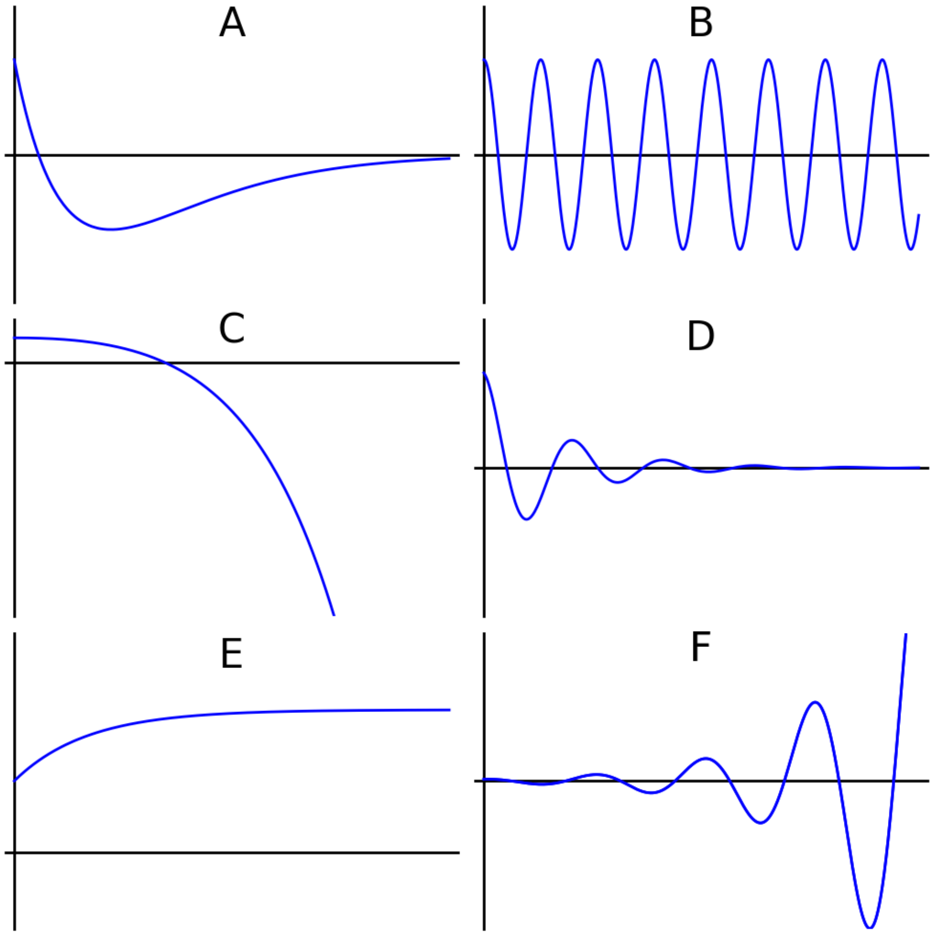
1. (10 points) Below are the graphs of six functions $y(t)$, with t and y being the horizontal and vertical axes respectively. The graphs are labeled A through F. The graphs are **not** all drawn to the same scale, and axis markings have been purposely omitted.



Each of the functions graphed above is a solution to exactly one of the six differential equations below. By analyzing the form of the equations' general solutions, write the letter of the graph next to the differential equation for which it is the solution. You do not need to show your work in this question to receive full credit.

1. $y'' - 3y' + 2y = 0$: C

   The characteristic equation for this DE is $r^2 - 3y + 2 = 0$, which has roots $r = 1$ and $r = 2$. Correspondingly the general solution to this differential equation is $y = c_1 e^t + c_2 e^{2t}$. Thus any nonzero solution must grow exponentially and not exhibit any oscillation. The only graph above that matches these criteria is graph number **C**.

2. $y'' + 16y = 0$: B

   The CE for this equation is $r^2 + 16 = 0$, which has roots $r = \pm 4i$. Correspondingly the general solution to this DE is $y = c_1 \cos(4t) + c_2 \sin(4t)$. Thus any nonzero solution must oscillate with constant amplitude. The only graph above exhibiting this behavior is graph number **B**.

3. $y'' - y' + \frac{3}{2}y = 0$: F

   The CE for this equation is $r^2 - r + \frac{3}{2} = 0$, which has roots $r = \frac{1}{2} \pm \frac{\sqrt{5}}{2}i$. Correspondingly the general solution to this DE is $y = e^{\frac{1}{2}t}(c_1 \cos(\frac{\sqrt{5}}{2}t) + c_2 \sin(\frac{\sqrt{5}}{2}t))$. Thus any nonzero solution must oscillate with exponentially growing amplitude. The only graph above exhibiting this behavior is graph number **F**.

4. $y'' + y' + \frac{3}{2}y = 0$: D

   The CE for this equation is $r^2 + r + \frac{3}{2} = 0$, which has roots $r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$. Correspondingly the general solution to this DE is $y = e^{-\frac{1}{2}t}(c_1 \cos(\frac{\sqrt{5}}{2}t) + c_2 \sin(\frac{\sqrt{5}}{2}t))$. Thus any nonzero solution must oscillate with exponentially decaying amplitude. The only graph above exhibiting this behavior is graph number **D**.

5. $y'' + y' + \frac{1}{4}y = 0$: A

   The CE for this equation is $r^2 + r + \frac{1}{4} = 0$, which has a double root at $r = -\frac{1}{2}$. Correspondingly the general solution to this DE is $y = (c_1 t + c_2)e^{-\frac{1}{2}t}$. Thus a nonzero solution to this differential equation may initially grow or it may cross the equilibrium point, but it must eventually decay to zero without exhibiting oscillation. The only graph above with this behavior is graph number **A**.

6. $y'' + 2y' = 0$: E

   The CE for this equation is $r^2 + 2r = 0$, which has a roots $r = 0$ and $r = -2$. Correspondingly the general solution to this DE is $y = c_1 + c_2 e^{-2t}$. Thus a nonzero solution to this differential equation must asymptote to a possibly nonzero constant without oscillating. The only graph above with this behavior is graph number **E**.

2. (10 points) Solve the following initial value problem:

$$y'' - 6y' + 9y = 36t, \qquad y(0) = 1,\ y'(0) = 0.$$

To solve this IVP, we must first obtain the general solution to the non-homogeneous solution, which in turn requires us to find the general solution to the homogenous equation $y'' - 6y' + 9y = 0$ and a specific solution to the full nonhomogeneous DE.

First, the homogeneous equation: $y'' - 6y' + 9y = 0$. This has characteristic equation $r^2 - 6r + 9 = 0$, which has a double root at $r = 3$. Consulting our knowledge of general solutions to homogenous systems, we deduce that the general solution to the homogeneous equation is

$$y = (c_1 + c_2t)e^{3t}$$

Now we find $Y(t)$, the particular solution to the full nonhomogeneous DE $y'' - 6y' + 9y = 36t$ using the method of undetermined coefficients. We observe that the forcing function $36t$ is a linear function i.e. a degree 1 polynomial, so we guess a generic degree 1 polynomial:

$$Y(t) = At + B$$

Then $Y' = A$ and $Y'' = 0$, so plugging Y back into the left hand side of the DE we get

$$Y'' - 6Y' + 9Y = 0 - 6A + 9(At + B) = 9At + (-6A + 9B)$$

We want this to be equal to the right hand side i.e. $36t + 0$; the only way this can happen is if the coefficients match up, giving us the system of equations

$$9A = 36 \qquad -6A + 9B = 0$$

Solving the system yields $A = 4$ and $B = \frac{8}{3}$. This gives us the specific solution $Y(t) = 4t + \frac{8}{3}$. Hence the general solution to the nonhomogeneous DE, being the sum of $Y(t)$ and the general solution to the homogeneous equation, is

$$y = (c_1 + c_2t)e^{3t} + 4t + \frac{8}{3}$$

Now we apply the initial conditions to solve for c_1 and c_2. $y(0) = 1$ gives us $c_1 + \frac{8}{3} = 1$, so $c_1 = -\frac{5}{3}$. To use the second condition we must differentiate the solution:

$$y' = 3c_1e^{3t} + c_2e^{3t} + 3c_2te^{3t} + 4$$

Hence $y'(0) = 0$ gives us $3c_1 + c_2 + 4 = 0$; using $c_1 = -\frac{5}{3}$, we obtain $c_2 = 1$.

Thus the solution to the initial value problem is

$$y = \left(t - \frac{5}{3}\right)e^{3t} + 4t + \frac{8}{3}$$

3. (10 total points) A certain vibrating system satisfies the differential equation

$$0.5y'' + 0.1y' + 2y = 3\cos(\omega_0 t)$$

where ω_0 is the natural frequency of the system.

(a) (5 points) Compute the amplitude of the system's steady-state solution.

There are two ways to approach solving this question. One way is to solve the equation fully and write the steady-state solution in the form $y = R\cos(\omega t - \delta)$ for constants R, ω and δ. However, we've done the full general case in class, and it's perfectly okay to just quote the formula for R in terms of the coefficients in the DE. To that effect, given the DE $my'' + \gamma y' + ky = F_0\cos(\omega t)$, we found in class that

$$R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}.$$

We have $m = \frac{1}{2}$, $\gamma = \frac{1}{10}$, $k = 2$ and $F_0 = 3$. Furthermore we know that for us $\omega = \omega_0 = \sqrt{\frac{k}{m}} = 2$. Hence

$$R = \frac{3}{\sqrt{(2 - \frac{1}{2}\cdot 2^2)^2 + \left(\frac{1}{10}\right)^2 \cdot 2^2}} = \frac{3}{\sqrt{0 + \frac{1}{25}}} = 15.$$

That is, the amplitude of the steady-state respond in this example is $R = 15$.

(b) (5 points) Suppose the forcing function's frequency is doubled to $2\omega_0$, but everything else remains the same. What does the amplitude of the steady-state solution now become?

Same setup as above, but now $\omega = 4$. Thusly:

$$R = \frac{3}{\sqrt{(2 - \frac{1}{2}\cdot 4^2)^2 + \left(\frac{1}{10}\right)^2 \cdot 4^2}} = \frac{3}{\sqrt{36 + \frac{4}{25}}} = \frac{15}{2\sqrt{226}} = 0.4989.$$

So the steady-state solution's amplitude is now much smaller, at $R = 0.4989$.

4. (10 total points) Consider the initial value problem

$$(\alpha-2)y''+(3\alpha)y'+(2\alpha+1)y=0, \qquad y(0)=1, y'(0)=0$$

for a given constant α.

(a) (5 points) Find the values of α for which the solution to the IVP exhibits oscillatory behavior. For which values will the solution's oscillations be damped, constant in amplitude or exponentially growing?

The characteristic equation corresponding to this differential equation is

$$(\alpha-2)r^2+(3\alpha)r+(2\alpha+1)=0.$$

The solution will exhibit oscillatory behavior if the CE has complex roots, which in turn happens when the discriminant ('b^2-4ac') is negative. Thus to get an oscillating solution to the DE we require

$$(3\alpha)^2-4(\alpha-2)(2\alpha+1)<0,$$

or, after simplifying, $\alpha^2+12\alpha+8<0$. Now $\alpha^2+12\alpha+8$ is a quadratic in α with positive coefficient in front of the α^2 term, so it will be negative between its two roots. The quadratic formula yields $\alpha^2+12\alpha+8=0$ when $\alpha=-6\pm2\sqrt{7}$. We therefore have that the DE exhibits oscillatory behavior for

$$-6-2\sqrt{7}<\alpha<-2+2\sqrt{7},$$

or $-11.292<\alpha<-0.708$.

Finally, for these values of α both the coefficients in front of the y'' and the y' term ($\alpha-2$ and 3α respectively) are negative; thus the CE has roots whose real parts ('$-\frac{b}{a}$') are negative. This translates into a general solution with sine and cosine terms multiplied by an exponentially decaying term. That is, the solution will be damped for all α for which the solution exhibits oscillatory behavior.

(b) (5 points) Let α be the value which maximizes the solution's quasi-frequency, and let $y(t)$ be the solution to the IVP for this value of α. Find a time t_0 beyond which the amplitude of y never exceeds 0.1, i.e. for which $|y(t)| \leq 0.1$ for all $t > t_0$.

[Note: Don't worry if this part of the question had you flummoxed: when I created this question I made an error when working out the solution; and this part is trickier algebraically than I intended it to be (when it comes to an exam I'll be sure to double check that the questions are all readily doable in the allotted amount of time). Nevertheless, it **is** solvable, and the solution is given below.]

By the above, we know now to restrict ourselves to $-11.2921 = -6-2\sqrt{7} < \alpha < -2+2\sqrt{7} = -0.708$; the differential equation as stated therefore has negative coefficients in front of all three terms.

The solution's quasi frequency is maximized when the the imaginary part of the roots of the characteristic equation are largest in magnitude. Using the quadratic formula, we note that the characteristic equation $(\alpha-2)r^2 + (3\alpha)r + (2\alpha+1) = 0$ has roots at

$$r = \frac{-3\alpha \pm \sqrt{\alpha^2+12\alpha+8}}{2(\alpha-2)}$$

Now we are looking at the case where we have oscillation in the roots (i.e. $-6-2\sqrt{7} < \alpha < -2+2\sqrt{7}$); the discriminant is therefore negative, so we can instead write the roots of the CE as $r = \lambda \pm \omega \cdot i$, where

$$\lambda = -\frac{3\alpha}{2(\alpha-2)} \qquad \text{and} \qquad \omega = \frac{\sqrt{-\alpha^2-12\alpha-8}}{2(\alpha-2)}$$

(we have factored out a -1 under the square root sign, which becomes i when square rooted). Thus to find the value of α that maximizes the quasi-frequency ω, we must maximizes the magnitude of $\frac{\sqrt{-\alpha^2-12\alpha-8}}{2(\alpha-2)}$.

This is done by finding turning points: differentiating ω with respect to α, setting the result equal to zero, and solving for α. We compute

$$\begin{aligned}
\frac{d\omega}{d\alpha} &= \frac{d}{d\alpha}\left[\frac{\sqrt{-\alpha^2-12\alpha-8}}{2(\alpha-2)}\right] \\
&= \frac{\frac{1}{2}(-\alpha^2-12\alpha-8)^{-\frac{1}{2}}(-2\alpha-12)\cdot(2\alpha-4) - (-\alpha^2-12\alpha-8)^{\frac{1}{2}}\cdot 2}{(2\alpha-4)^2} \\
&= \frac{\frac{1}{2}(-2\alpha-12)(2\alpha-4) - 2(-\alpha^2-12\alpha-8)}{(2\alpha-4)^2\sqrt{-\alpha^2-12\alpha-8}} \\
&= \frac{16\alpha+40}{(2\alpha-4)^2\sqrt{-\alpha^2-12\alpha-8}}
\end{aligned}$$

This is zero when the numerator is zero *i.e.*$16\alpha+40 = 0$, or $\alpha = -\frac{5}{2}$. We conclude that the quasi-frequency is maximized when $\alpha = -2.5$.

[That was the part that was algebraically tricky; I won't expect you to do something like the above in a 50-minute exam.]

Using this value of α, the IVP that we must therefore solve (after multiplying the DE by -1) is

$$\frac{9}{2}y'' + \frac{15}{2}y' + 4y = 0, \qquad y(0) = 1, y'(0) = 0.$$

To find out when the amplitude of the solution decays to less than 0.1, we will write the solution in the form $y = Re^{-ct}\cos(\omega t - \delta)$ for constants R, c, ω and δ, as then we know that the solution is at most Re^{-ct} in magnitude. The characteristic equation is $\frac{9}{2}r^2 + \frac{15}{2}r + 4 = 0$, which has roots $r = -\frac{5}{6} \pm \frac{\sqrt{7}}{6} \cdot i$, so the solution to this DE can be written in the form

$$y = e^{-\frac{5}{6}t}\left(A\cos\left(\frac{\sqrt{7}}{6}t\right) + B\sin\left(\frac{\sqrt{7}}{6}t\right)\right)$$

Using the initial value $y(0) = 1$ gives us $A = 1$, while the second initial value $y'(0) = 0$ gives us $-\frac{5}{6}A + \frac{\sqrt{7}}{6}B = 0$, so $B = \frac{5}{\sqrt{7}}$.

Now recall that to convert the solution to the form $y = Re^{-ct}\cos(\omega t - \delta)$ we use $R = \sqrt{A^2 + B^2}$, so

$$R = \sqrt{1^2 + \left(\frac{5}{\sqrt{7}}\right)^2} = \sqrt{\frac{32}{7}} = 4\sqrt{\frac{2}{7}} = 2.13809.$$

We therefore know that at time t the solution is at most $4\sqrt{\frac{2}{7}} \cdot e^{-\frac{5}{6}t}$ in magnitude. To find a time beyond which the solution is always less than $\frac{1}{10}$ in magnitude, we solve for t in the equation

$$\frac{1}{10} = 4\sqrt{\frac{2}{7}} \cdot e^{-\frac{5}{6}t}.$$

Squaring both sides we get $\frac{1}{100} = \frac{32}{7}e^{-\frac{5}{3}}t$. Solving for t yields

$$t = \frac{3}{5}\ln\left(\frac{3200}{7}\right) = 3.67499\ldots$$

We conclude that for $\alpha = -2.5$, the solution damps to magnitude less than 0.1 after about $t = 3.675$.

5. (10 total points) An object of unknown mass is placed on a flat surface and attached to a horizontal spring with spring constant 2.5 kg/s^2. The damping constant in the system is precisely 1 kg/s. When the object is pulled to the right of its equilibrium position and released, the damped oscillations of its subsequent motion are observed to have a quasi-period of $\frac{20}{7}\pi$ seconds.

   (a) (10 points) Suppose we know the object weighs more than 1 kg. What is the mass of the object? Justify your answer numerically.

Here we have the initial value problem

$$my'' + y' + \frac{5}{2}y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

The corresponding characteristic equation is

$$mr^2 + r + \frac{5}{2} = 0,$$

with roots

$$r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot m \cdot 5/2}}{2m} = \frac{-1}{2m} \pm \frac{1}{2m}\sqrt{1-10m}.$$

We are told the solution exhibits oscillatory behavior, so the thing under the square root sign must be negative. Hence the roots to the CE can be written as

$$\frac{-1}{2m} \pm \frac{\sqrt{10m-1}}{2m} \cdot i.$$

This means the solution will contain sine and cosine terms with radial quasi-frequency ω, where $\omega = \frac{\sqrt{10m-1}}{2m}$.

On the other hand, if T is the quasi-period, then $\omega = \frac{2\pi}{T}$; hence

$$\omega = \frac{2\pi}{\frac{20}{7}\pi} = \frac{7}{10}.$$

Thus we must have that $\frac{\sqrt{10m-1}}{2m} = \frac{7}{10}$. It now remains to solve for m. Cross-multiplying to clear denominators we get

$$5\sqrt{10m-1} = 7m,$$

so after squaring both sides we have $25(10m-1) = 49m^2$, or

$$49m^2 - 250m + 25 = 0.$$

This quadratic has the solutions $m = 5$ or $m = \frac{5}{49}$.

Going back to our original differential equation, we see that 5 and $\frac{5}{49}$ are both valid (non-negative) values for the object's mass such that the quasi-frequency of the solution's oscillations is $\frac{7}{10}$. However, we are told that the object's mass exceeds 1 kg, so we conclude that the object weighs 5 kg.

(b) (Bonus: 3 points) Compute the Wronskian of the system (i.e. compute the Wronskian of $y_1(t)$ and $y_2(t)$, where y_1 and y_2 are a fundamental basis of solutions to the differential equation obeyed by the object above).

Having found the mass of the object in part a, we note that the governing differential equation for the system is

$$5y'' + y' + \frac{5}{2}y = 0$$

This has corresponding characteristic equation $5r^2 + r + \frac{5}{2}r = 0$, which has roots

$$r = \frac{-1 \pm \sqrt{1 - 4\cdot 5\cdot \frac{5}{2}}}{2\cdot 5} = -\frac{1}{10} \pm \frac{7}{10}i$$

We conclude, using our knowledge of homogeneous second order systems, that the general solution to the DE can be written as

$$y = e^{-\frac{t}{10}}\left(c_1\cos\left(\frac{7}{10}t\right) + c_2\sin\left(\frac{7}{10}t\right)\right)$$

Thus we may take as a fundamental basis for the system

$$y_1(t) = e^{-\frac{t}{10}}\cos\left(\frac{7t}{10}\right) \text{ and } y_2(t) = e^{-\frac{t}{10}}\cos\left(\frac{7t}{10}\right)$$

It now remains to compute the Wronskian of y_1 and y_2. Recall that this is given by $y_1y_2' - y_2y_1'$. Computing the derivatives of y_1 and y_2 gives us

$$y_1' = -\frac{1}{10}e^{-\frac{t}{10}}\cos\left(\frac{7t}{10}\right) - \frac{7}{10}e^{-\frac{t}{10}}\sin\left(\frac{7t}{10}\right) = -\frac{1}{10}e^{-\frac{t}{10}}\left(\cos\left(\frac{7t}{10}\right) + 7\sin\left(\frac{7t}{10}\right)\right)$$
$$y_2' = -\frac{1}{10}e^{-\frac{t}{10}}\sin\left(\frac{7t}{10}\right) + \frac{7}{10}e^{-\frac{t}{10}}\cos\left(\frac{7t}{10}\right) = -\frac{1}{10}e^{-\frac{t}{10}}\left(-7\cos\left(\frac{7t}{10}\right) + \sin\left(\frac{7t}{10}\right)\right)$$

Hence

$$\begin{aligned}
y_1y_2' - y_2y_1' &= \left[e^{-\frac{t}{10}}\cos\left(\frac{7t}{10}\right)\right]\cdot\left[-\frac{1}{10}e^{-\frac{t}{10}}\left(-7\cos\left(\frac{7t}{10}\right) + \sin\left(\frac{7t}{10}\right)\right)\right] \\
&\quad - \left[e^{-\frac{t}{10}}\sin\left(\frac{7t}{10}\right)\right]\cdot\left[-\frac{1}{10}e^{-\frac{t}{10}}\left(\cos\left(\frac{7t}{10}\right) + 7\sin\left(\frac{7t}{10}\right)\right)\right] \\
&= \frac{1}{10}e^{-\frac{t}{10}}\left[7\cos^2\left(\frac{7t}{10}\right) - \sin\left(\frac{7t}{10}\right)\cos\left(\frac{7t}{10}\right) + \sin\left(\frac{7t}{10}\right)\cos\left(\frac{7t}{10}\right) + 7\sin^2\left(\frac{7t}{10}\right)\right] \\
&= \frac{7}{10}e^{-\frac{t}{10}}\left[\cos^2\left(\frac{7t}{10}\right) + \sin^2\left(\frac{7t}{10}\right)\right] \\
&= \frac{7}{10}e^{-\frac{t}{10}}
\end{aligned}$$

We conclude that the Wronskian of the system is $\frac{7}{10}e^{-\frac{t}{10}}$.

[Note: There are multiple valid fundamental bases $y_1(t)$ and y_2 for the above system; for example, we could just as well have used $y_1 = e^{\left(-\frac{1}{10}+\frac{7}{10}i\right)t}$ and $y_2 = e^{\left(-\frac{1}{10}-\frac{7}{10}i\right)t}$. However, no matter what basis you choose, you will always get the same Wronskian out, up to some constant factor. That is, the Wronskian will always be $Ae^{-\frac{t}{10}}$ for some constant value of A.]