Your Name

Student ID #

- **DON'T PANIC!** There are more questions in this exam than in the midterms, but you have more than twice the time to solve them. If you get stuck, take a deep breath and move on to something else. Return if you have time at the end.
- Cellphones off please!
- You are allowed one two-sided handwritten notesheet for this midterm.
- You may use a scientific calculator, although this exam has been written so that you don't need to. Graphing calculators and all other course-related materials may not be used.
- In order to receive full credit, you must **show your working** unless explicitly stated otherwise by the question. You may quote and use any formula you have seen in class to save time, but be sure to indicate with a word or two when you are doing so.
- **There is a table of Laplace transforms and rules at the back of this exam**. You may quote and use any of the formulas and rules in the table as is without having to derive them from scratch.
- Give your answers in exact form (for example $\pi/3$ or $e^{-5\sqrt{3}}$) unless explicitly stated otherwise by the question. Simplify your answers if possible.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10 points) Solve the following initial value problem:

$$\frac{1}{x} \cdot \frac{dy}{dx} = e^{x-y}, \qquad y(0) = 0$$

Your answer should be a function $y(x)$ with no undetermined constants in it.

2. (10 points) Consider the non-homogeneous differential equation

$$y'' + 4y' - 21y = g(t),$$

for some nonzero forcing function $g(t)$. For each of the following possibilities for $g(t)$, write down the form that the particular solution $Y(t)$ to the DE would take. Your answer should be in the form $Y = f(t)$, where f includes undetermined coefficients (A, B, C etc.). For example, if you thought the particular solution was a general linear function in t, you would write $Y = At + B$. **You don't need to compute the actual values of these coefficients**.

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

(a) $g(t) = e^{-t} + 1$

(b) $g(t) = \sin(t)$

(c) $g(t) = e^{3t} - e^{-7t}$

(d) $g(t) = e^{2t} \cos 3t$

(e) $g(t) = t^2 + 2t$

3. (10 points) Compute the inverse Laplace transform of the following function. Your answer should be a function $f(t)$. You may quote any formula or rule given in the Laplace transform formula sheet at the back of the exam paper.

$$F(s) = \frac{e^{-s} - e^{-3s}}{s^2 + s}$$

4. (10 total points) Consider the following linear first-order initial value problem:

$$\frac{dy}{dt} = \cos(t) \cdot y + \sin(t), \qquad y(0) = 0$$

(a) (2 points) Using the appropriate existence and uniqueness theorem for first-order differential equations, state the largest time interval for which the solution to the above IVP is guaranteed to exist.

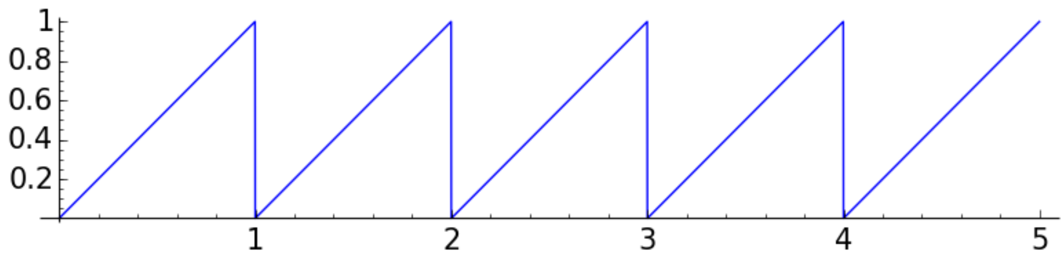
(b) (6 points) Use Euler's Method with a step size of $\frac{\pi}{2}$ to find an approximate value of the solution at $t = \frac{3}{2}\pi$. You may use decimals in this part of the question (although you don't need to); if you do be sure to maintain at least four digits of precision.

(c) (2 points) Does the differential equation $\frac{dy}{dt} = \cos(t) \cdot y + \sin(t)$ have any equilibrium solutions? That is, are there any constant solutions $y = C$ to this DE? Justify your answer.

5. (10 total points) Consider the unit sawtooth function $f(t)$, defined by :

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ t - n, & n \leq t < n+1 \end{cases}$$

for n any integer. A graph of $f(t)$ for t between 0 and 5 is given below:



(a) (7 points) Compute the Laplace transform of $f(t)$. Your answer should be a function $F(s)$.

(b) (3 points) Use your answer above to compute the Laplace transform of the solution to the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \quad y'(0) = 0$$

where $f(t)$ is the sawtooth function detailed above. Your answer should be a function $\Phi(s)$ with no undetermined constants in it. **You do not need to find the solution to the IVP to answer this question.**

6. (10 total points) An applied mathematician is investigating the motion of a particular object, and establishes that the function describing its motion $y(t)$ obeys the differential equation

$$y'' + by' + cy = 0$$

where a and b are constants. The mathematician doesn't initially know the values of b and c, but can show the following two facts:

- The function $y_1(t) = e^{-3t}$ is a solution to the differential equation.
- The Wronskian of the system is $W(t) = e^{2t}$.

(a) (7 points) Using the above two facts, find a second function $y_2(t)$, linearly independent from the first, that satisfies the differential equation. Your answer should be a function in t **with no undetermined coefficients in it.**

(b) (3 points) Given that the functions $y_1(t)$ and $y_2(t)$ both solve the DE, what are the constants b and c?

7. (10 total points) A 5 kg block is placed on a flat surface and attached to a long horizontal spring. When the block is pulled 0.2 meters to the right of its equilibrium position, the spring exerts a force of 2.5 Newtons to the left on the block. The surface imparts a frictional force on the block proportional to its velocity, such that when the block is traveling at 1 ms^{-1} the retarding force is 5 Newtons. Furthermore, the block is subjected to an external oscillating force of $g(t) = \cos(\omega t)$ Newtons, where t is in seconds and ω is a positive constant.

   (a) (2 points) Write down a differential equation describing the position of the block as a function of time.

   (b) (3 points) For what value of ω will the amplitude of the block's steady-state response be maximized?

   (c) (5 points) How much resonance is there in this system? To answer this question, compute and interpret the quantity R/R_0, where R is the amplitude of the steady-state response for the value of ω you found above, and R_0 is the amplitude of the response when the forcing function is a constant 1 Newton (i.e. when $\omega = 0$).

8. (10 points + 4 bonus points) Two state troopers are stationed in their cruiser at a speed trap on a long straight road. The cops have been reading a book on differential equations, and decide to use their newfound knowledge to come up with a chase strategy when pursuing a speeding vehicle.

   The cops agree that if a target is fleeing from them, they should accelerate towards is. The first cop argues that the acceleration of the cop car should be proportional to the distance between them and their target. The second cop instead argues that their acceleration should be proportional to the difference of velocities of the target and the police cruiser. In the end the cops agree that both ideas have merit, so decide to adopt the strategy that is a *weighted sum* of the two above: their acceleration will be the sum of [α times the distance between the vehicles] and [β times the difference of their velocities], for some constant values of α and β.

   After more discussion the cops decide to use the values $\alpha = \frac{1}{100}$ and $\beta = \frac{1}{5}$. For example, if their target was 100 meters ahead traveling 10 ms^{-1} faster than them, the cops would accelerate toward it at 3 ms^{-2}.

   At time t = 0 seconds a car comes speeding past the trap, traveling at a constant speed of 40 ms^{-1}. The cops are initially stationary at the trap, but immediately give chase using the strategy above.

   (a) (10 points) Formulate and solve an initial value problem to find a formula for the position of the cop cruiser for time $t \geq 0$.

[Question continued on the next page]

(b) (Bonus: 4 points) Compute or estimate how long it would take after the chase begins for the cops to close to within 20 meters of their speeding target.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$f(t)$ periodic with period T	$\frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}$	