Homework 2

Total: 20 points
Due: Wed 08 Oct 2014 at the beginning of class

If a question is taken from the textbook, the refence is given on the right of the page.

FIRST ORDER MODELING

1. Boyce 2.3 Q5

A tank contains 100 gal of water in which 50 oz of salt is dissolved. Water containing a salt concentration if $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

(a) Find the amount of salt in the tank at any time.

(b) Sketch a graph of the solution for a time period long enough so that you see the ultimate behaviour of the graph.

(c) The long-time behaviour of the graph should look like an oscillation about a certain constant level. What is this level, and what is the amplitude of oscillation?

2. Boyce 2.3 Q9

A college student borrows \$8000 to buy a car. The lender charges interest at a rate of 10% per annum. Assume that the interest is compounded continuously and that the student makes payments continuously at a constant annual rate k.

(a) Determine the payment rate k that is required to pay off the loan in 3 years.

(b) What is the total amount the student pays over the 3-year period?

3. Some science fiction authors observe that the rate at which scientific advances are made by the human race are ever-increasing, and predict that as a result humanity will reach a point where our scientific progress becomes infinitely rapid. This point in time is known as the *Singularity.*
Suppose that we have ascertained that the total scientific knowledge of humanity has increased by 12.5% between the years of 2000 and 2010. Furthermore, suppose that we determine that the rate at which scientific knowledge advances is proportional to the *square* of our current scientific knowledge. When will the Singularity occur?

What I'm testing for here is problem-solving ability. You are given less information here than usual, but you still have enough to answer the question. It's up to us to choose variables and units; the answer shouldn't be affected by the choice. So let $y(t)$ be the scientific knowledge of humanity at time t, where t is in years AD and y is some appropriate unspecified choice of units. We then have

$$\frac{dy}{dt} = ky^2$$

for some constant k.
This is separable, so

$$\frac{1}{y^2}\,dy = k\,dt$$

Antidifferentiating yields

$$-\frac{1}{y} = k(t - C)$$

(note that keeping k on the outside on the right is the most illuminating, but we can also have $kt - C$), and solving for y we then get

$$y = \frac{1}{k(C-t)}.$$

Now we know that $y(2010) = \frac{9}{8}y(2000)$; this is not enough to solve for both k and C, but it is enough to solve for C in terms of k, which in turn is enough to answer the question:

$$\frac{1}{k(C-2010)} = \frac{9}{8} \cdot \frac{1}{k(C-2000)},$$

so we can clear k and multiply through to get

$$C - 2000 = \frac{9}{8}(C - 2010)$$

Solving for C yields $C = 2090$.
We thus have that

$$y(t) = \frac{1}{k(2090-t)}.$$

This function asymptotes to infinity when $t \to 2090$. The model therefore suggests that the Singularity will occur in the year 2090.
Note that the choice of units of time is up to you; we could just as well define time to be years since 2000, for example. However, the final answer of the Singularity occurring in 2090 is the same regardless of what units you use for time.

4. Boyce 2.3 Q17

Heat transfer from a body to its surroundings through radiation is accurately described by Stefan-Boltzmann's law, which dictates that the rate of heat loss between the object and its surroundings is proportional to the difference between the 4th powers of their respective temperatures. However, if the object is much hotter than its surroundings, the system can be approximated by the differential equation

$$\frac{dy}{dt} = -\alpha y^4,$$

where $y(t)$ is the temperature of the object in degrees Kelvin, and α is a proportionality constant dependant on the physical parameters of the object in question.
Suppose that a slug of molten steel with an initial temperature 2000 K is placed in a room whose temperature is controlled at 300 K, and suppose that $\alpha = 2.0 \times 10^{-12} K^{-3}/s$.

(a) Determine the temperature of the metal slug for all t by solving the above differential equation.

(b) Plot the graph of $y(t)$.

(c) Determine the time when the slug has cooled to 600 K, twice the ambient temperature. It is interesting to note that even though we are using an approximite differential equation, up to this time the error from the solution to the true DE is less than 1%.

5. Boyce 2.3 Q23

A skydiver weighing 180 lb (including equipment) jumps from a plane at 5000 ft and falls vertically downward; after 10 seconds of free fall the skydiver's parachute opens. Assume that the force of air resistance acts proportional and opposite to velocity, with proportionality constants 0.75 when the parachute is closed and 12 when it is open respectively. Here v is measured in ft/sec.

(a) Find the speed of the skydiver when the parachute opens.

(b) Find the distance fallen when the parachute opens.

(c) What is the limiting velocity v_L after the parachute opens?

(d) Estimate to the nearest second how long the sky dive will take in its entirety i.e. from when the skydiver jumps from the plane until when they touch the ground.

6. Water hyacinth is a particularly aggressive invasive plant species in lakes in the southern US. One of the reasons is that it grows very quickly: under good conditions a population will grow at a rate proportional to its own size, with its biomass increasing by a factor of $e = 2.71828\ldots$ every 14 days. Suppose a water hyacinth population establishes itself in a large lake in Florida where conditions are close to ideal. When ecologists discover the population it has a biomass of 750kg.
Removal efforts begin immediately; however, because it takes some time to train local volunteers to remove the weed efficiently, the rate $R(t)$ at which water hyacinth can be removed from the lake is given by the function

$$R(t) = 600\left(1 - e^{-t}\right),$$

where t is measured in **weeks** since the beginning of the removal effort, and $R(t)$ is in kg/week.

(a) Establish an initial value problem and solve it to find an explicit formula for the biomass of water hyacinth in the lake at time t.

Let $y(t)$ be the biomass in kg of water hyacinth in the lake at a time t, where t is measured in weeks, and $t = 0$ corresponds to the beginning of removal efforts.
In the absence of any removal effort, the above paragraph tells us that y would grow at a rate proportional to y, i.e.

$$\frac{dy}{dt} = ry$$

for some $r > 0$.
Solving this preliminary DE with $y(0) = y_0$ yields

$$y = y_0 e^{rt}$$

We know if that left unchecked biomass increases by a factor of e every 2 weeks. In mathematics this statement is

$$\frac{y(2)}{y(0)} = e$$

But

$$\frac{y(2)}{y(0)} = \frac{y_0 e^{2r}}{y_0 e^0} = e^{2r}$$

so $e^{2r} = e$. Solving for r by taking logs on both sides yields $r = \frac{1}{2}$.
Now in reality the rate at which y grows is reduced by the rate of removal $R(t)$. Hence the water hyacinth in the lake obeys the differential equation

$$\frac{dy}{dt} = \frac{1}{2}y - 600\left(1 - e^{-t}\right)$$

subject to the initial condition $y(0) = 750$.
To solve this IVP, note that it is linear. In standard form we have

$$\frac{dy}{dt} - \frac{1}{2}y = -600\left(1 - e^{-t}\right),$$

i.e. $f(t) = -\frac{1}{2}$ and $g(t) = -600\left(1 - e^{-t}\right)$. The integrating factor is thus

$$\mu(t) = e^{\int f(t)\,dt} = e^{\int -\frac{1}{2}\,dt} = e^{-\frac{1}{2}t},$$

and so the general solution is given by

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(-600\int e^{-\frac{1}{2}t}(1 - e^{-t})\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(-600\int e^{-\frac{1}{2}t}\,dt + 600\int e^{-\frac{3}{2}t}\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(1200e^{-\frac{1}{2}t} - 400e^{-\frac{3}{2}t} + C\right)\\
&= 1200 - 400e^{-t} + Ce^{\frac{1}{2}t}.
\end{aligned}$$

Applying the initial condition $y(0) = 750$ yields

$$750 = 1200 - 400e^0 + Ce^0,$$

so $C = -50$. Hence the biomass in the lake at time t is given by the function

$$y(t) = 1200 - 400e^{-t} - 50e^{\frac{1}{2}t}.$$

(b) Will efforts to completely remove the water hyacinth from the lake be successful? Justify your answer.

We see that the solution

$$y(t) = 1200 - 400e^{-t} - 50e^{\frac{1}{2}t}$$

has an exponentially growing term with a negative coefficient; this term will eventually dominate the other two terms and so $lim_{t\to\infty} y(t) = -\infty$. Thus there must be some point in the future where $y(t) = 0$, i.e. there is no more water hyacinth in the lake.
Thus yes, the removal effort under the assumptions above will ultimately be successful.

(c) If the answer to the above question is yes, estimate how many weeks it will take for the water hyacinth to be removed completely from the lake. If the answer to the above question is no, estimate how many weeks it will take for the water hyacinth to reach 10000kg biomass. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

We seek the time where there is no more water hyacinth in the lake, i.e. the t for which $y(t) = 0$. Thus we must solve the equation

$$1200 - 400e^{-t} - 50e^{\frac{1}{2}t} = 0.$$

This equation can in fact be solved exactly through logs and cubic equations, but it's ugly; there's no way I'm expecting you to do this. The best simplifying observation we can make is that $50e^{\frac{1}{2}t}$ increases exponentially in size with time, while $400e^{-t}$ decreases exponentially. So if t is more than a few units in size we can expect the $-400e^{-t}$ term in the above expression to be negligible in size compared to the $-50e^{\frac{1}{2}t}$ term.
It thus makes sense to discard the $-400e^{-t}$ term and solve instead for t in the equation

$$1200 - 50e^{\frac{1}{2}t} = 0.$$

We thus get that $e^{\frac{t}{2}} = 24$ so $t = 2\ln(24) = 4\ln(6) = 6.3561\ldots$
Indeed, if we plug this t-value into $y(t) = 1200 - 400e^{-t} - 50e^{\frac{t}{2}}$ we get

$$1200 - 400e^{-4\ln 6} - 1200 = -\frac{25}{81} = -0.3086\ldots.$$

This is only a hair below zero (considering the starting value for y of 1200). We therefore expect the t-value for which $y(t) =$ to be just slightly less than our estimate

$$y = 4\ln(6) = 6.3561\ldots$$

(In fact, the true solution to the equation is

$$y = 6.3550\ldots,$$

so our estimate is accurate to 2 decimal places.)
In other words, the lake will be cleared of invasive water hyacinth in just under $6\,\frac{1}{2}$ weeks time.