Math 307 E - Summer 2011
Mid-Term Exam
July 20, 2011

Name: ______________________ Student number: ______________

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	3*	
Total	60+	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 60 minutes to complete the exam.

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1. Solve for $y(t)$:

(a) (5 points)

$$y' = 1 + t + y + ty, \qquad y(0) = 0.$$

This is linear and separable. To solve by separating, we factor $1 + t + y + ty = (1 + t)(1 + y)$. Then the equation reduces to

$$\frac{dy}{1 + y} = (1 + t)dt.$$

Integrating gives $\ln(y + 1) = t + t^2/2 + C$. Since $y(0) = 0$, we have that $C = 0$. Thus

$$y = e^{t+t^2/2} - 1.$$

(b) (5 points)

$$ty' = 1 - y. \qquad y(1) = 0, \quad t > 0.$$

This is linear and separable. To solve using linearity, we can rewrite the equation as

$$y' + \frac{1}{t}y = \frac{1}{t}.$$

Then the integrating factor $\mu = e^{\int t^{-1}dt} = t$, so that the equation is

$$\frac{d}{dt}(ty) = 1.$$

Integrating, we see that $ty = t + C$, and further that $C = -1$ since $y(1) = 0$. Finally,

$$y = 1 + \frac{-1}{t}.$$

2. Find the general solution to:

(a) (5 points)

$$x^2\frac{dy}{dx} = (x-y)^2 + 3xy, \quad x, y > 0.$$

This equation is homogeneous – rewriting we have

$$\frac{dy}{dx} = \frac{x^2+y^2+xy}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

Applying the substitution $u = \frac{y}{x}$, so that $\frac{dy}{dx} = u + x\frac{du}{dx}$, we get

$$u + x\frac{du}{dx} = 1 + u + u^2 \qquad \text{or} \qquad x\frac{du}{dx} = 1 + u^2.$$

Separating yields

$$\frac{du}{u^2+1} = \frac{1}{x}dx.$$

Integrating gives $\tan^{-1}(u) = \ln(x) + C$, so that $y = x\tan(\ln(x) + C)$.

(b) (5 points)

$$e^{y-x}(\cos(x) - \sin(x)) - \sin(x) + \sin(x)e^{y-x}\frac{dy}{dx} = 0.$$

Hint: Integrate $N(x, y)$ with respect to y.

We suspect the equation is exact; let's check! First, $M(x, y) = e^{y-x}(\cos(x) - \sin(x)) - \sin(x)$, while $N(x, y) = \sin(x)e^{y-x}$. Then

$$M_y = e^{y-x}(\cos(x) - \sin(x)) = N_x,$$

so it is exact. Thus we need to find the potential function $\phi(x, y)$. We know that $\phi_y(x, y) = N(x, y)$, or in otherwords

$$\phi(x, y) = \int N(x, y)dy + h(x) = \int e^{y-x}\sin(x)dy + h(x) = e^{y-x}\sin(x) + h(x).$$

Secondly, we know that $\phi_x(x, y) = M(x, y)$, so that

$$\frac{d}{dx}\left(e^{y-x}\sin(x)\right) + h'(x) = e^{y-x}(\cos(x) - \sin(x)) - \sin(x).$$

This implies that $h'(x) = -\sin(x)$; thus $h(x) = \int -\sin(x)dx = \cos(x)$. Finally, $\phi(x, y) = e^{y-x}\sin(x) + h(x) = e^{y-x}\sin(x) + \cos(x)$, so the solution is given implicitly by

$$e^{y-x}\sin(x) + \cos(x) = C.$$

(In this case, we could also solve for y explicitly:

$$y = x + \ln(C\csc(x) - \cot(x)).$$

3. A tank initially contains 9 gal of water with 5 lb of salt in solution. A solution containing a constant concentration of γ lb salt per gallon runs into the tank at a rate of 1 gal/min; the well-mixed solution drains from the tank at the rate of γ^2 gal/min.

   (a) (5 points) Set up the initial value problem that models $Q(t)$, the amount of salt in pounds at time t for $0 \leq t < \frac{9}{\gamma^2}$.

   The volume function $vol(t) = 9 + t - \gamma^2 t = 9 + (1-\gamma^2)t$, since each minute 1 gallon enters while γ^2 gallons leave. Thus the concentration function $c(t) = \frac{Q(t)}{vol(t)} = \frac{1}{9+(1-\gamma^2)t}Q(t)$. Finally,

$$\frac{dQ}{dt} = \gamma - \gamma^2 \cdot c(t) = \gamma - \frac{\gamma^2}{9+(1-\gamma^2)t}Q(t),$$

   with the initial condition that $Q(0) = 5$.

   (b) (5 points) Solve this linear differential equation for $Q(t)$ using an integrating factor.

   From the equation above, we have that $p(t) = \frac{\gamma^2}{9+(1-\gamma^2)t}$, so

$$\int p(t)dt = \frac{\gamma^2}{1-\gamma^2}\ln(9+(1-\gamma^2)t).$$

   Thus the integrating factor $\mu(t) = e^{\int p(t)dt} = (9+(1-\gamma^2)t)^{\frac{\gamma^2}{1-\gamma^2}}$. The differential equation is then

$$\frac{d}{dt}\left((9+(1-\gamma^2)t)^{\frac{\gamma^2}{1-\gamma^2}}Q(t)\right) = \gamma(9+(1-\gamma^2)t)^{\frac{\gamma^2}{1-\gamma^2}}.$$

   Integrating gives us that

$$\begin{aligned}(9+(1-\gamma^2)t)^{\frac{\gamma^2}{1-\gamma^2}}Q(t) &= \frac{\gamma}{1-\gamma^2}(9+(1-\gamma^2)t)^{\frac{\gamma^2}{1-\gamma^2}+1}\cdot\frac{1}{\frac{\gamma^2}{1-\gamma^2}+1}+C\\ &= \frac{\gamma}{1-\gamma^2}(9+(1-\gamma^2)t)^{\frac{1}{1-\gamma^2}}\cdot(1-\gamma^2)+C\\ &= \gamma(9+(1-\gamma^2)t)^{\frac{1}{1-\gamma^2}}+C.\end{aligned}$$

   Thus

$$\begin{aligned}Q(t) &= \gamma(9+(1-\gamma^2)t)^{\frac{1}{1-\gamma^2}-\frac{\gamma^2}{1-\gamma^2}}+C(9+(1-\gamma^2)t)^{-\frac{\gamma^2}{1-\gamma^2}}\\ &= \gamma(9+(1-\gamma^2)t)+C(9+(1-\gamma^2)t)^{-\frac{\gamma^2}{1-\gamma^2}}.\end{aligned}$$

   The initial condition gives that $C = (5-9\gamma)9^{\frac{\gamma^2}{1-\gamma^2}}$.

4. Jill has a saving's account with a balance of $S(t)$ dollars at time t. Assume that her credit is outstanding, so that any negative balance is acceptable to her bank and represents a loan. Assume that Jill's account earns interest at an annual rate of r compounded continuously. Furthermore, assume that Jill continuously withdraws money from her account at a rate proportional to the cube of her balance, with proportionality constant r^3.

(a) (4 points) Write down a differential equation modeling $S(t)$.

$$\frac{dS}{dt} = rS - r^3S^3.$$

(b) (4 points) Find the equilibrium solutions, and classify as stable, unstable or semi-stable.

The equilibrium solutions occur when $rS - r^3S^3 = rS(1-r^2s^2) = rS(1-rS)(1+rS) = 0$. This occurs when $S = 0, \frac{1}{r}, -\frac{1}{r}$. If we graph this function, we see that it looks like a "negative cubic" with a negative slope at $S = \pm\frac{1}{r}$ and a positive slope at $S = 0$. Thus the two equilibrium solutions $S = \pm\frac{1}{r}$ are stable, while the solution $S = 0$ is unstable.

(c) (2 points) If Jill has saved any money at $t = 0$, how much money do you expect her to have in her account for large values of t if $r = 1\%$?

Since $0 < S$ at $t = 0$, we expect S to increase (if less than $\frac{1}{r}$) towards the equilibrium solution, $S = \frac{1}{r} = 100$; likewise if $S > 100$ we expect S to decrease towards this stable equilibrium. So we expect her to have $\$100$ in the long run.

5. Find the general solution to the following second-order differential equations:

(a) (3 points)

$$\frac{1}{2}y'' + 2y' + \frac{5}{2}y = 0.$$

$$\frac{1}{2}r^2 + 2r + \frac{5}{2} = 0 \quad \Longrightarrow \quad r = -2 \pm i.$$

Then $e^{rt} = e^{-2t} \cdot e^{it} = e^{-2t}(\cos(t) + i\sin(t))$, and the real and imaginary parts are $e^{-2t}\cos(t)$ and $e^{-2t}\sin(t)$. The general solution is thus

$$y = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t).$$

(b) (3 points)

$$16y'' + 8y' + y = 0.$$

$$16r^2 + 8r + 1 = 16(r + \frac{1}{4})^2 = 0 \quad \Longrightarrow \quad r = -1/4.$$

Thus $e^{rt} = e^{-t/4}$. Since $r = -1/4$ is a double root, the other solution is $te^{-t/4}$, and the general solution is

$$y = c_1 e^{-t/4} + c_2 te^{-t/4}.$$

(c) (4 points)

$$y'' - y' - 2y = 0.$$

$$r^2 - r - 2 = (r-2)(r+1) = 0 \quad \Longrightarrow \quad r = -1, 2.$$

Thus the fundamental solutions are e^{-t} and e^{2t}, and the general solution is

$$y = c_1 e^{-t} + c_2 e^{2t}.$$

6. Consider the differential equation

$$y'' + 2y' + \left(1 - \frac{3}{4t^2}\right) y = 0 \qquad t > 0.$$

Following the reduction of order technique, assume that $y_1(t)$ is a solution and guess that $y_2(t) = w(t) \cdot y_1(t)$.

(a) (3 points) Show that if $y_2(t)$ is a solution, then $w(t)$ must satisfy

$$\frac{w''}{w'} = -2\left(\frac{y_1'}{y_1} + 1\right).$$

Hint: You do not need to know what $y_1(t)$ is, only that it is a solution.

$$y_2' = w'y_1 + wy_1' \qquad y_2'' = w''y_1 + 2w'y_1' + wy_1''.$$

Plugging this into the differential equation, we get

$$w''[y_1] + w'[2y_1' + 2y_1] + w[y_1'' + 2y_1' + \left(1 - \frac{3}{4t^2}\right) y_1] = 0.$$

The coefficient of w is zero, since y_1 is a solution, so the equation is actually

$$w''y_1 = -w'[2y_1' + 2y_1],$$

which is equivalent to the formula above.

(b) (7 points) It is true (and you may assume) that $y_1(t) = t^{3/2}e^{-t}$ satisfies the equation. Find $w(t)$ using the formula from part (a); write down the general solution as a linear combination of $y_1(t)$ and $y_2(t)$.
We calculate that $\frac{y_1'}{y_1} = \frac{3}{2t} - 1$. Thus

$$\begin{aligned}
\text{formula:} \quad & \frac{w''}{w'} = -2\left(\frac{3}{2t} - 1 + 1\right) = -\frac{3}{t} \\
\text{integrate:} \quad & \ln(w') = -3\ln(t) \\
\text{simplify:} \quad & w' = t^{-3} \\
\text{integrate:} \quad & w = -\frac{1}{2}t^{-2}.
\end{aligned}$$

It follows that $y_2 = w \cdot y_1 = -\frac{1}{2}t^{-2}t^{3/2}e^{-t} = -\frac{1}{2}t^{-1/2}e^{-t}$, so that

$$y(t) = c_1 t^{3/2}e^{-t} + c_2 t^{-1/2}e^{-t}.$$

7. (3 bonus points) Assume that $y_1(t) = e^{t^2/2}\cos(t)$ is a solution to the differential equation

$$y'' - 2ty' + t^2 y = 0.$$

Find another independent solution. (Just guessing will not earn all points.)
Hint: Derive a similar formula as in problem 6 part (a), or use a trick.

First, we use a formula similar to that of problem 6 part (a). We get that

$$\frac{w''}{w'} = -2\left(\frac{y_1'}{y_1}\right) + 2t.$$

We compute that

$$\frac{y_1'}{y_1} = t - \tan(t),$$

so that

$$\begin{aligned} \text{formula:} \quad & \frac{w''}{w'} = -2\left(t - \tan(t)\right) + 2t = 2\tan(t) \\ \text{integrate:} \quad & \ln(w') = -2\ln(\cos(t)) \\ \text{simplify:} \quad & w' = \cos(t)^{-2} = \sec^2(t) \\ \text{integrate:} \quad & w = \tan(t). \end{aligned}$$

It follows that $y_2 = w \cdot y_1(t) = \tan(t)\cos(t)e^{t^2/2} = \sin(t)e^{t^2/2}$.

Alternatively, you might have guessed that $y_1(t)$ was the real part of the complex valued function $y_c = e^{t^2/2+it}$. We want to see if this complex-valued function is actually a solution. So we compute $y_c' = (t+i)y_c$ and $y_c'' = (t+i)^2 y_c + y_c = [t^2 + 2it]y_c$. Plugging this into the equation, we get

$$[t^2 + 2it]y_c - 2t \cdot [t+i]y_c + t^2 \cdot y_c = [t^2 + 2it - 2t^2 - 2ti + t^2]y_c = 0 \cdot y_c = 0.$$

Since y_c and $y_1(t)$ are both solutions, their difference $y_d = y_c - y_1(t) = ie^{t^2/2}\sin(t)$ is also a solution. Multiplying y_d by the constant $-i$ we get another solution, which is

$$y_2 = -iy_d = e^{t^2/2}\sin(t).$$