Homework 3

Total: 20 points
Due: Wed 22 Oct 2014 at the beginning of class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **EXISTENCE & UNIQUENESS**

   (a) In each of the following problems, use the existence and uniqueness theorems to determine (without solving the DE) the largest interval in which the solution to the given initial value problem is guaranteed to exist. Write a sentence or two justifying your answer.

   i. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$ Boyce 2.4 Q1

   ii. $t(t-4)y' + y = 0, \quad y(2) = 1$ Q2

   iii. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$ Q3

   iv. $(\ln t)y' + y = \cot t, \quad y(2) = 3$ Q6

   (b) Find all solutions to the initial value problem

$$(y')^2 - y = 4, \quad y(2) = 0.$$

Clearly this function is nonlinear. We can't apply the existence/uniqueness theorem for nonlinear ODEs until we have it in the for $y' = f(x, y)$. Solving for y' in the above equation yields two distinct options:

$$\frac{dy}{dt} = \sqrt{y+4} \quad \text{and} \quad \frac{dy}{dt} = -\sqrt{y+4}$$

We must treat each of these separately.

Consider the first option: $\frac{dy}{dt} = \sqrt{y+4}$ with $y(2) = 0$. Both $\sqrt{y+4}$ and $\frac{\partial}{\partial y}\sqrt{y+4} = \frac{1}{2\sqrt{y+4}}$ are continuous about y=0, so the existince/uniqueness theorem guarantees that there is a unique solution to the IVP in some interval about $x = 2$.
We proceed to solve this DE. It is separable, so

$$\frac{1}{\sqrt{y+4}}\,dy = dt$$

Integrating yields

$$2\sqrt{y+4} = t + C,$$

or, solving for y,

$$y = \left(\frac{1}{2}t + \frac{1}{2}C\right)^2 - 4$$

Applying the initial value: $t = 2, y = 0$ yields either $C = 2$ or $C = -6$; however, only the value $C = 2$ obeys the original equation $2\sqrt{y+4} = t + C$ when $t = 2$ and $y = 0$ (by squaring both sides we've introduced an extra solution for C). The solution to the DE is thus, after simplifying,

$$y = \frac{1}{4}t^2 + t - 3.$$

For the second option $y' = -\sqrt{y+4}$ we proceed in much the same manner. Again we are guaranteed a single unique solution. Solving the separable DE yields

$$-2\sqrt{y+4} = t + C \quad \Longrightarrow \quad y = \left(-\frac{1}{2}t - \frac{1}{2}C\right)^2 - 4.$$

Applying the IC $y(2) = 0$ to the latter equation yields $C = 2$ or $C = -6$, but similar to before only $C = -6$ satisfies $-2\sqrt{y+4} = t + C$ when $y(2) = 0$. Hence, after simplifying, we have

$$y = \frac{1}{4}t^2 - 3t + 5.$$

The existence/uniqueness theorem guarantees that these are the only two solutions to the IVP.

(c) Consider the initial value problem

$$\frac{dy}{dt} = (y-1)^{\frac{1}{5}}, \quad y(0) = y_0$$

i. For which value of y_0 does the IVP *not* have a unique solution in some interval about $x = 0$?

The existence/uniqueness theorem for nonlinear ODEs in the form $y' = f(x, y),\ y(x_0) = y_0$ dictates that we may only not have a unique solution when either f or $\frac{\partial f}{\partial y}$ is discontinuous at (x_0, y_0).
For us $f(x, y) = (y-1)^{\frac{1}{5}}$; this function is continuous for all real x and y. However

$$\frac{\partial f}{\partial y} = \frac{1}{5}(y-1)^{-\frac{4}{5}}$$

is discontinuous at $y = 1$; It therefore follows that the only possible instance of non-uniqueness can occur when $y(0) = 1$.

ii. For this value of y_0, find all continuous differentiable functions which satisfy the differential equation.

This is a separable differential equation, so

$$(y-1)^{-\frac{1}{5}}\ dy = dt.$$

Antidifferentiating yields

$$\frac{5}{4}(y-1)^{\frac{4}{5}} = t + C;$$

solving for y then has

$$y = 1 + \left(\frac{4}{5}t + C\right)^{\frac{5}{4}},$$

where we absorb the constant multiplier $\frac{4}{5}$ into the C.
Applying the initial condition $y(0) = 1$ has $C = 0$, so

$$y = 1 + \left(\frac{4}{5}t\right)^{\frac{5}{4}}$$

is one solution to the DE.

Now note that $y = 1$ is also a solution to the IVP (this can be done by eyeballing the differential equation); further note that the equation

$$y = 1 - \left(\frac{4}{5}t\right)^{\frac{5}{4}}$$

also solve the IVP.

Next, note that $\frac{d}{dt}\ 1 + \left(\frac{4}{5}t\right)^{\frac{5}{4}} = 0$ when $y = 1$, as does $\frac{d}{dt}1 - \left(\frac{4}{5}t\right)^{\frac{5}{4}} = 0$; what this means is that we can splice together the above functions, suitably shifted, into a new function and create yet more solutions. Specifically, for any constants $c_1 \leq 0$ and $c_2 \geq 0$, the generalized function

$$y(t) = \begin{cases} 1 \pm \left(\frac{4}{5}(t - c_1)\right)^{\frac{5}{4}} & t < c_1 \\ 1 & c_1 \leq t \leq c_2 \\ 1 \pm \left(\frac{4}{5}(t - c_2)\right)^{\frac{5}{4}} & t > c_2 \end{cases}$$

is a continuous and continuously differentiable function that also solves the differential equation. This is in fact the most general solution to the DE.

2. **AUTONOMOUS EQUATIONS**

(a) In the following two autonomous equations $\frac{dy}{dt} = f(y)$, sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each equilibrium solution as asymptotically stable, unstable or semistable. Then sketch a graph of several solutions on the ty-plane, including the equilibrium solutions and a few other solutions to indicate asymptotic behaviour.

i. $\frac{dy}{dt} = y(y-1)(y-2)$ Boyce 2.5 Q3

ii. $\frac{dy}{dt} = e^{-y} - 1$ Boyce 2.5 Q5

(b) Boyce 2.5 Q18

A pond forms as water collects in a conical depression of radius a and depth h. Suppose that the water flows in at a constant rate k, and is lost through evaporation at a rate proportional to the pond's surface area.

i. Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}},$$

where α is the coefficient of evaporation.

ii. Find the equilibrium depth of the water in the pond. Is the equilibrium asymptotically stable or unstable?

iii. Find a condition relating k and α that must be satisfied if the pond is not to overflow.

3. **EULER'S METHOD** Consider the initial value problem

$$\frac{dy}{dt} = (t-1)(y+1), \quad y(1) = 1.$$

(a) Let $y = \phi(t)$ be the unique solution to this IVP. Estimate the value of $\phi(2)$ using Euler's method with a step size of $h = 1$. Then do the same for step sizes of $h = 0.5$ and $h = 0.2$.

Recall that Euler's method for the IVP $\frac{dy}{dt} = f(t,y), \;\; y(t_0) = y_0$ with step size h is given by the scheme

- Set t_0 and y_0 to be the given initial conditions
- for $n \geq 0$ set $t_{n+1} = t_n + h$ and $y_{n+1} = y_0 + h \cdot f(t_n, y_n)$.

For us we have $f(t,y) = (t-1)(y+1)$, $t_0 = 1$ and $y_0 = 1$, and we begin with $h = 1$. Thus we have

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = 2$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + 1 \cdot (1-1)(1+1) = 1$

At this point we stop, as we've reached $t_1 = 2$.
Our first estimate for $y(2)$ using $h = 1$ is thus $y = 1$.

Now we repeat the process with $h = \frac{1}{2}$:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = \frac{3}{2}$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + \frac{1}{2} \cdot (1-1)(1+1) = 1$

- $t_2 = 2$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + \frac{1}{2} \cdot (\frac{3}{2} - 1)(1 + 1) = \frac{3}{2}$

At this point we stop, as we've reached $t_2 = 2$.
Our estimate for $y(2)$ using $h = \frac{1}{2}$ is thus $y = \frac{3}{2}$.

Finally, we repeat the process with $h = \frac{1}{5}$:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = \frac{6}{5}$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + \frac{1}{5} \cdot (1 - 1)(1 + 1) = 1$
- $t_2 = \frac{7}{5}$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + \frac{1}{5} \cdot (\frac{6}{5} - 1)(1 + 1) = \frac{27}{25}$
- $t_3 = \frac{8}{5}$ and $y_3 = y_2 + h \cdot f(t_2, y_2) = \frac{27}{25} + \frac{1}{5} \cdot (\frac{7}{5} - 1)(\frac{27}{25} + 1) = \frac{779}{625}$
- $t_4 = \frac{9}{5}$ and $y_4 = y_3 + h \cdot f(t_3, y_3) = \frac{779}{625} + \frac{1}{5} \cdot (\frac{8}{5} - 1)(\frac{779}{625} + 1) = \frac{23687}{15625}$
- $t_5 = 2$ and $y_5 = y_4 + h \cdot f(t_4, y_4) = \frac{23687}{15625} + \frac{1}{5} \cdot (\frac{9}{5} - 1)(\frac{23687}{15625} + 1) = \frac{749423}{390625}$

At this point we stop, as we've reached $t_2 = 2$.
Our estimate for $y(2)$ using $h = \frac{1}{2}$ is thus $y = \frac{749423}{390625}$.

We repeat the last set of calculations using decimals (given to four decimal places, but calculations are done to higher precision), since this is the route that many people will choose:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = 1.2$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + 0.2 \cdot (1 - 1)(1 + 1) = 1$
- $t_2 = 1.4$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + 0.2 \cdot (1.2 - 1)(1 + 1) = 1.08$
- $t_3 = 1.6$ and $y_3 = y_2 + h \cdot f(t_2, y_2) = 1.08 + 0.2 \cdot (1.4 - 1)(1.08 + 1) = 1.2464$
- $t_4 = 1.8$ and $y_4 = y_3 + h \cdot f(t_3, y_3) = 1.2464 + 0.2 \cdot (1.6 - 1)(1.2464 = 1.5160$
- $t_5 = 2$ and $y_5 = y_4 + h \cdot f(t_4, y_4) = 1.5160 + 0.2 \cdot (1.8 - 1)(1.5160 + 1) = 1.9185$

Our estimates of $y(2)$ using $h = 1$, $h = 0.5$ and $h = 0.2$ are thus 1, 1.5 and 1.9185 respectively.

(b) Solve the IVP and state the true value of $\phi(2)$. Do your estimates underpredict or overpredict $\phi(2)$? Do they get more accurate as h decreases?

The equation $\frac{dy}{dt} = (t-1)(y+1)$ is both linear and separable; we'll solve it using separation of variables. We have

$$\frac{1}{y+1}\, dy = (t-1)\, dt$$

The left hand side integrates to $\ln|y+1|$, and the right hand side integrates to $\frac{1}{2}t^2 - t + C$. Setting these two expressions equal to each other and exponentiating both sides then yields

$$y + 1 = Ae^{\frac{1}{2}t^2 - t},$$

where $A = \pm e^C$ as necessary to deal with the absolute value signs. Hence $y = Ae^{\frac{1}{2}t^2 - t} - 1$.

Now we apply the initial condition $y(1) = 1$ to get

$$1 = Ae^{\frac{1}{2}\cdot 1^2 - 1} - 1,$$

so $A = 2e^{\frac{1}{2}}$. Thus

$$y = 2e^{\frac{1}{2} + \frac{1}{2}t^2 - t} - 1 = 2e^{\frac{1}{2}(t-1)^2} - 1.$$

The value of $y(2)$ is therefore

$$y(2) = 2e^{\frac{1}{2}(2-1)^2} - 1 = 2\sqrt{e} - 1 = 2.2974\ldots$$

Comparing this value with our estimates, we see that the estimates all underpredict the true value of $y(2)$ - this is because $\frac{dy}{dt}$ is increasing in the region of the ty-plane that we're considering, so tangent line approximations will always undershoot the graph of the true solution. However, we do note that the Euler's method approximations are getting more accurate as h gets smaller, as they are supposed to do.

4. **2ND ORDER LINEAR DIFFERENTIAL EQUATIONS**

(a) In each of the following problems, find the general solution to the given differential equation.

i. $y'' + 2y' - 3y = 0$ Boyce 3.1 Q1

ii. $y'' + 3y' + 2y = 0$ Boyce 3.1 Q2

iii. $y'' + 5y' = 0$ Boyce 3.1 Q5

iv. $y'' - 2y' - 2y = 0$ Boyce 3.1 Q8

(b) In each of the following problems, find the solution to the given initial value problem, and sketch a graph of th solution, indicating the behaviour as t increases.

i. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$ Boyce 3.1 Q9

ii. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$ Boyce 3.1 Q10

iii. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$ Boyce 3.1 Q12

(c) Boyce 3.1 Q23

Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0,$$

where α is a given constant. Determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.