Homework 5 Solutions

Total: 20 points
Due: Wed 5 November 2014 at the beginning of class

Remember to show all steps in your working. If a question is taken from the textbook, the refence is given on the right of the page.

1. CHARACTERISTIC EQUATIONS WITH EQUAL ROOTS

(a) Find the general solution to the following differential equations:

i. $9y'' + 6y' + y = 0$ Boyce 3.4 Q2

ii. $y'' - 6y' + 9y = 0$ Boyce 3.4 Q6

(b) Find the solution to each of the following initial value problems. Sketch the graph of the solution and describe its behaviour for increasing t:

i. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, y'(0) = -1$ Boyce 3.4 Q11

ii. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, y'(-1) = 1$ Boyce 3.4 Q14

(c) Boyce 3.4 Q16

Consider the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \qquad y(0) = 2, \quad y'(0) = b,$$

where b is a constant. Find the solution as a function of t for a given b, and determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.

(d) Boyce 3.4 Q24

Consider the Euler equation

$$t^2y'' + 2ty' - 2y = 0, \qquad t > 0.$$

We may use the method detailed in the previous homework (substituting $x = \ln(t)$) to find the general solution to this equation. However, if we already know a specific solution we may instead use the method of reduction in order to find the DE's general solution.
Given that $y_1(t) = t$ is a solution to this differential equation, use the method of reduction of order to find the general solution to this DE.

The method of reduction of order dictates that we suppose the general solution takes the form $y = v(t)y_1(t)$, where $v(t)$ is some undetermined function of t. We then plug y back into the DE to obtain a second differential equation in v and t; solving this tells us what v is, and thus what the general solution to the original DE is.

So let $y = v(t)y_1(t) = vt$, where we remember that v is a function of t. Hence by the product rule $y' = v't + v$; differentiating this a second time we get $y'' = v''t + v' + v' = v''t + 2v'$.

Plugging this in to the DE we get

$$\begin{aligned} t^2y'' + 2ty' - 2y &= t^2(v''t + 2v') + 2t(v't + v) - 2(vt) \\ &= t^3v'' + 2t^2v' + 2t^2v' + 2tv - 2tv \\ &= t^2(tv'' + 4v') \end{aligned}$$

Since we want $y(t)$ to solve the DE, this must be equal to 0. And since we are told $t > 0$, we must therefore conclude that

$$tv'' + 4v' = 0.$$

This is a 2nd-order linear DE, but it is a particularly straightforward one to solve. Let $w = v'$, then

$$tw' + 4w = 0.$$

This first-order DE is both linear and separable, so it can be solved either way; we solve it by separating variables here. Rearranging terms we get

$$t\frac{dw}{dt} = -4w,$$

which, once variables are separated, becomes

$$\frac{1}{w}\,dw = -\frac{4}{t}\,dt$$

Now integrate both sides. The integral on the left is $\ln|w|$, and on the right we get $-4\ln(t) + A$ (we don't need absilute value signs for this one, as t is positive). Exponentiating both sides and absorbing the absolute value signs into the constant gives us

$$w = Bt^{-4},$$

where $B = \pm e^A$ as necessary. Now $w = v'$, so the antiderivative of the above gives us v:

$$v = c_1 t^{-3} + c_2,$$

where $c_1 = -\frac{B}{3}$ and c_2 is the second integration constant introduced by antidifferentiating w. Thus we arrive at

$$y = vt = c_1 t^{-2} + c_2 t$$

for arbitrary constants c_1 and c_2. This is indeed the general solution to the DE.

2. **NONHOMOGENEOUS EQUATIONS**

(a) Find the general solution to the following differential equations:

i. $y'' - 2y' - 3y = 3e^{2t}$ Boyce 3.5 Q1

ii. $y'' + 2y' + 5y = 3\sin(2t)$ Boyce 3.5 Q2

iii. $y'' + 2y' + y = 2e^{-t}$ Boyce 3.5 Q8

iv. $y'' + y' + 4y = 2\sinh t$ [Hint: $\sinh t = \frac{1}{2}(e^t - e^{-t})$] Boyce 3.5 Q13

(b) Boyce 3.5 Q20

Solve the initial value problem

$$y'' + 2y' + 5y = 4e^{-t}\cos(2t), \qquad y(0) = 1, \quad y'(0) = 0.$$

(c) Consider the following initial value problem:

$$y'' + 4y' + 4y = \cos(t) + e^{-2t}, \qquad y(0) = 1, \quad y'(0) = 0.$$

The function $g(t) = \cos(t) + e^{-2t}$ is often called the *forcing function* in this DE, and the corresponding solution $y(t)$ to the IVP is called the *response*.

i. Find the solution to the above IVP.

We start by finding the general solution to the DE. Recall that the general solution to $L[y] = g(t)$ can be written in the form

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$

where $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions to the homogeneous DE $L[y] = 0$, and $Y(t)$ is a particular solution to the full non-homogenous DE.

Thus first we find the general solution to $y'' + 4y' + 4y = 0$. This has the characteristic equation

$$r^2 + 4r + 4 = 0,$$

which has a single double root at $r = -2$. Thus, looking at our notes, we see it has general solution

$$y = (c_1 + c_2 t)e^{-2t}.$$

In other words, we can choose $y_1(t) = e^{-2t}$ and $y_2 = te^{-2t}$ as our fundamental set of solutions to the homogeneous DE.

Next, we find the specific solution $Y(t)$ to the non-homogeneous DE. Recall the linearity of specific solutions: if $Y_1(t)$ is a specific solution to $L[y] = g_1(t)$ and $Y_2(t)$ is a specific solution to $L[y] = g_2(t)$, then $Y = Y_1(t) + Y_2(t)$ is a specific solution to $L[y] = g_1(t) + g_2(t)$. We note that we have two term on the RHS in our non-homogeneous DE; we will thus find the specific solutions corresponding to each of them separately, then add those two solutions together to get the specific solution to the full non-homogeneous DE.

Let $Y_1(t)$ be the specific solution to

$$y'' + 4y' + 4y = \cos(t).$$

Using the method of undetermined coefficients (and recalling that cos and sin functions hunt in packs), we guess Y_1 to be of the form

$$Y_1(t) = A\cos t + B\sin t.$$

Thus $Y_1' = -A\sin t + B\cos t$ and $Y_1'' = -A\cos t - B\sin t$; hence

$$\begin{aligned} Y_1'' + 4Y_1' + 4Y_1 &= (-A\cos t - B\sin t) + 4(-A\sin t + B\cos t) + 4(A\cos t + B\sin t) \\ &= (3A + 4B)\cos t + (-4A + 3B)\sin t \end{aligned}$$

If Y_1 solves $y'' + 4y' + 4y = cos(t)$, then we must have that the above is equal to $\cos t = 1\cot\cos 1 + 0 \cdot \sin t$ for any value of t; the only way this can be true is if the coefficients in front of the respective cos and sin terms are equal. Thus we deduce the system of equations in A and B:

$$\begin{aligned} 3A + 4B &= 1 \\ -4A + 3B &= 0 \end{aligned}$$

Solving this system of equations gives us

$$A = \frac{3}{25}, \qquad B = \frac{4}{25};$$

Thus

$$Y_1(t) = \frac{3}{25}\cos t + \frac{4}{25}\sin t.$$

Next, let $Y_2(t)$ be the specific solution to

$$y'' + 4y' + 4y = e^{-2t}.$$

Observing that both e^{-2t} and te^{-2t} are solutions to the homogeneous equation, we guess that Y_2 is of the form

$$Y_2(t) = (At^2 + Bt + C)e^{-2t}.$$

However, we already know that the $(Bt + C)e^{-t}$ part will give us zero when we plug it back into the DE, so we might as well set B and C equal to 0. In other words we have

$$Y_2(t) = At^2e^{-2t}$$

. Then

$$Y_2' = (2At)e^{-2t} - 2(At^2)e^{-2t} = (-2At^2 + 2At)e^{-t}$$

and

$$Y_2'' = (-4At + 2A)e^{-t} - 2(-2At^2 + 2At)e^{-2t} = (4At^2 - 8At + 2A)e^{-2t}$$

So

$$\begin{aligned} Y_2'' + 4Y_2' + 4Y_2 &= (4At^2 - 8At + 2A)e^{-2t} + 4(-2At^2 + 2At)e^{-t} + 4At^2e^{-2t} \\ &= 2Ae^{-2t}. \end{aligned}$$

This must equal e^{-2t} for Y_2 to solve $y'' + 4y' + 4y = e^{-2t}$, so we must have that $2A = 1$. Thus $A = \frac{1}{2}$, and we get

$$Y_2(t) = \frac{1}{2}t^2e^{-2t}.$$

We conclude that our particular solution to $y'' + 4y' + 4y = \cos t + e^{-2t}$ is

$$Y(t) = Y_1(t) + Y_2(t) = \frac{3}{25}\cos t + \frac{4}{25}\sin t + \frac{1}{2}t^2e^{-2t},$$

and that the general solution to the full non-homogeneous differential equation is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t) = \left(c_1 + c_2t + \frac{1}{2}t^2\right)e^{-2t} + \frac{3}{25}\cos t + \frac{4}{25}\sin t.$$

Finally, we apply the initial conditions to obtain the solution to the IVP. We have y(0) = 1, so

$$1 = \left(c_1 + c_2 \cdot 0 + \frac{1}{2} \cdot 0^2\right) \cdot 1 + \frac{3}{25} \cdot 1 + \frac{4}{25} \cdot 0 = c_1 + \frac{3}{25},$$

so $c_1 = \frac{22}{25}$. Next, we have $y'(0) = 0$. Now

$$y' = (c_2 + t)e^{-2t} - 2\left(c_1 + c_2t + \frac{1}{2}t^2\right)e^{-2t} - \frac{3}{25}\sin t + \frac{4}{25}\cos t,$$

so

$$0 = (c_2 + 0) \cdot 1 - 2\left(c_1 + c_2 \cdot 0 + \frac{1}{2} \cdot 0^2\right) \cdot 1 - \frac{3}{25} \cdot 0 + \frac{4}{25} \cdot 1 = c_2 - 2c_1 + \frac{4}{25}.$$

Using $c_1 = \frac{22}{25}$, we solve for c_2 to get $c_2 = \frac{40}{25}$

Hence at last we have arrived at the solution to the initial value problem:

$$\begin{aligned} y &= \left(\frac{22}{25} + \frac{40}{25}t + \frac{1}{2}t^2\right)e^{-2t} + \frac{3}{25}\cos t + \frac{4}{25}\sin t \\ &= \frac{1}{25}\left[\left(22 + 40t + \frac{25}{2}t^2\right)e^{-2t} + 3\cos t + 4\sin t\right]. \end{aligned}$$

ii. The limiting behaviour of the response, after exponential terms have decayed, can be written in the form $R\cos(t-\delta)$, where R and δ are called the *amplitude* and *phase shift* of the response respectively. Find R and δ.

Observe that in the solution as listed above, the $\left(\frac{22}{25}+\frac{40}{25}t+\frac{1}{2}t^2\right)e^{-2t}$ term decays to zero as t increases (exponential decay always eventually beats polynomial growth). Thus after a while, the solution looks like

$$y(t) = \frac{3}{25}\cos t + \frac{4}{25}\sin t$$

(this is known as the *steady state response* of the system). Recall that we have

$$A\cos t + B\sin t = R\cos(t-\delta),$$

where $R=\sqrt{A^2+B^2}$, and $\tan(\delta)=\frac{B}{A}$. Using these formulae we get

$$R = \sqrt{\left(\frac{3}{25}\right)^2+\left(\frac{4}{25}\right)^2} = \sqrt{\frac{25}{25^2}} = \frac{1}{5},$$

and

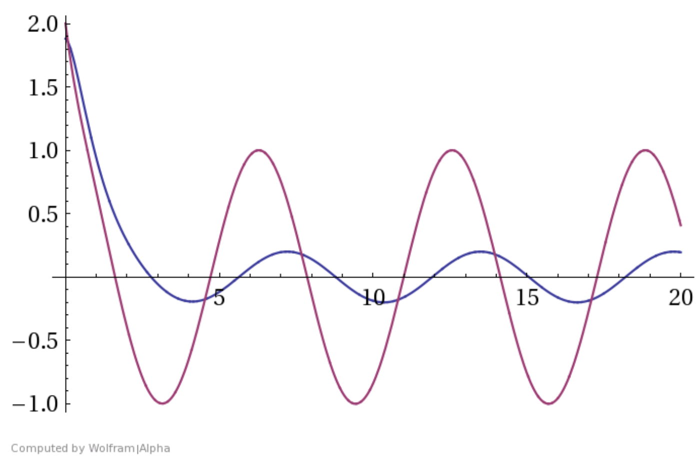
$$\tan(\delta) = \frac{4}{3}.$$

Thus $\delta = \arctan\frac{4}{3} == 0.927295\ldots$ or $\arctan\frac{4}{3}+\pi = 4.068887\ldots$ radians. However, we see that both A and B are positive, so δ must be in the first quadrant, i.e. between 0 and $\frac{\pi}{2}$; We conclude that

$$\delta = \arctan\frac{4}{3} = 0.927295\ldots \quad \text{rad.}$$

iii. Plot a graph of the response as a function of time. Also include on the graph a plot of $\cos(t)$, the oscillating part of the forcing function. Be sure to indicate on the graph the amplitude and phase shift of the response's limiting behaviour vs. that of $\cos(t)$.

A graph of the solution vs. the forcing function for $0 \le t \le 20$ looks as follows:



The response is plotted in blue, and the forcing function in red. Note that both the forcing function and the response rapidly converge to sinusoidal functions of the same frequency. However, the response has an amplitude of only one fifth of the forcing function, and its phase is offset with respect to the forcing function too (by $\arctan(4/3)$, which is about 15% of a cycle).