# Homework 2

Total: 20 points
Due: Wed 22 Jan 2014 09:30 in class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **FIRST ORDER MODELING**

(a) Boyce 2.3 Q5

A tank contains 100 gal of water in which 50 oz of salt is dissolved. Water containing a salt concentration if $\frac{1}{4}(1 + \frac{1}{2}\sin t)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

i. Find the amount of salt in the tank at any time.

ii. Sketch a graph of the solution for a time period long enough so that you see the ultimate behaviour of the graph.

iii. The long-time behaviour of the graph should look like an oscillation about a certain constant level. What is this level, and what is the amplitude of oscillation?

(b) Boyce 2.3 Q9

A college student borrows 8000 to buy a car. The lender charges interest at a rate of 10% per annum. Assume that the interest is compounded continuously and that the student makes payments continuously at a constant annual rate $k$.

i. Determine the payment rate $k$ that is required to pay off the loan in 3 years.

ii. What is the total amount the student pays over the 3-year period?

(c) Some science fiction authors observe that the rate at which scientific advances are made by the human race are ever-increasing, and predict that as a result humanity will reach a point where our scientific progress becomes infinitely rapid. This point in time is known as the *Singularity*.
Suppose we ascertain that humanity's rate at which scientific progress is made is proportional to the square of our current sum total scientific knowledge. Furthermore, suppose that scientists ascertain that the sum total scientific knowledge of humanity increased by 12.5% between the years of 2000 and 2010.
When will the Singularity occur?

What I'm testing for here is problem-solving ability. You are given less information here than usual, but you still have enough to answer the question. It's up to us to choose variables and units; the answer shouldn't be affected by the choice. So let $y(t)$ be the scientific knowledge of humanity at time $t$, where $t$ is in years AD and $y$ is some appropriate unspecified choice of units. We then have

$$\frac{dy}{dt} = ky^2$$

for some constant $k$.
This is separable, so

$$\frac{1}{y^2}\,dy = k\,dt$$

Antidifferentiating yields

$$-\frac{1}{y} = k(t - C)$$

(note that keeping $k$ on the outside on the right is the most illuminating, but we can also have $kt - C$), and solving for $y$ we then get

$$y = \frac{1}{k(C - t)}.$$

Now we know that $y(2010) = \frac{9}{8}y(2000)$; this is not enough to solve for both $k$ and $C$, but it is enough to solve for $C$ in terms of $k$, which in turn is enough to answer the question:

$$\frac{1}{k(C-2010)} = \frac{9}{8} \cdot \frac{1}{k(C-2000)},$$

so we can clear $k$ and multiply through to get

$$C - 2000 = \frac{9}{8}(C - 2010)$$

Solving for $C$ yields $C = 2090$.
We thus have that

$$y(t) = \frac{1}{k(2090-t)}.$$

This function asymptotes to infinity when $t \to 2090$. The model therefore suggests that the Singularity will occur in the year 2090.

Note that the choice of units of time is up to you; we could just as well define time to be years since 2000, for example. However, the final answer of the Singularity occurring in 2090 is the same regardless of what units you use for time.

(d) Boyce 2.3 Q17

Heat transfer from a body to its surroundings through radiation is accurately described by Stefan-Boltzmann's law, which dictates that the rate of heat loss between the object and its surroundings is proportional to the difference between the 4th powers of their respective temperatures. However, if the object is much hotter than its surroundings, the system can be approximated by the differential equation

$$\frac{dy}{dt} = -\alpha y^4,$$

where $y(t)$ is the temperature of the object in degrees Kelvin, and $\alpha$ is a proportionality constant dependant on the physical parameters of the object in question.
Suppose that a slug of molten steel with an initial temperature 2000 $K$ is placed in a room whose temperature is controlled at 300 $K$, and suppose that $\alpha = 2.0 \times 10^{-12} K^{-3}/s$.

i. Determine the temperature of the metal slug for all $t$ by solving the above differential equation.
ii. Plot the graph of $y(t)$.
iii. Determine the time when the slug has cooled to 600 $K$, twice the ambient temperature. It is interesting to note that even though we are using an approximite differential equation, up to this time the error from the solution to the true DE is less than 1%.

(e) Boyce 2.3 Q23

A skydiver weighing 180 lb (including equipment) jumps from a plane at 5000 ft and falls vertically downward; after 10 seconds of free fall the skydiver's parachute opens. Assume that the force of air resistance acts proportional and opposite to velocity, with proportionality constants 0.75 when the parachute is closed and 12 when it is open respectively. Here $v$ is measured in ft/sec.

i. Find the speed of the skydiver when the parachute opens.
ii. Find the distance fallen when the parachute opens.
iii. What is the limiting velocity $v_L$ after the parachute opens?
iv. Estimate to the nearest second how long the sky dive will take in its entirety i.e. from when the skydiver jumps from the plane until when they touch the ground.
Hint: In order to do so you'll end up needing to solve for $t$ in an equation that looks like $t = A + Be^{-Ct}$ for some constants $A, B$ and $C$ (that you know). There is no way to solve for $t$ explicitly in this equation, so you will have to find an approximate solution. To do this note that the $Be^{-Ct}$ term is very small compared to the constant $A$ term, so your answer should be very close to $t = A$.

2. **EXISTENCE & UNIQUENESS**

(a) In each of the following problems, use the existence and uniqueness theorems to determine (without solving the DE) the largest interval in which the solution to the given initial value problem is guaranteed to exist. Write a sentence or two justifying your answer.

i. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$ Boyce 2.4 Q1

ii. $t(t-4)y' + y = 0, \quad y(2) = 1$ Q2

iii. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$ Q3

iv. $(\ln t)y' + y = \cot t, \quad y(2) = 3$ Q6

(b) Find all solutions to the initial value problem

$$(y')^2 - y = 4, \quad y(2) = 0.$$

Clearly this function is nonlinear. We can't apply the existence/uniqueness theorem for nonlinear ODEs until we have it in the for $y' = f(x, y)$. Solving for $y'$ in the above equation yields two distinct options:

$$\frac{dy}{dt} = \sqrt{y+4} \quad \text{and} \quad \frac{dy}{dt} = -\sqrt{y+4}$$

We must treat each of these separately.

Consider the first option: $\frac{dy}{dt} = \sqrt{y+4}$ with $y(2) = 0$. Both $\sqrt{y+4}$ and $\frac{\partial}{\partial y}\sqrt{y+4} = \frac{1}{2\sqrt{y+4}}$ are continuous about y=0, so the existince/uniqueness theorem guarantees that there is a unique solution to the IVP in some interval about $x = 2$.
We proceed to solve this DE. It is separable, so

$$\frac{1}{\sqrt{y+4}}\, dy = dt$$

Integrating yields

$$2\sqrt{y+4} = t + C,$$

or, solving for $y$,

$$y = \left(\frac{1}{2}t + \frac{1}{2}C\right)^2 - 4$$

Applying the initial value: $t = 2, y = 0$ yields either $C = 2$ or $C = -6$; however, only the value $C = 2$ obeys the original equation $2\sqrt{y+4} = t + C$ when $t = 2$ and $y = 0$ (by squaring both sides we've introduced an extra solution for $C$). The solution to the DE is thus, after simplifying,

$$y = \frac{1}{4}t^2 + t - 3.$$

For the second option $y' = -\sqrt{y+4}$ we proceed in much the same manner. Again we are guaranteed a single unique solution. Solving the separable DE yields

$$-2\sqrt{y+4} = t + C \quad \Longrightarrow \quad y = \left(-\frac{1}{2}t - \frac{1}{2}C\right)^2 - 4.$$

Applying the IC $y(2) = 0$ to the latter equation yields $C = 2$ or $C = -6$, but similar to before only $C = -6$ satisfies $-2\sqrt{y+4} = t + C$ when $y(2) = 0$. Hence, after simplifying, we have

$$y = \frac{1}{4}t^2 - 3t + 5.$$

The existence/uniqueness theorem guarantees that these are the only two solutions to the IVP.

(c) Consider the initial value problem

$$\frac{dy}{dt} = (y-1)^{\frac{1}{5}}, \quad y(0) = y_0$$

i. For which value of $y_0$ does the IVP *not* have a unique solution in some interval about $x = 0$?

The existence/uniqueness theorem for nonlinear ODEs in the form $y' = f(x, y),\ y(x_0) = y_0$ dictates that we may only not have a unique solution when either $f$ or $\frac{\partial f}{\partial y}$ is discontinuous at $(x_0, y_0)$.
For us $f(x, y) = (y-1)^{\frac{1}{5}}$; this function is continuous for all real $x$ and $y$. However

$$\frac{\partial f}{\partial y} = \frac{1}{5}(y-1)^{-\frac{4}{5}}$$

is discontinuous at $y = 1$; It therefore follows that the only possible instance of non-uniqueness can occur when $y(0) = 1$.

ii. For this value of $y_0$, find all continuous differentiable functions which satisfy the differential equation.

This is a separable differential equation, so

$$(y-1)^{-\frac{1}{5}}\ dy = dt.$$

Antidifferentiating yields

$$\frac{5}{4}(y-1)^{\frac{4}{5}} = t + C;$$

solving for $y$ then has

$$y = 1 + \left(\frac{4}{5}t + C\right)^{\frac{5}{4}},$$

where we absorb the constant multiplier $\frac{4}{5}$ into the $C$.
Applying the initial condition $y(0) = 1$ has $C = 0$, so

$$y = 1 + \left(\frac{4}{5}t\right)^{\frac{5}{4}}$$

is one solution to the DE.

Now note that $y = 1$ is also a solution to the IVP (this can be done by eyeballing the differential equation); further note that the equation

$$y = 1 - \left(\frac{4}{5}t\right)^{\frac{5}{4}}$$

also solve the IVP.

Next, note that $\frac{d}{dt}\ 1 + \left(\frac{4}{5}t\right)^{\frac{5}{4}} = 0$ when $y = 1$, as does $\frac{d}{dt}1 - \left(\frac{4}{5}t\right)^{\frac{5}{4}} = 0$; what this means is that we can splice together the above functions, suitably shifted, into a new function and create yet more solutions. Specifically, for any constants $c_1 \leq 0$ and $c_2 \geq 0$, the generalized function

$$y(t) = \begin{cases} 1 \pm \left(\frac{4}{5}(t-c_1)\right)^{\frac{5}{4}} & t < c_1 \\ 1 & c_1 \leq t \leq c_2 \\ 1 \pm \left(\frac{4}{5}(t-c_2)\right)^{\frac{5}{4}} & t > c_2 \end{cases}$$

is a continuous and continuously differentiable function that also solves the differential equation. This is in fact the most general solution to the DE.