Friday 17 Jan — MATH 307 LECTURE 6

§1.5: EXISTENC & UNIQUNESS

BOYCE 2.4

AKA: DIFFERENCES BEWEEN LINEAR & NONLINEAR EQUATIONS

Often when analysing differential equations or systems thereof, it is useful to determine a priori if a solution is guaranteed to exist, and if this system is guaranteed to be unique. To this effect in this lecture we present two theorems on existence & uniqueness for solutions to first order differential equations, and show some applications & examples thereof.

> **Theorem 1.5.1** "Existence & Uniqueness for Linear ODEs"
> Consider the first order linear DE
> $$\frac{dy}{dt} + f(t)\cdot y = g(t)$$
> together with IC $y(t_0) = y_0$.
> If f and g are both continuous on the interval $\alpha < t < \beta$, and the interval contains $t = t_0$, then there exists a unique solution to the DE with $y(t_0) = y_0$, for all $\alpha < t < \beta$.

Corollary 1.5.2 If $f(t)$ & $g(t)$ are defined for all t, then a unique solution exists for any initial condition, and that solution is defined for all t.

Example 1 $\frac{dy}{dx} + 2xy = e^{-x^2}$, $y(x_0) = y_0$

Solution: $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$

So
$$y(x) = e^{-x^2}\left(\int e^{x^2}\cdot e^{-x^2}\,dx + C\right) = xe^{-x^2} + Ce^{-x^2}$$

IC: $y_0 = x_0 e^{-x_0^2} + Ce^{-x_0^2}$

$\Rightarrow C = y_0 e^{x_0^2} - x_0$

So $y(x) = xe^{-x^2} + (y_0 e^{x_0^2} - x_0)e^{-x^2}$

$$\Rightarrow y(x) = (x - x_0)e^{-x^2} + y_0 e^{x_0^2 - x^2}$$

Clearly this is valid for all x, given any x_0, y_0 □

PTO

Example 2: $(t^2-4t)\frac{dy}{dt} + (t-4)y - t = 0, \quad y(3)=y_0$

Standard form: $\frac{dy}{dt} + \frac{1}{t}\cdot y = \frac{1}{t-4}$

So $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = |t|$

But $y(3)=y_0$ means we are considering ~~at least some $t \geq 0$~~ t around 3,
so take $t > 0 \Rightarrow \mu(t) = t$.

Then
$$\begin{aligned} y(t) &= \frac{1}{t}\left(\int t \cdot \frac{1}{t-4}\, dt + C\right) \\ &= \frac{1}{t}\left(\int 1 + 4\cdot\frac{1}{t-4}\, dt + C\right) \\ &= 1 + \frac{4}{t}\cdot\ln|t-4| + \frac{C}{t} \end{aligned}$$

IC: $y(3)=y_0 \Rightarrow y_0 = 1 + \frac{4}{3}\ln 1 + \frac{C}{3}$
$\Rightarrow C = 3(y_0 - 1)$

So $$y(t) = 1 + \frac{4}{t}\cdot\ln|t-4| + \frac{3(y_0-1)}{t}$$

Now note that this is only continuous about $t=3$ for $0 < t < 4$, in which case $|t-4| = 4-t$.
So the final solution here is
$$y(t) = 1 + \frac{4}{t}\cdot\ln(4-t) + \frac{3(y_0-1)}{t}, \quad 0 < t < 4. \quad \square$$

Clearly we see that in general the above solution only exists for $0 < t < 4$, which is what Theorem 1.5.1 predicts:
$f(t) = \frac{1}{t}$ is continuous on $(0,\infty)$ and $(-\infty, 0)$
$g(t) = \frac{1}{t-4}$ is continuous on $(-\infty, 4)$ and $(4, \infty)$, so the largest interval containing $t_0 = 3$ on which both are continuous is $0 < t < 4$.

Note 1.5.3: Theorem 1.5.1 is only a guarantee on the minimum interval in which a unique solution must exist. Sometimes the solution to the IVP exists on a larger interval.
For example, in the above example, Having the IC as $y(3)=1$ yields the solution $y(t) = 1 + \frac{4}{t}\cdot\ln(4-t) - \frac{4\cdot\ln(4)}{t}$, which we can show is actually continuous on $(-\infty, 4)$. $-\frac{8}{3}\ln(2)$

PTO

Friday 17 Jan MATH 307 Lecture 6, cont...

The theorem governing existence & uniqueness for nonlinear ODEs is not quite as powerful:

Theorem 1.5.4 "Existence & Uniqueness for nonlinear ODES"

Consider the first-order DE $\frac{dy}{dx} = f(x, y)$, ~~...~~
with IC $y(x_0) = y_0$.
Suppose both f and $\frac{\partial f}{\partial y}$ are continuous for all $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (x_0, y_0);
Then for some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there exists a unique solution to the DE with $y(t_0) = y_0$

Contrast this with the theorem governing linear ODEs: here we are guaranteed only that ~~a~~ a unique~~a~~ solution exists on some interval about t_0, whereas with linear DEs the solution exists on the maximum possible interval. Moreover, for nonlinear ODEs the interval on which the solution is defined may depend on the y-value of the IC, unlike the general case with linear DEs.

Example 1 $\quad \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \qquad y(0) = -1$

Observe $f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}$ and $\frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y-1)^2}$ are both continuous everywhere except for the line $y = 1$; we may thus draw a box around the point $(0, -1)$ inside ~~for~~ which both f & $\frac{\partial f}{\partial y}$ are continuous. We are therefore guaranteed a unique solution to the ~~ME~~ IVP, valid for some interval about $x = 0$.

Solution: $\quad 2(y-1)\,dy = 3x^2 + 4x + 2$

$$\Rightarrow \quad (y-1)^2 = x^3 + 2x^2 + 2x + C$$

$$\Rightarrow \quad y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

IC: $y(0) = -1 \quad \Rightarrow \quad -1 = 1 \pm \sqrt{C} \quad \Rightarrow \quad C = 4$ & take negative root

$$\Rightarrow \quad y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

$$\Rightarrow \quad y = 1 - \sqrt{(x+2)(x^2+2)}$$

PTO

Note that this solution is valid for all $x > -2$.

Now suppose we had $y(0) = 1$ as the IC.
Since both $f(x,y)$ & $\frac{df}{dy}$ are not continuous at $y = 1$, the theorem makes no assertion as to the existence or uniqueness of a solution.

Indeed, solving the DE has
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C} \quad \text{with IC } y(0) = 1 \Rightarrow C = 0$$
And we see both
$$y = 1 + \sqrt{x^3 + 2x^2 + 2x} \quad \text{and} \quad y = 1 - \sqrt{x^3 + 2x^2 + 2x}$$
obey both the differential equation and the initial condition.

Example 2: Consider $y' = y^{\frac{1}{3}}$, $y(0) = 0$.
"When things go really wrong"

Note $f(t,y) = y^{\frac{1}{3}}$ is continuous everywhere, but
$$\frac{df}{dy} = \frac{1}{3} y^{-\frac{2}{3}} \quad \text{is discontinuous at } y = 0,$$
So Theorem 1.5.4 does not apply. We are therefore not guaranteed a unique solution to the IVP.

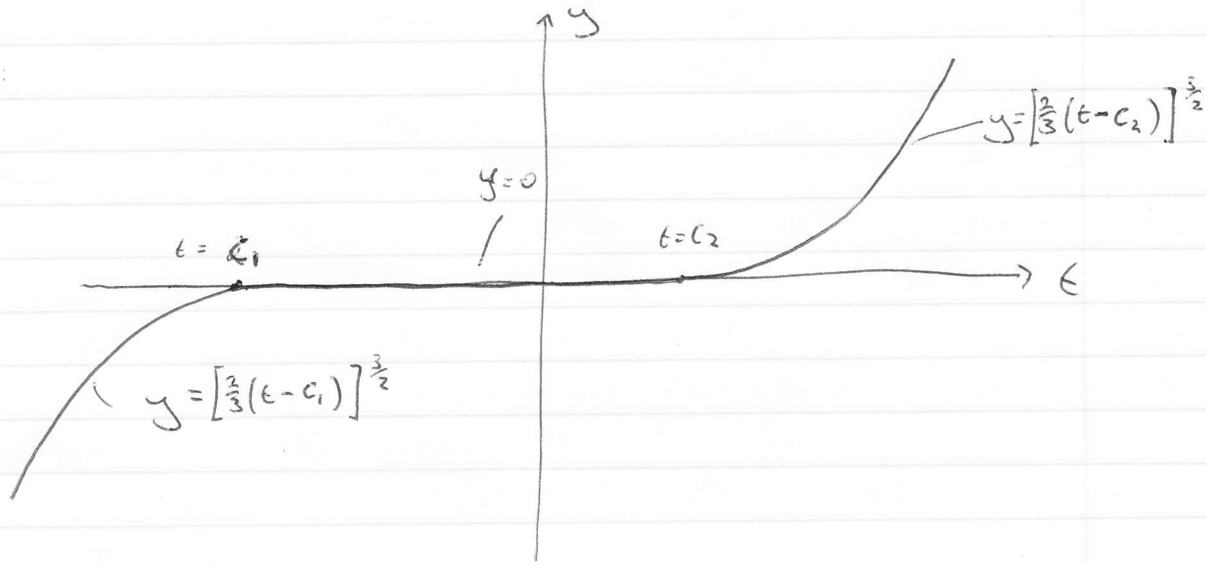
Solving:
$$y^{-\frac{1}{3}} dy = dt$$
$$\Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t + C$$
$$\text{So} \quad y = \left(\tfrac{2}{3}t + C\right)^{\frac{3}{2}} \quad \text{absorbing the } \tfrac{2}{3} \text{ into the } C$$
Applying the IC: $y(0) = 0 \Rightarrow C = 0$
$$\text{So} \quad y = \left(\tfrac{2}{3}t\right)^{\frac{3}{2}}$$

On the other hand, note that the function
$$y = -\left(\tfrac{2}{3}t\right)^{\frac{3}{2}} \quad \text{also satisfies the DE.}$$
Moreover $y = 0$ also works.

PTO

FRI 17 JAN MATH 307 Lecture 6, Cont...

Now here's the crazy part: We can splice together the above three functions to create even more solutions to the DE.
We can let $y(t) = 0$ for some interval $t < t_0$, then at t_0 switch to one of the other solutions, suitably shifted. Since $y = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$ has zero derivative for $y = 0$, such a function will have a continuous derivative.



The solutions to $y' = y^{\frac{1}{3}}$, $y(0) = 0$ in their most general form are then

$$y = \begin{cases} \pm\left[\frac{2}{3}(t - c_1)\right]^{\frac{3}{2}}, & t < c_1, \quad c_1 \le 0 \\ 0, & c_1 \le t \le c_2 \\ \pm\left[\frac{2}{3}(t - c_2)\right]^{\frac{3}{2}}, & c_2 < t, \quad c_2 \ge 0. \end{cases}$$

This problem has infinitely many solutions!