Mon 24 Feb MATH 307 A LECTURE 18

CHAPTER 3: THE LAPLACE TRANSFORM

§3.1: INTRO & DEFINITION BOYCE 6.1

3.1.0 Motivation

Consider the IVP $y'' + y' + 2y = g(t)$, $y(0) = 0$, $y'(0) = 0$,
where
$$g(t) = \begin{cases} 0, & 0 < t < 5 \\ 1, & 5 \le t < 20 \\ 0, & 20 \le t \end{cases}$$

i.e. the forcing function $g(t)$ is a (discontinuous) unit pulse between $t = 5$ & 20, and zero elsewhere.

It *is* possible to solve this DE using the methods we currently know. However, we would have to decompose the problem into 3 separate IVPs, one for each continuous segment of $g(t)$, and solve them in turn in order to get the ICs for the next part.

This is horribly tedious, and impractical when $g(t)$ gets even more complicated than above. This is where the Laplace Transform comes to the rescue: it is an example of an (integral) linear operator that transforms differential equations into algebraic equations. These tend to be much easier than DEs, and allow us to solve DEs like the one above much more quickly than we otherwise could.

3.1.1 Definition: The Laplace transform $\mathcal{L}$ is a linear operator on functions $f(t)$, $t>0$ (a "function of functions"). It produces as output a second function. Because of the way it's defined we use a second variable s for the output variable. It is defined by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

For this improper integral to make sense we require the following:

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i) $f(t)$ is piecewise continuous, i.e. we can break up f into pieces where it is continuous.

ii) $\exists\ K, a \geq 0$ and some $t_0 > 0$ such that
$$|f(t)| \leq K e^{at} \quad \text{for all } t > t_0.$$

That is, we require that f be "eventually at most exponential in growth."

3.1.2 Examples

1) If $f(t) = 1$ then
$$\begin{aligned}\mathcal{L}[f] &= \mathcal{L}[1] \\ &= \int_0^\infty 1 \cdot e^{-st}\, dt \\ &= -\tfrac{1}{s} e^{-st} \Big|_0^\infty \\ &= -\tfrac{1}{s}\left[0 - (-1)\right] \\ &= \tfrac{1}{s}\end{aligned}$$

So $\mathcal{L}[1] = \frac{1}{s}$

2) If $f(t) = 0$ then $\mathcal{L}[f] = \int_0^\infty 0 \cdot e^{-st}\, dt = 0$

3) $f(t) = e^{at}$ for some a:
$$\begin{aligned}\mathcal{L}[f] &= \int_0^\infty e^{at} \cdot e^{-st}\, dt \\ &= \int_0^\infty e^{(a-s)t}\, dt \\ &= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty \\ &= \begin{cases} \frac{1}{a-s}\left(0 - (-1)\right), & s > a \\ \infty, & \text{else.}\end{cases}\end{aligned}$$

So $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$.

3.5) $f(t) = e^{t^2} \rightarrow \mathcal{L}[f] = \int_0^\infty e^{t^2} e^{-st}\, dt = \int_0^\infty e^{t^2 - st}\, dt = \infty$ for all s; $\mathcal{L}[f]$ doesn't exist as f doesn't obey criterion ii).

4) $f(t) = \cos(at)$ for some a:
$$\begin{aligned}\mathcal{L}[f] &= \int_0^\infty \cos(at) e^{-st}\, dt \\ \text{IBP: } &= -\tfrac{1}{s}\cos(at) e^{-st}\Big|_0^\infty - \tfrac{a}{s}\int_0^\infty \sin(at) e^{-st}\, dt \\ \text{IBP: } &= \tfrac{1}{s} - \tfrac{a}{s}\left[-\tfrac{1}{s}\sin(at) e^{-st}\Big|_0^\infty + \tfrac{a}{s}\int_0^\infty \cos(at) e^{-st}\, dt\right] \\ &= -\tfrac{1}{s} - \tfrac{a^2}{s^2}\int_0^\infty \cos(at) e^{-st}\, dt.\end{aligned}$$

So $\mathcal{L}[f] = \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}[f] \Rightarrow \mathcal{L}[f]\left(1 + \frac{a^2}{s^2}\right) = \frac{1}{s}$

$$\Rightarrow \mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}.$$

5) $f(t) = \sin(at)$. We could use the previous method to compute $\mathcal{L}[f]$; however, it's faster to use complex numbers!

Let $h(t) = e^{iat} = \cos(at) + i \cdot \sin(at)$.

Then $\mathcal{L}[h] = \int_0^\infty e^{(ia)t} e^{-st}\, dt = \frac{1}{s-ia}$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s}{s^2+a^2} + i \cdot \frac{a}{s^2+a^2}.$$

On the other hand, $\mathcal{L}[h] = \int_0^\infty \cos(at)e^{-st} + i \int_0^\infty \sin(at) e^{-st}\, dt$

$$= \mathcal{L}[\cos(at)] + i \cdot \mathcal{L}[\sin(at)].$$

So we must have that $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$.

__Theorem 3.1.3__ The Laplace transform is a linear operator:

for any constants c_1, c_2 & functions $f_1(t)$, $f_2(t)$, we have

$$\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$$

__Proof:__ $\mathcal{L}[c_1 f_1 + c_2 f_2] = \int_0^\infty (c_1 f_1(t) + c_2 f_2(t)) e^{-st}\, dt$

$$= \int_0^\infty c_1 f_1(t) e^{-st} + c_2 f_2(t) e^{-st}\, dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-st}\, dt + c_2 \int_0^\infty f_2(t) e^{-st}\, dt$$

$$= c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$$

So $\mathcal{L}$ inherits its linearity from integration.

__Note 3.1.4__: • As we saw in the last example, c_1 & c_2 could be complex constants; $\mathcal{L}$ is a __complex linear operator__

(• We may also have that s may be a complex variable; however, we'll mostly assume it's real).

__Example 3.1.5__ $f(t) = 5e^{-2t} - 3\sin(4t)$, $t \geq 0$.

Then $\mathcal{L}[f] = 5\mathcal{L}[e^{-2t}] - 3\mathcal{L}[\sin(4t)]$ by linearity

$$= \frac{5}{s+2} - \frac{12}{s^2+16}$$ from previous examples.

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Theorem 3.1.6: for constant a & function $f(t)$,

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a).$$

Proof: $\mathcal{L}[e^{at}f] = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt$
$= \mathcal{L}[f](s-a).$

Example 3.1.7 $f(t) = e^{at}\cos(bt)$

Then $\mathcal{L}[F] = \int_0^\infty e^{at}\cos(bt)e^{-st}\,dt = \int \cos(bt)e^{-(s-a)t}\,dt$

So $\mathcal{L}[e^{at}\cos(bt)] = \dfrac{s-a}{(s-a)^2+b^2}$

Definition 3.1.8 The Gamma function $\Gamma(\alpha) : (-1,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$$

We'll most often use it in the form $\Gamma(1+a) = \int_0^\infty t^a e^{-t}\,dt$

Observe: $\Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = 1$

And for positive integers n,

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}$$

IBP $= \cancel{n} - t^n e^{-t}\Big|_0^\infty + n\int_0^\infty t^{n-1}e^{-t}\,dt$

$= n\Gamma(n)$

So $\Gamma(n+1) = n\Gamma(n)$.

Since $\Gamma(1) = 1 \Rightarrow \Gamma(2) = 1\cdot\Gamma(1) = 1$

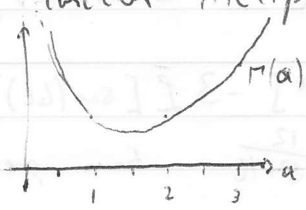
$\Gamma(3) = 2\cdot\Gamma(2) = 2$

$\Gamma(4) = 3\cdot\Gamma(3) = 6$ etc.

So $\Gamma(n+1) = n!$

Thus the Gamma function interpolates the factorial function.

Looks like:



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Example 3.1.9: $f(t) = t^a$ for some $a > -1$

Then $\mathcal{L}[f] = \int_0^\infty t^a e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u} \cdot \frac{1}{s}\,du$

$u = st$ $t=0$ $u=0$

$du = s\,dt$ $t=\infty$ $u=\infty$

or $\frac{1}{s}du = dt$

$= \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u}\,du$

$= \frac{\Gamma(1+a)}{s^{1+a}}$.

So $\mathcal{L}[t^a] = \frac{\Gamma(1+a)}{s^{1+a}}$

Suppose we know that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ (show if there's time)

Example 3.1.10 i) $f(t) = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}}$.

Then $\mathcal{L}[f] = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{1}{\sqrt{s}}\Gamma(\tfrac{1}{2})$ by example 3.1.9.

And $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du$

$u = \sqrt{t}$ $t=0$ $u=0$

$du = \frac{1}{2\sqrt{t}}dt$ $t=\infty$ $u=\infty$

$= 2 \cdot \frac{\sqrt{\pi}}{2}$ by our supposition

$= \sqrt{\pi}$

Hence $\mathcal{L}[t^{-\frac{1}{2}}] = \sqrt{\frac{\pi}{s}}$.

2) $f(t) = \sqrt{t} = t^{\frac{1}{2}}$

Then $\mathcal{L}[f] = \frac{\Gamma(\frac{3}{2})}{s^{3/2}}$ by Ex. 3.1.9

But $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$ by properties of the Γ-function

$= \frac{1}{2}\cdot\sqrt{\pi}$ by i).

So $\mathcal{L}[\sqrt{t}] = \sqrt{\frac{\pi}{2s^3}}$ or $\frac{\sqrt{\pi}}{2s^{3/2}}$.