WED 20 Feb

MATH 307A LECTURE 19:

§3.2 SOLVING IVPs WITH LAPLACE TRANSFORMS (BOYCE 6.2)

How do we use Laplace transforms to solve initial value problems? The fundamental insight comes from the following theorem:

Theorem 3.2.1 Let $f(t)$, $t \geq 0$ be a continuous, differentiable function whose Laplace transform exists, s.t. f' is piecewise continuous. Then

$$\boxed{\mathcal{L}[f'] = s\cdot\mathcal{L}[f] - f(0)}$$

for $s >$ some a.

Proof sketch:
$$\begin{aligned}\mathcal{L}[f'] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &\overset{IBP}{=} f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0) + s\mathcal{L}[f].\end{aligned}$$

Corollary 3.2.2
$$\begin{aligned}\mathcal{L}[f''] &= s\cdot\mathcal{L}[f'] - f'(0) \\ &= s(s\mathcal{L}[f] - f(0)) - f'(0)\end{aligned}$$

So $\boxed{\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)}$

when f is doubly differentiable & well behaved (i.e. its derivatives are at most exponentially growing etc.)

We can of course use the above methodology to obtain formulae for the Laplace transform of higher derivatives of f (if they exist).

General approach 3.2.3 The idea to solving IVPs using Laplace transforms is therefore as follows: Given a constant coefficient linear ODE with known initial values,

- Suppose $\phi(t)$ solves the ~~equation~~ IVP. Let $\Phi(s)$ be the Laplace transform of $\phi(t)$, i.e. $\mathcal{L}[\phi] = \Phi(s)$.

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- Take the Laplace transform of the DE, to obtain an equation in Φ, s, $\phi(0)$, $\phi'(0)$ etc.
- Solve for Φ as a function of s, all other quantities being known
- Look up in a table to see which $\phi(t)$ have Laplace transform $\Phi(s)$.

<u>Example 3.2.4</u> We start with an easy homogeneous equation:

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

[We know how to solve this conventionally: CE is $r^2 - r - 2 = 0$,
so solution has the form $y = c_1 e^{-t} + c_2 e^{2t}$
ICs: $y(0) = 1 \Rightarrow c_1 + c_2 = 1$
$y'(0) = 0 \Rightarrow -c_1 + 2c_2 = 0$
Solving yields $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$, so $y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$.]

Laplace transform way: Suppose $y = \phi(t)$ is the solution to this IVP. Then $\phi'' - \phi' - 2\phi = 0$, $\phi(0) = 1$, $\phi'(0) = 0$.

- Hit the DE with $\mathcal{L}$:

$$\mathcal{L}[\phi'' - \phi' - 2\phi] = \mathcal{L}[0]$$

$\Rightarrow \mathcal{L}[\phi''] - \mathcal{L}[\phi'] - 2\mathcal{L}[\phi] = 0$, as $\mathcal{L}[0] = 0$, & linearity.

$\Rightarrow \left(s^2\mathcal{L}[\phi] - s\phi(0) - \phi'(0)\right) - \left(\mathcal{L}[\phi] - \phi(0)\right) - 2\mathcal{L}[\phi] = 0$ using Theorems 3.2.1 & 3.2.2.

Suppose $\mathcal{L}[\phi] = \Phi$.

Then $(s^2 - s - 2)\Phi - s - 0 + 1 = 0$ using $\phi'(0) = 0$, $\phi(0) = 1$

So $\Phi(s) = \dfrac{s-1}{s^2 - s - 2} = \dfrac{s-1}{(s-2)(s+1)}$

Now we use partial fractions to get $\dfrac{s-1}{(s+1)(s-2)} = \dfrac{2/3}{s+1} + \dfrac{1/3}{s-2}$.

Hence

$$\Phi(s) = \frac{2}{3} \cdot \frac{1}{s+1} + \frac{1}{3} \cdot \frac{1}{s-2}.$$

Finally, we know $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ from previously, so we conclude that we must have

$$\phi(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}. \quad \square$$

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Note that ~~what~~ in the last step, where we go from $\Phi(s)$ to obtaining the ~~Φ~~ $\phi(t)$ whose Laplace transform is $\Phi(s)$, we have made some implicit assumptions, which are laid out below.

__Definition 3.2.5__ The __Inverse Laplace Operator__ $\mathcal{L}^{-1}$ is a linear (integral) operator that takes as input functions in s-space & returns functions in t-space. It is (for all practical purposes) the unique inverse operator of $\mathcal{L}$, i.e.

$$\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$$
$$\mathcal{L}[\mathcal{L}[F(s)]] = F(s)$$

for all $f(t)$ & $F(s)$ suitably well-behaved.

The Inverse Laplace Operator can be defined using a complex integral, and hence is a bit ~~outside the~~ outside the scope of the course. Thus what we usually do is consult a lookup table to see which functions $f(t)$ have Laplace transform $F(s)$.

__Note__: $\mathcal{L}^{-1}$ is linear, so
$$\mathcal{L}^{-1}[c_1 F_1(s) + c_2 F_2(s)] = c_1 \mathcal{L}^{-1}[F_1(s)] + c_2 \mathcal{L}^{-1}[F_2(s)]$$

The Laplace Transform is particularly effective when it comes to solving nonhomogeneous DEs:

__Example 3.2.6__: Find the solution to $y'' + y = \sin(2t)$, $y(0) = 2$, $y'(0) = 1$

So let $\phi(t)$ be the solution to the IVP, & let $\Phi(s) = \mathcal{L}[\phi(t)]$.

Then $\phi'' + \phi = \sin(2t)$

So $\mathcal{L}[\phi''] + \mathcal{L}[\phi] = \mathcal{L}[\sin(2t)]$

$\Rightarrow (s^2\Phi - s\phi(0) - \phi'(0)) + \Phi = \frac{2}{s^2+4}$ using $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$

$\Rightarrow (s^2+1)\Phi - 2s - 1 = \frac{2}{s^2+4}$

$\Rightarrow (s^2+1)\Phi = \frac{2}{s^2+4} + \frac{(2s+1)(s^2+4)}{s^2+4}$

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$$\Rightarrow \quad \Phi(s) = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}$$

We can use partial fractions again to decompose the ~~right~~ RHS:

$$\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

So $2s^3+s^2+8s+6 = (as+b)(s^2+4) + (cs+d)(s^2+1)$
$\qquad = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$.

Hence we must have $\quad a+c = 2, \quad b+d = 1$
$\qquad 4a+c = 8, \quad 4b+d = 6$

These are 2 independent systems of linear equations in 2 variables each, which we know how to solve. We get: $a = 2, \ c = 0, \ b = \frac{5}{3}$ and $d = -\frac{2}{3}$,

So
$$\Phi(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

Hence $\phi(t) = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \frac{5}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$

$$\Rightarrow \quad \underline{\phi(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t).} \qquad \square$$

We see then that the Laplace Transform method has advantages & disadvantages associated with it:

<u>Advantages</u>:
- Converts differential problem into purely algebraic problem.
- Can be used to solve higher-order IVPs.
- Works on NH equations with a large range of forcing functions

<u>Disadvantages</u>:
- Algebra can get messy
- Need a lookup table of Laplace (inverse) transforms
- Constant coefficient DEs only
  (can be extended to non-constant coeff. equation, but the algebra can become unsolvably messy).

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Friday 27 Feb MATH 307A LECTURE 20

SOLVING DEs WITH LAPLACE TRANSFORMS, PART 2 (BOYCE 6.2)

~~Example~~ We continue with a final example to show just how much easier solving an IVP can be with intelligent use of Laplace transforms.

Example 3.2.7 Solve the IVP

$$y'' + 4y' + 4y = (t^2 - 3t + 2)e^{-2t}, \quad y(0) = y'(0) = 0.$$

This would be a tough IVP to solve using the usual homogeneous solution + particular solution method.

- The homogeneous part of the DE has ~~a~~ repeated roots
- The forcing function is quite complicated
- Both independent solutions (e^{2t}, te^{2t}) to the homogeneous DE are already present in the forcing function

As such, what would we even guess for the particular solution?

However, consider solving it with Laplace transforms: Let $y = \phi(t)$ be the IVP's solution, and let $\Phi(s) = \mathcal{L}[\phi(t)]$

Note that $\mathcal{L}[\phi'] = s\mathcal{L}[\phi] - \phi(0) = s\Phi$,
while $\mathcal{L}[\phi''] = s^2\mathcal{L}[\phi] - s\phi(0) - \phi'(0) = s^2\Phi$
So we already have a simplification.

So: $\phi'' + 4\phi' + 4\phi = (t^2 - 3t + 2)e^{-2t}$
Becomes $\mathcal{L}[\phi'' + 4\phi' + 4\phi] = \mathcal{L}[(t^2 - 3t + 2)e^{-2t}]$ — ①

The LHS is thus
$$\begin{aligned}\mathcal{L}[\phi''] + 4\mathcal{L}[\phi'] + 4\mathcal{L}[\phi] &= s^2\Phi + 4s\Phi + 4\Phi \\ &= (s^2 + 4s + 4)\Phi \\ &= (s+2)^2\Phi\end{aligned}$$

To compute the Laplace transform of the RHS, recall that $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)](s-a)$

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So $\mathcal{L}[(t^2-3t+2)e^{-2t}] = \mathcal{L}[t^2-3t+2](s+2)$

And $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

So $\mathcal{L}[t^2-3t+2] = \mathcal{L}[t^2] - 3\mathcal{L}[t] + 2\mathcal{L}[1]$

$$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s}$$

$$= \frac{2s^2-3s+2}{s^3}.$$

Hence $\mathcal{L}[(t^2-3t+2)e^{-2t}] = \frac{2(s+2)^2-3(s+2)+2}{(s+2)^3}$

Thus, equating the LHS of ① with the RHS we get

$$\mathcal{L}[\phi''+4\phi'+4] = \mathcal{L}[(t^2-3t+2)e^{-2t}]$$

$$(s^2+2)^2 \cdot \Phi = \frac{2(s+2)^2-3(s+2)+2}{(s+2)^3}$$

So $\Phi = \frac{2(s+2)^2-3(s+2)+2}{(s+2)^5} = 2(s+2)^{-3} - 3(s+2)^{-4} + 2(s+2)^{-5}$

And $\phi(t) = \mathcal{L}^{-1}[\Phi(s)] = \mathcal{L}^{-1}\left[2(s+2)^{-3} - 3(s+2)^{-4} + 2(s+2)^{-5}\right]$

$$= 2\mathcal{L}^{-1}[(s+2)^{-3}] - 3\mathcal{L}^{-1}[(s+2)^{-4}] + 2\mathcal{L}^{-1}[(s+2)^{-5}]$$

Next recall again that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)](s-a)$, so going backwards we get $\mathcal{L}^{-1}[F(s-a)] = e^{at}\mathcal{L}^{-1}[F(s)]$.

In other words, $2\mathcal{L}^{-1}[(s+2)^{-3}] - 3\mathcal{L}^{-1}[(s+2)^{-4}] + 2\mathcal{L}^{-1}[(s+2)^{-5}]$

$$= 2e^{-2t}\mathcal{L}^{-1}[s^{-3}] - 3e^{-2t}\mathcal{L}^{-1}[s^{-4}] + 2e^{-2t}\mathcal{L}^{-1}[s^{-5}]$$

Finally, since $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$, we have $\mathcal{L}^{-1}[s^{-(n+1)}] = \frac{1}{n!}t^n$

Thus $2e^{-2t}\mathcal{L}^{-1}[s^{-3}] - 3e^{-2t}\mathcal{L}^{-1}[s^{-4}] + 2e^{-2t}\mathcal{L}^{-1}[s^{-5}]$

$$= e^{-2t}\left(2\cdot\frac{1}{2!}t^2 - 3\cdot\frac{1}{3!}t^3 + 2\cdot\frac{1}{4!}\cdot t^4\right)$$

$$= e^{-2t}\cdot t^2\left(1 - \frac{1}{2}t + \frac{1}{12}t^2\right)$$

So we arrive at the solution $y = \phi(t) = \frac{1}{12}e^{-2t}\cdot t^2(t^2-6t+12)$

□