1. (10 points) Find the general solution to the following second-order differential equation:

$$y'' + 16y = e^{-t}.$$

Your answer should be a function with two undetermined constants in it.

We recall that the general solution to non-homogeneous differential equation is given by the sum of the general solution to the corresponding homogeneous equation, and a particular solution to the full nonhomogeneous DE.

First, the homogeneous equation: $y'' + 16y = 0$. This has characteristic equation $r + 16 = 0$, which has the purely imaginary roots $r = \pm 4i$. Consulting our knowledge of general solutions to homogenous systems, we deduce that the general solution to the homogeneous equation is

$$y = c_1 \cos(4t) + c_2 \sin(4t)$$

Now we find $Y(t)$, the particular solution to the full nonhomogeneous DE $y'' + 16y = e^{-t}$ using the method of undetermined coefficients. We observe that the forcing function e^{-t} is an exponential that is itself not a solution to the homogeneous DE, so we guess

$$Y(t) = Ae^{-t}$$

for some as-yet unknown value of A. Then $Y' = -Ae^{-t}$ and $Y'' = Ae^{-t}$, so plugging Y back into the left hand side of the DE we get

$$Y'' + 16Y = Ae^{-t} + 16Ae^{-t} = 17Ae^{-t}$$

We want this to be equal to the right hand side i.e. e^{-t}; the only way this can happen is if the coefficients match up, giving us the equation $17A = 1$. Solving yields $A = \frac{1}{17}$.

Hence the general solution to the nonhomogeneous DE, being the sum of $Y(t)$ and the general solution to the homogeneous equation, is

$$y = c_1 \cos(4t) + c_2 \sin(4t) + \frac{1}{17}e^{-t}$$

2. (10 points) In each part of this question you are given a function $y(t)$ which is the general solution to a constant-coefficient homogeneous 2nd-order differential equation. Write down the differential equation that that function satisfies. Your answer should be a DE in the form $ay'' + by' + cy = 0$ for some values of a, b and c.

   Each part is worth 2 points. You don't need to show your working to get full credit for this question.

   (a) $y(t) = c_1 e^{4t} + c_2 e^{-t}$

   This corresponds to a DE whose characteristic equation has roots $r = 4$ and $r = -1$. That is, the characteristic equation is $(r-4)(r+1) = r^2 - 3r - 4 = 0$. By comparing coefficients we see that the differential equation that $y(t)$ satisfies is therefore

$$y'' - 3y' - 4y = 0.$$

   (b) $y(t) = c_1 e^{\frac{t}{2}} \cos(t) + c_2 e^{\frac{t}{2}} \sin(t)$

   Both parts of the general solution have exponential growth with growth rate constant $\lambda = \frac{1}{2}$, and oscillation with radial frequency $\omega = 1$. We recognize that this comes from a DE whose characteristic equation has roots $r = \frac{1}{2} \pm i$. Hence the CE is
   $(r - \frac{1}{2} + i)(r - \frac{1}{2} - i) = (r - \frac{1}{2})^2 - (i)^2 = r^2 - r + \frac{5}{4} = 0$, so we get the DE

$$y'' - y' + \frac{5}{4} y = 0.$$

   Alternatively we could multiply everything by 4 to obtain the equally valid $4y'' - 4y' + 7y = 0$.

   (c) $y(t) = c_1 + c_2 t$

   This is a general linear equation. Since the double derivative equal of any linear function is zero, we have that $y(t)$ satisfies the DE

$$y'' = 0.$$

   (d) $y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

   We recognize that this is the general solution of a DE whose characteristic has repeated roots. We note that since e^{-3t} is a solution the repeated root must be $r = -3$. In other words $(r+3)^2 = r^2 + 6r + 9 = 0$ is the characteristic equation, so we get the DE

$$y'' + 6y' + 9y = 0.$$

3. **(10 total points)** A 5 kg mass is placed on a fairly smooth horizontal surface and attached to a horizontal spring. The spring exerts a force of $-\frac{y}{20}$ Newtons on the mass, where y is the displacement in meters of the mass relative to its equilibrium position. The surface subjects the mass to a frictional force of $-\frac{y'}{10}$ Newtons, where y' is the mass's velocity in ms^{-1}. Furthermore, the mass is subjected to an external force of $2\cos(\omega t)$ Newtons, where t is time measured in seconds, and ω is a constant.

   (a) (2 points) Write down a differential equation that describes the position of the mass at time t.

   All quantities stated in the problem description are in SI units. We therefore may take the object's mass to be $m = 5$, the spring constant to be $k = \frac{1}{20}$, the damping constant to be $\gamma = \frac{1}{10}$, and the forcing function to be $g(t) = 2\cos(\omega t)$. Thus the motion of the object is governed by the constant coefficient 2nd-order differential equation

$$5y'' + \frac{1}{10}y' + \frac{1}{20}y = 2\cos(\omega t)$$

   (b) (4 points) For which of the four values of ω below will the amplitude of the steady-state response be greatest? Circle your answer, and write a sentence or two justifying your choice.

$$\omega = 0 \qquad \omega = \frac{1}{10} \qquad \omega = 1 \qquad \omega = 10$$

   Recalled for forced oscillation in damped systems, resonance only occurs if friction isn't too big. To this effect we compute the dimensionless value

$$\Gamma = \frac{\gamma^2}{km} = \frac{0.1^2}{0.05 \cdot 5} = 0.04$$

   This is small, so we conclude that we will observe maximum resonance occurring when the forcing function frequency is a hair less than the natural frequency of the system.

   Now the natural frequency is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.05}{5}} = 0.1$, which is exactly one of the forcing function frequencies listed above. We therefore conclude that of the four listed angular frequencies, $\omega = \frac{1}{10}$ will result in a forced response of greatest amplitude.

   (c) (4 points) For the value of ω you found above, compute the amplitude R of the steady-state response.

   This may be obtained by finding the particular solution $Y(t)$ to the DE $5y'' + \frac{1}{10}y' + \frac{1}{20}y = 2\cos\left(\frac{1}{10}t\right)$ (e.g. by guessing $Y = A\cos(t/10) + B\sin(t/10)$, and then finding the resulting amplitude by using the conversion formula $R^2 = A^2 + B^2$). However, this exact type of case was covered in class, so we may quote the formula for the forced response amplitude we derived there:

$$R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}$$

   For us $F_0 = 2, k = \frac{1}{20}, m = 5, \omega = \frac{1}{10}$ and $\gamma = \frac{1}{10}$; we note that $k - m\omega^2 = 0$. Thus, plugging the values above into the formula for R we get

$$R = 200 \text{ meters.}$$

4. (10 total points) Consider the differential equation

$$t^2y'' - 6ty' + 6y = 0, \quad t > 0$$

One can check that $y_1(t) = t$ is a solution to this DE.

(a) (10 points) Use the method of reduction of order or any other method of your choosing to find a second solution $y_2(t)$ to the DE that is linearly independent from $y_1(t)$. Your answer should be a function with no undetermined constants in it.

This equation can be solved by letting $x = \ln(t)$ and recasting the above as a differential equation in x and y; however we will solve it using the method of reduction of order. Let $y = v(t)y_1(t) = vt$ solve the differential equation, where v is an as-yet undetermined function of t. Symbolically differentiating we get

$$y' = v't + v \qquad \text{and} \qquad y'' = v''t + 2v'$$

Since y is a solution to the differential equation, so we must have that $t^2y'' - 6ty' + 6y = 0$. Now

$$\begin{aligned} t^2y_2'' - 6ty_2' + 6y_2 &= t^2(v''t + 2v') - 6t(v't + v) + 6vt \\ &= v''t^3 + 2v't^2 - 6v't^2 - 6vt + 6vt \\ &= t^2(tv'' - 4v') \end{aligned}$$

This must equal zero; since $t > 0$ we conclude that $tv'' - 4v' = 0$. This is a second-order linear differential equation in v – or better yet, a first-order linear differential equation in v'. So let $w = v'$; then after dividing through by t we have

$$w' - \frac{4}{t}w = 0$$

This equation is linear (and also separable), so we can solve it in the standard way we solve linear first-order DEs. The integrating factor is

$$\mu(t) = e^{\int -\frac{4}{t}\,dt} = e^{-4\ln t} = t^{-4}$$

(we may omit the absolute value signs, since we know $t > 0$). The solution is then

$$w = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right) = t^4\left(\int 0\,dt + C\right) = Ct^4$$

And $w = v'$, so antidifferentiating we obtain $v = c_1t^5 + c_2$, where $c_1 = \frac{C}{5}$ and c_2 are undetermined constants. We conclude that

$$y(t) = v(t)\cdot t = c_1t^6 + c_2t$$

Now this is in fact the general solution to the differential equation, so we must choose values of c_1 and c_2 to get a fundamental basis solution. We might as well choose $c_2 = 0$, since $y_1(t) = t$ will capture all parts of the general solution that are multiples of t; and we can choose $c_1 = 1$ for convenience. Hence we arrive at a second solution to the differential equation:

$$y_2(t) = t^6$$

5. (10 total points + 3 bonus points) A series circuit contains an inductor of inductance 0.2 henrys and a capacitor of capacitance 2×10^{-3} Farads. Resistance in the circuit is negligible, and the both current in the circuit and charge on the capacitor are initially zero. Starting at time $t = 0$ an external voltage of $0.49\cos(48t)$ volts is applied on the circuit.

   (a) (10 points) The capacitor is rated to hold a maximum charge of 0.03 Coulombs. Is the circuit safe, or will the capacitor fail? Be sure to justify your conclusion numerically.

   The capacitor will fail if at any point in time the charge on it exceeds 0.03 Coulombs; thus we must ascertain if the amplitude of the charge on the capacitor is less than 0.03 Coulombs or not. To this end, we must first establish an initial value problem satisfied by the charge on the capacitor.

   This system falls into the category of series circuits that we studied in class: those containing a resistor of resistance R, a capacitor of capacitance C, an inductor of inductance L, and an external voltage given by $E(t)$. For this model we derived the DE

$$LQ'' + RQ' + \frac{1}{C} \cdot Q = E(t)$$

   where $Q(t)$ is the charge in Coulombs on the capacitor at time t. For us $L = 0.5 = \frac{1}{5}$, $R = 0$, $C = 2 \times 10^{-3} = \frac{1}{500}$, and $E(t) = 0.49\cos(48t)$. furthermore we are given the initial conditions $Q(0) = Q'(0) = 0$. Thus the charge on the capacitor obeys the initial value problem

$$\frac{1}{5}Q'' + 500Q = \frac{49}{100}\cos(48t), \qquad Q(0) = 0, \;\; Q'(0) = 0.$$

   Now the natural frequency ω_0 of the system is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{\frac{1}{5}}} = 50 \;\text{ rad s}^{-1}$$

   This is precisely the case we examined in class where we observe beats occurring: since the forcing function's frequency is almost but not precisely equal to the natural frequency of the system, the solution $Q(t)$ will oscillate back and forth with a slowly varying amplitude. The IVP above is certainly solvable in the usual manner; however, since we derived it in class we may quote the formula for the solution to the system:

$$Q(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \cdot \sin\left(\frac{\omega_0 - \omega}{2} \cdot t\right)\right] \sin\left(\frac{\omega_0 + \omega}{2} \cdot t\right)$$

   where F_0 is the amplitude of the forcing function, ω its radial frequency, and m the coefficient in front of the Q'' term in the DE. For us $F_0 = \frac{49}{100}$, $\omega = 48$, $\omega_0 = 50$ and $m = \frac{1}{5}$; plugging these into the formula gives us

$$
\begin{aligned}
Q(t) &= \left[\frac{2 \cdot \frac{49}{100}}{\frac{1}{5}(50^2 - 48^2)} \cdot \sin\left(\frac{50-48}{2} \cdot t\right)\right] \sin\left(\frac{50+48}{2} \cdot t\right) \\
&= \left[\frac{49}{10 \cdot 196} \cdot \sin(t)\right] \sin(49t) \\
&= \frac{1}{40} \cdot \sin(t) \cdot \sin(49t)
\end{aligned}
$$

We note that since the $sin(t)$ and $\sin(49t)$ terms never exceed 1 in magnitude, $Q(t)$ will only ever be at most $\frac{1}{40} = 0.025$ in magnitude. That is, the charge on the capacitor never exceeds the stated max of 0.03 Coulombs.

We conclude that the capacitor will not fail, and that the circuit is safe.

(b) (Bonus: 3 points) If the capacitor could hold any amount of charge (i.e. it never fails), compute or estimate the maximum amount of current that occurs in the circuit given the above setup.

We could certainly attempt to solve this exactly by taking the above formula, differentiating it w.r.t. t to get the current $I(t)$, find the critical points, locate the one which corresponds to the maximum amplitude point, and then plug that time value back in to get the amplitude at that time. However, this is drawn-out and complicated, and there is an easier way to get a bound on the current:

Recall the other formula for the solution to the IVP:

$$
\begin{aligned}
Q(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cdot [\cos(\omega t) - \cos(\omega_0 t)] \\
&= \frac{1}{80}[\cos(50t) - \cos(48t)]
\end{aligned}
$$

where the constants are as described above. Since $I(t) = \frac{dQ}{dt}$, we thus have that

$$
I(t) = \frac{1}{80}[-50\sin(50t) + 48\sin(48t)] = \frac{3}{5}\sin(48t) - \frac{5}{8}\sin(50t)
$$

Since $\sin(48t)$ and $\sin(50t)$ oscillate between -1 and 1 but at different frequencies, we must have that $I(t)$ varies between $\frac{3}{5} + \frac{5}{8} = \frac{49}{40}$ and the negative of that. Furthermore, it's reasonable to conclude that at some point they will be in phase and the amplitudes will add up exactly.

We conclude that the maximum current in the circuit is $\frac{49}{40} = 1.225$ amps.