Wed Feb 5 MATH 307A LECTURE 11

§2.3 : CHARACTERISTIC EQUATIONS WITH REPEATED ROOTS

Setup: $ay''+by'+cy=0$,
with CE $ar^2+br+c=0$

In both the cases we've looked at so far we can write the general solution to the DE as
$$y = c_1e^{r_1t} + c_2e^{r_2t},$$
where r_1 & r_2 are the two different roots to the CE.

Question 2.3.1: What happens when $r_1 = r_2$?
Then the above collapses to
$$y = ce^{rt}, \quad (c = c_1+c_2), (r = r_1 = r_2)$$
where r is the repeated root to the CE.

Problem: For a linear 2nd-order ~~DE~~ homogeneous DE we always have 2 linearly independent solutions; here we only have one. So we need to find a 2nd solution that isn't a multiple of e^{rt}.

Example 2.3.2: $y''-2y'+y=0$
$\Rightarrow$ CE $r^2-2r+1 = (r-1)^2 = 0$
So $y = e^t$ is a solution
But what is the other solution to this DE?

The Method of ~~Variation of Parameters~~ Reduction of Order

This method can be used on more general linear DEs. The idea is to take your known solution and use it to find a 2nd solution.

Consider the example above.

Method 2.3.3: • ~~Replace~~ Guess a solution $y = v(t)e^t$ for some function $v(t)$.
• Then see what $v(t)$ works:

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$$y' = v'e^t + ve^t = (v+v')e^t$$
$$y'' = v''e^t + v'e^t + v'e^t + ve^t = (v''+2v'+v)e^t$$

So $y'' - 2\cancel{r}y' + y = (v''+2v'+v)e^t - 2(v+v')e^t + \cancel{a}ve^t = 0$
$\Rightarrow v''e^t = 0$

So $v'' = 0$

Hence $v = c_1 \cdot t + c_2$ for any constants $c_1, c_2 \in \mathbb{R}$.

Thus $y = c_1te^t + c_2e^t$ works as a solution to the DE

Now as with the integrating factor in 1st-order linear DEs, we only need *one* solution - as all others can be written as a ~~multiple~~ linear combination of this one and e^t - so might as well pick $c_1 = 1$, $c_2 = 0$

Hence we've found the basis solutions $y = e^t$ & $y = te^t$ to the DE $y'' - 2y' + y' = 0$

$\Rightarrow$ General solution is $y = c_1e^t + c_2te^t$.

General Case 23.4: Note that if the CE has repeated roots then we can write it as a perfect square.

i.e. $ay'' + by' + cy = 0$

has $ar^2 + br + c = 0$ with repeated roots, then ~~$(r-r_0)^2 = 0$~~ $b^2 - 4ac = 0$, and so

$r = -\frac{b}{2a}$ is a double root to the CE.

Thus $y = e^{-\frac{b}{2a} \cdot t}$ is one solution to the DE.

So let $y = v(t)e^{-\frac{b}{2a} \cdot t}$ be the 2nd solution

$$\Rightarrow y' = v'e^{-\frac{b}{2a} \cdot t} - \frac{b}{2a}ve^{-\frac{b}{2a}t}$$
$$y'' = v''e^{-\frac{b}{2a} \cdot t} - \frac{b}{a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}$$

So $a\left[\left(v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}v\right)e^{-\frac{b}{2a}t}\right] + b\left[\left(v' - \frac{b}{2a}v\right)e^{-\frac{b}{2a}t}\right] + c\left[ve^{-\frac{b}{2a}t}\right] = 0$

$$\Rightarrow av'' + (-b+b)v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v = 0$$

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But $\frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c$

$= -\frac{1}{4a}(b^2 - 4ac)$

$= 0$, since $b^2 - 4ac$ in this case.

Hence we have $av'' = 0$

$\Rightarrow v'' = 0$

~~$\Rightarrow$ $v = c_1 t$ as before~~

$\Rightarrow v = c_1 t + c_2$ as before

~~Al~~ So general solution is

$$y = c_0 e^{-\frac{b}{2a}t} + (c_1 t + c_2) e^{-\frac{b}{2a}\cdot t}$$

$$= c_1' e^{-\frac{b}{2a}t} + c_2' t e^{-\frac{b}{2a}t} \quad (c_1' = c_0 + c_2,\ c_2' = c_1).$$

So in the repeated roots case, $y = e^{-\frac{b}{2a}t}$ and $y = te^{-\frac{b}{2a}t}$ are the 2 independent solutions to the DE.

Example 2.3.5 Solve the IVP

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2,\ y'(0) = \frac{1}{3}.$$

Solution: CE: $r^2 - r + \frac{1}{4} = 0$

$\Rightarrow (r - \frac{1}{2})^2 = 0$

So $r = \frac{1}{2}$ is the double root.

$\Rightarrow y = e^{\frac{1}{2}t}$ & $y = te^{\frac{1}{2}t}$ are the two independent solutions to the DE:

$$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$ is the general solution.

ICs: $y(0) = 2 \Rightarrow c_1 = 2.$

$y' = \frac{1}{2}c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} + \frac{1}{2}c_2 t e^{\frac{1}{2}t}$

$\Rightarrow y'(0) = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{2}c_1 + c_2$

$= 1 + c_2 \Rightarrow c_2 = -\frac{2}{3}$

So solution is

$$y = 2e^{\frac{1}{2}t} - \frac{2}{3}te^{\frac{1}{2}t}$$

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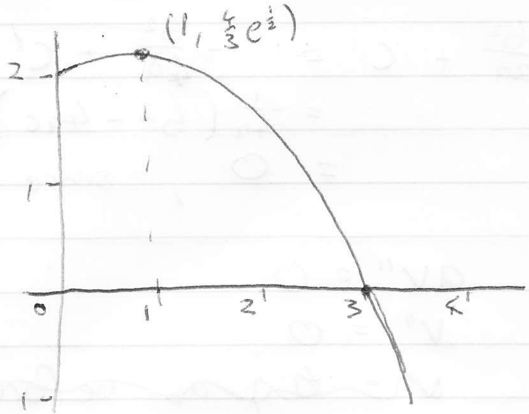
Plot:



$\square$

<u>Note 2.3.6</u>: The method of reduction of order can be used on linear 2nd-order DEs where the coefficients aren't constant: if we know one solution to the DE, we can find the 2nd.

<u>Example 2.3.7</u>: Consider the DE

$$2t^2y'' + 3ty' - y = 0 \quad , \quad t > 0.$$

Given that $y_1(t) = \frac{1}{t}$ is a solution to this DE, find ~~a 2nd solution that's linearly independent to $y_1(t)$~~ the general 2-parameter solution to the DE.

<u>Solution</u>: set $y = v(t)t^{-1}$

$$\Rightarrow y' = v't^{-1} - vt^{-2}$$

$$y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

So subbing into the DE we get

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - (vt^{-1}) = 0$$

$$\Rightarrow 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v = 0$$

$$\Rightarrow \quad 2tv'' - v' = 0$$

So let $w = v'$; then $2tw' - w = 0$

Separable: $\frac{dw}{dt} = \frac{w}{2t}$

$$\Rightarrow \frac{1}{w}\,dw = \frac{1}{2t}\,dt$$

$$\Rightarrow \ln|w| = \frac{1}{2}\ln(t) + c$$

$$\Rightarrow w = At^{\frac{1}{2}} \quad , \quad A = \pm e^c$$

$$\Rightarrow \frac{dv}{dt} = At^{\frac{1}{2}}$$

$$\Rightarrow v = Bt^{\frac{3}{2}} + D \qquad B = \tfrac{2}{3}A$$

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Hence the 2nd solution is $y = (Bt^{\frac{3}{2}} + D) \cdot t^{-1}$
$= Bt^{\frac{1}{2}} + Dt^{-1}$.

But again, we look for the simplest such function.
Note that every multiple of t^{-1} is already covered by $y_1(t) = t^{-1}$, so set $D = 0$.
Secondly we choose $B = 1$, since we're looking at all multiples of y anyway

$\Rightarrow$ General solution is

$$y = c_1 t^{-1} + c_2 t^{\frac{1}{2}} = c_1 \frac{1}{t} + c_2 \sqrt{t}$$