§2.6 - FORCED VIBRATIONS, PART 2 (BOYCE 3.8)

2.6.8 - Forced Vibrations With Damping

Here we consider a vibrating system with nontrivial damping and an oscillating forcing function, i.e.

$$my'' + \gamma y' + ky = F_0 \cos(\omega t), \quad m, \gamma, k > 0$$

All solutions to this DE will be of the following form:

$$y = u(t) + R\cos(\omega t - \delta),$$

where

- $u(t)$ is the solution to the homogeneous part of the DE, i.e. $mu'' + \gamma u' + ku = 0$.
  We know because m, γ & $k > 0$, that $u(t)$ decays exponentially over time with decay constant $\sim -\gamma/2m$.
- $Y(t) = R\cos(\omega t - \delta)$ is the particular solution to the full inhomogeneous DE.

Furthermore, note that $u(t)$ depends on the ICs, however, $Y(t)$ is the same regardless of the ICs.

Definition 2.6.9:
- $u(t)$ since it dies out over time, is known as the transient solution. It is often of little importance, as it may be undetectable after a short time.
- $Y(t) = R\cos(\omega t - \delta)$ is the steady-state solution or the forced response.

Example 2.6.10: Find the solution to the spring system described by

$$y'' + y' + \tfrac{5}{4} y = \tfrac{17}{8}\cos(t), \quad y(0) = 4, \; y'(0) = 0.$$

Solution: CE is $r^2 + r + \frac{5}{4} = 0$ which has roots $r = -\frac{1}{2} \pm i$

So general solution to homogeneous DE is $y = e^{-\frac{1}{2}t}(c_1\cos(t) + c_2\sin(t))$.

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Particular solution to the Nonhomogeneous DE:

guess $Y = A\cos(t) + B\sin(t)$

$\Rightarrow Y' = B\cos(t) - A\sin(t)$

$Y'' = -A\cos(t) - B\sin(t)$

So $Y'' + Y' + \frac{5}{4}Y = \left(\frac{1}{4}A + B\right)\cos(t) + \left(-A + \frac{1}{4}B\right)\sin(t)$

$= \frac{17}{8}\cos(t) + 0\sin(t)$,

so $\frac{1}{4}A + B = \frac{17}{8}$, $-A + \frac{1}{4}B = 0$

which has solution $A = \frac{1}{2}$, $B = 2$.

So $y = e^{-\frac{1}{2}t}\left(c_1\cos(t) + c_2\sin(t)\right) + \frac{1}{2}\cos(t) + 2\sin(t)$.

Apply ICs: $y(0) = 4 \Rightarrow c_1 = \frac{7}{2}$

$y' = e^{-\frac{1}{2}t}\left(\left(-\frac{1}{2}c_1 + c_2\right)\cos(t) + \left(-\frac{1}{2}c_2 - c_1\right)\sin(t)\right) - \frac{1}{2}\sin(t) + 2\cos(t)$

So $y'(0) = 0 \Rightarrow 0 = -\frac{1}{2}c_1 + c_2 + 2$

$\Rightarrow c_2 = -\frac{1}{4}$

So the solution is $y = e^{-\frac{t}{2}}\left(\frac{7}{2}\cos(t) - \frac{1}{4}\sin(t)\right) + \frac{1}{2}\cos(t) + 2\sin(t)$

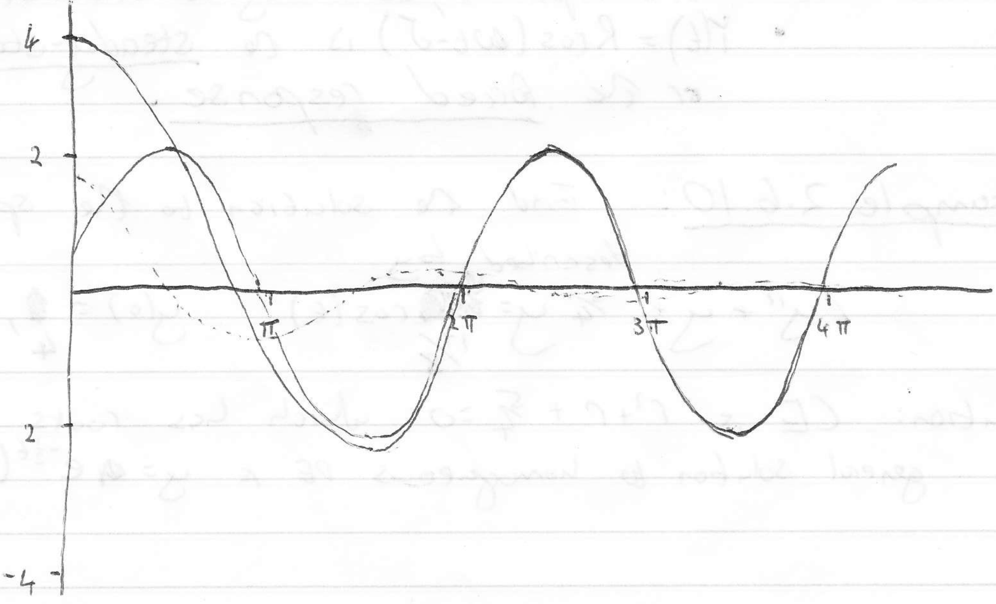
But we can write $\frac{1}{2}\cos(t) + 2\sin(t) = R\cos(t - \delta)$ using

$R^2 = 2^2 + \frac{1}{4} \Rightarrow R = \frac{\sqrt{17}}{2} \approx 2.062$

and $\tan(\delta) = 4 \Rightarrow \delta = 1.3258$

So $y = e^{-\frac{t}{2}}\left(\frac{7}{2}\cos(t) - \frac{1}{4}\sin(t)\right) + 2.062\cos(t - 1.3258)$.

Graph:



So we see that the solution decays rapidly to ~~that~~ $Y(t) = \sim 2.062\cos(t-1.3258)$. Beyond $\sim t = 2\pi$ the difference between the true solution & $Y(t)$ is negligible.

It therefore makes sense to investigate the forced response more carefully. Specifically we are interested in the amplitude R & phase shift δ of the forced response as a function of the forcing frequency ω & the damping constant γ.

General Case 2.6.11: $\underbrace{my''+\gamma y'+ky}_{Y(t)} = F_0\cos(\omega t)$.

Recall that the forced response doesn't depend on ICs. Let $\omega_0 = \sqrt{\frac{k}{m}}$ be the natural frequency of the undamped, unforced frequency.

We seek the particular solution $Y(t)$.

Guess: $Y = A\cos(\omega t) + B\sin(\omega t)$

$Y' = \omega B\cos(\omega t) - \omega A\sin(\omega t)$

$Y'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$.

$$\begin{aligned} \text{So } mY''+\gamma Y'+kY &= (-m\omega^2 A)\cos(\omega t) + (-m\omega^2 B)\sin(\omega t) \\ &+ (\gamma\omega B)\cos(\omega t) + (-\gamma\omega A)\sin(\omega t) \\ &+ (kA)\cos(\omega t) + (kB)\sin(\omega t). \\ &= (F_0)\cos(\omega t). \end{aligned}$$

So we have

$$-m\omega^2 A + \gamma\omega B + kA = F_0, \quad -m\omega^2 B - \gamma\omega A + kB = 0$$

Solving for A & B is tedious, but we can do it. The result is

$$A = \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+\gamma^2\omega^2}, \quad B = \frac{F_0\gamma\omega}{(k-m\omega^2)^2+\gamma^2\omega^2}$$

This is clearly not particularly enlightening, which is why we put the forced response in the $R\cos(\omega t-\delta)$ form.

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Now $R^2 = A^2 + B^2$, so $R = \dfrac{F_0}{\sqrt{(k-m\omega^2)^2 + \gamma^2\omega^2}}$

& $\tan\delta = B/A$, so $\delta = \arctan^*\left(\dfrac{\gamma\omega}{k-m\omega^2}\right)$,

where ~~we~~ must rember to pick δ in the correct quadrat according to the signs of A & B.

This is better, but it may still be opaque as to how R & δ behave as we vary ω, for example. Let's look at some specific cases.

2.6.12 Case 1: $\omega \to 0$

This corresponds to a ~~constant~~ forcing function with a very long period.
We see that $R \to F_0/k$ (which is exactly the amplitude of the response for a constant forcing function).
And $\delta \to 0$.

The interpretation of this is that for forcing ~~periods~~ functions with long periods, the response will be in phase with the forcing function, with amplitude $\frac{1}{k}$ that of the amplitude ~~of~~ the forcing function.
This makes sense: we are in the case where the system is essentially in a slowly changing equilibrium, where the spring force balances out the force imparted by the forcing function.

2.6.13 Case 2: $\omega \to \infty$

i.e. forcing function oscillating much more rapidly than ω_0.

We see $R \sim \dfrac{F_0}{m\omega^2} \to 0$ as $\omega \to \infty$,
while $\delta \sim$ ~~[illegible]~~ $\arctan^*(-\gamma/m\omega) \to -\frac{\pi}{2}$ as $\omega \to \infty$.
Thus the forced response goes to zero for large ω, while $\delta \sim -\frac{\pi}{2}$, implies that the response will be out of phase with the forcing function

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The most interesting case is, for $\omega \sim \omega_0$. Specifically, we can ask: when is R maximized?

2.6.14 <u>Case 3:</u> ω about the same size as ω_0

R is maximized w.r.t ω when the denominator in $\frac{F_0}{\sqrt{(k-m\omega^2)^2+\gamma^2\omega^2}}$ is minimized i.e. when $\frac{d}{d\omega}\left[(k-m\omega^2)^2 + \gamma^2\omega^2 \right] = 0$

$$\Rightarrow 2(k-m\omega^2)\cdot -2m\omega + 2\gamma^2\omega = 0$$

So $\omega = 0$ or

$$-2m(k-m\omega^2) = -\gamma^2$$
$$-k+m\omega^2 = \frac{-\gamma^2}{2m}$$
$$m\omega^2 = k - \frac{\gamma^2}{2m}$$
$$\omega = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$
$$= \omega_0\sqrt{1-\frac{\gamma^2}{2mk}}$$

Definition 2.6.15

Let $\Gamma = \frac{\gamma^2}{mk}$.

Γ is a dimensionless constant that governs the behaviour of the forced response.

We see then that the maximum amplitude response occurs at

$$\omega_{max} = \omega_0\sqrt{1-\Gamma/2}$$
$$\simeq \omega_0(1-\Gamma/4) \text{ for small } \Gamma.$$

Furthermore, the maximum response is then

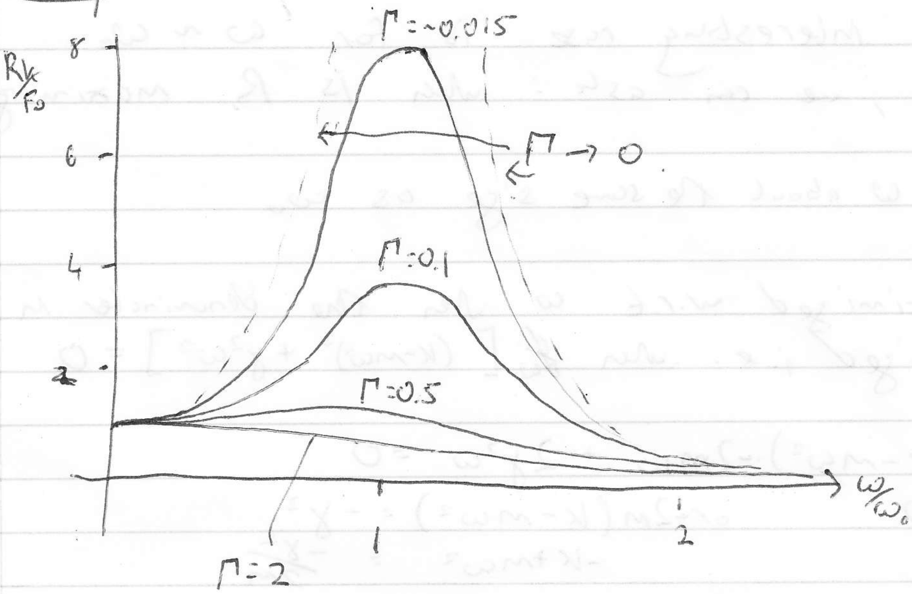
$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4km}}} = \frac{F_0}{\gamma\omega_0}\cdot\frac{1}{\sqrt{1-\Gamma/4}} \simeq \frac{F_0}{\gamma\omega_0}(1+\Gamma/8)$$

for small Γ.

To show this graphically, it is informative to look at Rk/F_0 vs ω/ω_0. Both are dimensionless quantities, and one can show

$$\frac{Rk}{F_0} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2}} \qquad \text{PTO}$$

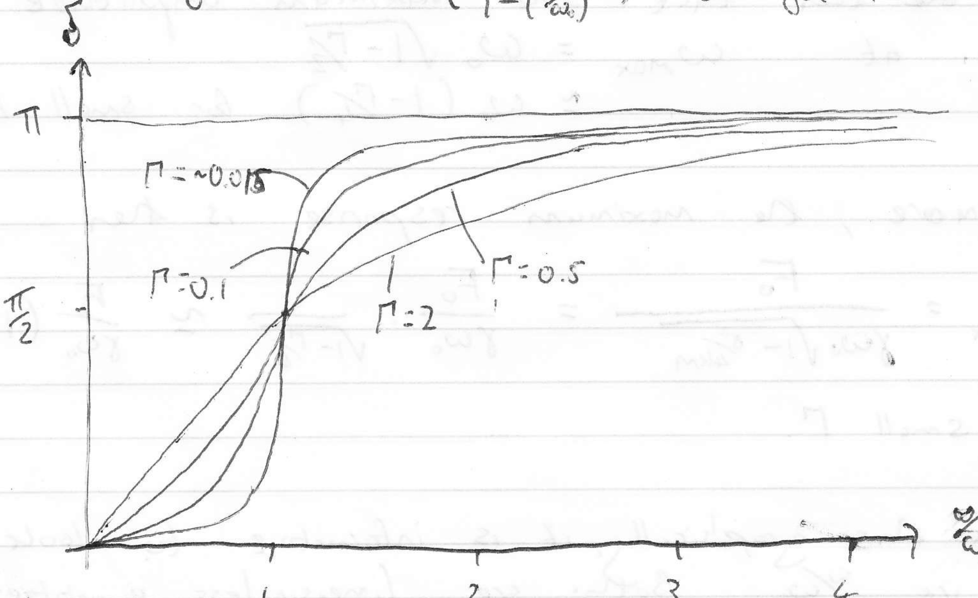
Graph



It's clear then that we get resonance (vs. F_0/k) for ω around ω_0; the strength of that resonance is a function of the quantity $\Gamma = \frac{\gamma^2}{mk}$

Furthermore, we can show

$$\delta = \arctan^*\left(\frac{\sqrt{\Gamma}\cdot(\omega/\omega_0)}{1-(\omega/\omega_0)^2}\right)$$ to get the following graph



Thus the switch from $\delta \sim 0$ to $\delta \sim \pi$ get more rapid as $\Gamma \to 0$.