§3.2 - The Wronskian

The Wronskian is a quantity associated to 2nd-order linear DEs that allow us to determine whether two given solutions can be used to generate *all* solutions to that DE; and in so doing, whether some linear combination thereof can be used to satisfy a given set of initial conditions.

To talk about the Wronskian it will be useful to define the notion of a Differential operator.

Let $p(t)$, $q(t)$ be continuous on ~~interval~~: $\alpha < t < \beta$.

Define 3.2.1 • A (2nd-order) differential operator is a "function of functions" that acts on any twice differentiable function $\phi(t)$ to produce another function.

• The differential operator $L[\phi] = \phi'' + p(t)\cdot\phi' + q(t)\cdot\phi$ takes function $\phi(t)$ to a function whose value at t is
$$\phi''(t) + p(t)\cdot\phi'(t) + q(t)\phi(t).$$

Example 3.2.2 Let $p(t) = t^2$, $q(t) = 1+t$ & $\phi(t) = \sin(3t)$.

$$\text{Then } L[\phi](t) = (\sin 3t)'' + t^2\cdot(\sin 3t)' + (1+t)\cdot(\sin(3t))$$
$$= -9\sin(3t) + 3t^2\cos(3t) + (1+t)\sin(3t).$$

Note: A differential operator L is often written in terms of the differentiation operator D $(= \frac{d}{dt})$, i.e. $L = D^2 + pD + q$

In the above example $L = D^2 + t^2 D + (1+t)$

Note too that a solution ~~of~~ the DE $y'' + p(t)y' + q(t)y = 0$ is a function $y(t)$ s.t. $L[y] = 0$.

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Start with this

Theorem 3.2.3 Existence & Uniqueness for 2nd-order DEs

Consider the IVP $y''+p(t)y'+q(t)y=g(t)$, $y(t_0)=y_0$, $y'(t_0)=y_0'$

Let I be the largest t-interval containing t_0 on which $p(t)$, $q(t)$ & $g(t)$ are all continuous. Then there exists a unique solution to the IVP $y=\phi(t)$ for all $t\in I$.

Corollary: The IVP $ay''+by'+cy=g$, a,b,c,g constant $(a\neq 0)$
$y(t_0)=y_0$, $y'(t_0)=y_0'$
has a unique solution that is defined for all t.

Theorem 3.2.4: Principle of superposition

If y_1 & y_2 are two solutions to $L[y]=0$ (i.e. $y''+p(t)y'+q(t)y=0$), then any linear combination $c_1y_1+c_2y_2$ is too.

Proof:
$$\begin{aligned}L[c_1y_1+c_2y_2] &= [c_1y_1+c_2y_2]''+p(t)\cdot[c_1y_1+c_2y_2]'+q(t)\cdot[c_1y_1+c_2y_2]\\ &= c_1y_1''+c_2y_2''+p(t)\cdot c_1y_1'+p(t)\cdot c_2y_2'+q(t)\cdot c_1y_1+q(t)\cdot c_2y_2\\ &= c_1\left(y_1''+p(t)y_1'+q(t)\right)+c_2\left(y_2''+p(t)y_2'+q(t)\right)\\ &= c_1L[y_1]+c_2L[y_2]\end{aligned}$$
$\Rightarrow =0$ if $L[y_1]=L[y_2]=0$ □.

Suppose $y_1(t)$ & $y_2(t)$ both solve the DE $L[y]=0$ and we want to find the solution to the IVP $L[y]=0$, $y(t_0)=y_0$, $y'(t_0)=y_0'$. By the above, the solution ~~is~~ should be $y=c_1y_1+c_2y_2$, for specific values of c_1 & c_2 s.t.
$$c_1y_1(t_0)+c_2y_2(t_0)=y_0$$
$$c_1y_1'(t_0)+c_2y_2'(t_0)=y_0'$$

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Solving for c_1 & c_2 symbolically we get

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} \qquad c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}.$$

This is ugly! Thankfully there is a much nicer way to write c_1 & c_2 in terms of determinants of 2×2 matrices:

$$c_1 = \frac{\det\begin{pmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0)\end{pmatrix}}{\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0)\end{pmatrix}} \qquad c_2 = \frac{\det\begin{pmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0'\end{pmatrix}}{\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0)\end{pmatrix}}$$

Note the matrix in the bottom is the same. If its determinant at the point t_0 is nonzero then we can solve for c_1 & c_2.

Definition 3.2.5 The determinant, as a function of t,

$$W = \det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t)\end{pmatrix} = y_1 y_2' - y_2 y_1'$$

is called the Wronskian of the solutions y_1 & y_2.

In other words, it is possible to solve the previous IVP with some linear combination of y_1 & y_2, so long as the Wronskian of y_1 & y_2 is not zero at t_0.

Theorem 3.2.6 Let $y_1(t)$ & $y_2(t)$ be two solutions to $L[y] = y'' + p(t)y' + q(t)y = 0$
The family $y = c_1 y_1(t) + c_2 y_2(t)$ comprises *all* solutions to the DE if and only if the Wronskian of y_1 & y_2 is nonzero somewhere.

Example 3.2.7 $y'' + 5y' + 6y = 0$ had solutions $y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$

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Then $W = \det\begin{pmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{pmatrix} = (e^{-2t})(-3e^{-3t}) - (e^{-3t})(-2e^{-2t})$
$= -3e^{-5t} + 2e^{-5t}$
$= -e^{-5t}$
$\neq 0$ for all t

$\Rightarrow$ Any set of ICs can be satisfied to yield a solution to the IVP.

<u>Definition 3.2.8</u>

- Any pair of solutions $y_1(t), y_2(t)$ that to the DE $y'' + p(t)y' + q(t)y = 0$ with nonzero Wronskian is called a <u>fundamental set of solutions</u> to the DE.

- $y = c_1 y_1(t) + c_2 y_2(t)$ is called the <u>general solution</u> to the DE

<u>Example 3.2.9</u>: Show that $y_1(t) = t^{\frac{1}{2}}$ & $y_2(t) = t^{-1}$ form a fundamental set of solutions to $2t^2y'' + 3ty' - y = 0, \quad t > 0$

<u>Solution</u>: • Verify $y_1(t)$ & $y_2(t)$ obey the DE:

$y_1'(t) = \frac{1}{2}t^{-\frac{1}{2}}, \quad y_1''(t) = -\frac{1}{4}t^{-\frac{3}{2}}$

$\Rightarrow 2t^2y_1'' + 3ty_1' - y_1 = 2t^2(-\frac{1}{4}t^{-\frac{3}{2}}) + 3t(\frac{1}{2}t^{-\frac{1}{2}}) - (t^{\frac{1}{2}})$
$= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}}$
$= 0$ ✓

$y_2' = -t^{-2}, \quad y_2'' = +2t^{-3}$

$\Rightarrow 2t^2y_2'' + 3ty_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - (t^{-1})$
$= 4t^{-1} - 3t^{-1} - t^{-1}$
$= 0$ ✓

• Compute Wronskian:

$W = \det\begin{pmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{pmatrix} = (t^{\frac{1}{2}})(-t^{-2}) - (t^{-1})(\frac{1}{2}t^{-\frac{1}{2}}) = -\frac{3}{2}t^{-\frac{3}{2}}$.

Since $W \neq 0$ for $t > 0$, conclude $y_1(t)$ & $y_2(t)$ form a fundamental set of solutions to the DE.

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Note 3.2.10: Given a DE $L[y]=y''+p(t)y'+q(t)y=0$, we can always find a fundamental set of solutions ~~by~~ $y_1(t)$ & $y_2(t)$ by solving

$$y_1(t_0)=1, \quad y_1'(t_0)=0$$
&
$$y_2(t_0)=0, \quad y_2'(t_0)=1$$

In this case, the solution to the IVP $L[y]=0$, $y(t_0)=y_0$, $y'(t_0)=y_0'$ is then just

$$y=y_0\cdot y_1(t)+y_0'\cdot y_2(t).$$

Example 3.2.11: Find the fundamental set of solutions to
$$y''-y=0, \quad t_0=0.$$

Recall we previously found solutions $y=e^t$ & $y=e^{-t}$
The solution to $y_1(0)=1$, $y_1'(0)=0$ is
$$y_1=\tfrac{1}{2}(e^t+e^{-t}) = \cosh(t)$$
While the solution to $y_2(0)=0$, $y_2'(0)=1$ is
$$y_2=\tfrac{1}{2}(e^t-e^{-t}) = \sinh(t)$$

$\Rightarrow$ $y_1(t)=\cosh(t)$, $y_2(t)=\sinh(t)$ form a fundamental set of solutions to $y''-y=0$.

Note 3.2.12. From the above we see that there can be more than one fundamental set of solutions to a given DE. So just pick the one that's easiest to work with in general.