The Bite of an Elliptic Curve

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Definition of the Bite

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- Modularity Theorem (Breuil, Conrad, Diamond, Taylor, Wiles): $L_E(s)$ admits analytic continuation to $\mathbb{C}$.
- $L_E(s)$ can be evaluated to k bits precision in $\tilde{O}(k \cdot \sqrt{N_E})$ time.

The Zeros of $L_E(s)$

Three flavors:

- A simple zero at $0, -1, -2, -3, \ldots$
- A zero of order r_{an} at $s = 1$; r_{an} is called the *analytic rank* of E
- Countably infinite zeros in the strip $0 < \Re(s) < 2$, symmetric about $\Re(s) = 1$ and x-axis.

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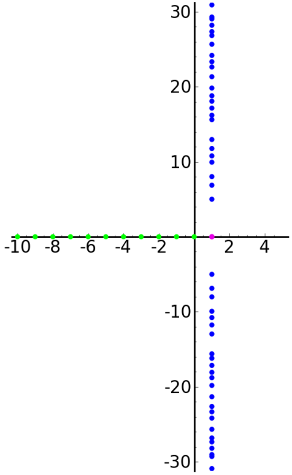

Figure: The zeros of $L_E(s)$ for $E = 37a$

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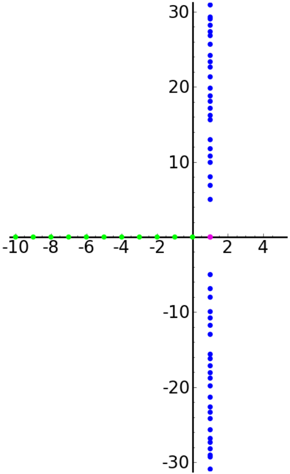
Definition

The *Bite* of E is

$$\beta(E) = \beta_E := \sum_{\gamma \neq 0} \frac{1}{\gamma^2}$$

where γ ranges over the imaginary parts of **noncentral** nontrivial zeros of $L_E(s)$.

Examples of the Bite



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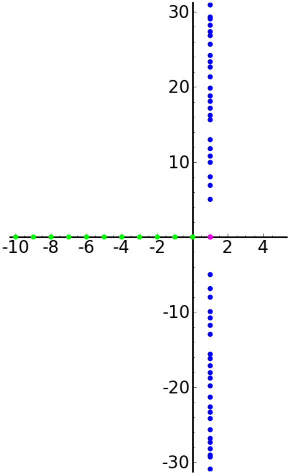

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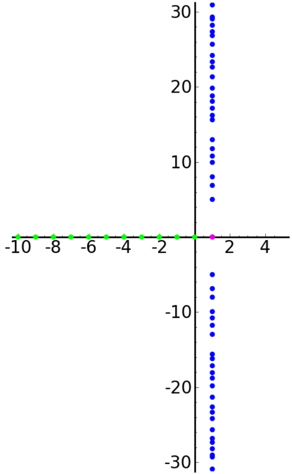

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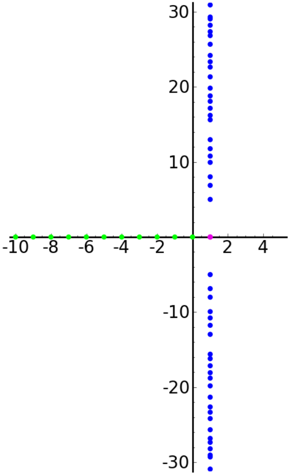

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- Can use Rubinstein's `lcalc` package to compute zeros.
- $E = 37\mathtt{a} : y^2 = x^3 - 16x + 16$ has nontrivial zeros at

$$\begin{array}{l} 0.000000000\ldots \\ 5.00317001\ldots \\ 6.87039122\ldots \\ 8.01433081\ldots \\ 9.93309835\ldots \\ 10.7751382\ldots \end{array}$$

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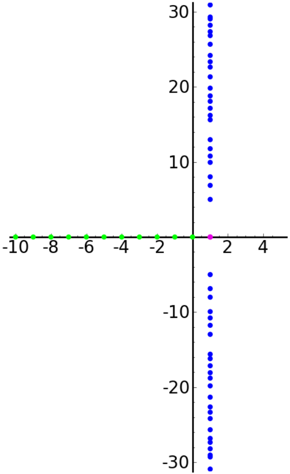

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- $\implies \beta_E = 0.3792\ldots$

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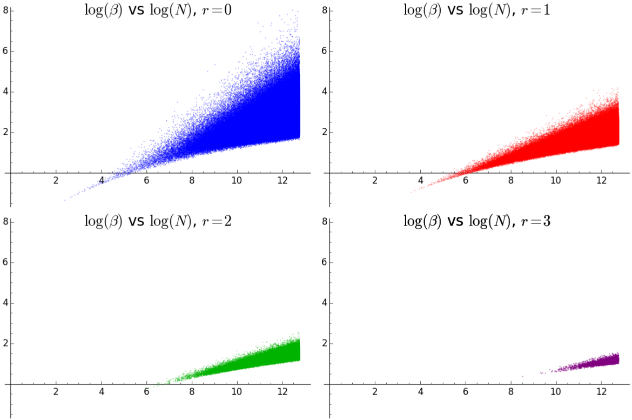

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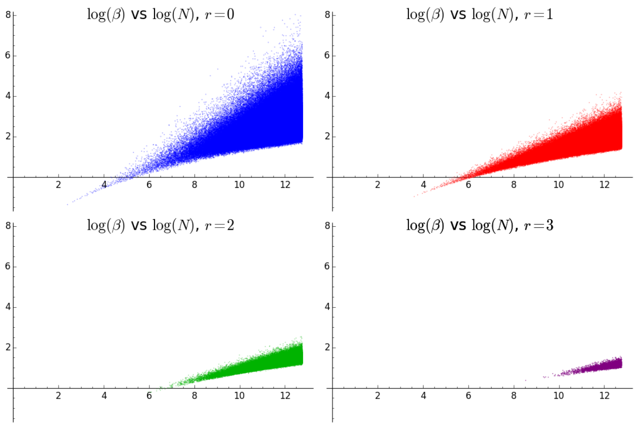

- Smallest bite: $E = 11\mathrm{a}$, $\beta_E = 0.2551\ldots$
- Largest bite: $E = 256944\mathrm{c}$, $\beta_E = 3056.1912\ldots$

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Proposition

Let $E/\mathbb{Q}$ have conductor N_E, L-function $L_E(s)$ with bite β_E and analytic rank r_E. Let $L_E(1+s)$ have Taylor expansion

$$L_E(1+s) = C \cdot s^{r_E} \cdot \left[1 + a \cdot s + b \cdot s^2 + O(s^3)\right]$$

Then

$$a = -\left[-\eta + \log\left(\frac{\sqrt{N_E}}{2\pi}\right)\right]$$

$$2b = \left[-\eta + \log\left(\frac{\sqrt{N_E}}{2\pi}\right)\right]^2 - \frac{\pi^2}{6} + \beta_E,$$

where η is the Euler-Mascheroni constant $= 0.5772\ldots$.

Why is the Bite interesting?

Proof.

- Define completed L-function:

$$\Lambda_E(1+s) := (N_E)^{\frac{1+s}{2}}(2\pi)^{-1-s}\Gamma(1+s)L_E(1+s)$$

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$$\Lambda_E(1+s) = C \cdot s^r \cdot \prod_{\gamma \neq 0}\left(1+\frac{s^2}{\gamma^2}\right)$$

So $\Lambda_E(1+s) = C \cdot s^r \cdot \left(1+\beta \cdot s^2 + O(s^4)\right)$

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- Gamma function has nice expansion about $s = 1$:
  $\Gamma(1+s) = 1 - \eta \cdot s + \frac{1}{12}\left(6\eta^2 + \pi^2\right) \cdot s^2 + O(s^3)$

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$\Gamma(1+s) = 1 - \eta \cdot s + \frac{1}{12}\left(6\eta^2 + \pi^2\right) \cdot s^2 + O(s^3)$
- Multiply out expansions and collect terms (tediously!)

$\square$

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Corollary

For E with conductor N and analytic rank r,

$$\beta(E) = \frac{2}{(r+1)(r+2)} \cdot \frac{L_E^{(r+2)}(1)}{L_E^{(r)}(1)} - \left[-\eta + \log\left(\frac{\sqrt{N}}{2\pi}\right)\right]^2 + \frac{\pi^2}{6}$$

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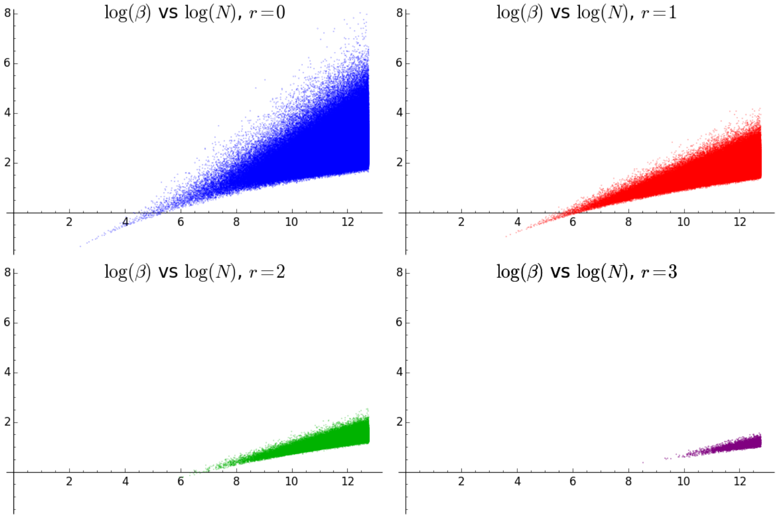
- Bites for all Cremona curves were computed using the above formula
- $\pm$ 1 week computation time on SMC.

The average size of β_E



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- It would be nice to have upper and lower bounds on β_E
- Lower bounds easier than upper bounds.

Zero Density

Define the zero counting function

$$M_E(t) = \#\left\{\gamma : \ |\gamma| \leq t\right\},$$

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where the error term is positive as often as it negative and contributes no net bias.

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- Expect β_E to grow with $\log N_E$ at least

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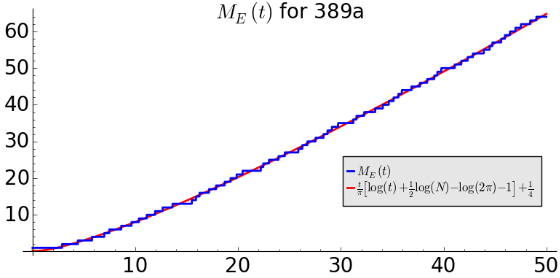

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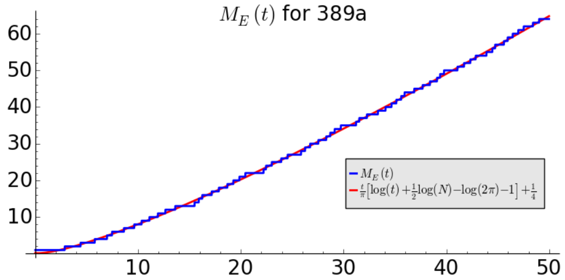

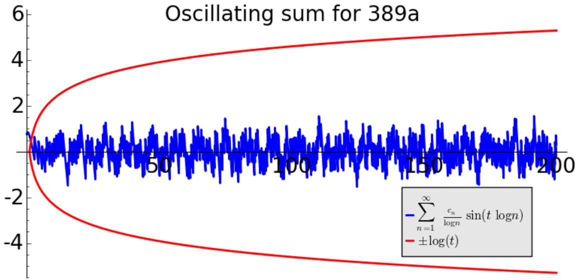

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Theorem (S.)

For all $\epsilon > 0$ there is a constant $K(\epsilon) > 0$ such that for all elliptic curves E, β_E obeys

$$\beta_E > \frac{1}{1+\epsilon} \log N_E - K(\epsilon).$$

A Lemma Regarding β_E

Lemma (1)

For any $\sigma > \frac{1}{2}$, the bite β_E and analytic rank r_E of a curve E obey

$$\sigma \cdot \beta_E + \frac{r_E}{\sigma} > \frac{1}{2}\log N_E + \frac{\Gamma'}{\Gamma}(1+\sigma) - \log(2\pi) - 2\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\sigma\right)$$

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Proof.

- Take logarithmic derivative of $\Lambda_E(1+s)$ to get

$$\frac{\Lambda_E'}{\Lambda_E}(1+s) = \sum_\gamma \frac{s}{s^2+\gamma^2} = \frac{1}{2}\log N_E + \frac{\Gamma'}{\Gamma}(1+s) - \log(2\pi) + \frac{L_E'}{L_E}(1+s)$$

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- $\left|\frac{L_E'}{L_E}(1+s)\right| < -2\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\Re(s)\right)$ from Hasse bound on Dirichlet coefficients of $\frac{\Lambda_E'}{\Lambda_E}(1+s)$

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- At $s=\sigma$,

$$\sum_\gamma \frac{\sigma}{\sigma^2+\gamma^2} = \frac{1}{\sigma}\left(r_E + \sum_{\gamma\neq 0}\frac{1}{1+\frac{\gamma^2}{\sigma^2}}\right) < \frac{1}{\sigma}\left(r_E + \sigma^2\cdot\beta_E\right)$$

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- At $s = \sigma$,

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- Combine inequalities and simplify

$\square$

Bounding Analytic Rank in terms of Conductor

Lemma (2)

r_E grows more slowly than $\log N_E$, i.e. for any $\epsilon > 0$ there exists $K(\epsilon)$ s.t. for any E,

$$r_E < \epsilon \cdot \log N_E + K(\epsilon)$$

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Two sentence proof sketch

From previous talk: For any $\Delta > 0$,

$$\sum_{\gamma} \operatorname{sinc}^2(\Delta\gamma) = \sum_{\gamma} \left(\frac{\sin(\Delta\gamma)}{\Delta\gamma}\right)^2 = \frac{1}{\pi\Delta} \log N_E + S(E, \Delta)$$

where $S(E, \Delta)$ is a finite sum with global bound that grows with Δ but is independent of E.

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where $S(E, \Delta)$ is a finite sum with global bound that grows with Δ but is independent of E.
And $\sum_{\gamma} \operatorname{sinc}^2(\Delta\gamma) > r_E$ always. So let $\Delta \to \infty$.

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- Choose $\sigma = \frac{1}{2} + \frac{\epsilon}{2}$

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$$\beta_E > \frac{1}{2\sigma} \log N_E - \frac{r_E}{\sigma^2} + \frac{1}{\sigma}\left[\frac{\Gamma'}{\Gamma}(1+\sigma) - \log(2\pi) + 2\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\sigma\right)\right]$$

- Choose $\sigma = \frac{1}{2} + \frac{\epsilon}{2}$
- Bound r_E by $\epsilon^2 \log N_E + K'(\epsilon^2)$

Lower Bounds on β_E

Theorem (S.)

$$\beta_E > \frac{1}{1+\epsilon} \log N_E - K(\epsilon) \quad \textit{for any } \epsilon > 0$$

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- Combine inequalities and let $\epsilon \to 0$

$\square$

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  - Sarnak: lowest zero approaches constant distribution as $N_E \to \infty$
  - $\implies$ bite grows more slowly than $(\log N_E)^2$

Upper bounds on β_E

Corollary to efficient bite formula

For E with analytic rank r_E, bite β_E and (completed L-function) leading central Taylor coefficient C_E,

$$\beta_E \cdot C_E = \frac{\Lambda_E^{(r_E+2)}(1)}{(r_E+2)!}$$

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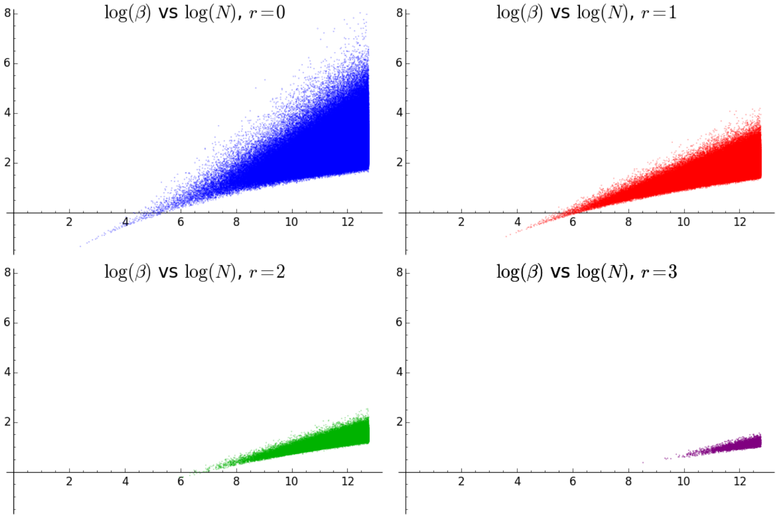
- $\Longrightarrow$ The bite and the leading central Taylor coefficient are intimately linked
- Behavior of $\beta_E \cdot C_E$ heavily constrained

The bite times the BSD coefficient



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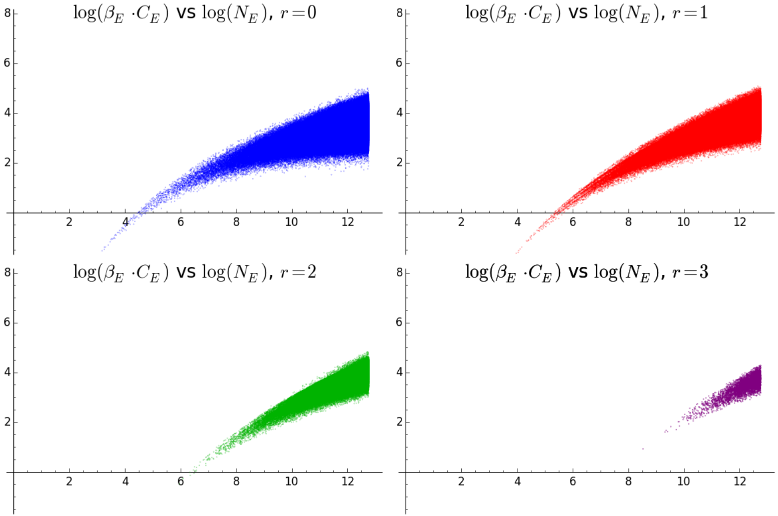

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- $$\begin{aligned}\{\text{Bounding } \beta_E \text{ from above}\} &\iff \{\text{Bounding } \gamma_0 \text{ from below}\} \\ &\iff \{\text{Bounding } C_E \text{ from below}\}\end{aligned}$$

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Proposition

The analytic rank of E is the largest integer less than

$$\frac{1}{\sqrt{\beta_E}}\left[\left(-\eta + \log\left(\frac{\sqrt{N_E}}{2\pi}\right)\right) + \frac{1}{2\sqrt{\beta_E}}\left(\frac{\pi^2}{6} - Li_2\left(e^{-2\sqrt{\beta_E}}\right)\right)\right.$$
$$\left. + \sum_{\log n < 2\sqrt{\beta_E}} c_n \cdot \left(1 - \frac{\log n}{2\sqrt{\beta_E}}\right)\right] \quad \textit{where}$$

- $\eta = 0.5772\ldots$ *is the Euler-Mascheroni constant*
- $Li_2(s)$ *is the dilogarithm function on* $\mathbb{C}$
- $c_n = c_n(E)$ *is the nth Dirichlet coefficient of* $\frac{L_E'}{L_E}(1+s)$

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$$r_{al}(E) = \lim_{X\to\infty} \frac{1}{\log X}\left[\sum_{p<X} \frac{-a_p(E)\log p}{p}\right]$$

- Convergence rate of this sum should scale with $e^{\sqrt{\beta_E}}$

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- Is there any arithmetic significance to β_E?

Thank You