1. (10 total points)

   (a) (4 points) Find the expicit general solution to the following first-order differential equation. Your answer should be a function in the form $y = g(x,C)$, where C is an integration constant parameterizing the family of solutions to the DE.

$$\frac{dy}{dx} - 2xy - x = 0$$

This problem is linear, so we put it in standard form: $y' + f(x) \cdot y = g(x)$. Hence

$$\frac{dy}{dx} + (-2x)y = x$$

Our integrating factor is thus $\mu(x) = e^{\int f(x)\,dx}$, so

$$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$$

The general solution to the DE is then

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)\,dx + C\right),$$

so here we have

$$\begin{aligned} y(x) &= e^{x^2}\left(\int e^{-x^2} \cdot x\,dx + C\right) \\ &= e^{x^2}\left(-\frac{1}{2}e^{-x^2} + C\right) \\ &= -\frac{1}{2} + Ce^{x^2}. \end{aligned}$$

So the general solution to the DE is

$$y = -\frac{1}{2} + Ce^{x^2}.$$

(b) (6 points) Solve the following initial value problem (your answer should be in the form $y = g(t)$, where there is no undetermined constant in g). State for what t-interval the solution is defined.

$$\frac{dy}{dt} = e^{2t-3y}, \quad y(1) = 2.$$

This equation is separable. Separating the variables we get

$$e^{3y}\,dy = e^{2t}\,dx.$$

Antidifferentiating both dies gives us

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2t} + C,$$

or, after solving for y,

$$y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2t} + C\right),$$

where we've absorbed the factor of 3 into the C. Substituting in the initial conditions $y(1) = 2$ gives us

$$2 = \frac{1}{3}\ln\left(\frac{3e^2}{2} + C\right),$$

so $C = e^6 - \frac{3}{2}e^2$. Hence the solution to the IVP is

$$y = \frac{1}{3}\ln\left(\frac{3}{2}e^{2t} + e^6 - \frac{3}{2}e^2\right).$$

Looking at the above solution, we see that $\frac{3}{2}e^{2t} + e^6 - \frac{3}{2}e^2 > 0$ for all t, hence the natural logarithm of that quantity is defined and continuous for all t. We conclude that the solution is defined for all t.

2. (10 total points) Consider the following initial value problem:

$$\frac{dy}{dx} = \sqrt{\frac{y}{x}}, \qquad y(x_0) = y_0$$

where (in order to avoid having to make sense of square roots of negative numbers or infinity) we may assume that $y_0 \geq 0$ and $x_0 > 0$.

(a) (2 points) Using the existence and uniqueness theorem for nonlinear differential equations, state for which values of y_0 the DE is **not** guaranteed to have a unique solution.

Here we have $f(x,y) = \sqrt{\frac{y}{x}}$; thus

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{xy}}.$$

Observe that f is continuous for any non-negative value of y, but for $\frac{\partial f}{\partial y}$ the function is undefined (and thus not continuous) for $y = 0$. Everywhere else the functions f and $\frac{\partial f}{\partial y}$ are both continuous. The existence and uniqueness theorem for nonlinear first-order DEs thus states that we are not guaranteed a unique solution to the IVP if $y_0 = 0$, but we are if $y_0 > 0$.

(b) (8 points) There is more than one distinct solution to the above DE for the initial condition $y(1) = 0$. One such solution is $y = 0$. Solve the differential equation to find a second different solution.

This is a separable equation. Sticking the y stuff on the left and the x stuff on the right gives us

$$\frac{1}{\sqrt{y}}\,dy = \frac{1}{\sqrt{x}}\,dx$$

Integrating both sides gives us

$$2\sqrt{y} = 2\sqrt{x} + C$$

Or $\sqrt{y} = \sqrt{x} + B$ if we let $B = 2C$. Thus

$$y = \left(\sqrt{x} + B\right)^2$$

Plugging in the initial condition $y(1) = 0$ gives us

$$\begin{aligned} 0 &= \left(\sqrt{1} + B\right)^2 \\ \implies B + 1 &= 0 \\ \implies B &= -1 \end{aligned}$$

Thus the function

$$y = \left(\sqrt{x} - 1\right)^2$$

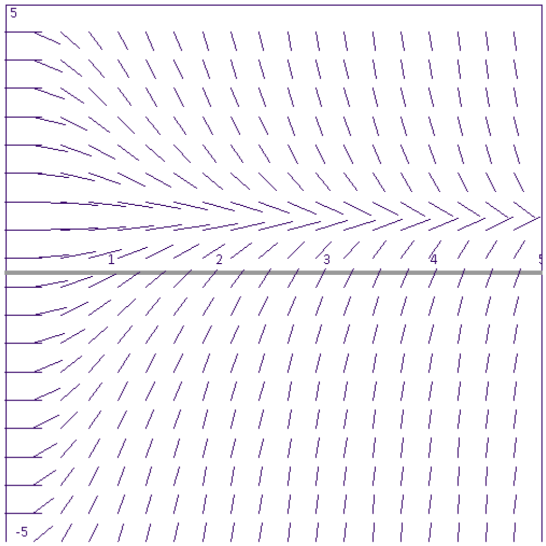
also satisfies the initial value problem.

Note that we may splice together the zero solution and the nonzero solution to obtain more solutions to the DE. That is, the function

$$y = \begin{cases} 0 & x \leq 1 \\ (\sqrt{x} - 1)^2 & x > 1 \end{cases}$$

Also obeys the DE and the initial conditions etc.

3. (10 total points)
The slope field to the differential equation $\frac{dy}{dx} = f(x,y)$ is plotted below for $0 \le x \le 5, -5 \le y \le 5$:



(a) (4 points) Circle the differential equation that corresponds to the above slope field (you do not need to show your working to receive full grade for this part of the question).

$$\frac{dy}{dx} = x(y+1) \qquad \frac{dy}{dx} = -x(y+1) \qquad \frac{dy}{dx} = x(y-1) \qquad \frac{dy}{dx} = -x(y-1)$$

We see that $\frac{dy}{dx}$ has an equilibrium solution at $y = 1$. This is the case for $\frac{dy}{dx} = x(y-1)$ and $\frac{dy}{dx} = -x(y-1)$ but not so for the other two equations, ruling them out. Moreover we see from the slope field that $\frac{dy}{dx}$ is negative for $t > 0$ and $y > 1$ and positive for $t > 0$ and $y < 1$. This is the case when $f(x,y) = -x(y-1)$ but not so for $f(x,y) = x(y-1)$. We conclude that the slope field is that of the differential equation

$$\frac{dy}{dx} = -x(y-1).$$

(b) (6 points) Let $y = \phi(x)$ be the solution to the differential equation you circled above that satisfies the initial condition $y(0) = 0$. Use Euler's method with a step size of $h = 0.5$ to estimate the value of the solution at $x = 1.5$. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

Recall that Euler's method for the IVP $\frac{dy}{dx} = f(x,y), \; y(x_0) = y_0$ with step size h is given by the scheme

- Set x_0 and y_0 to be the given initial conditions
- for $n \ge 0$ set $x_{n+1} = x_n + h$ and $y_{n+1} = y_0 + h \cdot f(x_n, y_n)$.

For us we have $h = 0.5$ and $f(x,y) = -x(y-1)$, so we have

- $x_0 = 0$ and $y_0 = 0$
- $x_1 = \frac{1}{2}$ and $y_1 = y_0 + h \cdot f(x_0, y_0) = 0 + \frac{1}{2} \cdot [-0(0-1)] = 0$
- $x_2 = 1$ and $y_2 = y_1 + h \cdot f(x_1, y_1) = 0 + \frac{1}{2} \cdot \left[-\frac{1}{2}(0-1)\right] = \frac{1}{4}$
- $x_2 = \frac{3}{2}$ and $y_3 = y_2 + h \cdot f(x_2, y_2) = \frac{1}{4} + \frac{1}{2} \cdot \left[-1(\frac{1}{4} - 1)\right] = \frac{5}{8}$

At this point we stop, as we've reached $x_3 = 1.5$. Our estimate for $\phi(1.5)$ is thus $y_3 = \frac{5}{8} = 0.625$.

4. (10 total points) Consider the autonomous differential equation

$$\frac{dy}{dt} = sin^2(y) - K,$$

where K is a constant such that $y = \frac{\pi}{3}$ is an equilibrium solution.

(a) (5 points) Find K, and state whether $y = \frac{\pi}{3}$ is a stable, unstable or semistable equilibrium solution. Be sure to justify your answer.

Recall that an autonomous equation $\frac{dy}{dt} = f(y)$ has an equilibrium solution $y = \alpha$ if $f(\alpha) = 0$. So here we know

$$f\left(\frac{\pi}{3}\right) = sin^2\left(\frac{\pi}{3}\right) - K = 0.$$

Hence

$$K = sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Now recall that to ascertain whether $y = C$ is a stable, unstable or semistable equilibrium solution to $\frac{dy}{dt} = f(y)$ we can evaluate $\frac{df}{dy}$ at C; if the derivative is negative we have a stable solution, if the derivative is positve we have an unstable solution, and if the derivative is zero we need to do more work.

Here we have $\frac{df}{dy} = 2\sin(y)\cos(y) = \sin(2y)$, which evaluated at $y = \frac{\pi}{3}$ is $\frac{\sqrt{3}}{2}$, ie. positive. Thus the equilibrium solution $y = \frac{\pi}{3}$ is unstable.

Alternately you can draw a picture of $f(y)$ versus y, and indicate that $f(y) < 0$ to the immediate left of $y = \frac{\pi}{3}$, and positive to the immediate right.

(b) (5 points) Suppose $y = \phi(t)$ is the unique solution to the differential equation satisfying the initial condition $y(0) = 0$. Find $\lim_{t\to\infty} \phi(t)$.

To answer this question one can draw a graph of $f(y)$ vs. y. We see that $f(y)$ is negative for $y = 0$, so $\phi(t)$ is thus decreasing initially. In fact, $\phi(t)$ will continue decreasing as it approaches the first negative y-value for which $f(y) = 0$. We are guaranteed that the solution $\phi(t)$ to the DE will never cross this y-value, since that corresponds to a (stable) equilibrium solution, and two separate solutions having the same y-value at the same time would violate uniqueness.

Our conclusion then is that $\phi(t)$ asymptotes to the first negative root of $f(y) = sin^2(y) - \frac{3}{4}$. To find this we set

$$sin^2(y) - \frac{3}{4} = 0$$

and solve for y. The first negative solution is $y = -\frac{\pi}{3}$.

Hence

$$\lim_{t\to\infty} \phi(t) = -\frac{\pi}{3}.$$

5. (10 total points) A reservoir on a farm initially contains 10000 liters of water, in which 200kg nitrate fertilizer is dissolved. The owner of the reservoir decides the amount of dissolved nitrate needs to be increased, so starts pumping in a 1kg nitrate:1 liter water solution at a rate of 100 l/min. However, unbeknownst to the farmer the reservoir simultaneously develops a leak, and starts draining at a rate of 200 l/min.

   (a) (7 points) Assuming the solution remains perfectly mixed at all times, find the mass of nitrate in the reservoir at time t.

   Let $y(t)$ be the mass in kg of nitrate in the reservoir at time t, where t is in minutes, and let $V(t)$ be the volume of water in liters in the reservoir at time t. We collect the following facts from the problem description:

   - $V(0) = 10000$
   - $\frac{dV}{dt} = -100$
   - $y(0) = 200$
   - The rate at which nitrate is entering the tank is 100 l/min $\times$1kg/l $=$ 100kg/min
   - We assume the solution in the reservoir is well-mixed at all times, so the rate at which nitrate is exiting the tank is the outflow rate times the nitrate concentration, i.e. $200 \times \frac{y(t)}{V(t)}$.

   Now we know the rate at which the tank is draining is linear, i.e. combining the first two bullet points gives us the volume in the tank at time t:

$$V(t) = 10000 - 100t.$$

   Note that this is valid for $0 \le t \le 100$; at that point the tank has emptied, so both V and y are zero from then on.

   Finally, this is a mixing problem: we know that $\frac{dy}{dt} =$ rate in $-$ rate out. Thus, combining all the info above we get the IVP

$$\frac{dy}{dt} = 100 - 200 \cdot \frac{y}{10000 - 100t} \qquad y(0) = 200.$$

   The differential equation is linear; in standard form after simplifying it is

$$\frac{dy}{dt} + \frac{2}{100-t}y = 100.$$

   To solve it we determine the integrating factor to be $\mu(t) = e^{\int \frac{2}{100-t}} = (100-t)^{-2}$; since we are only considering the case $0 < t < 100$ we don't need the absolute value signs when we integrate $\frac{2}{100-t}$. The general solution is then

$$\begin{aligned} y(t) &= (100-t)^2 \left(\int (100-t)^{-2} \cdot 100\, dt + C \right) \\ &= (100-t)^2 \left(100(100-t)^{-1} + C\right) \\ &= 100(100-t) + C(100-t)^2. \end{aligned}$$

   We plug in the initial condition $y(0) = 200$ to get $200 = 100^2 + C(100)^2$, so $C = -\frac{49}{50}$. The solution to the differential equation is thus

$$y(t) = -\frac{49}{50}(100-t)^2 + 100(100-t) = -\frac{49}{50}t^2 + 96t + 200.$$

   Note that this is valid only for $0 \le t \le 100$; after this the reservoir has emptied, so $y = 0$.

(b) (3 points) What is the maximum amount of nitrate in the reservoir, and when does it occur?

The solution to the IVP is

$$y(t) = -\frac{49}{50}t^2 + 96t + 200.$$

This is a quadratic function with a negative coefficient in front of the t^2 term (a "sad quadratic"), so we know its turning point is a global max. The time at which it occurs is

$$t = \frac{-96}{2 \cdot \frac{49}{50}} = \frac{2400}{49} \simeq 48.9796,$$

and the corresponding y-value is thus

$$y = -\frac{49}{50}\left(\frac{2400}{49}\right)^2 + 96\left(\frac{2400}{49}\right) + 200 = \frac{125000}{49} \simeq 2551.0204.$$

In other words, to the nearest kilogram and minute respectively the maximum amount of nitrate in the tank is about 2551kg, occurring at around the 49 minute mark.