1. (8 points) A mass of 100 kg stretches a (very big) spring by 2 m. Suppose that the mass is in a medium that exerts a viscous resistance of 100 N when the mass has a velocity of $\frac{1}{2}$ m/s. An external force of $2500t + 1000\cos(t)$ N is applied to the mass. Suppose that, initially, the mass is in its equilibrium position, and that its initial velocity is zero.

   Find the position of the mass for $t > 0$. You can take the standard gravity to be $g = 10\,\text{m/s}^2$.

The setup tells us that $m = 100$, $L = 2$, so since $kL - mg = 0$,

$$k = \frac{mg}{L} = \frac{100 \cdot 10}{2} = 500.$$

Next, $100 = F_d = \gamma|u'| = \gamma\frac{1}{2}$, so $\gamma = 200$. Finally, $F_e = 2500t + 1000\cos(t)$, so the position of the mass satisfies

$$100u'' + 200u' + 500u = 2500t + 1000\cos(t),$$

or

$$u'' + 2u' + 5u = 25t + 10\cos(t).$$

The initial conditions are $u(0) = u'(0) = 0$.

The characteristic polynomial associated to the homogeneous solution is $r^2 + 2r + 5$ which has roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 2 \cdot 1}}{2} = -1 \pm 2i.$$

Hence, the homogeneous equation has solution

$$u_h(t) = e^{-t}\left(A\cos(2t) + B\sin(2t)\right).$$

To determine a particular solution to the inhomogeneous equation, we treat the terms $25t$ and $10\cos(t)$ separately. For the polynomial term, we guess $u_{c_1} = At + B$. Then $u'_{c_1} = A$ and $u''_{c_1} = 0$, so

$$25t = u''_{c_1} + 2u'_{c_1} + 5u_{c_1} = 2A + 5(At + B) = 5At + (2A + 5B).$$

Hence, $25 = 5A$ and $0 = 2A + 5B$, which implies $A = 5$ and $B = -2$, so $u_{c_1} = 5t - 2$.

For the trigonometric term, we guess $u_{c_2} = A\cos(t) + B\sin(t)$. Then $u'_{c_2} = -A\sin(t) + B\cos(t)$ and $u''_{c_2} = -A\cos(t) - B\sin(t)$, and

$$\begin{aligned}
10\cos(t) &= u''_{c_2} + 2u'_{c_2} + 5u_{c_2} \\
&= (-A\cos(t) - B\sin(t)) + 2\left(-A\sin(t) + B\cos(t)\right) + 5\left(A\cos(t) + B\sin(t)\right) \\
&= (4A + 2B)\cos(t) + (4B - 2A)\sin(t).
\end{aligned}$$

Thus $4A + 2B = 10$ and $4B - 2A = 0$ which is easily solved for $B = 1$ and $A = 2$, so that $u_{c_2} = 2\cos(t) + \sin(t)$. Therefore the general solution to the inhomogeneous solution is given by

$$u(t) = e^{-t}\left(A\cos(2t) + B\sin(2t)\right) + 5t - 2 + 2\cos(t) + \sin(t).$$

Evaluate this equation at 0 and use the initial condition $u(0) = 0$ to deduce

$$0 = u(0) = A - 2 + 2,$$

so $A = 0$, and

$$u(t) = Be^{-t}\sin(2t) + 5t - 2 + 2\cos(t) + \sin(t).$$

Differentiation yields

$$u'(t) = -Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t) + 5 - 2\sin(t) + \cos(t).$$

so evaluating at 0 and using $u'(0) = 0$,

$$0 = u'(0) = 2B + 5 + 1,$$

and $B = -3$. Therefore,

$$u(t) = -3e^{-t}\sin(2t) + 5t - 2 + 2\cos(t) + \sin(t).$$

2. (6 total points) On Thanksgiving day, you take the turkey out of the oven in your kitchen. At that time, the temperature of the Turkey is 240° C. The temperature in the kitchen is kept constant at 20° C. After one hour, the turkey is put into the refrigerator which is set to a temperature of 10° C. Finally, it is known that the cooling constant of a Turkey equals 4.

   (a) (4 points) Use the Laplace transform to determine the temperature of the turkey for any time $t > 0$.

By Newton's law of cooling,

$$T' = -4(T - T_S),$$

where

$$T_S(t) = \begin{cases} 20 & 0 \le t \le 1 \\ 10 & t > 1 \end{cases} = 20H_0(t) - 10H_1(t).$$

The initial condition is $T(0) = 240$. We can rewrite the differential equation as

$$T' + 4T = 80H_0 - 40H_1.$$

Apply the Laplace transform to both sides to deduce

$$\mathscr{L}[T' + 4T] = \mathscr{L}[80H_0 - 40H_1] = 80\mathscr{L}[H_0] - 40\mathscr{L}[H_1] = \frac{80}{s} - \frac{40e^{-s}}{s}.$$

Since

$$\mathscr{L}[T' + 4T] = s\mathscr{L}[T] - T(0) + 4\mathscr{L}[T] = (s+4)\mathscr{L}[T] - 240,$$

we deduce that

$$\mathscr{L}[T] = \frac{80}{s(s+4)} - \frac{40e^{-s}}{s(s+4)} + \frac{240}{s+4} = \left(80 - 40e^{-s}\right)\frac{1}{s(s+4)} + \frac{240}{s+4}.$$

To simplify this expression, compute the partial fraction decomposition of $\frac{1}{s(s+4)}$:

$$\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4} \Rightarrow 1 = (A+B)s + 4A$$

so $1 = 4A$ and $0 = A + B$, so $A = \frac{1}{4}$ and $B = -\frac{1}{4}$. We can thus write

$$\begin{aligned} \mathscr{L}[T] &= \left(80 - 40e^{-s}\right)\left(\frac{1}{4s} - \frac{1}{4(s+4)}\right) + \frac{240}{s+4} \\ &= \frac{20}{s} - \frac{10e^{-s}}{s} + \frac{10e^{-s}}{s+4} + \frac{220}{s+4}. \end{aligned}$$

The only term which is not straightforward to deal with is $\frac{e^{-s}}{s+4}$. Write

$$\frac{e^{-s}}{s+4} = e^{-cs}\mathscr{L}[f] = \mathscr{L}[H_c f_c],$$

the we see that $c = 1$ and $f(t) = e^{-4t}$, so that $H_c(t)f_c(t) = H_1(t)e^{-4(t-1)}$. We conclude that

$$\begin{aligned} T(t) &= 20 - 10H_1(t) + 10H_1(t)e^{-4(t-1)} + 220e^{-4t} \\ &= 20 + 220e^{-4t} + H_1(t)\left(10e^{-4(t-1)} - 10\right). \end{aligned}$$

(b) (1 point) What is the temperature of the Turkey after two hours?

Since $H_1(2) = 1$,

$$T(2) = 20 + 220e^{-8} + \left(10e^{-4} - 10\right) = 10 + 10e^{-4} + 220e^{-8}.$$

(c) (1 point) Assume that you want to serve the turkey precisely when it is 40° C warm. At what time does that happen? (*Hint:* It happens within one hour after you took the turkey out of the oven).

Since said time satisfies $0 \le t_0 \le 1$ by the hint, we know that $H_1(t_0) = 0$, and thus

$$40 = T(t_0) = 20 + 220e^{-4t_0}.$$

Solving this equation for t_0 yields

$$t_0 = -\frac{1}{4}\ln\left(\frac{1}{11}\right) \Leftrightarrow t_0 = \frac{\ln(11)}{4}.$$

3. (6 points) A series circuit has a resistor of $4\,\Omega$, a capacitor of $\frac{1}{5}$ F and an inductor of 1 H. The initial charge on the capacitor is zero. Assume that up to time zero, the circuit is closed, so that the current at time $t = 0$ is zero, too. At time $t = 0$, an electromotive force is connected to the circuit which impresses a voltage of $8\cos(t)$.

   Use the Laplace transform to determine the charge at time $t > 0$.

By Ohm's law, $E_R = 4I$, by Faraday's law $E_L = 1\frac{\mathrm{d}I}{\mathrm{d}t}$, and by the Capacitance law $E_C = \frac{1}{1/5}Q$. Since the impressed voltage is given by $E(t) = 8\cos(t)$, Kirchhoff's law implies that

$$8\cos(t) = E = E_R + E_L + E_C = 4I + \frac{\mathrm{d}I}{\mathrm{d}t} + 5Q$$

The initial conditions are $Q(0) = I(0) = 0$. Since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, we can rewrite the initial value problem as

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 4\frac{\mathrm{d}Q}{\mathrm{d}t} + 5Q = 8\cos(t)$$

with initial conditions $Q(0) = \frac{dQ}{dt}(0) = 0$.
Apply the Laplace transform to deduce

$$\mathscr{L}[Q'' + 4Q' + 5Q] = \mathscr{L}[8\cos(t)] = \frac{8s}{1+s^2}.$$

and

$$\begin{aligned}
\mathscr{L}[Q'' + 4Q' + 5Q] &= \mathscr{L}[Q''] + 4\mathscr{L}[Q'] + 5\mathscr{L}[Q] \\
&= s^2\left(s^2\mathscr{L}[Q] - sQ(0) - Q'(0)\right) + 4\left(s\mathscr{L}[Q] - Q(0)\right) + 5\mathscr{L}[Q] \\
&= (s^2 + 4s + 5)\mathscr{L}[Q].
\end{aligned}$$

Therefore,

$$\mathscr{L}[Q] = \frac{8s}{(s^2+1)(s^2+4s+5)}.$$

To simplify the expression, compute the partial fraction decomposition:

$$\frac{8s}{(s^2+1)(s^2+4s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5},$$

so

$$8s = (A+C)s^3 + (4A+B+D)s^2 + (5A+4B+C)s + (5B+D).$$

Comparing coefficients yields $0 = A+C$, $0 = 4A+B+D$, $8 = 5A+4B+C$, and $0 = 5B+D$. Solving this system of equations gives $A = 1$, $B = 1$, $C = -1$, and $D = -5$, so that

$$\begin{aligned}
\mathscr{L}[Q] &= \frac{-s-5}{s^2+4s+5} + \frac{s+1}{s^2+1} \\
&= -\frac{s+5}{(s+2)^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1} \\
&= -\frac{s+2}{(s+2)^2+1} - \frac{3}{(s+2)^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}.
\end{aligned}$$

We conclude that

$$Q(t) = -e^{-2t}\cos(t) - 3e^{-2t}\sin(t) + \cos(t) + \sin(t).$$

4. (5 points) You want to launch a missile straight up into the sky. Assume you start the missile at time $t = 1$, at that the missile is in rest at that time. You know that due to the consumption of fuel, the mass of the missile decays, and that its mass can be well approximated by the function $\frac{1}{t}$ for times $t > 1$. Moreover, you know that the force due to air resistance equals $2|v|$ where v is the velocity of the missile. The force due to the missile's drive propulsion is constant equal to 30 N. You can take the standard gravity to be $g = 10\,\text{m/s}^2$.

   Determine the velocity of the missile for all times $t > 1$ in terms of the function

   $$F(t) = \int_1^t e^{s^2}\,ds.$$

   Hint: By the fundamental theorem of calculus, $F'(t) = e^{t^2}$.

By Newton's second law,

$$mv' = F = mg - 2v - 30,$$

where $m = \frac{1}{t}$ and the initial condition is $v(1) = 0$. Thus,

$$v' + 2tv = 10 - 30t.$$

This equation is linear, so we use the integrating factor method: Multiply by an integrating factor μ, so

$$\mu v' + 2t\mu v = \mu(10 - 30t)$$

For the right hand side to be the derivative of the product mv we require $\mu' = 2t\mu$. This is a separable differential equation which is easily solved,

$$\frac{\mu'}{\mu} = 2t \Rightarrow \ln\mu = t^2 \Rightarrow \mu = e^{t^2}.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{t^2} v = e^{t^2} v' + 2te^{t^2} v = 10e^{t^2} - 30te^{t^2}.$$

To integrate this equation, use the hint for the left integral, and the substitution $u = t^2$ for the right integral,

$$e^{t^2} v = 10\int e^{t^2}\mathrm{d}t - 30\int te^{t^2}\mathrm{d}t + C = 10F(t) - 30\int \frac{1}{2}e^u \mathrm{d}u + C = 10F(t) - 15e^{t^2} + C.$$

Solving for v yields

$$v(t) = 10e^{-t^2}F(t) - 15 + Ce^{-t^2}.$$

Evaluate this equation at $t = 1$, so that

$$0 = v(1) = 10e^{-1}F(1) - 15 + Ce^{-1} = -15 + \frac{C}{e} \Rightarrow C = 15e.$$

Note that we have used that

$$F(1) = \int_1^1 e^{t^2}\mathrm{d}t = 0.$$

We conclude that

$$v(t) = 10e^{-t^2}F(t) + 15 + 15e^{1-t^2}.$$

5. (5 points) Initially, there are 2000 lions in the savanna. At that time, 1000 antelopes are released into the savanna. Based on previous experience, it is known that while antelopes are present, the population of the lions increases proportionally to the number of antelopes, while the population of antelopes declines proportionally to the number of lions.

   Determine the number of lions in the savanna at the moment when the antelopes have disappeared. Simplify your answer; your final answer should only involve very simple functions such as powers or roots. *Note:* Your answer will depend on certain unknown parameters.

Let A denote the number of Antelopes and L be the number of lions, both in thousands. Then

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \lambda A \tag{1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\alpha L(t), \tag{2}$$

where $\alpha > 0$ and $\lambda > 0$ are proportionality constants, and $L(0) = 2, A(0) = 1$. To get a differential equation, differentiate equation (2) and plug in equation (1):

$$A'' = -\alpha L' = -\alpha\lambda A.$$

One initial condition is given by $A(0) = 1$, and to get the second initial condition we evaluate equation (2) at time 0,

$$A'(0) = -\alpha L(0) = -2\alpha.$$

The characteristic polynomial of the differential equation is $r^2 + \alpha\lambda$ which has roots

$$r = \pm i\sqrt{\alpha\lambda}.$$

Therefore,

$$A(t) = C\sin\left(\sqrt{\alpha\lambda}t\right) + D\cos\left(\sqrt{\alpha\lambda}t\right).$$

Evalute this equation at zero to obtain $1 = A(0) = D$. The derivatite is

$$A'(t) = C\sqrt{\alpha\lambda}\cos\left(\sqrt{\alpha\lambda}t\right) - \sqrt{\alpha\lambda}\sin\left(\sqrt{\alpha\lambda}t\right),$$

and evaluating the derivative at 0 yields $-2\alpha = A'(0) = C\sqrt{\alpha\lambda}$, so $C = -2\sqrt{\frac{\alpha}{\lambda}}$. Therefore,

$$A(t) = -2\sqrt{\frac{\alpha}{\lambda}}\sin\left(\sqrt{\alpha\lambda}t\right) + \cos\left(\sqrt{\alpha\lambda}t\right).$$

To find the time t_0 when the antelopes are gone, we need $A(t_0) = 0$. Thus

$$\begin{aligned}
& 0 = A(t_0) = -2\sqrt{\frac{\alpha}{\lambda}}\sin\left(\sqrt{\alpha\lambda}t_0\right) + \cos\left(\sqrt{\alpha\lambda}t_0\right) \\
\Rightarrow\quad & \tan\left(\sqrt{\alpha\lambda}t_0\right) = \frac{\sin\left(\sqrt{\alpha\lambda}t_0\right)}{\cos\left(\sqrt{\alpha\lambda}t_0\right)} = \frac{1}{2}\sqrt{\frac{\lambda}{\alpha}} \\
\Rightarrow\quad & t_0 = \frac{1}{\sqrt{\alpha\lambda}}\arctan\left(\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\right).
\end{aligned}$$

Now by equation (2),

$$\begin{aligned}
L(t) &= -\frac{1}{\alpha}A'(t) \\
&= -\frac{1}{\alpha}\left(-2\alpha\cos\left(\sqrt{\alpha\lambda}t\right) - \sqrt{\alpha\lambda}\sin\left(\sqrt{\alpha\lambda}t\right)\right) \\
&= 2\cos\left(\sqrt{\alpha\lambda}t\right) + \sqrt{\frac{\lambda}{\alpha}}\sin\left(\sqrt{\alpha\lambda}t\right),
\end{aligned}$$

so that

$$L(t_0) = 2\cos\left(\arctan\left(\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\right)\right) + \sqrt{\frac{\lambda}{\alpha}}\sin\left(\arctan\left(\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\right)\right).$$

To simplify those quantities, we need to compute $\cos(\arctan(\delta))$ and $\sin(\arctan(\delta))$. Think of $\arctan(\delta)$ being an angle θ, so

$$\frac{\delta}{1} = \tan(\theta) = \frac{\text{opp}}{\text{adj}},$$

so we can consider a right triangle with an angle θ whose adjacent has length 1 and whose opposite has length δ:

By the Pythagorean theorem, the hypothenuse has length $\sqrt{1+\delta^2}$, and we can therefore compute

$$\cos(\arctan(\delta)) = \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+\delta^2}},$$

and similarly

$$\sin(\arctan(\delta)) = \sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{\delta}{\sqrt{1+\delta^2}}.$$

We conclude that the quantity in question is

$$\begin{aligned}
L(t_0) &= 2\cos\left(\arctan\left(\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\right)\right) + \sqrt{\frac{\lambda}{\alpha}}\sin\left(\arctan\left(\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\right)\right) \\
&= 2\frac{1}{\sqrt{1+\frac{1}{4}\frac{\lambda}{\alpha}}} + \sqrt{\frac{\lambda}{\alpha}}\frac{\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}}{\sqrt{1+\frac{1}{4}\frac{\lambda}{\alpha}}} \\
&= 2\frac{1+\frac{\lambda}{4\alpha}}{\sqrt{1+\frac{\lambda}{4\alpha}}} \\
&= 2\sqrt{1+\frac{\lambda}{4\alpha}}.
\end{aligned}$$