§1.4: FIRST-ORDER MODELING (BOYCE 2.3)

We now have enough machinery to be able to look at ~~some real world~~ examples of using DEs to solve ~~real~~ some real-world problems.

Note 1.4.0: CAVEAT As with any situation where mathematics is applied to model physical reality, many assumptions and simplifications must be made up front.
As such, any result must be taken in context, and will only be an ~~exact~~ approximate solution to the true problem at best. Furthermore the solution will only be valid given the assumed constraints hold
- Many problems have arisen when the latter caveat is neglected, e.g. the Black-Scholes model.

1.4.1 Mixing Problems

~~Example 1.1~~ In general, these problems are about trying to find the concentration/amount of some substance in a large volume of liquid, where the volume has stuff both entering & exiting it.
If $y(t)$ describes the substance in question, then the differential equation describing the process generally takes the form

$$\frac{dy}{dt} = \text{rate in} - \text{rate out},$$

with some known initial condition given.

Examples

1) Initially, a tank contains 50 gal of water, in which 30 lbs of salt is dissolved. Fresh pure water runs into the tank at 2 gal/min, and runs out at the same rate. Assuming the mixture is homogenous at all times, how much salt remains in the tank after 20 mins?

PTO

Solution Let $y(t)$ be the #lbs salt in the tank after t mins.

Then
$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$
$$= 0 - 2 \cdot \frac{y}{50}$$
$$\Rightarrow \frac{dy}{dt} = -\frac{1}{25} y$$

Separable: $\frac{1}{y}\, dy = -\frac{1}{25}\, dt$
$$\ln|y| = -\frac{1}{25} t + C$$
$$y = Ae^{-\frac{1}{25}t}$$

IC: $y(0) = 30 \Rightarrow A = 30$

So
$$y(t) = 30 e^{-\frac{t}{25}}$$

Hence
$$y(20) = 30 \cdot e^{-\frac{20}{25}} = 30 e^{-\frac{4}{5}}$$
$$= 13.480 \text{ lbs.}$$
$\square$.

Example 2: A 1000 ~~gallon~~ l tank initially contains ~~gallons~~ 400 liters of water, in which 50 kg of ammonia is dissolved. A 30% by weight solution of ammonia is poured into the tank at a rate of 30 ~~gal~~ l/min, while the tank drains at a rate of 10 l/min at the bottom. Assuming mixing is instantaneous, how much ammonia is in the tank when it reaches capacity?

Solution: Let $y(t)$ be the #kgs of ammonia in the tank at time t.

Then
$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$
$$= 9 - 10 \cdot \frac{y}{400+20t} \qquad \text{with } y(0) = 50$$
$$= 9 - \frac{y}{40+2t} \qquad \text{And capacity } t = 30 \text{ min}$$

So
$$\frac{dy}{dt} + \frac{1}{40+2t} y = 9$$

Linear FOLDE:
$$\mu(t) = e^{\int \frac{1}{40+2t} dt} = e^{\frac{1}{2}\ln|40+2t|}$$
$$= \sqrt{40+2t}$$

PTO

Mon 13 Jan MATH 307 LECTURE 4 cont...

So
$$y(t) = (40+2t)^{-\frac{1}{2}}\left(\int 9\cdot(40+2t)^{\frac{1}{2}}\,dt + C\right)$$
$$= (40+2t)^{-\frac{1}{2}}\left(3(40+2t)^{\frac{3}{2}} + C\right)$$
$$= 3(40+2t) + C(40+2t)^{-\frac{1}{2}}.$$

And $y(0) = 50 \Rightarrow 50 = 3(40) + \frac{C}{\sqrt{40}}$
$$\Rightarrow \frac{C}{\sqrt{40}} = -70$$
$$C = -140\sqrt{10}.$$

So
$$y(t) = 3(40+2t) - 140\sqrt{10}(40+2t)^{-\frac{1}{2}}$$
$$= 120 + 2t - \frac{140}{\sqrt{4+\frac{1}{5}t}}$$

So at capacity i.e. $t = 30$ we have
$$y = 120 + 2\cdot 30 - \frac{140}{\sqrt{4+6}}$$
$$= 180 - \frac{140}{\sqrt{10}} \simeq 135.728 \text{ kg ammonia.}$$

1.4.2 Cooling problems

Newton's Law of cooling:

"The rate at which an object cools is proportional to the temperature difference between that object and its surroundings."

If $y(t)$ measures the temperature of the object in question, then the DE generally looks like
$$\frac{dy}{dt} = -k(y-a),$$
a the ambient temperature, k the cooling constant.

2 sets of known conditions can be given to account for the unknown cooling constant.

Example: I take pizza out of my 350°F oven and place it on my counter to cool. The thermostat is set to 65°F. After 2~~0~~ minutes I get impatient and try the pizza, but

but discover it is 255°F — still to hot to eat. I decide that 180°F is when I can comfortably tuck in; how much longer (to the nearest 30 seconds) should I wait?

<u>Solution:</u> Let $y(t)$ be the temp. of the pizza at time t in mins.
Have $y(0) = 350$, $y(2) = 255$,
and $\frac{dy}{dt} = -k(y-65)$ by Newton.

Separable: $\frac{1}{y-65}\,dy = -k\,dt$

$\Rightarrow \ln|y-65| = -kt + C$

$\Rightarrow y - 65 = Ae^{-kt}$

$y = 65 + Ae^{-kt}$

IC:

@ $t=0, y=350 \Rightarrow 350 = 65 + A$

$A = 285$

So $y = 65 + 285e^{-kt}$

And @ $t=2$, $y=255$: $\Rightarrow 255 = 65 + 285e^{-k2}$

$\Rightarrow e^{-2k} = \frac{2}{3}$

$2k = \ln\left(\frac{3}{2}\right)$

$k = \frac{1}{2}\ln\left(\frac{3}{2}\right)$

So $y = 65 + 285\,e^{-\frac{1}{2}\ln\left(\frac{3}{2}\right)t} = 65 + 285\cdot\left(\frac{2}{3}\right)^{\frac{1}{2}t}$

So when is $y = 180$?

$180 = 65 + 285\left(\frac{2}{3}\right)^{\frac{1}{2}t}$

$\Rightarrow \frac{115}{285} = \left(\frac{2}{3}\right)^{\frac{1}{2}t}$

$\Rightarrow \frac{1}{2}t\cdot\ln\left(\frac{2}{3}\right) = \ln\left(\frac{23}{57}\right)$

So $t = 2\,\frac{\ln\left(\frac{23}{57}\right)}{\ln\left(\frac{2}{3}\right)} \approx 4.477$ minutes.

Thus the pizza will be cool enough $\sim 4\frac{1}{2}$ mins after I take it out the oven, or $2\frac{1}{2}$ mins after I try it the first time. □

PTO