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MATH 307 LECTURE 5

§1.4 FIRST ORDER MODELING, CONTINUED...

1.4.3 Escape Velocity

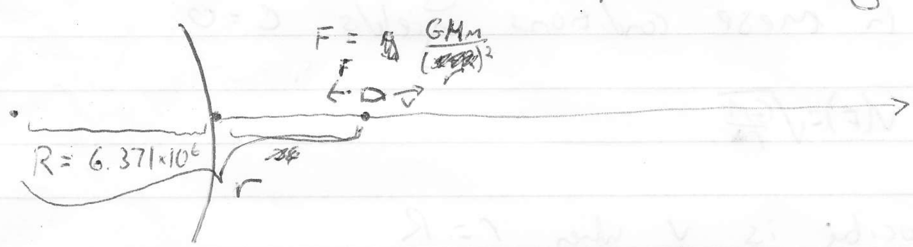
Given that the mass of the earth is 5.972×10^{24} kg $\quad M$

radius of the earth is 6.371×10^{6} m, $\quad R$

the gravitational constant is $6.674 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$, $\quad G$

compute earth's escape velocity.

That is, calculate the minimum speed that an artillery shell would have to be fired straight up at so that, neglecting air resistence, it would escape earth's gravitational field.



- We know $F = ma = m\frac{dv}{dt}$,

  while $F = \frac{-GMm}{r^2}$, since force acts opposite to velocity here,

So setting equal we get $\quad m\frac{dv}{dt} = \frac{-GMm}{r^2}$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{-GM}{r^2}$$

There are too many variables in this equation! v, r & t.
But here's a neat trick to simplify the equation:
By the chain rule, $\quad \frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt} = \frac{dv}{dr}\cdot v$

So $\quad v\frac{dv}{dr} = -\frac{GM}{r^2}$, separable equation.

$\Rightarrow \quad v\,dv = -GMr^{-2}\,dr$

$\Rightarrow \quad \frac{1}{2}v^2 = GMr^{-1} + C$

So $\quad v = \pm\sqrt{C + \frac{GM}{r}}$

$\Rightarrow \quad v = \sqrt{c + \frac{GM}{r}}$; we can choose +ve root, since we know velocity is positive in this problem.

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Now escape velocity is the least positive initial velocity v_0 such that velocity remains +ve from then on.
We see $v = \sqrt{C + \frac{GM}{r}}$ is positive for any $C \geq 0$, so we can choose $C = 0$ to get least v_0.

Another way to describe this is to say that escape velocity results in $v = 0$ at infinity, i.e. $v(\infty) = 0$. Plugging in these conditions yields $C = 0$.

Hence $v(r) = \sqrt{\frac{GM}{r}}$

Escape velocity is v when $r = R$
$\Rightarrow$
$$V_0 = \sqrt{\frac{GM}{R}}$$
$$= 11185.7 \quad ms^{-1}$$

or $\underline{V_0 \simeq 11.2 \ kms^{-1}}$ $\square$.

1.4.4 Drag and terminal Velocity

Consider an object of mass m dropped from a height h_0 near the earth's surface. Without air resistance the object would accelerate linearly, with force proportional to the objects' mass,

i.e. $F = ma = -mg$

$\Rightarrow \frac{dv}{dt} = -g$. defining up as positive direction

where g = acceleration due to gravity near earth's surface $\approx 9.8\,ms^{-2}$
However, in reality drag forces act to counter velocity; a reasonable model is to assume that drag is proportional to velocity.

Then $m \cdot \frac{dv}{dt} = -mg - k'v$ for (Drag) proportionality constant k'

or $\frac{dv}{dt} = -g - kv$, $k = \frac{k'}{m}$

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Problem 1:

Knowing that $v(0) = v_0$, say, we can solve for $v(t)$.

This is a linear ODE:

$$\frac{dv}{dt} + kv = -g.$$

Integrating factor $\mu(t) = e^{\int k\,dt} = e^{kt}$

Then

$$v(t) = e^{-kt}\left(\int -g e^{kt}\,dt + C\right)$$
$$= e^{-kt}\left(-\frac{g}{k}e^{kt} + C\right)$$

$$\Rightarrow \quad v(t) = -\frac{g}{k} + Ce^{-kt}$$

IC: $v(0) = v_0 \Rightarrow C = v_0 + \frac{g}{k}$

So $$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}$$

or

$$v(t) = v_0 e^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right) \quad \square.$$

Problem 2:

Knowing that $h(0) = h_0$, find $h(t)$.

We have $\frac{dh}{dt} = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}$

Integrating

$$h(t) = -\frac{g}{k}\cdot t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + D$$

ICs: $h(0) = h_0 \Rightarrow D = h_0 + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)$

So

$$h(t) = -\frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + h_0 + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)$$

or $$h(t) = h_0 - \frac{g}{k}\cdot t + \frac{1}{k}\left(v_0 + \frac{g}{k}\right)\left(1 - e^{-kt}\right) \quad \square.$$

Example: A mortar shell is fired straight up (by rather inexperienced army recruits). It is known that the mortar weighs 1kg, has initial muzzle velocity $50\,ms^{-1}$ And drag constant (k) $= 0.1\,kgs^{-1}$. Take $g = 9.8\,ms^{-1}$. What is the max height reached by the mortar?

Solution: max height occurs when $v = 0$

So $$-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt} = 0$$

$$e^{-kt} = \frac{g/k}{v_0 + g/k} = \frac{1}{1+\frac{k}{g}v_0}$$

$$\Rightarrow \quad -kt = \ln\left(\left(1+\frac{k}{g}v_0\right)^{-1}\right)$$

$$\Rightarrow \quad t = \frac{1}{k}\cdot\ln\left(1+\frac{k}{g}v_0\right)$$

Plugging this in to $h(t)$ we find

$$h = h_0 - \frac{g}{k}\cdot\left[\frac{1}{k}\cdot\ln\left(1+\frac{k}{g}v_0\right)\right] + \frac{1}{k}\left(v_0+\frac{g}{k}\right)\left(1-e^{-k\left[\frac{1}{k}\ln\left(1+\frac{k}{g}v_0\right)\right]}\right)$$

$$= h_0 - \frac{g}{k^2}\ln\left(1+\frac{k}{g}v_0\right) + \frac{1}{k}\left(v_0+\frac{g}{k}\right)\cdot\left(1-\frac{1}{\left(1+\frac{k}{g}v_0\right)}\right)$$

$$= h_0 - \frac{g}{k^2}\cdot\ln\left(1+\frac{k}{g}v_0\right) + \frac{v_0}{k} + \frac{g}{k^2} - \frac{1}{k}\cdot\frac{\left(v_0+\frac{g}{k}\right)}{\left(1+\frac{k}{g}v_0\right)}$$

$$= h_0 - \frac{g}{k^2}\ln\left(1+\frac{k}{g}v_0\right) + \frac{v_0}{k} + \cancel{\frac{g}{k^2}} - \cancel{\frac{g}{k^2}}$$

$$h = h_0 - \frac{g}{k^2}\cdot\ln\left(1+\frac{k}{g}v_0\right) + \frac{v_0}{k}.$$

Plugging in $h_0 = 0$

$g = 9.8$

$k = \frac{0.1}{1} = 0.1 \quad \left(k = \frac{k'}{m}\right)$

$v_0 = 50$

gives

$$h = 0 - \frac{9.8}{0.01}\cdot\ln\left(1+\frac{0.1}{9.8}\cdot 50\right) + \frac{50}{0.1}$$

$$= 500 - 980\cdot\ln\left(\frac{74}{49}\right)$$

$$\Rightarrow \quad h = 96.0001 \text{ m}.$$

1.~~4~~.5 Economics Problems

Consider the sprocket market; let $P(t)$, $D(t)$ & $S(t)$ be the price of sprockets, demand & supply over time respectively. These can be assumed to be continuous functions of t.

If demand exceeds supply we expect price to increase; if supply exceeds demand we expect price to fall. It makes sense to suppose then that the rate of change of price is proportional to $D-S$. We therefore have the DE

$$\frac{dP}{dt} = k\left(D(t) - S(t)\right) \quad \text{for some } k>0.$$

<u>Example</u>: Suppose that demand is a linear function of price, i.e. $D(t) = a + b\cdot P(t)$ for some $a,b \in \mathbb{R}$, $(a>0, b<0)$

And that supply has both a linear component dependent on price, and a seasonal cyclic component, i.e. $S(t) = q + r\cdot P(t) + s\cdot\cos(t)$, $q,r,s \in \mathbb{R}$, $q>0$, $r>0$?

- Find the price $P(t)$ for any t.
- What is the limiting behaviour?

<u>Solution</u>: - We have no IC, so solution will involve constant C.

$$\begin{aligned}\text{Now } \frac{dP}{dt} &= k(D-S) \\ &= k\left((a+bP) - (q + rP + s\cdot\cos(t))\right) \\ &= k(b-r)P + k(a-q+s\cos(t))\end{aligned}$$

This is a FOLDE: $\frac{dP}{dt} - k(b-r)\cdot P = k(a-q+s\cdot\cos(t))$

So

$$\mu(t) = e^{\int -k(b-r)\,dt} = e^{-k(b-r)t}$$

And

$$P(t) = e^{k(b-r)t}\left(\int e^{-k(b-r)t}\cdot k(a-q+s\cdot\cos(t))\,dt + C\right)$$

PTO

Now $k \int e^{-k(b-r)t}(a-q+s\cdot\cos(t))\,dt$

$$= k(a-q)\int e^{-k(b-r)t}\,dt + k\cdot s\int e^{-k(b-r)t}\cos(t)\,dt$$

IBP $\rightarrow$

$$= -\left(\frac{a-q}{b-r}\right)\cdot e^{-k(b-r)t} - \frac{ks}{1+k^2(b-r)^2}\, e^{-k(b-r)t}\left(k(b-r)\cos(t)-\sin(t)\right)$$

Hence $\underline{P(t) = \left(\frac{q-a}{b-r}\right) - \frac{ks}{1+k^2(b-r)^2}\left[k(b-r)\cos(t)-\sin(t)\right] + Ce^{k(b-r)t}}$

Now if $b > r$ i.e. if demand is more sensitive to price than supply, the term $Ce^{(b-r)k\cdot t}$ will $\to \infty$ exponentially as $t \to \infty$;
Since other terms are bounded we then have that $P(t)$ grows exponentially.
This isn't realistic behaviour.
However, if $b < r$ then the $Ce^{k(b-r)t}$ term decays over time, and the solution approaches

$$P(t) = \cancel{\left(\frac{q-a}{b-r}\right)} \left(\frac{a-q}{r-b}\right) + \text{oscillating part.}$$

Note that this is realistic if $a > q$ i.e. if the base demand level exceeds the base supply level, since then $\left(\frac{a-q}{r-b}\right) > 0$. $\square$

This is a good example of how DEs can be used to answer qualitative questions rather than quantitative questions; this is perhaps more common in economics & finance than, say, engineering problems.