Wed 22 Jan MATH307 LECTURE 7

§1.6 : AUTONOMOUS EQUATIONS

Definition 1.6.1: An autonomous differential equation is a first-order DE that can be written in the form

$$\frac{dy}{dt} = f(y)$$

i.e. where there is no explicit time-dependence in f.

Note that autonomous equations are separable, so we can always at least implicitly solve them:

$$\int \frac{1}{f(y)}\, dy = t + C$$

Example 1.6.2: Exponential growth

Let $y(t)$ measure the ~~population~~ size of a population of organisms. When y is small compared the amount food in its environment and conditions are ideal, it is reasonable to expect that y grows proportional to the size of y,

i.e. $$\frac{dy}{dt} = r \cdot y \quad , \quad r > 0,\ y \geq 0$$

IC $y(0) = y_0$

where r is the growth rate

Solving $\frac{1}{y}\, dy = r\, dt$

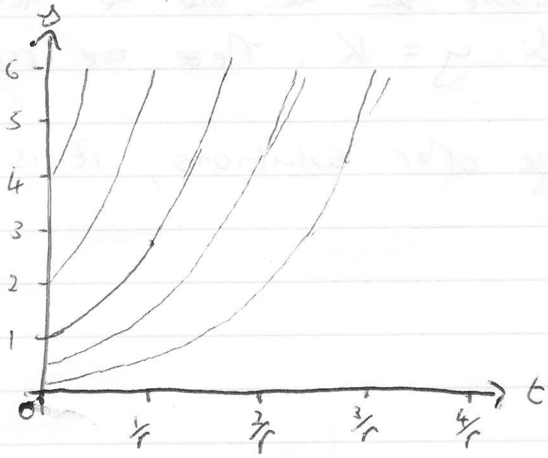
$\Rightarrow \ln(y) = rt + c$

$y = Ae^{rt}$

using $y(0) = y_0$

$\Rightarrow y = y_0 e^{rt}$

Picture:



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Things can get more difficult if we cannot explicitly evaluate the integral $\int \frac{1}{f(y)} dy$. In that case, autonomous equations allow for some powerful qualitative methods to analyse the behaviour of solutions to the DE, without having to solve the DE explicitly.

<u>Example 1.6.3.</u> <u>The logistic Equation</u>

When a population grows, one cannot expect ideal conditions to last forever. Eventually competition for space & food will slow down growth. A better DE for population growth should then be

$$\frac{dy}{dt} = h(y) \cdot y,$$

where $h \approx r$ when y is small, h should be decreasing as ~~y~~ y increases, and $h<0$ when y is sufficiently large. Easiest such function:

$$h(y) = ~~ar~~ \; r - ay$$

for some $a > 0$.

It's actually easier to have it in the form

$$h(y) = r\left(1 - \frac{y}{K}\right), \quad K > 0$$

because then we know $\frac{dy}{dt} = 0$ when $y = K$.

So then we have $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y$, $\quad r>0, K>0, \; y \geq 0, \; y(0) = y_0$

From the above DE we see we have 2 constant solutions: $y = 0$ & $y = K$. These are called <u>equilibrium solutions</u>

To visualize other solutions, it is helpful to plot $f(y)$:

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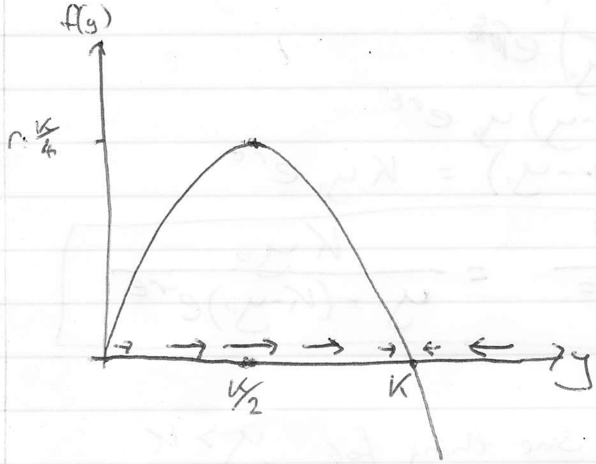
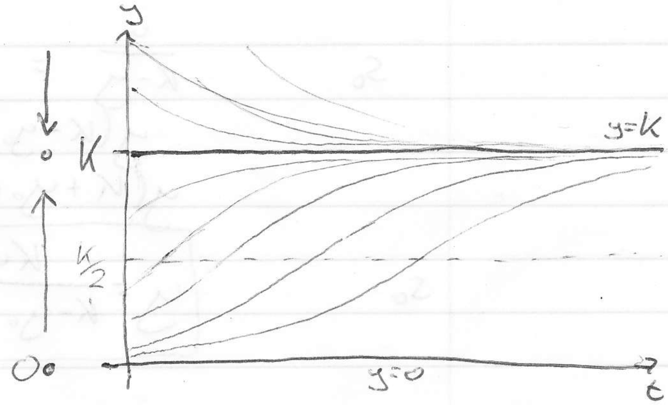
Autonomous Equations, cont...





We see then that any nonzero solution will approach $y=K$ over time.
$y=K$ is called the <u>stable solution</u>, while $y=0$ is <u>unstable</u>

- K is known as the <u>saturation level</u> or <u>environmental carrying capacity</u>

<u>Note 1.6.4</u>: The uniqueness theorem for nonlinear ODEs applies, since both f & $\frac{\partial f}{\partial y}$ are continuous for all t & y. Hence no two solutions will cross; therefore any solution starting below $y=K$ will remain so.

<u>Note 1.6.5</u>: We can actually solve the logistic equation to confirm our analysis:

$$\frac{1}{(1-\frac{y}{K})y}\,dy = r\,dt$$

Partial fractions: $$\frac{1}{y} + \frac{\frac{1}{K}}{1-\frac{y}{K}}\,dy = r\,dt$$

$$\Rightarrow \quad \ln|y| - \ln|K-y| = rt + C$$

Case: $0<y<K$

$$\Rightarrow \quad \ln\left(\frac{y}{K-y}\right) = rt + C$$

$$\Rightarrow \quad \frac{y}{K-y} = Ae^{rt}$$

IC: $t=0$, $y=y_0$ $\quad \frac{y}{K-y_0} = A \Rightarrow A=$

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So $$\frac{y}{K-y} = \left(\frac{y_0}{K-y_0}\right) e^{rt}$$

$$y(K-y_0) = (K-y)\, y_0 e^{rt}$$

$$y(K + y_0 e^{rt} - y_0) = K y_0 e^{rt}$$

So $$\boxed{y = \frac{K y_0 e^{rt}}{K - y_0 + y_0 e^{rt}} = \frac{K y_0}{y_0 + (K-y_0)e^{-rt}}}$$

<u>Exercise:</u> Show you get the same thing for $y > K$.

<u>Example:</u> A bacterial colony is initialised on a petri dish. The colony initially has a mass of 0.001 g, and is observed to ~~grow at a rate~~ double in mass in 3 hours.
A second colony started a while back under the same conditions is observed to be stable at 0.12 g mass.
When will the new colony reach 0.1 g mass?

Here $K = 0.12$

$y_0 = 0.001$

$r = ? \quad \Rightarrow e^{r\cdot 3} = 2 \Rightarrow r = \frac{1}{3}\log 2$

So $$\frac{y(t)}{K} = \frac{y_0}{y_0 + (K-y_0)\cdot 2^{-t/3}}$$

$$\Rightarrow \frac{5}{6} = \frac{0.001}{0.001 + (0.119)\cdot 2^{-t/3}} = \frac{1}{1 + 119\cdot 2^{-t/3}}$$

$$\Rightarrow 5 + 595\cdot 2^{-t/3} = 6$$

$$2^{-t/3} = \frac{1}{595}$$

$$2^{t} = 595^3$$

$$t = \frac{3\ln(595)}{\ln(2)} = 27.65$$

So the new colony reaches 0.1 g after 27.65 hours. $\square$