Your Name

Student ID #

- **DON'T PANIC!** There are more questions in this exam than in the midterms, but you have more than twice the time to solve them. If you get stuck, take a deep breath and move on to something else. Return if you have time at the end.
- Cellphones off please!
- You are allowed one two-sided handwritten notesheet for this midterm. You may use a scientific calculator; graphing calculators and all other course-related materials may not be used.
- In order to receive full credit, you must **show your working** unless explicitly stated otherwise by the question. You may quote and use any formula you have seen in class to save time, but be sure to indicate with a word or two when you are doing so.
- Give your answers in exact form (for example $\pi/3$ or $e^{-5\sqrt{3}}$) unless explicitly stated otherwise by the question. Simplify your answers if possible
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- **There is a table of Laplace transforms and rules at the back of this exam**. You may quote and use any of the formulas and rules in the table as is without having to derive them from scratch.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10 points) Use the method of undetermined coefficients to find the **particular solution** to the following differential equation:

$$y'' + 9y = te^{-t} - 1.$$

Your answer should be a function $Y(t)$ with no undetermined constants in it.

2. (10 points) Recall that the Wronskian of a second-order linear system is $W(t) = y_1 y_2' - y_1' y_2$, where $y_1(t)$ and $y_2(t)$ form a fundamental basis of solutions to that system (the Wronskian is always the same up to multiplication by a constant, no matter which pair of basis of solutions you choose).

   Compute the Wronskian of the following system:

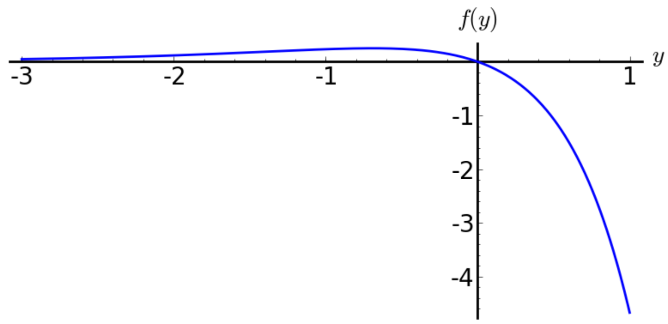
$$y'' - 2y' + 10y = 0$$

   Your answer should be a function $W(t)$ that contains no undetermined constants. Remember to simplify your answer if possible.

3. (10 total points) Consider the autonomous differential equation

$$y' = e^{y} - e^{2y},$$

where y is a function of t. Below is a graph of $f(y)$ versus y:



(a) (4 points) Find all equilibrium solutions to this differential equation, and classify them according to their stability (stable, unstable or semistable). Be sure to justify your answer.

(b) (6 points) Suppose we are now looking at the solution to the above DE subject to the initial condition $y(0) = -1$. Use a single iteration of Euler's method to find an approximate value of the solution at $t = 0.5$. You may use decimals in this part of the question, but be sure to maintain at least four digits of precision.

4. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{s^2 + 2s - 2}{s^3 - s}$$

5. (10 points) My buddy is coming over to watch the game. Unfortunately my fridge has broken down, so I have to resort to alternative measures to cool our drinks down. The drinks are initially at 20 degrees Celsius; one hour before the game starts I place the drinks in an ice box. I note that the rate of cooling of the drinks is proportional to the temperature difference between the drinks and the ice box; moreover, I observe that the proportionality constant's magnitude is precisely $\frac{1}{50}$ when the units of time are minutes and the units of temperature are degrees Celsius.

   However, the icebox itself is slowing heating up. One hour before the game the icebox is at 0 degrees Celsius, but its temperature is increasing linearly at a rate of 1 degree Celsius every 10 minutes.

   Formulate and solve an initial value problem to find the temperature of the drinks when the game begins.

6. (10 total points) A $\frac{1}{2}$ kg mass is placed on a surface and attached to a horizontal spring with spring constant β kg/s^2, where β is a positive constant. Friction acts on the mass such that when the mass is traveling at 1 m/s it experiences a frictional force of 1 Newton.

   (a) (2 points) Establish a differential equation that the mass obeys.

   (b) (5 points) For what values of β will the system will be overdamped, critically damped and underdamped respectively? Justify your answer.

   (c) (3 points) Find the value of β for which the radial quasi-frequency of the mass's damped oscillation is exactly 4 radians/sec. [Note: I'm referring to the *angular frequency* ω, not the cyclic frequency f.]

7. (10 total points) A series circuit contains an inductor of inductance 0.1 henrys, a resistor of resistance 2 ohms, and a capacitor of capacitance 0.01 farads. At time $t = 0$ seconds there is no current in the circuit, but the charge on the capacitor is 0.01 coulombs.

   Initially no external voltage is applied on the circuit. At $t = 1$ seconds a source is switched on which exerts a constant voltage of 10 volts on the circuit. At $t = 4$ seconds the voltage source is switched off, and no external voltage acts on the circuit from thereon.

   (a) (3 points) Let $E(t)$ be the forcing function in the above system. Express $E(t)$ using Heaviside functions $u_c(t)$. Your answer should be a function that is a linear combination of $u_c(t)$'s.

   (b) (2 points) Write down the initial value problem that models the charge on the capacitor for any time $t \geq 0$.

   (c) (5 points) Let $Q = \phi(t)$ be the solution to the IVP above. Compute the Laplace transform $\Phi(s)$ of the solution as a function of s.

   [NB: you do not need to fully solve the IVP to answer this part of the question.]

8. (10 total points + 4 bonus points) A two-way pump attached attached to a reservoir pumps water into and then out of the reservoir at a rate of $2000\sin(t)$ liters per hour, where t is measured in hours. At time $t = 0$ a valve at the bottom of the reservoir is opened and it begins to drain at a rate proportional to the amount of water in the reservoir. The reservoir initially contains 10000 liters of water, and the initial outflow rate is measured to be 1000 liters per hour.

   (a) (3 points) Establish an initial value problem that models the volume of water in the reservoir at time t.

   (b) (7 points) Solve the initial value problem to find the number of liters of water in the reservoir at time t.

[Question continued on the next page]

(c) (4 bonus points) Estimate the point in time when the reservoir first runs dry.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}(f(t))$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	