Mon 10 March MATH307A LECTURE 22

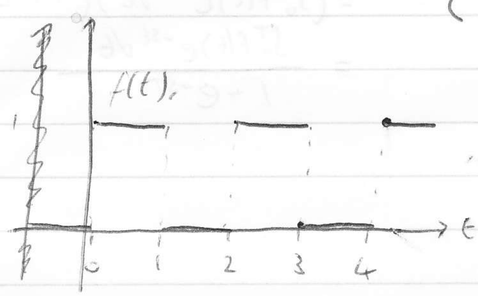
LAPLACE & DISCONTINUOUS FUNCTIONS, PART 2 (BOYCE 6.3)

An important class of functions which may be discontinuous but are still nicely Laplace transformable are the periodic functions

Definition 3.3.10 A function $f(t)$ is periodic if there is a $T > 0$ such that $f(t+T) = f(t)$ for all t. T is called the period of f.

Example 3.3.11 The square wave $f(t) = \begin{cases} 1, & 2n \le t < 2n+1 \\ 0, & 2n+1 \le t < 2n+2 \end{cases}$, $n \in \mathbb{Z}$.

is periodic. Graph:



Has period 2.

We find $\mathcal{L}[f(t)]$.

Observe
$$\begin{aligned}
\mathcal{L}[f] &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \int_0^1 e^{-st}\,dt + \int_2^3 e^{-st}\,dt + \int_4^5 e^{-st}\,dt + \dots \\
&= -\tfrac{1}{s}e^{-st}\Big|_0^1 + -\tfrac{1}{s}e^{-st}\Big|_2^3 + -\tfrac{1}{s}e^{-st}\Big|_4^5 + \dots \\
&= \tfrac{1}{s}\left[e^{-st}\Big|_1^0 + e^{-st}\Big|_3^2 + e^{-st}\Big|_5^4 + \dots \right] \\
&= \tfrac{1}{s}\left[1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - e^{-5s} + \dots \right] \\
&= \tfrac{1}{s}\left(1 + (-e^{-s})^1 + (-e^{-s})^2 + (-e^{-s})^3 + \dots \right) \\
&= \frac{1}{s} \cdot \frac{1}{1+e^{-s}} \quad \text{by geometric series formula.}
\end{aligned}$$

Thus $\boxed{\mathcal{L}[f] = \frac{1}{s(1+e^{-s})} = \frac{e^s}{s(e^s+1)}}$.

□.

And in general, we have a nice formula for the Laplace transform of a periodic function in terms of a (definite) integral.

Theorem 3.3.12: If $f(t)$ is periodic with period T, then
$$\boxed{\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}}$$

PTO.

Proof: $\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}\,dt$

$$= \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \int_{2T}^{3T} f(t)e^{-st}\,dt + \dots$$

$$= \int_0^T f(t)e^{-st}\,dt + \int_0^{T} f(t+T)e^{-s(t+T)}\,dt + \int_0^T f(t+2T)e^{-s(t+2t)}\,dt + \dots$$

using the transform of integration variable $t \longmapsto t+n\cdot T$

$$= \int_0^T f(t)e^{-st}\,dt + \int_0^T f(t)e^{-s(t+T)}\,dt + \int_0^T f(t)e^{-s(t+2t)}\,dt + \dots$$

using periodicity $f(t+nT) = f(t)$,

$$= \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^T f(t)e^{-st}\,dt + e^{-s\cdot 2T}\int_0^T f(t)e^{-st}\,dt + \dots$$

$$= \left(\int_0^T f(t)e^{-st}\,dt\right)\left(1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \dots\right)$$

$$= \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}$$ again by geometric series formula.

$\square$.

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§3.4 : DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS FORCING FUNCTIONS (BOYCE 6.4)

We now use Laplace to solve a DE with a discontinuous forcing function.

Example 3.4.1: Find the solution to the IVP

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $g(t) = \begin{cases} 0, & 0 \le t < 5 \\ 1, & 5 \le t < 20 \\ 0, & 20 \le t. \end{cases}$

This is the example we put up right at the beginning of chapter 3.

Solution: Note $g(t) = u_5(t) - u_{20}(t)$.
Let $\phi(t)$ solve the DE, and let $\Phi(s) = \mathcal{L}[\phi(t)]$
Hence taking Laplace transforms of both sides gives us:

$$2\mathcal{L}[\phi''] + \mathcal{L}[\phi'] + 2\mathcal{L}[\phi] = \mathcal{L}[u_5(t)] - \mathcal{L}[u_{20}(t)]$$
$$\Rightarrow 2(s^2\Phi - s\phi(0) - \phi'(0)) + (s\Phi - \phi(0)) + 2\Phi = \frac{e^{-5s} - e^{-20s}}{s}$$
$$\Rightarrow \Phi(2s^2 + s + 2) = \frac{1}{s}(e^{-5s} - e^{-20s})$$

So $$\Phi = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}$$

So now all we need to do is find $\mathcal{L}^{-1}\left[\frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}\right]$

To do this we use the transformation rules we've already covered.
Let $H(s) = \frac{1}{s(2s^2+s+2)}$. Then $\Phi = e^{-5s}H(s) - e^{-20s}H(s)$.
Let $h(t) = \mathcal{L}^{-1}[H(s)]$
Hence
$$\phi(t) = \mathcal{L}^{-1}[\Phi] = \mathcal{L}^{-1}[e^{-5s}H(s)] - \mathcal{L}^{-1}[e^{-20s}H(s)]$$
$$= u_5(t)h(t-5) - u_{20}(t)h(t-20)$$
So we have described $\phi(t)$ in terms of the inverse Laplace transform of $\frac{1}{s(2s^2+s+2)}$.

PTO.

Now we find $\mathcal{L}^{-1}\left[\frac{1}{s(2s^2+s+2)}\right]$:

Partial fractions: $\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$.

Solving in the usual way we get $A=\frac{1}{2}$, $B=-1$, $C=-\frac{1}{2}$,

so $H(s) = \frac{1}{s(2s^2+s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}$

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left(\frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

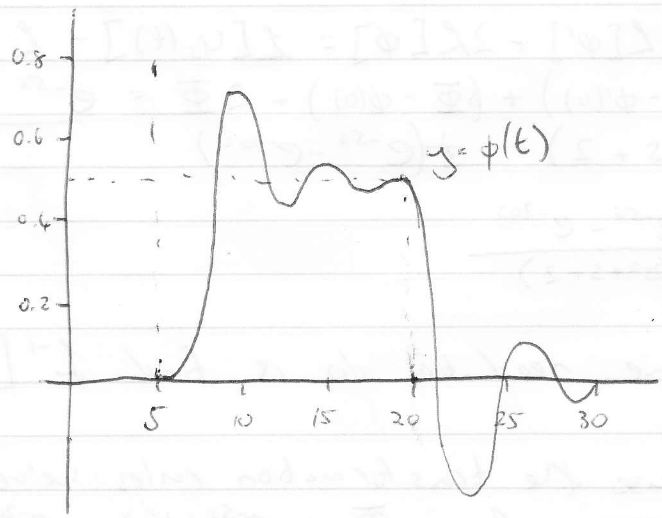
$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2} - \frac{1}{2\sqrt{15}}\cdot\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}$$

So $h(t) = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right)$

or $h(t) = \frac{1}{2}\left[1 - e^{-\frac{1}{4}t}\left(\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right)\right]$.

Hence $\phi(t) = u_5 h(t-5) - u_{20}h(t-2)$, with $h(t)$ as above.

Graph



We see then that $y=\phi(t)$ is 0 until $t=5$ as expected, then accelerates upwards and begins to stabilize at $\frac{1}{2}$, but before it does so fully drops back down to 0 and stabilizes there.