Homework 5 Solutions

Total: 20 points
Due: Wed 19 Feb 2014 09:30 in class

Remember to show all steps in your working. If a question is taken from the textbook, the refence is given on the right of the page.

1. **NONHOMOGENEOUS EQUATIONS**

(a) Solve the following initial value problem.

$$y'' + 3y' - 4y = 4t - 5e^{-4t}, \qquad y(0) = 1, \quad y'(0) = 0.$$

[Note: This is the same IVP as the one mentioned at the end of class on Monday Feb 10, except that one of the coefficients in the forcing function on the RHS of the equation has been changed to make the arithmetic a bit nicer.]

First we find the general solution to the homogeneous equation

$$y'' + 3y' - 4y = 0.$$

The corresponding characteristic equation is

$$r^2 + 3r - 4 = 0,$$

which has roots $r_1 = 1, r_2 = -4$. Hence the general solution is

$$y = c_1 e^t + c_2 e^{-4t}.$$

Next we find a specific solution $Y(t)$ to the full nonhomogeneous DE. To do this, we recall that $Y(t) = Y_1(t) + Y_2(t)$, where $Y_1(t)$ is a specific solution to the DE

$$y'' + 3y' - 4y = 4t$$

and $Y_2(t)$ is a specific solution to

$$y'' + 3y' - 4y = -5e^{-4t}.$$

We will use the method of undetermined coefficients to find $Y_1(t)$ and $Y_2(t)$.

For the first case $y'' + 3y' - 4y = 4t$, we note that the forcing function $g(t) = 4t$ is a polynomial of degree 1, so we guess that $Y(t)$ will also be a degree 1 polynomial in t, i.e.

$$Y_1(t) = At + B$$

for some A and B. Doing this we compute that $Y_1' = A$ and $Y_1'' = 0$, so then

$$Y_1'' + 3Y_1' - 4Y_1 = 0 + 3A - 4(At + B) = -4At + (3A - 4B).$$

We must therefore have that $-4At + (3A - 4B) = 4t$, i.e.

$$-4A = 4 \quad \text{and} \quad 3A - 4B = 0.$$

Solving yields $A = -1$ and $B = -\frac{3}{4}$, so we have obtained

$$Y_1(t) = -t - \frac{3}{4}.$$

Next up we find $Y_2(t)$, the specific solution to $y'' + 3y' - 4y = -5e^{-4t}$. The forcing function $g(t) = -5e^{-4t}$ is exponential, so ordinarily we would guess $Y_2(t) = Ae^{-4t}$ and solve for A. However, e^{-4t} is a solution to the homogeneous DE $y'' + 3y' - 4y = 0$, so we must bump up our guess by a power of t. That is, we guess

$$Y_2 = Ate^{-4t}$$

for some constant A. We then have $Y_2' = e^{-4t}(A - 4At)$ and $Y_2'' = e^{-4t}(-8A + 16At)$. Plugging these into the LHS of the DE and setting equal to the RHS yields

$$e^{-4t}(-8A + 16At) + e^{-4t}(3A - 12At) - 4Ate^{-4t} = -5e^{-4t};$$

after canceling the e^{-4t} term and simplifying we get $-5A = -5$, so $A = 1$. We therefore have

$$Y_2(t) = te^{-4t}.$$

Therefore the full specific solution is $Y(t) = -\frac{3}{4} - t + te^{-4t}$. The full general solution to $y'' + 3y' - 4y = 4t - 5e^{-4t}$ is the sum of the general solution to the homogeneous DE and the specific solution, so we obtain

$$y = -\frac{3}{4} - t + te^{-4t} + c_1e^t + c_2e^{-4t}.$$

Finally, we apply the initial conditions. We have $y = 1$ when $t = 0$, so $1 = -\frac{3}{4} - 0 + 0 + c_1 + c_2$, or $c_1 + c_2 = \frac{7}{4}$. The other condition is on y', so we calculate $y' = -1 + e^{-4t} - 4te^{-4t} + c_1e^t - 4c_2e^{-4t}$. We have $y' = 0$ when $t = 0$, so $0 = -1 + 0 - 0 + c_1 - 4c_2$, or $c_1 - 4c_2 = 0$. Solving the system of 2 linear equations yields

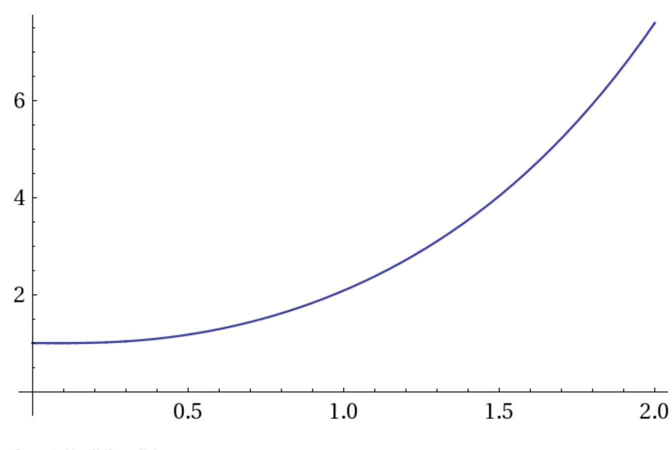
$$c_1 = \frac{7}{5}, \qquad c_2 = \frac{7}{20}.$$

Therefore the solution to the IVP is

$$y = -\frac{3}{4} - t + te^{-4t} + \frac{7}{5}e^t + \frac{7}{20}e^{-4t}.$$

(b) Plot a graph of the solution to the above IVP for $t \geq 0$. What is $\lim_{t\to\infty} y(t)$?

A plot for the solution for $0 \leq t \leq 2$ looks as follows:



We can see immediately that $\lim_{t\to\infty} y(t) = +\infty$. This can also be seen from analyzing the solution $y = -\frac{3}{4} - t + te^{-4t} + \frac{7}{5}e^t + \frac{7}{20}e^{-4t}$: the terms te^{-4t} and $\frac{7}{20}e^{-4t}$ decay to 0 over time while $-\frac{3}{4}$ (obviously) remains constant, so all three will be dominated by the other two terms, which grow in magnitude with time. Now The $-t$ term goes to $-\infty$ while $\frac{7}{5}e^t$ goes to $+\infty$; however, exponential growth always beats polynomial growth, so the $\frac{7}{5}e^t$ term is the dominant term in the solution function. We therefore have the solution going to $+\infty$ with time.

2. **2nd-ORDER MODELING**

(a) In the following two equations, determine ω_0, R and δ so as to write them in the form $y = R\cos(\omega_0 t - \delta)$:

i. $y = -\cos(t) + \sqrt{3}\sin(t)$ Boyce 3.7 Q2

ii. $y = -2\cos(\pi t) - 3\sin(\pi t)$ Boyce 3.7 Q4

(b) Boyce 3.7 Q5

A mass weighing 2 lb hangs from a spring, stretching it 6 inches. If the mass is pulled down an additional 3 in and then released, and there is no damping, determine the vertical position y of the mass at any time t. Also find the frequency, period and amplitude of the motion.

[Hint: You will need to use the fact that that the 2 lb weight stretches the spring 6 inches when at rest to find the spring constant k. This is the point where the downward gravitational force on the mass exactly balances the upwards force imparted by the spring. You may take $g = 32\ ft/s^2$, and remember that in imperial units mass m is given by $m = w/g$, where w is the weight of the object in pounds. If you need extra guidance on this question, Boyce section 3.7 has a number of similar examples, and details the method on how to set up the DE.]

(c) A 2 kg block is placed on a smooth surface attached to a horizontally-acting spring with spring constant $k = 9\,\frac{1}{16}$ kg/s^2. We have seen in class that the displacement in meters $y(t)$ of the block from its rest position is then modeled by the differential equation

$$2y'' + \frac{145}{16}y = 0.$$

Suppose that the surface is not actually frictionless, but instead imparts a force of $-\gamma v$ on the block, where v is the horizontal velocity of the block, and γ is a constant.

i. The block is released from a position of $y = 0.5$ m with zero initial velocity. Find the position of the block for all time $t \geq 0$ if $\gamma = 0.5$ kg/s.

We must solve the IVP

$$2y'' + 0.5y' + 9.0625y = 0, \qquad y(0) = 0.5, \quad y'(0) = 0.$$

This has the characteristic equation

$$2r^2 + \frac{1}{2}r + \frac{145}{16} = 0,$$

or $32r^2 + 8r + 145 = 0$ after clearing denominators. This has roots

$$r = \frac{-8 \pm \sqrt{8^2 - 4\cdot 32\cdot 145}}{2\cdot 32} = \frac{-8 \pm \sqrt{-18496}}{64},$$

so (after simplifying) the two distinct roots to the characteristic equation are

$$r_1 = -\frac{1}{8} + \frac{17}{8}\cdot i, \quad -\frac{1}{8} - \frac{17}{8}\cdot i$$

The fact that the CE has complex roots means that the general solution can be written in the form

$$y = e^{\lambda t}\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right),$$

where in this case $\lambda = -\frac{1}{8}$ and $\omega = \frac{17}{8}$. We thus have the general solution to the DE as

$$y = e^{-\frac{1}{8}t}\left(c_1\cos\left(\frac{17}{8}t\right) + c_2\sin\left(\frac{17}{8}t\right)\right).$$

Applying the initial condition $y(0) = \frac{1}{2}$ yields $c_1 = \frac{1}{2}$. The second initial condition $y'(0) = 0$ requires us to find y':

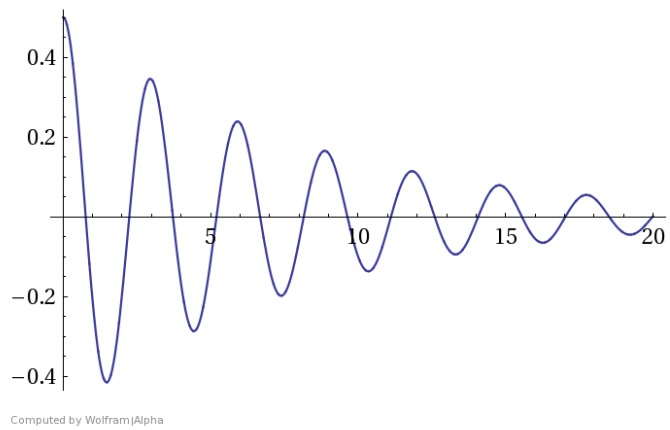
$$y' = e^{-\frac{1}{8}t}\left(\left(-\frac{1}{8}c_1 + \frac{17}{8}c_2\right)\cos\left(\frac{17}{8}t\right) + \left(-\frac{1}{8}c_2 - \frac{17}{8}c_1\right)\sin\left(\frac{17}{8}t\right)\right);$$

So $y'(0) = 0$ implies that $-\frac{1}{8}c_1 + \frac{17}{8}c_2 = 0$, or $-c_1 + 17c_2 = 0$ after clearing denominators. Since $c_1 = \frac{1}{2}$ we thus have that $c_2 = \frac{1}{34}$. Hence the position of the block at time t is given by the function

$$y(t) = e^{-\frac{1}{8}t}\left(\frac{1}{2}\cos\left(\frac{17}{8}t\right) + \frac{1}{34}\sin\left(\frac{17}{8}t\right)\right)$$

ii. Plot a graph of the above solution. Find the quasi-period of the oscillations of the block, and a time t_0 for which $|y(t)| < 0.05$ m for all $t > t_0$.

A plot for the solution for $0 \le t \le 20$ looks as follows:



The quasi-period T of the oscillation is the time taken for the cos and sin parts of the solution equation to complete one full cycle, i.e. the T for which $\frac{17}{8}T = 2\pi$. Solving for t, we see that the quasi-period is

$$T = \frac{16}{17}\pi \approx 2.9568 \text{ seconds.}$$

To find a time t_0 for which $|y(t)| < 0.05$ m for all $t > t_0$, we must first ascertain the amplitude of the oscillations. Recall that if $y = A\cos(\omega t) + B\sin(\omega t)$, then the amplitude of y is $R = \sqrt{A^2 + B^2}$. Therefore in our example the oscillations have amplitude

$$R = e^{-\frac{1}{8}t}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{17}\right)^2} = e^{-\frac{1}{8}t} \cdot \frac{\sqrt{293}}{34} \approx 0.5034e^{-\frac{1}{8}t}$$

As our graph indicates, the amplitude of the oscillations is decreasing exponentially. Therefore we must find the time t_0 when $0.5034e^{-\frac{1}{8}t_0} = 0.05$. Solving for t_0 yields

$$t_0 = -8 * \ln\left(\frac{34}{20\sqrt{293}}\right) = 4\ln\left(\frac{29300}{289}\right) \approx 18.4757 \text{ seconds.}$$

An acceptable approximation is to note that the amplitude of the sin part of the oscillation $(\frac{1}{17})$ is much smaller than that of the cos part of the oscillation $(\frac{1}{2})$. The total amplitude of the oscillation is thus approximately $\frac{1}{2}$. If we make this simplification, then we end up solving for t_0 when $e^{-\frac{1}{8}t_0} = 0.1$. We then have that

$$t_0 = 8\ln(10) = 18.4207 \text{ seconds.}$$

This agrees with the true answer to 1 decimal place, so it's a decent approximation.

iii. Given the same starting conditions as above, what is the smallest value of γ which will result in the block never crossing it rest position?

We require the smalles value of γ so that the solution to the DE does not exhibit oscillating behaviour. Oscillating behaviour occurs when the characteristic equation $2r^2 + \gamma r + \frac{145}{16} = 0$ has complex roots. In other words, we must find the smallest γ for which the discriminant of the quadratic is not negative. Hence we must solve the equation

$$\gamma^2 - 4 \cdot 2 \cdot \frac{145}{16} = 0,$$

or $\gamma^2 = \frac{145}{2}$. Taking the positive root (we know that $\gamma > 0$), we have

$$\gamma = \sqrt{\frac{145}{2}} \approx 8.5147 \; kg\, s^{-1}.$$

(d) Boyce 3.7 Q8

A series circuit has a capacitor of capacitance 0.25×10^{-6} F, an inductor of inductange 1 H, and negligible resistence. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t.

(e) Boyce 3.7 Q12

A series circuit contains a capacitor of 10^{-5} F, an inductor of 0.2 H, and a resistor of 3×10^2 Ω. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

(f) Boyce 3.7 Q13

A certain vibrating system satisfies the differential equation

$$y'' + \gamma y' + y = 0.$$

Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

(g) Boyce 3.7 Q24

The position of a vibrating system satisfies the initial value problem

$$\frac{3}{2}y'' + ky = 0, \qquad y(0) = 2, y'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3 respectively, determine the values of k and v.