Fri 23 Jan    MATH307A LECTURE 8:

# §1.7: EULER'S METHOD    (BOYCE 2.7)

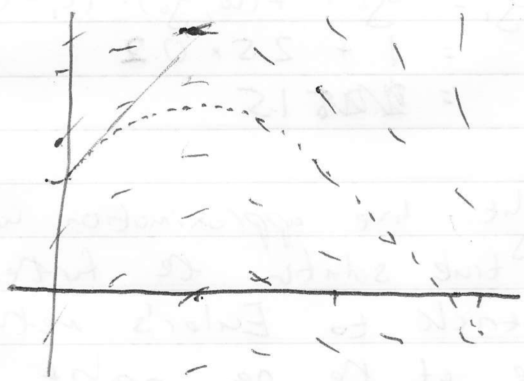
Euler's Method is a useful numerical technique for approximating the solution to a first-order DE when we can't solve the DE directly.

It is easiest to understand the method through an example:

**Example:** $\frac{dy}{dt} = \overbrace{3 - 2t - \frac{y}{2}}^{f(t,y)}, \quad y(0) = 1$

(This DE is quite readily solveable, but suppose for the moment that it isn't).

Now this DE isn't autonomous, so we can't use the techniques dissussed last lecture to get qualitative info on how solutions might behave.
However, we can still draw a slope field of $f(t,y)$.



DEMO: SLOPE FIELD.

This allows us to visually sketch the trajectory of our solution from its starting point, based on the "flow lines" i.e. the slopes given by $f(t,y)$ at regular points in the $(t,y)$ plane.

Now we know by the existence & uniqueness theorem that a solution to this IVP exists. We don't know what it is, but we do know what its slope is at $t_0$:
$\left.\frac{dy}{dt}\right|_{t_0} = f(t_0, y_0) = 3 - 2\cdot 0 - \frac{1}{2} = 5/2$.

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Since $f(t,y)$ here is a nice smooth continuous function, it's therefore reasonable to assume that the true solution to the DE has slope approximately $\approx 2.5$ for $t$ near $0$.

In otherwords, the solution should be close to the tangent line at $(t_0, y_0)$.

<u>Recall</u>: Tangent line approximation:

$$y = y_0 + \left.\frac{dy}{dt}\right|_{t_0} \cdot (t - t_0)$$
$$= y_0 + f(t_0, y_0)(t - t_0)$$

So here

$$y = 1 + 2.5\,t$$

So we can therefore approximate the true solution's value at, say, $t_1 = 0.2$?

$$y_1 = y_0 + f(t_0, y_0) \cdot (t_1 - t_0)$$
$$= 1 + 2.5 \times 0.2$$
$$= 1.5$$

Now the straight line approximation will get ~~worse~~ further away from the true solution the further we get from $t_0$, however, the trick to Euler's method is to realise that the slope at the new point $(0.2, 1.5) = (t_1, y_1)$ can be used to draw a new tangent line approximation there, and hence get a 2nd new approximation etc:

$$y_2 = y_1 + \left.\frac{dy}{dt}\right|_{y_1} \cdot (t_2 - t_1)$$
$$= y_1 + f(t_1, y_1)(t_2 - t_1).$$

wash, rinse, repeat. In general, we can obtain the $(n+1)$th y-value by linear extrapolation from the $n$th point:

$$y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n)$$

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Most of the time it's easiest to use constant time steps i.e. $t_{n+1} - t_n = h$, for some $h > 0$, say.
In that case Euler's Method is the numerical scheme to approximate the DE.

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0 \quad , \text{ given by}$$

Definition 1.7.1:

$$\boxed{\begin{aligned} &y(t_0) = y_0 \\ \text{And} \quad &y_{n+1} = y(t_{n+1}) = y_n + f(t_0 + n\cdot h,\ y_n)\cdot h \end{aligned}}$$

Example: Use Euler's Method with $h = 0.2$ to approximate the solution to $\frac{dy}{dt} = 3 - 2t - \frac{y}{2}$, $y(0) = 1$ ~~on the interval for $0 \le t \le 4$,~~ for 5 steps.

Easiest to make a table:

| $t_n$ | Tangent line | Euler: $y_n$ | (Exact) |
|---|---|---|---|
| 0 | $y = 1 + 2.5t$ | 1 | 1 |
| 0.2 | $y = 1.13 + 1.85t$ | 1.5 | 1.43711 |
| 0.4 | $y = 1.364 + 1.265t$ | 1.87 | 1.75650 |
| 0.6 | $y = 1.6799 + 0.735t$ | 2.123 | 1.96936 |
| 0.8 | $y = 2.05898 + 0.26465t$ | 2.27070 | 2.08584 |
| 1.0 | | 2.32363 | 2.11510 |

Note 1.7.2: ~~The~~ How good of an approximation is this?
Fortunately we can solve the DE exactly here:

$$\frac{dy}{dt} = 3 - 2t - \frac{y}{2}$$

$$\Rightarrow \frac{dy}{dt} + \frac{1}{2}y = 3 - 2t$$

$$\mu(t) = e^{\frac{t}{2}}$$

$$\begin{aligned} \Rightarrow y(t) &= e^{-\frac{t}{2}}\left(\int e^{\frac{t}{2}}(3-2t)\,dt + C\right) \\ &= e^{-\frac{t}{2}}\left((14-4t)e^{\frac{t}{2}} + C\right) \\ &= 14 - 4t + Ce^{-\frac{t}{2}} \end{aligned}$$

IC: $t = 0, y = 1 \Rightarrow C = -13$, so $y(t) = 14 - 4t - 13e^{-\frac{t}{2}}$

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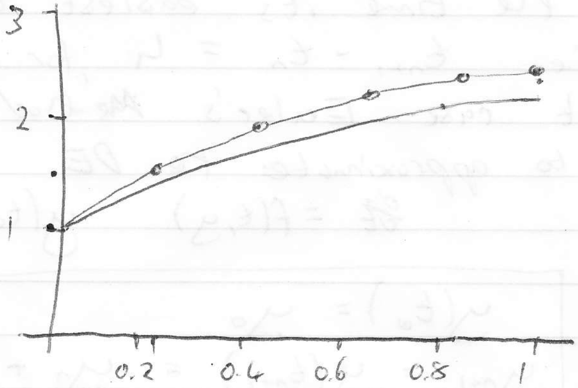
We can fill in values in the table above, and draw a graph:




The approximation is decent, but not great.

Note 1.7.3: The approximation will in general get better if you take more steps of smaller size.
(In general: for a given t value, if you double the number of intermediate steps to reach an approximation for $y(t)$, you halve the error between the approximation & the true solution.

In fact here are some numerical results:

| Exact @ $t=1$: | $h=0.1$ | $h=0.05$ | $h=0.025$ | $h=0.01$ |
|---|---|---|---|---|
| 2.1151 | 2.2164 | 2.1651 | 2.1399 | 2.1250 |

Things to Note: For this DE, every solution is of the form $y(t) = 14-4t+Ce^{-\frac{t}{2}}$, so every solution $\to 14-4t$ as $t\to\infty$.

Thus: the numerical approximation will never get too far from the true solution if we keep iterating forward.

However: This is most often not the case.
In general, numerical approximations diverge from the true solution exponentially.

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Example 1.7.3  $\frac{dy}{dt} = 4 - t + 2y$, $y(0)$

Has true solution $y(t) = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$

Here's a table of numerical solutions for $h = 0.1$, $0.025$ & $0.01$.

| $t$ | Exact | $h=0.1$ | $h=0.025$ | $h=0.01$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 19.07 | 15.8 | 18.1 | 18.7 |
| 2 | 149.4 | 104.7 | 135.6 | 143.6 |
| 3 | 1109.2 | 652.5 | 959.3 | 1045.4 |
| 4 | 8197.9 | 4042.1 | 6755.2 | 7575.6 |

Things to note:

- For a given $t$, numerical approx. at $t$ becomes better as ~~$t \to \infty$~~ $h \to 0$
- For a given $h$, numerical approx. at $t$ gets worse as $t \to \infty$.

Translation: Small changes in IC data yield large changes in numerical prediction for sufficiently large $t$.

This is why the predicting power of, say, weather models, are limited to a few days ahead: exponentially growing error terms.

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Example 1.7.4

Consider the DE $\frac{dy}{dx} = x^2 + y^2$ $y(0) = 0$.

a) • Estimate the value of the solution at $x = 1$ using Euler's method and $h = \frac{1}{3}$.

b) • Is this an underestimate or an overestimate?
  • Justify your answer.

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a) We have $y_1 = y_0 + f(t_0, y_0) \cdot \frac{1}{3}$

$\Rightarrow \quad = 0 + (0^2 + 0^2) \cdot \frac{1}{3}$

$= 0$

So $t_1, y_1 = \frac{1}{3}, 0$.

Then $y_2 = y_1 + f(\frac{1}{3}, 0) \cdot \frac{1}{3}$

$= 0 + (\frac{1}{9} + 0) \frac{1}{3}$

$= \frac{1}{27}$.

So $t_2, y_2 = \frac{2}{3}, \frac{1}{27}$

And finally $y_3 = y_2 + f(t_2, y_2) \cdot \frac{1}{3}$

$= \frac{1}{27} + \left( (\frac{2}{3})^2 + (\frac{1}{27})^2 \right) \cdot \frac{1}{3}$

$= \frac{325}{2187} \simeq 0.1486$

So $y(1) \simeq 0.1486$.

b) This is an underestimate;
Since $f(x,y) = x^2 + y^2$ is concave up, we are always understooting the true solution with our tangent line approximations
$\Rightarrow$ Numerical approx. at each step will be less than true solution value: