~~Fri 7 Feb.~~
Mon 10 Feb

MATH 307A LECTURE 13, ~~contin~~.

§2.4 NONHOMOGENEOUS EQUATIONS, PART 2.

..., Ex. 2.4.9: $y''-3y'-4y=-8e^t(\cos 2t)$ $\Rightarrow$ Guess $Y=e^t(A\cos 2t+B\sin 2t)$.

Then $Y' = e^t((A+2B)\cos 2t + (-2A+B)\sin 2t)$

And $Y'' = e^t[(-3A+4B)\cos 2t + (-4A-3B)\sin 2t]$

So $Y''-3Y'-4Y = -8e^t\cos 2t$

$\Rightarrow e^t[(-3A+4B)\cos 2t + (-4A-3B)\sin 2t] + e^t[(A+2B)\cos 2t + (-2A+B)\sin 2t]$
$+ e^t[A\cos 2t + B\sin 2t] = e^t\cdot(-8)\cos 2t$

After simplifying, we get

$$10A+2B=8 \quad \& \quad 2A-10B=0$$

$\Rightarrow A=10/13,\ B=2/13$. Thus a particular solution to the DE is

$$Y(t)=e^t\left[\tfrac{10}{13}\cos 2t + \tfrac{2}{13}\sin 2t\right]$$

Theorem 2.4.10 If $Y_1(t)$ is a particular solution to
$$y''+p(t)y'+q(t)y=g_1(t),$$
and $Y_2(t)$ is a particular solution to $y''+p(t)y'+q(t)y=g_2(t)$, then $c_1Y_1(t)+c_2Y_2(t)$ is a particular solution to
$$y''+p(t)y'+q(t)y=c_1g_1(t)+c_2g_2(t),$$
where c_1 & c_2 are constants.

That is, if the function on the right $g(t)$ is a linear combination of (simpler) functions, we can find a particular solution to the non-homogeneous DE by finding particular solutions to DEs where we replace $g(t)$ with the individual simpler functions, and then take the results and combine them to get the solution to the full equation.

Example 2.4.11: Find a particular solution to
$$y''-3y'-4y=3e^{2t}+2\sin t-8e^t\cos 2t$$

Solution:
By splitting the right side into 3 functions we get the 3 following DEs:

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$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$

particular

We've already computed the solutions to these 3 DEs: $Y_1 = -\frac{1}{2}e^{2t}$, $Y_2 = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$, $Y_3 = e^t[\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t]$ respectively.

Hence a particular solution to the full DE is
$$Y = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + e^t[\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t].$$

Sometimes the guess-and-check approach as we've seen it so far comes up short:

Example 2.4.12: Find a particular solution to $y'' - 3y' - 4y = 2e^{-t}$

$\Rightarrow$ Guess $Y = Ae^{-t}$

$\Rightarrow$ $Y' = -Ae^{-t}$, $Y'' = Ae^{-t}$

So $Y'' - 3Y' - 4Y = (Ae^{-t}) - 3(-Ae^{-t}) - 4(Ae^{-t}) = (A + 4A - 4A)e^{-t} = 0$

Problem: $Y'' - 3Y' - 4Y = 0$ regardless of the choice of A, so no value of A will make $Y'' - 3Y' - 4Y = 2e^{-t}$.

Why is this? This is because e^{-t} is already a solution to the homogeneous equation $y'' - 3y' - 4y = 0$; (i.e. $L[Ae^{-t}] = 0$, $L = \frac{d^2}{dt^2} - 3\frac{d}{dt} - 4$).

Solution: As in the case with characteristic equations with repeated roots, try $Y = v(t)e^{-t}$

Then $Y' = (v' - v)e^{-t}$

$Y'' = (v - 2v' + v'')e^{-t}$

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So $Y'' - 3Y' - 4Y = (v - 2v' + v'')e^{-t} - 3(v' - v)e^{-t} - 4ve^{-t}$

$= e^{-t}(v'' - 5v')$

set $= 2e^{-t}$

Thus we must have $v'' - 5v' = 2$

Let $w = v'$; then we get the first-order Linear DE

$$w' - 5w = 2$$

To solve, let $\mu(t) = e^{\int -5\,dt} = e^{-5t}$

$\Rightarrow$ $w = \frac{1}{\mu(t)}\left(\int \mu(t) g(t)\,dt + C\right)$

$= e^{5t}\left(\int 2e^{-5t}\,dt + C\right)$

$= e^{5t}\left(-\frac{2}{5}e^{-5t} + C\right)$

$= -\frac{2}{5} + Ce^{5t}$

And $v' = w \Rightarrow v = -\frac{2}{5}t + C'e^{5t} + D$, $C' = \frac{C}{5}$

And $Y = ve^{-t} \Rightarrow Y = -\frac{2}{5}te^{-t} + Ce^{4t} + De^{-t}$

This is in fact the full general solution to $y'' - 3y' - 4y = 2e^{-t}$.
A specific solution is therefore given by $C = D = 0$,
i.e. $Y = -\frac{2}{5}t$

Careful: unlike with repeated roots, $v(t)$ is not always equal to t:

Example 2.4.13: Find a particular solution to

$$y'' - 8y' + 16y = 3e^{4t}$$

Guess $Y = v(t)e^{4t}$

Then $Y' = (v' + 4v)e^{4t}$

$Y'' = (v'' + 8v' + 16v)e^{4t}$

So $Y'' - 8Y' + 16Y = (v'' + 8v' + 16v)e^{4t} - 8(v' + 4v)e^{4t} + 16(ve^{4t})$

$= e^{4t}(v'')$

set $= 3e^{4t}$

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Hence we must have $V'' = 3$

$$\Rightarrow \quad V = 3t^2 + At + B$$

And $Y = ve^{4t}$

So $y = 3t^2 e^{4t} + Ate^{4t} + Be^{4t}$ is the full general solution to the DE;

A specific solution is $Y = 3t^2 e^{4t}$.

<u>To Sum</u>

- Sometimes for the DE $\cancel{y'' + p(t)y' + q(t)y}$ $ay'' + by' + cy = g(t)$, guessing $Y = Ag(t)$ doesn't work, as $y = g(t)$ is already a solution to the DE $\cancel{y'' + p(t)y' + q(t) = 0}$. $ay'' + by' + cy = 0$
- In that case, guess $Y = At \cdot g(t)$.
- Sometimes both $g(t)$ and $tg(t)$ are solutions to the DE $ay'' + by' + cy = 0$ (i.e. when the CE has equal roots).
- In this case to find a particular solution to $ay'' + by' + cy = g(t)$, guess $Y = At^2 g(t)$.
- We are guaranteed that for linear 2nd-order DE's with constant coefficients, we will never need to go beyond multiplying $g(t)$ by t^2 to find a particular solution.

- Again, this technique (guess $Y = v(t)g(t)$) is more general and can be applied to DEs where the coefficients are not constant. Guessing $Y = V_1(t)y_1(t) + V_2(t)y_2(t)$, where $y_1(t)$ & $y_2(t)$ are ~~the~~ a fundamental basis to the homogeneous DE $y'' + p(t)y' + q(t)y = 0$, is ~~This is touching on~~ known as the <u>method of variation of parameters</u> (Boyce 3.6).

Finally, let's use the method of undetermined coefficients to solve an IVP.

<u>Example 2.4.14:</u> Solve the IVP

$$y'' + 3y' - 4y = 4t - 5e^{4t}, \quad y(0) = 1, \quad y'(0) = 0$$

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Solution: • First, solve the homogeneous DE $y''+3y'-4=0$

CE: $r^2+3r-4=0$

So $r=-4$ or $r=1$

Hence the general solution is $y=c_1e^{t}+c_2e^{-4t}$.

• Next, Let $g_1(t)=4t$, $g_2(t)=-5e^{-4t}$ & $g(t)=g_1(t)+g_2(t)$

we find a particular solution of $y''+3y'-4y=g_1(t)=3t$.

Guess $Y_1=At+B$, since $4t$ is a degree 1 polynomial

$\Rightarrow Y_1'=A$

$Y_1''=0$

$\Rightarrow Y_1''+3Y_1'-4Y_1=3A-4At-4B=4t$

So $-4A=4 \Rightarrow A=-1$

& $3A-4B=0 \Rightarrow B=-\frac{3}{4}$

Hence $Y_1(t)=-t-\frac{3}{4}$

• Now we find a particular solution to

$$y''+3y'-4y=g_2(t)=-5e^{-4t}$$

Since e^{-4t} is a solution to the homogeneous equation $y''+3y'-4y=0$,

Guess $Y_2(t)=A\cdot t\cdot e^{-4t}$

$\Rightarrow Y_2'=e^{-4t}(A-4At)$

$Y_2''=e^{-4t}(-8A+16At)$

Hence $Y_2''+3Y_2'-4Y_2=e^{-4t}(-8A+16At+3(A-4At)-4At)$

$=e^{-4t}(-5A+0)$

$=-5Ae^{-4t}$

set $=-5e^{-4t}$

So $A=1$

And hence $Y_2(t)=te^{-4t}$

Thus the full general solution to $y''+3y'-4y=4t-5e^{-4t}$

is $y=Y_1(t)+Y_2(t)+c_1e^{r_1t}+c_2e^{r_2t}=-t-\frac{3}{4}+te^{-4t}+c_1e^{t}+c_2e^{-4t}$.

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Now apply ICs:

$y(0)=1 \Rightarrow 1 = -\frac{3}{4} + C_1 + C_2$

$\Rightarrow C_1 + C_2 = \frac{7}{4}$

And $y' = -1 + e^{-4t} - 4te^{-4t} + C_1e^t - 4C_2e^{-4t}$

So $y'(0)=0 \Rightarrow 0 = -1 + 1 + C_1 - 4C_2$

$\Rightarrow C_1 - 4C_2 = 0$

Solving the system of linear equations in C_1 & C_2 yields

$C_1 = \frac{7}{5}, \quad C_2 = \frac{7}{20}$

Hence the solution to the IVP is

$$y = -\frac{3}{4} - t + te^{-4t} + \frac{7}{5}e^t + \frac{7}{20}e^{-4t}$$