~~Monday 27 March~~ wed 12 MATH3074 LECTURE 23, ~~part 2~~

§3.4 SOLVING DEs WITH DISCONTINUOUS FORCING FUNCTIONS PART 2

(BOYCE 6.4)

<u>Some things to note:</u> Continuing from the previous example,

- $\phi(t)$ and $\phi'(t)$ are both continuous for all t.
- $\phi''(t)$ has jump discontinuities as $t=5$ and $t=20$, as ϕ'' is proportional to the forcing function $g(t)$ which has jump discontinuities there

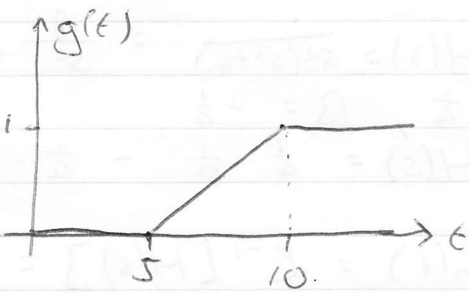
In general, if $y=\phi(t)$ is the solution to the DE $y''+p(t)y'+q(t)y=g(t)$, then the $(n+2)$th derivative of $\phi(t)$ is discontinuous wherever the nth derivative of $g(t)$ is discontinuous.

<u>Example 3.4.2</u> Consider the IVP

$$y''+4y=g(t),\ y(0)=0,\ y'(0)=0,\quad g(t)=\begin{cases}0, & 0\le t<5\\ \frac{1}{5}(t-5), & 5\le t<10\\ 1, & t\ge 10\end{cases}$$

a) Describe the qualitative nature of the solution.
b) Solve the IVP.

a) Graph of $g(t)$:



Since $y(0)=y'(0)=0$, we know the solution will be $y=0$ for $0\le t<5$, since there is no forcing function for $t\le 5$.

Now observe that $y=A\cos(2t-\delta)$ is the general solution to the homogeneous DE, while for $t>10$ $Y=\frac{1}{4}$ is a particular solution. We therefore expect the solution to oscillate about $y=\frac{1}{4}$ with frequency $=2$ for $t\ge 10$.

And for $5\le t<10$ the particular solution is a linear function increasing in t; we therefore expect the solution

Sketch Graph:

to oscillate about this linear increase for $5 \le t < 10$ until the median point hits $y = \frac{1}{4}$.

We might be interested in the amplitude of the oscillations, however, and for this we'll need to solve the DE.

b) Observe that $g(t) = \frac{1}{5}u_5(t-5) - \frac{1}{5}u_{10}(t-10)$.
Letting $y = \phi(t)$ be the solution and $\Phi(s) = \mathcal{L}[\phi(t)]$, we thus have that

$$(s^2+4)\,\Phi = \frac{1}{5}\cdot\frac{e^{-5s}}{s^2} - \frac{1}{5}\cdot\frac{e^{-10s}}{s^2}$$

or $$\Phi = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)}$$

Write $H(s) = \frac{1}{s^2(s^2+4)}$, then $\Phi = \frac{1}{5}e^{-5s}H(s) - \frac{1}{5}e^{-10s}H(s)$,
so $\phi(t) = \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10)$,
where $h(t) = \mathcal{L}^{-1}[H(s)]$.

Now $H(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s^2+4}$ by partial fractions
$\Rightarrow A = \frac{1}{4}$, $B = -\frac{1}{4}$,
so $H(s) = \frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s^2+4}$.

Hence $h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{4}t - \frac{1}{8}\sin(2t)$.

That is, $$\phi(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{20}(t-5) - \frac{1}{40}\sin(2(t-5)), & 5 \le t < 10 \\ \frac{1}{4} - \frac{1}{40}\left(\cancel{\sin(2(t-5)) - \sin(2(t-10))}\right), & t \ge 10 \end{cases}$$
$\frac{\sin(5)}{20}\cdot\cos(2t-15)$

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Example 3.4.3: A certain ~~elec~~ series circuit has a capacitor whose charge is subject to the DE $y'' + 6y' + 5y = 0$.
The current is initially zero, as is the charge on the capacitor. ~~At t~~ At $t=0$ an external square wave voltage of $E(t)$ is applied, where $E(t) = \begin{cases} 1, & 2n \le t < 2n+1 \\ 0, & 2n+1 \le t < 2n+2 \end{cases}$, n a positive integer.

a) Find the charge on the capacitor at time t.
b) What is the asymptotic $\overset{\text{min}}{\wedge}$ max $\overset{\text{average}}{\wedge}$ charge on the capacitor?

a) Let $y = \phi(t)$ be the solution, and $\Phi(s) = \mathcal{L}[\phi(t)]$. Then taking Laplace transforms of the DE we get

$$(s^2 + 6s + 5)\Phi = \mathcal{L}\left[1 - \sum_{n=1}^{\infty} u_n(t)\right] = \frac{1}{s(1+e^{-s})}$$

$$\Rightarrow \Phi = \frac{1}{s(s+5)(s+1)(1+e^{-s})} = \frac{1}{s(s+1)(s+5)}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - \dots\right)$$

So $\phi(t) = h(t) - u_1(t)h(t-1) + u_2(t)h(t-2) - u_3(t)h(t-3) + \dots$
$= \sum_{n=0}^{\infty} (-1)^n u_n(t) h(t-n)$,
where $h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]$

Now $H(s) = \frac{1}{s(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+5}$ by partial fractions.

$\Rightarrow 1 = A(s+1)(s+5) + Bs(s+5) + C(s)(s+1)$
@ $s=0$: $1 = 5A \Rightarrow A = \frac{1}{5}$
@ $s=-1$ $1 = -4B \Rightarrow B = -\frac{1}{4}$
@ $s=-5$ $1 = 20C \Rightarrow C = \frac{1}{20}$
So $H(s) = \frac{1}{20}\left(4 \cdot \frac{1}{s} - 5 \cdot \frac{1}{s+1} + \frac{1}{s+5}\right)$
$\Rightarrow h(t) = \frac{1}{20}\left(4 - 5e^{-t} + e^{-5t}\right)$
with $\phi(t) = \sum_{n=0}^{\infty} (-1)^n u_n(t) h(t-n)$. □

To calculate the max charge on the capacitor, we note that asymptotically this will occur just before the voltage switches off for a second; at that point the charge

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will be $\left(\frac{1}{5} - \frac{1}{4}e^{-1} + \frac{1}{20}e^{-5}\right) - \left(\frac{1}{5} - \frac{1}{4}e^{-2} + \frac{1}{20}e^{-2\cdot 5}\right) + \left(\frac{1}{5} - \frac{1}{4}e^{-3} + \frac{1}{20}e^{-3\cdot 5}\right) - \dots$

$$= \frac{1}{5} - \frac{1}{4}\left(e^{-1} - e^{-2} + e^{-3} - \dots\right) + \frac{1}{20}\left(e^{-5} - (e^{-5})^2 + (e^{-5})^3 + \dots\right)$$

$$= \frac{1}{5} - \frac{1}{4}\cdot\frac{e^{-1}}{1+e^{-1}} + \frac{1}{20}\cdot\frac{e^{-5}}{1-e^{-5}}$$

$$= \frac{1}{5} - \frac{1}{4(e+1)} + \frac{1}{20(e^5+1)}$$

$$\approx 0.1331.$$

The asymptotic min is $\frac{1}{5}$ - this number, i.e. ≈ 0.0669.
And the asymptotic average charge is 0.1. $\square$

Graph: