~~Wed 12 Feb~~
Fri 14 Feb ♥

MATH 307a LECTURE 15, cont...

MECHANICAL & ELECTRICAL VIBRATIONS CONT...

(BOYCE 3.7)

Example continued: $0.1y'' + 14.4y = 0, \quad y(0) = -0.1, \quad y'(0) = 1.5$

<u>Solution</u> we have $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ with

$$A = -\tfrac{1}{10} \quad B = \tfrac{1}{8}, \quad \omega_0 = 12.$$

~~RM~~ $A = R\cos(\delta), \; B = R\sin(\delta)$,

so $R^2 = A^2 + B^2, \quad \tan(\delta) = B/A.$

$$R = \sqrt{\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{1}{8}\right)^2} = \tfrac{1}{2}\sqrt{\left(\tfrac{1}{5}\right)^2 + \left(\tfrac{1}{4}\right)^2} = \tfrac{1}{2}\sqrt{\tfrac{1}{25} + \tfrac{1}{16}} = \tfrac{1}{2}\sqrt{\tfrac{16+25}{400}} = \tfrac{1}{2}\sqrt{\tfrac{41}{400}} = \tfrac{\sqrt{41}}{40} \approx 0.16008\ldots$$

$$\tan(\delta) = \frac{\frac{1}{8}}{-\frac{1}{10}} = -\tfrac{5}{4}$$

so $\delta = \arctan^*\left(-\tfrac{5}{4}\right) = -0.89606 + n\cdot\pi$

But $\cos\delta < 0, \; \sin\delta > 0 \Rightarrow \delta$ 2nd Quadrant

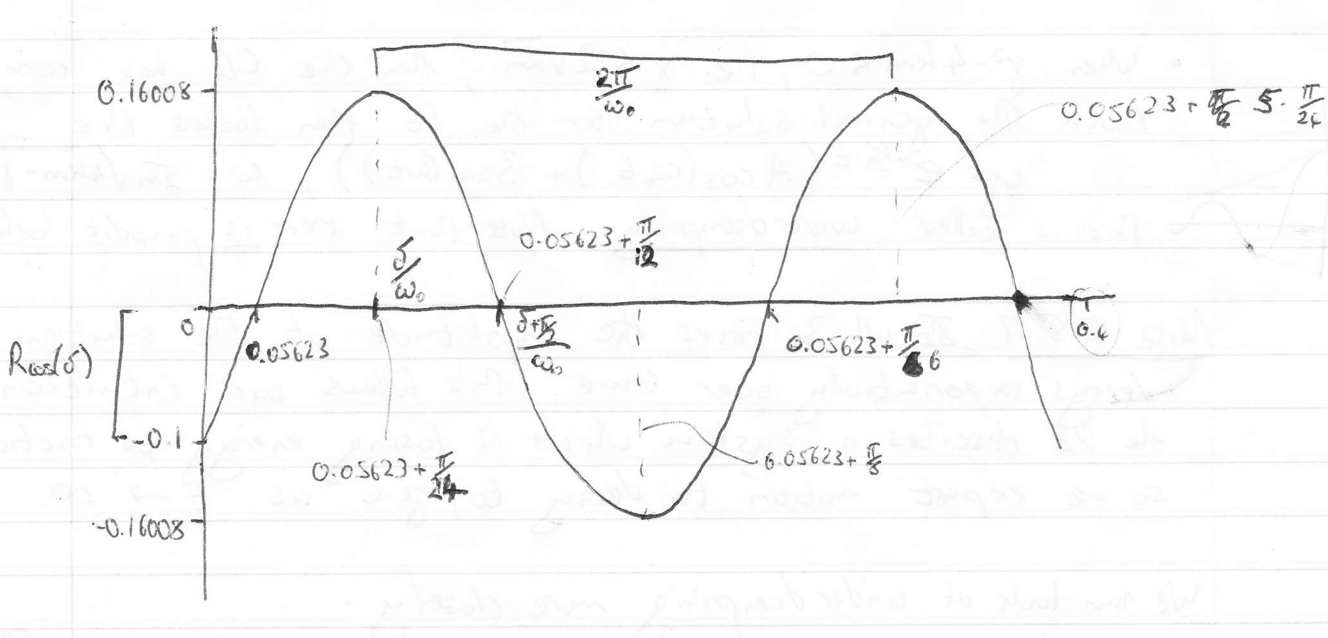
$\Rightarrow \delta = \pi - 0.89606 = 2.24553.$ ($\tfrac{\pi}{2} < \delta < \pi$)

Thus, to 5 d.p., we have

$$y = 0.16008\cos(12t - 2.24553)$$

We see then that the amplitude of the oscillation is ≈ 16.008 cm, and the first time when the block crosses its rest position is when $12t - 2.24553 = -\tfrac{\pi}{2}$

$$\Rightarrow t = \tfrac{1}{12}\left(2.24553 - \tfrac{\pi}{2}\right) = 0.05623 \text{ s.}$$



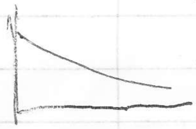
2.5.6 Damped Free Vibrations

Now suppose the block is on a surface with friction. Then we get the DE $my'' + \gamma y' + ky = 0$, $m, \gamma, k > 0$.
The corresponding CE is $mr^2 + \gamma r + k = 0$, which has roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

The big thing to realize is that depending on the discriminant $\gamma^2 - 4km$ the solution to the DE may look like one of the following three forms:

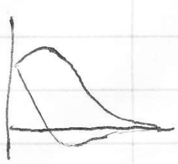
- If $\gamma^2 - 4km > 0$, i.e. $\gamma > 2\sqrt{km}$, then we have 2 real roots to the CE. Note that since $\sqrt{\gamma^2 - 4km}$ is smaller than γ in magnitude, both roots are negative. Here the solution looks like



$$y = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}, \quad \lambda_1 = -\frac{\gamma}{2m}\left(1 - \sqrt{1 - \frac{4km}{\gamma^2}}\right), \ \lambda_2 = -\frac{\gamma}{2m}\left(1 + \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

  This situation is called overdamping. Note: no periodic behaviour.

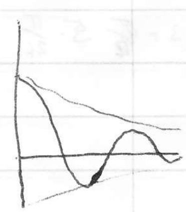
- If $\gamma^2 - 4km = 0$, i.e. $\gamma = 2\sqrt{km}$, then the CE has repeated roots. The general solution to the DE looks like



$$y = (A + Bt)e^{-\frac{\gamma}{2m}t}$$

  This is called critical damping. Note: no periodic behaviour

- When $\gamma^2 - 4km < 0$, i.e. $\gamma < 2\sqrt{km}$, then the CE has complex roots. The general solution to the DE then looks like



$$y = e^{-\frac{\gamma}{2m}t}\left(A\cos(\omega_0 t) + B\sin(\omega t)\right), \quad \omega = \frac{1}{2m}\sqrt{4km - \gamma^2}$$

  This is called underdamping. Note that there is periodic behaviour.

Note 2.5.7 In all 3 cases the amplitude of the solution always decays exponentially over time. This follows our intuition: the DE describes a system which is losing energy to friction, so we expect motion to decay to zero as $t \to \infty$.

We now look at underdamping more closely.

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We investigate the case of underdamping more closely.

Recall we can rewrite $A\cos(\omega t) + B\sin(\omega t)$ as $R\cos(\omega t - \delta)$, so another way to write the general solution to an underdamped system is

$$y = Re^{-\frac{\gamma}{2m}t}\cos(\omega t - \delta), \quad \omega = \frac{1}{2m}\sqrt{4km - \gamma^2}$$

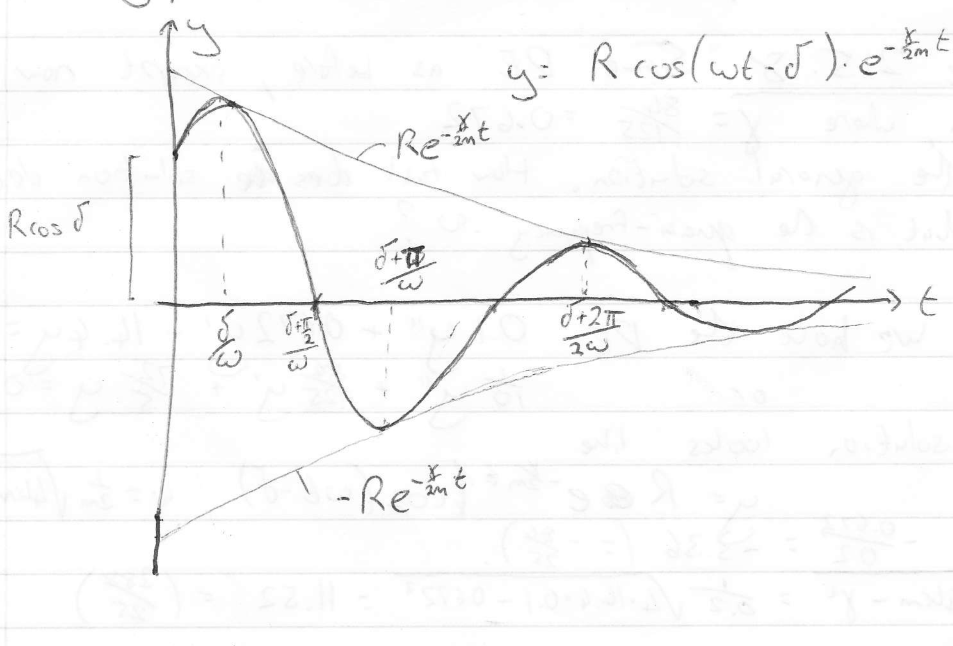
We can think of this as a sinusoidal function with exponentially decaying amplitude.

Notation: • ω is called the quasi-(cyclic) frequency

[Note: there is some ambiguity as to whether the term 'quasi-frequency' refers to the ω above, - the number of radians per second of the oscillatory part of the solution - or $\frac{\omega}{2\pi}$, the number of cycles per second. In any ~~written~~ homework or exam I will always be explicit as to which one I am referring to.]

• $T = \frac{2\pi}{\omega}$ is called the quasi-period. This is the time between successive peaks of the function.

The graph of such a solution will look as follows:



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The reason they have a 'quasi' in front is because the solution isn't actually periodic - only the oscillating part is.

- We may compare the quasi-frequency ω to the natural frequency ω_0 - this is what the frequency would be if there was no damping.
  Recall $\omega_0 = \sqrt{\frac{k}{m}}$, so
  $$\frac{\omega}{\omega_0} = \frac{\frac{1}{2m}\sqrt{4km-\gamma^2}}{\sqrt{\frac{k}{m}}} = \sqrt{1-\frac{\gamma^2}{4km}}$$

  When $\frac{\gamma^2}{4km}$ is small, we may use the approximation $\sqrt{1-x} \simeq 1-\frac{x}{2}$ to say that $\frac{\omega}{\omega_0} \simeq 1 - \frac{\gamma^2}{8km}$.
  That is, as friction increases, the quasi-frequency <u>decreases</u>; so damping not only influences how fast the amplitude $\to 0$, but also the rate at which the solution oscillates back & forth.

- Similarly, we may compare the quasi-period with the natural period:
  $$\frac{2\pi/\omega}{2\pi/\omega_0} = \frac{\omega_0}{\omega} = \frac{1}{\sqrt{1-\frac{\gamma^2}{4km}}}$$
  Analogous to above this is $\simeq 1 + \frac{\gamma^2}{8km}$ when $\frac{\gamma^2}{4km}$ is $<< 1$.
  Thus as expected, quasi-period increases as $\frac{\gamma^2}{4km}$ increases.

<u>Example 2.5.8</u> Same DE as before, except now with friction, where $\gamma = \frac{84}{125} = 0.672$

- Find the general solution. How fast does a solution decay, and what is the quasi-frequency ω?

<u>Solution</u>: We have the DE $0.1y'' + 0.672y' + 14.4y = 0$,
or $\frac{1}{10}y'' + \frac{84}{125}y' + \frac{72}{5}y = 0$.

Recall the solution looks like
$$y = Re^{-\frac{\gamma}{2m}t}\cos(\omega t - \delta), \quad \omega = \frac{1}{2m}\sqrt{4km-\gamma^2}.$$

Now $-\frac{\gamma}{2m} = -\frac{0.672}{0.2} = -3.36 \ (= -\frac{84}{25})$.

And $\frac{1}{2m}\sqrt{4km-\gamma^2} = \frac{1}{0.2}\sqrt{4\cdot 14.4\cdot 0.1 - 0.672^2} = 11.52 = \left(\frac{288}{25}\right)$

So solution looks like $y = Re^{-3.36t}\cos(11.52t - \delta)$

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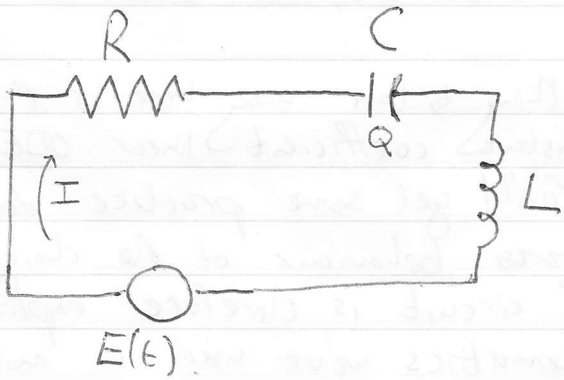
So the solution decays like $e^{-3.36t}$, and has quasi-frequency $\omega = 11.52$.

Contrast this with the natural frequency of $\omega_0 = 12$. – the damped oscillations are a bit slower as expected.
In fact, $\omega/\omega_0 = \frac{11.52}{12} = 0.96$,

while $1 - \frac{\gamma^2}{8km} = 1 - \frac{0.672^2}{8 \cdot 14.4 \cdot 0.1} = 0.9608$, so $1 - \frac{\gamma^2}{8km}$ is a good approximation to ω/ω_0 in this case.

2.5.9 Electric Circuits

2nd-order DE's also appear in electrical circuit modeling. Consider the following setup:



A closed circuit containing:

- A resistor of resistance R (measured in ohms Ω)
- A capacitor of capacitance C (measured in Farads F)
- An inductor of inductance L (measured in henrys H)
- An applied known voltage (impressed) $E(t)$ (measured in volts V).

We can construct a DE using Kirhoff's second Law: In a closed circuit the applied voltage equals the sum of the voltage drops across the rest of the components in the circuit.

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Thus $E(t)$ = (drop across inductor) + (drop across resistor) + (drop across capacitor).

Let • $Q(t)$ be the charge on the capacitor at time t (measured in ~~amps A~~ Coulombs C)
• $I(t)$ be the current in the circuit (measured in amperes A).

By the theory of circuits we have the following:

- $I(t) = \frac{dQ}{dt}$
- voltage across resistor $= IR$
- voltage across capacitor $= Q/C$
- voltage across inductor $= L \cdot \frac{dI}{dt}$.

Hence we get the DE $L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$, or using $I = \frac{dQ}{dt}$,

$$\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)}$$

Usually we have initial conditions $Q(t_0) = Q_0$, $Q'(t_0) = I(t_0) = I_0$.

The important thing to take away here is that this is just another 2nd-order constant coefficient linear ODE - which we know how to solve. You'll get some practice doing this in the homework, but the ~~theory is~~ behaviour of the charge on the capacitor in the above circuit is therefore explainable using exactly the same mathematics we've used to analyze mechanical vibrations.

And that's why abstraction is cool.