Homework 2

Total: 20 points
Due: Wed 08 Oct 2014 at the beginning of class

If a question is taken from the textbook, the refence is given on the right of the page.

FIRST ORDER MODELING

1. Boyce 2.3 Q5
A tank contains 100 gal of water in which 50 oz of salt is dissolved. Water containing a salt concentration if $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.
   (a) Find the amount of salt in the tank at any time.
   (b) Sketch a graph of the solution for a time period long enough so that you see the ultimate behaviour of the graph.
   (c) The long-time behaviour of the graph should look like an oscillation about a certain constant level. What is this level, and what is the amplitude of oscillation?

2. Boyce 2.3 Q9
A college student borrows \$8000 to buy a car. The lender charges interest at a rate of 10% per annum. Assume that the interest is compounded continuously and that the student makes payments continuously at a constant annual rate k.
   (a) Determine the payment rate k that is required to pay off the loan in 3 years.
   (b) What is the total amount the student pays over the 3-year period?

3. Some science fiction authors observe that the rate at which scientific advances are made by the human race are ever-increasing, and predict that as a result humanity will reach a point where our scientific progress becomes infinitely rapid. This point in time is known as the *Singularity*.
Suppose that we have ascertained that the total scientific knowledge of humanity has increased by 12.5% between the years of 2000 and 2010. Furthermore, suppose that we determine that the rate at which scientific knowledge advances is proportional to the *square* of our current scientific knowledge. When will the Singularity occur?

4. Boyce 2.3 Q17
Heat transfer from a body to its surroundings through radiation is accurately described by Stefan-Boltzmann's law, which dictates that the rate of heat loss between the object and its surroundings is proportional to the difference between the 4th powers of their respective temperatures. However, if the object is much hotter than its surroundings, the system can be approximated by the differential equation
$$\frac{dy}{dt} = -\alpha y^4,$$
where $y(t)$ is the temperature of the object in degrees Kelvin, and α is a proportionality constant dependant on the physical parameters of the object in question.
Suppose that a slug of molten steel with an initial temperature 2000 K is placed in a room whose temperature is controlled at 300 K, and suppose that $\alpha = 2.0 \times 10^{-12} K^{-3}/s$.
   (a) Determine the temperature of the metal slug for all t by solving the above differential equation.
   (b) Plot the graph of $y(t)$.
   (c) Determine the time when the slug has cooled to 600 K, twice the ambient temperature. It is interesting to note that even though we are using an approximite differential equation, up to this time the error from the solution to the true DE is less than 1%.

NB: More questions overleaf!

5. Boyce 2.3 Q23

A skydiver weighing 180 lb (including equipment) jumps from a plane at 5000 ft and falls vertically downward; after 10 seconds of free fall the skydiver's parachute opens. Assume that the force of air resistance acts proportional and opposite to velocity, with proportionality constants 0.75 when the parachute is closed and 12 when it is open respectively. Here v is measured in ft/sec.

(a) Find the speed of the skydiver when the parachute opens.

(b) Find the distance fallen when the parachute opens.

(c) What is the limiting velocity v_L after the parachute opens?

(d) Estimate to the nearest second how long the sky dive will take in its entirety i.e. from when the skydiver jumps from the plane until when they touch the ground.
Hint: In order to do so you'll end up needing to solve for t in an equation that looks like $t = A + Be^{-Ct}$ for some constants A, B and C (that you know). There is no way to solve for t explicitly in this equation, so you will have to find an approximate solution. To do this note that the Be^{-Ct} term is very small compared to the constant A term, so your answer should be very close to $t = A$.

6. Water hyacinth is a particularly aggressive invasive plant species in lakes in the southern US. One of the reasons is that it grows very quickly: under good conditions a population will grow at a rate proportional to its own size, with its biomass increasing by a factor of $e = 2.71828\ldots$ every 14 days. Suppose a water hyacinth population establishes itself in a large lake in Florida where conditions are close to ideal. When ecologists discover the population it has a biomass of 750kg.
Removal efforts begin immediately; however, because it takes some time to train local volunteers to remove the weed efficiently, the rate $R(t)$ at which water hyacinth can be removed from the lake is given by the function

$$R(t) = 600\left(1 - e^{-t}\right),$$

where t is measured in **weeks** since the beginning of the removal effort, and $R(t)$ is in kg/week.

(a) Establish an initial value problem and solve it to find an explicit formula for the biomass of water hyacinth in the lake at time t.

(b) Will efforts to completely remove the water hyacinth from the lake be successful? Justify your answer.

(c) If the answer to the above question is yes, estimate how many weeks it will take for the water hyacinth to be removed completely from the lake. If the answer to the above question is no, estimate how many weeks it will take for the water hyacinth to reach 10000kg biomass. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).
Hint: Again, you might not be able to solve explicitly for t in the relevant equation to answer this question exactly. As with Question 5, you will have to approximate: analyze the equation term-by-term. See which terms are of negligible size; if you discard those the equation should become solvable.