Homework 6

Total: 20 points
Due: Wed 19 November 2014 at the beginning of class

Remember to show all steps in your working. If a question is taken from the textbook, the refence is given on the right of the page.

1. **2nd-ORDER MODELING**

   (a) In the following two equations, determine ω_0, R and δ so as to write them in the form $y = R\cos(\omega_0 t - \delta)$:
      i. $y = -\cos(t) + \sqrt{3}\sin(t)$ — Boyce 3.7 Q2
      ii. $y = -2\cos(\pi t) - 3\sin(\pi t)$ — Boyce 3.7 Q4

   (b) Boyce 3.7 Q5
   A mass weighing 2 lb hangs from a spring, stretching it 6 inches. If the mass is pulled down an additional 3 in and then released, and there is no damping, determine the vertical position y of the mass at any time t. Also find the frequency, period and amplitude of the motion.

   [Hint: You will need to use the fact that that the 2 lb weight stretches the spring 6 inches when at rest to find the spring constant k. This is the point where the downward gravitational force on the mass exactly balances the upwards force imparted by the spring. You may take $g = 32\ ft/s^2$, and remember that in imperial units mass m is given by $m = w/g$, where w is the weight of the object in pounds. If you need extra guidance on this question, Boyce section 3.7 has a number of similar examples, and details the method on how to set up the DE.]

   (c) A 2 kg block is placed on a smooth surface attached to a horizontally-acting spring with spring constant $k = 9\,\frac{1}{16}$ kg/s^2. We have seen in class that the displacement in meters $y(t)$ of the block from its rest position is then modeled by the differential equation

   $$2y'' + \frac{145}{16}\,y = 0.$$

   Suppose that the surface is not actually frictionless, but instead imparts a force of $-\gamma v$ on the block, where v is the horizontal velocity of the block, and γ is a constant.
      i. The block is released from a position of $y = 0.5$ m with zero initial velocity. Find the position of the block for all time $t \geq 0$ if $\gamma = 0.5$ kg/s.
      ii. Plot a graph of the above solution. Find the quasi-period of the oscillations of the block, and a time t_0 for which $|y(t)| < 0.05$ m for all $t > t_0$.
      iii. Given the same starting conditions as above, what is the smallest value of γ which will result in the block never crossing it rest position?

   (d) Boyce 3.7 Q8
   A series circuit has a capacitor of capacitance 0.25×10^{-6} F, an inductor of inductange 1 H, and negligible resistence. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t.

   (e) Boyce 3.7 Q12
   A series circuit contains a capacitor of 10^{-5} F, an inductor of 0.2 H, and a resistor of $3 \times 10^2\ \Omega$. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

(f) Boyce 3.7 Q13

A certain vibrating system satisfies the differential equation

$$y'' + \gamma y' + y = 0.$$

Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

(g) Boyce 3.7 Q24

The position of a vibrating system satisfies the initial value problem

$$\frac{3}{2}y'' + ky = 0, \qquad y(0) = 2, y'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3 respectively, determine the values of k and v.

2. **FORCED VIBRATIONS**

(a) In the following two problems, Use the compound angle formulae for $\cos(A+B)$ and $\cos(A-B)$ to rewrite the given expressions as the product of two trigonometric functions of different frequencies.

i. $\cos(9t) - \cos(7t)$ Boyce 3.8 Q1

ii. $\sin(3t) + \sin(4t)$ Boyce 3.8 Q4

(b) The vibration of strings of a string instrument can be thought as idealized spring-mass systems when the amplitude of vibration is small. Specifically, if $y(t)$ is the displacement of the center of the string from its position of rest, then y is governed by the differential equation

$$my'' + \gamma y' + ky = g(t),$$

where m is mass of the playable part of the string, γ the damping constant due to air resistence, k the spring constant arising from the elasticity of the spring, and $g(t)$ a given external forcing function.

Consider the A string on a double bass in an otherwise still room. When tuned correctly this string vibrates at a frequency of exactly 55 Hertz (i.e. cycles per second, **not** radians per second). Suppose that the playable part of the A string on a bass weighs 0.01 kg, and that friction is negligible.

i. What is the spring constant k in this situation?

ii. The string is initially stationary in its equilibrium position. Starting at time $t = 0$ a speaker in the room plays a loud tone at precisely 56 Hertz, subjecting the bass's A string to a force of $g(t) = \frac{\pi^2}{50}\cos(56 \cdot 2\pi \cdot t)$ Newtons, where t is in seconds. Formulate an initial value problem describing the motion of the string for all $t \geq 0$. Remember to state what units your variables are in.

iii. Solve the initial value problem you formulated above to find the position of the string at time t. Using the compound angle formulae for $\cos(A + B)$ and $\cos(A - B)$, write your answer in the form

$$y = [R\sin(\omega_1 t)] \cdot \sin(\omega_2 t),$$

where the $\sin(\omega_1 t)$ term oscillates much more slowly than the $\sin(\omega_2 t)$ term.

iv. Using your answer above, determine the maximum displacement of the center of the A string from its equilibrium position, and the cyclic frequency of the **beat** $\frac{\omega_1}{2\pi}$. If you were standing next to the bass, this is the frequency at which you'd hear the loudness of the A string's vibration oscillate over time.

(c) Boyce 3.8 Q5 & Q7

A mass weighing 4 lb hangs from a spring, stretching it 1.5 inches. The mass is given a positive displacement of 2 inches from its equilibrium position and released with no initial velocity. Assume that there is no damping and that the mass is acted on by an external force of $2\cos(3t)$ lb.

i. Formulate an initial value problem describing the motion of the mass. Remember to show your work, and state what units your variables are in.
ii. Solve the initial value problem to find the position of the mass at time t.
iii. Plot a graph of the solution.
iv. If the given external force is replaced by a force of $4\sin(\omega t)$ of frequency ω, find the value of ω for which resonance occurs.

(d) Boyce 3.8 Q6 & Q8

A mass of 5 kg hangs from a spring, stretching it 10 cm. The mass is acted on by an external force of $10\sin(\frac{1}{2}t)$ Newtons, and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. The mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s.

i. Formulate an initial value problem describing the motion of the mass. Remember to show your work, and state what units your variables are in.
ii. Solve the initial value problem to find the position of the mass at time t.
iii. Identify the transient and steady state parts of the solution.
iv. Plot the graph of the **steady state solution**.
v. If the given external force is replaced by a force of $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is a maximum.
vi. For the value of ω you have just found, calculate the amplitude R and the phase shift δ of the forced response.

(e) Boyce 3.8 Q16

A series circuit contains a capacitor of 0.25×10^{-6} F, an inductor of 1 H, and a resistor of 5×10^3 Ω. The initial charge on the capacitor is zero and there is no initial current. A 12 V battery is connected to the circuit and the circuit is closed at $t = 0$.

i. Find the charge on the capacitor at any time t.
ii. Determine the charge on the capacitor at $t = 0.0001$ s, $t = 0.001$ s and $t = 0.01$ s. Give your answer to four significant figures (e.g. 1.2345×10^{-7} C).
iii. What is the limiting charge on the capacitor as $t \to \infty$?

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