begin{aligned} \text{So } \cos(\omega t) - \cos(\omega_0 t) &= \cos(A - B) - \cos(A + B) \\ &= \cos A \cos B + \sin A \sin B \\ &\quad - (\cos A \cos B - \sin A \sin B) \\ &= 2 \sin A \sin B \\ &= 2 \sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right) \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right). \end{aligned}$$

Hence we can also write the solution to the IVP as

$$y = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right) \right] \sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right).$$

If ω is very close to ω_0, the $\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$ term will oscillate very slowly compared to the $\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$ term. We can therefore think of the solution function as one which oscillates rapidly with frequency $\frac{1}{2}(\omega_0 + \omega)$, but the amplitude of the oscillation varies slowly in a sinusoidal fashion: amplitude $= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$

Definition 2.6.3: • If a rapidly oscillating function exhibits a periodically varying amplitude, that function is said to possess a beat.

• The ~~Ang~~ radial beat frequency is the frequency (in rad/sec) of the variation of amplitude, i.e. $\frac{1}{2}(\omega_0 - \omega)$ in our case to the left.

• The cyclic beat frequency and beat period are $\frac{\frac{1}{2}(\omega_0-\omega)}{2\pi}$ (in Hertz or cycles per second) and $\frac{2\pi}{\frac{1}{2}(\omega_0-\omega)}$ (in seconds) respectively.

This phenomenon is also known as amplitude modulation in electronics. In acoustics beats are readily noticeable, for example when two side by side instruments play two very slightly out of tune notes.

Example 2.6.4: Solve the IVP $y''+y=\frac{1}{2}\cos(\frac{4}{5}t)$, $y(0)=0$, $y'(0)=0$.

We've already worked down the ~~solutn~~ form of the solution on the previous page. Here we have $F_0=\frac{1}{2}$, $m=1$, $k=1$, $\omega_0=1$, $\omega=\frac{4}{5}$. So $\frac{1}{2}(\omega_0+\omega)=\frac{9}{10}$ & $\frac{1}{2}(\omega_0-\omega)=\frac{1}{10}$,

& $\frac{2F_0}{m(\omega_0^2-\omega^2)} = \frac{2\cdot\frac{1}{2}}{1(1^2-(\frac{4}{5})^2)} = \frac{25}{9} = 2.777\ldots$

So the solution to the IVP is

$$y=\left[\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\tfrac{1}{2}(\omega_0-\omega)t\right)\right]\sin\left(\tfrac{1}{2}(\omega_0+\omega)t\right)$$

or $$y=\left(\tfrac{25}{9}\sin\left(\tfrac{1}{10}t\right)\right)\sin\left(\tfrac{9}{10}t\right)$$

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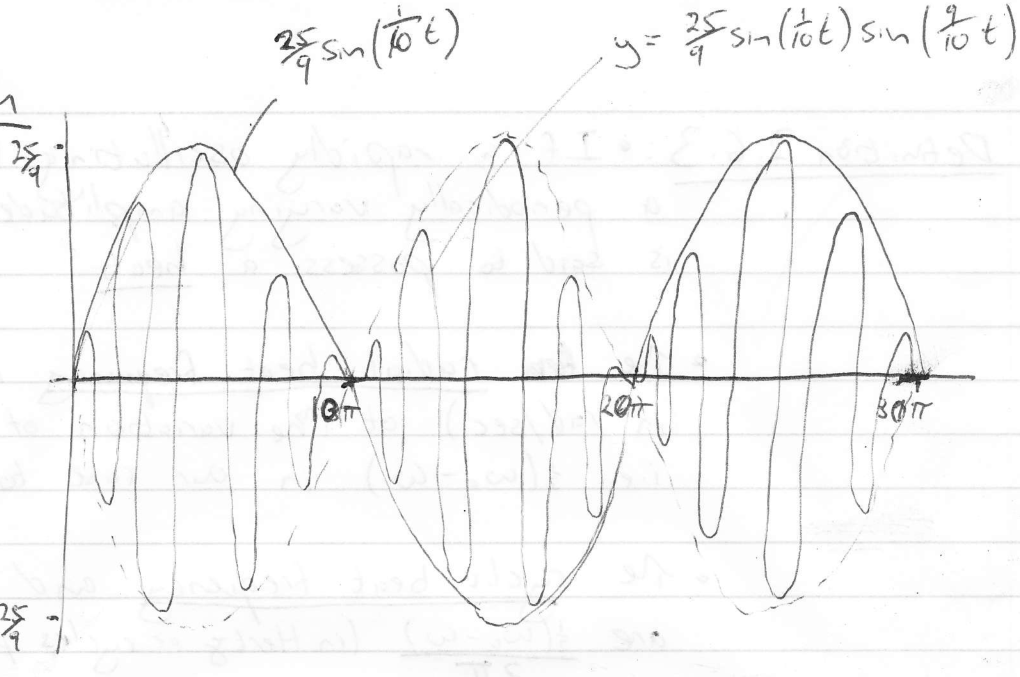
Graph



What happens if ω is increased to 0.9?

$\Rightarrow$

- $\frac{1}{2}(\omega_0+\omega) = \frac{19}{20} = 0.95$
- $\frac{1}{2}(\omega_0-\omega) = \frac{1}{20} = 0.05$
- $\frac{2F_0}{m(\omega_0^2-\omega^2)} = \frac{100}{19} \approx 5.263.$

So if the difference between ω & ω_0 is halved

- the rapid oscillation $\frac{1}{2}(\omega_0+\omega)$ is relatively unchanged
- the beat frequency $\frac{1}{2}(\omega_0-\omega)$ is halved
- the oscillation amplitude $\frac{2F_0}{m(\omega_0^2-\omega^2)}$ about doubles.

As $\omega \to \omega_0$ we get the second case.

2.6.5 Case 2: $\omega = \omega_0$

So now $my''+ky = F_0\cos(\omega_0 t)$;

Since $\cos(\omega_0 t)$ is a solution to the homogeneous solution, we must multiply our particular solution guess by t

The particular solution now becomes $Y = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$

Check: Write $Y = t\cdot f(t)$, where we know $mf''+kf = 0$

then $Y' = f + tf'$

$Y'' = 2f' + tf''$

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So
$$\begin{aligned} mY'' + kY &= m(2f' + tf'') + k\cdot tf \\ &= t(\cancel{mf'' + kf}) + 2mf' \\ &= 2mf' \\ &= 2m \tfrac{d}{dt}\left(\tfrac{F_0}{2m\omega_0}\sin(\omega_0 t)\right) \\ &= F_0\cos(\omega_0 t) \quad \checkmark \end{aligned}$$

So the general solution to the DE is
$$y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

The thing to realise here is that the $t\sin(\omega_0 t)$ term grows in amplitude in time.

__Definition 2.6.6__ The case where $\omega = \omega_0$ is called __resonance__. In this case the forcing function continues to pour energy into the system at just the right frequency. As such the amplitude of the resulting oscillation grows unchecked over time.

[Contrast this with $\omega_0 \neq \omega$, where eventually the solution will oscillate out of phase with the forcing function. The forcing function will then work in opposition to the vibrating object, reducing for a time the energy of the system until the oscillations line up again]

__Example 2.6.7__ $y'' + y = \frac{1}{2}\cos(t)$, $y(0) = 0$, $y'(0) = 0$
i.e. same as before, except $\omega = \omega_0 = 1$

We have $\frac{F_0}{2m\omega_0} = \frac{1}{4}$, so the solution looks like
$$y = c_1\cos(t) + c_2\sin(t) + \frac{1}{4}t\sin(t)$$
The initial conditions give $c_1 = c_2 = 0$, so the solution is
$$y = \frac{1}{4}t\sin(t).$$

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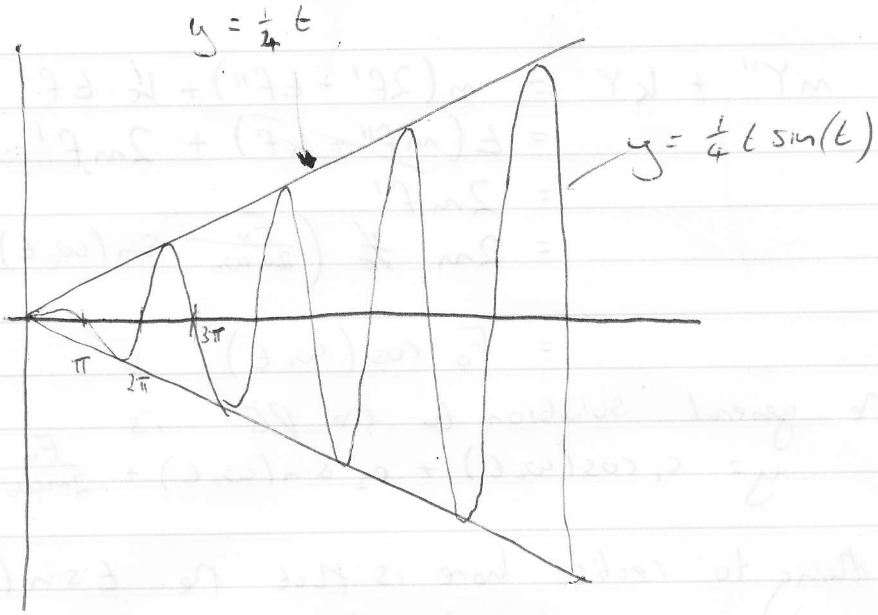
<u>Graph</u>



If you calculate the total energy of the object:

$$E = \underbrace{\tfrac{1}{2}my'^2}_{\text{kinetic}} + \underbrace{\tfrac{1}{2}ky^2}_{\text{spring}}$$

here

$$\begin{aligned}
&= \tfrac{1}{2}\left(\sin(t) + t\cos(t)\right)^2 + \tfrac{1}{2}\left(t\sin(t)\right)^2 \\
&= \tfrac{1}{2}\left[\sin^2(t) + 2t\sin(t)\cos(t) + t^2\cos^2(t)\right] + \tfrac{1}{2}t^2\sin^2(t)\,] \\
&= t^2 + \tfrac{1}{2}\sin^2(t) + t\sin(2t)
\end{aligned}$$

we see that the energy is growing unbounded over time

Note that in reality, of course, the mathematical model will break down before we reach infinity. At some point friction will become significant or the spring response will stop being linear. Thus the case of resonance is only valid for small t, when the amplitude of the oscillations is still relatively small.