Homework 3

Total: 20 points
Due: Wed 5 Feb 2014 09:30 in class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **AUTONOMOUS EQUATIONS**

(a) In the following two autonomous equations $\frac{dy}{dt} = f(y)$, sketch the graph of $f(y)$ versus y, determin the critical (equilibrium) points, and classify each equilibrium solution as asymptotically stable, unstable or semistable. Then sketch a graph of several solutions on the ty-plane, including the equilibrium solutions and a few other solutions to indicate asymptotic behaviour.

i. $\frac{dy}{dt} = y(y-1)(y-2)$ Boyce 2.5 Q3

ii. $\frac{dy}{dt} = e^{-y} - 1$ Boyce 2.5 Q5

(b) Boyce 2.5 Q18

A pond forms as water collects in a conical depression of radius a and depth h. Suppose that the water flows in at a constant rate k, and is lost through evaporation at a rate proportional to the pond's surface area.

i. Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}},$$

where α is the coefficient of evaporation.

ii. Find the equilibrium depth of the water in the pond. Is the equilibrium asymptotically stable or unstable?

iii. Find a condition relating k and α that must be satisfied if the pond is not to overflow.

2. **EULER'S METHOD** Consider the initial value problem

$$\frac{dy}{dt} = (t-1)(y+1), \quad y(1) = 1.$$

(a) Let $y = \phi(t)$ be the unique solution to this IVP. Estimate the value of $\phi(2)$ using Euler's method with a step size of $h = 1$. Then do the same for step sizes of $h = 0.5$ and $h = 0.2$.

Recall that Euler's method for the IVP $\frac{dy}{dt} = f(t, y), \;\; y(t_0) = y_0$ with step size h is given by the scheme

- Set t_0 and y_0 to be the given initial conditions
- for $n \geq 0$ set $t_{n+1} = t_n + h$ and $y_{n+1} = y_0 + h \cdot f(t_n, y_n)$.

For us we have $f(t, y) = (t-1)(y+1)$, $t_0 = 1$ and $y_0 = 1$, and we begin with $h = 1$. Thus we have

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = 2$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + 1 \cdot (1-1)(1+1) = 1$

At this point we stop, as we've reached $t_1 = 2$.
Our first estimate for $y(2)$ using $h = 1$ is thus $y = 1$.

Now we repeat the process with $h = \frac{1}{2}$:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = \frac{3}{2}$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + \frac{1}{2} \cdot (1-1)(1+1) = 1$
- $t_2 = 2$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + \frac{1}{2} \cdot (\frac{3}{2} - 1)(1+1) = \frac{3}{2}$

At this point we stop, as we've reached $t_2 = 2$.
Our estimate for $y(2)$ using $h = \frac{1}{2}$ is thus $y = \frac{3}{2}$.

Finally, we repeat the process with $h = \frac{1}{5}$:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = \frac{6}{5}$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + \frac{1}{5} \cdot (1-1)(1+1) = 1$
- $t_2 = \frac{7}{5}$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + \frac{1}{5} \cdot (\frac{6}{5} - 1)(1+1) = \frac{27}{25}$
- $t_3 = \frac{8}{5}$ and $y_3 = y_2 + h \cdot f(t_2, y_2) = \frac{27}{25} + \frac{1}{5} \cdot (\frac{7}{5} - 1)(\frac{27}{25} + 1) = \frac{779}{625}$
- $t_4 = \frac{9}{5}$ and $y_4 = y_3 + h \cdot f(t_3, y_3) = \frac{779}{625} + \frac{1}{5} \cdot (\frac{8}{5} - 1)(\frac{779}{625} + 1) = \frac{23687}{15625}$
- $t_5 = 2$ and $y_5 = y_4 + h \cdot f(t_4, y_4) = \frac{23687}{15625} + \frac{1}{5} \cdot (\frac{9}{5} - 1)(\frac{23687}{15625} + 1) = \frac{749423}{390625}$

At this point we stop, as we've reached $t_2 = 2$.
Our estimate for $y(2)$ using $h = \frac{1}{2}$ is thus $y = \frac{749423}{390625}$.

We repeat the last set of calculations using decimals (given to four decimal places, but calculations are done to higher precision), since this is the route that many people will choose:

- $t_0 = 1$ and $y_0 = 1$
- $t_1 = 1.2$ and $y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + 0.2 \cdot (1-1)(1+1) = 1$
- $t_2 = 1.4$ and $y_2 = y_1 + h \cdot f(t_1, y_1) = 1 + 0.2 \cdot (1.2-1)(1+1) = 1.08$
- $t_3 = 1.6$ and $y_3 = y_2 + h \cdot f(t_2, y_2) = 1.08 + 0.2 \cdot (1.4-1)(1.08+1) = 1.2464$
- $t_4 = 1.8$ and $y_4 = y_3 + h \cdot f(t_3, y_3) = 1.2464 + 0.2 \cdot (1.6-1)(1.2464 = 1.5160$
- $t_5 = 2$ and $y_5 = y_4 + h \cdot f(t_4, y_4) = 1.5160 + 0.2 \cdot (1.8-1)(1.5160+1) = 1.9185$

Our estimates of $y(2)$ using $h = 1$, $h = 0.5$ and $h = 0.2$ are thus 1, 1.5 and 1.9185 respectively.

(b) Solve the IVP and state the true value of $\phi(2)$. Do your estimates underpredict or overpredict $\phi(2)$? Do they get more accurate as h decreases?

The equation $\frac{dy}{dt} = (t-1)(y+1)$ is both linear and separable; we'll solve it using separation of variables. We have

$$\frac{1}{y+1}\,dy = (t-1)\,dt$$

The left hand side integrates to $\ln|y+1|$, and the right hand side integrates to $\frac{1}{2}t^2 - t + C$. Setting these two expressions equal to each other and exponentiating both sides then yields

$$y+1 = Ae^{\frac{1}{2}t^2-t},$$

where $A = \pm e^C$ as necessary to deal with the absolute value signs. Hence $y = Ae^{\frac{1}{2}t^2-t} - 1$.

Now we apply the initial condition $y(1) = 1$ to get

$$1 = Ae^{\frac{1}{2}\cdot 1^2 - 1} - 1,$$

so $A = 2e^{\frac{1}{2}}$. Thus

$$y = 2e^{\frac{1}{2}+\frac{1}{2}t^2-t} - 1 = 2e^{\frac{1}{2}(t-1)^2} - 1.$$

The value of $y(2)$ is therefore

$$y(2) = 2e^{\frac{1}{2}(2-1)^2} - 1 = 2\sqrt{e} - 1 = 2.2974\ldots$$

Comparing this value with our estimates, we see that the estimates all underpredict the true value of $y(2)$ - this is because $\frac{dy}{dt}$ is increasing in the region of the ty-plane that we're considering, so tangent line approximations will always undershoot the graph of the true solution. However, we do note that the Euler's method approximations are getting more accurate as h gets smaller, as they are supposed to do.

3. **2ND ORDER LINEAR DIFFERENTIAL EQUATIONS**

(a) In each of the following problems, find the general solution to the given differential equation.

i. $y'' + 2y' - 3y = 0$ Boyce 3.1 Q1

ii. $y'' + 3y' + 2y = 0$ Boyce 3.1 Q2

iii. $y'' + 5y' = 0$ Boyce 3.1 Q5

iv. $y'' - 2y' - 2y = 0$ Boyce 3.1 Q8

(b) In each of the following problems, find the solution to the given initial value problem, and sketch a graph of th solution, indicating the behaviour as t increases.

i. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$ Boyce 3.1 Q9

ii. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$ Boyce 3.1 Q10

iii. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$ Boyce 3.1 Q12

(c) Boyce 3.1 Q23

Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0,$$

where α is a given constant. Determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.