Homework 2

Total: 20 points
Due: Wed 22 Jan 2014 09:30 in class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **FIRST ORDER MODELING**

   (a) Boyce 2.3 Q5

   A tank contains 100 gal of water in which 50 oz of salt is dissolved. Water containing a salt concentration if $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

   i. Find the amount of salt in the tank at any time.
   ii. Sketch a graph of the solution for a time period long enough so that you see the ultimate behaviour of the graph.
   iii. The long-time behaviour of the graph should look like an oscillation about a certain constant level. What is this level, and what is the amplitude of oscillation?

   (b) Boyce 2.3 Q9

   A college student borrows 8000 to buy a car. The lender charges interest at a rate of 10% per annum. Assume that the interest is compounded continuously and that the student makes payments continuously at a constant annual rate k.

   i. Determine the payment rate k that is required to pay off the loan in 3 years.
   ii. What is the total amount the student pays over the 3-year period?

   (c) Some science fiction authors observe that the rate at which scientific advances are made by the human race are ever-increasing, and predict that as a result humanity will reach a point where our scientific progress becomes infinitely rapid. This point in time is known as the *Singularity*.
   Suppose we ascertain that humanity's rate at which scientific progress is made is proportional to the square of our current sum total scientific knowledge. Furthermore, suppose that scientists ascertain that the sum total scientific knowledge of humanity increased by 12.5% between the years of 2000 and 2010.
   When will the Singularity occur?

   (d) Boyce 2.3 Q17

   Heat transfer from a body to its surroundings through radiation is accurately described by Stefan-Boltzmann's law, which dictates that the rate of heat loss between the object and its surroundings is proportional to the difference between the 4th powers of their respective temperatures. However, if the object is much hotter than its surroundings, the system can be approximated by the differential equation

   $$\frac{dy}{dt} = -\alpha y^4,$$

   where $y(t)$ is the temperature of the object in degrees Kelvin, and α is a proportionality constant dependant on the physical parameters of the object in question.
   Suppose that a slug of molten steel with an initial temperature 2000 K is placed in a room whose temperature is controlled at 300 K, and suppose that $\alpha = 2.0 \times 10^{-12} K^{-3}/s$.

   i. Determine the temperature of the metal slug for all t by solving the above differential equation.
   ii. Plot the graph of $y(t)$.
   iii. Determine the time when the slug has cooled to 600 K, twice the ambient temperature. It is interesting to note that even though we are using an approximite differential equation, up to this time the error from the solution to the true DE is less than 1%.

   NB: More questions overleaf!

(e) Boyce 2.3 Q23

A skydiver weighing 180 lb (including equipment) jumps from a plane at 5000 ft and falls vertically downward; after 10 seconds of free fall the skydiver's parachute opens. Assume that the force of air resistance acts proportional and opposite to velocity, with proportionality constants 0.75 when the parachute is closed and 12 when it is open respectively. Here v is measured in ft/sec.

i. Find the speed of the skydiver when the parachute opens.

ii. Find the distance fallen when the parachute opens.

iii. What is the limiting velocity v_L after the parachute opens?

iv. Estimate to the nearest second how long the sky dive will take in its entirety i.e. from when the skydiver jumps from the plane until when they touch the ground.
Hint: In order to do so you'll end up needing to solve for t in an equation that looks like $t = A + Be^{-Ct}$ for some constants A, B and C (that you know). There is no way to solve for t explicitly in this equation, so you will have to find an approximate solution. To do this note that the Be^{-Ct} term is very small compared to the constant A term, so your answer should be very close to $t = A$.

2. **EXISTENCE & UNIQUENESS**

(a) In each of the following problems, use the existence and uniqueness theorems to determine (without solving the DE) the largest interval in which the solution to the given initial value problem is guaranteed to exist. Write a sentence or two justifying your answer.

i. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$ Boyce 2.4 Q1

ii. $t(t-4)y' + y = 0, \quad y(2) = 1$ Q2

iii. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$ Q3

iv. $(\ln t)y' + y = \cot t, \quad y(2) = 3$ Q6

(b) Find all solutions to the initial value problem

$$(y')^2 - y = 4, \quad y(2) = 0.$$

(c) Consider the initial value problem

$$\frac{dy}{dt} = (y-1)^{\frac{1}{5}}, \quad y(0) = y_0$$

i. For which value of y_0 does the IVP *not* have a unique solution in some interval about $x = 0$?

ii. For this value of y_0, find all continuous differentiable functions which satisfy the differential equation.