Fri 7 March MATH3074 LECTURE 21

§3.3 LAPLACE & DISCONTINUOUS FUNCTIONS

(BOYCE 6.3)

The Laplace transform comes into its own when considering DEs with forcing functions that are discontinuous.

In order to talk about functions with discontinuities, it is useful to introduce some terminology.

Example 3.3.1: Let
$$f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \ge 9. \end{cases}$$

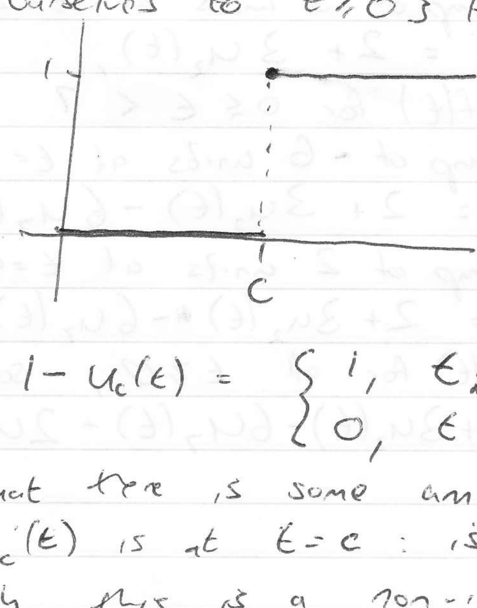
This is a cumbersome way to write f. There is a more compact way of expressing $f(t)$, for which we need a special function.

Definition 3.3.2 The Heaviside function or unit step function $u_c(t)$ is defined as

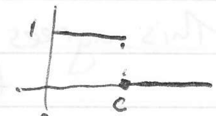
$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \ge c \end{cases}$$

(we restrict ourselves to $t \ge 0$) for some positive parameter c.

Graph:



Note that $1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \ge c \end{cases}$



* Note also that there is some ambiguity for what the value of $u_c(t)$ is at $t = c$: is it 0 or is it 1? For us, though, this is a non-issue; we will be using $u_c(t)$ to describe forcing functions, ~~like~~ for which the value at a point of discontinuity doesn't actually matter with regards to the solution.

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__Example 3.3.3:__ Consider $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \geq 2\pi. \end{cases}$

Then $f(t) = u_\pi(t) - u_{2\pi}(t)$.

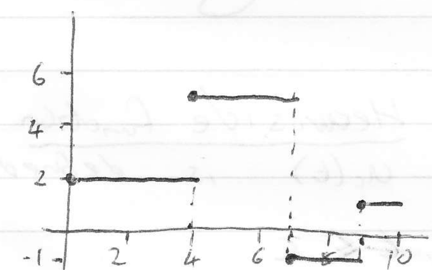
This is because $u_\pi(t) = 0$ & $u_{2\pi}(t) = 0$ for $0 \le t < \pi$

$u_\pi(t) = 1$ & $u_{2\pi}(t) = 0$ for $\pi \le t < 2\pi$

and $u_\pi(t) = 1$ & $u_{2\pi}(t) = 1$ for $2\pi \le t$.

Back to __Example 3.3.1__ : $f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \geq 9. \end{cases}$

Graph:



Start with $f_1(t) = 2$; this agrees with $f(t)$ for $0 \le t < 2$.

Then we need a jump of 3 units at $t = 2$, so get

$$f_2(t) = 2 + 3u_2(t),$$

this agrees with $f(t)$ for $0 \le t < 7$.

Then we need a jump of -6 units at $t = 7$, so get

$$f_3(t) = 2 + 3u_2(t) - 6u_7(t).$$

Finally we need a jump of 2 units at $t = 9$, so get

$$f_4(t) = 2 + 3u_2(t) - 6u_7(t) + 2u_9(t).$$

This agrees with $f(t)$ for all $t \geq 0$, so we have found that

$$f(t) = 2 + 3u_2(t) - 6u_7(t) + 2u_9(t).$$

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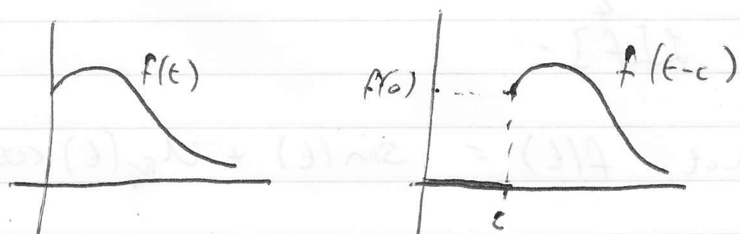
The reason why we would want to write step functions in terms of $u_c(t)$ is that it has a particularly nice Laplace transform:

Definition 3.3.4: $\mathcal{L}[u_c(t)] = \int_0^\infty u_c(t)e^{-st}\,dt$
$$= \int_c^\infty e^{-st}\,dt$$
$$= -\frac{1}{s}e^{-st}\Big|_c^\infty$$
$$\Rightarrow \boxed{\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}}$$

The Heaviside function also allows us to talk about the Laplace transform of the translate of a given function. Given a function $f(t)$, it will be useful to consider $f_c(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$

Graph:



But note that we have
$$f_c(t) = u_c(t)f(t-c).$$
The following theorem tells us how translated functions transform:

Theorem 3.3.5 Let $f(t)$ be such that $F(s) = \mathcal{L}[f(t)]$ exists for $s > a \geq 0$ and let c be a positive constant.

Then
$$\boxed{\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}F(s)}$$
Conversely, if $f(t) = \mathcal{L}^{-1}[F(s)]$, then
$$\boxed{\mathcal{L}^{-1}[e^{-cs}F(s)] = u_c(t)f(t-c)}$$

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Proof: $\mathcal{L}[u_c(t)f(t-c)] = \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt$

$= \int_c^\infty f(t-c)e^{-st}\,dt$

$= \int_0^\infty f(x)e^{-(x+c)s}\,dx$ using

the substitution $x = t-c$,

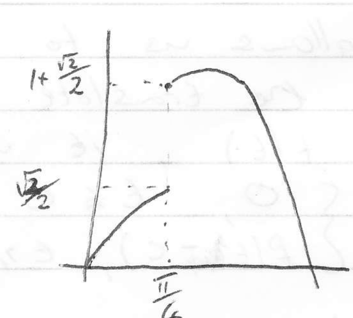
$= e^{-cs}\int_0^\infty f(x)e^{-sx}\,dx$

$= e^{-cs}\mathcal{L}[f]$.

The converse follows from taking the inverse Laplace transform of both sides.

Example 3.3.6: Let $f(t) = \begin{cases} \sin(t), & 0 \le t < \frac{\pi}{4} \\ \sin(t)+\cos(t-\frac{\pi}{4}), & t \geq \frac{\pi}{4} \end{cases}$

Graph:



We find $\mathcal{L}[f]$:

Note that $f(t) = \sin(t) + u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})$

Hence $\mathcal{L}[f(t)] = \mathcal{L}[\sin(t)] + \mathcal{L}[u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})]$

$= \mathcal{L}[\sin(t)] + e^{-\frac{\pi}{4}s}\mathcal{L}[\cos(t)]$ by Theorem 3.3.5

So $\mathcal{L}[f(t)] = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\cdot\frac{s}{s^2+1}$

$\Rightarrow \mathcal{L}[f(t)] = \frac{1+se^{-\frac{\pi}{4}s}}{s^2+1}$.

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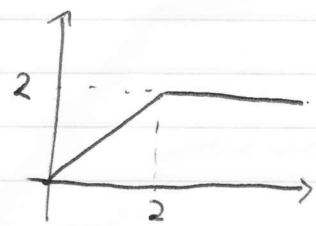
Example 3.3.7: Find $\mathcal{L}^{-1}[F(s)]$, where $F(s) = \frac{1-e^{-2s}}{s^2}$.

Solution: $\mathcal{L}^{-1}\left[\frac{1-e^{-2s}}{s^2}\right] = \mathcal{L}^{-1}[s^{-2}] - \mathcal{L}^{-1}\left[e^{-2s}\cdot\frac{1}{s^2}\right]$
$= t - u_2(t)\cdot(t-2)$ by the rules of Laplace.

Graph:



i.e. $\mathcal{L}^{-1}[F(s)] = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \geq 2. \end{cases}$

An analogy of Theorem 3.3.5 in the opposite direction:

Theorem 3.3.8: Let $f(t)$ be such that $F(s)$ exists for $s > a \geq 0$, and let c be a constant. Then $\mathcal{L}[e^{ct}f(t)] = F(s-c)$ for $s > a+c$.

– This is just the property covered in Example 3.1.7 of §3.1! It serves to highlight the duality between transformations in t-space vs. transformations in s-space.

Finally:

Theorem 3.3.9: Same setup as above with $c>0$, then

$$\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right) \quad \text{for } s > ca.$$

Proof: $\mathcal{L}[f(ct)] = \int_0^\infty f(ct)e^{-st}\,dt = \int_0^\infty f(x)e^{-s\left(\frac{x}{c}\right)}\cdot\frac{1}{c}\,dx$
using the transform $x = ct$, $= \frac{1}{c}\int_0^\infty f(x)\,e^{-\left(\frac{s}{c}\right)x}\,dx = \frac{1}{c}\mathcal{L}[f]\left(\frac{s}{c}\right)$.

$\square$