Exercise: Solve $y'' - 29y' + 100y = 0$ $\quad y(0) = 2$, $y'(0) = 29$

(A: $y(t) = e^{4t} + e^{25t}$.

Example: ~~So~~ Find the general solution to $y'' + y' + y = 0$.

Char. eqn: $r^2 + r + 1 = 0$

$$\rightarrow r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

What does $e^{\frac{-1}{2} + i\frac{\sqrt{3}}{2}}$ even mean??

Complex Numbers.

Defn: A complex number z is given by a pair of real numbers x & y and is written $z = x + iy$ w/ $i^2 = -1$.

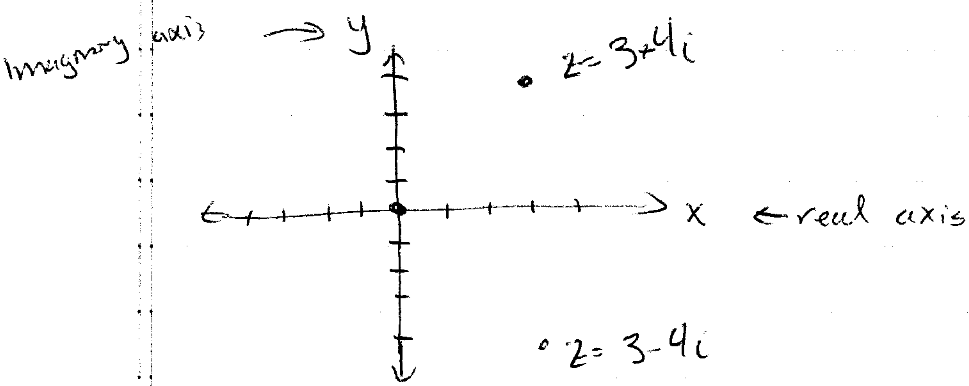
Real part of z

$\boxed{\text{BOTH REAL NUMBERS}}$ ← Imaginary part of z

Ex] $1 \leftrightarrow 1 + 0 \cdot i$

$i \leftrightarrow 0 + 1 \cdot i$

Complex Plane



+/− : Just like ~~addition of~~ add/sub of vectors:

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2).$$

× : $\underbrace{(x_1+iy_1)(x_2+iy_2)}_{\text{Foil!}} = x_1x_2 + iy_1x_2 + iy_2x_1 + \overset{-1}{i^2}y_1y_2$

$$= x_1x_2 - y_1y_2 + i(x_2y_1 + x_1y_2).$$

÷ : Wait.

Defn: If θ is a real number, then

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Prop: $e^{i\theta_1+i\theta_2} = e^{i\theta_1}e^{i\theta_2}$

Pf] $e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= \underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\underbrace{(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)}_{\sin(\theta_1+\theta_2)}$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= e^{i(\theta_1+\theta_2)}.$$

$\boxed{/\!/}$

So this is like usual exponent rules...

Constant w.r.t θ

Note: $\frac{d}{d\theta} e^{i\theta} = \frac{d}{d\theta}(\cos\theta + i\sin\theta)$

$$= -\sin\theta + i\cos\theta$$
$$= i^2\sin\theta + i\cos\theta$$ sneaky!
$$= i(\cos\theta + i\sin\theta)$$
$$= ie^{i\theta}$$

So $e^{i\theta}$ does really behave how we think it should...

Ex| $e^{\frac{\pi}{2}i} = i$

$e^{\pi i} = -1$ ← Amazeballs!

$e^{\frac{3\pi}{2}i} = -i$

$e^{2\pi i} = 1$

Polar Coordinates:

$(x,y) \longleftrightarrow (r,\theta)$

$x = r\cos\theta$ $\quad$ $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\quad$ $\theta = \arctan y/x$

Say $z = x+iy = r\cos\theta + ir\sin\theta$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$ ← more than one θ can work.

Ex| $z = -4+4i \rightarrow r = \sqrt{16+16} = 4\sqrt{2}$

$\theta = \arctan(-1) \implies \theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

B/c $x<0$.

So $-4+4i = 4\sqrt{2}\, e^{\frac{3\pi i}{4}} = 4\sqrt{2}\, e^{\frac{11\pi i}{4}} = \ldots$

With Polar Coordinates, $\times$ + $\div$ are easy!

$\underline{\times}$: $z_1 \times z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$

$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$\div$: $\frac{z_1}{z_2}$ $(z_2 \neq 0)$ $= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

<u>De Moivre's Thm</u>: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

<u>Pf</u> $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

$$= e^{in\theta}$$

$$= \cos n\theta + i\sin n\theta.$$ $\boxtimes$

<u>Finally</u>: $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$.

<u>Now</u>: Back to:

Find the general solution to $y'' + y' + y = 0$.

<u>Chr eqn</u>: $r^2 + r + 1 = 0$

$$r_{1,2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

So $z_1(t) = e^{r_1 t} = e^{(-\frac{1}{2}+i\frac{\sqrt{3}}{2})t} = e^{-\frac{1}{2}t} e^{i\frac{\sqrt{3}}{2}t}$

$$= e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right)$$

$z_2(t) = e^{r_2 t} = e^{-\frac{1}{2}t}\left[\cos\left(-\frac{\sqrt{3}}{2}t\right) + i\sin\left(-\frac{\sqrt{3}}{2}t\right)\right]$

$$= e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t\right)$$

↑ even ↑ odd

But... $z_1(t)$ & $z_2(t)$ are not real valued. ☹.

By the Principle of Superposition, we know

$$z_1(t) + z_2(t) = 2e^{-\frac{1}{2}t} \cos \tfrac{\sqrt{3}}{2}t$$

AND

$$z_1(t) - z_2(t) = 2i\, e^{-\frac{1}{2}t} \sin \tfrac{\sqrt{3}}{2}t$$

are solutions. But multiplying by a scalar also preserves solutions so

$$y_1(t) = \frac{z_1(t) + z_2(t)}{2} = e^{-\frac{1}{2}t} \cos \tfrac{\sqrt{3}}{2} t$$

$$y_2(t) = \frac{z_1(t) + z_2(t)}{2i} = e^{-\frac{1}{2}t} \sin \tfrac{\sqrt{3}}{2} t$$

are solutions too!

<u>Exercise:</u> ~~Use the Wronskin to check~~ Check this & use Wronskin to show $\{y_1, y_2\}$ is a fundamental set of solutions.

<u>In General:</u> If $ar^2 + br + c = 0$ has complex conjugate roots $r_{1,2} = \lambda \pm i\mu$, then

$$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$$

is a general solution to

$$ay'' + by' + cy = 0.$$

Exercise: Find general solution to $y'' - 2y' + 2y = 0$.

Answer: $y = C_1 e^t \cos t + C_2 e^t \sin t$

Exercise: Solve $y'' + 36y = 0$ $y(0) = 2$ $y'(0) = 12$

Answer: $y = 2(\cos 6t + \sin 6t)$