Sat 4 Jan — Math 307 Lecture 1

Chapter §0: Review

§ 0.0 Course Info:

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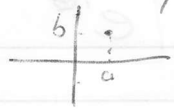
58	~~30~~ 42%	Final Exam:	Wed 19 March 2013, 08:30-10:20 CDH 110B
15	20%	Midterm 1:	Wed 29 Jan 09:30-10:20 CDH110B (wk 4)
15	20%	Midterm 2:	Wed 5 March 09:30-10:20 CDH 110B (wk 8)
12	~~20~~ 18%	Homeworks ×7	Due beginning of class on weeks 2,3, 5,6,7, 9,10. No homework due on midterm weeks.

- Assume you're familiar with differentiation, partial differentiation & integration.

§0.2 Complex Numbers

Define 0.2.1 Complex numbers are numbers of the form
$z = a + b\cdot i$ where a, b are real $\in \mathbb{R}$
& $i^2 = -1$.



"rectangular coordinates"

The set of complex numbers is denoted $\mathbb{C}$.
It is a field: you may add, subtract & multiply any two complex numbers to get a third, & every complex number other than 0 has a multiplicative inverse.

Examples: 0.2.2

- $(3+2i) + (-7+i) = (3-7) + (2+1)\cdot i = -4+3i$
- $(3+2i)\cdot(-7+i) = 3\cdot -7 + 3\cdot i + 2\cdot -7\cdot i + 2\cdot i\cdot i = -21 + 3i - 14i - 2 = -23 - 11i$

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In general: • $(a+bi) + (c+di) = (a+c) + (b+d)i$
• $(a+bi)\cdot(c+di) = (ac-bd) + (ad+bc)i$

0.2.3 Inverses: Observe $\frac{1}{3+2i} = \frac{1}{3+2i} \cdot \frac{3-2i}{3-2i}$

$$= \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{3^2+2^2}$$

$$= \frac{3}{13} - \frac{2}{13} \cdot i$$

In general: $\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} \cdot i$

Another way to write complex numbers

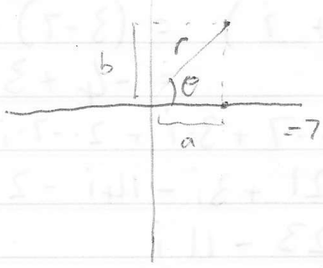
Observe 0.2.4 The Taylor Series for e^{ix} at 0 is:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$
$$= 1 + i\cdot x - \frac{x^2}{2!} - i\cdot\frac{x^3}{3!} + \frac{x^4}{4!} + i\,\frac{x^5}{5!} - \dots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$
$$= \cos(x) + i\cdot\sin(x)$$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

Specifically $e^{i\pi} = \cos(\pi) + i\cdot\sin(\pi)$
$= -1$

So $e^{i\pi} + 1 = 0$ — Euler's identity



$z = a + ib$, $a = r\cos\theta$
$b = r\sin\theta$

$\Rightarrow z = r\cos\theta + i\cdot r\sin\theta = r(\cos\theta + i\sin\theta)$
$= r e^{i\theta}$

magnitude   argument   "Polar Coordinates"

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Sat 5 Jan   Math307 Lecture 1 Revision pt. 2

§0.1 Taylor Series

Consider the function $f(x) = (x-3)^3 + 3(x^2+1) - 4$
This is a cubic polynomial, so we could write it in standard form:
$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

<u>Observe 0.1.1:</u>
$$f(0) = a_0$$
$$f'(0) = a_1 + 2a_2 x + 3a_3 x^2 \big|_0 = a_1$$
$$f''(0) = 2a_2 + 6a_3 x \big|_0 = 2a_2$$
$$f'''(0) = 6a_3$$

So $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = \frac{f''(0)}{2}$, $a_3 = \frac{f'''(0)}{6}$.
or $a_n = \frac{f^{(n)}(0)}{n!}$, i.e. $f(x) = f(0) + \frac{f^{(1)}(0)}{1} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3$

So for our example
$$a_0 = (x-3)^3 + 3(x^2+1) - 4 \big|_0 = -28$$
$$a_1 = 3(x-3)^2 + 6x \big|_0 = 27$$
$$a_2 = \tfrac{1}{2}\left(6(x-3) + 6\right)\big|_0 = -6$$
$$a_3 = \tfrac{1}{6}(6)\big|_0 = 1$$

We can do the same for nice* infinitely differentiable functions, where we keep going indefinitely to get the <u>Taylor Series</u> for that function.

<u>Define 0.1.2:</u> If $g(x)$ is real analytic* at $x=0$, then the <u>Taylor Series</u> for $g(x)$ at $x=0$ is

$$g(x) = g(0) + \frac{g^{(1)}(0)}{1!} x + \frac{g^{(2)}(0)}{2!} x^2 + \frac{g^{(3)}(0)}{3!} x^3 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \cdot x^n$$

<u>Examples 0.1.3:</u> 1) $g(x) = e^x$
Then $g^{(n)}(x) = e^x$ for any n, so $g^{(n)}(0) = 1$

$$\Rightarrow e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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2) $g(x) = \sin(x)$ $\quad g'(x) = \cos(x) \Rightarrow g'(0) = 1$

$\Rightarrow g(0) = 0$ $\quad g''(x) = -\sin(x) \Rightarrow g''(0) = 0$

$g'''(x) = -\cos(x) \Rightarrow g'''(0) = -1$

$g^{(4)}(x) = \sin(x) \Rightarrow g^{(4)}(0) = 0$

$\vdots$

$$\begin{aligned}\text{So } \sin(x) &= 0 + \frac{1}{1}\cdot x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}\end{aligned}$$

3) <u>You do:</u> $g(x) = \cos(x)$.

$$\begin{aligned}\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

4) $g(x) = \frac{1}{1-x} = (1-x)^{-1}$ $\quad g(0) = 1$

$g'(x) = (1-x)^{-2}$ $\quad g'(0) = 1$

$g''(x) = 2(1-x)^{-3}$ $\quad g''(0) = 2$

$g'''(x) = 6(1-x)^{-4}$ $\quad g'''(0) = 6$

$\vdots$

$g^{(n)}(x) = n!(1-x)^{-(n+1)}$ $\quad g^{(n)}(0) = n!$

$$\begin{aligned}\text{So } \frac{1}{1-x} &= \frac{1}{1} + \frac{1}{1}\cdot x + \frac{2}{2}\cdot x^2 + \frac{6}{6}\cdot x^3 + \dots \\ &= 1 + x + x^2 + x^3 + \dots \\ &= \sum_{n=0}^{\infty} x^n\end{aligned}$$

<u>Note</u> 0.1.4 The Taylor series for $g(x)$ will converge for x lying in some interval about 0, called the <u>Interval of Convergence</u> for $g(x)$ at 0.

- The Taylor series for $\frac{1}{1-x}$ at 0 converges for $x \in (-1,1)$
- The Taylor series for e^x, $\sin(x)$ & $\cos(x)$ at $x=0$ converge for $x \in (-\infty, \infty)$ i.e. for all x.

Sun 5 Jan

MATH 307 Lecture 1. Part 3

Advantage of writing complex numbers in polar ~~coord~~ coordinates:
Multiplication is easy:

$$(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$
$$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

"To multiply complex numbers, you multiply the magnitudes & add the arguments."