Tues 07 Jan MATH 307 Lecture 2

CHAPTER 1: FIRST ORDER D.E's (Boyce Ch. 2)

§1.0: Intro

Define 1.0.1:

- A first order differential equation is an equation in two variables of the form

$$\frac{dy}{dx} = f(x,y)$$

(First order because only the first derivative appears) (continuous, differentiable)

- A solution to a d.e. is a function $y = g(x)$ which obeys (or satisfies) that particular d.e. (for some values of x, y)

- ~~Often~~ Sometimes we will ~~can~~ find an implicit solution: A relation $h(x,y) = 0$ that we can show obeys the d.e. (e.g. by implicitly differentiating), but we cannot solve for y explicitly.

Examples:

- The function $y = e^{-\frac{1}{2}x^2}$ satisfies the differential equation

$$\frac{dy}{dx} = -xy$$

- The (first-order) d.e. $\frac{dy}{dx} = \frac{1-e^x}{1+e^y}$ has the implicit solution

$$y - x - e^x - e^y = 0;$$

here you cannot solve for y.

Note 1.0.2: A given d.e. can have infinitely many solutions

Observe $y = 2e^{-\frac{1}{2}x^2}$, $y = -7e^{-\frac{1}{2}x^2}$ and $y = Ce^{-\frac{1}{2}x^2}$ for any C satisfy $\frac{dy}{dx} = -xy$.

Extra information can be supplied to ensure that there is a unique solution to a d.e. This is most often a specified point on the solution curve ("initial conditions").

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__Example:__ The unique solution to $\frac{dy}{dx} = -xy$ subject to $y(0) = 3$ is the curve $y = 3e^{-\frac{1}{2}x^2}$.

~~The Goal for thi~~

Unfortunately there is no uniform technique for solving first-order d.e.'s

The goal for this section is to provide you with the techniques to solve ~~a larger subclass~~ of first-order d.e.'s.
the most common types

__Things To Remember__

- In general first-order d.e's have ∞-ly many solutions spanned by 1 paremeter, usually denoted c (the "integration constant")

- When IC's are specified, we sometimes only get a solution ~~A~~ in a limited interval about that point i.e. not for all x
  we'll see an example soon.

- Often we have $\frac{dx}{dt} = f(x,t)$ with $x(0) = x_0$, where t is time - hence initial conditions name

- Sometimes Bad stuff happens wrt. uniqueness.

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§1.1 : Separable Differential Equations (Boyce 2.2)

Definition 1.1.1: A separable Differential equation is a First-order d.e. of the form

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

How to solve:

$$\frac{dy}{dx} = f(x)\, g(y)$$

$$\Rightarrow \frac{1}{g(y)}\, dy = f(x)\, dx$$

$$\Rightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx + C$$, solve for y if poss.

Essentially the same technique for all separable d.e's.

Examples: 1) $\frac{dy}{dx} = -xy$, $y(0) = 3$.

$$\Rightarrow \frac{1}{y}\, dy = -x\, dx$$

$$\Rightarrow \int \frac{1}{y}\, dy = \int -x\, dx + C$$

$$\Rightarrow \ln(y) = -\tfrac{1}{2}x^2 + C$$

$$\Rightarrow y = e^{-\frac{1}{2}x^2 + C} = Ae^{-\frac{1}{2}x^2}, \quad A = e^C$$

Now plug in I.C's:

$y(0) = 3$, $3 = Ae^{-\frac{1}{2}\cdot 0^2} = A$

So $A = 3$.

Hence $y = 3e^{-\frac{1}{2}x^2}$

2) $\frac{dy}{dx} = 2xy^2$, $y(2) = \frac{1}{5}$.

So $\int \frac{1}{y^2}\, dy = \int 2x\, dx + C$

$\Rightarrow -\frac{1}{y} = x^2 + C$

ICs: $x = 2$, $y = \frac{1}{5}$ $\Rightarrow -5 = 4 + C$

$\Rightarrow C = -9$

So $y = \frac{1}{9 - x^2}$, valid for $-3 < x < 3$

3) Find the general explicit solution to $\frac{dy}{dx} = \frac{y^2-1}{2x}$

$\Rightarrow \frac{2}{y^2-1}\, dy = \frac{1}{x}\, dx$

$\Rightarrow \frac{1}{y-1} - \frac{1}{y+1}\, dy = \frac{1}{x}\, dx$

$\Rightarrow \ln\left|\frac{y-1}{y+1}\right| = \ln|x| + c$

$\Rightarrow \left|\frac{y-1}{y+1}\right| = A \cdot |x|$, $A = e^c$

$\Rightarrow \frac{y-1}{y+1} = B \cdot x$, $B = \pm A$ as necessary.

$\Rightarrow y - 1 = Bx(y+1)$

$y - Bxy = 1 + Bx$

$y(1 - Bx) = 1 + Bx$

So $y = \frac{1+Bx}{1-Bx}$

4) Solve $\frac{dy}{dx} = \frac{5x^4+1}{7y^6-2y}$, $y(-2) = 1$

$\Rightarrow (7y^6 - 2y)\, dy = (5x^4+1)\, dx$

$\Rightarrow y^7 - y^2 = x^5 + x + C$

ICs: $x = -2$, $y = 1$

$\Rightarrow 1 - 1 = -32 - 2 + C$

$\Rightarrow C = 34$

So $y^7 - y^2 - x^5 - x - 34$ is the implicit solution to this DE.

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1.1.2 : Substitution / Change of variables

Not all first-order DEs are separable. However, sometimes a first-order d.e. can be made separable by a change of variables.

Example: i) $\frac{dy}{dx} = 2x - 5y$

This is not separable a priori.
However, let $v = 2x - 5y$
Then

$$\frac{dv}{dx} = 2 - 5\frac{dy}{dx} = 2 - 5(2x - 5y) = 2 - 5v$$

which is separable!
So

$$\frac{1}{2-5v} = dx$$

$$-\frac{1}{5}\ln|2-5v| = x + C$$

$$\ln|2-5v| = -5x + A, \quad A = -5C$$

$$2 - 5v = Be^{-5x}, \quad B = \pm e^{A}$$

$$2 - 5(2x - 5y) = Be^{-5x}$$

$$2 - 10x + 25y = Be^{-5x}$$

$$25y = Be^{-5x} + 10x - 2.$$

So

$$y = De^{-5x} - \frac{2}{25} + \frac{2}{5}x, \quad D = \frac{B}{25}.$$

1.1.3 Homogenous ~~Equations~~ Functions Equations *

If the first order DE $\frac{dy}{dx} = f(x,y)$ has $f(x,y) = F\left(\frac{y}{x}\right)$ for some F, then we can always make the substitution $v = \frac{y}{x}$ to make the DE separable. The ~~function~~ equation $\frac{dy}{dx} = f(x,y)$ is then called homogenous *

* Not to be confused with Homogenous linear DE's studied later.

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Example: Solve $x\frac{dy}{dx} - y + xe^{\frac{y}{x}} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - e^{\frac{y}{x}}$$

$$= F\left(\frac{y}{x}\right), \text{ where } F(v) = v - e^{v} \quad \checkmark$$

So let $v = \frac{y}{x}$ or $y = xv$

Implicit differentiation: $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So $\cancel{v} + x\frac{dv}{dx} = \cancel{v} - e^{v}$

$$\Rightarrow \quad x\frac{dv}{dx} = -e^{v}$$

$$\Rightarrow \quad -e^{-v}\,dv = \frac{1}{x}\,dx$$

$$e^{-v} = \ln|x| + C$$

$$-v = \ln(\ln|x| + C)$$

or $v = -\ln(\ln|x| + C)$

$$\Rightarrow \quad \frac{y}{x} = -\ln(\ln|x| + C)$$

So $y = -x\ln(\ln|x| + C)$