Fri 31 Jan MATH 307 A LECTURE 9

CHAPTER 2: 2ND ORDER DE'S

§2.0: INTRO Boyce 3.1

Definition 2.0.1: • A 2nd-order ODE is an equation of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

• If $f(t, y, \frac{dy}{dt})$ is of the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)\cdot y,$$

then the DE is said to be linear;
in this case we can write the linear ODE in standard form:

$$\frac{d^2y}{dt^2} + p(t)\cdot\frac{dy}{dt} + q(t)\cdot y = g(t).$$

Occasionally we'll see linear 2nd-order DEs in the form

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)\cdot y = G(t)$$

• If the 2nd-order ODE is not writeable in one of the above 2 forms, it is called nonlinear.

In general, solving nonlinear 2nd order ODEs is very hard, or impossible to do analytically, and doing so requires a more advanced set of techniques beyond the scope of this course.
In MATH 307 we will restrict ourselves to solving linear 2nd-order ODEs; these, however, comprise a large percentage of the 2nd-order DEs we see in the real world.

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§2.1: Homogeneous Equations with Constant Coefficients

Definition 2.1.1 A 2nd-order linear DE $P(t)y'' + Q(t)y' + R(t)y = G(t)$ is called homogeneous if $G(t) = 0$ for all t.
It is called nonhomogeneous if $G(t) \neq 0$.

We'll consider non-homogeneous equations in a later section.

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The simplest homogeneous equations we can consider are when $P(t)$, $Q(t)$ & $R(t)$ are all constant, i.e.

$$ay'' + by' + cy = 0$$

for constants a, b, c $(a \neq 0)$; these DEs are always solveable in terms of elementary functions of t.

~~Example 2.1.2~~ <u>Note 2.1.2</u>

In general, when solving a 2nd-order DE we'll end up integrating twice; the result is that we introduce 2 independent free constants in the general solution. We therefore need a set of two conditions to specify a particular solution to a 2nd-order DE.

<u>Definition 2.1.3</u> An <u>initial value problem</u> for a 2nd-order DE is a 2nd-order differential equation along with a pair of conditions, usually of the form

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0$$

Sometimes we'll also see ICs in the form

$$y(t_0) = y_0 \quad \text{and} \quad y(t_1) = y_1$$

<u>Example 2.1.4</u> Solve the IVP

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

This is a linear homogenous 2nd-order ODE with constant coefficients. We have $y'' = y$; 2 functions that satisfy this DE immediately spring to mind:

$$y = e^{t} \quad \underline{\text{and}} \quad y = e^{-t}$$

Note, however, that any multiple of e^{t} also satisfies the DE, as does any multiple of e^{-t}.
Now note that we can add e^{t} to e^{-t} and get another solution: $y = e^{t} + e^{-t}$ obeys the DE.

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In fact, any multiple of e^t can be added to any multiple of e^{-t}, and we still have a solution to $y''-y=0$.

Thus a general solution to $y''-y=0$ is
$$y = c_1e^t + c_2e^{-t}$$
Now we can apply ICs:
$$y(0)=2 \Rightarrow 2 = c_1 + c_2$$
And $y'(t) = c_1e^t - c_2e^{-t}$, so $y'(0) = -1$
$$\Rightarrow -1 = c_1 - c_2$$
Solving simultaneously for c_1 & c_2 yields
$$c_1 = \tfrac{1}{2}, \quad c_2 = \tfrac{3}{2}$$
So we get
$$y = \tfrac{1}{2}e^t + \tfrac{3}{2}e^{-t}$$ □

How does we solve the general case
$$ay'' + by' + cy = 0 \; ?$$

Tactic 3.1.5: "Guess" a solution $y = e^{rt}$ for some r.
Then $y' = re^{rt}$ & $y'' = r^2e^{rt}$,
So $ay''+by'+cy = ar^2e^{rt} + bre^{rt} + ce^{rt}$
$$= (ar^2+br+c)e^{rt}$$
Now for this to $=0$, since e^{rt} is never zero, we must have $ar^2+br+c=0$

Definition 3.1.6 The equation $ar^2+br+c=0$ is called the characteristic equation of the DE

So if r is a number such that $ar^2+br+c=0$, then $y=e^{rt}$ will satisfy the DE.

Note 3.1.7 $ar^2+br+c=0$ is a quadratic equation, which may have 2 real but different roots, 2 real & equal roots, or 2 complex but different roots

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The behaviour of the solutions to the DE turn out to depend heavily on what type of roots the characteristic polynomial have.

Today we'll consider the case where $ar^2+br+c=0$ has real & different roots; the other cases we consider in later sections.

So let r_1 & r_2 be the real roots to $ar^2+br+c=0$, with $r_1 \neq r_2$. Note that $y=e^{r_1t}$ and $y=e^{r_2t}$ are both solutions to $ay''+by'+cy=0$.
In fact, so is any linear combination of e^{r_1t} & e^{r_2t}, i.e. $y = c_1e^{r_1t} + c_2e^{r_2t}$ is a general solution to the DE. Note that we can show that this is the full general solution to the DE.

Exercise:
If you then have ICs $y(t_0)=y_0$, $y'(t_0)=y_0'$, show that the unique solution to the IVP is
$y = c_1e^{r_1t}+c_2e^{r_2t}$, where $c_1 = \frac{y_0'-y_0r_2}{r_1-r_2}e^{-r_1t_0}$, $c_2 = \frac{y_0r_1-y_0'}{r_1-r_2}e^{-r_2t_0}$

Example 2.1.8 Find the General solution to $y''+5y'+6y=0$.
$\Rightarrow$ Suppose $y=e^{rt}$ solves the DE
Then $0=y''+5y'+6y = r^2e^{rt}+5re^{rt}+6e^{rt} = (r^2+5r+6)e^{rt}$,
so $r^2+5r+6=0$
Thus $r=-2$ or $r=-3$.
The general solution is thus
$$y = c_1e^{-2t}+c_2e^{-3t}$$

Example 2.1.9: Solve the IVP: $y''+5y'+6y=0$, $y(0)=2$, $y'(0)=3$
We have that $y=c_1e^{-2t}+c_2e^{-3t}$
Setting $y(0)=2 \Rightarrow 2=c_1+c_2$
And $y'(0)=3 \Rightarrow 3=-2c_1-3c_2$
Solving gets $c_1=9$, $c_2=-7$.
Hence the solution is $y = 9e^{-2t}-7e^{-3t}$.

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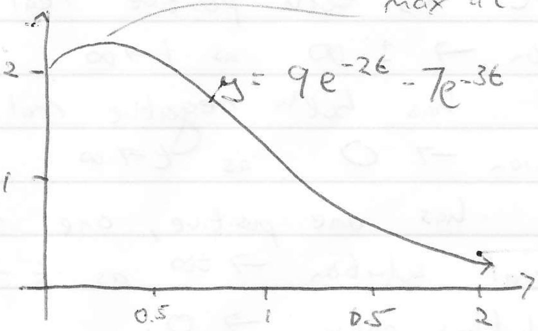
Plot of the solution:

max at $t = \ln(\frac{7}{6}) = 0.15415\ldots$

$y = \frac{108}{49} = 2.20408$



<u>Example 2.1.10</u> Find the solution to $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$.
And compute the t for which it is a max.

get $4r^2 - 8r + 3 = 0$

$\Rightarrow r = \frac{3}{2}$ or $r = \frac{1}{2}$

So $y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$

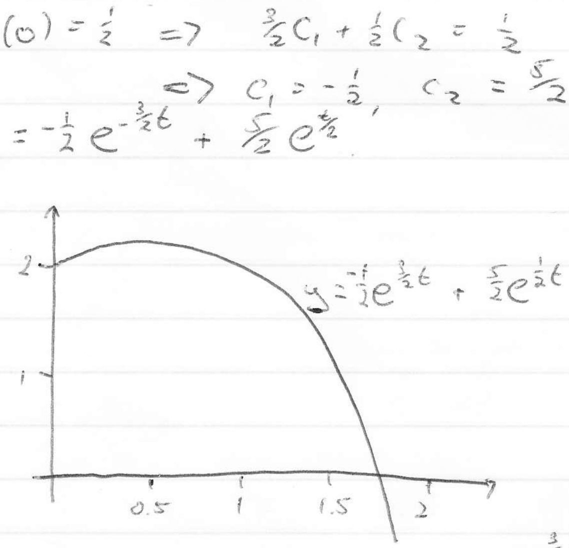
ICs: $y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = \frac{1}{2} \Rightarrow \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$

$\Rightarrow c_1 = -\frac{1}{2}, \quad c_2 = \frac{5}{2}$.

So $y = -\frac{1}{2}e^{-\frac{3}{2}t} + \frac{5}{2}e^{\frac{t}{2}}$

Plot



Max occurs when $y' = 0 \Rightarrow -\frac{3}{4}e^{\frac{3}{2}t} + \frac{5}{4}e^{\frac{1}{2}t} = 0$

$\Rightarrow 5e^{\frac{1}{2}t} = 3e^{\frac{3}{2}t}$

$\Rightarrow e^t = \frac{5}{3} \Rightarrow t = \ln(\frac{5}{3})$

So max y is $-\frac{1}{2}(\frac{5}{3})^{\frac{3}{2}} + \frac{5}{2}(\frac{5}{3})^{\frac{1}{2}}$.

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Summary: When ~~ay''~~ considering the DE $ay'' + by' + cy = 0$, a, b, c constant, the general solution can have the following behaviour:

- If $ar^2 + br + c$ has both positive real roots, then solution $\to \pm\infty$ as $t \to \infty$
- If " " has both negative real roots, then solution $\to 0$ as $t \to \infty$
- If " " has one positive, one negative root, then in general solution $\to \pm\infty$ as $t \to \infty$; however some specific solutions may $\to 0$. (w/ the other $\to 0$)
- If one root of characteristic equation $= 0$, then solution $\to$ a constant

(If other root is positive, then solution $\to \pm\infty$ in general).