The Explicit Formula for Elliptic Curves

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June 7, 2013

1 Introduction

This work comes out of a challenge problem from Barry Mazur in the paper 'How Explicit is the Explicit Formula?' [M&S], which is joint work with William Stein. In it they ask how explicit can we make the explicit formula when applied to elliptic curve L-function?

To further qualify this question we will need a few definitions.

Let $g(s) = \sum_n a_n n^{-s}$ be a Dirichlet series with meromorphic continuation to $\mathbb{C}$, and let σ_0 be the mantissa of g, i.e. $\sum_n a_n n^{-s}$ converges absolutely for $\mathrm{Re}(s) > \sigma_0$.

Definition 1.1. For $x > 1$, let

$$\psi_g(x) := \sum_{n=1}^{\lfloor x \rfloor} a_n \tag{1.1}$$

be the cumulative sum function of the Dirichlet coefficients of g, extended to $\mathbb{R}_{>0}$, and let

$$\psi_{g,0}(x) := \begin{cases} \psi_g(x) & x \notin \mathbb{Z} \\ \frac{1}{2}\left[\psi_g(x) + \psi_g(x+1)\right] & x \in \mathbb{Z}, \end{cases} \tag{1.2}$$

i.e. the same as $\psi_g(x)$, except at integer values, where $\psi_{g,0}(x)$ takes a value halfway between $\psi_g(x)$ and $\psi_g(x+1)$.

The Explicit Formula is a method of writing $\psi_{g,0}(x)$ as a sum over the poles of g. Specifically, by Perron's Formula we have that

$$\psi_{g,0}(x) = \lim_{T\to\infty} \frac{-1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} g(s) \frac{x^s}{s}\, ds \tag{1.3}$$

for $\sigma > \sigma_0$ (See [M&V, 5.1] for a good reference).

The idea is then to pull this integral to the left, picking up residues as we go. If the poles of g are all simple, then we obtain

$$\psi_{g,0}(x) = -\operatorname{Res}\left(g(s)\frac{x^s}{s}, 0\right) - \sum_\rho \frac{x^\rho}{\rho}, \tag{1.4}$$

where ρ runs over the nonzero poles of g. Note that the case at zero must be dealt with separately, since $g(s)\frac{x^s}{s}$ may have a pole of order 1 or 2 at zero.

In fact Bernhard Riemann did exactly this for the zeta function which now bears his name, and it was his attempt to understand and quantify the sum across zeros which led to his famous hypothesis, which remains unproven to this day. It is the statement that all the nontrivial zeros of an L-function lie along a single vertical line in the complex plane that allows us to rewrite the explicit formula more explicitly, namely as the sum of a constant term, a well-understood error term, and an oscillatory term whose statistical behavior is at east conjecturally understood.

There is some work that needs to be done to show that explicit formula for an arbiterry Dirichlet series is indeed valid. Specifically, one must show that the other legs of the contour integral one ends up taking (e.g. $\int_{\sigma+iT}^{-\infty+iT} g(s)\frac{x^s}{s}\, ds$) go to zero as $T \to \infty$.

In the context of this paper we will use the explicit formula on the logarithmic derivative of elliptic curve L-functions, as the sum over poles of the logarithmic derivative corresponds to a sum over zeros of the original L-function. It is thus provides information on how the distribution of nontrivial zeros of the L-function relates to the Dirichlet coefficients of the logarithmic derivative. Specifically, Mazur argues how we can manipulate the explicit formula into something that looks like the following:

$$\text{Sum of local data} = \text{Global Data} + \text{Error Term} + \text{Oscillatory Term} \tag{1.5}$$

This gives a direct link between local and global data on an elliptic curve, and shows how the distribution of zeros of the curve's L-function is intimately linked to the various invariants of the curve, both local and global.

The aim of this paper is to threefold: to provide a more complete description of the logarithmic derivative of L-functions attached to elliptic curves; to prove the explicit formula for these L-functions, and in so doing provide explicit bounds on the error when truncated sums of zeros are taken; and to fully understand the constant and error terms that arise from the residue at zero and sums over trivial zeros.

2 Definitions and Preliminary Statements

First, some notation. For the purpose of this paper, we will always have that

- E is an elliptic curve over the rational numbers with conductor N (a positive integer)
- p is a (rational) prime number
- x is a real number > 1
- s and z are generic complex numbers
- η is the Euler-Mascheroni constant $= 0.5772156649\ldots$
- ρ will always be used to denote complex zeros of an L-function
- γ will always be used to denote the imaginary parts of nontrivial zeros of an L-function.

Let $f(z) = \sum a_n(f) \cdot q^n \in S_k(\Gamma)$ be a weight k cusp form with respect to congruence subgroup $\Gamma_0(N)$.

Definition 2.1. The *L-function* attached to f is the Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} a_n(f) n^{-s}, \tag{2.1}$$

Definition 2.2. The *completed L-function* attached to f is

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s), \tag{2.2}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^s \frac{dt}{t}$ is the usual gamma function on $\mathbb{C}$.

Remark 2.3. There is an important generalization of the above definition, that of L-functions of cusp forms with nontrivial character χ (or equivalently of cusp forms over $\Gamma_1(N)$). We will not discuss them here, but many of the results in this paper should generalize to modular L-functions of nontrivial character, and the hope is to prove these results in future work. See [S&R, Section 16] for definitions and some basic results.

For the duration of this paper we will subsume into subscript the modular form (and later elliptic curve) attached to L and Λ, i.e. write $L_f(s) := L(f,s)$ and $\Lambda_f(s) := \Lambda(f,s)$ respectively.

We may realize $\Lambda_f(s)$ as a Mellin transform of f:

$$\Lambda_f(s) = N^{s/2} \int_0^\infty f(it) y^t \frac{dt}{t}; \tag{2.3}$$

this integral converges for $\mathrm{Re}(s) > 1 + \frac{k}{2}$. As such, the Dirichlet series for $L_f(s)$ converges for $\mathrm{Re}(s) > 1 + \frac{k}{2}$. A proof of this can be found in most standard texts on modular L-functions.

Proposition 2.4. *$L_f(s)$ admits holomorphic continuation to $\mathbb{C}$ via a functional equation relating $\Lambda_f(s)$ with $\Lambda_{W(f)}(k-s)$, where $W(f) \in S_k(\Gamma)$ is the image of f under the Atkin-Lehner operator W (see [S&R, Section 16.1.1] for the complete statement). Specifically, when $k = 2$ we have that $\Lambda_f(s)$ obeys the functional equation*

$$\Lambda_f(s) = w\Lambda_f(2-s), \tag{2.4}$$

where $w = -1$ if the order of vanishing of the zero of $\Lambda_f(s)$ at $s = 1$ is odd, and $w = 1$ otherwise.

Proposition 2.5. *If f is a weight k newform on level N, then the Dirichlet series for $L_f(s)$ converges for* $\mathrm{Re}(s) > \frac{k+1}{2}$. *Furthermore, $L_f(s)$ can be written in terms of an Euler Product:*

$$L_f(s) = \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + p^{k-1}p^{2s}}, \tag{2.5}$$

and this product converges (absolutely) for $\mathrm{Re}(s) > \frac{k+1}{2}$. *Equivalently we may write*

$$L_f(s) = \prod_{p|N} \frac{p^s}{F(p^s)} \cdot \prod_{p\nmid N} \frac{p^{2s}}{F(p^s)}, \tag{2.6}$$

where $F(X) = X - a_p$ if $p \mid N$ and $F(X) = X^2 - a_pX + p^{k-1}$ if $p \nmid N$ is the characteristic polynomial of Frobenius at p acting on any ℓ-adic representation attached to f (with $\ell \neq p$).

Definition 2.6. Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N. Let $a_p = a_p(E) = p+1-\#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ is the number of points on E modulo p.
We define the L-function attached to E by the Euler product

$$L_E(s) = \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + p^{k-1}p^{2s}}. \tag{2.7}$$

If we expand the above product out into a Dirichlet series, i.e. $L_E(s) = \sum_n a_n n^{-s}$, then the a_n are given by the following relations: $a_1 = 1$; $a_{mn} = a_m a_n$ when $\gcd(m,n) = 1$; and for $r \geq 2$, $a_{p^r} = (a_p)^r$ when $p \mid N$ and $a_{p^r} = a_{p^{r-1}} a_p - p a_{p^{r-2}}$ when $p \nmid N$..

Theorem 2.7 (Breuille, Conrad, Diamond, Taylor, Wiles et al, 1999,2001)**.** *There exists an integral newform $f = \sum_n a_n q^n$ of of weight $k = 2$ and level N such that $L_E(s) = L_f(s)$. As such $L_E(s)$ extends to an entire function on $\mathbb{C}$ (that is, $L_E(s)$ has no poles on $\mathbb{C}$).*

Hence any theorem about elliptic curve L-functions are really theorems about L-functions of weight 2 newforms in disguise (this is essentially the converse of the theorem by Shimura in the 1960s: if f is a weight 2 newform of level N, then there exists some elliptic curve $E/\mathbb{Q}$ of conductor N such that $L_f(s) = L_E(s)$).

The nature of the zeros of L-functions is a topic of intense interest in number theory, to the point where they are the subject of two of the seven Millennium Prize Problems. For our purposes, the zeros of $L_f(s)$ can be divided into three categories:

- A countably infinite set of simple zeros at $\mathbb{Z}_{\leq 0}$, i.e. $\{0, -1, -2, -3, \ldots\}$; these are denoted the *trivial zeros* of $L_f(s)$, since their existence is guaranteed by the functional equation of $L_f(s)$.
- A potential zero at $s = k/2$; the order of this zero is denoted r_{an}, the analytic rank of $L_f(s)$; this zero, if it exists, is denoted the *central zero.*
- A countably infinite set of zeros in the strip $0 \leq \mathrm{Re}(s) \leq k$; these are called the *nontrivial zeros*, and their distribution is symmetric about both the real axis and the line $\mathrm{Re}(s) = \frac{k}{2}$.

Note that by convention the central zero is included in the nontrivial zeros; we will always assume so unless we state otherwise.

Conjecture 2.8 (Generalized Riemann Hypothesis)**.** *All nontrivial zeros for $L_f(s)$ are simple, and occur on the line* $\mathrm{Re}(s) = k/2$.

Remark 2.9. While GRH is not necessary to state the explicit formula for elliptic curves in its most general form, to be of any practical use we must know the location of the zeros of $L_E(s)$; as such from hereon we will always assume the Generalized Riemann Hypothesis unless we state otherwise.

The other Millennium Problem relating to zeros of L-functions is of course the *Birch and Swinnerton-Dyer Conjecture*. While it is not the topic of this paper, we state (one form of) it here for interest's sake:

Conjecture 2.10 (Birch, Swinnerton-Dyer)**.** *Let E be an elliptic curve over $\mathbb{Q}$. Then the expansion of $L_E(s)$ about the central point $s = 1$ is*

$$L_E(s) = (s-1)^r \left[\left(\frac{\Omega_E \cdot \mathrm{Reg}_E \cdot \#Ш(E/\mathbb{Q}) \cdot \prod_p c_p}{(\#E_{Tor}(\mathbb{Q}))^2} \right) + O((s-1)^1) \right], \tag{2.8}$$

where

- r *is the algebraic rank of* $E(\mathbb{Q})$*,*
- Ω_E *is the real period of (an optimal model of)* E*,*
- Reg_E *is the regulator of* E*,*
- $\#Ш(E/\mathbb{Q})$ *is the order of the Shafarevich-Tate group attached to* $E/\mathbb{Q}$*,*
- $\prod_p c_p$ *is the product of the Tamagawa numbers of* E*, and*
- $\#E_{Tor}(\mathbb{Q})$ *is the number of rational torsion points on* E*.*

For an excellent description of the conjecture and a breakdown of the arithmetic invariants mentioned above, see Andrew Wiles' official description of the BSD Conjecture on the Clay Math website [WilBSD].

3 The Logarithmic Derivative $\frac{L_E'}{L_E}(s)$

In this section we provide an explicit description for the Dirichlet series of the logarithmic derivative $\frac{L_f'}{L_f}(s)$, since this is necessary to both write down and prove the explicit formula for elliptic curve L-functions.

Let $f(z) = \sum a_n(f) \cdot q^n \in S_k(\Gamma)$ be a weight k newform w.r.t. congruence subgroup Γ of level N, and let $L_f(s)$ be the L-function attached to f.

Recall that for $p \nmid N$, the characteristic polynomial of Frobenius w.r.t. f at p is $x^2 - a_p(f)x + p^{k-1}$. Let this quadratic polynomial split as $(x - \alpha_p)(x - \beta_p)$ in $\overline{\mathbb{Q}}$ (where the dependance on f is understood).

Definition 3.1. For $n \in \mathbb{N}$, let

$$b_n(f) := \begin{cases} -\left(\alpha_p^e + \beta_p^e\right) \cdot \log(p), & n = p^e \text{ a prime power } (e \geq 1), \text{ and } p \nmid N \\ -a_p^e \cdot \log(p), & n = p^e \text{ and } p \mid N \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

Lemma 3.2. *The Dirichlet series for $\frac{L_f'}{L_f}(s)$ is given by*

$$\frac{L'(f,s)}{L(f,s)} := \frac{L_f'}{L_f}(s) = \sum_{n=1}^{\infty} b_n(f)\, n^{-s} \tag{3.2}$$

where the coefficients $b_n(f)$ are defined as in Definition 3.1.

Proof. Recall $L_f(s)$ is defined as the Euler product

$$L(f,s) = \prod_{p \mid N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s}}, \tag{3.3}$$

which converges absolutely for $\mathrm{Re}(s) > \frac{k+1}{2}$.

The negative logarithmic derivative of $L(f,s)$ is thus, using the product rule,

$$\begin{aligned} -\frac{L_f'}{L_f}(s) &= -\sum_{p \mid N} \left(1 - a_p p^{-s}\right) \cdot \frac{d}{ds}\left[\left(1 - a_p p^{-s}\right)^{-1}\right] \\ &\quad - \sum_{p \nmid N} \left(1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s}\right) \cdot \frac{d}{ds}\left[\left(1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s}\right)^{-1}\right] \\ &= \sum_{p \mid N} \frac{a_p \log(p) \cdot p^{-s}}{1 - a_p p^{-s}} + \sum_{p \nmid N} \frac{a_p \log(p) \cdot p^{-s} - 2p^{k-1} \log(p) \cdot p^{-2s}}{1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s}}. \end{aligned} \tag{3.4}$$

We can expand each summand in terms of powers of p^{-s}; since these do not interact in any way we can analyze each one separately.

For $p \mid N$ we have

$$\begin{aligned}
-\frac{a_p \log(p) \cdot p^{-s}}{1 - a_p p^{-s}} &= -\left(a_p \log(p) \cdot p^{-s}\right)\left(1 + p^{-s} + p^{-2s} + p^{-3s} + \dots\right) \\
&= -a_p \log(p) \sum_{e=1}^{\infty} (p^e)^{-s} \\
&= \sum_{e=1}^{\infty} b_{p^e}(f) \cdot (p^e)^{-s}.
\end{aligned}$$

For $p \nmid N$, observe $1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s} = (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})$, so

$$\begin{aligned}
-\frac{a_p \log(p) \cdot p^{-s} - 2p^{k-1} \log(p) \cdot p^{-2s}}{1 - a_p p^{-s} + p^{k-1} \cdot p^{-2s}} &= -\log(p) \cdot p^{-s}\left(a_p - 2p^{k-1} \cdot p^{-s}\right)\left(1 - \alpha_p p^{-s}\right)^{-1}\left(1 - \beta_p p^{-s}\right)^{-1} \\
&= -\log(p) \cdot p^{-s}\left((\alpha_p + \beta_p) - 2\alpha_p \beta_p p^{-s}\right)\left(\sum_{j=0}^{\infty} \alpha_p^j p^{-js}\right)\left(\sum_{k=0}^{\infty} \beta_p^j p^{-js}\right) \\
&= -\log(p) \cdot p^{-s}\left((\alpha_p + \beta_p) - 2\alpha_p \beta_p p^{-s}\right)\left(\sum_{e=0}^{\infty}\left(\sum_{m=0}^{e} \alpha_p^m \beta_p^{e-m}\right)(p^e)^{-s}\right).
\end{aligned}$$

We get telescoping cancellation in the above double sum, and the result is

$$\begin{aligned}
&-\log(p) \cdot p^{-s} \sum_{e=0}^{\infty} (\alpha^{e+1} + \beta^{e+1}) \log(p) \cdot (p^e)^{-s} \\
= &\sum_{e=1}^{\infty} -(\alpha^e + \beta^e) \log(p) \cdot (p^e)^{-s}, \ \text{ rebasing the sum,} \\
= &\sum_{e=1}^{\infty} b_{p^e}(f) \cdot (p^e)^{-s}.
\end{aligned}$$

So finally we have that

$$\begin{aligned}
\frac{L_f'}{L_f}(s) &= \sum_p \sum_{e \geq 1} b_{p^e}(f) \cdot (p^e)^{-s} \\
&= \sum_n b_n(f)\, n^{-s},
\end{aligned}$$

since all other terms are zero. □

It is often easier to work with the shifted logarithmic derivative $\frac{L_f'}{L_f}(s+1)$, so we define notation for its Dirichlet series below:

Definition 3.3. The logarithmic derivative of the shifted L-function $L_f(s+1)$ is given by Dirichlet series

$$\frac{L_f'}{L_f}(s+1) := \sum_n c_n n^{-s} = \sum_n \frac{b_n}{n} n^{-s}, \tag{3.5}$$

where the b_n are defined as above.

Proposition 3.4. *When $f = f_E$ is the new form attached to elliptic curve $E/\mathbb{Q}$, we have*

$$b_n(E) := b_n(f_E) = \begin{cases} -\left(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e})\right) \cdot \log(p), & n = p^e \text{ a prime power,} \\ 0, & \text{otherwise} \end{cases} \tag{3.6}$$

where $\#\widetilde{E}(\mathbb{F}_{p^e})$ is the number of points over $\mathbb{F}_{p^e}$ on the (possibly singular) projective curve obtained by reducing a minimal model of E modulo p.

Proof. It is a standard result that if $(x - \alpha_p)(x - \beta_p)$ is the characteristic polynomial for Frobenius on E at prime p of good reduction, then

$$\#E(\mathbb{F}_{p^e}) = p^e + 1 - \alpha_p^e - \beta_p^e \tag{3.7}$$

(see [Silv, pp. 134-136] for a proof), from which the result at $p \nmid N$ follows.

For primes of bad reduction, recall

$$a_p(E) := \begin{cases} +1, & E \text{ has split multiplicative reduction at } p \\ -1, & E \text{ has non-split multiplicative reduction at } p \\ 0, & E \text{ has additive multiplicative reduction at } p \end{cases} \tag{3.8}$$

Let $E_{\rm ns}(\mathbb{F}_{p^e})$ be the group of nonsingular points on $\widetilde{E}(\mathbb{F}_{p^e})$.
When E has additive reduction at p, $E_{\rm ns}(\mathbb{F}_{p^e}) \simeq (\mathbb{F}_{p^e}, +)$, so together with the singular point $\#\widetilde{E}(\mathbb{F}_{p^e}) = p^e+1$; Hence $(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e}))\log(p) = 0 = a_p^e \log(p)$.
When E has split multiplicative reduction at p, $E_{\rm ns}(\mathbb{F}_{p^e}) \simeq (\mathbb{F}_{p^e}^*, \times)$, so together with the singular point $\#\widetilde{E}(\mathbb{F}_{p^e}) = (p^e - 1) + 1 = p^e$; So $(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e}))\log(p) = 1 \cdot \log(p) = a_p^e \log(p)$.
When E has non-split multiplicative reduction at p, let $L/\mathbb{F}_{p^e}$ be the quadratic extension obtained by adjoining to $\mathbb{F}_{p^e}$ the slopes of the tangent lines at the singular point; then $E_{\rm ns}(\mathbb{F}_{p^e}) \simeq \ker(\mathrm{Norm}_{L/\mathbb{F}_{p^e}})$.
Some thought should convince you that there are $p^e - (-1)^e$ elements in L with norm 1, so together with the singular point $\#\widetilde{E}(\mathbb{F}_{p^e}) = p^e + 1 - (-1)^e$;
Hence $(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e}))\log(p) = (-1)^e \cdot \log(p) = a_p^e \log(p)$. See [Silv, pg. 180, Prop. 5.1] for the proofs of the above isomorphisms.

□

4 The Completed Logarithmic Derivative $\frac{\Lambda_E'}{\Lambda_E}(s)$

It turns out that the completed logarithmic derivative $\frac{\Lambda_E'}{\Lambda_E}(s)$ has a particularly elegant representation as a sum over the nontrivial zeros of $L_E(s)$; this section states and proves that representation, as well as the surprising corollary thereof about the series expansion of $L_E(s)$ about $s = 1$.

From hereon fix an elliptic curve $E/\mathbb{Q}$ of conductor N. Recall that the completed L-function for E is given by

$$\Lambda_E(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s).$$

Logarithmic differentiating the above gives a simple relation between $\frac{L_E'}{L_E}(s)$ and $\frac{\Lambda_E'}{\Lambda_E}(s)$:

$$\frac{\Lambda_E'}{\Lambda_E}(s) = \log\left(\frac{\sqrt{N}}{2\pi}\right) + F(s) + \frac{L_E'}{L_E}(s), \tag{4.1}$$

where $F(s) = \frac{\Gamma'}{\Gamma}(s)$ is the digamma function on $\mathbb{C}$.

Proposition 4.1. *$\frac{\Lambda_E'}{\Lambda_E}(s+1)$ is an odd function on $\mathbb{C}$. That is, $\frac{\Lambda_E'}{\Lambda_E}(s+1) = -\frac{\Lambda_E'}{\Lambda_E}(-s+1)$.*

Proof. This follows directly from logarithmically the functional equation $\Lambda_E(s) = w\Lambda_E(2-s)$ and shifting the results one unit to the left. □

Lemma 4.2.

$$\frac{\Lambda_E'}{\Lambda_E}(s+1) = \sum_\gamma \frac{s}{s^2+\gamma^2}, \tag{4.2}$$

where the sum is taken over all nontrivial zeros (including the central zero with multiplicity), and the sum converges absolutely for any given s outside the set of nontrivial zeros for $L_E(s+1)$.

Proof. Observe that $\Lambda_E(s+1)$ has a zero of order r_{an} at the origin, and by GRH all other zeros of $\Lambda_E(s+1)$ are simple, lie on the imaginary axis, and are symmetric about the origin.

Now since $\Lambda_E(s+1)$ is an entire function of finite order, we may express it as a Hadamard product over its zeros; as with the Hadamard product for the Riemann Zeta function this product is particularly simple:

$$\Lambda_E(s+1) = As^{r_{an}} \prod_{\gamma\neq 0}\left(1 - \frac{s}{i\gamma}\right), \tag{4.3}$$

where A is some constant (whose value is of course the subject of the BSD conjecture), and for convergence the product should be taken over conjugate pairs of zeros.
Logarithmic differentiation then yields

$$\begin{aligned}\frac{\Lambda_E'}{\Lambda_E}(s+1) &= \frac{r_{an}}{s} + \sum_{\gamma\neq 0}\frac{-i\gamma}{1-\frac{s}{i\gamma}} \\ &= \frac{r_{an}}{s} + \sum_{\gamma\neq 0}\frac{1}{s-i\gamma},\end{aligned}$$

where again the sum converges if taken over pairs of nontrivial zeros. Now

$$\frac{1}{s-i\gamma} + \frac{1}{s+i\gamma} = \frac{2s}{s^2+\gamma^2},$$

so if we rewrite the sum by averaging over pairs of zeros and incorporating the central zero $\gamma = 0$ with multiplicity, we get

$$\frac{\Lambda'_E}{\Lambda_E}(s+1) = \sum_{\gamma} \frac{s}{s^2+\gamma^2}$$

as desired, and the sum converges absolutely for any s not coinciding with a nontrivial zero. $\square$

Corollary 4.3. *The Laurent expansion of $\frac{\Lambda'_E}{\Lambda_E}(s+1)$ about zero is given by*

$$\frac{\Lambda'_E}{\Lambda_E}(s+1) = \frac{r_{an}}{s} + \left(\sum_{\gamma\neq 0}\gamma^{-2}\right) s - \left(\sum_{\gamma\neq 0}\gamma^{-4}\right) s^3 + \left(\sum_{\gamma\neq 0}\gamma^{-6}\right) s^5 - \dots \tag{4.4}$$

$$= \frac{r_{an}}{s} + \sum_{k=1}^{\infty}\left[(-1)^{k-1}\left(\sum_{\gamma\neq 0}\frac{1}{\gamma^{2k}}\right)\right] s^{2k-1} \tag{4.5}$$

and this converges for $|s| < \gamma_0$ the first nontrivial zero of $L_E(s+1)$ on the imaginary axis.

The proof of this follows immediately by expanding the sum in Equation 4.2 and collecting terms.

Corollary 4.4. *Let a, b and c be the first three coefficients in the expansion of $L_E(s)$ about $s = 1$, i.e.*

$$L_E(s+1) = s^{r_{an}}\left[a + b\cdot s + c\cdot s^2 + O(s^3)\right] \tag{4.6}$$

(here a is the coefficient whose value is given by the BSD conjecture).
Then we may express the ratios b/a and c/a in terms of simple formulae, namely

$$\frac{b}{a} = \eta + \log\left(\frac{2\pi}{\sqrt{N}}\right) \tag{4.7}$$

$$2\frac{c}{a} = \left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right]^2 - \frac{\pi^2}{6} + \sum_{\gamma\neq 0}\gamma^{-2} \tag{4.8}$$

Proof. We note that the digamma function has a particularly elegant expansion about $s = 1$:

$$\digamma(s+1) = -\eta - \sum_{k=1}^{\infty}(-1)^k\zeta(k+1)s^k, \tag{4.9}$$

where η is the Euler-Mascheroni constant, and $\zeta(s)$ is the Riemann zeta function.
Thus by equation 4.1 and Corollary 4.3 we have that

$$\frac{L'_E}{L_E}(s+1) = \frac{r_{an}}{s} + \left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right] + \left[-\zeta(2) + \sum_{\gamma\neq 0}\gamma^{-2}\right]\cdot s + O(s^2)$$

But if $L_E(s+1) = s^{r_{an}}\left[a + b\cdot s + c\cdot s^2 + O(s^3)\right]$, then careful logarithmic differentiation yields

$$\frac{L'_E}{L_E}(s+1) = \frac{r_{an}}{s} + \frac{b}{a} + \left[-\frac{b^2}{a^2} + 2\frac{c}{a}\right]\cdot s + O(s^2)$$

Comparing terms and solving for $\frac{b}{a}$ and $\frac{2c}{a}$ produces the desired formulae. $\square$

We may continue in the same vein to produce formulae for higher order coefficients of $L_E(s)$. As can be seen from above, these can in general be written in terms of sums of powers of $\left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right]$, inverse sums of even powers of the nontrivial zeros, and values of the zeta function at the integers.

What's also worth pointing out here is that the above suggests that the Taylor expansion about $s = 1$ of the L-series attached to E essentially contains no new information about the curve's attached invariants beyond that which can be found in the first nonzero coefficient.

5 The Explicit Formula for Elliptic Curve L-Functions

We will derive two separate explicit formula for elliptic curve L-functions: one using the $\frac{L_E'}{L_E}(s)$, and the other $\frac{L_E'}{L_E}(s+1)$.

Let $b_n = b_n(E)$ and $c_n = b_n/n$ be defined as in Definitions 3.1 and 3.3.

Definition 5.1. For $x > 1$, let

$$\psi(x) := \sum_{n=1}^{\lfloor x \rfloor} b_n \tag{5.1}$$

i.e. the cumulative sum function of the b_n's, extended to $\mathbb{R}_{>0}$, and let

$$\psi_0(x) := \begin{cases} \psi(x) & x \notin \mathbb{Z} \\ \frac{1}{2}\left[\psi(x) + \psi(x+1)\right] & x \in \mathbb{Z}, \end{cases} \tag{5.2}$$

i.e. $\psi_0 = \psi$, except at integer points where it takes a value halfway between $\psi(x)$ and $\psi(x+1)$.
Similarly, let

$$\widetilde{\psi}(x) := \sum_{n=1}^{\lfloor x \rfloor} c_n = \sum_{n=1}^{\lfloor x \rfloor} \frac{b_n}{n}, \tag{5.3}$$

and define $\widetilde{\psi}_0(x)$ analogously to $\psi_0(x)$.

Observe that $\psi(x)$ and $\widetilde{\psi}(x)$ are the cumulative sum functions of the Dirichlet coefficients of $\frac{L_E'}{L_E}(s)$ and $\frac{L_E'}{L_E}(s+1)$ respectively.

We are now ready to state the two explicit formulae for elliptic curve L-functions.

Theorem 5.2. *Let $x > 1$ be a real number greater than 1. We have*

$$\psi_0(x) = -\operatorname{Res}\left(\frac{L_E'}{L_E}(s)\,\frac{x^s}{s}, 0\right) - \sum_{\rho \neq 0} \frac{x^\rho}{\rho} \quad \textit{and} \tag{5.4}$$

$$\widetilde{\psi}_0(x) = -\operatorname{Res}\left(\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}, 0\right) - \sum_{\rho \neq 1} \frac{x^{(\rho-1)}}{\rho - 1}, \tag{5.5}$$

where ρ runs over the zeros of $L_E(s)$, excepting for $\rho = 0$ in the first instance and (potentially) $\rho = 1$ in the second, and summands are counted with multiplicity corresponding to the degree of the zero.

Note that the statements above do not require GRH.

This is just a direct application of the explicit formula to the Dirichlet series $\frac{L_E'}{L_E}(s)$ and $\frac{L_E'}{L_E}(s+1)$ respectively, and as such are not particularly enlightening. The explicit formula in this generality has been proven for a wide class of L-functions, including modular L-functions and thus those attached to elliptic curves; the statements above follow, for example, from Theorem 5.11 in [I&K] (however, all versions I've found in the literature only seem to be 'big-Oh'-type proofs, i.e. ones without explicit constants in front of the error terms). As such we will defer a proof for now, although we will end up reproving the explicit formula for $\frac{L_E'}{L_E}(s+1)$ in the next section when we obtain concrete error bounds for truncated sums.

The explicit formula becomes more concrete when we obtain explicit descriptions of the residue at zero term, and divide the infinite sums on the right into sums over trivial and nontrivial zeros. Under GRH we thus have:

Theorem 5.3.

$$\psi_0(x) = -C_0(E) - \log(x-1) - x \cdot \left(r_{an} + \sum_{\gamma \neq 0} \frac{x^{i\gamma}}{1+i\gamma} \right), \quad \textit{and} \tag{5.6}$$

$$\widetilde{\psi}_0(x) = -\widetilde{C}_0(E) - \log(1-x^{-1}) - r_{an}\log(x) - \sum_{\gamma \neq 0} \frac{x^{i\gamma}}{i\gamma}, \tag{5.7}$$

where

- $C_0(E)$ *and* $\widetilde{C}_0(E)$ *are the constant terms in the expansions of* $\frac{L_E'}{L_E}(s)$ *about* $s = 0$ *and* $s = 1$ *respectively and are thus independent of* x
- γ *runs over the nonzero imaginary parts of nontrivial zeros on the line* $\mathrm{Re}(s) = 1$.

For the proof of the above we will need a lemma.

Lemma 5.4.

$$\mathrm{Res}\left(\frac{L_E^{'}}{L_E}(s)\frac{x^s}{s}, 0 \right) = C_0(E) + \log(x), \quad \textit{and} \tag{5.8}$$

$$\mathrm{Res}\left(\frac{L_E^{'}}{L_E}(s+1)\frac{x^s}{s}, 0 \right) = \widetilde{C}_0(E) + r_{an}\log(x), \tag{5.9}$$

where $C_0(E)$ *and* $\widetilde{C}_0(E)$ *are defined as above.*

Proof. If $f(s)$ is an analytic function on $\mathbb{C}$ with a pole of order r at the origin, we may write $h(s) = s^r(a + bs + O(s^2))$ for some constants a and b. Logarithmic differentiation of the Taylor series yields that $\frac{h'}{h}(s) = r \cdot s^{-1} + \frac{b}{a} + O(s)$.
Since $L_E(s)$ has a simple zero at $s = 0$, we thus have

$$\frac{L_E'}{L_E}(s) = s^{-1} + C_0(E) + O(s).$$

where $C_0(E)$ is defined to be the coefficient of the constant term in the expansion of $\frac{L_E'}{L_E}(s)$ about $s = 0$.
We also have

$$\frac{L_E'}{L_E}(s+1) = r_{an} \cdot s^{-1} + \widetilde{C}_0(E) + O(s),$$

where $\widetilde{C}_0(E)$ is defined analogously, since $L_E(s)$ has a zero of order r_{an} at $s = 1$.

Now

$$\begin{aligned} \frac{x^s}{s} &= s^{-1} \cdot e^{\log(x)\cdot s} \\ &= s^{-1}\left(1 + \log(x)\cdot s + O(s^2)\right) \\ &= s^{-1} + \log(x) + O(s). \end{aligned}$$

So

$$\begin{aligned} \frac{L_E'}{L_E}(s) \cdot \frac{x^s}{s} &= \left(s^{-1} + C_0(E) + O(s)\right) \cdot \left(s^{-1} + \log(x) + O(s)\right) \\ &= s^{-2} + (C_0(E) + \log(x)) \cdot s^{-1} + O(1) \end{aligned}$$

and

$$\frac{L_E'}{L_E}(s+1)\cdot\frac{x^s}{s} = r_{an}s^{-2} + \left(\widetilde{C}_0(E)+\log(x)\right)\cdot s^{-1} + O(1)$$

analogously. And since residue of $h(s)$ at 0 is just the coefficient of s^{-1} in the expansion of $h(s)$ about zero, so this completes the proof.

□

Proof of Theorem 5.3. The explicit formula (Theorem 5.2 above) states that

$$\begin{aligned}\psi_0(x) &= -\operatorname{Res}\left(\frac{L_E'}{L_E}(s)\,\frac{x^s}{s},0\right) - \sum_{\rho\neq 0}\frac{x^\rho}{\rho}\\ &= -C_0(E) - \log(x) - \sum_{\rho\neq 0}\frac{x^\rho}{\rho} \quad \text{by Lemma 5.4.}\end{aligned}$$

We can split up the sum over zeros into a sum over the negative integers (omitting $s=0$) and a sum over the nontrivial zeros on the line $\mathrm{Re}(s)=1$. Now by GRH, all zeros of L_E are simple with the exception of the potential zero at $s=1$, which has multiplicity r_{an}. Accordingly, if γ runs over the nonzero imaginary parts of nontrivial zeros, we get

$$\begin{aligned}\sum_{\rho\neq 0}\frac{x^\rho}{\rho} &= \sum_{k=-1}^{-\infty}\frac{x^k}{k} + r_{an}\cdot x + \sum_{\gamma\neq 0}\frac{x^{1+i\gamma}}{1+i\gamma}\\ &= \log\left(1-\frac{1}{x}\right) + x\left(r_{an} + \sum_{\gamma\neq 0}\frac{x^{i\gamma}}{1+i\gamma}\right),\end{aligned}$$

since $x>1$ always. We thus get equation 5.6 once the two log terms are combined.

For equation 5.7, note that the zeros of $L_E(s+1)$ are identical to those of $L_E(s)$ but shifted one unit to the left. As such, the sum over zeros is modified accordingly; also, we can now add the zero which was previously at $s=0$ to again get the same sum over the negative integers, and hence the $\log(1-x^{-1})$ term.

However, the central zero now sits over the origin, so we must include the residue of $\frac{L_E'}{L_E}(s+1)\frac{x^s}{s}$ at zero. But by Lemma 5.4 this is $\widetilde{C}_0(E) + r_{an}\log(x)$; so we have accounted for all the terms in Equation 5.7. □

Further to the statements in Theorem 5.3, the constant terms $C_0(E)$ and $\widetilde{C}_0(E)$ can be explicitly described. We start with the second first.

Corollary 5.5.

$$\widetilde{C}_0(E) = \eta + \log\left(\frac{2\pi}{\sqrt{N}}\right), \tag{5.10}$$

where η is the Euler-Mascheroni constant $= 0.5772156649\ldots$ and N the conductor of E.

Proof. This follows directly from the proof of Corollary 4.4: the constant term in the expansion of $\frac{L_E'}{L_E}(s+1)$ at zero is precisely $\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)$. □

Lemma 5.6.

$$C_0(E) = 2\eta - 1 + \log\left(\frac{4\pi^2}{N}\right) + \sum_{p\|N}\left(\frac{a_p}{p^2-a_p}\right)\log(p) + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p), \tag{5.11}$$

where η is the Euler-Mascheroni constant $= 0.5772156649\ldots$ and N the conductor of E, and the final (infinite) sum converges absolutely.

Proof. The above may be obtained via the functional equation for E. By Equations 2.2 and 2.4 we have

$$L_E(s) = w \cdot \left(\frac{4\pi^2}{N}\right)^{s-1} \Gamma(2-s)\Gamma(s)^{-1} L_E(2-s). \tag{5.12}$$

Logarithmically differentiating yields

$$\frac{L_E'}{L_E}(s) = \log\left(\frac{4\pi^2}{N}\right) - F(2-s) - F(s) - \frac{L_E'}{L_E}(2-s), \tag{5.13}$$

where $F(s) = \frac{\Gamma'}{\Gamma}(s)$ is the digamma function on $\mathbb{C}$.
Now $F(2) = 1-\gamma$, while the Taylor expansion for F at $s=0$ is $F(s) = -s^{-1} - \gamma + O(s)$; hence

$$\begin{aligned}\lim_{s\to 0}\left(\frac{L_E'}{L_E}(s) - \frac{1}{s}\right) &= \log\left(\frac{4\pi^2}{N}\right) - F(2) - \frac{L_E'}{L_E}(2) + \lim_{s\to 0}\left(-F(s) - \frac{1}{s}\right) \\ &= 2\gamma - 1 + \log\left(\frac{4\pi^2}{N}\right) - \frac{L_E'}{L_E}(2), \end{aligned} \tag{5.14}$$

giving an alternate representation for the constant term for the expansion of $\frac{L_E'}{L_E}(s)$ at zero.
Finally, the Dirichlet series for $\frac{L_E'}{L_E}(s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{3}{2}$, so we may evaluate $\frac{L_E'}{L_E}(2)$ directly as an infinite sum. From equation 3.4, we have

$$\begin{aligned}-\frac{L_E'}{L_E}(2) &= \sum_{p\mid N} \frac{a_p \log(p)\cdot p^{-2}}{1-a_p p^{-2}} + \sum_{p\nmid N} \frac{a_p \log(p)\cdot p^{-2} - 2p\log(p)\cdot p^{-4}}{1 - a_p p^{-s} + p\cdot p^{-4}} \\ &= \sum_{p\mid N}\left(\frac{a_p}{p^2 - a_p}\right)\log(p) + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p).\end{aligned}$$

We note that when E has additive reduction at p i.e. $p^2 \mid N$, then $a_p = 0$, yielding final sum

$$\frac{L_E'}{L_E}(2) = -\sum_{p\| N}\left(\frac{a_p}{p^2 - a_p}\right)\log(p) - \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p), \tag{5.15}$$

completing the proof. □

Lemma 5.7. *For any $E/\mathbb{Q}$ with conductor N, $\sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p)$ is bounded by a constant; namely*

$$-2.3510\ldots < \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p) < 3.0103\ldots \tag{5.16}$$

Proof. This is just a manual computation using the Hasse bound on the a_p:

$$\begin{aligned}\sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p) &< \sum_{p}\left(\frac{2\sqrt{p}\cdot p - 2}{p^3 - 2\sqrt{p}\cdot p + 1}\right)\log(p) \quad \text{using } a_p \le 2\sqrt{p}\ \forall\, p \\ &= \sum_p \frac{2(p^{3/2}-1)}{(p^{3/2}-1)^2}\cdot\log(p) \\ &= \sum_p \frac{2\log(p)}{p^{3/2}-1}.\end{aligned}$$

The last line above is in fact $-2\frac{\zeta'}{\zeta}\left(\frac{3}{2}\right)$; the sum converges (absolutely) to a value $\approx 3.01047\ldots$

For the other side, using $-a_p \leq 2\sqrt{p}$ yields

$$\sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\log(p) > \sum_{p}\frac{-2\log(p)}{p^{3/2}+1},$$

and the latter sum converges absolutely to $-2.3510\ldots$ □

Corollary 5.8. *For any $E/\mathbb{Q}$ with conductor N we have*

$$-\frac{6}{5}\log(N) + \frac{4}{3} < C_0(E) < -\frac{2}{3}\log(N) + 7. \tag{5.17}$$

Proof. We have by Lemma 5.6

$$\begin{aligned} C_0(E) &= 2\gamma - 1 + \log\left(\frac{4\pi^2}{N}\right) + \sum_{p\|N}\left(\frac{a_p}{p^2 - a_p}\right)\log(p) + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right) \\ &= \left[-1 + \sum_{p|N}\left(\frac{a_p}{p^2 - a_p}\right)\frac{\log(p)}{\log(N)}\right]\log(N) + \left[2\gamma - 1 + 2\log 2\pi + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\right]. \end{aligned}$$

Now since $a_p = -1, 0$ or 1 for bad p, we have for individual terms in the sum on the left

$$-\frac{1}{5} \leq \frac{a_p}{p^2 - a_p} \leq \frac{1}{3}, \tag{5.18}$$

with the bounds being obtained for $p = 2$; hence

$$-\frac{6}{5} \leq \left[\sum_{p|N}\left(\frac{a_p}{p^2 - a_p}\right)\frac{\log(p)}{\log(N)}\right] \leq -\frac{2}{3}. \tag{5.19}$$

And by lemma 5.7

$$2\gamma - 1 + 2\log 2\pi + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right) \leq 2\gamma - 1 + 2\log 2\pi + 3.0103\ldots = 6.8405\ldots < 7,$$

while

$$\frac{4}{3} < 1.4791\ldots = 2\gamma - 1 + 2\log 2\pi - 2.3510\ldots < 2\gamma - 1 + 2\log 2\pi + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right),$$

i.e.

$$\frac{4}{3} < \left[2\gamma - 1 + 2\log 2\pi + \sum_{p\nmid N}\left(\frac{a_p p - 2}{p^3 - a_p p + 1}\right)\right] < 7. \tag{5.20}$$

Combining inequalities 5.19 and 5.20 yields the desired result. □

Note that the value $\frac{4}{3}$ and 7 were picked just because they are simple fractions close to the real numbers obtained by combining the various other constants in the above sums. As such they could be replaced with better approximations by, for example, looking at the continue fractions of the numbers $1.4791\ldots$ and $6.8405\ldots$.

6 The Explicit Formula for $\frac{L_E'}{L_E}(s+1)$ with Error Bounds

From the previous section we have for $x > 1$ that

$$\widetilde{\psi}_0(x) = -\widetilde{C}_0(E) - r_{an}\log(x) - \log(1-x^{-1}) - \sum_{\gamma\neq 0}\frac{x^{i\gamma}}{i\gamma};$$

If we take the result from Lemma 5.5, and combine the terms in the sum from conjugate pairs of zeros we obtain the "most explicit form of the explicit formula" yet:

$$\widetilde{\psi}_0(x) = -\left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right] - \log(1-x^{-1}) - r_{an}\log(x) - \sum_{\gamma>0}\frac{\sin(\gamma\log(x))}{\gamma}. \tag{6.1}$$

However, this statement of the explicit formula provides no information as to how fast the sum over nontrivial zeros converges.

Let $T > 2$ be a positive real number. If we truncate the sum over nontrivial zeros we get an error term that depends on T and x, i.e.

$$\epsilon(T,x) = \widetilde{\psi}_0(x) + \left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right] + \log(1-x^{-1}) + r_{an}\log(x) + \sum_{0<\gamma<T}\frac{\sin(\gamma\log(x))}{\gamma}.$$

The aim of this section is to provide explicit bounds on $\epsilon(T,x)$ in terms of easily computable functions in T and x. Note that the proof of the explicit formula is equivalent to showing $\epsilon(T,x) \to 0$ as $T \to \infty$ for any fixed x.

As per section 1, recall by Perron's formula we have that

$$\widetilde{\psi}_0(x) = \lim_{T\to\infty}\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds \tag{6.2}$$

for $\sigma > \frac{1}{2}$ (since the Dirichlet series for $\frac{L_E'}{L_E}(s)$ converges absolutely for $\mathrm{Re}(s) > \frac{3}{2}$, the Dirichlet series for $\frac{L_E'}{L_E}(s+1)$ converges absolutely for $\mathrm{Re}(s) > \frac{1}{2}$). We will obtain explicit bounds on $\epsilon(T,x)$ by estimating the integral $\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds$ in two different ways.

Proposition 6.1. *Let $T > 2$, $x > 1$ and $\sigma > \frac{1}{2}$, and let*
$\epsilon(T,x) = \widetilde{\psi}_0(x) + \left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right] + \log(1-x^{-1}) + r_{an}\log(x) + \sum_{0<\gamma<T}\frac{\sin(\gamma\log(x))}{\gamma}$.
Then we may express $\epsilon(T,x)$ as the sum of the following collection of path integrals:

$$\begin{aligned}\epsilon(T,x) = \frac{1}{\pi}\,\mathrm{Im}\Bigg[&\sum_{\gamma}\int_{-\sigma+iT}^{\sigma+iT}\frac{x^s}{s^2+\gamma^2}\,ds\\ &- \widetilde{C}_0(E)\left(\int_{\sigma+iT}^{\infty+iT}\frac{x^{-s}}{s}\,ds + \int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s}\,ds\right)\\ &- \sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{\sigma+iT}^{\infty+iT}\frac{x^{-s}}{s+k}\,ds + \int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s+k}\,ds\right)\\ &- \sum_{n<x}c_n\int_{-\infty+iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s} + \sum_{n>x}c_n\int_{\sigma+iT}^{\infty+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s}\\ &- \sum_{n=1}^{\infty}c_n\int_{\sigma+iT}^{\infty+iT}(nx)^{-s}\,\frac{ds}{s}\Bigg] + \delta_{x,n}\frac{c_n}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{T}{\sigma}\right)\right),\end{aligned}$$

where δ is the Kronecker Delta function, i.e. the last term in the sum is only included if x is an integer.

Proof. Consider the path integral

$$\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds.$$

Since the integral occurs wholly within the half plane of absolute convergence for the Dirichlet series for $\frac{L_E'}{L_E}(s+1)$, we may replace $\frac{L_E'}{L_E}(s+1)$ by its Dirichlet series and swap the sum and integral signs to get

$$\sum_{n=1}^{\infty}\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}c_n n^{-s}\frac{x^s}{s}\,ds=\frac{1}{\pi}\sum_{n=1}^{\infty}c_n\cdot\frac{-1}{2i}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}.$$

Now when $n<x$ we have by the Cauchy Residue Theorem that

$$\frac{1}{2\pi i}\left(\int_{\sigma-iT}^{\sigma+iT}+\int_{\sigma+iT}^{-\infty+iT}+\int_{-\infty-iT}^{\sigma-iT}\right)\left(\frac{x}{n}\right)^s\,\frac{ds}{s}=1,\tag{6.3}$$

since $(x/n)^s$ decays rapidly in the left half plane, and $\frac{(x/n)^s}{s}$ has a single simple pole with residue 1 at the origin.
Hence

$$\begin{aligned}c_n\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}&=-c_n-\frac{c_n}{2\pi i}\left(\int_{-\infty+iT}^{\sigma+iT}-\int_{-\infty-iT}^{\sigma-iT}\right)\left(\frac{x}{n}\right)^s\,\frac{ds}{s}\\&=-c_n-\frac{c_n}{\pi}\operatorname{Im}\left[\int_{-\infty+iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}\right],\end{aligned}$$

since we are integrating along complex conjugate paths.

When $n>x$ we have $x/n<1$, so we complete the contour integral in the right half plane instead. Note that now we do not encircle the pole at the origin, so the result is

$$c_n\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}=\frac{c_n}{\pi}\operatorname{Im}\left[\int_{\sigma+iT}^{\infty+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}\right].$$

And when $x=n$ is an integer are just integrating $\frac{ds}{s}$, so we can compute the path integral explicitly:

$$\begin{aligned}c_n\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{ds}{s}&=\frac{-c_n}{2\pi i}\log(s)\Big|_{\sigma-iT}^{\sigma+iT}\\&=\frac{-c_n}{\pi}\arctan\left(\frac{T}{\sigma}\right)\\&=-\frac{c_n}{2}+\frac{c_n}{\pi}\left[\frac{\pi}{2}-\arctan\left(\frac{T}{\sigma}\right)\right],\end{aligned}$$

which we write in the final form since the second term in the line above goes to zero as $T\to\infty$.

Thus overall we have that

$$\begin{aligned}\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds=-\widetilde{\psi}_0(x)&\\+\frac{1}{\pi}\operatorname{Im}\left[-\sum_{n<x}c_n\int_{-\infty+iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}+\sum_{n>x}c_n\int_{\sigma+iT}^{\infty+iT}\left(\frac{x}{n}\right)^s\,\frac{ds}{s}\right]&\\+\delta_{x,n}\frac{c_n}{\pi}\left[\arctan\left(\frac{T}{\sigma}\right)-\frac{\pi}{2}\right].&\end{aligned}\tag{6.4}$$

On the other hand, by Equation 4.1 we have that

$$\frac{L_E'}{L_E}(s+1) = \log\left(\frac{2\pi}{\sqrt{N}}\right) - \digamma(s) + \frac{\Lambda_E'}{\Lambda_E}(s+1)$$

Now the digamma function has the expansion

$$\digamma(s+1) = -\eta + \sum_{k=1}^{\infty} \frac{s}{k(k+s)} \quad \text{for } s \neq -1, -2, -3, \ldots, \tag{6.5}$$

so by the above we have

$$\frac{L_E'}{L_E}(s) = \widetilde{C}_0(E) - \sum_{k=1}^{\infty} \frac{s}{k(k+s)} + \sum_{\gamma} \frac{s}{s^2+\gamma^2}, \tag{6.6}$$

and the sums converge absolutely for any s not a negative integer or nontrivial zero (this is in fact just the logarithmic derivative of the Hadamard product for $L_E(s+1)$).

We thus have that

$$\begin{aligned}\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds = {} & \widetilde{C}_0(E)\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s}\,ds \\ & - \sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s+k}\,ds \\ & + \frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{\Lambda_E'}{\Lambda_E}(s+1)\,\frac{x^s}{s}\,ds\end{aligned}$$

We proceed as before by replacing each integral with path integrals to the left, plus any residues picked up. Specifically, we get

$$\widetilde{C}_0(E)\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s}\,ds = -\widetilde{C}_0(E) - \frac{1}{\pi}\,\mathrm{Im}\left[\widetilde{C}_0(E)\int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s}\,ds\right] \quad \text{and}$$

$$\frac{1}{k}\cdot\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{x^s}{s+k}\,ds = \frac{x^{-k}}{k} + \frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{k}\int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s+k}\,ds\right]$$

To compute $\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{\Lambda_E'}{\Lambda_E}(s+1)\,\frac{x^s}{s}$ we use the Cauchy Residue Theorem, Equation 4.2 and the reflection formula for $\frac{\Lambda_E'}{\Lambda_E}(s+1)$ to get

$$\begin{aligned}\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{\Lambda_E'}{\Lambda_E}(s+1)\,\frac{x^s}{s} = {} & -r\log x - \sum_{0<|\gamma|<T}\frac{x^{1\gamma}}{i\gamma} \\ & + \frac{1}{\pi}\,\mathrm{Im}\left[\sum_{\gamma}\int_{-\sigma+iT}^{\sigma+iT}\frac{x^s}{s^2+\gamma^2}\,ds - \int_{\sigma+iT}^{\infty+iT}\frac{\Lambda_E'}{\Lambda_E}(s+1)\,\frac{x^{-s}}{s}\,ds\right]\end{aligned}$$

Finally, we again write $\frac{\Lambda_E'}{\Lambda_E}(s+1) = -\log\left(\frac{2\pi}{\sqrt{N}}\right) + \digamma(s) + \frac{L_E'}{L_E}(s+1)$ and use the same method to write $\int_{\sigma+iT}^{\infty+iT}\frac{\Lambda_E'}{\Lambda_E}(s+1)\,\frac{x^{-s}}{s}\,ds$ as a sum of path integrals from $\sigma+iT$ to $\infty+iT$.

Setting the two representations of $\frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{L_E'}{L_E}(s+1)\,\frac{x^s}{s}\,ds$ equal to each other and collecting terms completes the proof.

$\square$

Proposition 6.2. *We have the following bounds regarding terms included in $\epsilon(T,x)$:*

- $$\left|\frac{1}{\pi}\operatorname{Im}\left[-\sum_{n<x} c_n \int_{-\infty+iT}^{\sigma+iT}\left(\frac{x}{n}\right)^s \frac{ds}{s}+\sum_{n>x} c_n \int_{\sigma+iT}^{\infty+iT}\left(\frac{x}{n}\right)^s \frac{ds}{s}\right]+\delta_{x,n}\frac{c_n}{\pi}\left(\frac{\pi}{2}-\arctan\left(\frac{T}{\sigma}\right)\right)\right| < \frac{1}{\pi T}\left[\delta_{x,n}\cdot\pi\sigma+x^\sigma\sum_{n\neq x}\frac{1}{n^\sigma}\left|\frac{c_n}{\log(\frac{x}{n})}\right|\right]$$

- $$\left|\frac{1}{\pi}\operatorname{Im}\left[-\sum_{n=1}^{\infty} c_n \int_{-\infty+iT}^{\sigma+iT}(nx)^{-s}\frac{ds}{s}\right]\right| < \frac{2\zeta(\sigma+\frac{1}{2})x^{-\sigma}}{\pi T\log x} \tag{6.7}$$

- $$\left|\frac{1}{\pi}\operatorname{Im}\left[-\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{\sigma+iT}^{\infty+iT}\frac{x^{-s}}{s+k}\,ds+\int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s+k}\,ds\right)\right]\right| < \frac{(\log T+\frac{3}{2})(x^\sigma+x^{-\sigma})}{\pi T\log x}$$

- $$\left|\frac{1}{\pi}\operatorname{Im}\left[-\widetilde{C}_0(E)\left(\int_{\sigma+iT}^{\infty+iT}\frac{x^{-s}}{s}\,ds+\int_{-\infty+iT}^{\sigma+iT}\frac{x^s}{s}\,ds\right)\right]\right| < \frac{\widetilde{C}_0(E)(x^\sigma+x^{-\sigma})}{\pi T\log x}$$

Proof. Long and tedious, involving bounding the terms meticulously. Unfortunately I ran out of time in typing this writeup; however, I have the derivations of each of the above statements written down on paper. □

Editorial Note

Unfortunately I'm still working on bounding the imaginary part of the integral across the critical strip $\frac{1}{\pi}\operatorname{Im}\left[\sum_\gamma \int_{-\sigma+iT}^{\sigma+iT}\frac{x^s}{s^2+\gamma^2}\,ds\right]$. I can show the following:

Proposition 6.3. *For any given $\gamma > 0$ we have*

$$\frac{1}{\pi}\operatorname{Im}\left[\int_{-\sigma+iT}^{\sigma+iT}\frac{x^s}{s^2+\gamma^2}\,ds\right] = \frac{\cos(T\log x)}{2\pi\gamma}\int_0^\sigma \sinh(s\log x)\left(\frac{s}{s^2+(T+\gamma)^2}-\frac{s}{s^2+(T+\gamma)^2}\right)ds + \frac{\sin(T\log x)}{2\pi\gamma}\int_0^\sigma \cosh(s\log x)\left(\frac{T+\gamma}{s^2+(T+\gamma)^2}-\frac{T-\gamma}{s^2+(T+\gamma)^2}\right)ds \tag{6.8}$$

and thus

$$\left|\frac{1}{\pi}\operatorname{Im}\left[\sum_{\gamma\neq 0}\int_{-\sigma+iT}^{\sigma+iT}\frac{x^s}{s^2+\gamma^2}\,ds\right]\right| < \frac{x^\sigma}{\pi}\sum_{\gamma\neq 0}\frac{1}{|\gamma|\sqrt{1+\left(\frac{T-\gamma}{\sigma}\right)^2}}.$$

Empirically we see that the sum on the right is $O(x^\sigma\log^2(T)\cdot T^{-1})$; however, to get an explicit bound we need explicit theorems about the asymptotic density of zeros along the critical line for modular L-functions. Right now there only seem to be Big-Oh theorems in the literature regarding modular L-function zero density, so this is this is something I need to work on to fully complete this proof.

If this is true, we can make the following statement:

Proposition 6.4. *Suppose there exists a constant D such that*

$$\left| \frac{1}{\pi} \operatorname{Im} \left[\sum_{\gamma \neq 0} \int_{-\sigma+iT}^{\sigma+iT} \frac{x^s}{s^2+\gamma^2} \, ds \right] \right| < D \frac{x^\sigma \log^2(T)}{T}.$$

Then for $T >> 1$ there exists a constant $M > 0$ such that

$$\epsilon(T,x) < M \cdot \frac{\log^2 T}{T} \cdot \frac{x^\sigma + x^{-\sigma}}{\log x} \cdot \left(1 + \sum_{n \neq x} \frac{1}{n^\sigma} \left| \frac{c_n}{\log(\frac{x}{n})} \right| \right) \tag{6.9}$$

Clearly for any given $x >$ this bound goes to zero as $T \to \infty$, so this implies the validity of the explicit formula for $L_E(s+1)$.

However, what's also worth noting is that the $\left(1 + \sum_{n \neq x} \frac{1}{n^\sigma} \left| \frac{c_n}{\log(\frac{x}{n})} \right| \right)$ term correctly models behaviour when x is close to a prime power, since then $\log(\frac{x}{n})$ is very small for one n; in that case the whole term is large, reflecting the fact that we get worse convergence close to prime powers, where discontinuities occur for the cumulative sum function $\widetilde{\psi}_0(x)$. Fourier analysis predicts that the Gibbs phenomenon occurs around discontinuities, and this is exactly what we observe when we truncate the explicit formula to a finite sum of nontrivial zeros. It is definitely worth further investigating the behaviour of the truncated sum close to prime powers.

7 Applications

There are plenty of exciting things one can do that use the explicit formula. Below is a list of potential applications (by no means complete) that either use the explicit formula, or could benefit from knowledge of explicit error bounds on the truncated sum in the oscillatory term:

- Currently the best algorithm for computing zeros of modular L-functions, due to Michael Rubenstein [Rub98], are polynomial in the modular level N. The explicit formula shows that, at least formally, knowledge of an elliptic curve's a_p values is equivalent to knowledge of the location of its L-function's nontrivial zeros. As such, there is hope that a deeper understanding of the explicit formula will lead to a more efficient algorithm for computing zeros. Top prize would be for one that is polynomial in $\log(N)$, which at face value doesn't seem infeasible.

- Jonathan Bober has results [Bob11] that allow for efficient computing bounds on analytic rank of elliptic curves of large rank (conditional on GRH). For example, he has shown that Elkie's rank 28 curve has analytic rank either 28 or 30. His method involves replacing the sum over nontrivial zeros in the explicit formula with an easily computable integral along the critical line. A faster algorithm for computing zeros would allow us improve upon his bounds - and conditionally bound ranks of curves with large conductors in general - by computing the locations of the zeros.

- The location of the first (noncentral) nontrivial zero for an L-function is of interest, as it corresponds to the dominant term in the oscillating term of the explicit formula, and thus governs asymptotic behaviour thereof. Much has been published on the statistical behavior of the location of the imaginary part of the first nontrivial zero $\gamma_1(E)$ as E varies over a given family of curves, for example in [Mil99]. Specifically, γ_1 seems to be smaller to the real axis for curves with larger conductor, and for curves of a given conductor, larger when E has larger analytic rank. However, the relationship is not well understood, and currently no theorems exist giving explicit constraints on the location of the first nontrivial zero. Knowledge of the explicit formula could potentially be used to provide bounds on the location of γ_1 as a function of analytic rank and conductor.

- New invariants! There are some invariants which pop up over and over again when investigating the explicit formula for elliptic curves. For example, In this paper, the term $\widetilde{C}_0(E) = \eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)$ appears in a number of places. And Mazur in [M&S, pg. 14] defines the *bite* of an elliptic curve β_E to be $\sum_{\gamma\neq 0}\gamma^{-2}$, which is the standard deviation of the oscillatory term viewed as a random variable on x. Indeed, the bite makes an appearance in section 4 of this paper. There is much work yet to be done investigating further these invariants and potentially others yet undiscovered, and seeing if and how they influence the behaviour of the elliptic curve.

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May 1999