1. (10 points) Solve the following initial value problem:

$$\frac{1}{x} \cdot \frac{dy}{dx} = e^{x-y}, \qquad y(0) = 0$$

Your answer should be a function $y(x)$ with no undetermined constants in it.

This is a separable equation. Separating variables gives us

$e^y dy = xe^x dx$, so $\int e^y dy = \int xe^x dx$

Now $\int e^y dy = e^y$, while for $\int xe^x dx$ we use integration by parts:

$u = x \quad dv = e^x$

$du = dx \quad dv = e^x dx$

so $\int xe^x dx = xe^x - \int e^x dx$

$= xe^x - e^x + C$

$= (x-1)e^x + C$

Thus $e^y = (x-1)e^x + C$

taking logs: $\Rightarrow y = \ln((x-1)e^x + c)$

Applying the IC $y(0) = 0$: $0 = \ln((0-1)e^0 + c)$

$\Rightarrow 1 = -1 + C$

$\Rightarrow C = 2$

Thus $\boxed{y = \ln((x-1)e^x + 2)}$

2. (10 points) Consider the non-homogeneous differential equation

$$y'' + 4y' - 21y = g(t),$$

for some nonzero forcing function $g(t)$. For each of the following possibilities for $g(t)$, write down the form that the particular solution $Y(t)$ to the DE would take. Your answer should be in the form $Y = f(t)$, where f includes undetermined coefficients (A, B, C etc.). For example, if you thought the particular solution was a general linear function in t, you would write $Y = At + B$. **You don't need to compute the actual values of these coefficients.**

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

Homogeneous equation $y''+4y'-21y=0$ has CE $r^2+4r-21=0$

$\Rightarrow (r+7)(r-3)=0$

So $y_1 = e^{-7t}$ & $y_2 = e^{3t}$ are solutions to the homog. equation

(a) $g(t) = e^{-t} + 1$

- e^{-t} isn't a solution to the homog. equation
$\Rightarrow$ Guess $Y_1(t) = Ae^{-t}$
- 1 is a degree 0 polynomial
$\Rightarrow$ Guess constant function $Y_2(t) = B$

$\Rightarrow$ $\boxed{Y(t) = Ae^{-t} + B}$

(b) $g(t) = \sin(t)$

"cos & sin functions hunt in packs"

$\boxed{Y(t) = A\cos(t) + B\sin(t)}$

(c) $g(t) = e^{3t} - e^{-7t}$

Both e^{3t} & e^{-7t} are solutions to the homogeneous equation
$\Rightarrow$ multiply guess by t.

$\Rightarrow$ $\boxed{Y(t) = Ate^{3t} + Bte^{-7t}}$

(d) $g(t) = e^{2t}\cos 3t$

Derivatives of $e^{2t}\cos 3t$ will include $e^{2t}\cos 3t$ & $e^{2t}\sin 3t$ terms. Same for $e^{2t}\sin 3t$

$\Rightarrow$ $\boxed{Y(t) = Ae^{2t}\cos 3t + Be^{2t}\sin 3t}$

(e) $g(t) = t^2 + 2t$

t^2+2t is a degree-2 polynomial $\Rightarrow$ guess a general quadratic polynomial

$\boxed{Y(t) = At^2 + Bt + C}$

3. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{e^{-s} - e^{-3s}}{s^2 + s}$$

Note that $F(s) = e^{-s}H(s) - e^{-3s}H(s)$, where $H(s) = \frac{1}{s^2+s}$

Thus $\mathcal{L}^{-1}[F(s)] = u_1(t)\,h(t-1) - u_3(t)\,h(t-3)$,

where $h(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+s}\right]$

Now $\frac{1}{s^2+s} = \frac{1}{s} - \frac{1}{s+1}$

So $\mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$

$= 1 - e^{-t}$

And so $\mathcal{L}^{-1}[F(s)] = u_1(t)\cdot(1-e^{-(t-1)}) - u_3(t)\cdot(1-e^{-(t-3)})$

or $\boxed{\mathcal{L}^{-1}[F(s)] = u_1(t)\cdot(1-e^{1-t}) - u_3(t)\cdot(1-e^{3-t})}$

Note: there are probably multiple equivalent ways to write the above function.

4. (10 total points) Consider the following linear first-order initial value problem:

$$\frac{dy}{dt} = \overbrace{\cos(t)}^{-f(t)} \cdot y + \overbrace{\sin(t)}^{g(t)}, \qquad y(0) = 0$$

(a) (2 points) Using the appropriate existence and uniqueness theorem for first-order differential equations, state the largest time interval for which the solution to the above IVP is guaranteed to exist.

$y' - \cos(t)\cdot y = \sin(t)$

The DE is linear, so a unique solution exists on the largest t-interval containing $t_0 = 0$ for which both $-\cos(t)$ and $\sin(t)$ are continuous.

But $-\cos(t)$ & $\sin(t)$ are continuous for all t

$\Rightarrow$ $\boxed{\text{Conclude unique solution exists for all } t.}$

(b) (6 points) Use Euler's Method with a step size of $\frac{\pi}{2}$ to find an approximate value of the solution at $t = \frac{3}{2}\pi$. You may use decimals in this part of the question (although you don't need to); if you do be sure to maintain at least four digits of precision.

3 steps with $h = \frac{\pi}{2}$ $\quad y' = f(t,y)$, where $f(t,y) = \cos(t)\cdot y + \sin(t)$.

$t_0 = 0 \quad y_0 = 0$

$t_1 = \frac{\pi}{2} \quad y_1 = y_0 + hf(t_0, y_0) = 0 + \frac{\pi}{2}\left(\cos(0)\cdot 0 + \underbrace{\sin(0)}_{0}\right) = 0$

$t_2 = \pi \quad y_2 = y_1 + hf(t_1, y_1) = 0 + \frac{\pi}{2}\left(\underbrace{\cos(\tfrac{\pi}{2})}_{0}\cdot 0 + \underbrace{\sin(\tfrac{\pi}{2})}_{1}\right) = \frac{\pi}{2}$

$t_3 = \frac{3}{2}\pi \quad y_3 = y_2 + hf(t_2, y_2) = \frac{\pi}{2} + \frac{\pi}{2}\left(\underbrace{\cos(\pi)}_{-1}\cdot\frac{\pi}{2} + \underbrace{\sin(\pi)}_{0}\right) = \frac{\pi}{2} - \frac{\pi^2}{4}.$

STOP HERE

$\boxed{\text{So an approximate value for } y(\tfrac{3}{2}\pi) \text{ is } y_3 = \tfrac{\pi}{2} - \tfrac{\pi^2}{4} = -0.8966}$

(c) (2 points) Does the differential equation $\frac{dy}{dt} = \cos(t)\cdot y + \sin(t)$ have any equilibrium solutions? That is, are there any constant solutions $y = C$ to this DE? Justify your answer.

$\boxed{\text{No.}}$ The easiest way to see this is to suppose $y = C$ is an equilibrium solution for some C. Then for that solution $\frac{dy}{dt} = 0$ always, so plugging the solution into the DE we get

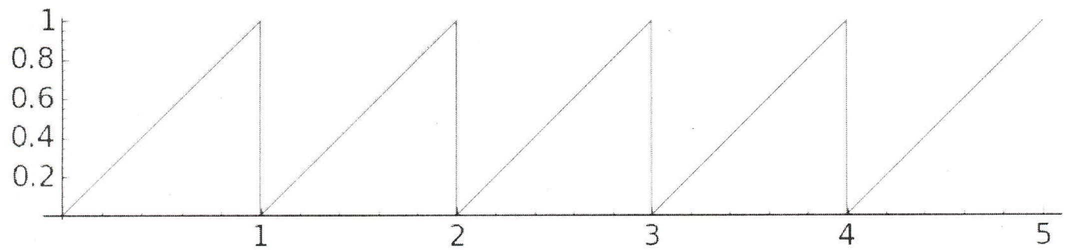
$0 = \cos(t)\cdot C + \sin(t)$ for all t. $\Rightarrow$ $C = -\tan(t)$ for all t

But no such C exists, as $-\tan(t)$ takes on infinitely many values. Contradiction, so conclude that the DE has no equilibrium solutions.

5. (10 total points) Consider the unit sawtooth function $f(t)$, defined by :

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-n, & n \le t < n+1 \end{cases}.$$

for n any integer. A graph of $f(t)$ for t between 0 and 5 is given below:



(a) (7 points) Compute the Laplace transform of $f(t)$. Your answer should be a function $F(s)$.

$f(t)$ is periodic with period $T=1$, so by the formula $\mathcal{L}[f] = \dfrac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{\int_0^1 te^{-st}dt}{1-e^{-s}}$$

To compute $\int_0^1 te^{-st}dt$, use IBP:

$u = t \qquad v = -\frac{1}{s}e^{-st}$

$du = dt \qquad dv = e^{-st}dt$

$$\Rightarrow \int_0^1 te^{-st}dt = -\frac{1}{s}te^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}$$
$$= -\frac{1}{s}e^{-s} + 0 - \frac{1}{s^2}e^{-st}\Big|_0^1$$
$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}$$

So $\int_0^1 te^{-st} = \frac{1}{s^2}\left(1-(s+1)e^{-s}\right)$

Hence $\boxed{\mathcal{L}[f(t)] = \dfrac{1-(s+1)e^{-s}}{s^2(1-e^{-s})}}$

(b) (3 points) Use your answer above to compute the Laplace transform of the solution to the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0$$

where $f(t)$ is the sawtooth function detailed above. Your answer should be a function $\Phi(s)$ with no undetermined constants in it. **You do not need to find the solution to the IVP to answer this question.**

Let $\phi(t)$ solve the IVP, $\Phi(s) = \mathcal{L}[\phi(t)]$.

Then $\mathcal{L}[\phi''] = s^2\Phi$ and $\mathcal{L}[\phi'] = s\Phi$, since $\phi(0) = \phi'(0) = 0$

$$\Rightarrow \mathcal{L}[\phi'' + 4\phi] = \mathcal{L}[f(t)]$$
$$\Rightarrow s^2\Phi + 4\Phi = \frac{1-(s+1)e^{-s}}{s^2(1-e^{-s})}$$
$$\Rightarrow (s^2+4)\Phi = \frac{1-(s+1)e^{-s}}{s^2(1-e^{-s})}$$

So $\boxed{\Phi(s) = \dfrac{1-(s+1)e^{-s}}{s^2(s^2+4)(1-e^{-s})}}$

6. (10 total points) An applied mathematician is investigating the motion of a particular object, and establishes that the function describing its motion $y(t)$ obeys the differential equation

$$y'' + by' + cy = 0$$

where a and b are constants. The mathematician doesn't initially know the values of b and c, but can show the following two facts:

- The function $y_1(t) = e^{-3t}$ is a solution to the differential equation.
- The Wronskian of the system is $W(t) = e^{2t}$.

(a) (7 points) Using the above two facts, find a second function $y_2(t)$, linearly independent from the first, that satisfies the differential equation. Your answer should be a function in t **with no undetermined coefficients in it.**

We have $e^{2t} = W(t) = y_1 y_2' - y_1' y_2 = e^{-3t} y_2' - (-3e^{-3t}) y_2$

So $e^{-3t} y_2' + 3e^{-3t} y_2 = e^{2t}$

$\Rightarrow \quad y_2' + 3y_2 = e^{5t}$

This is a first-order linear DE:

$\mu(t) = e^{\int 3\,dt} = e^{3t}$

So $y_2(t) = e^{-3t}\left(\int e^{5t} \cdot e^{+3t}\,dt + C\right)$

$= Ce^{-3t} + e^{-3t} \int e^{8t}\,dt$

$= Ce^{-3t} + e^{-3t} \cdot \frac{1}{8} e^{8t}$

$= \frac{1}{8} e^{5t} + Ce^{-3t}$

Now we can set $C = 0$, since $y_1(t) = e^{-3t}$ ~~that~~ is a fundamental basis solution that already covers the e^{-3t} part of any solution to the DE

$\Rightarrow \quad \boxed{y_2 = \frac{1}{8} e^{5t}}$ works.

Note: Any multiple of the above also works, so we could use

$\boxed{y_2(t) = e^{5t}}$ just as good.

(b) (3 points) Given that the functions $y_1(t)$ and $y_2(t)$ both solve the DE, what are the constants b and c?

We see e^{-3t} & e^{5t} are both solutions to the DE;

CE must then have roots $r = -3$ & $r = 5$.

$\Rightarrow \quad (r+3)(r-5) = 0$

$\Rightarrow \quad r^2 - 2r - 15 = 0$

So DE is $y'' - 2y' - 15y = 0$

i.e. $\boxed{b = -2, \quad c = -15}$

7. (10 total points) A 5 kg block is placed on a flat surface and attached to a long horizontal spring. When the block is pulled 0.2 meters to the right of its equilibrium position, the spring exerts a force of 2.5 Newtons to the left on the block. The surface imparts a frictional force on the block proportional to its velocity, such that when the block is traveling at 1 ms^{-1} the retarding force is 5 Newtons. Furthermore, the block is subjected to an external oscillating force of $g(t) = \cos(\omega t)$ Newtons, where t is in seconds and ω is a positive constant.

   (a) (2 points) Write down a differential equation describing the position of the block as a function of time. $my'' + \gamma y' + ky = g(t)$.

   For us: $m = 5$, $\gamma = 5$, $k = \frac{2.5}{0.2} = \frac{25}{2}$ and $g(t) = \cos(\omega t)$.

   ICs: Not present or relevant.

   So DE is $\boxed{5y'' + 5y' + \frac{25}{2}y = \cos(\omega t)}$

   (b) (3 points) For what value of ω will the amplitude of the block's steady-state response be maximized?

   We quote the resonant ~~ampli~~ frequency formula for forced damped systems:

   $$\omega_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}} = \sqrt{\frac{25/2}{5} - \frac{5^2}{2\cdot 5^2}} = \sqrt{\frac{5}{2} - \frac{1}{2}} = \sqrt{2}$$

   So $\boxed{\omega_{max} = \sqrt{2} \text{ rad sec}^{-1} = 1.414\ldots}$

   (c) (5 points) How much resonance is there in this system? To answer this question, compute and interpret the quantity R/R_0, where R is the amplitude of the steady-state response for the value of ω you found above, and R_0 is the amplitude of the response when the forcing function is a constant 1 Newton (i.e. when $\omega = 0$).

Recall that $R_0 = \frac{F_0}{k} = \frac{2}{25}$ for us.

$R/R_0 = \frac{Rk}{F_0}$ is therefore given by the formula

$$\frac{Rk}{F_0} = \frac{1}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + \Gamma(\frac{\omega}{\omega_0})^2}},$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the system and $\Gamma = \frac{\gamma^2}{km}$.

For us $\omega_0 = \sqrt{\frac{5}{2}}$, so $\left(\frac{\omega}{\omega_0}\right)^2 = \left(\frac{\sqrt{2}}{\sqrt{5/2}}\right)^2 = \frac{4}{5}$, and $\Gamma = \frac{5^2}{(\frac{25}{2})\cdot 5} = \frac{2}{5}$.

Thus $$R/R_0 = \frac{Rk}{F_0} = \frac{1}{\sqrt{(1-\frac{4}{5})^2 + \frac{2}{5}\cdot\frac{4}{5}}} = \frac{1}{\sqrt{\frac{1}{25} + \frac{8}{25}}} = \frac{1}{\sqrt{\frac{9}{25}}} = \boxed{\frac{5}{3} = 1.6. = \frac{R}{R_0}}$$

Thus the resonant amplitude is 1.6 times as large as the "basis" amplitude of $\frac{2}{25}$ when $\omega = 0$. That is, there is some resonance present, but not very much. This is because ~~damp~~ friction is quite large in the system ($\Gamma = 0.4$), but not ~~too~~ so large as to induce critical damping.

8. (10 points + 4 bonus points) Two state troopers are stationed in their cruiser at a speed trap on a long straight road. The cops have been reading a book on differential equations, and decide to use their newfound knowledge to come up with a chase strategy when pursuing a speeding vehicle.

The cops agree that if a target is fleeing from them, they should accelerate towards is. The first cop argues that the acceleration of the cop car should be proportional to the distance between them and their target. The second cop instead argues that their acceleration should be proportional to the difference of velocities of the target and the police cruiser. In the end the cops agree that both ideas have merit, so decide to adopt the strategy that is a *weighted sum* of the two above: their acceleration will be the sum of [α times the distance between the vehicles] and [β times the difference of their velocities], for some constant values of α and β.

After more discussion the cops decide to use the values $\alpha = \frac{1}{100}$ and $\beta = \frac{1}{5}$. For example, if their target was 100 meters ahead traveling 10 ms^{-1} faster than them, the cops would accelerate toward it at 3 ms^{-2}.

At time t = 0 seconds a car comes speeding past the trap, traveling at a constant speed of 40 ms^{-1}. The cops are initially stationary at the trap, but immediately give chase using the strategy above.

(a) (10 points) Formulate and solve an initial value problem to find a formula for the position of the cop cruiser for time $t \geq 0$.

Let $y(t)$ be the position of the cop car at time t, t in seconds since the speeding car goes past the trap. We have ICs: $y(0)=0$, $y'(0)=0$.
The position of the speeding vehicle is given by the function $40t$
$\Rightarrow$ its speed is thus 40 ms^{-1} for all t.

Now $y'' = \frac{1}{100}[\text{distance between cars}] + \frac{1}{5}\cdot[\text{difference in cars' velocities}]$
$= \frac{1}{100}(40t - y) + \frac{1}{5}(40 - y')$, making sure to choose the correct signs in front of each term so that acceleration ~~always~~ is in the right direction.

$\Rightarrow y'' + \frac{1}{5}y' + \frac{1}{100}y = \frac{2}{5}t + 8$, $y(0)=0$, $y'(0)$ is the IVP we must solve.

Particular solution: Guess $Y(t) = At + B$. $\Rightarrow Y' = A$ & $Y'' = 0$
So $\frac{1}{5}\cdot A + \frac{1}{100}(At+B) = \frac{2}{5}t + 8$.
$\Rightarrow \frac{1}{100}A = \frac{2}{5}$ and $\frac{1}{5}A + B = 8$
$\Rightarrow A = 40$, $B = 0$
So $Y(t) = 40t$
(This makes sense, since the cop car wouldn't accelerate or decelerate if it was on top of the speeding vehicle).

PTO

[Question continued on the next page]

General solution to the homogeneous DE $y'' + \frac{1}{5}y' + \frac{1}{100}y = 0$:

CE is $r^2 + \frac{1}{5}r + \frac{1}{100} = 0$

$\Rightarrow \quad r = \dfrac{-\frac{1}{5} \pm \sqrt{(\frac{1}{5})^2 - 4\cdot1\cdot\frac{1}{100}}}{2\cdot1}$ "critical damping!"

$= -\frac{1}{10}$ double root. i.e. $(r + \frac{1}{10})^2 = 0$

So homog. equation has solution $y = (c_1 + c_2 t)e^{-\frac{1}{10}t}$

Hence full nonhomogeneous DE has general solution

$y = (c_1 + c_2 t)e^{-t/10} + 40t$

Now apply ICs: $y(0) = 0 \Rightarrow c_1 = 0$ So $y = c_2 t e^{-\frac{t}{10}} + 40t$

While $y' = c_2 e^{-\frac{t}{10}} + -\frac{c_2}{10} t e^{-\frac{t}{10}} + 40$

So $y'(0) = 0 \Rightarrow c_2 = -40$.

Hence $y(t) = -40te^{-\frac{t}{10}} + 40t$.

That is, the position of the cop car is given by

$$\boxed{y(t) = 40t(1 - e^{-\frac{t}{10}})}$$

(b) (Bonus: 4 points) Compute or estimate how long it would take after the chase begins for the cops to close to within 20 meters of their speeding target.

The cops close to within 20 meters of the target when $y(t) = 40t - 20$

So we must solve for t in the equation

$$40t(1-e^{-\frac{t}{10}}) = 40t - 20$$

$$\Rightarrow \cancel{40t} - 40te^{-\frac{t}{10}} = \cancel{40t} - 20$$

$$\Rightarrow te^{-\frac{t}{10}} = \frac{1}{2}$$

This is not an equation that can be solved explicitly, so we must resort to estimation techniques to find an approximate value of t that works.

By plugging in sample values, we see $40e^{-\frac{40}{10}} \approx 0.733 > 0.5$ and $50e^{-\frac{50}{10}} \approx 0.337 < 0.5$,

so the time we're looking for is somewhere between 40 & 50 seconds. (In fact, $45e^{-\frac{45}{10}} = 0.49990$, so the value is *very* close to 45 seconds).

We could improve the above guess by using Newton's method (Recall NM: to find the root of $f(t)$, guess a value t_0 near the root, then set $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$; this sequence will converge on the root.

So if we set $t_0 = 45$ and $f(t) = te^{-\frac{t}{10}} - \frac{1}{2} \Rightarrow f'(t) = e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}$

$$\Rightarrow \frac{f}{f'} = \frac{te^{-\frac{t}{10}} - \frac{1}{2}}{e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}} = \frac{t - \frac{1}{2}e^{\frac{t}{10}}}{1 - \frac{1}{10}t}$$

then $t_1 = 45 - \frac{45 - \frac{1}{2}e^{4.5}}{1 - 4.5} = 45 + \frac{1}{3.5}(45 - \frac{1}{2}e^{4.5}) = \boxed{44.9976 \text{ seconds is our estimate}}$

This is likely more accurate than we'll ever need, so we can stop here.

Note There are *many* different ways to estimate the time when the two cars are 20 m apart; what I'm looking for here is your ability to come up with an answer in the face of an equation that can't be solved exactly.