~~Thurs~~ Fri 10 Jan MATH 307 LECTURE 3

§1.2 : Slope Fields (Boyce 1.1)

For a first-order DE, $\frac{dy}{dx} = f(x,y)$, note that the slope of any solution curve is given by the function $f(x,y)$

We may get a good idea of the behaviour of solution curves by drawing slope fields:
A section of the x-y plane where we draw in lots of shortish lines indicating slopes at regularly spaced points (x_i, y_i), where the slope is given by $f(x_i, y_i)$.

* Demo slope field for $\frac{dy}{dt} = -ty$ *

This allows us to discern the general behaviour for solutions to first-order DEs without solving them exactly, including asymptotic behaviour (i.e. $\lim_{t\to\infty} y(t)$).

Example: for $\frac{dy}{dt} = -ty$, $y(0) = k$, we see that $\lim_{t\to\infty} y(t) = 0$ no matter the initial value k.

Slope fields are thus a useful tool to getting a qualitative sense of the behaviour of solutions to DEs before solving the DE, and can be a good check on whether you have arrived at the ~~good~~ correct solution.

We will revisit slope fields in sections to come.

A link to one slope field plotter is on my website; there are many more that can be found by googling for slope/direction fields.

□.

§1.3: Integrating Factors and FOLDEs

Definition 1.3.1: A first-order linear differential equation (FOLDE) is a first-order DE that can be written in the form

$$\frac{dy}{dt} = f(t) \cdot y + g(t).$$

(Because $ax+b$ is a linear function of x, and the RHS of the above equation looks similar w.r.t. y, these are known as linear DEs)

Note 1.3.2:
- It's most often useful to put FOLDEs in the following form:

$$\frac{dy}{dt} + \tilde{f}(t) \cdot y = g(t)$$

i.e. by subtracting $f(t) \cdot y$ from both sides $\Rightarrow \tilde{f}(t) = -f(t)$

- we'll also see them in this form:

$$f(t) \cdot \frac{dy}{dt} + g(t) \cdot y = h(t)$$

we can get back to the previous form by dividing by $f(t)$ on both sides.

Example: $t^2 \cdot \frac{dy}{dt} + 2ty = \cos(t)$

Observe 1.3.3: The LHS of the above equation is exactly $\frac{d}{dt}(t^2 \cdot y)$! So we have

$$\frac{d}{dt}(t^2 \cdot y) = \cos(t)$$

$$\Rightarrow \quad t^2 y = \sin(t) + C$$

So $$y = \frac{\sin(t)}{t^2} + \frac{C}{t^2}$$

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"The trick to solving FOLDEs is to write the LHS as $\frac{d}{dt}(y \cdot \text{(some function of } t))$, then integrate."

<u>Example</u>: $\frac{dy}{dt} = 2ty + e^{t^2}$

Rewriting: $\frac{dy}{dt} - 2t \cdot y = e^{t^2}$

Now we are stuck: there is no way to write $\frac{dy}{dt} - 2ty$ as the derivative of $y \cdot$ some function of t.

<u>However</u>, observe if we multiply everything by e^{-t^2}:

$$e^{-t^2} \cdot \frac{dy}{dt} - 2te^{-t^2} \cdot y = 1$$

$$\Rightarrow \quad \frac{d}{dt}\left(y \cdot e^{-t^2}\right) = 1$$

So $\quad y \cdot e^{-t^2} = t + C$

$$\Rightarrow \quad y = te^{t^2} + Ce^{t^2}$$

<u>Definition 1.3.4</u>: An <u>integrating factor</u> is a function $\mu(t)$, such that when we multiply the FOLDE $\frac{dy}{dt} + f(t) \cdot y = g(t)$ by $\mu(t)$, we can then write the LHS as the derivative of $\mu(t) \cdot y$.

<u>Example</u>: $\frac{dy}{dt} = y + \sin(t)$

So $\quad \frac{dy}{dt} - y = \sin(t)$

Multiply by $\mu(t)$ (as yet undetermined) to get

$$\mu(t)\frac{dy}{dt} - \mu(t) \cdot y = \mu(t) \cdot \sin(t)$$

We want $\mu(t)\frac{dy}{dt} - \mu(t) \cdot y = \frac{d}{dt}\left(\mu(t) \cdot y\right)$

$$= \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt} \cdot y$$

Hence this is true if $\frac{d\mu}{dt} = -\mu$

PTO

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This is a separable DE!

Solution: $\mu(t) = e^{-t} + C$; can pick any function which works, so choose $\mu(t) = e^{-t}$

Thus $e^{-t}\cdot\frac{dy}{dt} - e^{-t}y = \sin(t)e^{-t}$

$\Rightarrow \quad \frac{d}{dt}(e^{-t}\cdot y) = \sin(t)e^{-t}$

So $\quad e^{-t}\cdot y = \int \sin(t)e^{-t}\,dt$

IBP:
$$\int \overset{u}{\sin(t)}\overset{v'}{e^{-t}}\,dt = -\overset{u}{\sin(t)}\overset{v}{e^{-t}} - \int \overset{u'}{\cos(t)}(\overset{v}{-e^{-t}})\,dt$$
$$= -\sin(t)e^{-t} + \int \cos(t)e^{-t}\,dt$$
$$= -\sin(t)e^{-t} - \cos(t)e^{-t} - \int \sin(t)e^{-t}\,dt + C$$

So $\quad 2\int \sin(t)e^{-t}\,dt = -e^{-t}(\sin(t)+\cos(t)) + C$

$\Rightarrow \quad \int \sin(t)e^{-t} = -\frac{1}{2}e^{-t}(\sin(t)+\cos(t)) + C$

Hence $\quad e^{-t}\cdot y = -\frac{1}{2}e^{-t}(\sin(t)+\cos(t)) + C$

So $\quad y = -\frac{\cos(t)+\sin(t)}{2} + Ce^{t}$ □

1.3.5 How to Solve in General

- Get FOLDE into the form
  $$\frac{dy}{dt} + f(t)\cdot y = g(t)$$
- Multiply by $\mu(t)$:
  $$\mu(t)\frac{dy}{dt} + \mu(t)\cdot f(t)\cdot y = \mu(t)g(t)$$
- We want LHS $= \frac{d}{dt}(\mu(t)\cdot y) = \mu(t)\cdot\frac{dy}{dt} + \cancel{\mu'}\frac{d\mu}{dt}\cdot y$, so
  $$\frac{d\mu}{dt} = \mu\cdot f(t)$$
- Separable equation: $\frac{1}{\mu}\,d\mu = f(t)\,dt$
  $$\Rightarrow \quad \mu(t) = e^{\int f(t)\,dt}$$
- Then: $\frac{d}{dt}(\mu(t)\cdot y) = \mu(t)g(t)$

  So $\quad \mu(t)\cdot y = \int \mu(t)g(t)\,dt + C$
- So finally $\quad y = \frac{1}{\mu(t)}\int \mu(t)g(t)\,dt + C,$

  where $\quad \mu(t) = e^{\int f(t)\,dt}$.

PTO

Example: Solve $t\frac{dy}{dt} + 2y - 4t^2 = 0$, $y(1) = 2$.

$$\Rightarrow \quad \frac{dy}{dt} + \underbrace{\frac{2}{t}}_{f(t)} \cdot y = \underbrace{4t}_{g(t)}$$

So $\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = |t|^2 = t^2$

And
$$y = \overset{\frac{1}{\mu}}{\frac{1}{t^2}} \int \overset{\mu \cdot g}{t^2 \cdot 4t}\,dt$$
$$= \frac{1}{t^2}\int 4t^3\,dt$$
$$= \frac{1}{t^2}\left(t^4 + c\right)$$

So $y = t^2 + c \cdot t^{-2}$

Finally $y(1) = 2 \Rightarrow 2 = 1 + c$
$\Rightarrow c = 1$

Hence
$$y = t^2 + t^{-2}, \text{ valid for } 0 < t < \infty \qquad \square.$$

Note: 1.3.6: Sometimes we can only express the solution in integral form:

Example: $\frac{dy}{dt} + \frac{t}{2} \cdot y = 1$

Then $\mu(t) = e^{\int \frac{t}{2}\,dt} = e^{t^2/4}$

And
$$y = e^{-t^2/4}\left(\int_0^t e^{\frac{s^2}{4}}\,ds + c\right), \text{ say}$$

$\int e^{s^2/4}\,ds$ has no nice antiderivative, so we just leave it in that form. This is more of an inconvenience than a game-stopper, since a computer can still compute $y(t)$ to any given precision given the above formula.