Homework 6 Solutions

Total: 20 points
Due: Wed 19 November 2014 at the beginning of class

Remember to show all steps in your working. If a question is taken from the textbook, the refence is given on the right of the page.

1. **2nd-ORDER MODELING**

   (a) In the following two equations, determine ω_0, R and δ so as to write them in the form $y = R\cos(\omega_0 t - \delta)$:

   i. $y = -\cos(t) + \sqrt{3}\sin(t)$ Boyce 3.7 Q2

   ii. $y = -2\cos(\pi t) - 3\sin(\pi t)$ Boyce 3.7 Q4

   (b) Boyce 3.7 Q5

   A mass weighing 2 lb hangs from a spring, stretching it 6 inches. If the mass is pulled down an additional 3 in and then released, and there is no damping, determine the vertical position y of the mass at any time t. Also find the frequency, period and amplitude of the motion.

   [Hint: You will need to use the fact that that the 2 lb weight stretches the spring 6 inches when at rest to find the spring constant k. This is the point where the downward gravitational force on the mass exactly balances the upwards force imparted by the spring. You may take $g = 32\ ft/s^2$, and remember that in imperial units mass m is given by $m = w/g$, where w is the weight of the object in pounds. If you need extra guidance on this question, Boyce section 3.7 has a number of similar examples, and details the method on how to set up the DE.]

   (c) A 2 kg block is placed on a smooth surface attached to a horizontally-acting spring with spring constant $k = 9\,\frac{1}{16}$ kg/s^2. We have seen in class that the displacement in meters $y(t)$ of the block from its rest position is then modeled by the differential equation

   $$2y'' + \frac{145}{16}y = 0.$$

   Suppose that the surface is not actually frictionless, but instead imparts a force of $-\gamma v$ on the block, where v is the horizontal velocity of the block, and γ is a constant.

   i. The block is released from a position of $y = 0.5$ m with zero initial velocity. Find the position of the block for all time $t \geq 0$ if $\gamma = 0.5$ kg/s.

   We must solve the IVP

   $$2y'' + 0.5y' + 9.0625y = 0, \qquad y(0) = 0.5, \quad y'(0) = 0.$$

   This has the characteristic equation

   $$2r^2 + \frac{1}{2}r + \frac{145}{16} = 0,$$

   or $32r^2 + 8r + 145 = 0$ after clearing denominators. This has roots

   $$r = \frac{-8 \pm \sqrt{8^2 - 4\cdot 32\cdot 145}}{2\cdot 32} = \frac{-8 \pm \sqrt{-18496}}{64},$$

   so (after simplifying) the two distinct roots to the characteristic equation are

   $$r_1 = -\frac{1}{8} + \frac{17}{8}\cdot i, \quad -\frac{1}{8} - \frac{17}{8}\cdot i$$

The fact that the CE has complex roots means that the general solution can be written in the form

$$y = e^{\lambda t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right),$$

where in this case $\lambda = -\frac{1}{8}$ and $\omega = \frac{17}{8}$. We thus have the general solution to the DE as

$$y = e^{-\frac{1}{8}t}\left(c_1 \cos\left(\frac{17}{8}t\right) + c_2 \sin\left(\frac{17}{8}t\right)\right).$$

Applying the initial condition $y(0) = \frac{1}{2}$ yields $c_1 = \frac{1}{2}$. The second initial condition $y'(0) = 0$ requires us to find y':

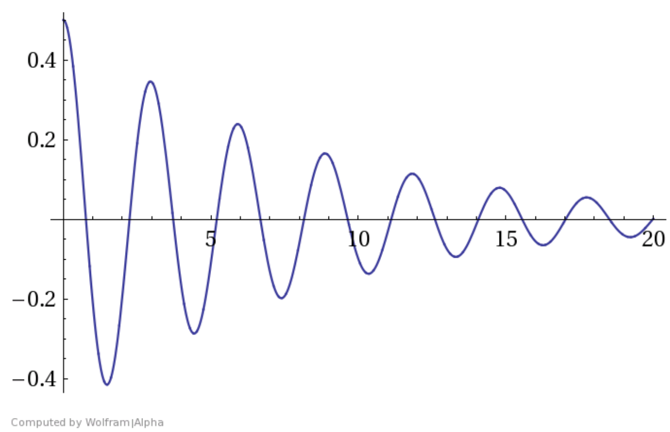
$$y' = e^{-\frac{1}{8}t}\left(\left(-\frac{1}{8}c_1 + \frac{17}{8}c_2\right)\cos\left(\frac{17}{8}t\right) + \left(-\frac{1}{8}c_2 - \frac{17}{8}c_1\right)\sin\left(\frac{17}{8}t\right)\right);$$

So $y'(0) = 0$ implies that $-\frac{1}{8}c_1 + \frac{17}{8}c_2 = 0$, or $-c_1 + 17c_2 = 0$ after clearing denominators. Since $c_1 = \frac{1}{2}$ we thus have that $c_2 = \frac{1}{34}$. Hence the position of the block at time t is given by the function

$$y(t) = e^{-\frac{1}{8}t}\left(\frac{1}{2}\cos\left(\frac{17}{8}\,t\right) + \frac{1}{34}\sin\left(\frac{17}{8}\,t\right)\right)$$

ii. Plot a graph of the above solution. Find the quasi-period of the oscillations of the block, and a time t_0 for which $|y(t)| < 0.05$ m for all $t > t_0$.

A plot for the solution for $0 \le t \le 20$ looks as follows:



The quasi-period T of the oscillation is the time taken for the cos and sin parts of the solution equation to complete one full cycle, i.e. the T for which $\frac{17}{8}T = 2\pi$. Solving for t, we see that the quasi-period is

$$T = \frac{16}{17}\pi \approx 2.9568 \text{ seconds.}$$

To find a time t_0 for which $|y(t)| < 0.05$ m for all $t > t_0$, we must first ascertain the amplitude of the oscillations. Recall that if $y = A\cos(\omega t) + B\sin(\omega t)$, then the amplitude of y is $R = \sqrt{A^2 + B^2}$. Therefore in our example the oscillations have amplitude

$$R = e^{-\frac{1}{8}t}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{17}\right)^2} = e^{-\frac{1}{8}t}\cdot\frac{\sqrt{293}}{34} \approx 0.5034e^{-\frac{1}{8}t}$$

As our graph indicates, the amplitude of the oscillations is decreasing exponentially. Therefore we must find the time t_0 when $0.5034e^{-\frac{1}{8}t_0} = 0.05$. Solving for t_0 yields

$$t_0 = -8\ln\left(\frac{34}{20\sqrt{293}}\right) = 4\ln\left(\frac{29300}{289}\right) \approx 18.4757 \text{ seconds.}$$

An acceptable approximation is to note that the amplitude of the sin part of the oscillation ($\frac{1}{17}$) is much smaller than that of the cos part of the oscillation ($\frac{1}{2}$). The total amplitude of the oscillation is thus approximately $\frac{1}{2}$. If we make this simplification, then we end up solving for t_0 when $e^{-\frac{1}{8}t_0} = 0.1$. We then have that

$$t_0 = 8\ln(10) = 18.4207 \text{ seconds.}$$

This agrees with the true answer to 1 decimal place, so it's a decent approximation.

iii. Given the same starting conditions as above, what is the smallest value of γ which will result in the block never crossing it rest position?

We require the smalles value of γ so that the solution to the DE does not exhibit oscillating behaviour. Oscillating behaviour occurs when the characteristic equation $2r^2 + \gamma r + \frac{145}{16} = 0$ has complex roots. In other words, we must find the smallest γ for which the discriminant of the quadratic is not negative. Hence we must solve the equation

$$\gamma^2 - 4 \cdot 2 \cdot \frac{145}{16} = 0,$$

or $\gamma^2 = \frac{145}{2}$. Taking the positive root (we know that $\gamma > 0$), we have

$$\gamma = \sqrt{\frac{145}{2}} \approx 8.5147 \; kg\, s^{-1}.$$

(d) Boyce 3.7 Q8

A series circuit has a capacitor of capacitance 0.25×10^{-6} F, an inductor of inductange 1 H, and negligible resistence. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t.

(e) Boyce 3.7 Q12

A series circuit contains a capacitor of 10^{-5} F, an inductor of 0.2 H, and a resistor of 3×10^2 Ω. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

(f) Boyce 3.7 Q13

A certain vibrating system satisfies the differential equation

$$y'' + \gamma y' + y = 0.$$

Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

(g) Boyce 3.7 Q24

The position of a vibrating system satisfies the initial value problem

$$\frac{3}{2}y'' + ky = 0, \qquad y(0) = 2, y'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3 respectively, determine the values of k and v.

2. **FORCED VIBRATIONS**

(a) In the following two problems, Use the compound angle formulae for $\cos(A+B)$ and $\cos(A-B)$ to rewrite the given expressions as the product of two trigonometric functions of different frequencies.

i. $\cos(9t) - \cos(7t)$ Boyce 3.8 Q1

ii. $\sin(3t) + \sin(4t)$ Boyce 3.8 Q4

(b) The vibration of strings of a string instrument can be thought as idealized spring-mass systems when the amplitude of vibration is small. Specifically, if $y(t)$ is the displacement of the center of the string from its position of rest, then y is governed by the differential equation

$$my'' + \gamma y' + ky = g(t),$$

where m is mass of the playable part of the string, γ the damping constant due to air resistence, k the spring constant arising from the elasticity of the spring, and $g(t)$ a given external forcing function.

Consider the A string on a double bass in an otherwise still room. When tuned correctly this string vibrates at a frequency of exactly 55 Hertz (i.e. cycles per second, **not** radians per second). Suppose that the playable part of the A string on a bass weighs 0.01 kg, and that friction is negligible.

i. What is the spring constant k in this situation?

Recall that the fundamental frequency ω_0 of the system is given by

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Solving for k yields $k = m \cdot \omega_0^2$.

Now we know that the cyclic frequency $f = 55$ Hz, so $f = \frac{\omega_0}{2\pi}$, so

$$\omega = 2\pi f = 2\pi \cdot 55 = 110\pi.$$

Hence

$$k = 0.01(110\pi)^2 = 121\pi^2 \approx 1194.222 \ \text{rad/sec}.$$

ii. The string is initially stationary in its equilibrium position. Starting at time $t = 0$ a speaker in the room plays a loud tone at precisely 56 Hertz, subjecting the bass's A string to a force of $g(t) = \frac{\pi^2}{50}\cos(56 \cdot 2\pi \cdot t)$ Newtons, where t is in seconds. Formulate an initial value problem describing the motion of the string for all $t \geq 0$. Remember to state what units your variables are in.

We have

$$my'' + \gamma y' + ky = g(t).$$

For us $m = 0.01$ kg, $\gamma = 0$ (as friction is negligible), $k = 121\pi^2$ kg/s^2 from the previous part of the question, and $g(t) = \frac{\pi^2}{50}\cos(56 \cdot 2\pi \cdot t) = \frac{\pi^2}{50}\cos(112\pi t)$ N. Since 1 Newton = 1 kg m/s^2, all the quantities are in compatible SI units. Hence we have the DE

$$\frac{1}{100}y'' + 121\pi^2 y = \frac{\pi^2}{50}\cos(112\pi t),$$

where y is in meters and t is in seconds.

Finally, the string is at rest in the equilibrium condition, so we have the initial conditions $y(0) = 0$, $y'(0) = 0$.

iii. Solve the initial value problem you formulated above to find the position of the string at time t. Using the compound angle formulae for $\cos(A + B)$ and $\cos(A - B)$, write your answer in the form

$$y = [R\sin(\omega_1 t)] \cdot \sin(\omega_2 t),$$

where the $\sin(\omega_1 t)$ term oscillates much more slowly than the $\sin(\omega_2 t)$ term.

First we find the general solution to the homogeneous equation

$$\frac{1}{100}y'' + 121\pi^2 y = 0$$

We see this is undamped spring motion. We know that the general solution is the superposition of cos and sin terms oscillating at the natural frequency $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{121\pi^2}{0.01}} = 110\pi$. Hence the general solution to the homogeneous solution is

$$y = c_1 \cos(110\pi t) + c_2 \sin(110\pi t).$$

Next, we obtain the particular solution $Y(t)$ to the full nonhomogeneous equation. It's perfectly fine to consult the class notes at this point, as the particular solution is of a known form; however we'll find it from scratch for good measure.

We make the guess $Y(t) = A\cos(112\pi t)$. We omit the the $B\sin(112\pi t)$ term, as the first derivative doesn't appear in the homogeneous part of the DE. Hence $Y'' = -112^2\pi^2 A\cos(112\pi t)$; thus

$$\frac{1}{100}Y'' + 121\pi^2 Y = \frac{-112^2\pi^2 A}{100}\cos(112\pi t) + 121\pi^2 A\cos(112\pi t) = -\frac{111}{25}\pi^2 A\cos(112\pi t).$$

This must be equal to $\frac{\pi^2}{50}\cos(112\pi t)$, so we must have that

$$-\frac{111}{25}\pi^2 A = \frac{\pi^2}{50}.$$

Solving for A yields $A = -\frac{1}{222}$.
Hence the general solution to the DE is

$$y = c_1 \cos(110\pi t) + c_2 \sin(110\pi t) - \frac{1}{222}\cos(112\pi t).$$

Applying the initial condition $y(0) = 0$ gives us $c_1 = \frac{1}{222}$, while the second initial condition $y'(0) = 0$ yields $c_2 = 0$. Hence the solution to the differential equation is given by

$$y = \frac{1}{222}\left(\cos(110\pi t) - \cos(112\pi t)\right).$$

Finally we write $110\pi t = A - B$ and $112\pi t = A + B$, where $A = 111\pi t$ and $B = \pi t$. Then

$$\begin{aligned}
\frac{1}{222}\left(\cos(110\pi t) - \cos(112\pi t)\right) &= \frac{1}{222}\left(\cos(A - B) - \cos(A + B)\right) \\
&= \frac{1}{222}\left((\cos(A)\cos(B) + \sin(A)\sin(B)) - (\cos(A)\cos(B) - \sin(A)\sin(B))\right) \\
&= \frac{1}{111}\sin(A)\sin(B) \\
&= \frac{1}{111}\sin(111\pi t)\sin(\pi t).
\end{aligned}$$

So finally (by swapping the *sin* terms) we can write the solution in the desired form:

$$y = \left[\frac{1}{111}\sin(\pi t)\right]\sin(111\pi t).$$

iv. Using your answer above, determine the maximum displacement of the center of the A string from its equilibrium position, and the cyclic frequency of the **beat** $\frac{\omega_1}{2\pi}$. If you were standing next to the bass, this is the frequency at which you'd hear the loudness of the A string's vibration oscillate over time.

We can read these off from the answer above. The maximum displacement of the string's center is the maximum amplitude of oscillation:

$$\frac{1}{111}\text{ m} \approx 0.90\text{ cm}.$$

The cyclic beat frequency is then $\frac{\omega_1}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$ Hz.

[Note: I'll also accept an answer of 1 Hz for the beat frequency. In fact, the time between successive maxima of the amplitude is actually 1 second, so this is in some sense more correct. The discrepancy arises from the fact that the amplitude function $\frac{1}{111}\sin(\pi t)$ is periodic with period 2 seconds, but the absolute value thereof, $\left|\frac{1}{111}\sin(\pi t)\right|$ is periodic with period 1 second. So depending how you define amplitude when it itself is periodic (i.e the value of the coefficient function vs. the absolute value thereof), either values is valid. This definitional ambiguity of frequency arises in real world situations too, for example in how frequency is defined in AC circuitry.]

(c) Boyce 3.8 Q5 & Q7

A mass weighing 4 lb hangs from a spring, stretching it 1.5 inches. The mass is given a positive displacement of 2 inches from its equilibrium position and released with no initial velocity. Assume that there is no damping and that the mass is acted on by an external force of $2\cos(3t)$ lb.

i. Formulate an initial value problem describing the motion of the mass. Remember to show your work, and state what units your variables are in.

ii. Solve the initial value problem to find the position of the mass at time t.

iii. Plot a graph of the solution.

iv. If the given external force is replaced by a force of $4\sin(\omega t)$ of frequency ω, find the value of ω for which resonance occurs.

(d) Boyce 3.8 Q6 & Q8

A mass of 5 kg hangs from a spring, stretching it 10 cm. The mass is acted on by an external force of $10\sin(\frac{1}{2}t)$ Newtons, and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. The mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s.

i. Formulate an initial value problem describing the motion of the mass. Remember to show your work, and state what units your variables are in.

ii. Solve the initial value problem to find the position of the mass at time t.

iii. Identify the transient and steady state parts of the solution.

iv. Plot the graph of the **steady state solution**.

v. If the given external force is replaced by a force of $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is a maximum.

vi. For the value of ω you have just found, calculate the amplitude R and the phase shift δ of the forced response.

(e) Boyce 3.8 Q16

A series circuit contains a capacitor of 0.25×10^{-6} F, an inductor of 1 H, and a resistor of 5×10^3 Ω. The initial charge on the capacitor is zero and there is no initial current. A 12 V battery is connected to the circuit and the circuit is closed at $t = 0$.

i. Find the charge on the capacitor at any time t.

ii. Determine the charge on the capacitor at $t = 0.0001$ s, $t = 0.001$ s and $t = 0.01$ s. Give your answer to four significant figures (e.g. 1.2345×10^{-7} C).

iii. What is the limiting charge on the capacitor as $t \to \infty$?