Homework 4

Total: 20 points
Due: Wed 29 October 2014 at the beginning of class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **THE WRONSKIAN**

   (a) For the following two initial value problems, determine the longest interval in which a unique twice-differentiable solution is guaranteed to exist. Do not attempt to find the solution.
      i. $(t-1)y'' - 3ty' + 4y = \sin(t), \quad y(-2) = 2, \quad y'(-2) = 1$ Boyce 3.2 Q8
      ii. $(t-3)y'' + ty' + (\ln|t|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$ Boyce 3.2 Q11

   (b) In each of the following, find the Wronskian of the given pair of functions. Be sure to simplify your solution if possible.
      i. $\cos(t), \quad \sin(t)$ Boyce 3.2 Q2
      ii. $e^{-2t}, \quad te^{-2t}$ Boyce 3.2 Q3
      iii. $e^t \sin(t), \quad e^t \cos(t)$ Boyce 3.2 Q5

   (c) Verify that the functions y_1 and y_2 are solutions of the following differential equation, and ascertain whether they constitute a fundamental set of solutions for the DE: Boyce 3.2 Q26

   $$t^2y'' - t(t+2)y' + (t+2)y = 0, \ \ t > 0, \quad y_1(t) = t, \ \ y_2(t) = te^t$$

   (d) If the Wronskian W of f and g is t^2e^t and $f(t) = t$, find $g(t)$. Boyce 3.2 Q18

2. **CHARACTERISTIC EQUATIONS WITH COMPLEX ROOTS**

   (a) For the following pairs of complex numbers z_1 and z_2, compute $z_1 + z_2$, $z_1 - z_2$, $z_1 \times z_2$ and $\frac{z_1}{z_2}$. Write your answers in the form $a + ib$, where a and b are real numbers; simplify your answers if possible.
      i. $z_1 = 1 + i, \quad z_2 = 1 - i$
      ii. $z_1 = 3 - 4i, \quad z_2 = 7 + 24i$
      iii. $z_1 = \frac{1}{2} + 2i, \quad z_2 = -1 + \frac{1}{3}i$

   (b) Use Euler's Formula: $e^{ix} = \cos(x) + i\sin(x)$ to rewrite the following complex numbers in the form $a + ib$.
      i. e^{1+2i} Boyce 3.3 Q1
      ii. $e^{2-\frac{\pi}{2}i}$ Boyce 3.3 Q4
      iii. 2^{1-i} Boyce 3.3 Q5

   (c) Give the general solution to the following differential equations:
      i. $y'' - 2y' + 2y = 0$ Boyce 3.3 Q7
      ii. $y'' + 6y' + 13y = 0$ Boyce 3.3 Q11
      iii. $y'' + 4y' + 6.25y = 0$ Boyce 3.3 Q16

[More questions on the next page!]

(d) Find the solution to each of the following initial value problems. Sketch the graph of the solution and describe its behaviour for increasing t:

i. $y'' + 4y = 0, \quad y(0) = 0, y'(0) = 1$ — Boyce 3.3 Q17

ii. $y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0, y'(\frac{\pi}{2}) = 2$ — Boyce 3.3 Q19

iii. $y'' + 2y' + 2y = 0, \quad y(\frac{\pi}{4}) = 2, y'(\frac{\pi}{4}) = -2$ — Boyce 3.3 Q22

(e) Boyce 3.3 Q34 & 42

An equation of the form

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \qquad t > 0,$$

where α and β are real constants, is called an Euler equation.

i. Let $x = \ln(t)$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

ii. Use the result of part (a) to transform the above differential equation to

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

Observe that this equation has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to this equation, then $y_1(\ln(t))$ and $y_2(\ln(t))$ form a fundamental set of solutions to the original Euler equation.

iii. Use parts (a) and (b) to find the general solution to

$$t^2y'' + 7ty' + 10y = 0.$$