1. (10 points) Use the method of undetermined coefficients to find the **particular solution** to the following differential equation:
$$y'' + 9y = te^{-t} - 1.$$
Your answer should be a function $Y(t)$ with no undetermined constants in it.

Particular solutions are additive, so we know the particular solution will be the sum of the particular solutions to $y''+9y=-1$ and $y''+9y=te^{-t}$. For the first equation we have a constant forcing function, and since constants don't obey the homogeneous part of the DE we therefore guess a constant for this part of the particular solution. That is, guess

$$Y_1(t) = A$$

for some value of A. Then $Y_1' = Y_1'' = 0$, so when we plug Y_1 back into the equation we get

$$0 + 9 \cdot 0 + 9A = -1.$$

Solving for A gives us $A = -\frac{1}{9}$, so $Y_1 = -\frac{1}{9}$.

The second equation $y'' + 9y = te^{-t}$ has a forcing function that is a linear polynomial times an exponential. Since this doesn't obey the homogeneous part of the DE, we guess a general linear polynomial times an exponential of the same exponent, i.e.

$$Y_2(t) = (At + B)e^{-t}.$$

Note that we will need two undetermined constants in our guess: when we take derivatives of the guess we will get [constant]$\times e^{-t}$ terms appear, and if we don't have one in our guess we won't be able to balance coefficients.

Now $Y_2' = ((-A)t + (A - B))\,e^{-t}$ and $Y_2'' = ((A)t + (-2A + B))\,e^{-t}$, so plugging Y_2 back into the DE gives us

$$\begin{aligned} Y_2'' + 9Y_2 &= (At + (-2A + B))\,e^{-t} + 9(At + B)e^{-t} \\ &= 10Ate^{-t} + (-2A + 10B)e^{-t}. \end{aligned}$$

Since this must equal $te^{-t} = 1 \cdot te^{-t} + 0 \cdot e^{-t}$ we therefore have that $10A = 1$ and $-2A + 10B = 0$. Solving this system of equations gives us $A = \frac{1}{10}$ and $B = \frac{1}{50}$. Hence

$$Y_2(t) = \left(\frac{1}{10}t + \frac{1}{50}\right)e^{-t}.$$

Finally, combining the two particular solutions gives us the solution $Y(t)$ to the complete nonhomogeneous differential equation. That is,

$$Y(t) = -\frac{1}{9} + \frac{1}{50}(5t + 1)\,e^{-t}.$$

2. (10 points) Recall that the Wronskian of a second-order linear system is $W(t) = y_1 y_2' - y_1' y_2$, where $y_1(t)$ and $y_2(t)$ form a fundamental basis of solutions to that system (the Wronskian is always the same up to multiplication by a constant, no matter which pair of basis of solutions you choose).

   Compute the Wronskian of the following system:

   $$y'' - 2y' + 10y = 0$$

   Your answer should be a function $W(t)$ that contains no undetermined constants. Remember to simplify your answer if possible.

To compute the Wronskian of the system we must first obtain a fundamental basis of solutions to the DE $y'' - 2y' + 10y = 0$ i.e. find the general solution to this homogeneous differential equations. The corresponding characteristic equation $r^2 - 2r + 10 = 0$; solving for r using the quadratic formula gives us

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$$

That is, $r = \lambda \pm \omega i$, where $\lambda = 1$ and $\omega = 3$. Consulting our knowledge of solutions to homogeneous 2nd-order constant coefficients, we know we're in the case where the CE has complex roots, and thus the general solution to the differential equation is correspondingly $y = e^{\lambda t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$. Thus for us we have the general solution

$$y = e^t (c_1 \cos(3t) + c_2 \sin(3t))$$

We may therefore take as a fundamental basis for the system the functions

$$y_1(t) = e^t \cos(3t), \quad y_2(t) = e^t \sin(3t)$$

Therefore by the product rule

$$\begin{aligned} y_1' &= e^t \cos(3t) - 3e^t \sin(3t) = e^t (\cos(3t) - 3\sin(3t)) \qquad \text{and} \\ y_2' &= e^t \sin(3t) + 3e^t \cos(3t) = e^t (\sin(3t) + 3\cos(3t)) \end{aligned}$$

Thus

$$\begin{aligned} W(t) &= y_1 y_2' - y_1' y_2 \\ &= e^t \cos(3t) \cdot e^t (\sin(3t) + 3\cos(3t)) - e^t \sin(3t) \cdot e^t (\cos(3t) - 3\sin(3t)) \\ &= e^{2t} \left(\sin(3t)\cos(3t) + 3\cos^2(3t)\right) - e^{2t} \left(\sin(3t)\cos(3t) - 3\sin^2(3t)\right) \\ &= e^{2t} \left(\sin(3t)\cos(3t) - \sin(3t)\cos(3t) + 3\cos^2(3t) + 3\sin^2(3t)\right) \\ &= 3e^{2t} \left(\cos^2(3t) + \sin^2(3t)\right) \\ &= 3e^2 t \end{aligned}$$

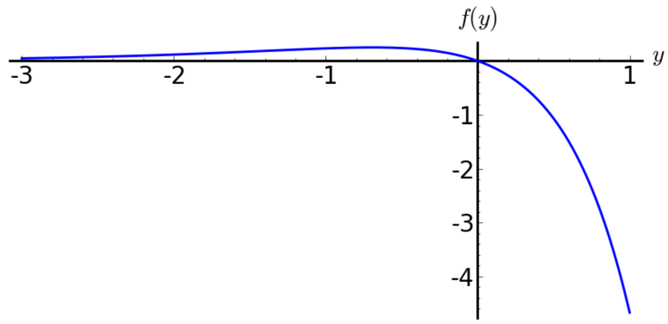
We conclude that the Wronskian of the system is $W(t) = 3e^{2t}$.

[Note: depending on your choice of fundamental basis functions you may end up with the Wronskian being a different multiple of e^{2t}. However, the answer will *always* be some constant times e^{2t} for this system.]

3. **(10 total points)** Consider the autonomous differential equation

$$y' = e^y - e^{2y},$$

where y is a function of t. Below is a graph of $f(y) = e^y - e^{2y}$ versus y:



(a) **(4 points)** Find all equilibrium solutions to this differential equation, and classify them according to their stability (stable, unstable or semistable). Be sure to justify your answer.

Recall that equilibrium solutions in an autonomous system are constant solutions, so to find them we must solve for y in the equation

$$f(y) = e^y - e^{2y} = 0.$$

Adding e^{2y} to both sides and dividing by e^y gives us $e^y = 1$; taking logs gives us the unique solution $y = 0$. This corresponds to the sole x-intercept of the graph above.

To ascertain the equilibrium solution's stability, note that $f(y)$ is negative to the left of $y = 0$; this means that y' is negative for any solution that starts out a bit above $y = 0$, so solutions bigger than zero tend to zero. Similarly $f(y) > 0$ for $y < 0$, so solutions that start out a bit below zero increase, and thus head towards the equilibrium point. We conclude that $y = 0$ is a stable equilibrium point. Alternatively we could calculate $\frac{df}{dy}$ at $y = 0$ (or just look at the graph above) and make note that it negative; this corresponds to a stable solution.

(b) **(6 points)** Suppose we are now looking at the solution to the above DE subject to the initial condition $y(0) = -1$. Use a single iteration of Euler's method to find an approximate value of the solution at $t = 0.5$. You may use decimals in this part of the question, but be sure to maintain at least four digits of precision.

Our stated initial conditions are $t_0 = 0$, $y_0 = -1$. Recall that Euler's method is given by the scheme

$$y_{n+1} = y_n + hf(t_n, y_n)$$

for $n \geq 1$. We will a single step with $h = \frac{1}{2}$ and $f(t_n, y_n) = e^{y_n} - e^{2y_n}$. Thus we have

$$y_1 = -1 + \frac{1}{2}\left(e^{-1} - e^{-2}\right) = \frac{-2 + \frac{1}{e} - \frac{1}{e^2}}{2} = \frac{-2e^2 + e - 1}{2e^2} \approx -0.8837.$$

Note that the approximation at $t = 0.5$ is closer to zero than the initial value of -1; this is as expected, since $y = 0$ is a stable equilibrium solution.

4. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{s^2+2s-2}{s^3-s}$$

We need to rewrite $F(s)$ using partial fractions so that we can get it into a form where we can take recognizable inverse Laplace transforms of all the constituent parts.

Observe that $s^2 - s = s(s-1)(s+1)$, so we can write

$$\frac{s^2+2s-2}{s^3-s} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}.$$

Clearing denominators gives us

$$s^2+2s-2 = A(s-1)(s+1) + Bs(s+1) + Cs(s-1).$$

When we evaluate the above equation at $s=0$ we get $-2=-A$, so $A=2$.
When we evaluate the above equation at $s=1$ we get $1=2B$, so $B=\frac{1}{2}$.
When we evaluate the above equation at $s=-1$ we get $-3=2C$, so $C=-\frac{3}{2}$. Hence

$$\frac{s^2+2s-2}{s^3-s} = 2\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s-1} - \frac{3}{2}\cdot\frac{1}{s+1}.$$

Finally, recall that $\mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1$, while $\mathscr{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$. Thus by linearity of the inverse Laplace transform we arrive at

$$\mathscr{L}^{-1}\left[\frac{s^2+2s-2}{s^3-s}\right] = 2 + \frac{1}{2}e^{t} - \frac{3}{2}e^{-t}.$$

[Note: many of you might also come up with the answer of

$$\mathscr{L}^{-1}\left[\frac{s^2+2s-2}{s^3-s}\right] = 2 - \cosh(t) + 2\sinh(t).$$

This is also correct, as $\cosh(t) = \frac{e^t+e^{-t}}{2}$ and $\sinh(t) = \frac{e^t-e^{-t}}{2}$.]

5. (10 points) My buddy is coming over to watch the game. Unfortunately my fridge has broken down, so I have to resort to alternative measures to cool our drinks down. The drinks are initially at 20 degrees Celsius; one hour before the game starts I place the drinks in an ice box. I note that the rate of cooling of the drinks is proportional to the temperature difference between the drinks and the ice box; moreover, I observe that the proportionality constant is precisely $\frac{1}{50}$ when the units of time are minutes and the units of temperature are degrees Celsius.

   However, the icebox itself is slowing heating up. One hour before the game the icebox is at 0 degrees Celsius, but its temperature is increasing linearly at a rate of 1 degree Celsius every 10 minutes.

   Formulate and solve an initial value problem to find the temperature of the drinks when the game begins.

Let y be the temperature of the drinks in degrees Celsius as a function of time, and let t be time since I place the drinks in the icebox, (i.e. $t = 0$ is one hour before the game starts), where t is measured in **minutes**. This is the natural choice of units, as the proportionality constant mentioned in the problem statement is given in terms of °C and minutes.

We know that the drinks are initially at 20 °C, so this gives us the IC $y(0) = 20$. Next, we know that the temperature of the icebox is increasing linearly at a rate of 1 °C every 10 minutes, starting at 0 °C when $t = 0$; hence the temperature of the icebox as a function of time is given by $\frac{t}{10}$.

Now we invoke Newton's Law of Cooling to set up a differential equation. According to our observations the rate of cooling of the drinks is precisely $\frac{1}{50}$th of the temperature difference between the drinks and the icebox; translating this into mathematics we get the equation

$$\frac{dy}{dt} = -\frac{1}{50}\left(y - \frac{t}{10}\right).$$

We must be sure to include the minus sign, as if the drinks are warmer than the icebox we expect their temperature to decrease. After multiplying out and shuffling things around we get the first-order initial value problem

$$y' + \frac{1}{50}y = \frac{1}{500}t, \qquad y(0) = 20.$$

We can solve this linear DE using integrating factors, but Laplace transforms work just as well. Let $y = \phi(t)$ be the solution to this IVP, and let $\Phi(s) = \mathscr{L}[\phi(t)]$. Taking the Laplace transform of both sides of the equation gives us

$$s\Phi - 20 + \frac{1}{50}\Phi = \frac{1}{500} \cdot \frac{1}{s^2},$$

using $\phi(0) = 20$. Thus, after solving for Φ we get

$$\Phi = \frac{1}{500} \cdot \frac{1}{s^2(s + \frac{1}{50})} + 20 \cdot \frac{1}{(s + \frac{1}{50})}$$

Using partial fractions in the usual way we find that $\frac{1}{500} \cdot \frac{1}{s^2(s+\frac{1}{50})} = \frac{1}{10} \cdot \frac{1}{s^2} - 5 \cdot \frac{1}{s} + 5 \cdot \frac{1}{(s+\frac{1}{50})}$

[Continued on the next page]

Thus

$$\Phi = \frac{1}{10} \cdot \frac{1}{s^2} - 5 \cdot \frac{1}{s} + 25 \cdot \frac{1}{(s + \frac{1}{50})}.$$

Taking inverse Laplace transforms we find that

$$y = \phi(t) = \frac{t}{10} - 5 + 25e^{-\frac{1}{50}t}.$$

The temperature of the drinks are at the beginning of the game is therefore

$$\phi(60) = \frac{60}{10} - 5 + 25e^{-\frac{60}{50}} = 1 + 25e^{-\frac{6}{5}} \approx 8.53\ ^\circ\text{C}.$$

Cold enough to quench thirst.

6. (10 total points) A $\frac{1}{2}$ kg mass is placed on a surface and attached to a horizontal spring with spring constant β kg/s^2, where β is a positive constant. Friction acts on the mass such that when the mass is traveling at 1 m/s it experiences a frictional force of 1 Newton.

   (a) (2 points) Establish a differential equation that the mass obeys.

   Let t be measured in seconds, and let y be the horizontal position of the mass at time t, where y is in meters increasing to the right, and $t = 0$ is some unspecified zero point in time.

   We assume that frictional force is proportional to velocity, acting in the opposite direction. Because the object experiences a force of 1 N when its velocity is 1 m/s, we conclude that the friction coefficient is $\gamma = 1$ kgs^{-1}. There is no forcing function mentioned in the problem outline, so we can assume that forcing is zero. Hence according to the standard setup we have the homogeneous system

   $$\frac{1}{2}y'' + y' + \beta y = 0.$$

   That's it for this part.

   (b) (5 points) For what values of β will the system will be overdamped, critically damped and underdamped respectively? Justify your answer.

   The characteristic equation corresponding to this differential equation is $\frac{1}{2}r^2 + r + \beta = 0$; this has roots

   $$r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot \frac{1}{2} \cdot \beta}}{2 \cdot \frac{1}{2}} = -1 \pm \sqrt{1 - 2\beta}.$$

   The nature of solutions to the DE depends on whether the discriminant $1 - 2\beta$ is positive, zero or negative.

   - Overdamping occurs when the CE has real and unequal roots, which corresponds to the discriminant being positive, i.e. $1 - 2\beta > 0$. Since we know $\beta > 0$, we conclude that overdamping occur for $0 < \beta < \frac{1}{2}$.
   - Critical damping occurs when the CE has equal roots, which corresponds to a discriminant of zero, i.e. $1 - 2\beta$. This critical damping occurs for $\beta = \frac{1}{2}$ exactly.
   - Underdamping occurs when the CE has complex roots, which corresponds to a negative discriminant, i.e. $1 - 2\beta < 0$. Thus we get underdamping for $\beta > \frac{1}{2}$.

   (c) (3 points) Find the value of β for which the quasi-frequency of the mass's damped oscillation is exactly 4 radians/sec. [Note: I'm referring to the *angular frequency* ω, not the cyclic frequency.]

   Recall that in a unforced underdamped system, if the roots to the characteristic equation are $r = -\alpha + \omega i$, then a general solution looks like $y = Re^{-\alpha t}\cos(\omega t - \delta)$ for some amplitude R and phase shift δ. Thus we seek β such that solutions to the CE have imaginary part $4i$. But by part (a) the solutions to the CE are $r = -1 \pm \sqrt{1 - 2\beta} = -1 \pm i \cdot \sqrt{2\beta - 1}$; we thus must have that $\sqrt{2\beta - 1} = 4$. Solving for β gives us the solution $\beta = \frac{17}{2}$.

7. (10 total points) A series circuit contains an inductor of inductance 0.1 henrys, a resistor of resistance 2 ohms, and a capacitor of capacitance 0.01 farads. At time $t = 0$ seconds there is no current in the circuit, but the charge on the capacitor is 0.01 coulombs.

   Initially no external voltage is applied on the circuit. At $t = 1$ seconds a source is switched on which exerts a constant voltage of 10 volts on the circuit. At $t = 4$ seconds the voltage source is switched off, and no external voltage acts on the circuit from thereon.

   (a) (3 points) Let $E(t)$ be the forcing function in the above system. Express $E(t)$ using Heaviside functions $u_c(t)$. Your answer should be a function that is a linear combination of $u_c(t)$'s..

   We write $E(t)$ in terms of Heaviside functions by building up a series of functions $E_i(t)$ which agree with $E(t)$ for larger and larger time intervals.

   The function $E(t)$ is identically zero until $t = 1$, so start with $E_1(t) = 0$; thus $E(t) = E_1(t)$ for $0 \leq t < 1$.
   Then $E(t)$ becomes 10 at $t = 1$, so let $E_2(t) = E_1(t) + 10u_1(t) = 10u_1(t)$. We then have $E(t) = E_2(t)$ for $0 \leq t < 4$.
   Finally $E(t)$ drops by 10 units to become 0 for $t \geq 4$, so let $E_3(t) = E_2(t) - 10u_4(t)$. Now $E(t) = E_3(t)$ for all $t \geq 0$, so we have found that (after simplifying)

$$E(t) = 10u_1(t) - 10u_4(t).$$

   (b) (2 points) Write down the initial value problem that models the charge on the capacitor for any time $t \geq 0$.

   The standard inductor-resistor-capacitor series circuit differential equation is

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

   where Q is the charge on the capacitor, L the inductance of the inductor, R resistor's resistance, C the capacitance of the capacitor, and $E(t)$ the externally applied voltage. The latter is exactly the $E(t)$ we've found in the first part of the question; the inductance, resistance and capacitance are given as $L = \frac{1}{10}, R = 2$ and $C = \frac{1}{100}$ respectively.

   Finally, there is no current in the circuit and a charge of 0.01 coulombs on the capacitor, so we have the initial conditions $Q(0) = \frac{1}{100}$, $Q'(0) = 0$. Putting these all together we arrive at the IVP

$$\frac{1}{10}Q'' + 2Q' + 100Q = 10u_1(t) - 10u_4(t), \qquad Q(0) = \frac{1}{100},\ Q'(0) = 0.$$

(c) (5 points) Let $Q = \phi(t)$ be the solution to the IVP above. Compute the Laplace transform $\Phi(s)$ of the solution as a function of s.

[NB: you do not need to fully solve the IVP to answer this part of the question.]

$\phi(t)$ satisfies the DE established in the previous part of the question, so we have

$$\frac{1}{10}\phi'' + 2\phi' + 100\phi = 10\,u_1(t) - 10\,u_4(t), \qquad \phi(0) = \frac{1}{100},\ \phi'(0) = 0.$$

To compute $\Phi(s) = \mathscr{L}[\phi]$, we take Laplace transforms of both sides of the above DE. Using our list of known rules, we transform the left hand side as follows:

$$\begin{aligned}
\mathscr{L}\left[\frac{1}{10}\phi'' + 2\phi' + 100\phi\right] &= \frac{1}{10}\mathscr{L}\left[\phi''\right] + 2\mathscr{L}\left[\phi'\right] + 100\mathscr{L}\left[\phi\right] \\
&= \frac{1}{10}\left(s^2\Phi - s\phi(0) - \phi'(0)\right) + 2\left(s\Phi - \phi(0)\right) + 100\Phi \\
&= \left(\frac{1}{10}s^2 + 2s + 100\right)\Phi - \frac{1}{10}\cdot s\cdot\frac{1}{100} - \frac{1}{10}\cdot 0 - 2\cdot\frac{1}{100} \\
&= \left(\frac{1}{10}s^2 + 2s + 100\right)\Phi - \frac{s}{1000} - \frac{1}{50}
\end{aligned}$$

Hitting the right hand side with Laplace yields

$$\begin{aligned}
\mathscr{L}\left[10\,u_1(t) - 10\,u_4(t)\right] &= 10\mathscr{L}\left[u_1(t)\right] - 10\mathscr{L}\left[u_4(t)\right] \\
&= 10\frac{e^{-s}}{s} - 10\frac{e^{-4s}}{s}
\end{aligned}$$

Finally, by equating the left and right hand sides of the transformed equation we get

$$(\frac{1}{10}s^2 + 2s + 100)\Phi - \frac{s}{1000} - \frac{1}{50} = \frac{10(e^{-s} - e^{-4s})}{s}$$

Before solving for Φ, it's probably easiest to multiply everything by 1000 to clear denominators:

$$100(s^2 + 20s + 1000)\Phi - s - 20 = \frac{10000(e^{-s} - e^{-4s})}{s}$$

Then

$$\begin{aligned}
\Phi &= \frac{10000(e^{-s} - e^{-4s})}{100s(s^2 + 20s + 1000)} + \frac{s}{100(s^2 + 20s + 1000)} + \frac{20}{100(s^2 + 20s + 1000)} \\
\Longrightarrow \Phi(s) &= \frac{100(e^{-s} - e^{-4s})}{s(s^2 + 20s + 1000)} + \frac{1}{100}\cdot\frac{s + 20}{(s^2 + 20s + 1000)}
\end{aligned}$$

Note: we could try simplify the above equation even more, but this is a game that could go on forever. However, it looks decent as is, so I'm happy if you leave it there.

8. (10 total points + 4 bonus points) A two-way pump attached attached to a reservoir pumps water into and then out of the reservoir at a rate of $2000\sin(t)$ liters per hour, where t is measured in hours. At time $t = 0$ a valve at the bottom of the reservoir is opened and it begins to drain at a rate proportional to the amount of water in the reservoir. The reservoir initially contains 10000 liters of water, and the initial outflow rate is measured to be 1000 liters per hour.

   (a) (3 points) Establish an initial value problem that models the volume of water in the reservoir at time t.

   This is a flow rate problem. Let $y(t)$ be the number of liters of water in the reservoir at time t, where t is measured in hours since the the valve is opened. Recall that the rate of change of y equals rate in minus rate out. In this instance the "rate in" is the pump flow rate, i.e. $2000\sin(t)$ liters per hour, while the rate out is the rate at which water escapes through the valve at the bottom of the tank.

   We are told the outflow rate through the valve is proportional to the number of liters currently in the reservoir; that is, outflow rate $= -ky$, where k is some proportionality constant. We therefore have that

$$\frac{dy}{dt} = 2000\sin(t) - ky.$$

   We are not told what k is, but we are told that the initial outflow rate is 1000 liters per hour. Since the pump flow rate is $2000\sin(0) = 0$, this must mean that the initial flow rate through the valve is -1000 liters per hour. We therefore have that at time zero $-1000 = -k \cdot 10000$, so $k = \frac{1}{10}$.

   Putting this and the initial condition $y(0) = 10000$ together, we get the IVP

$$\frac{dy}{dt} = -\frac{1}{10}y + 2000\sin(t), \qquad y(0) = 10000.$$

   (b) (7 points) Solve the initial value problem to find the number of liters of water in the reservoir at time t.

   This equation is linear. In standard form it is $y' + \frac{1}{10}y = 2000\sin(t)$. We could solve this IVP using integrating factors, but since we've already used that method in a previous question, we'll use Laplace transforms here instead.

   Let $y = \phi(t)$ be the solution to this IVP, and let $\Phi(s) = \mathscr{L}[\phi(t)]$. We now take Laplace transform of both sides of the equation. On the left hand we get

$$\begin{aligned}\mathscr{L}[\phi'] + \frac{1}{10}\mathscr{L}[\phi] &= (s\Phi - \phi(0)) + \frac{1}{10}\Phi \\ &= \left(s + \frac{1}{10}\right)\Phi - 10000.\end{aligned}$$

On the right hand side we get

$$\mathscr{L}[2000\sin(t)] = 2000 \cdot \frac{1}{s^2+1}.$$

Equating the two sides and solving for Φ gives us

$$\Phi = 10000 \cdot \frac{1}{(s+\frac{1}{10})} + 2000 \cdot \frac{1}{(s+\frac{1}{10})(s^2+1)}.$$

All that remains then is to compute the inverse Laplace transform of $\Phi(s)$. Note that we can use partial fractions on $\frac{1}{(s+\frac{1}{10})(s^2+1)}$ to break it up into a sum of terms with only $s+\frac{1}{10}$ or s^2+1 in the denominator. That is

$$\frac{1}{(s+\frac{1}{10})(s^2+1)} = \frac{A}{s+\frac{1}{10}} + \frac{Bs+C}{s^2+1}$$

for some values of A, B and C. Clearing denominators we get

$$1 = A(s^2+1) + (Bs+C)\left(s+\frac{1}{10}\right).$$

Evaluating at $s = -\frac{1}{10}$ yields $1 = A \cdot (\frac{1}{100}+1)$, so $A = \frac{100}{101}$.
Evaluating at $s = 0$ yields $1 = A + \frac{1}{10}C$, so $C = \frac{10}{101}$;
Evaluating at $s = 1$ yields $1 = 2A + \frac{11}{10}(B+C)$, so solving for B we get $B = -\frac{100}{101}$. That is, we have

$$\frac{1}{(s+\frac{1}{10})(s^2+1)} = \frac{10}{101}\left(10 \cdot \frac{1}{s+\frac{1}{10}} - 10 \cdot \frac{s}{s^2+1} + \frac{1}{s^2+1}\right).$$

We therefore have that the inverse Laplace transform of Φ is

$$\begin{aligned}\mathscr{L}^{-1}[\Phi] &= 10000\,\mathscr{L}^{-1}\left[\frac{1}{s+\frac{1}{10}}\right] + \frac{20000}{101}\left(10\,\mathscr{L}^{-1}\left[\frac{1}{s+\frac{1}{10}}\right] - 10\,\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]\right)\\ &= 10000e^{-\frac{1}{10}t} + \frac{20000}{101}\left(10e^{-\frac{1}{10}t} - 10\cos(t) + \sin(t)\right).\end{aligned}$$

So after collecting terms we conclude that the amount of water in the reservoir at time t is given by

$$y = \phi(t) = \frac{10000}{101}\left(121e^{-\frac{1}{10}t} - 20\cos(t) + 2\sin(t)\right) \text{ liters.}$$

[Question continued on the next page]

(c) (4 bonus points) Estimate the point in time when the reservoir first runs dry.

We want to find the first t such that $\phi(t) = 0$. That is, solve for t when

$$121e^{-\frac{1}{10}t} - 20\cos(t) + 2\sin(t) = 0.$$

Note that we may divide through by that ugly $\frac{10000}{101}$ coefficient, since it won't affect where the function hits zero.

Now this is not an equation where one can solve for t explicitly, so we'll have to make some approximations. Note that the exponential decay constant is small competed to the oscillation frequency; we therefore expect that in the time take for the oscillation part to go between its minimum and maximum values, the exponential part of the function will not have decayed very much. In other words, we can think of the function as relatively rapid oscillation about some slowly decreasing equilibrium.

The result is that the first root of the equation should be near the point in time where $121e^{-\frac{1}{10}t} - R$ first dips below zero, where R is the amplitude of the oscillation part. Recall that $R = \sqrt{A^2 + B^2}$, where A and B are the amplitudes of the cos and sin functions, so we have $R = \sqrt{20^2 + 2^2} = 2\sqrt{101}$. We thus should solve for when

$$121e^{-\frac{1}{10}t} - 2\sqrt{101} = 0.$$

Solving for t in this equation gives us

$$t = 10\log\left(\frac{121}{2\sqrt{101}}\right) = 5\log\left(\frac{14641}{404}\right) \approx 17.95 \text{ hours.}$$

We conclude that the reservoir empties at just under the 18 hour mark.

In reality the first time $y(t) = 0$ happens a bit later than this - as the oscillation doesn't hit its minimum value at exactly the moment when the decay part of the equation comes within R of zero. Using Mathematica to numerically find the root we get $t \approx 18.44$ hours, so about a half hour after our prediction. Our estimate is therefore within 3 percent of the actual value, which for most on-the-fly real world problems is good enough.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}(f(t))$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	