Lecture Notes for Math 307.

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Part 1

First Order Equations

1. Chapter 1

A differential equation is an equation that involves a function and its derivatives. Let us look at some examples.

Example 1.1.

$$\frac{dy}{dx} = y.$$

This is a simple looking differential equation. Recall from calculus that the function $y(x) = e^x$ satisfies this differential equation. In fact, all solutions to this differential equation are of the form $y(x) = Ae^x$ for some real number A.

Example 1.2.

$$\frac{d^2h}{dt^2} = -g$$

where $g = 9.8$ meters per second squared. This differential equation models the height of an object close to the surface of the earth with no air resistance. The solution to this differential equation can be found by taking antiderivatives.

$$h(t) = \frac{-g}{2}t^2 + v_0 t + h_0$$

where v_0 is the initial velocity and h_0 is the initial height.

Example 1.3. Suppose we have a string of length L with its ends attached to two rigid objects. Let $u(x,t)$ be the vertical displacement of the string at position x from the left endpoint and at time t. One learns that $u(x,t)$ satisfies the following differential equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is a constant that depends on the string. The wave equation is similar:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Remark. Look at the first three examples. Notice that the first two examples only involve a function of one variable and its derivatives while the third example involves a function of two variables and its partial derivatives. We will only consider differential equations of functions of one variable in this course.

Definition 1.1. An *Ordinary Differential Equation*, or ODE for short, is an equation involving a function of one variable and its derivatives as well as the independent variable itself. A solution to an ODE is a function of one variable defined on some interval $I = (a,b)$ such that $a < b$ and which satisfies the differential equation.

To make the distinction, differential equations of the type in example 1.3 are called *partial differential equations*. Here are some more examples of differential equations.

Example 1.4.

(1)

$$\frac{dp}{dt} = F.$$

This is Newton's second law of motion. The force on an object is equal to the time rate of change of its momentum $p = mv$, where m is the mass of the object and v is its velocity. Note that $v = \frac{dx}{dt}$ where $x(t)$ is the objects position at time t. F is the force which tends to be a function of $x(t)$ and its derivatives as well as t itself. Consider a mass m_i centered at the origin and a mass m_2 at a point $x(t)$. Newton's law of gravitation gives $F = \frac{-Gm_1m_2}{x(t)^2}$ where G is the gravitational constant.

(2)

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi.$$

This is the time-independent Schrodinger equation for the one-dimensional harmonic oscillator. Here $\hbar$ is Planck's constant, m is the mass of the object, ω is the "frequency" and E is the energy.

(3)

$$R\frac{dI}{dt} + \frac{1}{C}I = 0$$

This equation describe the current I through a circuit with a constant voltage source, a resistor of resistance R and a capacitor of capacitance C.

Section 1.3 of the text gives a discussion of the classification of some differential equations. I recommend that you glance over it, but it is not crucial for the class. Suffice it to say that we will be concerned with first order and second order linear ODE's.

First Order equations and Direction fields. The first section of this course deals with first order differential equations. In particular, will be interested with differential equations that can be written as

$$\frac{dy}{dx} = f(x, y)$$

where f is a function of the two variables x and y. Note that we will generally use x and t for the independent variables whereas the dependent variables will vary more often. Typical examples of equations of this form are $y' = x^2 + y^2$ or $y' = 2xy$.

Consider the equation $y' = y^2$. The solutions of this equation are either $y = 0$ or $y = \frac{1}{c-x}$ for some real number c. This is an example of an equation that not only has a solution, but we can actually write the solution in a simply manner. However, for the equation $y' = x^2 + y^2$, it is known that a solution exists, but it cannot be written down using the elementary functions that one sees in say, a calculus class. How might we get around this problem? It turns out that even though we cannot write down the solution to $y' = x^2 + y^2$, we can still learn something about it. This brings us to the notion of **direction field** or slope field. Let y be a differentiable function of x. Then we know that at any given x-value, the slope of the tangent line at the point $(x, y(x))$ is given by $m = y'(x)$. If y is a solution to the differential equation $y' = f(x, y)$ then we see that the slope of the tangent is given by $m = y'(x) = f(x, y(x))$. This allows us to get a picture of what the solutions to the equation $y' = f(x, y)$ are doing. We will motivate this idea with an example.

Example 1.5. Consider the differential equation

$$\frac{dy}{dx} = 2y.$$

We can solve this to get $y(x) = Ce^{2x}$ where C is an arbitrary real number. Suppose we do not know what the solutions are. Let us denote a solution as $y(x)$. At any given point $(x, y(x))$ in the xy-plane, we know the slope of the tangent line is given by $y'(x) = 2y(x)$. So consider the point $(0, 1)$ If $y(x)$ is a solution that satisfies $y(0) = 1$ (this is the case $C = 1$) then we know $y'(0) = 2y(0) = 2$. Let us mark a little line segment based at the point $(0, 1)$ of slope 2 (DRAW THE PICTURE). Now do the same thing at the point $(1/2, 3)$, say. In fact, we can do this at all the points in the xy-plane to get the slope field. Now if $y(x)$ is a solution, then we know $y(x)$ must follow these little line segments as x increases. (GO THROUGH THE MOTIONS)

EXAMPLE 1.6. Draw a slope field for $y' = y^2$.

It does not take too many examples for one to learn that drawing out slope fields is tedious. Luckily, there are nice programs that will do it for us and can do it better. The website http://math.rice.edu/˜dfield/dfpp.html has a nice application for doing this. Use the DFIELD option. These two programs are based on Matlab. Other options are Maple, Mathematica or Sage and probably more I do not know about. I will primarily use Sage for the examples that involve plotting or numerics.

EXAMPLE 1.7. Use some software to plot slope fields for the following differential equations.

(1)
$$\frac{dy}{dx} = 2y$$

(2)
$$\frac{dy}{dx} = y^2$$

(3)
$$\frac{dy}{dx} = x^2 + y^2$$

(4)
$$\frac{dy}{dx} = 4y(2 - y)$$

(5)
$$\frac{dy}{dx} = (x^2 - 1)e^{-y}$$

Consider the differential equation $y' = 2x + 3$. We know that all the solutions to this are just $y = x^2 + 3x + c$ for some constant c. What information would we have to know in order to be able to determine c? If we knew that the solution had to pass through the point $(1, 1)$ then this determines c since $1 = 1^2 + 3 * 1 + c$ gives $c = -3$. From this discussion we find that although there are infinitely many solutions to the equation $y' = 2x + 3$, there is only one solution to $y' = 2x + 3$ that satisfies $y(1) = 1$. The extra condition $y(1) = 1$ is called an **initial condition**

EXAMPLE 1.8. Consider the equation $y' = y$. All the solutions to this are given by $y(x) = Ce^x$ for some constant C. If we also require $y(1) = -2$ then $-2 = y(1) = Ce^1$. This tells us $C = -2e^{-1}$ and we find $y(x) = -2e^{-1}e^x = -2e^{x-1}$. In this situation we get a unique solution.

DEFINITION 1.2. A differential equation $y' = f(x,y)$ along with an initial condition, $y(a) = b$, is called an **initial value problem**. A *solution* to an initial value problem is a solution $y = y(x)$ of the ODE such that $y(a) = b$ and y is defined on some small interval about a. In other words, there is some positive number ε such that y is a function defined on the interval $(a-\varepsilon, a+\varepsilon)$ with $y(a) = b$ and $y' = f(x,y)$.

EXAMPLE 1.9. Consider the initial value problem

$$\frac{dy}{dx} = \frac{2y}{x}, \qquad y(0) = 0.$$

It is easy to check that $y(x) = Cx^2$ satisfies this initial value problem for all constants C. Thus, in this case there are infinitely many solutions.

EXAMPLE 1.10. Consider the initial value problem $y' = 1/x$ and $y(0) = 0$. The general solution to the differential equation is $y(x) = \ln(x) + c$ for some constant c. We can see however that we cannot even plug in 0 for x since the logarithm is not defined at 0. Therefore, this initial value problem has no solution.

We find that any of the following situations can arise for a given initial value problem

(1) No solution exists,
(2) Infinitely many solutions exist,
(3) A unique solution exists.

Generally speaking, the hope is that when one comes across an initial value problem, the third condition above holds. This lead mathematicians to ask the question "Under what conditions can one guarantee that the initial value problem, $y' = f(x,y)$, $y(a) = b$, has a unique solution." It turns out there is a simple answer to this.

THEOREM 1.1. *Suppose both the function $f(x,y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R = \{(x,y) \mid x_0 < x < x_1,\ y_0 < y < y_1\}$ in the xy-plane that contains the point (a,b). Then there is an interval $I = \{x \mid c < x < d\}$ that contains a where the initial value problem*

$$\frac{dy}{dx} = f(x,y), \qquad y(a) = b$$

has a unique solution.

REMARK. Discuss about 1.2, 1.3 and 1.4, move on to 2.2.

2. Separable Equations, 2.2.

Let start out with an example.

EXAMPLE 2.1. Consider the differential equation $y' = 2y + 3$. We can solve this using the methods learned in MATH 125,

$$\frac{dy}{dx} = 2y+3 \qquad \Rightarrow \qquad \frac{dy}{dx} = 2(y+\frac{3}{2}) \quad \Rightarrow \quad \frac{dy}{y+\frac{3}{2}} = 2dx \quad \Rightarrow$$

$$\int \frac{dy}{y+\frac{3}{2}} = \int 2dx = 2x + c \qquad \Rightarrow \qquad \ln(|y+\frac{3}{2}|) = 2x+c \quad \Rightarrow \quad |y+\frac{3}{2}| = e^{2x+c}$$

Note $e^{2x+c} = e^c e^{2x} = Ce^{2x}$. We can remove the absolute values and write $y + \frac{3}{2} = Ae^{2x}$ where A can be either C or $-C$, this would be determined by any initial conditions. In either case, we find solutions to $y' = 2y+3$ are given by $y(x) = -\frac{3}{2} + Ae^{2x}$.

The ODE above is an example of a separable differential equation. It turns out that separable equations can often be solved implicitly. We will see this below, but first, let us define a separable differential equation.

Definition 2.1. Consider a first order differential equation $y' = f(x, y)$. This equation is called separable if it can be written in the following form,

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

The notation above is meant that M is just a function of x and N is just a function of y. Let $H_1(x)$ be an antiderivative of $M(x)$ and $H_2(y)$ be an antiderivative of $N(y)$. Recall that if y is a function of x then

$$\frac{d}{dx}H_2(y) = H_2'(y)\frac{dy}{dx}$$

by the chain rule. Since $\frac{d}{dx}H_1(x) = M(x)$ we can write

$$0 = M(x) + N(y)\frac{dy}{dx} = H_1'(x) + H_2'(y)\frac{dy}{dx} = \frac{d}{dx}(H_1(x) + H_2(y)).$$

By integrating this equation once we find we get $H_1(x) + H_2(y) = C$ for some constant C. This exhibits y implicitly as a function of x. If it is possible to solve this equation for y in terms of x then we get a solution. It is not always possible to do this though and so we just get an implicit solution. Look back at the example above. We can rewrite the equation $y' = 2x + 3$ as

$$-2 + \frac{1}{y + \frac{3}{2}}\frac{dy}{dx} = 0.$$

If we let $H_1(x) = -2x$ and $H_2(y) = \ln(y + \frac{3}{2})$ then we get $-2x + \ln(y + \frac{3}{2}) = c$ for some constant c. We can solve this equation to get the equation above in example 2.1.

Example 2.2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5} \qquad y(1) = 3.$$

First note, this equation is separable since we can write it as

$$(4 - 2x) + (5 - 3y^2)\frac{dy}{dx} = 0.$$

This tells us to let $H_1(x) = 4x - x^2$ and $H_2(y) = 5y - y^3$ to get $4x - x^2 + 5y - y^3 = c$. To determine c we must use the initial condition. If we plug in 1 for x then we must have 3 for y. This gives $c = 4(1) - (1)^2 + 5(3) - (3)^3 = -9$ Plugging this in for c and getting all the y's on one side we get

$$y^3 - 5y = 4x - x^2 + 9.$$

Use Sage to graph some solution curves.

We must be careful when we solve separable equations. It may turn out that a solution to a given differential equation does not show up when we transform a differential equation into the form $M(x) + N(y)y' = 0$ and find implicit solutions. This tends to happen when we divide things. This is analogous to the case of introducing extraneous solutions when solving algebraic equations. For example, when we solved the differential equation $y' = 2y + 3$ we divided by $y+\frac{3}{2}$. It turns out, however, the solution $y=-\frac{3}{2}$ still turned up in the solution by letting the constant be zero. The next example shows a case when this does not happen.

EXAMPLE 2.3. Solve the differential equation

$$\frac{dy}{dx}=6x(y-1)^{\frac{2}{3}}.$$

If we separate the variables we can get the equation

$$-2x+\frac{1}{3(y-1)^{\frac{2}{3}}}\frac{dy}{dx}=0.$$

By taking antiderivatives we obtain

$$-x^2+(y-1)^{\frac{1}{3}}=c$$

for some constant c. This equation can be solved to get $y=1+(x^2+c)^3$. Note that for no value of the constant c can we get the equation $y=1$. However, it is easy to see $y=1$ is a solution to this differential equation. Also note that for the initial condition $y(0)=1$, both $y=1$ and $y=1+x^6$ are solutions to the initial value problem. This is another example of non uniqueness of solutions that can occur. Note that in Theorem 1 we needed a rectangle R about the point $(0,1)$ where $f(x,y)$ and $f_y(x,y)$ are continuous. We see $f_y(x,y)$ is not continuous at $y=1$.

We will finish up this section with some examples.

EXAMPLE 2.4. *Newton's law of cooling.* Newton's law of cooling states that if an object of temperature $T(t)$ is immersed in a medium of constant temperature A then the time rate of change of T is proportional to the difference $A-T$. A 4-lb roast is initially 50°F. The roast is placed in an oven of temperature 375°F. After 75 minutes the temperature of the roast is measured to be 125°F. When will the roast be 150°F?
Newton's law of cooling tells us

$$\frac{dT}{dt}=k(375-T)$$

for some constant k. The statement of the problem implies $T(0)=50$. First of all, we need to solve the ODE. We can do this by separating the variables to get

$$-k+\frac{1}{375-T}\frac{dT}{dt}=0.$$

By finding antiderivatives we get $-kt-\ln(|375-T|)=c$. From here we see $|375-T(t)|=Ce^{-kt}$. Since $T(0)=50$, $|375-50|=Ce^0=C$ which gives $C=325$. When we remove the absolute values we know initially $375-T$ is positive. Since $|375-T(t)|=325e^{-kt}$ we know $375-T(t)$ can never be zero and must therefore always be positive. Hence, $375-T(t)=325e^{-kt}$ which gives $T(t)=375-325e^{-kt}$. We do not know what the value of k is. However, we do know that $T(75)=125$. Using this we find $125=T(75)=375-325e^{-75k}$. We can solve this equation to get $e^{-75k}=\frac{250}{325}$ which gives $k=-\frac{1}{75}\ln(\frac{250}{325})$. Therefore, to find out when the roast is 150°F we just have to solve the equation $375-325e^{-kt}=150$. This gives $e^{-kt}=\frac{225}{325}$ from which we get $t=-\frac{1}{k}\ln(\frac{225}{325})$, plugging in the known value of k gives

$$t=75\frac{\ln(\frac{225}{325})}{\ln(\frac{250}{325})}\cong 105.144.$$

So the roast will reach a delicious 150°F after about 105 minutes.

EXAMPLE 2.5. From Torrecilli's Law one can derive the formula

$$A(y)\frac{dy}{dt} = -a\sqrt{2g}\sqrt{y}$$

which describes the draining of water from a hole at the bottom of a container. Here, $y(t)$ is the height of water from the bottom of the container, $A(y)$ is the area of the water surface at height y, a is the area of the hole and $g = 32$ feet per second squared is the acceleration due to gravity.

Suppose there is a hemispherical tank of radius 4 feet full of water and at time $t = 0$, a small hole of diameter 1 in is drilled in the bottom. How long will it take for the water to drain out of the tank.

From the picture (draw picture), we see $A(y) = \pi r^2 = \pi(16 - (4 - y)^2) = \pi(8y - y^2)$. Thus, the differential equation becomes

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi(\frac{1}{24})^2\sqrt{2 \cdot 32}\sqrt{y} = -\pi\frac{1}{72}\sqrt{y}$$

with initial data $y(0) = 4$. If we separate this equation we get

$$\frac{1}{72} + (8y^{\frac{1}{2}} - y^{\frac{3}{2}})\frac{dy}{dt} = 0.$$

Upon taking the appropriate antiderivatives we find $\frac{t}{72} + \frac{16}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = c$. Upon plugging in 4 for y and 0 for t we see $c = \frac{16}{3}4^{\frac{3}{2}} - \frac{2}{5}4^{\frac{5}{2}} = \frac{448}{15}$ giving

$$\frac{t}{72} + \frac{16}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = \frac{448}{15}.$$

The condition that the tank be empty is $y = 0$. So if we plug in $y = 0$ in our implicit solution we find $\frac{t}{72} = \frac{448}{15}$ to get $t = 72\frac{448}{15} = \frac{32256}{15} = 2150.4$ seconds. This is about 36 minutes.

3. First Order Linear Equations and Integrating Factors 2.1

We want to consider differential equations of the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

where $p(t)$ and $q(t)$ are some known functions.

EXAMPLE 3.1. Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-\frac{x}{3}}, \qquad y(0) = -1.$$

This equation is not separable so we cannot solve it using previous methods. Recall from calculus that if $f(x)$ and $g(x)$ are differentiable functions of x then

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x).$$

Let us see if this can be useful for us. We are dealing with an equation of the form $y' - y$, is there some function $\mu(x)$ such that

$$\mu(x)\frac{dy}{dx} - \mu(x)y = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y?$$

If there is then we would guess that $\mu'(x) = -\mu(x)$. This equation has a simple solution, $\mu(x) = e^{-x}$ that can be obtained by separation of variables. Using this we see

$$\frac{d}{dx}(e^{-x}y(x)) = e^{-x}\frac{dy}{dx} - e^{-x}y.$$

Let us consider the equation

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\frac{11}{8}e^{-\frac{x}{3}} = \frac{11}{8}e^{-\frac{4x}{3}}.$$

We can rewrite this equation as

$$\frac{d}{dx}(e^{-x}y(x)) = \frac{11}{8}e^{-\frac{4x}{3}}$$

which can be solved by taking an anti-derivative to get

$$e^{-x}y = -\frac{33}{32}e^{-\frac{4x}{3}} + C.$$

Upon multiplication by e^x we find

$$y = Ce^x - \frac{33}{32}e^{-\frac{x}{3}}.$$

It can be checked that this is a solution to the original equation(check it). To find what C is we plug in the initial condition to get $-1 = y(0) = Ce^0 - \frac{33}{32}e^0$ which tells us that $C = \frac{1}{32}$. So we finally have the solution to the initial value problem given by

$$y = \frac{1}{32}e^x - \frac{33}{32}e^{-\frac{x}{3}}.$$

Let us consider a general first order linear equation

$$\frac{dy}{dt} + p(t)y = q(t).$$

We want to see if we can find a function $\mu(t)$ such that

$$\mu\frac{dy}{dt} + \mu p(t)y = \frac{d}{dt}(\mu y).$$

Expanding out the right hand side we get

$$\frac{d}{dt}(\mu y) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y.$$

Hence, if

$$\frac{d\mu}{dt} = p(t)\mu$$

then

$$\mu\frac{dy}{dt} + \mu p(t)y = \frac{d}{dt}(\mu y).$$

The equation $\frac{d\mu}{dt} = p(t)\mu$ is a separable equation that can be solved to get

$$\mu(t) = e^{\int p(t)}.$$

Here, $\int p(t)$ is just an anti-derivative of $p(t)$. You should check that this μ solves $\frac{d\mu}{dt} = p(t)\mu$. Using this, let us multiply the original differential equation by $\mu = e^{\int p(t)}$, we get

$$e^{\int p(t)}\frac{dy}{dt} + e^{\int p(t)}p(t)y = e^{\int p(t)}q(t).$$

The whole reason for finding μ is that now the left hand side of the equation directly above is $\frac{d}{dt}(e^{\int p(t)}y)$. So we can rewrite our modified differential equation as

$$\frac{d}{dt}(e^{\int p(t)}y) = e^{\int p(t)}q(t).$$

Since the right side only depends on t and the left side is a derivative with respect to t, we can take anti-derivatives to get

$$e^{\int p(t)}y = \int e^{\int p(t)}q(t) + C$$

where C is some integration constant and $\int e^{\int p(t)dt}q(t)$ is an anti-derivative of $e^{\int p(t)dt}q(t)$. Since exponentials are never zero we can divide by $e^{\int p(t)}$ to get

$$y(t) = e^{-\int p(t)}\int e^{\int p(t)}q(t) + Ce^{-\int p(t)}. \tag{3.1}$$

From a certain point of view we are done. The problem of solving an equation of the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

has been reduced to computing a couple of anti-derivatives (for nice enough function $p(t)$ and $q(t)$). I do not recommend that you memorize this formula. The important aspect is the *integrating factor* $\mu(t) = e^{\int p(t)}$. I think it is much safer to just remember the integrating factor and the trick of multiplying

$$\frac{dy}{dt} + p(t)y = q(t)$$

by μ to get

$$e^{\int p(t)}\frac{dy}{dt} + e^{\int p(t)}p(t)y = \frac{d}{dt}(e^{\int p(t)}y) = e^{\int p(t)}q(t)$$

and then taking anti-derivatives and solving for y. Before going on to examples lets consider what we can do with any initial conditions. Suppose we have the initial condition $y(t_0) = y_0$. When we choose an integrating factor $\mu(t)$, we do not care about which anti-derivative we find. Let us take the anti-derivative given by $\int_{t_0}^t p(s)ds$, this anti-derivative is zero at t_0. This gives us an integrating factor of the form

$$\mu(t) = e^{\int_{t_0}^t p(s)ds}.$$

Note that $\mu(t_0) = e^0 = 1$. The other anti-derivative is coming from $e^{\int_{t_0}^t p(s)ds}q(t)$. Let us choose the anti-derivative given by

$$\int_{t_0}^t e^{\int_{t_0}^s p(u)du}q(s)ds.$$

This anti-derivative is zero when $t = t_0$. Using these two anti-derivatives, we find

$$e^{\int_{t_0}^t p(s)ds}y = \int_{t_0}^t e^{\int_{t_0}^s p(u)du}q(s)ds + C$$

for some constant C. Plugging in t_0 into both sides of the equation gives $y_0 = y(t_0) = C$. So by choosing these special anti-derivatives we can write the solution to the initial value problem

$$\frac{dy}{dt} + p(t)y = q(t), \qquad y(t_0) = y_0$$

as

$$y(t) = e^{-\int_{t_0}^{t} p(s)ds} \int_{t_0}^{t} e^{\int_{t_0}^{s} p(u)du} q(s)ds + y_0 e^{-\int_{t_0}^{t} p(s)ds} \tag{3.2}$$

Now we can move on to some examples.

Example 3.2. Solve the first order differential equation

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x.$$

In order to solve this using the method of integrating factors we must first write it in the appropriate form. Notice that to do this we must divide by x^2+1. Since x^2+1 is never zero this will not present any problems. So let us rewrite the ODE as

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}.$$

The first step is to find the integrating factor $\mu(x)$. Since $p(x) = \frac{3x}{x^2+1}$ we need to find an anti-derivative for $\frac{3x}{x^2+1}$. So we are faced with the problem

$$\int \frac{3x}{x^2+1}dx.$$

We can do this anti-derivative by a u substitution, let $u = x^2+1$. Then $du = 2xdx$ and we get the new integral

$$\frac{3}{2}\int \frac{1}{u}du = \frac{3}{2}\ln(|u|).$$

We are not worried about integration constants cause we just need to find an anti-derivative. Plugging x^2+1 back in for u we find the integral gives $\frac{3}{2}\ln(x^2+1) = \ln((x^2+1)^{\frac{3}{2}})$, we do not need to worry about the absolute value bars since x^2+1 is always positive. Thus, our integrating factor is

$$\mu(x) = e^{\ln((x^2+1)^{\frac{3}{2}})} = (x^2+1)^{\frac{3}{2}}.$$

We know we can write our differential equation as

$$\frac{d}{dx}((x^2+1)^{\frac{3}{2}}y) = (x^2+1)^{\frac{3}{2}}\frac{6x}{x^2+1} = 6x(x^2+1)^{\frac{1}{2}}.$$

We need to find

$$\int 6x(x^2+1)^{\frac{1}{2}}dx.$$

Again, let $u = x^2+1$, then $du = 2xdx$ and the new integral is

$$\int 3u^{\frac{1}{2}}du = 2u^{\frac{3}{2}}.$$

Hence, the original integral gives $2(x^2+1)^{\frac{3}{2}}$. Putting this altogether we find

$$\frac{d}{dx}((x^2+1)^{\frac{3}{2}}y) = 6x(x^2+1)^{\frac{1}{2}}$$

gives

$$(x^2+1)^{\frac{3}{2}}y = 2(x^2+1)^{\frac{3}{2}} + C$$

which can be solved for y to get

$$y(x) = 2 + C(x^2+1)^{-\frac{3}{2}}.$$

EXAMPLE 3.3. Solve the initial value problem for $t > 0$.

$$\frac{dy}{dt} + \frac{1}{t}y = \frac{\sin(t)}{t^2}, \qquad y(1) = 1.$$

Since we have an initial condition let us use equation (0.2). For this differential equation we have $p(t) = \frac{1}{t}$, therefore, we want to compute the integral

$$\int_1^t \frac{1}{s}ds = \ln(t) - \ln(1) = \ln(t).$$

This gives an integrating factor $\mu(t) = e^{\ln(t)} = t$. Therefore,

$$\frac{d}{dt}(ty) = t\frac{\sin(t)}{t^2} = \frac{\sin(t)}{t}.$$

We need to determine the anti-derivative

$$\int_1^t \frac{\sin(s)}{s}.$$

It turns out that we cannot write down an anti-derivative of $\sin(s)/s$ even though we know one exists. So we will just have to leave it as it is. By the choices of anti-derivatives we know when we anti-differentiate

$$\frac{d}{dt}(ty) = \frac{\sin(t)}{t}$$

the integration constant will be $y_0 = 1$. Therefore, we end up with

$$ty = \int_1^t \frac{\sin(s)}{s} + 1$$

which gives the solution to the initial value problem,

$$y = \frac{1}{t}\int_1^t \frac{\sin(s)}{s} + \frac{1}{t}.$$

You should check to see that this is indeed a solution of the initial value problem.

4. Mathematical Modeling, 2.3.

Mixing Problems. Suppose there is some container of solution and an inflow of solution at some rate r_i into the container. The subscript i stand for solution flowing in. Also, suppose the concentration of the inflow is c_i. After being mixed the solution in the container leaves at a rate r_o where the o subscript means flowing out and let c_o be the concentration of the solution flowing out. Note that r_i, r_o, c_i and c_o may depend on t. Let $Q(t)$ denote the amount of material in the solution. If we let Δt be a small interval, then approximately

$$\Delta Q = Q(t + \Delta t) - Q(t) \cong r_i c_i \Delta t - r_o c_o \Delta t.$$

If we divide the above equation by Δt we get

$$\frac{\Delta Q}{\Delta t} \cong r_i c_i - r_o c_o.$$

In the limit as $\Delta t \to 0$ the approximations become better and better. The left hand side of the equation tends to $\frac{dQ}{dt}$ whereas the right hand side is independent of Δt. This gives us the mixing equation

$$\frac{dQ}{dt} = r_i c_i - r_o c_o.$$

A good way to remember this is $\frac{dQ}{dt}$ = rate in $-$ rate out. We can determine a little more about this differential equation. For example, we know c_o is the concentration of the solution in the container which tells us that $c_o = \frac{Q(t)}{V(t)}$. This allows us to write

$$\frac{dQ}{dt} = r_i c_i - r_o c_o = \frac{dQ}{dt} = r_i c_i - r_o \frac{Q(t)}{V(t)}.$$

Note that the volume can depend on time. In fact, the volume of solution changes according to the equation

$$\frac{dV}{dt} = r_i - r_o.$$

IF both r_i and r_o are constant then we can solve for V to get $V(t) = (r_i - r_o)t + V_0$ where V_0 is the initial volume. Note that if $r_i = r_o$, then $V(t) = V_0$ is constant. Lets look at some examples.

Example 4.1. Assume Lake Erie has a volume of 480 km^3 and that its rate of inflow from Lake Huron and outflow to Lake Ontario are both 350 km^3/yr. Suppose at time $t = 0$, the pollutant concentration of Lake Erie is five times that of Lake Huron. Assuming that the outflow is perfectly mixed, how long will it take to reduce the pollutant concentration in Lake Erie to twice that of lake Huron?

The differential equation modeling the amount of pollutant in the lake is given by

$$\frac{dQ}{dt} = 350c - \frac{350}{480} Q,$$

where c is the concentration of pollutant in Lake Huron. The initial condition is $Q(0) = 5 \cdot c \cdot V = 5 \cdot c \cdot 480$. Let us first solve the differential equation. Rearranging we have

$$\frac{dQ}{dt} + \frac{350}{480} Q = 350c.$$

Therefore, our integrating factor is given by $\mu(t) = e^{\frac{350}{480}t}$ and we get

$$\frac{d}{dt}(e^{\frac{350}{480}t}Q) = 350ce^{\frac{350}{480}t}.$$

Taking anti-derivatives we find

$$e^{\frac{350}{480}t}Q = 350ce^{\frac{350}{480}t}\frac{480}{350} + K = 480ce^{\frac{350}{480}t} + K$$

which gives

$$Q = 480c + Ke^{-\frac{350}{480}t}.$$

Using the initial condition we get

$$480 \cdot 5c = Q(0) = 480c + Ke^0 = 480c + K$$

which gives $K = 480 \cdot 4c$ and finally

$$Q(t) = 480c(1 + 4e^{-\frac{350}{480}t}).$$

We want to know when is $Q(t) = 2cV = 2c \cdot 480$. Setting this equal to our solution we have

$$2c \cdot 480 = 480c(1 + 4e^{-\frac{350}{480}t}) \quad \Rightarrow \quad 2 = 1 + 4e^{-\frac{350}{480}t} \quad \Rightarrow \quad 4e^{-\frac{350}{480}t} = 1$$
$$\Rightarrow \quad -\tfrac{350}{480}t = \ln(\tfrac{1}{4}) \quad \Rightarrow \quad t = -\tfrac{480}{350}\ln(\tfrac{1}{4}) \cong 1.901 \text{ years.}$$

Example 4.2. A 120 gallon tank initially contains 90 lbs of salt in 90 gallons of water. Brine(saltwater) of concentration 2 lbs/gallon enters the tank at a rate of 4 gal/min. The well stirred solution leaves that tank at a rate of 3 gal/min. How much salt does the tank contain when it is full?

In this example, the rate of inflow and outflow are not the same which tells us that the volume is not constant. Since the inflow and outflow rates are constant in time we know that $V(t) = (r_i - r_o)t + V_0 = (4-3)t + 90 = t + 90$. Therefore, our initial value problem is

$$\frac{dS}{dt} = 4 \cdot 2 - \frac{3}{t+90}S, \qquad S(0) = 90.$$

Rearranging the equation we find

$$\frac{dS}{dt} + \frac{3}{t+90}S = 8.$$

Since $p(t) = \frac{3}{t+90}$, we find that the anti-derivative is given by $3\ln(t+90)$. We do not have to worry about absolute values since we are only concerned about $t \geq 0$. This gives an integrating factor $\mu(t) = e^{3\ln(t+90)} = (t+90)^3$. So upon multiplying our differential equation by μ we get

$$\frac{d}{dt}((t+90)^3 S) = 8(t+90)^3.$$

We can easily take anti-derivatives to get

$$(t+90)^3 S = 2(t+90)^4 + C.$$

for which we can solve to get

$$S(t) = 2(t+90) + C(t+90)^{-3}.$$

The initial condition $S(0) = 90$ gives

$$90 = 2(0 + 90) + C(0 + 90)^{-3} = 180 + \frac{C}{90^3}$$

which can easily be solved to give $C = -90^4$. This gives the solution

$$S(t) = 2(t + 90) - \frac{90^4}{(t + 90)^3}.$$

The original question is how much salt will be in the tank when it is full. We know the tank holds 120 gallons and $V(t) = t + 90$, so the tank will be full after 30 minutes. The amount of salt after 30 minutes is given by

$$S(30) = 2(30 + 90) - \frac{90^4}{(30 + 90)^3} \cong 202 \text{ lbs.}$$

Drag forces. In the absence of air resistance, Newtons second law tells us that the differential equation determining the vertical speed of an object is given by

$$m\frac{dv}{dt} = -mg$$

where $g = 9.8$m/s or 32ft/s is the acceleration due to gravity and m is the mass of the object. This is a relatively good model if the velocity of the object is small and the object itself small and passes easily through air. However, anyone that has ever stuck there hand out of a moving car window notices there is quite a bit of drag. Moreover, it is most likely that you have noticed that you do not feel as strong a drag force when walking down the street on your hand as you do when driving in the car. Therefore, it seems reasonable to assume that the drag force on an object moving through the air is somehow dependent on its velocity. We will consider two different models here.

Example 4.3. Consider an object moving vertically and suppose the drag force it feels is proportional to its velocity. Determine its equation of motion if it is launched with velocity $v(0) = v_0$ and initial height $h(0) = h_0$ where $h(t)$ is its height. Find its maximum height.

Since the drag force is opposite the direction that the object is traveling we have $F_d = -k'v$ for some constant $k' > 0$. Here F_d means drag force. Taking positive velocity to be in the direction moving away from the earths surface we find

$$m\frac{dv}{dt} = -mg - k'v$$

which implies that we have the equation describing the velocity given by

$$\frac{dv}{dt} = -g - \frac{k'}{m}v = -g - kv$$

where $k = \frac{k'}{m}$. Rearranging gives the first order linear equation $v' + kv = -g$. Since $p(t) = k$ is constant we see easily that the integrating factor is given by $\mu(t) = e^{kt}$. Thus, upon multiplying the differential equation by μ we get

$$\frac{d}{dt}(e^{kt}v) = -ge^{kt}.$$

Taking antidervatives we get

$$e^{kt}v = -\frac{g}{k}e^{kt} + C.$$

Plugging in $t = 0$ we get $v_0 = -\frac{g}{k} + C$ which gives $C = v_0 + \frac{g}{k}$. So we find the solution to our equation is

$$v(t) = -\frac{g}{k} + (v_0 + \frac{g}{k})e^{-kt}.$$

Now that we know $v(t)$ we can use this to find $h(t)$ since $\frac{dh}{dt} = v$. Taking the appropriate antidervatives we get

$$h(t) = -\frac{g}{k}t - \frac{(v_0 + \frac{g}{k})}{k}e^{-kt} + K.$$

We can find K via $h_0 = h(0) = -\frac{(v_0+\frac{g}{k})}{k} + K$. Hence, $K = h_0 + \frac{(v_0+\frac{g}{k})}{k}$. This implies the height of the object t seconds after it is launched is determined by

$$h(t) = -\frac{g}{k}t - \frac{(v_0 + \frac{g}{k})}{k}e^{-kt} + h_0 + \frac{(v_0 + \frac{g}{k})}{k}.$$

The maximum height occurs when the derivative of h is zero. The derivative is just the velocity so we just need to know when the velocity is zero. This leads to the equation

$$-\frac{g}{k} + (v_0 + \frac{g}{k})e^{-kt} = 0.$$

We can solve this easily for

$$t = -\frac{1}{k}\ln\left(\frac{g}{kv_0 + g}\right).$$

Plugging this into $h(t)$ we find that the maximum height is given by

$$\frac{g}{k^2}\ln\left(\frac{g}{kv_0 + g}\right) - \frac{g}{k^2} + h_0 + \frac{(v_0 + \frac{g}{k})}{k} = \frac{g}{k^2}\ln\left(\frac{g}{kv_0 + g}\right) + h_0 + \frac{v_0}{k}.$$

Now the next question is how do we determine k. This needs to be done empirically and the values can range. Try computing these numbers with varying k and v_0 values to see what happens.

EXAMPLE 4.4. Now assume the drag force is proportional to the square of the velocity. Suppose we have an object launched from the ground vertically upward. The differential equation that determines the velocity during the upward motion is given by (assuming up is the positive direction)

$$\frac{dv}{dt} = -g - kv^2 \tag{4.1}$$

whereas the differential equation for the downward motion is given by

$$\frac{dv}{dt} = -g + kv^2. \tag{4.2}$$

The reason for the two different equations is that the drag force is always opposite the direction of motion but since we have v^2 involved the direction of motion (determined by the sign of v) is squared out. Lets take as our initial condition $v(0) = v_0 > 0$.

We will need to solve both of these equations so lets work on the first. Both of these are separable equations but are not linear so we have to use the separation of variables method. Lets rewrite equation 4.1 as

$$\frac{1}{(\sqrt{\frac{k}{g}}v)^2 + 1}\frac{dv}{dt} = -g.$$

If we do the substitution $u = \sqrt{\frac{k}{g}}v$ we find that an anti-derivative is given by

$$\int \frac{1}{(\sqrt{\frac{k}{g}}v)^2 + 1} dv = \sqrt{\frac{g}{k}} \arctan(\sqrt{\frac{k}{g}}v).$$

Therefore, we get the equation

$$\sqrt{\frac{g}{k}} \arctan(\sqrt{\frac{k}{g}}v) = -gt + C.$$

Plugging in $t = 0$ we get $\sqrt{\frac{g}{k}} \arctan(\sqrt{\frac{k}{g}}v_0) = C$, but we will still just write C. We can now solve the equation for $v(t)$ to get

$$v(t) = \sqrt{\frac{g}{k}} \tan\left(\sqrt{\frac{k}{g}}C - \sqrt{\frac{k}{g}}gt\right).$$

Note that $\sqrt{\frac{k}{g}}C = \arctan(\sqrt{\frac{k}{g}}v_0)$. To make this equation look nicer let us write $\rho = \sqrt{\frac{k}{g}}$ to get

$$v(t) = \frac{1}{\rho} \tan\left(\arctan(\rho v_0) - \rho g t\right).$$

We can integrate this using $-\ln|\cos(u)| + B$ for an anti-derivative of $\tan(u)$ to get (do this as an exercise)

$$h(t) = h_0 + \frac{1}{k} \ln \left| \frac{\cos(\arctan(\rho v_0) - \rho g t)}{\cos(\arctan(\rho v_0))} \right|.$$

For the downward motion, we see the differential equation changes to

$$\frac{dv}{dt} = -g + kv^2.$$

We can go through the same process as above, the only difference is that one comes across integrals of the form

$$\int \frac{1}{1 - u^2} du = \tanh^{-1}(u) + C$$

and

$$\int \tanh(u) du = \ln(\cosh(u)) + C.$$

You should go through the procedure. You should end up with (assuming I did it right)

$$v(t) = \frac{1}{\rho} \tanh\left(\tanh^{-1}(\rho v_0) - \rho g t\right)$$

for the velocity and

$$h(t) = h_0 - \frac{1}{k} \ln \left| \frac{\cosh(\tanh^{-1}(\rho v_0) - \rho g t)}{\cosh(\tanh^{-1}(\rho v_0))} \right|$$

for the height.

5. Differences Between Linear and Nonlinear Equations, 2.4.

Recall that during the first week we discussed the following theorem.

THEOREM 5.1. *Suppose both the function $f(x, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R = \{(x, y) \mid x_0 < x < x_1,\ y_0 < y < y_1\}$ in the xy-plane that contains the point (a, b). Then there is an interval $I = \{x \mid c < x < d\}$, that contains a, where the initial value problem*

$$\frac{dy}{dx} = f(x, y), \qquad y(a) = b$$

has a unique solution.

In the case where we have

$$\frac{dy}{dt} + p(t)y = q(t),$$

we find $f(t, y) = q(t) - p(t)y$. We can see this f is continuous and has continuous partial derivative $\frac{\partial f}{\partial y} = -p(t)$ whenever both the functions $p(t)$ and $q(t)$ are continuous. The theorem does not tell us much about the interval I over which the solution is known to exist. In the case of first order linear equations though, we can say something a little better.

THEOREM 5.2. *If the functions $p(t)$ and $q(t)$ are continuous on an interval $I = \{x \mid a < t < b\}$ containing the point t_0, then there exists a unique function $y(t)$ that satisfies the differential equation*

$$\frac{dy}{dt} + p(t)y = q(t)$$

for all t in I, and also satisfies the initial condition $y(t_0) = y_0$ for any value of y_0.

The extra bit of strength this theorem holds lies in the statement of existence on the whole interval I where both $p(t)$ and $q(t)$ are continuous.

EXAMPLE 5.1. Use theorem 2 to determine the interval over which the initial value problem

$$(t^2 - 1)\frac{dy}{dt} + 2y = \ln(t), \qquad y(\frac{1}{3}) = 4.$$

has a unique solution.

We need to write the equation in the standard form

$$\frac{dy}{dt} + \frac{2}{t^2 - 1}y = \frac{ln(t)}{t^2 - 1}.$$

To determine the interval of existence we need to find the common values of t for which $p(t)$ and $q(t)$ are continuous. Because of the $\ln(t)$ term in $q(t)$, we see t needs to be greater than 0 for $q(t)$ even to be defined. If we look at the term $\frac{1}{t^2-1}$ that is in both p and q we see this function is continuous everywhere except ± 1. So $p(t) = \frac{2}{t^2-1}$ is continuous everywhere except ± 1 and $q(t) = \frac{\ln(t)}{t^2-1}$ is continuous for $t > 0$ except $t = 1$. Therefore, the common intervals over which p and q are both continuous are $(0, 1)$ and $(1, \infty)$. Since the initial condition is $y(\frac{1}{3}) = 4$, theorem 5.2 tells us there exists a unique solution to the initial value problem on the whole interval $(0, 1)$.

The method in the last example does not change from problem to problem. If you are given a first order linear equation then you just find the largest interval containing t_0 where both $p(t)$ and $q(t)$ are continuous and theorem 2 tells us that a unique solution exists on that whole interval.

6. Autonomous Equations, 2.5

DEFINITION 6.1. A first order differential equation is said to be *autonomous* if it can be written as

$$\frac{dy}{dt} = f(y).$$

In other words, the variable t does not appear in the differential equation explicitly.

Note that autonomous equations are separable. We can often learn a lot about the solutions of an autonomous equation by studying the function $f(y)$. For example, if we can learn about the zeros of $f(y)$ then we know what the constant solutions are. If a is a zero of $f(y)$ then $y(t) = a$ solves the autonomous differential equation. The zeros of $f(y)$ (if they exist) are called *equilibrium solutions* or sometimes called *critical points*.

EXAMPLE 6.1. **Exponential Growth** Consider the differential equation

$$\frac{dy}{dt} = ky$$

where k is some nonzero constant. This equation is used to model things such as radioactive decay or the growth of bacteria in a controlled environment. The solution is easily obtained and is given by $y(t) = y_0 e^{k(t-t_0)}$ where $y(t_0) = y_0$ is the initial condition.

EXAMPLE 6.2. Consider the differential equation given by

$$\frac{dy}{dt} = y^2 - y - 2 = (y+1)(y-2).$$

We see the zeros of $f(y)$ are -1 and 2. Therefore, we know the equilibrium solutions are $y = -1$ and $y = 2$. Let us see if we can if we can determine the behavior of other solutions without actually finding them. Notice that $f(y) = (y+1)(y-2)$ is continuous and has continuous derivative with respect to y on the whole xy-plane. Therefore, we know a unique solution exists on some small time interval around any initial condition. In particular, we know that no two distinct solutions of this equation can cross at any point since that would violate uniqueness of solution at that point. From this we can conclude that if $y(t_0) = y_0 > 2$, then $y(t) > 2$ for all time in which the solution exists. Similarly, if $-1 < y_0 < 2$ then $-1 < y(t) < 2$ and $y(t) < -1$ if $y_0 < 1$. Lets analyze the function $f(y)$ in these different intervals. If $y > 2$ then $(y+1) > 0$ and $(y-2) > 0$ which tells us $y' = f(y) > 0$. Thus, y is an increasing function of t. If $-1 < y < 2$ then we see $y + 1 > 0$ and $y - 2 < 0$ which gives $y' = f(y) < 0$. Hence, y is a decreasing function of t. For $y < -1$ we see $y' = f(y) > 0$ and y is again an increasing function. We can arrange this data on a line which is called a *phase diagram* or a *phase line.*

$$\longrightarrow y = -1 \longleftarrow y = 2 \longrightarrow$$

Looking at the phase diagram we see solutions appear to be approaching $y = -1$ if y_0 is close to -1 and y appears to be moving away from the equilibrium solution $y = 2$. In this situation we can actually solve the differential equation and see if our intuition is correct. To solve the equation, first separate it to get

$$\frac{1}{(y+1)(y-2)}\frac{dy}{dt} = 1$$

To find an antiderivative for $\frac{1}{(y+1)(y-2)}$ we use partial fractions. We split it up as

$$\frac{1}{(y+1)(y-2)} = \frac{A}{y+1} + \frac{B}{y-2}.$$

Multiplying both sides by $(y+1)(y-2)$ we get $1 = A(y-2)+B(y+1)$. If we plug in $y=-1$ we get $1 = A(-3)$ which tells us $A = \frac{-1}{3}$ and similarly, plugging in $y=2$ gives $1 = B(3)$ which implies $B = \frac{1}{3}$. Therefore, we can write

$$\frac{1}{(y+1)(y-2)} = \frac{1}{3}\left(\frac{-1}{y+1} + \frac{1}{y-2}\right).$$

An antidervative is

$$\frac{1}{3}(\ln(|y-2|) - \ln(|y+1|)) = \frac{1}{3}\ln\left(\left|\frac{y-2}{y+1}\right|\right).$$

From this we find

$$\ln\left(\left|\frac{y-2}{y+1}\right|\right) = 3t + c$$

which implies $\frac{y-2}{y+1} = Ce^{3t}$ for some constant C. Plugging in the initial condition $y(t_0) = y_0$ (assuming $y_0 \neq 2$ or -1) we find $C = \frac{y_0-2}{y_0+1}e^{-3t_0}$. We can solve this equation for y to get

$$y(t) = \frac{2+Ce^{3t}}{1-Ce^{3t}} = \frac{(y_0+1)}{(y_0+1)}\frac{2+Ce^{3t}}{1-Ce^{3t}} = \frac{2(y_0+1)+(y_0-2)e^{3(t-t_0)}}{(y_0+1)-(y_0-2)e^{3(t-t_0)}}.$$

Notice that if we plug in -1 or 2 for y_0 then we get back the equilibrium solutions -1 and 2. Lets examine the behavior of the solutions as y_0 varies. If $y_0 < -1$ then $y(t) < -1$ for all time in which the solution is defined. In this situation note that both y_0+1 and y_0-2 are negative. Therefore, we find the denominator of $y(t)$ can be zero, this happens when

$$3(t-t_0) = \ln\frac{y_0+1}{y_0-2}.$$

Notice that both y_0+1 and y_0-2 are negative, in fact, since $y_0-2 < y_0+1 < 0$ we find $1 > \frac{y_0+1}{y_0-2} > 0$ which tells us

$$t = t_0 + \frac{1}{3}\ln\frac{y_0+1}{y_0-2}$$

and this is less than t_0 since the logarithm of a number between zero and one is negative. In the situation where $y_0 < -1$ we find that the solution is defined for all $t \geq t_0$. Moreover, since $e^{3(t-t_0)} \to \infty$ as $t \to \infty$ we find

$$\lim_{t\to\infty} y(t) = \frac{(y_0-2)}{-(y_0-2)} = -1.$$

Now assume $-1 < y_0 < 2$. Then $y_0 - 2 < 0 < y_0 + 1$ implying the denominator has a term that is always positive. Therefore, the solution is defined for all time and we see using the same reasoning as above that

$$\lim_{t\to\infty} y(t) = \frac{(y_0 - 2)}{-(y_0 - 2)} = -1.$$

Now suppose $2 < y_0$. Then the denominator is again zero when

$$t = t_0 + \frac{1}{3}\ln\frac{y_0 + 1}{y_0 - 2}.$$

since $\frac{y_0+1}{y_0-2} > 1$ the logarithm is a positive number and $t > t_0$. If we let $T = t_0 + \frac{1}{3}\ln\frac{y_0+1}{y_0-2}$, then we see the solution is defined for all $t < T$ and that as $t \to T$, the numerator is well behaved whereas the denominator approaches zero. Since the numerator and denominator are always positive for $t < T$, we see that as $t \to T$, $y(t) \to \infty$.

Whenever the solution starts out at a value less than 2 we find that the solution always approaches -1. In particular, if the solution starts close to -1 then it stays close to -1 for all time. We say that $y = -1$ is an *asymptotically stable* equilibrium solution. The idea behind a stable equilibrium solution is that if you start close to it then you stay close to it. On the other hand, we see that if the solution starts greater than 2 then the solution blows up to infinity. In particular, we find that no matter how close to 2 the solution starts, as long as it does not start at two, the solution moves further away from two. We say that $y = 2$ is an *unstable equilibrium solution.* Again the intuitive notion is that solutions that start close to 2 but are not exactly 2 eventually move away from 2.

We can see in the example above that the phase line determined the behavior of solutions quite nicely. We see that the phase line indicates that solutions that start out less than 2 approach the solution $y = -1$ and solutions that start above $y = 2$ move away. Note, however, that the phase line did not indicate to us the existence of vertical asymptotes. We had to do some analysis of the solution itself to find them. If we had not been able to solve the equation then we would have to resort to some other kind of analysis to determine the existence of vertical asymptotes.

7. Numerical Approximations: Euler's Method, 2.7

Quite often, a given differential equation cannot not be solved explicitly. An example of such is given by the ODE

$$\frac{dy}{dt} = t^2 + y^2.$$

The solution to this ODE cannot be written down using any of the elementary functions such as polynomials, trig functions, exponentials or logarithms. However, the existence and uniqueness theorem tells us that a solution does exist and is in fact unique since the function $f(t, y) = t^2 + y^2$ is continuous and has continuous partial derivative with respect to y on the whole ty-plane. One way to try and understand the solutions of this ODE is to solve it numerically. This yields an approximate solution to the ODE. The idea behind Euler's method is the tangent line approximation to a function at a point. Recall from calculus that if we have a differentiable function $y(t)$ then we can approximate $y(t)$ near the point $(t_0, y(t_0))$ as

$$y(t) \simeq y(t_0) + y'(t_0)(t - t_0).$$

We will use this to develop Euler's method. Suppose we have an initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

Since $y'(t_0) = f(t_0, y_0)$ we see from the tangent line approximation that $y(t) \simeq y(t_0) + y'(t_0)(t - t_0) = y_0 + f(t_0, y_0)(t - t_0)$. From this we find that if t_1 is a point close to t_0, then

$$y(t_1) \simeq y_0 + f(t_0, y_0)(t_1 - t_0).$$

Let $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$. We can continue using this idea to proceed further. If t_2 is close to t_1, then the tangent line approximation gives

$$y(t_2) \simeq y(t_1) + y'(t_1)(t_2 - t_1) = y(t_1) + f(t_1, y(t_1))(t_2 - t_1).$$

Unfortunately, we do not know what $y(t_1)$ is exactly. However, we do have the approximation y_1 so we can use that instead. Let

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1).$$

Then y_2 is an approximation to $y(t_2)$. We can continue this process to get a general formula. Suppose we have determined y_n, which is an approximation to $y(t_n)$. Then we find

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

is an approximation to $y(t_{n+1})$. It should be relatively clear that the smaller the steps, $t_{i+1} - t_i$, the better the approximation, however, this presents a computational problem. Suppose we have an initial value problem $y' = f(t, y)$ with initial condition $y(0) = 1$ and we want to know $y(1)$. If we want a good approximation we need to take small steps but the smaller the steps the more computations we need to do. This is where computers become very handy. In doing the computations of the y_i we have had to choose the values t_i. It is convenient to make the step sizes the same. If we determine our t_i by the relation $t_{i+1} - t_i = h$ for some fixed h, then we can write

$$y_{i+1} = y_i + f(t_i, y_i) * h.$$

Moreover, $t_1 - t_0 = h$ implies $t_1 = t_0 + h$. Similarly, we see $t_2 = t_1 + h = t_0 + h + h = t_0 + 2h$ and more generally that $t_i = t_0 + ih$. Hence, we get

$$y_{i+1} = y_i + f(t_0 + ih, y_i)h.$$

In this form it is easy to do computations or to program a computer to do the computations for you. Usually we will call h the step size and sometimes denote h by Δt.

Example 7.1. Use Euler's method with step sizes $h = 1/2$ and $h = \frac{1}{5}$ to approximate $y(1)$ where y is the solution to

$$\frac{dy}{dt} = ty, \qquad y(0) = 1.$$

First notice that $t_i = t_0 + ih = ih$. Therefore, we find

$$\begin{aligned} y_1 &= y_0 + f(t_0, y_0)h = 1 + 0 \cdot 1 \cdot \frac{1}{2} = 1, \\ y_2 &= y_1 + f(t_1, y_1)h = 1 + \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{5}{4} \end{aligned}$$

This gives the approximation $y(1) \simeq 1.25$. Using a step size $h = \frac{1}{5}$ we get $t_i = \frac{i}{5}$ which gives

$$\begin{aligned}
y_1 &= y_0 + f(t_0, y_0)h = 1 + 0 \cdot 1 \cdot \frac{1}{5} = 1, \\
y_2 &= y_1 + f(t_1, y_1)h = 1 + \frac{1}{5} \cdot 1 \cdot \frac{1}{5} = 1.04, \\
y_3 &= y_2 + f(t_2, y_2)h = 1.04 + \frac{2}{5} \cdot 1.04 \cdot \frac{1}{5} = 1.1232, \\
y_4 &= y_3 + f(t_3, y_3)h = 1.1232 + \frac{3}{5} \cdot 1.1232 \cdot \frac{1}{5} = 1.257984, \\
y_5 &= y_4 + f(t_4, y_4)h = 1.257984 + \frac{4}{5} \cdot 1.257984 \cdot \frac{1}{5} = 1.45926144.
\end{aligned}$$

This give the approximation $y(1) \simeq 1.46$. This initial value problem is separable and can easily be solved to yield

$$y(t) = e^{t^2/2}.$$

Using this we find the exact value is $y(1) = e^{1/2} \simeq 1.65$. We can see that our approximations are not very good. This is why using a computer helps a lot. I had maple do the computation with 100 steps and then with 1000 steps to get

$$\begin{aligned}
y_{100} &= 1.6378 \\
y_{1000} &= 1.6476
\end{aligned}$$

which are much better approximations.

Example 7.2. Use computer software to analyze the IVP's

$$y' = t + \frac{1}{5}y, \qquad y(0) = -3$$

and

$$y' = \cos(t) \cdot y, \qquad y(0) = 1.$$

Part 2

Second Order Linear Equations.

8. Homogeneous Equations.

Recall that the order of a differential equation is the order of the highest derivative that appears in the equation. We will focus on second order differential equation of the form

$$\frac{d^2y}{dt^2} = f(t, \frac{dy}{dt}, y).$$

EXAMPLE 8.1. Consider the equation given by

$$\frac{d^2y}{dt^2} = y\frac{dy}{dt}.$$

If we take the derivative of the equation

$$\frac{dy}{dt} = \frac{1}{2}y^2$$

then we get the previous equation. So a twice differentiable solution of the second equation will be a solution of the first equation. The second equation is first order separable and can be solved to get $y(t) = (C - \frac{t}{2})^{-1}$. It is easy to check that this solves the second order equation.

Solving the general second order differential equation is a difficult task. Generally, there is no known way to solve second order differential equations just as there were none for first order equations. However, there is a class of second order differential equations that can be tackled, these are the constant coefficient linear second order differential equations to be defined later. A *linear* second order ODE is one that can be written in the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t) \tag{8.1}$$

where P, Q, R and G depend only on the independent variable t. If the function P is identically zero then the equation is first order. Therefore, we will always assume that $P(t)$ is not identically zero. The ODE of example 8.1 is not linear. If the function $G(t)$ is always zero on the interval in question then we say that equation 8.1 is *homogeneous*, otherwise, it is called *non homogeneous*. If $G(t)$ is not identically zero then we call

$$P(t)y'' + Q(t)y' + R(t)y = 0 \tag{8.2}$$

the homogeneous equation associated to 8.1. Here is an example of a linear second order ODE.

EXAMPLE 8.2. $e^t y'' + \cos(t)y' + (t^2 + t - 1)y = \ln(t)$. I do not no how to solve this one. It may not even be solvable.

Consider a homogeneous second order ODE such as the one in equation 8.2. Second order linear equations satisfy a nice property given below.

LEMMA 8.1. *Let*

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

be a second order homogeneous ODE. If y_1 and y_2 are solutions then so is $y = c_1y_1 + c_2y_2$ for any real numbers c_1 and c_2.

PROOF. Using the fact that the derivative of a constant times a function is just the constant times the derivative of the function and that the derivative of a sum is the sum of the derivatives we find $y' = c_1y_1' + c_2y_2'$ and $y'' = c_1y_1'' + c_2y_2''$. Plugging this into the ODE we find

$$\begin{aligned} Py'' + Qy' + Ry &= P(c_1y_1'' + c_2y_2'') + Q(c_1y_1' + c_2y_2') + R(c_1y_1 + c_2y_2) \\ &= Pc_1y_1'' + Pc_2y_2'' + Qc_1y_1' + Qc_2y_2' + Rc_1y_1 + Rc_2y_2 \\ &= c_1(Py_1'' + Qy_1' + Ry_1) + c_2(Py_2'' + Qy_2' + Ry_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

This shows $y = c_1y_1 + c_2y_2$ is also a solution to the ODE. □

EXAMPLE 8.3. We can that $y_1 = \sin(t)$ and $y_2 = \cos(t)$ are both solutions of the differential equation

$$\frac{d^2y}{dt^2} + y = 0.$$

Therefore, the lemma above tells us $c_1 \sin(t) + c_2 \cos(t)$ is also a solution for any real numbers c_1 and c_2. In fact, every solution of this differential equation can be written in this way.

When one is confronted with a differential equation, the first question should be "Does it have a solution?" The second question should be "Is the solution unique?" In order to answer this question let us rewrite equation 8.1 in the form

$$y'' + p(t)y' + q(t)y = g(t)$$

that is obtained by dividing by $P(t)$.

THEOREM 8.1. *Suppose the functions p, q and g are continuous on the open interval I containing the point t_0. Then given any two numbers y_0 and y_0', the equation*

$$y'' + p(t)y' + q(t)y = g(t)$$

has a unique solution on the entire interval I that satisfies the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

EXAMPLE 8.4. We saw in example 3 that $\sin(t)$ and $\cos(t)$ are both solutions to $y''+y = 0$. From theorem 1 we know there is a unique solution to the initial value problem $y''+y = 0$ with $y(0) = 1$ and $y'(0) = -1$. We can see $\cos(t)$ satisfies $\cos(0) = 1$ but $\cos'(0) = -\sin(0) = 0$ so $\cos(t)$ is not a solution. On the other hand we find that $\sin(0) = 0$ and $\sin'(0) = \cos(0) = 1$ so $\sin(t)$ is also not a solution. However, if we take $\cos(t) - \sin(t)$ then this is a solution to the differential equation that satisfies the initial conditions.

EXAMPLE 8.5. Verify that e^t and te^t satisfy the differential equation $y'' - 2y' + y = 0$. Then find some combination of these two solutions that satisfy the initial conditions $y(0) = 3$ and $y'(0) = 1$.

$(e^t)'' = (e^t)' = e^t$ which shows $(e^t)'' - 2(e^t)' + e^t = 0$. Also, $(te^t)' = te^t + e^t$ and $(te^t)'' = (te^t + e^t)' = te^t + 2e^t$. Hence,

$$te^t + 2e^t - 2(te^t + e^t) + te^t = 0.$$

This shows both functions are solutions to the differential equation. In order to put them together to get a solution that satisfies the initial conditions let us write $y = c_1e^t + c_2te^t$. The first condition gives $y(0) = c_1 = 3$. The derivative is $y'(t) = c_1e^t + c_2(te^t + e^t)$ which must satisfy $y'(0) = 1$ therefore, $1 = y'(0) = c_1 + c_2$. Since $c_1 = 3$ we find $c_2 = -2$ and the function $y = 3e^t - 2te^t$ is a solution of the differential equation that satisfies the initial conditions.

Definition 8.1. **Linear Independence of two Functions, Linear Combinations.** Let f_1 and f_2 be two functions defined on a common interval I. If c_1 and c_2 are real numbers then we call $c_1 f_1 + c_2 f_2$ a linear combination of f_1 and f_2. A linear combination $c_1 f_1 + c_2 f_2$ is called *trivial* if $c_1 = c_2 = 0$. f_1 and f_2 are said to be *linearly independent* if $c_1 f_1 + c_2 f_2 = 0$ implies $c_1 = c_2 = 0$. In other words, the only linear combination which is 0 is the trivial linear combination. Otherwise, they are *linearly dependent*.

The two solutions $\cos(t)$ and $\sin(t)$ of example 4 are linearly independent since $\cos(t) \neq c\sin(t)$ for any $c \in \mathbb{R}$. Similarly, e^t and te^t are linearly independent as well. If we look at the last two examples note that we had two linearly independent solutions and we combined them in a way to get a solution that satisfied the initial conditions. This turns out to be a general method for second order linear ODEs. The Wronskian is a tool that has been developed for the study of these notions.

Definition 8.2. Let f and g be two differentiable functions defined on some interval I. The *Wronskian* of f and g is defined to be

$$W(f, g) = fg' - f'g.$$

We can use the Wronskian to help us determine whether two solutions are linearly independent or not.

Theorem 8.2. *Suppose y_1 and y_2 are solutions to*

$$y'' + p(t)y' + q(t)y = 0$$

on some interval I with p and q continuous on I.

(1) *If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) = 0$ on I.*
(2) *If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ on I.*

The theorem above will be useful when we want to check that two solutions are linearly independent. All we have to do is show that the Wronskian is non-zero at some point on I. If this is true then they cannot be linearly dependent since this implies the Wronskian is zero. The Wronskian is also used to prove:

Theorem 8.3. *Let y_1 and y_2 be linearly independent solutions of the differential equation*

$$y'' + p(t)y' + q(t)y = 0$$

with p and q continuous on some interval I. If y is any solution whatsoever of the above equation then there exists real numbers c_1 and c_2 such that

$$y = c_1 y_1 + c_2 y_2.$$

The usefulness of this theorem is that when we want to solve a homogeneous equation we can just look for two linearly independent solutions y_1 and y_2. Then we can form appropriate linear combinations of these two solutions to get a solution that satisfies the initial conditions. Any two linearly independent solutions will be called a *fundamental set of solutions* since any other solution is a linear combination of the two fundamental solutions.

9. Linear Homogeneous Second Order ODEs with Constant Coefficients, 3.1.

We now want to focus on equations of the form

$$ay'' + by' + cy = 0 \tag{9.1}$$

with $a \neq 0$, b and c constants. Recall that the function e^{rt} satisfies $(e^{rt})' = re^{rt}$ and $(e^{rt})'' = r^2e^{rt}$. Using this observation let us see if we can find a solution to equation 9.1 of the form e^{rt}. Plugging this into equation 9.1 we find

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} = 0.$$

The exponential e^{rt} is never zero, so if the equation above is to hold then we see r must satisfy

$$ar^2 + br + c = 0. \tag{9.2}$$

Equation 9.2 is called the characteristic equation of the homogeneous linear ODE. Recall that a quadratic polynomial can have two distinct solutions, one real solution or two complex conjugate solutions. We will tackle these three different cases separately.

10. Two Distinct Real Roots, still 3.1.

First assume the characteristic equation has two distinct real roots r_1 and r_2. Then by construction we find e^{r_1t} and e^{r_2t} are both solutions to equation 9.1. The Wronskian is given by

$$W(e^{r_1t}, e^{r_2t}) = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0.$$

By theorem 8.3 then, we find any solution of equation 9.1 has the form $c_1e^{r_1t} + c_2e^{r_2t}$ for some constants c_1 and c_2.

Example 10.1. Solve the initial value problem

$$y'' + y' - 6y = 0, \qquad y(0) = 1, \quad y'(0) = 7.$$

The characteristic equation is $r^2 + r - 6 = 0$. Since $r^2 + r - 6 = (r - 2)(r + 3)$ the roots are 2 and -3. Therefore, we get a set of fundamental solutions given by $\{e^{2t}, e^{-3t}\}$. In order to find the correct linear combination that satisfies the initial conditions let us write $y = c_1e^{2t} + c_2e^{-3t}$. Then $y' = 2c_1e^{2t} - 3c_2e^{-3t}$. Using the first initial condition we find $y(0) = c_1 + c_2 = 1$ and the second condition gives $y'(0) = 2c_1 - 3c_2 = 7$. So we get a system of linear equations. If we take 3 times the first and add it to the second we get $5c_1 = 10$ which gives $c_1 = 2$. Then from the first equation we see $c_2 = -1$. Therefore, the solution to the initial value problem is given by

$$y(t) = 2e^{2t} - e^{-3t}.$$

Example 10.2. Solve the initial value problem

$$y'' - 4y = 0, \qquad y(0) = 0, \quad y'(0) = 4.$$

The characteristic equation for this ODE is $r^2 - 4$. The two roots are ± 2. Therefore, we get the two solutions e^{2t} and e^{-2t}. By theorem 2 the solution to the initial value problem can be written as

$$y = ae^{2t} + be^{-2t}.$$

Since $y(0) = 0$ we get $a+b = 0$ which gives $b = -a$. Now $y' = 2ae^{2t} - 2be^{-2t} = 2a(e^{2t}+e^{-2t})$. Thus, $4 = y'(0) = 4a$ which gives $a = 1$. Therefore, $b = -1$ and we find

$$y = e^{2t} - e^{-2t}.$$

11. Complex Conjugate Roots, 3.3.

Now suppose the characteristic equation

$$ar^2 + br + c = 0$$

has two complex roots. We know from the quadratic formula that the two complex roots will be conjugates of each other. Therefore, we can write $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for real numbers α and $\beta \neq 0$. From this we find $e^{r_1 t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$ and $e^{r_2 t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$. Therefore, any linear combination

$$c_1 e^{\alpha t}(\cos(\beta t)+i\sin(\beta t))+c_2 e^{\alpha t}(\cos(\beta t)-i\sin(\beta t)) = (c_1+c_2)e^{\alpha t}\cos(\beta t)+i(c_1-c_2)e^{\alpha t}\sin(\beta t)$$

is a solution. Letting $c_1 = c_2 = \frac{1}{2}$ shows $e^{\alpha t}\cos(\beta t)$ is a solution. If we let $c_1 = -\frac{i}{2} = -c_2$ then we get $e^{\alpha t}\sin(\beta t)$ as another solution. Let us calculate the Wronskian of these two solutions,

$$\begin{aligned}
W &= e^{\alpha t}\cos(\beta t)(\alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t)) - e^{\alpha t}\sin(\beta t)(\alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t)) \\
&= e^{2\alpha t}\left(\alpha\sin(\beta t)\cos(\beta t) + \beta\cos^2(\beta t) - \alpha\sin(\beta t)\cos(\beta t) + \beta\sin^2(\beta t)\right) \\
&= \beta e^{2\alpha t}(\cos^2(\beta t) + \sin^2(\beta t)) = \beta e^{2\alpha t}.
\end{aligned}$$

This shows W is nonzero, therefore, $\{e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)\}$ is a fundamental set of solutions. By theorem 8.3 then we know any solution to $ay'' + by' + cy = 0$ is a linear combination of $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$.

Example 11.1. Find the general solution to

$$y'' - 2y' + 5y = 0.$$

The characteristic equation is given by

$$r^2 - 2r + 5 = 0.$$

The quadratic formula gives the two roots

$$r = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

Therefore, $\{e^t\cos(2t), e^t\sin(2t)\}$ is a fundamental set of solutions. The general solution to $y'' - 2y' + 5y = 0$ is

$$y = c_1 e^t\cos(2t) + c_2 e^t\sin(2t).$$

Example 11.2. Solve the initial value problem

$$y'' + 3y = 0, \qquad y(0) = 1, \quad y'(0) = 2$$

The two roots of the characteristic equation are $\pm\sqrt{3}i$. Therefore, $\{\cos(\sqrt{3}t), \sin(\sqrt{3}t)\}$ forms a fundamental set of solutions, the general solution is

$$y = c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t).$$

The initial condition $y(0) = 1$ gives $1 = c_1\cos(0) + c_2\sin(0) = c_1$. The derivative of y is given by $y' = \sqrt{3}(c_2\cos(\sqrt{3}t) - c_1\sin(\sqrt{3}t))$. Therefore, the second initial condition gives

$2 = y'(0) = \sqrt{3}(c_2\cos(0) - \sin(0)) = c_2\sqrt{3}$. Thus, the solution to the initial value problem is given by

$$y = \cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t).$$

12. Repeated Roots and Reduction of Order, 3.4.

Suppose the characteristic equation has one repeated real root. If we write the characteristic equation as

$$ar^2 + br + c = 0$$

this means that in the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the $b^2 - 4ac$ term is zero and $r = -b/2a$. Since the equation only has one root we only get the one solution e^{rt}. We need to find another way to find a linearly independent solution. We will use the method of *reduction of order*. Let us try a solution of the form $y = v(t)e^{rt}$. Taking derivatives we get $y' = rve^{rt} + v'e^{rt}$ and $y'' = r^2ve^{rt} + rv'e^{rt} + rv'e^{rt} + v''e^{rt}$. Plugging this into the ODE we have

$$a(r^2ve^{rt} + 2rv'e^{rt} + v''e^{rt}) + b(rve^{rt} + v'e^{rt}) + cve^{rt} = 0$$

We can rearrange terms to get

$$\begin{aligned} &ar^2ve^{rt} + brve^{rt} + cve^{rt} + (2ar + b)v'e^{rt} + v''e^{rt} \\ &= (ar^2 + br + c)ve^{rt} + (2ar + b)v'e^{rt} + v''e^{rt} = 0. \end{aligned}$$

Since r is the root of the characteristic equation we find that the first term is zero. The second term is zero since $r = -b/2a$ and so $v''e^{rt} = 0$. Since the exponential is never zero we see $v'' = 0$ giving $v = c_1t + c_2$ for some constants c_1 and c_2. This shows $y = (c_1t + c_2)e^{rt}$ is a solution to the ODE for any c_1 and c_2. Consider the set $\{e^{rt}, te^{rt}\}$, calculating the Wronskian we get

$$W = e^{rt}(e^{rt} + rte^{rt}) - re^{rt}(te^{rt}) = e^{2rt} \neq 0.$$

Therefore, we see $\{e^{rt}, te^{rt}\}$ forms a fundamental set of solutions.

EXAMPLE 12.1. Find the general solution to the differential equation

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is given by $r^2 - 4r + 4 = (r-2)^2 = 0$. Therefore, a fundamental set of solutions is given by $\{e^{2t}, te^{2t}\}$. Hence, the general solution is given by

$$y = c_1e^{2t} + c_2te^{2t}.$$

Consider now a differential equation of the form

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose we know of one solution y_1. We can use a similar reduction of order to find a second. Let us assume that another solution is given by $y_2(t) = v(t)y_1(t)$. Then taking derivatives we find $y' = v'y_1 + vy_1'$ and $y'' = v''y_1 + 2v'y_1' + vy_1''$. Plugging this into the ODE we get

$$v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + qvy_1 = v''y_1 + 2v'y_1' + pv'y_1 + v(y_1'' + py_1' + qy_1) = 0.$$

Since y_1 is a solution we find that the last term is zero. Therefore, we get the equation

$$y_1 v'' + (2y_1' + py_1)v' = 0.$$

If we let $u = v'$ then this is a linear first order equation for u,

$$y_1 u' + (2y_1' + py_1)u = 0.$$

Thus, we can use the methods of first order equations to solve this for u and hence for v. Note that we have reduced a second order equation to a first order equation, this is where the term reduction of order comes from. Note that in order for this method to work we have to find a solution to the original second order equation first. Let see this in action.

Example 12.2. Consider the second order differential equation

$$ty'' - y' + 4t^3 y = 0, \qquad t > 0,$$

given that $\sin(t^2)$ is a solution, find a second solution.

We want to use the method of reduction of order, but in order to do this we have to rewrite the equation in the form

$$y'' - \frac{1}{t}y' + 4t^2 y = 0.$$

We see here that $p(t) = -\frac{1}{t}$. Since $y_1 = \sin(t^2)$ we need to solve the first order equation

$$\sin(t^2)u' + (4t\cos(t^2) - \frac{1}{t}\sin(t^2))u = 0.$$

Dividing by $\sin(t^2)$ we get

$$u' + \left(4t\cot(t^2) - \frac{1}{t}\right)u = 0.$$

Let $P(t) = \left(4t\cot(t^2) - \frac{1}{t}\right)$, then $u' + P(t)u = 0$ which can be solved by integrating factors. A solution is given by

$$e^{-\int P(t)dt}.$$

We know that an antiderivative of $1/t$ is just $\ln(t)$. For the $4t\cot(t^2)$ we do a substitution $w = t^2$ to get $dw = 2tdt$ and we have the integral

$$\int 2\cot(w)dw = 2\ln(\sin(w)) + C.$$

Thus, we find our solution to be

$$u = exp\left(-2\ln|\sin(t^2)| + C + \ln(t)\right) = \frac{At}{\sin^2(t^2)}$$

Let us choose $A = -2$ for now, constants will be taken care of when we form the general solution. Since we want to find the function v we need to integrate u once, doing the same substitution $w = t^2$ we get

$$v = \int \frac{-2t}{\sin^2(t^2)}dt = \int -2t\csc^2(t^2) = \cot(t^2).$$

Therefore, a second solution to $ty'' - y' + 4t^3 y = 0$ is given by $y_2 = vy_1 = \cot(t^2)\sin(t^2) = \cos(t^2)$. So we have two solutions $\{\sin(t^2), \cos(t^2)\}$. It is probably not surprising that $\cos(t^2)$ turns out to be the other solution.

13. Non homogeneous Equations and Undetermined Coefficients, 3.5.

We now want to consider second order linear ODEs that are not homogeneous. So consider the differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{13.1}$$

where $g(t)$ is not constantly zero. Let y_1 and y_2 be a fundamental set of solutions to the associated homogeneous equation, i.e the same equation but with g replaced by zero, and let y_p be a solution to the original equation. Let c_1 and c_2 be two real numbers and let $y = c_1y_1 + c_2y_2 + y_p$. Then

$$\begin{aligned} & y'' + p(t)y' + q(t)y = c_1y_1'' + c_2y_2'' + y_p'' + p(t)(c_1y_1' + c_2y_2' + y_p') + q(t)(c_1y_1 + c_2y_2 + y_p) \\ = \; & c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) + y_p'' + p(t)y_p' + q(t)y_p = 0 + 0 + g(t) = g(t). \end{aligned}$$

This shows that $c_1y_1 + c_2y_2 + y_p$ is also a solution to equation 13.1. In fact, it turns out that every solution can be written in this way.

Theorem 13.1. *Suppose $Y_1(t)$ and $Y_2(t)$ are two solutions to equation 13.1 and $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the associated homogeneous equation. Then we can write*

$$Y_1 - Y_2 = c_1y_1 + c_2y_2$$

for some real numbers c_1 and c_2.

Proof. Let $Y = Y_1 - Y_2$. Then

$$\begin{aligned} Y'' + p(t)Y' + q(t)Y &= Y_1'' - Y_2'' + p(t)(Y_1' - Y_2') + q(t)(Y_1 - Y_2) \\ &= Y_1'' + p(t)Y_1' + q(t)Y_1' - (Y_2'' + p(t)Y_2' + q(t)Y_2) = g(t) - g(t) = 0. \end{aligned}$$

This shows Y is a solution of the homogeneous equation $y'' + py' + qy = 0$. Since y_1 and y_2 form a fundamental set of solutions we know we can write $Y = Y_1 - Y_2 = c_1y_1 + c_2y_2$ for some real numbers c_1 and c_2. □

Using theorem 13.1 we can determine what every solution to equation 13.1 looks like.

Theorem 13.2. *Suppose $Y(t)$ is a solution to equation 13.1 and $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the associated homogeneous equation. Then the general solution to equation 13.1 can be written in the form*

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t).$$

Proof. Let $y(t)$ be any solution to equation 13.1. Then by theorem 13.1, $y(t) - Y(t) = c_1y_1(t) + c_2y_2(t)$ for some real numbers c_1 and c_2. Therefore, $y = c_1y_1 + c_2y_2 + Y$. □

This theorem is what will determine our methods for solving non homogeneous ODEs. If we want to solve an ODE such as that in equation 13.1 the first step is to solve the associated homogeneous equation to get a fundamental set of solutions $\{y_1, y_2\}$. Then we will try to find a *particular solution* $Y(t)$ to equation 13.1. After this has been done we can form the combination $y = c_1y_1 + c_2y_2 + Y$ to get the general solution. If there are any initial conditions we will use them to find c_1 and c_2. Lets look at an example of this method.

Example 13.1. Find the general solution of

$$y'' + 3y' - 4y = 3t + 2.$$

Then find the solution that satisfies $y(0) = 1$ and $y'(0) = 0$.

The first step is to solve the associated homogeneous equation $y'' + 3y' - 4y = 0$. The characteristic equation is $r^2 + 3r - 4 = 0$. This factors as $(r+4)(r-1) = 0$ so we see that $\{e^{-4t}, e^t\}$ forms a fundamental set of solutions. The next step is to try and find a function $Y(t)$ that solves the equation. Since $g(t) = 3t + 2$ is a polynomial and derivatives of polynomials are again polynomials let us try $Y(t) = at + b$. Then we find $Y' = a$ and $Y'' = 0$. Plugging this into our ODE we find

$$Y'' + 3Y' - 4Y = 0 + 3a - 4(at + b) = 3t + 2.$$

which gives the equation $-4at + (3a - 4b) = 3t + 2$. The only way this can be true for all t is if $-4a = 3$ and $3a - 4b = 2$. This gives $a = -\frac{3}{4}$ and $4b = -\frac{9}{4} - \frac{8}{4} = -\frac{17}{4}$. Therefore, $Y(t) = -\frac{3}{4}t - \frac{17}{16}$. The general solution is then given by

$$y(t) = c_1 e^{-4t} + c_2 e^t - \frac{3}{4}t - \frac{17}{16}.$$

To find the solution which satisfies the initial conditions we get $1 = y(0) = c_1 + c_2 - \frac{17}{16}$. Also, we find $y'(t) = -4c_1 e^{-4t} + c_2 e^t - \frac{3}{4}$ which gives $1 = y'(0) = -4c_1 + c_2 - \frac{3}{4}$. It is easy to check that $c_1 = \frac{1}{16}$ and $c_2 = 2$ solve this system. Therefore, our solution is given by

$$y(t) = \frac{1}{16}e^{-4t} + 2e^t - \frac{3}{4}t - \frac{17}{16}.$$

We already have had some practice solving homogeneous equations. Therefore, let us focus on finding particular solutions to non homogeneous second order constant coefficient ODE's.

Example 13.2. Find a particular solution to

$$y'' - 4y = 2e^{3t}.$$

Exponentials behave nicely with respect to differentiation so lets try $y = Ae^{3t}$. Taking derivatives we find $y'' = 9Ae^{3t}$. Therefore,

$$9Ae^{3t} - 4Ae^{3t} = 2e^{3t}.$$

which gives $5A = 2$ or $A = \frac{2}{5}$. So a particular solution is given by $y(t) = \frac{2}{5}e^{3t}$.

The last two examples suggest that if we have a non homogeneous equation then we should try a function with a similar form as $g(t)$ for a particular solution. Consider the next two examples.

Example 13.3. Find a particular solution of

$$3y'' + y' - 2y = 2\cos(t).$$

As in the last two examples let us try $y(t) = A\cos(t)$. Then $y' = -A\sin(t)$ and $y'' = -A\cos(t)$. Therefore,

$$-3A\cos(t) - A\sin(t) - 2A\cos(t) = 2\cos(t).$$

We see that we have a problem here. There is nothing that will cancel out the $\sin(t)$ term. Therefore, we must have $A = 0$ but this gives us the zero solution which does not work. We need to modify our initial guess and try again. Note that the presence of a sine term in the first derivative is what is throwing us off. Let us see if we can rectify this by trying $y = A\cos(t) + B\sin(t)$. Taking derivatives we get $y' = -A\sin(t) + B\cos(t)$ and $y'' = -A\cos(t) - B\sin(t)$. Plugging this into our equation we find

$$\begin{aligned}3(-A\cos(t) - B\sin(t)) - A\sin(t) + B\cos(t) - 2(A\cos(t) + B\sin(t))\\ = (-3A + B - 2A)\cos(t) + (-3B - A - 2B)\sin(t) = 2\cos(t).\end{aligned}$$

Equating coefficients gives the system of equation $-5A + B = 2$ and $-A - 5B = 0$ whose solution is $A = -\frac{5}{13}$ and $B = \frac{1}{13}$. Therefore, our particular solution is

$$y(t) = -\frac{5}{13}\cos(t) + \frac{1}{13}\sin(t).$$

Example 13.4. Find a particular solution to

$$y'' - 4y = 2e^{2t}.$$

As before, let us try $y = Ae^{2t}$. Then $y'' = 4Ae^{2t}$ giving

$$4Ae^{2t} - 4(Ae^{2t}) = 0 \neq 2e^{2t}.$$

The associated homogeneous equation is $y'' - 4y = 0$, it is easily seen that $\{e^{2t}, e^{-2t}\}$ forms a fundamental set of solutions. Since the particular solution we tried is a solution to the homogeneous equation we need to find another solution. Maybe we can try something of the form $A(t)e^{2t}$ with A now being a function of t. The derivatives are given by $y' = A'e^{2t} + 2Ae^{2t}$ and $y'' = A''e^{2t} + 4A'e^{2t} + 4Ae^{2t}$. If we put this into our ODE we end up with

$$A''e^{2t} + 4A'e^{2t} + 4Ae^{2t} - 4Ae^{2t} = 2e^{2t}.$$

Dividing the equation above by e^{2t} we get $A'' + 4A' = 2$. Now, we just need to find one particular solution we do not need to find any general class of solutions. We found that letting A be a constant did not work so we cannot have $A' = 0$. However, lets try letting $A'' = 0$, then we get $4A' = 2$ which gives $A' = \frac{1}{2}$ or $A = \frac{1}{2}t + c$. Again, we can just choose $c = 0$ to get $A = \frac{1}{2}t$. Then we see that $\frac{1}{2}te^{2t}$ is a particular solution.

Suppose we have an ODE of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where we can write $g(t) = g_1(t) + g_2(t)$. Let Y_1 be a particular solution to $y'' + p(t)y' + q(t)y = g_1(t)$ and Y_2 be a particular solution to $y'' + p(t)y' + q(t)y = g_2(t)$. Then

$$\begin{aligned}(Y_1 + Y_2)'' + p(t)(Y_1 + Y_2)' + q(t)(Y_1 + Y_2)\\ = (Y_1'' + p(t)Y_1' + q(t)Y_1) + (Y_2'' + p(t)Y_2' + q(t)Y_2) = g_1 + g_2 = g\end{aligned}$$

This shows $Y_1 + Y_2$ is a particular solution to the original equation.

Example 13.5. Find a general solution to

$$y'' - 4y = 2e^{3t} + 2e^{2t}.$$

We already found that $\frac{2}{5}e^{3t}$ is a particular solution to $y'' - 4y = 2e^{3t}$ and $\frac{1}{2}te^{2t}$ is a particular solution to $y'' - 4y = 2e^{2t}$. Therefore,

$$\frac{2}{5}e^{3t} + \frac{1}{2}te^{2t}$$

is a particular solution to the original equation.

Example 13.6. Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-t} - 10\cos(3t), \qquad y(0) = 1, \quad y'(0) = 2.$$

The associated homogeneous equation is given by $y'' - 3y' + 2y = 0$. The characteristic equation is $r^2 - 3r + 2 = 0$ which has $r = 1$ and $r = 2$ as roots. Therefore, $\{e^t, e^{2t}\}$ form a fundamental set of solutions for the associated homogeneous equation. Notice that neither of the fundamental solutions show up in the non homogeneous term of the ODE so we can just try $y = Ae^{-t} + B\cos(3t) + C\sin(3t)$ for a particular solution. Thus,

$$\begin{aligned} y' &= -Ae^{-t} - 3B\sin(3t) + 3C\cos(3t)\text{and} \\ y'' &= Ae^{-t} - 9B\cos(3t) - 9C\sin(3t), \end{aligned}$$

which gives

$$Ae^{-t} - 9B\cos(3t) - 9C\sin(3t) - 3(-Ae^{-t} - 3B\sin(3t) + 3C\cos(3t)) + 2(Ae^{-t} + B\cos(3t) + C\sin(3t)) = 3e^{-t} - 10\cos(3t).$$

Collecting like terms and equating coefficients we find the system of equations $6A = 3$, $-7B - 9C = -10$ and $9B - 7C = 0$. The solutions are $A = \frac{1}{2}$, $B = \frac{7}{13}$ and $C = \frac{9}{13}$. Therefore, our particular solution is

$$y_p = \frac{1}{2}e^{-t} + \frac{7}{13}\cos(3t) + \frac{9}{13}\sin(3t).$$

Hence, the general solution and its derivatives are

$$\begin{aligned} y &= c_1e^t + c_2e^{2t} + \frac{1}{2}e^{-t} + \frac{7}{13}\cos(3t) + \frac{9}{13}\sin(3t) \\ y' &= c_1e^t + 2c_2e^{2t} - \frac{1}{2}e^{-t} - \frac{21}{13}\sin(3t) + \frac{27}{13}\cos(3t), \end{aligned}$$

which gives the system

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + \frac{1}{2} + \frac{7}{13} \\ 2 &= y'(0) = c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13}. \end{aligned}$$

You should check that the solutions to this are given by $c_1 = -\frac{1}{2}$ and $c_2 = \frac{6}{13}$. Therefore, the solution to the initial value problem is

$$y = -\frac{1}{2}e^t + \frac{6}{13}e^{2t} + \frac{1}{2}e^{-t} + \frac{7}{13}\cos(3t) + \frac{9}{13}\sin(3t).$$

I will finish up this section with one last example.

Example 13.7. Find a particular solution to

$$y'' + y' - 3y = e^t\cos(t).$$

Lets try a particular solution that looks like $e^t\cos(t)$. Recall example 3 where we ran into trouble with the $\sin(t)$ term coming from the derivatives. Taking heed of the problems there lets try $y = Ae^t\cos(t) + Be^t\sin(t)$. Then

$$\begin{aligned} y' &= Ae^t\cos(t) - Ae^t\sin(t) + Be^t\sin(t) + Be^t\cos(t) = (A+B)e^t\cos(t) + (B-A)e^t\sin(t) \\ y'' &= (A+B)e^t\cos(t) - (A+B)e^t\sin(t) + (B-A)e^t\sin(t) + (B-A)e^t\cos(t) \\ &= 2Be^t\cos(t) - 2Ae^t\sin(t). \end{aligned}$$

Therefore,

$$2Be^t\cos(t)-2Ae^t\sin(t)+(A+B)e^t\cos(t)+(B-A)e^t\sin(t)-3(Ae^t\cos(t)+Be^t\sin(t)) = e^t\cos(t).$$

Equating coefficients we get $-2A+3B=1$ and $-3A-2B=0$. The solutions are $A=-\frac{2}{13}$ and $B=\frac{3}{13}$. Hence,

$$y = -\frac{2}{13}e^t\cos(t) + \frac{3}{13}e^t\sin(t)$$

is a particular solution.

14. Variation of Parameters, 3.6.

The method described above for solving equations of the form

$$y'' + p(t)y'(t) + q(t)y = g(t) \tag{14.1}$$

are nice when g has a particularly nice form. However, the method can fail. An example is the equation $y''+y=\tan(t)$, the derivatives of $\tan(t)$ are linearly independent and will therefore not allow us to determine a particular solution. We need another method. This is where variations of parameters comes in. This method will always work in principle. The difficulty coming from whether certain antiderivatives can be found or not. Lets describe this method. Suppose we have a fundamental set of solutions $\{y_1, y_2\}$ for the associated homogeneous equation of 14.1. Then we know any function of the form $c_1y_1+c_2y_2$ will also be a solution to the associated homogeneous equation. Instead, lets see if it is possible to write a particular solution as $y=u_1(t)y_1+u_2(t)y_2$ for functions $u_1(t)$ and $u_2(t)$. Now, there is nothing telling us a priori that this will work. We are kind of just walking blindly at the moment, but lets push on. Taking a derivative we get

$$y' = u_1'y_1 + u_1y_1 + u_2'y_2 + u_2y_2' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2).$$

Remember that all we need to do is to come up with some particular solution, so lets try to make things as easy as possible. When we take a second derivative of y we are going to get some u_1'' and u_2'' terms from the second set of parenthesis above. Since this would turn a second order ODE of one function into one with two functions this does not seem like it would be helping much. Therefore, lets impose the condition that $u_1'y_1+u_2'y_2=0$. Again, there is no guarantee that this will work, but lets keep our fingers crossed. Using this fact we find that the second derivative is given by

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Plugging this all into equation 14.1 we end up with

$$\begin{aligned} y''+py'+qy &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) \\ &= u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2' \end{aligned}$$

Now, we want the equation to be equal to $g(t)$. Therefore, we need $u_1'y_1'+u_2'y_2'=g(t)$. This gives us a condition on the functions u_1 and u_2. We also imposed the condition $u_1'y_1+u_2'y_2=0$ so we get the following system of equations for u_1' and u_2':

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g(t). \end{aligned}$$

Remember that y_1, y_2 and g are known functions. The only unknowns above are u_1' and u_2'. If we multiply the first equation by y_2' and the second by y_2 we get

$$\begin{aligned} u_1'y_1y_2' + u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2 + u_2'y_2y_2' &= y_2g(t). \end{aligned}$$

Now subtract these two equations to find $u_1'(y_1y_2' - y_1'y_2) = -y_2g(t)$. Note that $y_1y_2' - y_1'y_2 = W(y_1, y_2)$, the Wronskian. Since $\{y_1, y_2\}$ forms a fundamental set of solutions we know $W(y_1, y_2) \neq 0$ on some interval I. Therefore, we can write

$$u_1' = -\frac{y_2(t)g(t)}{W(y_1, y_2)}.$$

Similarly, if we multiply the first equation by y_1' and the second equation by y_1 and subtract the first equation from the second we get

$$u_2' = \frac{y_1(t)g(t)}{W(y_1, y_2)}.$$

Notice that there is no minus sign on this one. We are now in a position to solve for u_1 and u_2, we just need to integrate both equations above. This gives,

$$u_1 = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)}dt, \text{ and } \qquad u_2 = \int \frac{y_1(t)g(t)}{W(y_1, y_2)}dt.$$

Remember that this whole procedure was developed to find a particular solution given by $y = u_1y_1 + u_2y_2$. Therefore, we have

THEOREM 14.1. *Suppose $\{y_1, y_2\}$ is a fundamental set of solutions to the associated homogeneous equation of equation 14.1. Then*

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)}dt$$

is a particular solution to equation 14.1.

Now lets do some examples.

EXAMPLE 14.1. Find the general solution to

$$y'' + y = \tan(t).$$

The associated homogeneous equation is given by $y'' + y = 0$ which has $r^2 + 1 = 0$ for a characteristic equation. The two roots are $\pm\imath$ which tells us that $\{\cos(t), \sin(t)\}$ is a fundamental set of solutions. Therefore, the Wronskian W is given by

$$W = W(\cos(t), \sin(t)) = \cos(t)(\cos(t)) - (-\sin(t))\sin(t) = \cos^2(t) + \sin^2(t) = 1.$$

Therefore, we need to compute

$$\int \sin(t)\tan(t)dt \text{ and } \qquad \int \cos(t)\tan(t) = \int \sin(t) = -\cos(t).$$

Now

$$\sin(t)\tan(t) = \frac{\sin^2(t)}{\cos(t)} = \frac{1 - \cos^2(t)}{\cos(t)} = \sec(t) - \cos(t)$$

so we get

$$\int \sin(t)\tan(t)dt = \int \sec(t) - \cos(t) = \ln|\sec(t) + \tan(t)| - \sin(t).$$

Therefore,

$$y_p(t) = -\cos(t)(\ln|\sec(t) + \tan(t)| - \sin(t)) + \sin(t)(-\cos(t)) = -\cos(t)\ln|\sec(t) + \tan(t)|$$

is a particular solution. Hence, the general solution is given by

$$y = c_1\cos(t) + c_2\sin(t) - \cos(t)\ln|\sec(t) + \tan(t)|.$$

EXAMPLE 14.2. Find a particular solution to

$$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}, \qquad t > 0.$$

The characteristic equation of the associated homogeneous equation is given by $r^2 + 4r + 4 = (r+2)^2 = 0$. Therefore, $\{e^{-2t}, te^{-2t}\}$ is a fundamental set of solutions. Recall that the Wronskian for the case of repeated roots is given by $W = e^{2r_1t} = e^{-4t}$. Therefore, the two integrals we have to compute are

$$\int te^{-2t}\frac{e^{-2t}}{t^2}\frac{1}{e^{-4t}}dt = \int \frac{1}{t}dt = \ln(t) \quad \text{and} \quad \int e^{-2t}\frac{e^{-2t}}{t^2}\frac{1}{e^{-4t}}dt = \int \frac{1}{t^2}dt = -\frac{1}{t}.$$

This gives

$$y_p(t) = -\ln(t)e^{-2t} - \frac{1}{t}te^{-2t} = -\ln(t)e^{-2t} - e^{-2t}.$$

We find that e^{-2t} is already apart of a fundamental set of solutions to the associated homogeneous equation. Therefore, we can just use

$$y_p(t) = -\ln(t)e^{-2t}$$

for a particular solution.

15. Mechanical Vibrations, 3.7.

If we consider the situation where a spring of natural length l is hanging with a mass m attached to it then we can model this system with a second order linear ODE. Lets ignore the mass of the spring. To a good approximation, the force that the spring exerts is proportional to the amount that it is stretched or compressed. Therefore, the spring force will be given by $F_s = -k\Delta x$ where Δx is the amount stretched or compressed and k is a positive constant. Notice that the force is opposite the direction the spring is pulled. Since the spring is hanging the mass will fill the affects of gravity and stretch the spring a little bit. Let L be the amount that the spring is stretched. Since the mass is motionless we find that L is determined by the equation

$$mg - kL = 0.$$

Here, g is the acceleration due to gravity. Let $u(t)$ be the distance from the equilibrium position $l + L$ of the mass at time t, we will take the downward direction as positive. Then we see that the force that the spring exerts on the mass is given by $F_s = -k(L + u)$. Since the mass will be moving through some medium, such as air, there are likely to be drag forces. Or, the mass may be attached to some kind of dampening apparatus. We will always assume that all the drag forces, or damping forces will be proportional to the velocity of the mass, i.e, $F_d = -\gamma u'(t)$. Also, there may be some external driving force acting on the mass, which we will denote by $F(t)$. Using Newton's second law we find that the motion of the spring is determined by

$$mu'' = mg - k(L+u) - \gamma u' + F(t).$$

Using the fact that $mg - kL = 0$ and rearranging we get the second order linear ODE

$$mu'' + \gamma u' + ku = F(t) \tag{15.1}$$

describing the motion of the mass. Note that all the constants in the equation are positive.

Undamped Free Vibrations. Lets consider the simplest case where there is no damping and no forcing. Equation 15.1 simplifies to

$$mu'' + ku = 0.$$

The characteristic equation for this is $mr^2 + k = 0$ which has $\pm i\sqrt{k/m}$ as the two roots. Let $\omega_0 = \sqrt{k/m}$. Then $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$ forms a fundamental set of solutions and the general solution is given by

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

If we have the initial conditions $u(0) = u_0$ and $u'(0) = v_0$ then we find that $c_1 = u_0$ and $c_2 = \frac{v_0}{\omega_0}$ and we can write

$$u(t) = u_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t).$$

Example 15.1. Suppose we have a spring mass system where $m = 5$ kilograms and the mass stretches the spring 0.1 meters when hanging. Suppose the mass is pulled down 0.3 meters and pushed with an initial velocity of 0.2 meters per second. Assuming there are no damping or driving forces find the motion of the mass as a function of time.

We need to know what the spring constant is. Since the mass stretches the spring by $0.1m$ we find $L = 0.1$. Therefore, we see from the condition $mg - kL = 0$ gives

$$k = \frac{mg}{L} = \frac{5 * 9.8}{0.1} = 490N/m.$$

From this we get $\omega_0 = \sqrt{k/m} = \sqrt{98} \simeq 9.899$. Therefore, $u_0 = 0.3$ and $v_0 = 0.2$ giving

$$u(t) = 0.3\cos(\sqrt{98}t) + \frac{0.2}{\sqrt{98}} \sin(\sqrt{98}t).$$

When working with functions of the form $A\cos(\omega_0 t) + B\sin(\omega_0 t)$ it is convenient to write it in the form $R\cos(\omega_0 t - \delta)$. Using the angle addition formula we find that $R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta)$. Therefore, $R\cos(\delta) = A$ and $R\sin(\delta) = B$. This allows us to find R and δ using the equations $A^2 + B^2 = R^2$ and $\tan(\delta) = B/A$. Also, we know that the period of cosine and sine are 2π. To determine the period of $\cos(\omega_0 t)$ we just need to find the smallest nonnegative number T for which $\cos(\omega_0 T) = 1$. We know that $\cos(t) = 1$ whenever $t = 2k\pi$ for some integer k so we find that $\omega_0 T = 2\pi$ which gives $T = \frac{2\pi}{\omega_0}$ as the period. Also, we call ω_0 the natural frequency. The nice thing about writing $A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$ is that the amplitude of oscillations, R, is easy to see.

EXAMPLE 15.2. From example 1

$$u(t) = 0.3\cos(\sqrt{98}t) + \frac{0.2}{\sqrt{98}}\sin(\sqrt{98}t).$$

Therefore,

$$R = \sqrt{0.3^2 + \frac{0.2^2}{98}} = \sqrt{0.09 + \frac{0.04}{98}} = \sqrt{\frac{8.86}{98}} \simeq 0.3007$$

and

$$\delta = \tan^{-1}(\frac{0.2}{0.3\sqrt{98}}) \simeq 0.067 \text{ rads.}$$

Thus, we can write

$$u(t) \simeq 0.3007\cos(\sqrt{98}t - 0.067).$$

Damped Free Vibrations. Now consider the case where there is some type of damping and or drag forces. Then the ODE that determines $u(t)$ is now given by

$$mu'' + \gamma u' + ku = 0.$$

The characteristic equation is $mr^2 + \gamma r + k = 0$ which has

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

Lets consider the discriminant $\gamma^2 - 4km$.

15.0.1. $\gamma^2 - 4km > 0$.

In the case where the discriminant is greater than zero we have two distinct real roots, call them r_1 and r_2. Then

$$u(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

is the solution. Notice $0 < \gamma^2 - 4km < \gamma^2$. Therefore, the two roots are negative. This shows e^{r_1t} and e^{r_2t} both approach zero as $t \to \infty$ and the motion of the mass dies. In this situation, no oscillation occurs and the mass just creeps back to its equilibrium position. This situation is called **overdamped**.

EXAMPLE 15.3. Consider a mass spring system governed by the equation

$$u'' + 3u' + 2 = 0, \qquad u(0) = 1, \quad u'(0) = 1.$$

the roots of the characteristic equation are -2 and -1. Thus, the solution is given by

$$u(t) = c_1e^{-t} + c_2e^{-2t}.$$

From the initial conditions, $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 1$. The solutions are easily found to be $c_1 = 3$ and $c_2 = -2$. A plot of this solution will show that the mass moves further away a bit then approaches the equilibrium position without ever crossing it.

15.0.2. $\gamma^2 - 4km = 0$.

In the case where the discriminant is zero we get one negative real root $r = -\frac{\gamma}{2m}$. The solutions in this case are of the form

$$u(t) = (c_1 + c_2 t)e^{-\frac{\gamma}{2m}t}$$

We can see $u(t) \to 0$ as $t \to \infty$ in this case as well. This situation is called **critically damped**. The behavior is similar to the case of overdamping except that there may be cases where the mass crosses the equilibrium solution once and then steadies back to the equilibrium solution.

Example 15.4. Consider a mass spring system governed by the equation

$$u'' + 2u' + u = 0, \qquad u(0) = 1, \quad u'(0) = -2.$$

The root of the characteristic equation in this case is going to be -1 and we get $u(t) = (c_1 + c_2 t)e^{-t}$. Using the first initial condition we find $c_1 = 1$. Now, $u' = -(1 + c_2 t)e^{-t} + c_2 e^{-t}$ which gives $-2 = u'(0) = -1 + c_2$ and we find $c_2 = -1$. This give the solution

$$u(t) = (1 - t)e^{-t}.$$

Notice that $u(1) = 0$ and this is the only zero which shows the mass crosses the equilibrium solution once and then approaches the equilibrium solution.

15.0.3. $\gamma^2 - 4km < 0$.

The case where the discriminant is negative is the most interesting. The two complex conjugate roots are given by

$$-\frac{\gamma}{2m} \pm i\frac{\sqrt{4km - \gamma^2}}{2m}.$$

Let $\mu = \frac{\sqrt{4km-\gamma^2}}{2m}$, then we can write the general solution as

$$u(t) = Re^{-\frac{\gamma}{2m}t}\cos(\mu t - \delta).$$

A series expansion will show that

$$\frac{\mu}{\omega_0} \simeq 1 - \frac{\gamma^2}{8km}$$

and

$$\frac{T_d}{T} \simeq 1 + \frac{\gamma^2}{8km}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency, T is the natural period and T_d is the damped period. We can see from these two approximations that damping causes the period to increase. Thus, it takes longer for the mass to oscillate in the presence of damping.

Example 15.5. Suppose there is a spring of spring constant $k = 65$ Newtons per meter, a mass $m = 16$ kilograms and a coefficient of damping $\gamma = 8$ Newtons second per meter. If the mass is pulled 1 meter past equilibrium and let go, find the equation of motion.

The ODE that determines u is given by

$$16u'' + 8u' + 65u = 0,$$

with initial position $u(0) = 1$. Since the mass is let go, it has no initial velocity, therefore, $u'(0) = 0$. The characteristic equation is given by $16r^2 + 8r + 65 = 0$ and so the roots are

$$\frac{-8 \pm \sqrt{64 - 4(16)(65)}}{32} = \frac{-1 \pm 8i}{4} = -\frac{1}{4} \pm 2i.$$

We can then write the solution as

$$u(t) = Re^{-\frac{t}{4}} \cos(2t - \delta).$$

Now, the initial condition $u(0) = 1$ gives $R\cos(\delta) = 1$. We find $u' = -\frac{1}{4}Re^{-t/4}\cos(2t - \delta) - 2Re^{-t/4}\sin(2t - \delta)$ so we get $0 = u'(0) = -(1/4)R\cos(\delta) + 2R\sin(\delta)$ which implies $8\sin(\delta) = \cos(\delta)$ or $\delta = \tan^{-1}(1/8) \simeq 0.124$ radians. Then $1 = R\cos(0.124)$ gives $R \simeq 1.0078$. Therefore, the solution is

$$u(t) = 1.0078e^{-t/4}\cos(2t - 0.124).$$

16. Forced Vibrations, 3.8.

Consider now the situation where there is an external driving force acting on the mass. The equation is given by

$$mu'' + \gamma u' + ku = F(t).$$

We will be mainly interested in the case where the forcing function is periodic, say, $F(t) = F_0\cos(\omega t)$.

16.1. Undamped Forced Vibrations. Lets consider the case where there is no damping and periodic driving force. Then the equation is given by

$$mu'' + ku = F_0\cos(\omega t).$$

Let $\omega_0 = \sqrt{k/m}$ and divide the equation above by m to get $u'' + \omega_0^2 u = F_0/m\cos(\omega t)$. We know $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$ is a fundamental set of solutions for the associated homogeneous equation. First, assume that $\omega \neq \omega_0$. Then we can try $u_p(t) = A\cos(\omega t)$ as a particular solution. Since there is no u' term in the equation we can ignore the presence of a sine term in the particular solution. Taking the second derivative we get $u_p'' = -\omega^2 A\cos(\omega t)$ which gives

$$-\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) = \frac{F_0}{m}\cos(\omega t).$$

This shows

$$u_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t)$$

and we get

$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos(\omega t) \tag{16.1}$$

as a general solution.

EXAMPLE 16.1. Suppose that $m = 1$, $k = 9$, $F_0 = 80$ and $\omega = 5$. Find $u(t)$ given $u(0) = u'(0) = 0$.

With these numbers we see that $\omega_0 = 3$ and the general solution is

$$u(t) = R\cos(3t - \delta) - 5\cos(5t)$$

Using the initial conditions we get $0 = R\cos(\delta) - 5$ and $3R\sin(\delta) = 0$. This is easily solved to give $R = 5$ and $\delta = 0$. Therefore, the solution is

$$u(t) = 5\cos(3t) - 5\cos(5t).$$

It is easy to show in fact that with the initial conditions $u(0) = u'(0) = 0$, we see that equation 16.1 gives the solution

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos(\omega t) - \cos(\omega_0 t)).$$

If we use the identity $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$ we see that $\cos(\omega t) - \cos(\omega_0 t) = 2\sin\frac{1}{2}(\omega_0 - \omega)t\sin\frac{1}{2}(\omega_0 + \omega)t$. This allows us to write

$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\frac{1}{2}(\omega_0 - \omega)t\sin\frac{1}{2}(\omega_0 + \omega)t.$$

We can consider this in the context that the mass is oscillating via $u(t) = A(t)\sin\frac{1}{2}(\omega_0 + \omega)t$ with a varying amplitude $A(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\frac{1}{2}(\omega_0 - \omega)t$. This leads to the notion of **beats**. If $\omega_0 \simeq \omega$ then the mass oscillates quickly with a frequency roughly equal to $2\omega_0$ but with a slowly (in comparison with the other frequency) varying amplitude.

EXAMPLE 16.2. With $m = 0.1$, $F_0 = 50$, $\omega_0 = 55$ and $\omega = 45$ we get

$$u(t) = \sin(5t)\sin(50t).$$

Plot this function to see the "beats."

Lets now consider the case where $\omega = \omega_0$. Then $A\cos(\omega_0 t)$ is a homogeneous solution, thus we need to modify our guess for a particular solution. Lets try $u_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$, notice that we now have to include the sine term. Going through the usual routine it is easy to find that

$$u_p(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$$

is a particular solution. Notice the factor of t, we can see that this solution has the property that it oscillates but with an amplitude that keeps growing. This phenomena is referred to as **resonance**.

EXAMPLE 16.3. Suppose $m = 1$, $F_0 = 80$ and $\omega = \omega_0 = 5$. Find $u(t)$ given $u(0) = u'(0) = 0$.

u will be given by $u(t) = R\cos(5t - \delta) + 8t\sin(5t)$ and $u'(t) = -5R\sin(5t - \delta) + 8\sin(5t) + 8t\cos(5t)$. The initial conditions give $R\cos(\delta) = 0$ and $-5R\sin(\delta) = 0$. This shows $R = 0$ and we get

$$u(t) = 8t\sin(5t).$$

Plot this solution on a computer and notice the growing amplitude.

16.2. Damped Forced Vibrations. Consider the case where there is some damping along with the external forcing,

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t). \tag{16.2}$$

Let $\{u_1, u_2\}$ be a fundamental set of solutions of the associated homogeneous equation, then we can write the general solution as

$$y(t) = c_1 u_1 + c_2 u_2 + u_p = u_h + u_p$$

where $u_h = c_1 u_1 + c_2 u_2$ is the *homogeneous solution*. Note that the roots of the characteristic equation of equation 16.2 are never purely imaginary, there is always a nonzero negative real part. Therefore, regardless of whether the system is overdamped, critically damped or underdampded we know $u_h \to 0$ as $t \to \infty$ due to the presence of exponentials with negative exponents. The long term behavior is thus determined by the particular solution u_p. Sometimes, u_h is also called the transient solution. Note that $i\omega$ is not a root of the characteristic equation which indicates that the particular solution will have the form $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Taking derivatives we get

$$\begin{aligned} u_p' &= -\omega A \sin(\omega t) + \omega B \cos(\omega t) \\ u_p'' &= -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t). \end{aligned}$$

Plugging this in and equating coefficients we get the equations

$$m(\omega_0^2 - \omega^2)A + \gamma\omega B = F_0, \qquad -\gamma\omega A + m(\omega_0^2 - \omega^2)B = 0.$$

The solutions are

$$A = \frac{m(\omega_0^2 - \omega^2)F_0}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \quad \text{and} \quad B = \frac{\gamma\omega F_0}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}.$$

We want to write the particular solution as $R\cos(\omega t - \delta)$. Recall that R is determined to be $\sqrt{A^2 + B^2}$ which gives

$$R = \sqrt{\frac{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)F_0^2}{(m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2)^2}} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}.$$

and we find delta to be given by

$$\delta = \arctan\frac{B}{A} = \arctan\frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$$

assuming that $\omega \neq \omega_0$. We need to be careful about taking the inverse tangent above, if $\omega_0 < \omega$ then A is negative which puts us in the second quadrant so we need to add π to whatever the calculator gives. If $\omega_0 = \omega$ then $\delta = \pi/2$.

The book has a discussion about what value of ω makes the amplitude

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}.$$

maximal. I will leave it as an exercise to do the differentiation of R with respect to ω and find the max. The answer turns out to be $\omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}$ which gives a maximal amplitude of

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - \frac{\gamma^2}{4mk}}}.$$

Lets look at a concrete example to see what is happening here.

EXAMPLE 16.4. Consider a damped mass-spring system with external forcing where $m = 1$, $\gamma = 2$, $k = 26$ and $F(t) = 82\cos(4t)$. The initial conditions are $u(0) = 6$ and $u'(0) = 0$

We want to solve the initial value problem

$$u'' + 2u' + 26u = 82\cos(4t), \qquad u(0) = 6, \quad u'(0) = 0.$$

The associated characteristic equation is given by $r^2 + 2r + 26 = 0$ which has roots

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(26)}}{2} = \frac{-2 \pm 2\sqrt{1 - 26}}{2} = -1 \pm 5i.$$

Therefore, $\{e^{-t}\cos(5t), e^{-t}\sin(5t)\}$ is a fundamental set of solutions. We know a particular solution is given by

$$u_p(t) = R\cos(4t - \delta)$$

where

$$R = \frac{82}{\sqrt{(26 - 16)^2 + 64}} = \frac{2 * 41}{\sqrt{4 * 41}} = \sqrt{41}$$

and

$$\delta = \arctan(8/10) \simeq 0.6747.$$

Hence, we get

$$u_p(t) = \sqrt{41}\cos(4t - 0.6747).$$

We can write the general solution as

$$\begin{aligned} u(t) &= e^{-t}(c_1\cos(5t) + c_2\sin(5t)) + \sqrt{41}\cos(4t - 0.6747) \\ u'(t) &= = -e^{-t}(c_1\cos(5t) + c_2\sin(5t)) + e^{-t}(-5c_1\sin(5t) + 5c_2\cos(5t)) - 4\sqrt{41}\sin(4t - 0.6747). \end{aligned}$$

An application of the identity $\cos(\arctan(x)) = 1/\sqrt{1 + x^2}$ gives $\cos(\delta) = 5/\sqrt{41}$, a similar identity yields $\sin(\delta) = 4/\sqrt{41}$. Therefore, $6 = u(0) = c_1 + \sqrt{41}\cos(\delta) = c_1 + 5$ which gives $c_1 = 1$. The other initial condition yields $0 = u'(0) = -c_1 + 5c_2 - 4\sqrt{41}\sin(\delta)$ which shows $0 = -1 + 5c_2 + 16$ giving $c_2 = -3$. Thus,

$$u(t) = e^{-t}(\cos(5t) - 3\sin(5t)) + \sqrt{41}\cos(4t - 0.6747)$$

Plot this solution along with the individual functions u_h and u_p to see the behavior.

Part 3

The Laplace Transform

17. Improper Integrals, 6.1.

In order to talk about the Laplace transform we need to discuss improper integrals. In particular, we need to discuss integrals of the form

$$\int_a^\infty f(t)dt.$$

This is a particular type of improper integral and the only type that we will be interested in. We make sense of this type of improper integral in the following way.

Definition 17.1. Let f be a function defined on some infinite interval (a, ∞). Then we define the improper integral of f over $(a, \infty)$to be the limit

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt.$$

If the limits exists then the improper integral is said to converge. If the improper integral fails to exist or is $\pm\infty$ then the improper integral is said to diverge.

Lets look at some examples.

Example 17.1. Evaluate

$$\int_0^\infty te^{-t}dt.$$

We can use integration by parts to find

$$\int_0^A te^{-t}dt = -(A+1)e^{-A} + 1.$$

So we need to compute the limit of this expression as A tends to infinity. Since the exponential term e^{-A} will kill of the polynomial term we see

$$\lim_{A\to\infty} -(A+1)e^{-A} + 1 = 1.$$

Therefore

$$\int_0^\infty te^{-t}dt = 1.$$

Example 17.2. Evaluate

$$\int_0^\infty \frac{dt}{1+t^2}.$$

$$\int_0^A \frac{dt}{1+t^2} = \arctan A - \arctan 0 = \arctan(A).$$

Therefore,

$$\int_o^\infty \frac{dt}{1+t^2} = \lim_{A\to\infty} \arctan(A) = \frac{\pi}{2}.$$

We want to be able to talk about second order linear equations $ay'' + by' + cy = F(t)$ with the possibility that $F(t)$ will only be *piecewise continuous.* A function f is said to be piecewise continuous on the interval $a \le t \le b$ if we can break up the interval as $a = t_0 < t_1 < \cdots < t_n = b$ such that f is continuous on each subinterval $t_i < t_{i+1}$ and such that the left and right limits of f as t approaches t_i is a finite number. If a function f is piecewise continuous on an interval $a \leq t \leq b$, then we can still integrate as

$$\int_a^b f(t)dt = \int_a^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \cdots + \int_{t_{n-1}}^{b} f(t)dt.$$

If f is piecewise continuous on $a \leq t \leq b$ for all $b > a$ then f is called piecewise continuous on $t \geq a$. Not all integrable functions have convergent improper integrals. Consider the simple function $f(t) = t$. Then for any real number a, it can be shown that

$$\int_a^\infty t dt = \infty.$$

In principal, it can be very difficult to determine whether or not a certain improper integral converges based solely on the integrand $f(t)$. However, in certain situations, the improper integral $\int_a^\infty f(t)dt$ can be determined by using a simpler function $g(t)$.

Theorem 17.1. *If f is piecewise continuous for $t \geq a$, and if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges. Conversely, if $0 \leq g(t) \leq f(t)$ for $t \geq M$ and $\int_M^\infty g(t)dt$ diverges then $\int_a^\infty f(t)dt$ also diverges.*

This theorem can be useful when considering whether or not an integral converges or diverges. However, this theorem does not tells us what the improper integral converges to if it does converge.

18. The Laplace Transform, 6.1.

We can now define the Laplace transform of a function. All of this may seem unnecessary for a differential equations class but there is a point to it. We will find that the Laplace transform will allow us to transform a differential equation into an algebraic equation and solve using ordinary algebraic rules. After this is done we can undo the Laplace transform to get a solution to the original differential equation.

Definition 18.1. Let $f(t)$ be a function defined on $t \geq 0$. The Laplace transform of f, if it exists, is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

There is no guarantee that this integral will converge for any function $f(t)$. However, using theorem 17.1 we will see there are lots of functions with a well defined Laplace transform. I also want to point out that the notation $F(s)$ is justified since the integral does depend on s. For different values of s we could potentially get different values of the integral. Before we compute some Laplace transform we need to discuss a theorem.

Theorem 18.1. *Suppose $f(t)$ is a piecewise continuous function defined on $t \geq 0$. If there exists positive numbers K and M and a real number a such that*

$$|f(t)| \leq K e^{at}, \qquad \text{for } t \geq M$$

then the Laplace transform $F(s)$ exists for all $s > a$.

PROOF. We need to show the integral

$$\int_0^\infty e^{-st}f(t)dt = \lim_{A\to\infty}\int_0^A e^{-st}f(t)dt$$

converges. If $A > M$ we can write

$$\int_0^A e^{-st}f(t)dt = \int_0^M e^{-st}f(t)dt + \int_M^A e^{-st}f(t)dt.$$

Since $f(t)$ is piecewise continuous, we know that the first integral on the right above exists as a finite number. Since this does not depend on A we see that the integral on the left will converge if the second integral on the right converges. Recall from calculus that the absolute value of an integral is alway less than or equal to the integral of the absolute value of the integrand. For $t \geq M$ we have by hypothesis that $|e^{-st}f(t)| \leq e^{-st}Ke^{at} = Ke^{(a-s)t}$. Therefore, we get

$$|\int_M^A e^{-st}f(t)dt| \leq \int_M^A |e^{-st}f(t)|dt \leq K\int_M^A e^{(a-s)t}dt = \left\{ \begin{array}{rl} K(A-M) & \text{if } s = a \\ \frac{K}{a-s}\left(e^{(a-s)A} - e^{(a-s)M}\right) & \text{if } s \neq a \end{array} \right.$$

From this we see that if $s = a$, then we are taking the limit of $K(A-M)$ as $A \to \infty$ which is just ∞. If $s < a$ then we are taking the limit of $\frac{K}{a-s}\left(e^{(a-s)A} - e^{(a-s)M}\right)$. This goes to ∞ since $e^{(a-s)A}$ has a positive exponent that keeps growing. So we find in these two cases, the comparison does not tell us about the convergence of $F(s)$. However, if $s > a$ then the limit of $\frac{K}{a-s}\left(e^{(a-s)A} - e^{(a-s)M}\right)$ tends to $-\frac{K}{a-s}e^{(a-s)M}$ as A tends to ∞ which shows the limit of $\int_M^A e^{-st}f(t)dt$ as A goes to ∞ is less than or equal to $-\frac{K}{a-s}e^{(a-s)M}$ and hence converges. This shows the Laplace transform of $f(t)$ exists for $s > a$. $\square$

The nice thing about this theorem is that we now know that any polynomial function has a Laplace transform, as do any exponential functions or sine or cosine functions. In fact, probably most of the functions you can think of have a Laplace transform.

EXAMPLE 18.1. Compute the Laplace transform of $f(t) = 1$.

We just need to compute the integral

$$\int_0^\infty e^{-st}dt = \lim_{A\to\infty}\int_0^A e^{-st}dt = \lim_{A\to\infty}\frac{1}{-s}\left(e^{-sA} - 1\right) = \frac{1}{s} \quad s > 0.$$

EXAMPLE 18.2. Compute the Laplace transform of $f(t) = e^{at}$.

We get

$$F(s) = \mathcal{L}\{e^{at}\} = \lim_{A\to\infty}\int_0^A e^{-st}e^{at} = \lim_{A\to\infty}\left\{ \begin{array}{rl} K(A-M) & \text{if } s = a \\ \frac{K}{a-s}\left(e^{(a-s)A} - 1\right) & \text{if } s \neq a \end{array} \right.$$

The only time this limit is finite is in the case where $s > a$ and we get

$$F(s) = \frac{1}{s-a} \quad \text{for } s > a.$$

EXAMPLE 18.3. Compute the Laplace transform of

$$f(t) = \left\{ \begin{array}{ll} 1 & 0 \leq t < 1 \\ 0 & t \geq 1. \end{array} \right.$$

By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt.$$

As soon as A is larger than 1 we get

$$\int_0^A e^{-st} f(t)dt = \int_0^1 e^{-st}dt = \frac{1}{-s} e^{-st}|_0^1 = \frac{1-e^{-s}}{s}.$$

Hence,

$$\mathcal{L}\{f(t)\} = \frac{1-e^{-s}}{s}.$$

There is a very important property that the Laplace transform has. Suppose $f_1(t)$ and $f_2(t)$ are functions defined on $t \geq 0$ which have Laplace transforms and c_1, c_2 are real numbers. Then

$$\begin{aligned}
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))dt = \lim_{A\to\infty} \int_0^A (c_1 e^{-st} f_1(t) + c_2 e^{-st} f_2(t))dt \\
&= \lim_{A\to\infty} \left(c_1 \int_0^A e^{-st} f_1(t)dt + c_2 \int_0^A e^{-st} f_2(t)dt \right) \\
&= c_1 \lim_{A\to\infty} \int_0^A e^{-st} f_1(t)dt + c_2 \lim_{A\to\infty} \int_0^A e^{-st} f_2(t)dt \\
&= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.
\end{aligned}$$

This shows the Laplace transform is a *linear operator*. It comes in handy when trying to compute Laplace transforms of functions that can be written as linear combinations of simpler functions. It should be mentioned at this time that there is an inverse Laplace transform. This is much more difficult to work with and requires doing contour integrals over the complex plane. The important thing for us is the fact that it exists which tells us that if we have a function $F(s)$ that satisfies nice properties then we can find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. We will put this into practice to solve initial value problems.

19. Solutions to Initial Value Problems, 6.2

Now we will see how the Laplace transform can be used to solve initial value problems. To see this we need to determine how the Laplace transform affects derivatives. Suppose that $f(t)$ is a differentiable function on $t \geq 0$. Is there a nice relation between $\mathcal{L}\{f'(t)\}$ and $\mathcal{L}\{f(t)\}$? The answer is yes as the next theorem shows.

Theorem 19.1. *Suppose $f(t)$ is a differentiable function on $t \geq 0$ and $f'(t)$ is piecewise continuous on $t \geq 0$. Suppose further that there are positive constants K and M and a real number a such that*

$$|f(t)| \leq Ke^{at} \quad \text{for } t \geq M.$$

Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof. Integration by parts tells us

$$\int_0^A e^{-st} f'(t)dt = f(A)e^{-sA} - f(0) + s\int_0^A e^{-st} f(t)dt.$$

Since $|f(A)| \leq Ke^{at}$ for $t \geq M$ we find for $A \geq M$, $|f(A)e^{-sA}| \leq Ke^{(a-s)A} \to 0$ as $A \to \infty$ as long as $s > a$. Therefore,

$$\lim_{A\to\infty} \int_0^A e^{-st} f'(t)dt = \lim_{A\to\infty} \left(f(A)e^{-sA} - f(0) + s\int_0^A e^{-st} f(t)dt \right) = s\mathcal{L}\{f(t)\} - f(0).$$

as long as $s > a$. □

We can continue this procedure to find the Laplace transform for higher derivatives of f if they exists. Suppose $f'(t)$ and $f''(t)$ satisfy the conditions imposed on f and f' in the previous theorem, then

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s(\mathcal{L}\{f(t)\} - f(0)) - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

This formula can be generalized to find the Laplace transform of the n-th derivative of f if suitable conditions hold.

Lets put this machinery to use with some examples.

EXAMPLE 19.1. Solve the initial value problem $y'' + 5y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = 1$.

We do not know a priori that the solution to this differential equation has a Laplace transform. However, we will just start working and hopefully things will work out. Let $Y(s)$ denote the Laplace transform of $y(t)$. Using the formulas for the Laplace transform of derivatives and the fact that taking Laplace transforms is linear we find

$$\begin{aligned} \mathcal{L}\{y'' + 5y' + 6y\} &= \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) \\ &= (s^2 + 5s + 6)Y(s) - (s+5)y(0) - y'(0) = (s^2 + 5s + 6)Y(s) - 2(s+5) - 1 = 0. \end{aligned}$$

Therefore, we can solve this for $Y(s)$ to obtain

$$Y(s) = \frac{2s + 11}{s^2 + 5s + 6} = \frac{2s + 11}{(s+2)(s+3)}.$$

We are now in a situation where we would need to take the inverse Laplace transform to find $y(t)$. However, if we look at the table on page 317 of the text then we may be able to fiddle with $Y(s)$ to get it in a form for which we know what it is the Laplace transform of. This is something that just comes with lots of practice. We can write

$$Y(s) = \frac{-5s - 10}{(s+2)(s+3)} + \frac{7s + 21}{(s+2)(s+3)} = -5\frac{s+2}{(s+2)(s+3)} + 7\frac{s+3}{(s+2)(s+3)} = \frac{-5}{s+3} + \frac{7}{s+2}.$$

Looking at the table we recall that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. Therefore, we can recognize $Y(s)$ as the Laplace transform of

$$y(t) = -5e^{-3t} + 7e^{-2t}.$$

EXAMPLE 19.2. Solve the initial value problem $y'' + 4y = \cos(t)$ with $y(0) = 1$ and $y'(0) = -1$.

Taking the Laplace transform of both sides of this ODE we get the equation

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{s}{s^2 + 1}.$$

This is solved to obtain

$$Y(s) = \frac{sy(0) + y'(0)}{s^2+4} + \frac{s}{(s^2+4)(s^2+1)} = \frac{s-1}{s^2+4} + \frac{s}{(s^2+4)(s^2+1)}.$$

Lets perform some partial fraction expansion on the second part of the right hand side. We get

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}.$$

From this we get the equation $s = (As+B)(s^2+1)+(Cs+D)(s^2+4)$. Plugging in $s = 0$ we get $0 = B+4D$. Plugging in $s = 1$ gives $1 = 2(A+B)+5(C+D)$. If we plug in $s = -1$ then we get $-1 = 2(-A + B) + 5(-C + D)$. We need one more equation to determine A, B, C and D. Lets try $s = 2$, we get $2 = 5(2A + B) + 8(2C + D)$. We get the four equations

$$\begin{aligned} B + 4D &= 0 \\ 2A + 2B + 5C + 5D &= 1 \\ -2A + 2B - 5C + 5D &= -1 \\ 10A + 5B + 16C + 8D &= 2. \end{aligned}$$

If we add the second and third equations together we get $4B + 10D = 0$. Along with the condition $B + 4D = 0$ we see that $B = D = 0$. Therefore, the second and fourth equations give $2A + 5C = 1$ and $10A + 16C = 2$. Therefore, $-9C = 10A + 16C - 5(2A + 5C) = 2 - 5(1) = -3$ which gives $c = 1/3$ and $2A = 1 - 5C = 1 - 5(1/3) = -1/3$. Therefore,

$$Y(s) = \frac{s-1}{s^2+4} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1} = \frac{2}{3}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1}.$$

Referring to the table on page 317 we find that $Y(s)$ is the Laplace transform of

$$y(t) = \frac{2}{3}\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\cos(t)$$

which is the solution to the initial value problem.

These two examples indicate that the method of Laplace transforms does not seem to help much. In fact, the second example would probably be easier to solve using the older methods discussed in chapter 3. The usefulness of the Laplace transform will be seen when we start solving differential equations of the form $ay'' + by' + cy = f(t)$ where $f(t)$ is a piecewise continuous function such as a step function.

20. Step Functions and Translations, 6.3

We want to be able to solve second order linear differential equations with discontinuous forcing functions. Before we can do such we need to obtain a better understanding of the step functions and their Laplace transforms. There is a class of very important step functions called the **Heaviside step functions** or, **unit step functions**, that are defined for all real numbers c by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c. \end{cases}$$

Lets look at some examples of piecewise defined functions to see the usefulness of the unit step function.

Example 20.1. Use unit step functions to describe the piecewise defined function

$$f(t) = \begin{cases} 1 & t < c \\ 0 & t \geq c. \end{cases}$$

We see that if $t < c$ then $1 - u_c(t) = 1 - 0 = 1$ whereas, if $t \geq c$ then $1 - u_c(t) = 1 - 1 = 0$. This shows $f(t) = 1 - u_c(t)$.

Example 20.2. Use step functions to describe the function

$$f(t) = \begin{cases} \sin(t) & 0 \leq t < \pi \\ t & \pi \leq t < 2\pi \\ e^t & 2\pi \leq t \end{cases}$$

Consider the function $1 - u_\pi(t)$. For $t \geq \pi$ we get $1 - u_\pi(t) = 1 - 1 = 0$ while for $0 \leq t < \pi$ we have $1 - u_\pi(t) = 1 - 0 = 1$. Similar arguments will show that $u_\pi(t) - u_{2\pi}(t)$ is zero outside of $\pi \leq t < 2\pi$ and 1 on that interval. From this we see

$$(1 - u_\pi(t))\sin(t) + (u_\pi(t) - u_{2\pi}(t))t + u_{2\pi}(t)e^t$$

agrees with $f(t)$ everywhere so they are equal.

We can use the step functions to give nice descriptions of translates of a function. Suppose we have a function $f(t)$ defined for $t \geq 0$ and c is a positive real number. Then we can define a new function as follows:

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

This is just the function f but shifted to the right by c and with value zero between 0 and c. This has a nice description as $g(t) = u_c(t)f(t-c)$. Luckily, the Laplace transform of a shifted function is easy to compute.

Theorem 20.1. *If $\mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and if c is a positive number then,*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}, \qquad s > a.$$

Proof. We can see this by the definition of the Laplace transform. We compute,

$$\mathcal{L}\{u_c(t)f(t-c)\} = \lim_{A\to\infty}\int_0^A e^{-st}u_c(t)f(t-c)dt = \lim_{A\to\infty}\int_c^A e^{-st}f(t-c)dt.$$

and this holds whenever $A > c$. If we do a change of variables $u = t - c$ then $du = dt$. Also, we find that $u = 0$ when $t = c$ so we get

$$\mathcal{L}\{u_c(t)f(t-c)\} = \lim_{A\to\infty}\int_0^{A-c} e^{-sc-su}f(u)du = e^{-sc}\lim_{A\to\infty}\int_0^{A-c} e^{-su}f(u)du = e^{-sc}\mathcal{L}\{f(t)\}.$$

$\square$

Example 20.3. Compute the Laplace transform of

$$f(t) = \begin{cases} t^2 & 0 \leq t < 1 \\ t^2 + e^{t-1} & t \geq 1. \end{cases}$$

We can see by inspection that $f(t) = t^2 + u_1(t)e^{t-1}$. Using the theorem above and a table of transforms we get

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{e^{-s}}{s-1}.$$

EXAMPLE 20.4. Find the inverse Laplace transform of $F(s) = \frac{1-e^{-2s}}{s^2}$.

We can see from the table on page 317 that $\mathcal{L}\{t\} = \frac{1}{s^2}$. Now the term e^{-2s} that comes with the $1/s^2$ suggests that we have t but shifted by 2. In particular, we find $\mathcal{L}\{u_2(t)(t-2)\} = \frac{e^{-2s}}{s^2}$. Therefore, we get the inverse Laplace transform

$$t + u_2(t)(t-2).$$

THEOREM 20.2. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and c is a constant, then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad s > a + c.$$

PROOF. Just note that

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)dt = \int_0^\infty e^{-(s-c)t}f(t)dt = F(s-c)$$

when $s - c > a$. □

EXAMPLE 20.5. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2 - 2s + 2}.$$

We find that $s^2 - 2s + 2 = s^2 - 2s + 1 + 1 = (s-1)^2 + 1$. Therefore, we need the inverse transform for

$$\frac{1}{(s-1)^2 + 1}.$$

We know that $\sin(t)$ has transform $\frac{1}{s^2+1}$, therefore, the inverse Laplace transform of $F(s)$ is just a sine multiplied by an exponential. Hence, we get

$$f(t) = e^t \sin(t).$$

21. Differential Equations with Discontinuous Forcing Functions, 6.4

Now we are in a situation where we can solve second order linear constant coefficient differential equations with discontinuous forcing function. Actually, this can be done using the methods learned in chapter 3, but it can be very tedious. This is where the Laplace transform method helps.

EXAMPLE 21.1. Solve the initial value problem

$$y'' - 3y' + 2y = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}, \qquad y(0) = y'(0) = 0.$$

Using example 20.1 we see the forcing function here is just $1 - u_1(t)$. Taking the Laplace transform of this equation and using the initial values we get

$$s^2Y(s) - 3sY(s) + 2Y(s) = \frac{1-e^{-s}}{s}$$

which is easily solved to yield

$$Y(s) = \frac{1-e^{-s}}{s(s^2-3s+2)} = \frac{1-e^{-s}}{s(s-1)(s-2)}$$

We can perform partial fractions on the $1/s(s-1)(s-2)$ to yield

$$Y(s) = (1-e^{-s})(\frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2}).$$

Now, the inverse Laplace transform of the stuff in the right set of parenthesis is $\frac{1}{2} - e^t + \frac{1}{2}e^{2t}$. The factor of e^{-s} is going to just shift the function by 1 so we get

$$y(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t} - u_1(t)\left(\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}\right).$$

Example 21.2. Solve the initial value problem $y'' + y' + \frac{5}{4}y = g(t)$ with initial conditions $y(0) = 0$, $y'(0) = 0$ where

$$g(t) = \begin{cases} \sin(t) & 0 \le t < \pi \\ 0 & t \ge \pi. \end{cases}$$

The first thing to note is that $g(t) = (1 - u_\pi(t))\sin(t)$. We want to use theorem 20.1 to compute the Laplace transform but $g(t)$ is not in the correct form. Notice that $\sin(t - \pi) = \sin(t)\cos(-\pi) + \sin(-\pi)\cos(t) = -\sin(t)$. Therefore, we can write $g(t) = \sin(t) - u_\pi(t)\sin(t) = \sin(t) + u_\pi(t)\sin(t-\pi)$, giving

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

Taking the transform of the equation we get

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) + \frac{5}{4}Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

This is solved for $Y(s)$ to obtain

$$Y(s) = \frac{1+e^{-\pi s}}{(s^2+s+5/4)(s^2+1)}$$

We need to do some partial fractions on this to take the inverse Laplace transform. We can write

$$\frac{1}{(s^2+s+5/4)(s^2+1)} = \frac{As+B}{s^2+s+5/4} + \frac{Cs+D}{s^2+1}$$

and upon multiplying everything by $(s^2+s+5/4)(s^2+1)$ we get

$$1 = (As+B)(s^2+1)+(Cs+D)(s^2+s+5/4) = (A+C)s^3+(B+C+D)s^2+(A+5/4C+D)s+(B+5/4D).$$

This gives a system of four equations in four unknowns. The solution is $A = 16/17$, $B = 12/17$, $C = -16/17$ and $D = 4/17$. Therefore,

$$Y(s) = \frac{4}{17}(1+e^{-\pi s})\left(\frac{4s+3}{s^2+s+5/4} + \frac{-4s+1}{s^2+1}\right)$$

We can factor $s^2+s+5/4 = s^2+s+1/4+1 = (s+1/2)^2+1$, using this we get

$$Y(s) = \frac{4}{17}(1+e^{-\pi s})\left(4\frac{s+1/2}{(s+1/2)^2+1} + \frac{1}{(s+1/2)^2+1} - 4\frac{s}{s^2+1} + \frac{1}{s^2+1}\right)$$

We know $\frac{s}{s^2+1}$ is the transform of $\cos(t)$, therefore, using theorem 20.2 we see $\frac{s+1/2}{(s+1/2)^2+1}$ is the transform of $e^{-t/2}\cos(t)$. Similarly, we find that $\frac{1}{(s+1/2)^2+1}$ is the transform of $e^{-t/2}\sin(t)$.

Therefore, we can conclude $Y(s)$ is the Laplace transform of

$$\begin{aligned} y(t) &= \frac{4}{17}(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)) \\ &\quad + \frac{4}{17}u_\pi(t)(4e^{-t/2+\pi/2}\cos(t-\pi) + e^{-t/2+\pi/2}\sin(t-\pi) - 4\cos(t-\pi) + \sin(t-\pi)) \end{aligned}$$

which is the solution to the initial value problem.