MATH 307A
LECTURE 14

Wed 12 Feb

§2.5 2nd-ORDER MODELING (BOYCE 3.7)

AKA MECHANICAL & ELECTRICAL VIBRATIONS

In this section we examine two real-world applications of ~~namely~~ modeling with 2nd-order linear DEs with constant coefficients, namely vibrations & oscillations in mechanical and electrical systems.

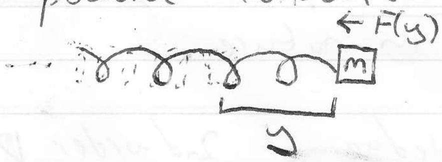
2.5.1 Mechanical Vibrations - Derivation

Here we are looking at the motion of an object about some equilibrium point when there is both friction present and some ~~known~~ external ~~system~~ force acting on the system.

The simplest such setup is to look at an object with mass m attached to a (massless) spring.
Let y be the displacement of the mass from its position of rest as a function of t. Recall Hooke's Law: the force pulling on the mass is proportional to its displacement. In other words,

$$F_s(y) = -ky$$

for some positive constant k



Since $F = my''$, we thus have a 2nd-order DE $my'' = -ky$, or

$$my'' + ky = 0.$$

Next, we'll look at where the mass experiences a frictional force proportional to its velocity (recall that friction is more closely proportional to velocity squared, but this assumption is valid for a large range of velocities and makes the mathematical analysis a *lot* easier). In other words,

$$F_f(y) = -\gamma y'$$

for some positive constant γ (we need the -ve sign, since friction always acts opposite to the direction of motion).
In this case $F = my'' = F_s(y) + F_f(y)$, or

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$$my'' + \gamma y' + ky = 0.$$

Finally, we'll examine where there is a third known force $g(t)$ acting on the mass (for example, if the object we're studying is a violin string being played, then $g(t)$ might be the force imparted on the string by the violinist's bow). In this case we then have that $F = my'' = F_s(y) + F_f(y) + g(t)$, or

$$\boxed{my'' + \gamma y' + ky = g(t).} \quad (1)$$

This is a 2nd-order linear non-homogeneous DE with constant coefficients in front of the y terms - exactly what we've been studying in the past few lectures!

☆ Warning: This is only an approximation: springs have finite length etc.

Define 2.5.2 In the above equation we give names to the various parameters:

- m is the mass of the object, always > 0
- γ is the damping constant, always > 0
- k is the spring constant, always > 0
- $g(t)$ is the forcing function, which could be anything, but is usually periodic in nature.

We'll now use our knowledge of 2nd-order linear DEs to investigate what the solutions to the equation $my'' + \gamma y' + ky = g(t)$ look like for varying values of m, γ, k & $g(t)$.

2.5.3 Frictionless System with no applied Force

The simplest case is when there's no friction ($\gamma = 0$) and no external applied force on the system ($g(t) = 0$). In that case we have the DE $my'' + ky = 0$, $m, k > 0$.

This has the characteristic equation $mr^2 + k = 0$, which has the complex roots $r = \pm i\sqrt{\frac{k}{m}}$.

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Going back over our notes on 2nd-order constant-coefficient DEs where the CE has complex roots, we see that the general solution to $my''+ky=0$ is given by

$$y = A\cos(\omega_0 t) + B\sin(\omega_0 t), \quad (2)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is called the natural frequency of the system.

* Interpretation: heavier objects vibrate more slowly, stiffer springs $\Rightarrow$ faster vibration

It is often more useful to write the solution in the form

$$y = R\cos(\omega_0 t - \delta), \quad (3)$$

(this can*always* be done),

where
- R is called the amplitude of the solution, and
- δ is the phase shift of the solution.

(usually have $-\pi < \delta \leq \pi$, sometimes $0 \leq \delta < 2\pi$)

(up here.
- frequency $f = \omega_0 / 2\pi$
- period $T = 2\pi / \omega_0$)

To see how to get from one form to the other, note that we can use the compound angle formula

$$\cos(P-Q) = \sin P \sin Q + \cos P \cos Q$$

to rewrite $R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta)$

Comparing this with the equation $A\cos(\omega_0 t) + B\sin(\omega_0 t)$, we see that

$$A = R\cos(\delta) \quad \& \quad B = R\sin(\delta)$$

So

$$R^2 = A^2 + B^2, \quad \tan(\delta) = \frac{B}{A}.$$

However, care must be taken when solving for δ in the final equation above: we must choose δ in the correct quadrant according to the signs of $\cos(\delta)$ & $\sin(\delta)$, as $\delta = \arctan(\frac{B}{A})$ only ever returns a value between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$.

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Example 2.5.4 A 100g iron weight is placed on a frictionless flat surface and attached to a (sideways-acting) spring. If we pull the weight 20cm to the right from its position of rest, we measure a leftward pulling force of 2.88 N on the block. Suppose at $t=0$ we release the block 10cm to the left of its rest position with a velocity of 150 cms^{-1} to the right. Find the position of the block at time t.

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in meters

<u>Solution</u> We Let $y(t)$ be the (horizontal) position of the block at time t. We know:

- $y(0) = -0.1$
- $y'(0) = 1.5$
- $my'' + ky = 0$, $m = 0.1$, $k = ?$

By Hooke's law $F_s(y) = -ky$.
we have $F_s(0.2) = -2.88$.
So $-2.88 = -k \cdot 0.2$
$\Rightarrow$ $k = 14.4$

Hence the DE for the system is $0.1y'' + 14.4y = 0$
Solution (general): $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$, where
$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{14.4}{0.1}} = \sqrt{144} = 12.$
$\Rightarrow$ $y = A\cos(12t) + B\sin(12t).$

Apply ICs: $y(0) = -0.1$
$\Rightarrow$ $-0.1 = A$ $\left(= \frac{-1}{10}\right)$
And $y' = -12A\sin(12t) + 12B\cos(12t)$
So $y'(0) = 1.5 \Rightarrow 1.5 = 12B$
$\Rightarrow B = 0.125$ $\left(= \frac{1}{8}\right)$.

So solution is $\underline{y(t) = \frac{-1}{10}\cos(12t) + \frac{1}{8}\sin(12t)}$.

[Insert previous page's stuff on $y = R\cos(\omega_0 t - \delta)$ here]

<u>Example 2.5.5</u> Write the solution to the previous example in the form $y = R\cos(\omega_0 t - \delta)$, and in so doing find
- the maximum displacement of the block from its rest position
- the first positive t when the block crosses its rest position

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Solution: we have $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ with

$$A = -\frac{1}{10}, \quad B = \frac{1}{8}, \quad \omega_0 = 12.$$

~~RM~~ $A = R\cos(\delta)$, $B = R\sin(\delta)$,

so $R^2 = A^2 + B^2$, $\tan(\delta) = B/A$.

$$\begin{aligned}
R &= \sqrt{\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{1}{8}\right)^2} \\
&= \tfrac{1}{2}\sqrt{\left(\tfrac{1}{5}\right)^2 + \left(\tfrac{1}{4}\right)^2} \\
&= \tfrac{1}{2}\sqrt{\tfrac{1}{25} + \tfrac{1}{16}} \\
&= \tfrac{1}{2}\sqrt{\tfrac{16+25}{400}} \\
&= \tfrac{1}{2}\sqrt{\tfrac{41}{400}} \\
&= \tfrac{\sqrt{41}}{40} \simeq 0.16008\ldots
\end{aligned}$$

$$\begin{aligned}
\tan(\delta) &= \frac{\frac{1}{8}}{-\frac{1}{10}} \\
&= -\frac{5}{4}
\end{aligned}$$

so $\delta = \arctan^*\left(-\frac{5}{4}\right) = -0.89606 + n\cdot\pi$

But $\cos\delta < 0$, $\sin\delta > 0 \Rightarrow \delta$ 2nd Quadrant

$\Rightarrow \delta = \pi - 0.89606 = 2.24553.$

$\frac{\pi}{2} < \delta < \pi$

Thus, to 5 d.p., we have

$$y = 0.16008\cos(12t - 2.24553)$$

We see then that the amplitude of the oscillation is $\simeq 16.008$ cm, and the first time when the block crosses its rest position is when $12t - 2.24553 = -\frac{\pi}{2}$

$$\Rightarrow t = \frac{1}{12}\left(2.24553 - \frac{\pi}{2}\right) = 0.05623 \text{ s}.$$