1. (10 points) Solve the following initial value problem explicitly. Your answer should be a function in the form $y = g(x)$, where there is no undetermined constant in g.

$$\frac{dy}{dx} = \frac{(x^2+1)(y^2+1)}{xy}, \quad y(1) = -1.$$

This equation is separable: note that we can write it as $\frac{dy}{dx} = \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)$. Sticking the y stuff on the left and the x stuff on the right gives us

$$\frac{y}{y^2+1}\,dy = \frac{x^2+1}{x}\,dx$$

Now we integrate both sides. To evaluate the integral on the left we can use u-substitution ($u = y^2+1$) to get

$$\int \frac{1}{y^2+1}\cdot y\,dy = \int \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(y^2+1)$$

Note that we don't need absolute value signs in the final expression, since $y^2+1>0$ always.

To evaluate the x integral, note that $\frac{x^2+1}{x} = x + \frac{1}{x}$. Thus

$$\int \frac{x^2+1}{x}\,dx = \int x + \frac{1}{x}\,dx = \frac{1}{2}x^2 + \ln|x| + C$$

Hence

$$\begin{aligned} \frac{1}{2}\ln(y^2+1) &= \frac{1}{2}x^2 + \ln|x| + C \\ \Longrightarrow \ln(y^2+1) &= x^2 + 2\ln|x| + 2C \\ \Longrightarrow y^2+1 &= A|x|^2e^{x^2} = Ax^2e^{x^2}, \end{aligned}$$

where $A = e^{2C}$ (and we can drop the absolute value signs, as anything squared is always nonnegative). Hence solving for y we get

$$y = \pm\sqrt{Ax^2e^{x^2}-1}$$

Note that we must include the $\pm$ sign on the right to have the full general solution.

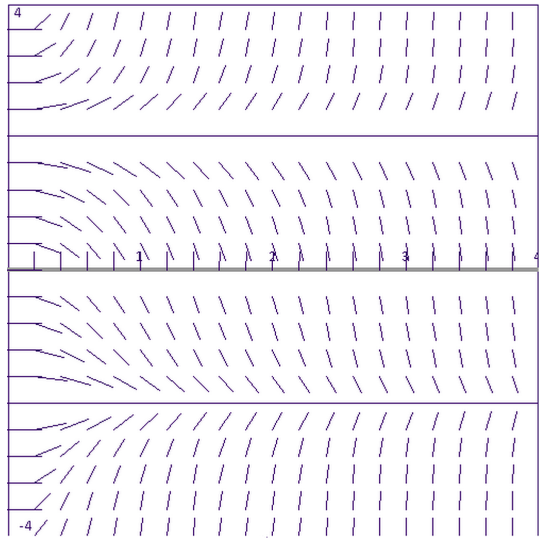
Now apply the IC to solve for A. We note that $y(1) = -1$ means that we must choose the negative root, as the positive square root solution will never be able to hit -1. Hence we have

$$\begin{aligned} -1 &= -\sqrt{A\cdot 1^2\cdot e^{1^2}-1}, \\ \Longrightarrow 1 &= A\cdot e - 1 \\ \Longrightarrow A &= \frac{2}{e} \end{aligned}$$

Hence the solution to the IVP is

$$y(x) = -\sqrt{\frac{2}{e}\cdot x^2e^{x^2}-1} = -\sqrt{2x^2e^{x^2-1}-1}$$

2. (10 total points)
The slope field to the differential equation $\frac{dy}{dx} = f(x,y)$ is plotted below for $0 \le x \le 4, -4 \le y \le 4$:



(a) (4 points) Circle the differential equation that corresponds to the above slope field.

$$\frac{dy}{dx} = (x^2-4)y \qquad \frac{dy}{dx} = x(y^2-4) \qquad \frac{dy}{dx} = -x(y^2-4) \qquad \frac{dy}{dx} = -(y^2-4)$$

We see that $\frac{dy}{dx}$ has equilibrium solutions at $y = -2$ and $y = 2$. This is not the case for $\frac{dy}{dx} = (x^2-4)y$, ruling it out. We also see that $\frac{dy}{dx} = 0$ at $x = 0$ regardless of the y-value, but not so for larger values of x. This means that $\frac{dy}{dx}$ must have some x-dependence, ruling out $\frac{dy}{dx} = -(y^2-4)$. Finally, we note that the slopes are negative when y is between -2 and 2, so we must have $\frac{dy}{dx} < 0$ for $-2 < y < 2$. The only DE which matches this criteria is thus

$$\frac{dy}{dx} = x(y^2-4).$$

(b) (6 points) Let $y = h(x)$ be the solution to the differential equation you circled above that satisfies the initial condition $y(1) = 1$. Use Euler's method with a step size of $h = 0.5$ to estimate the value of the solution at $x = 2$. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

Recall that Euler's method scheme for the IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$ with step size h is given by

- Set x_0 and y_0 to be the given initial conditions
- for $n \ge 0$ set $x_{n+1} = x_n + h$ and $y_{n+1} = y_0 + h \cdot f(x_n, y_n)$.

For us we have $h = 0.5$, $f(x,y) = x(y^2-4)$ and $(x_0,y_0) = (1,1)$, so we have

- $x_0 = 1$ and $y_0 = 1$
- $x_1 = \frac{3}{2}$ and $y_1 = y_0 + h \cdot f(x_0,y_0) = 1 + \frac{1}{2} \cdot \left[1(1^2-4)\right] = -\frac{1}{2}$
- $x_2 = 2$ and $y_2 = y_1 + h \cdot f(x_1,y_1) = -\frac{1}{2} + \frac{1}{2} \cdot \left[\frac{3}{2}((-\frac{1}{2})^2-4)\right] = -\frac{53}{16}$

At this point we stop, as we've reached $x_2 = 2$. Our estimate for $h(2)$ is thus $y_2 = -\frac{53}{16} = -3.3125$.

3. (10 total points)

(a) (2 points) Using the existence and uniqueness theorem for linear first-order differential equations, state the minimum t-interval for which a unique solution is guaranteed to exist for the following initial value problem. **You do not need to solve the DE to answer this question**.

$$t \cdot \frac{dy}{dt} = t\tan(t)\cdot y - \cos(t), \qquad y(-1) = 0$$

Recall that as per the existence and uniqueness theorem for linear first order DEs, the IVP

$$\frac{dy}{dt} + f(t)\cdot y = g(t), \qquad y(t_0) = y_0$$

is guaranteed to have a unique solution on any t interval containing t_0 for which both $f(t)$ and $g(t)$ are continuous. Putting our DE in standard form gives us

$$\frac{dy}{dt} - \tan(t) = \frac{\cos(t)}{t}$$

Thus we are looking for the largest t interval containing $t_0 = -1$ such that both $-\tan(t)$ and $\frac{\cos(t)}{t}$ are continuous.

Now $\tan(t)$ has vertical asymptotes (i.e. discontinuities) at $t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\ldots$ (where $\frac{\pi}{2} = 1.5707\ldots$), so the biggest interval containing $t = -1$ for which $\tan(t)$ is continuous is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Secondly, $\cos(t)$ is continuous for all t, so $\frac{\cos(t)}{t}$ is only discontinuous at $t = 0$. Thus the biggest interval containing $t = -1$ for which $\frac{\cos(t)}{t}$ is continuous is $(-\infty, 0)$. Combining these two we see that the biggest interval on which both functions are continuous is $\left(-\frac{\pi}{2}, 0\right)$. We conclude that the IVP has a unique solution for

$$t \in \left(-\frac{\pi}{2}, 0\right).$$

(b) (8 points) The temperature in my kitchen is 21°C and the temperature inside my fridge is 4°C. I pull a cold soda from the fridge and place it on the kitchen counter. I don't like my drinks too cold, so I decide to wait for it to warm up a bit before drinking it. Suppose that the rate of heating of the drink is proportional to the difference between its temperature and that of the kitchen; furthermore suppose I have ascertained that the proportionality constant involved is $k = \frac{1}{20}$ min^{-1}. How long must I wait until my drink warms to 15°C? You may provide your answer in exact form or as a decimal for this question.

Let t be time elapsed since the drink is removed from the fridge, and let $y(t)$ be the temperature of the drink in degrees Celsius. The rate of change of the drink's temperature is proportional to the difference between the drink's temperature and that of its surroundings. In other words,

$$\frac{dy}{dt} = -k(y-A)$$

where k is the proportionality constant and A the ambient temperature (this is just Newton's Law of Cooling). Recall that the minus sign is required, since if an object is cooler than its surroundings $y-A$ is negative, but we expect the object's temperature to increase. We are given the values of 21°C and $k = \frac{1}{20}$ min^{-1}, along with the initial condition of $y = 4$°C when the drink leave's the fridge. Thus we have the IVP

$$\frac{dy}{dt} = -\frac{1}{20}(y-21), \qquad y(0) = 4.$$

This equation is both separable and linear; we'll solve it the separable way. Sticking the y stuff on the left and the t stuff on the right gives us

$$\frac{1}{y-21}\,dy = -\frac{1}{20}\,dt$$

Now we integrate both sides. The integral on the right is just $-\frac{t}{20}+C$. The integral on the left is $\int \frac{1}{y-21}\,dy = \ln|t-21|$. Hence

$$\begin{aligned} \ln|y-21| &= -\frac{t}{20}+C \\ \implies |y-21| &= e^{-\frac{t}{20}+C} \\ \implies y &= 21+Ae^{-\frac{t}{20}}, \end{aligned}$$

where $A = \pm e^C$ as necessary to account for the absolute value signs. Now apply the IC $y(0) = 4$ to solve for A:

$$\begin{aligned} 4 &= 21+Ae^{-\frac{0}{20}} \\ \implies A &= -17 \end{aligned}$$

So the temperature of the drink at time t is given by

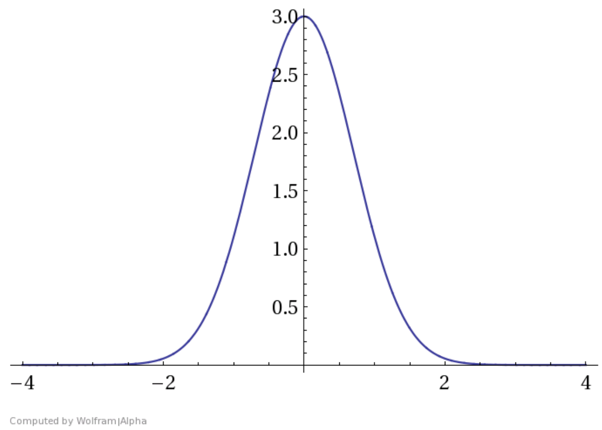
$$y(t) = 21-17e^{-\frac{t}{20}}$$

Finally, we want the t such that $y(t) = 15$. Setting $y = 15$ and solving for t gives us

$$t = 20\cdot\ln\left(\frac{17}{6}\right) \approx 20.829 \text{ minutes.}$$

That is, I must wait just under 21 minutes for my drink to warm up to 15°C.

4. **(10 total points)** The graph of $h(y) = 3e^{-y^2}$ looks as follows:



The graph is always positive, as $e^{-y^2} > 0$ for any value of y.

Now consider the autonomous differential equation

$$\frac{dy}{dt} = h(y) + \beta = 3e^{-y^2} + \beta,$$

where β is a constant.

(a) (4 points) For what value(s) of β does the DE have two different equilibrium solutions? Classify each such solution according to whether it stable, unstable or semistable.

Adding β to the above graph shifts it up or down. Recall that an autonomous equation $\frac{dy}{dt} = f(y)$ has an equilibrium solution wherever $f(y) = 0$; thus we must ask ourselves: for which values of β does $3e^{-y^2} + \beta$ have two distinct roots? In other words, for which values of β does $3e^{-y^2} = -\beta$ have two distinct solutions?

The answer is $-3 < \beta < 0$: shift the above graph down by between 0 and 3 units, and it will cut the horizontal axis at two distinct points.

Furthermore, the negative equilibrium solution will always be unstable and the positive solution stable. This is because $f(y)$ is positive to the right of the left root (so $\frac{dy}{dt} > 0$ there, and thus a solution is increasing away from the equilibrium solution); to the left of the equilibrium solution $f(y)$ is negative - meaning a solution there will decrease away from the equilibrium solution. The opposite is the case for the positive equilibrium solution, making it stable.

(b) (2 points) For what value(s) of β does the DE have only one equilibrium solution? Classify each such solution according to whether it stable, unstable or semistable.

Looking at the above graph again, we see that the only value of β for which $3e^{-y^2}+\beta$ has exactly one solution is when $\beta=-3$. In that case $y=0$ is the single equilibrium solution. It is semistable: solutions that start out just above $y=0$ will decrease down toward zero; however, solutions that start out just below $y=0$ will also decrease, taking them away from zero.

(c) (4 points) Suppose we are given the initial value problem

$$\frac{dy}{dt}=3e^{-y^2}-1, \quad y(0)=1.$$

Without solving the DE, compute the limiting value of the solution as $t\to\infty$. [That is, compute $\lim_{t\to\infty}h(t)$.]

This is in the case where there are two equilibrium solutions, with the the lower one being unstable and the upper one stable. Graphically we see that $y(0)$ is less than the upper (stable) equilibrium solution but greater than the lower unstable solution (or we compute that $\frac{dy}{dt}=3\cdot 1-1=2>0$ when $t=0$, so it must increase toward the stable solution. What remains is to determine the actual numerical value of that solution.

To do that we set

$$3e^{-y^2}-1=0$$

and solve for y. Thus $e^{-y^2}=\frac{1}{3}$, so $e^{y^2}=3$. We therefore have that

$$y=\pm\sqrt{\ln(3)}=\pm 1.048147\ldots$$

The positive value corresponds to the stable equilibrium solution, so we conclude that the limiting value of the solution to our DE is $\sqrt{\ln(3)}\approx 1.048147$.

5. (10 total points + 3 bonus points) A skydiver with a total mass of 100kg jumps from a plane, falling vertically downward. Gravity acts on the skydiver – you may take gravitational acceleration to be a constant $g = 10$ ms^{-2}. Air resistance also acts on the skydiver with a force proportional (and opposite in direction) to the skydiver's velocity.

   At time $t = 1$ second into the jump the skydiver's downward velocity is 9 ms^{-1}. However, at this time the skydiver starts progressively tucking in her arms, effectively decreasing the force of air resistance acting on her. The drag coefficient k for the skydiver is therefore no longer constant, and is instead given by

   $$k = \frac{25}{t} \text{ kg s}^{-1},$$

   where t is in seconds since the skydive began.

   (a) (10 points) Establish an initial value problem and solve it to find an explicit formula for the velocity of the the skydiver at any point after she start tucking in her arms.

   Define up to be positive. Let $v(t)$ be the velocity of the skydiver in ms^{-1}, where t is measured in seconds since the skydive began. Ordinarily the DE modeling the skydiver's velocity would be given by

   $$m\frac{dv}{dt} = -mg - kv$$

   where $m = 100$ kg, $g = 10$ ms^{-2} and k is a constant. Now, however, k is decreasing with time: $k = \frac{25}{t}$. The same differential equations still holds, but instead we therefore have

   $$m\frac{dv}{dt} = -mg - \frac{k'}{t}v$$

   where $k' = 25$, along with the initial condition $v(1) = -9$ (note the minus sign).

   This is a first-order linear differential equation, so we can go ahead and solve it (we'll do so symbolically here, and plug in numbers right at the end). In standard form it is

   $$\frac{dv}{dt} + \frac{k'}{mt}v = -g$$

   i.e. if have $\frac{dv}{dt} + F(t) \cdot v = G(t)$, where $F(t) = \frac{k'}{mt}$ and $G(t) = -g$. The integrating factor is then

   $$\mu(t) = e^{\int F(t)\,dt} = e^{\int \frac{k'}{mt}\,dt} = e^{\frac{k'}{m}\ln(t)} = t^{\frac{k'}{m}}$$

   (we may drop the absolute value signs immediately, since we are only interested in $t > 0$ here).

The general solution is then given by

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)G(t)\,dt + C\right)\\
&= t^{-\frac{k'}{m}}\left(\int t^{\frac{k'}{m}}\cdot(-g)\,dt + C\right)\\
&= t^{-\frac{k'}{m}}\left(-g\int t^{\frac{k'}{m}}\,dt + C\right)\\
&= t^{-\frac{k'}{m}}\left(-g\cdot\frac{t^{\frac{k'}{m}+1}}{\frac{k'}{m}+1} + C\right)\\
&= -\frac{g}{\frac{k'}{m}+1}\cdot t + Ct^{-\frac{k'}{m}}\\
&= -\frac{gm}{k'+m}\cdot t + Ct^{-\frac{k'}{m}}
\end{aligned}$$

Now we plug in $g = 10, m = 100, k = 25$ and simplify to get

$$v(t) = -8t + Ct^{-\frac{1}{4}}$$

Applying the initial condition $y(1) = -9$ yields

$$\begin{aligned}
-9 &= -8\cdot 1 + C\cdot 1^{-\frac{1}{4}}\\
\implies C &= -1
\end{aligned}$$

Hence the solution to this initial value problem is

$$y(t) = -8t - t^{-\frac{1}{4}}.$$

Note that this is only valid for $t \geq 1$, as that is when the skydiver starts tucking in her arms (and so when the drag coefficient starts decreasing).

(b) (3 bonus points) Estimate how long it takes after jumping for the skydiver's speed to reach 60 ms^{-1}. You may provide an approximate answer, but be sure to justify any such approximation.

We want to find the t value such that $v(t) = -60$. In other words, solve for t in

$$-8t - t^{-\frac{1}{4}} = -60$$

or $8t + t^{-\frac{1}{4}} = 60$.

This can actually be solved exactly using quartic polynomials, but I'm not expecting you to pull out something that complex here. The best simplifying observation we can make here is that the $8t$ term is increasing in size as t increases, while $t^{-\frac{1}{4}}$ is getting smaller. At $t = 1$ the former term is already eight times the magnitude of the latter, so we therefore expect it to dominate the left hand side for $t > 1$. It therefore makes sense to ignore the $t^{-\frac{1}{4}}$ term and just solve for t in

$$8t = 60$$

which yields $t = 7.5$ seconds.

The true solution is at about $t = 7.42427\ldots$, so this is a pretty good approximation.