Fri Feb 7 MATH 307A LECTURE 12

§2.4 NON-HOMOGENEOUS LINEAR EQUATIONS

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We will be looking at non-homogeneous linear 2nd-order DEs of the form

$$y'' + p(t)y' + q(t)y = g(t) \qquad (NH)$$

We'll also talk about the DE

$$y'' + p(t)y' + q(t)y = 0 \qquad (H),$$

the homogeneous equation corresponding to NH, when it comes to finding a solution to NH.

Definition 2.4.1: A (2nd-order) linear differential operator L is a "function of functions" which takes as input a function $y(t)$ (that is at least twice differentiable) and outputs a linear combination of y, y' and y'', i.e. another function of t.

Example 2.4.2 a) Let $L = \frac{d^2}{dt^2} + 2\frac{d}{dt} - 3$ and $y = e^{-t}$.

Then $L[y] = \left(\frac{d^2}{dt^2} + 2\frac{d}{dt} - 3\right)y = \frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y$

$= e^{-t} - 2e^{-t} - 3e^{-t}$

$= -4e^{-t}$

b) Let $L = t\frac{d^2}{dt^2} - \frac{1}{t}\frac{d}{dt} + 2$ and $y = 2t^2 + 1$

Then $L[y] = ty'' - \frac{1}{t}y' + 2y$

$= 4t - 4 + 4t^2 + 2$

$= 4t^2 + 4t - 2$.

Note 2.4.3: Linear operators are linear:

- $L[y_1 + y_2] = L[y_1] + L[y_2]$
- $L[cy] = c \cdot L[y]$ for any constant c

Hence to solve the NH: DE $y'' + p(t)y' + q(t) = g(t)$, we seek a function $y(t)$ s.t. $L[y] = g(t)$, where ~~L[y]~~ $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$.

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Theorem 2.4.4: The general solution to the ~~DE~~ nonhomogeneous DE $y''+p(t)y'+q(t) = g(t)$ is given by $y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$, where
- $Y(t)$ is any specific solution to the non-homogeneous DE
- y_1 & y_2 are a fundamental set of solutions to the homogeneous DE $y''+p(t)y'+q(t) = 0$.

Proof: Let $Y(t)$, $y_1(t)$ & $y_2(t)$ be as above, and let
$L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$.

Then by definition $L[Y] = Y'' + p(t)Y' + q(t)Y = g(t)$

And
$$L[c_1 y_1(t) + c_2 y_2(t)] = c_1 L[y_1(t)] + c_2 L[y_2(t)] = 0, \text{ as they solve } y''+p(t)y'+q(t)y=0.$$

Hence
$$L[c_1 y_1(t) + c_2 y_2(t) + Y(t)] = L[c_1 y_1(t) + c_2 y_2(t)] + L[Y] = 0 + g(t) = g(t).$$

So $c_1 y_1(t) + c_2 y_2(t)$ is a general solution to the NH DE. Furthermore, we know general solutions to 2nd-order linear DEs have 2 free constants, so we know any solution to the NH DE can be written in this form. □

Tactic 2.4.5: We therefore have the following tactic for finding the general solution to the nonhomogeneous DE $y''+p(t)y'+q(t)y = g(t)$:

1) Find a general solution $c_1 y_1(t) + c_2 y_2(t)$ to the homogeneous DE $y''+p(t)y'+q(t) = 0$.
2) Find a specific solution $Y(t)$ to the nonhomogeneous DE.
3) The general solution to the NH DE is then $y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

Good news! We already know how to do step 1) (~~in most~~ or at least when $p(t)$ & $q(t)$ are constant or easy functions of t). Therefore we'll spend the rest of this section focusing on how to find specific solutions to nonhomogeneous linear 2nd-order DEs.

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The Method of Undetermined Coefficients

Basically, this is an intelligent application of the 'guess and check' method.

Idea: • Guess a function $Y = f(t)$ 'similar to' the function $g(t)$, ~~multiplied by~~ involving some constant(s), as yet undetermined.
• Plug in $Y = f(t)$ into the DE and see what values of those constants make the output equal $g(t)$.

(Upsides: Straightforward. Downsides: Only works for limited range of linear DEs.)

Example 2.4.6 Find a particular solution to $y'' - 3y' - 4y = 3e^{2t}$

Solution: "Exponential function duplicate themselves under differentiation", so let's guess $Y = Ae^{2t}$ for some constant A.

Then $Y'' - 3Y' - 4Y = 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t}$
$= -6Ae^{2t}$.

We want this to $= 3e^{-2t} \Rightarrow A = -\frac{1}{2}$ works.

Hence $Y(t) = -\frac{1}{2}e^{2t}$ is a specific solution to the DE.

Example 2.4.7: Find a particular solution to $y'' - 3y' - 4y = 2\sin t$.

Solution: Try similar tactic to before: guess $Y = A\sin t$

Then $Y'' - 3Y' - 4Y = -A\sin(t) - 3A\cos(t) - 4A\sin(t) = 2\sin(t)$

$\Rightarrow (2 + 5A)\sin(t) + 3A\cos(t) = 0$

This must hold for all t, so can set specific t and solve for A

i.e. @ $t = 0 \Rightarrow 3A = 0 \Rightarrow A = 0$

@ $t = \frac{\pi}{2} \Rightarrow 2 + 5A = 0 \Rightarrow A = -\frac{2}{5}$

Contradiction! So no A works in this case.

We conclude therefore that our guess for the type of solution function wasn't correct.

How can we fix this? Observe that when we plugged Y into the DE, $\sin(t)$ & $\cos(t)$ functions appeared. Perhaps $Y(t)$ should

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therefore be some combination of sin(t) & cos(t)?

$\Rightarrow$ New guess: $Y(t) = A\sin(t) + B\cos(t)$.

Then $Y' = A\cos(t) - B\sin(t)$

& $Y'' = -A\sin(t) - B\cos(t)$.

Hence
$$\begin{aligned} Y'' - 3Y' - 4Y &= (-A\sin(t) - B\cos(t)) - 3(A\cos(t) - B\sin(t)) \\ &\quad - 4(A\sin(t) + B\cos(t)) \\ &= (-A + 3B - 4A)\sin(t) + (-B - 3A - 4B)\cos(t) \\ &= (-5A + 3B)\sin(t) + (-3A - 5B)\cos(t) \end{aligned}$$

We require this to $= 2\sin(t)$

Hence we must solve
$$-5A + 3B = 2$$
$$-3A - 5B = 0$$

Linear system of equations in 2 variables – we can solve!

$\Rightarrow$ Get $A = -\frac{5}{17}$, $B = \frac{3}{17}$.

Hence
$$Y(t) = -\tfrac{5}{17}\sin(t) + \tfrac{3}{17}\cos(t)$$
is a specific solution to the DE.

<u>Note 2.4.8</u>: We can use the same methodology when $g(t)$ is a polynomial. For example to find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$, guess $Y = At^2 + Bt + C$, then plug back in to the DE and solve for A, B & C.

In general, we can also use the same tactic for products of polynomials, exponentials and for sin & cos functions.

<u>Example 2.4.9</u>: Find a particular solution to
$$y'' - 3y' - 4y = -8e^t\cos 2t$$

<u>Solution</u>: Going by the previous example, guess
$$Y(t) = e^t(A\cos 2t + B\sin 2t).$$

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