The Explicit^2 Formula and Zero Density of Elliptic Curve L-Functions

Simon Spicer

University of Washington

mlungu@uw.edu

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Overview

- Recap: define EC L-functions, state explicit formula

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- The lowest nontrivial zero γ_0

Elliptic Curve L-Functions

Definition

The L-function attached to E is the analytic continuation to $\mathbb{C}$ of

$$L_E(s) := \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} = \sum_{n=1}^{\infty} a_n n^{-s}$$

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Definition

The *completed* L-function attached to E is

$$\Lambda_E(s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s)$$

The Zeros of $L_E(s)$

Three flavors:

- A simple zero at $0, -1, -2, -3, \ldots$
- A zero of order r_{an} at $s = 1$; r_{an} is called the *analytic rank* of E
- Countably infinite zeros in the strip $0 < \Re(s) < 2$, symmetric about $\Re(s) = 1$ and x-axis.

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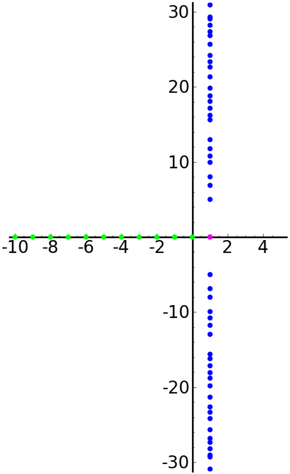

Figure: The zeros of $L_E(s)$ for $E = 37a$

Significant Caveat

IMPORTANT

We assume GRH for the all of the following work.

The Shifted Logarithmic Derivative

To state the explicit formula for elliptic curves, we will need to characterize $\frac{L_E'}{L_E}(s+1)$.

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Lemma 1 (S.)

$\frac{L_E'}{L_E}(s+1) = \frac{d}{ds}\log(L_E)(s+1) = \sum_n c_n(E)n^{-s}$, where

$$c_n(E) := \begin{cases} -\left(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e})\right) \cdot \frac{\log(p)}{p^e}, & n = p^e \text{ a prime power,} \\ 0, & \text{otherwise} \end{cases}$$

and

- $\#\widetilde{E}(\mathbb{F}_{p^e})$ is the number of points on over $\mathbb{F}_{p^e}$ on the (possibly singular) projective curve obtained by reducing E modulo p
- this Dirichlet series converges absolutely for $\Re(s) > \frac{1}{2}$

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Let

- $$S_E(x,T) := \sum_{|\gamma|<T} \frac{x^{i\gamma}}{i\gamma} = \sum_{0<\gamma<T} \frac{2\sin(\gamma \log x)}{\gamma}$$

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- $$\psi_E(x) := \sum_{n \leq x}{}' c_n(E)$$

  i.e. $\psi_E(x)$ is the cumulative sum function of the Dirichlet coefficients of $\frac{L_E'}{L_E}(s+1)$

  (Recall $c_n(E) = \left(p^e + 1 - \#\widetilde{E}(\mathbb{F}_{p^e})\right) \cdot \frac{\log(p)}{p^e}$ for $n = p^e$ and 0 otherwise)

The Explicit Formula for Elliptic Curves

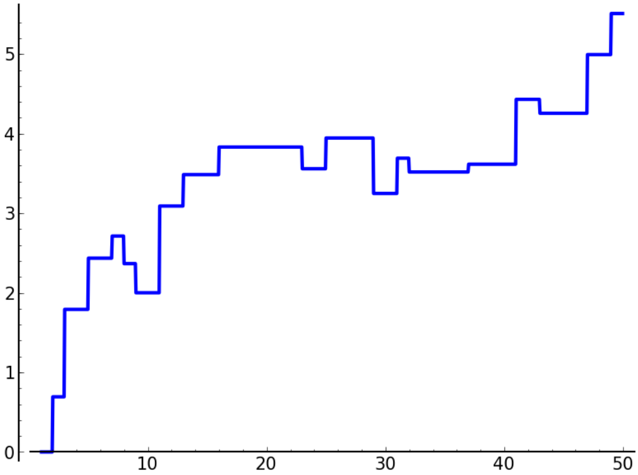

Figure: $\psi_E(x)$ for $E = 37a$

The Explicit Formula for Elliptic Curves

Theorem

For any any $E/\mathbb{Q}$ with conductor N and for any $x > 1$ the partial sum function $S_E(x, T)$ converges as $T \to \infty$. Specifically,

$$\lim_{T\to\infty} S_E(x, T) = \sum_{\gamma>0} \frac{2\sin(\gamma \log x)}{\gamma}$$

$$= -\eta - \log\left(\frac{2\pi}{\sqrt{N}}\right) - r_{an} \log x - \log(1 - x^{-1}) + \psi_E(x)$$

where η is the Euler-Mascheroni constant $= 0.5772156649\ldots$

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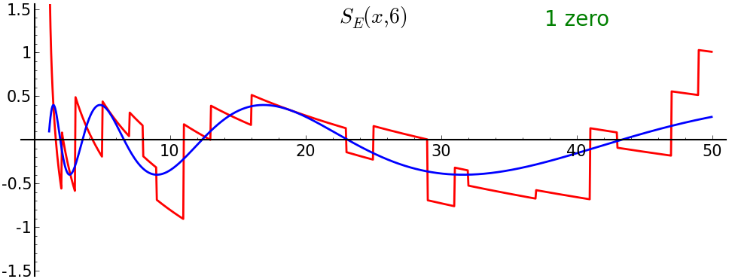

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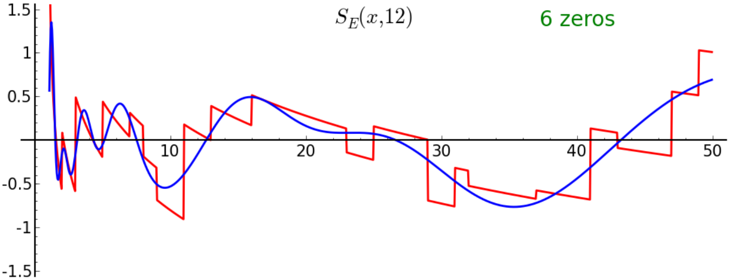

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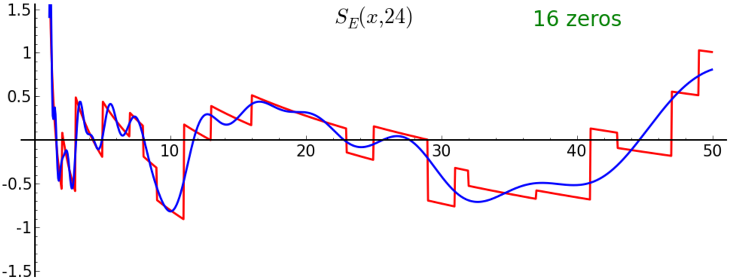

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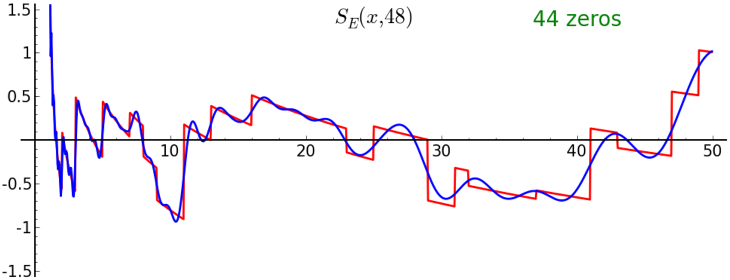

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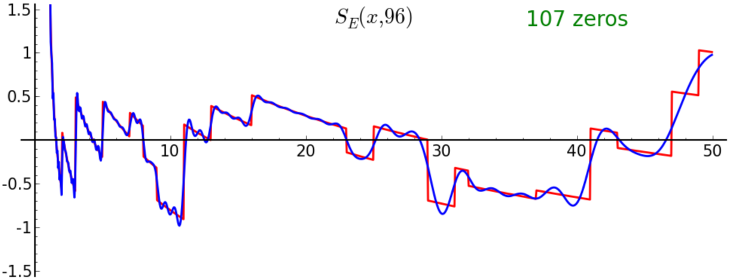

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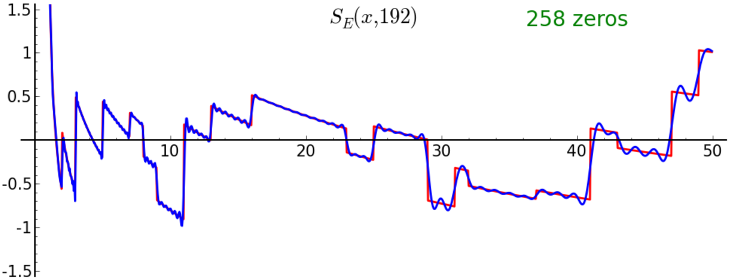

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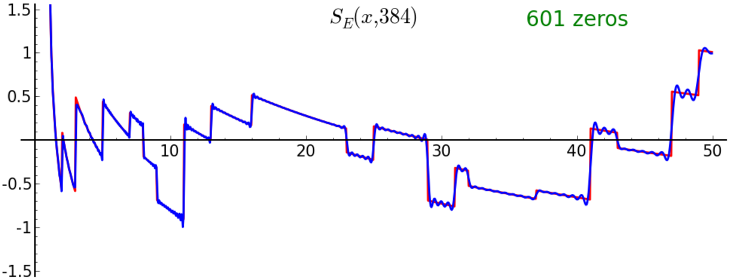

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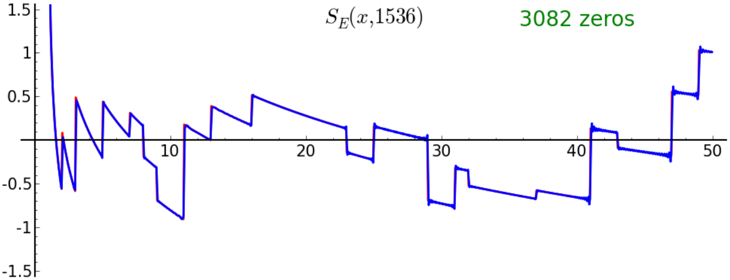

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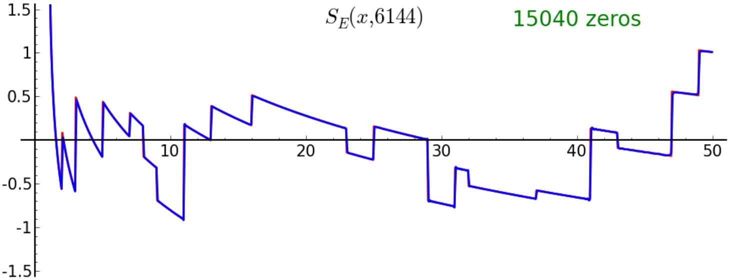

The Gibbs Phenomenon

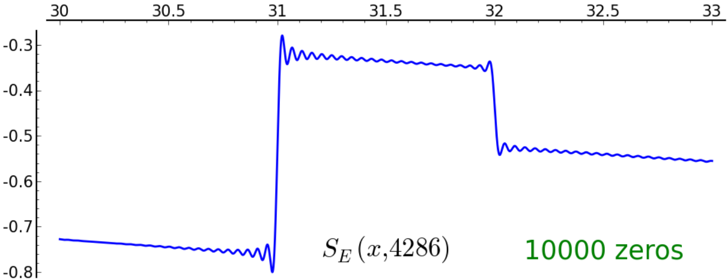

Figure: The Gibbs Phenomenon clearly visible at jump discontinuities for 37*a*.

Proving the Explicit Formula

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- By Perron's Formula we have
  $\psi_E(x) = \lim_{T\to\infty} \frac{-1}{2\pi i} \int_{1-iT}^{1+iT} \frac{L_E'}{L_E}(s+1)\, \frac{x^s}{s}\, ds$

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- Replace $\frac{L_E'}{L_E}(s+1)$ with two different series representations and distribute
- Replace each integral with contour integral on $\mathbb{C}$ plus residues
- Contour integrals $\to 0$ as $T \to \infty$.

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- In the literature: asymptotic arguments for the above.
- Can be proven explicitly.

Proving the Explicit Formula Explicitly

Let E be an elliptic curve with conductor N. Let $x > 1$ and $T \geq 2$, and let γ range over the imaginary parts of the zeros of $L_E(s)$.

Definition

Define the truncated zero sum error term

$$\epsilon_E(T,x) = \psi_E(x) + \left[\eta + \log\left(\frac{2\pi}{\sqrt{N}}\right)\right] + \log(1 - x^{-1}) + r_{an}\log(x) + \sum_{0<\gamma\leq T} \frac{2\sin(\gamma\log(x))}{\gamma}.$$

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Explicit formula is proven if $\lim_{T\to\infty} \epsilon_E(T,x) \to 0$ for any fixed x.

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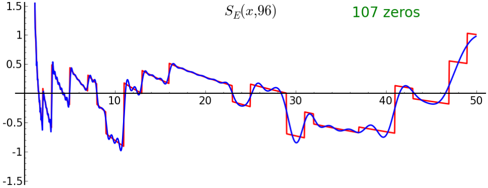

Figure: Worse convergence for increasing x.

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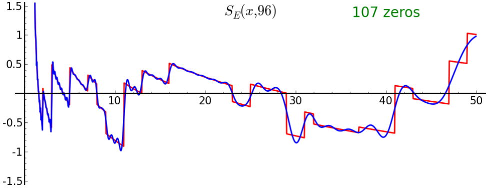

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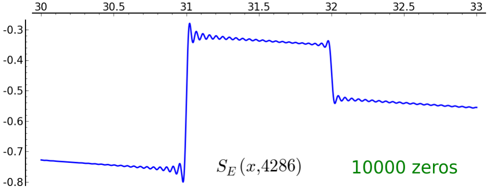

Figure: Worse convergence near jump discontinuities i.e. prime powers.

"The Explicit Explicit Formula"

Theorem (S.)

For $T \geq 2$ and $x > 1$,

$$\epsilon_E(T,x) < \left(\frac{1}{\pi} \cdot \frac{(3 + \log N + 2\log T)^2}{T} \cdot \frac{(x + x^{-1})}{\min\{1, \log x\}}\right) + \frac{x}{\pi T}\left(1 + \sum_{n \neq x} \left|\frac{c_n}{n \log(\frac{x}{n})}\right|\right)$$

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- Error term behaves as expected w.r.t x
- Note error term scales with conductor N

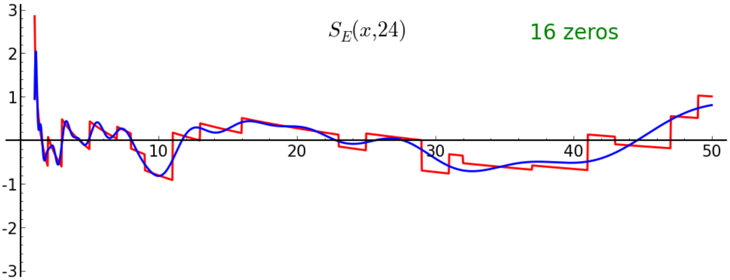
A more Easily Digestible Version

Theorem (S.)

Let $T \geq 2$ and $x \geq 2$. Let $\langle x \rangle$ be the distance from x to the closest prime power not equal to x. Then

$$\epsilon_E(T, x) < \frac{(4 + \log N + 2\log T)^2}{3T} \cdot x + \frac{4}{T} \cdot \frac{\sqrt{x}\log x}{\langle x \rangle}$$

Visualizing the Explicit Explicit Formula



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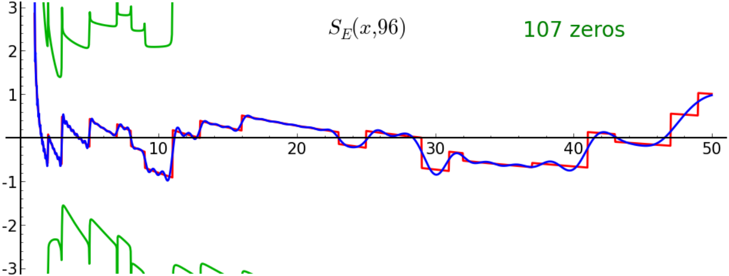

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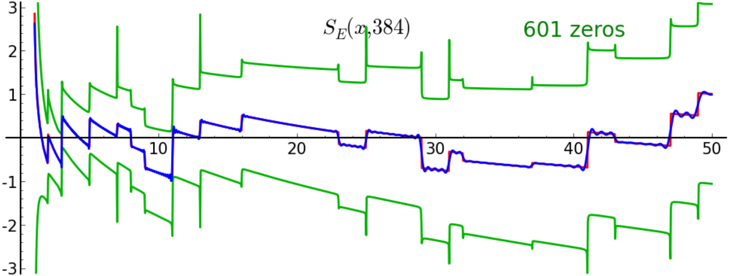

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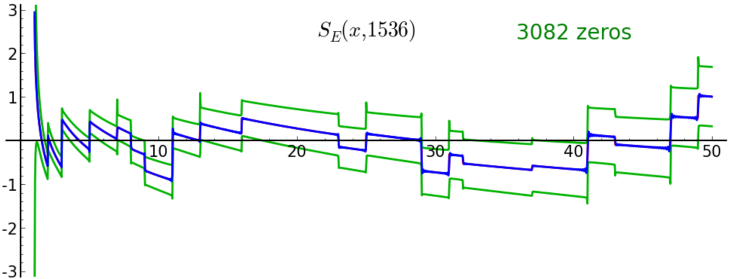

Integrating Across the Critical Strip

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Integrating Across the Critical Strip

- To prove the previous theorem, we must bound $\int \frac{L_E'}{L_E}(s+1)\,ds$ along various paths, including across critical strip
- $\implies$ We must know behavior of $\frac{L_E'}{L_E}(s)$ on the critical strip
- Since $\frac{L_E'}{L_E}(s)$ has simple poles wherever $L_E(s) = 0$,
  $\implies$ estimate the number of zeros on the the critical line for a given E.

The Shifted Completed Logarithmic Derivative

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Lemma 2 (S.)

$$\sum_\gamma \frac{s}{s^2+\gamma^2} = -\eta + \log\left(\frac{\sqrt{N}}{2\pi}\right) + \sum_{k=1}^{\infty} \frac{s}{k(k+s)} + \frac{L_E'}{L_E}(1+s)$$

where η is the Euler-Mascheroni constant $= 0.5772156649\ldots$
and γ ranges over the imaginary parts of all nontrivial zeros of L_E.

Workhorse Theorem

Theorem (S.)

Let E be an elliptic curve with conductor N and L-function $L_E(s)$. Let $\sigma > \frac{1}{2}$ and $\tau \in \mathbb{R}$, and let γ range over the imaginary parts of the nontrivial zeros of $L_E(s)$. Then

$$\left| \sum_{\gamma} \frac{\sigma}{\sigma^2 + (\gamma - \tau)^2} - \log\left(\frac{\sqrt{N}}{2\pi}\right) - \Re \frac{\Gamma'}{\Gamma}(1 + \sigma + i\tau) \right| < -2\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \sigma\right)$$

where

- $\frac{\Gamma'}{\Gamma}(1+s) = \sum_{k=1}^{\infty} \frac{s}{k(k+s)}$ *is the shifted digamma function, and*
- $\zeta(s)$ *is the Riemann zeta function*

Visualizing the Workhorse Theorem

We have $\sum_\gamma \frac{\sigma}{\sigma^2+\gamma^2} = \log\left(\frac{\sqrt{N}}{2\pi}\right) + \frac{\Gamma'}{\Gamma}(\sigma) + \sum_n c_n n^{-\sigma}$:

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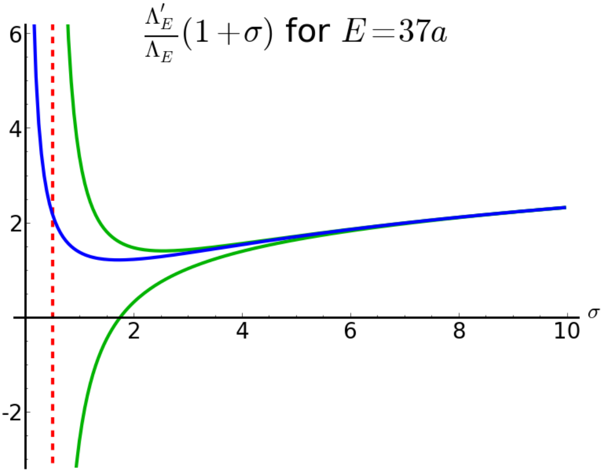

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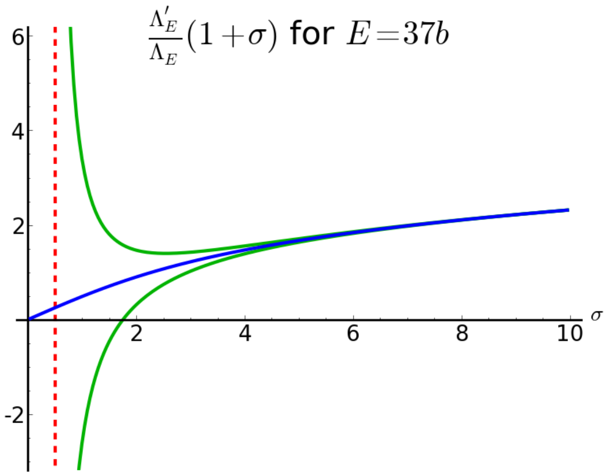

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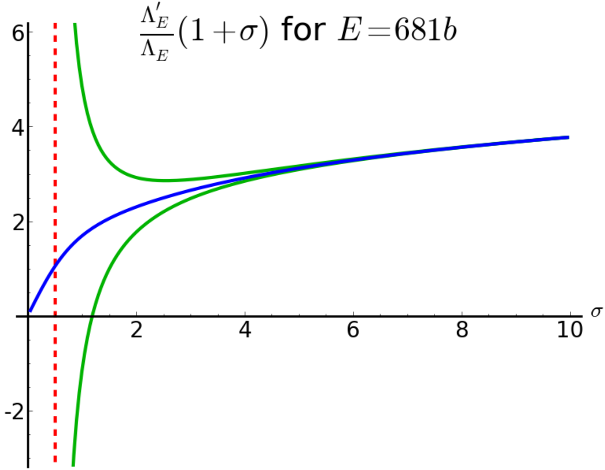

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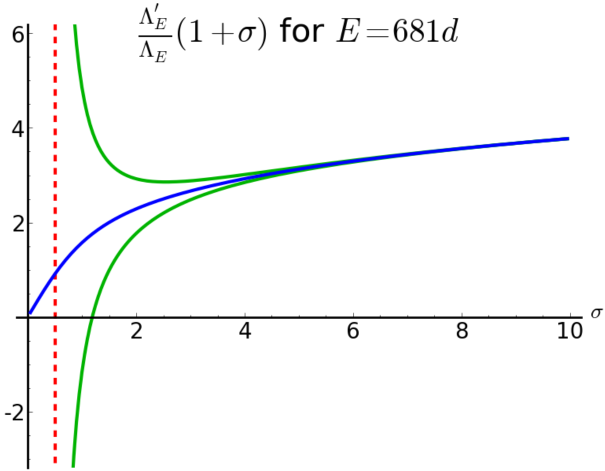

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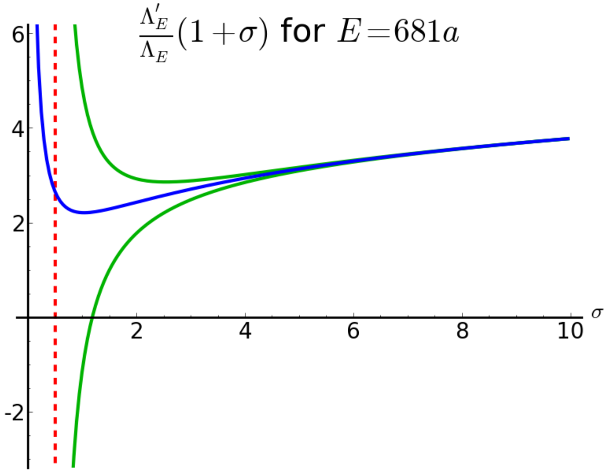

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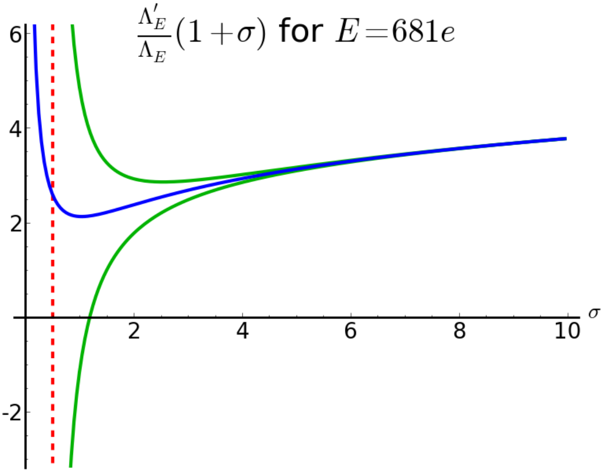

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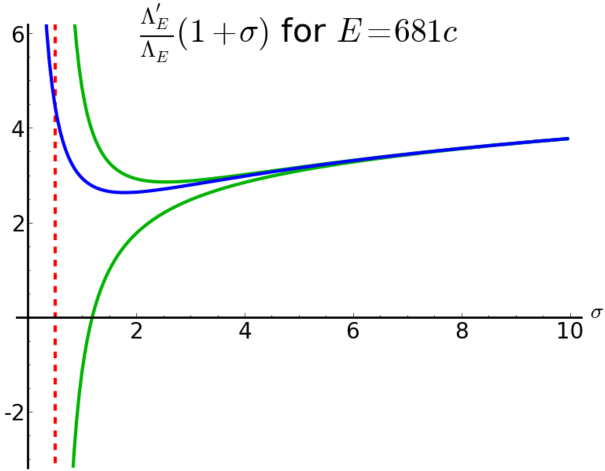

The Number of Zeros up to T

Let $M_E(T)$ be the counting function denoting the number of zeros of $L_E(s)$ with imaginary part at most T in magnitude.

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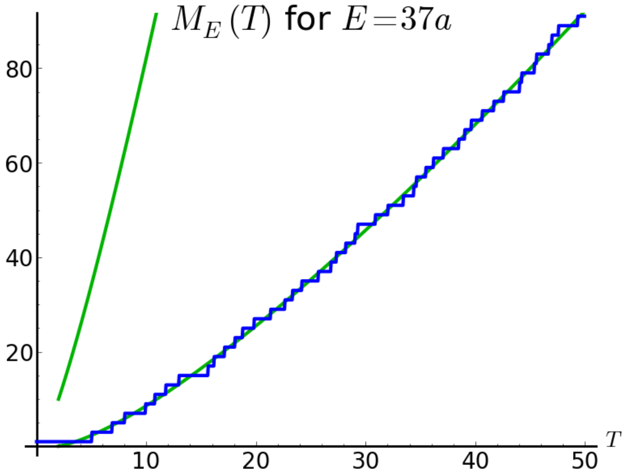
In reality, $M_E(T) << T(\log N + 2\log T)$.

Zero Density for Different Curves

$M_E(T) \leq T(\log N + 2\log T)$



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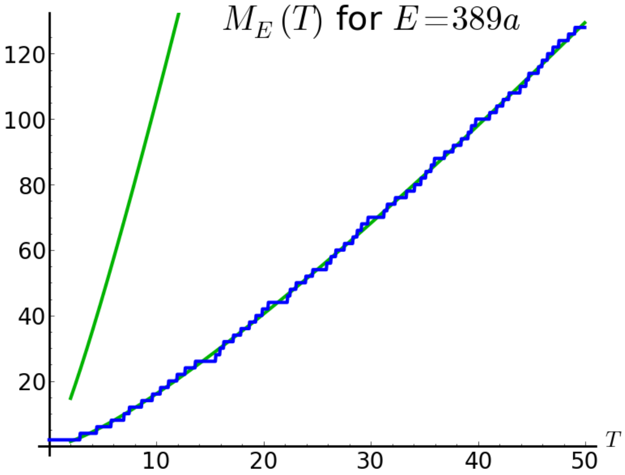

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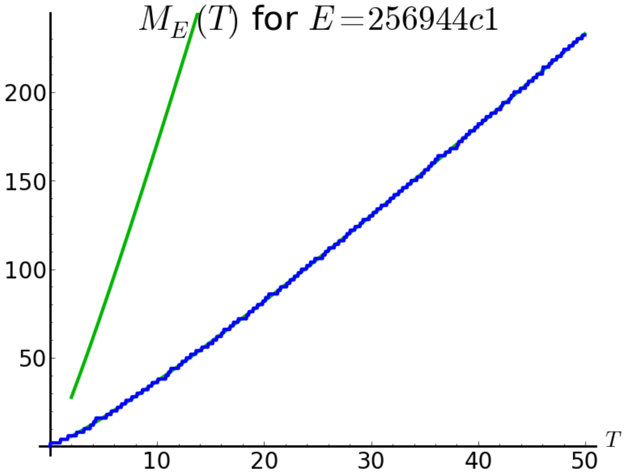

A Better Zero Density Estimate

Conjecture (S.)

$$M_E(T) = \frac{2}{\pi} \cdot \left(-\eta + \log\left(\frac{\sqrt{N}}{2\pi} \right) \right) \cdot T + \frac{2}{\pi} \cdot \sum_{k=1}^{\infty} \left[\frac{T}{k} - \arctan\left(\frac{T}{k} \right) \right] + O\left(\log^2 T \right)$$

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- Here $\sum_{k=1}^{\infty} \left[\frac{T}{k} - \arctan\left(\frac{T}{k}\right)\right] = \operatorname{Im}\left[\frac{\Gamma'}{\Gamma}(1+iT)\right] = O(T \log T)$.

Estimating Analytic Rank

Can we use the zero density results to estimate analytic rank?

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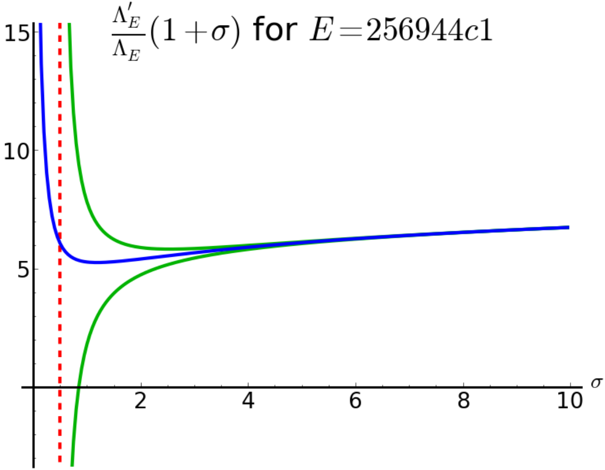

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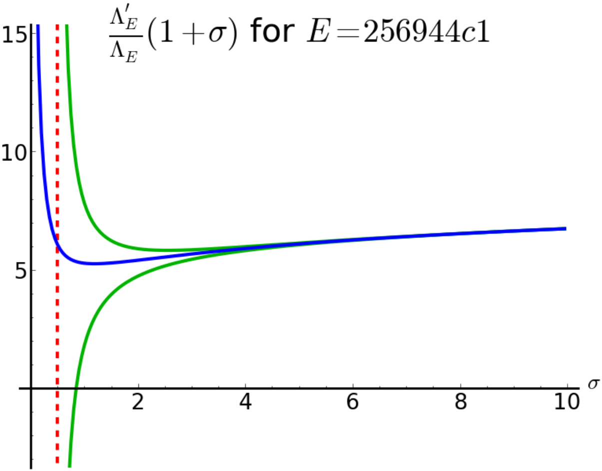

- This curve actually has rank zero!

Estimating Analytic Rank

```
E = EllipticCurve("256944c1")
print(E.rank())
L = E.lseries()
print(L.zeros(3))
```

```
0
[0.0256012097, 0.953965385, 1.67816734]
```

Bober's Method for Bounding Analytic Rank

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- Bober uses $f(t) = \left(\frac{\sin(\pi t)}{\pi t}\right)^2$.

Problems with Estimating Analytic Rank

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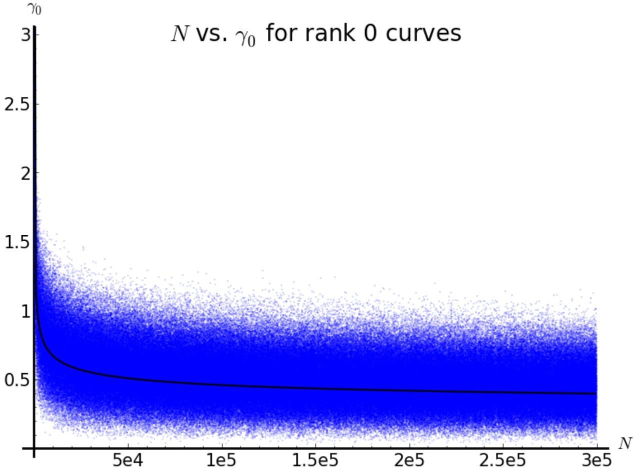
- Problem: such methods are sensitive to low-lying zeros
- In general, one cannot numerically determine if a zero is at the origin, or just very close
- No theorems currently exist bounding the lowest noncentral zero away from the origin

Visualizing the Lowest Zero



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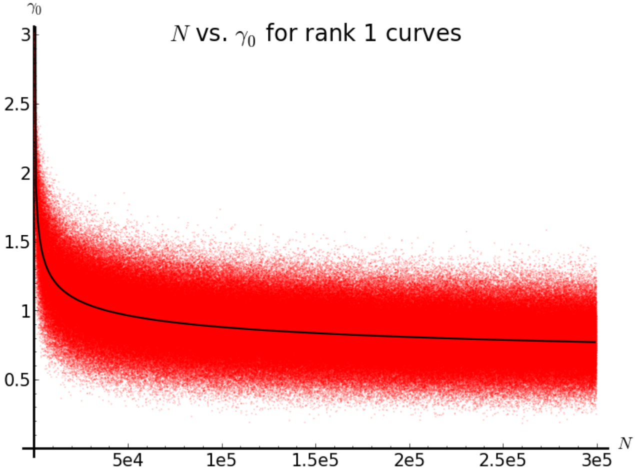

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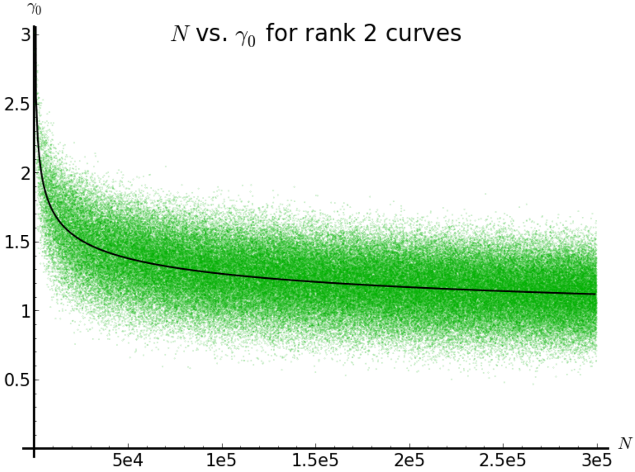

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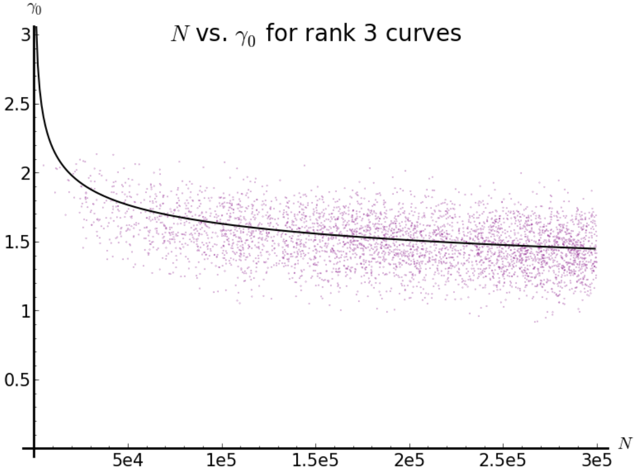


The Lowest Lying Zero Conjecture

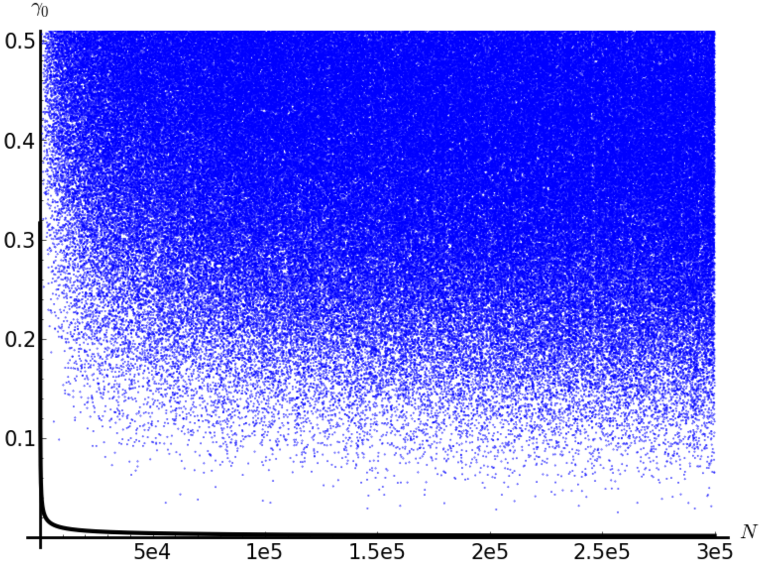
Let E have conductor N, and let $\gamma_0(E)$ denote the imaginary part of the lowest-lying noncentral nontrivial zero of $L_E(s)$ in the upper half plane.

Conjecture

There exist positive constants K and ϵ with $\epsilon < \frac{1}{2}$ such that

$$\gamma_0(E) > K \cdot N^{-\epsilon}.$$

Proof by Picture



Fin

Thank you!