Mon FEB 3 MATH307 LECTURE 10

§2.2: ~~Homogen~~ CHARACTERISTIC EQUATIONS WITH COMPLEX ROOTS

BOYCE 3.3

Recall 2.2.1 We are looking at linear 2nd-order homogeneous DEs with constant coefficients: ~~with~~

$$ay'' + by' + cy = 0$$

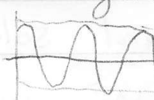
What happens when the roots of the characteristic equation $ar^2 + br + c = 0$ are complex?

Example 2.2.2: Consider $y'' + y = 0$

For the general solution, we want two functions whose double derivative is $-$(the original function).

$\Rightarrow$ $\sin(t)$ & $\cos(t)$ fit, so general solution is

$y = c_1 \sin(t) + c_2 \cos(t)$.


solution plot.

Let's try the method from the last lecture:

$y = e^{rt} \Rightarrow r^2 e^{rt} + e^{rt} = 0$

So $r^2 + 1 = 0$

$\Rightarrow r = \pm\sqrt{-1}$, i.e. $r = \pm i$

Thus $y = c_1 e^{it} + c_2 e^{-it}$.

How do we reconcile these two different-looking solutions?

- They are actually the same function, just written differently!

Recall 2.2.3: $e^{it} = \cos(t) + i\sin(t)$: Euler's Formula.

$\Rightarrow e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$.

Hence $y = c_1 e^{it} + c_2 e^{-it} = c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t))$

$= (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t)$

$= c_1' \cos(t) + c_2' \sin(t)$,

where $c_1' = c_1 + c_2$ & $c_2' = i(c_1 - c_2)$.

So the "guess-and-check" method with e^{rt} still works; however, we have to use complex numbers in both our exponents & our coefficients.

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$= 9.25$

<u>Example 2.2.4:</u> Solve the ~~IDE~~ IVP

$$y'' + y' + \frac{37}{4} y = 0, \quad y(0) = 2, \quad y'(0) = 8$$

we can't just guess $\cos(t)$ & $\sin(t)$ anymore, so let's look at the characteristic equation:

$$r^2 + r + \frac{37}{4} = 0$$

Quadratic formula: $r = \dfrac{-1 \pm \sqrt{1 - 37}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-36}$

$$= -\frac{1}{2} \pm \sqrt{-9}$$

$$= -\frac{1}{2} \pm 3i$$

So the two roots are $r_1 = -\frac{1}{2} + 3i$, $r_2 = -\frac{1}{2} - 3i$

$\Rightarrow$ the general solution is

$$\begin{aligned}
y &= c_1 e^{(-\frac{1}{2}+3i)t} + c_2 e^{(-\frac{1}{2}-3i)t} \\
&= \cancel{e^{A}}\, c_1 e^{-\frac{1}{2}t} \cdot e^{i3t} + c_2 e^{-\frac{1}{2}t} \cdot e^{-3it} \\
&= e^{-\frac{1}{2}t}\left(c_1 e^{i3t} + c_2 e^{-i3t}\right) \\
&= e^{-\frac{1}{2}t}\left(c_1(\cos(3t) + i\sin(3t)) + c_2(\cos(3t) - i\sin(3t))\right) \\
&= e^{-\frac{1}{2}t}\left((c_1+c_2)\cos(3t) + i(c_1-c_2)\sin(3t)\right) \\
&= e^{-\frac{1}{2}t}\left(c_1'\cos(3t) + c_2'\sin(3t)\right),
\end{aligned}$$

where $c_1' = c_1 + c_2$, $c_2' = i(c_1 - c_2)$

<u>ICs</u>: $y(0) = 2 \Rightarrow 2 = 1 \cdot (c_1 \cancel{1} + 0) \Rightarrow c_1 = 2$

$$y' = -\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1\cos(3t) + c_2\sin(3t)\right) + 3e^{-\frac{1}{2}t}\left(-c_1 \overset{\sin}{\cancel{\cos}}(3t) + c_2\cos(3t)\right)$$

$$= e^{-\frac{1}{2}t}\left(\left[-\frac{1}{2}c_1 + 3c_2\right]\cos(3t) + \left[-3c_1 - \frac{1}{2}c_2\right]\sin(3t)\right)$$

So $y'(0) = 0 \Rightarrow 8 = 1 \cdot \left[\left(-\frac{1}{2}c_1 + 3c_2\right)1 + 0\right]$

$\Rightarrow 8 = -1 + 3c_2$

$\Rightarrow c_2 = 3.$

Hence the solution is

$$y = e^{-\frac{1}{2}t}\left(2\cos 3t + 3\sin 3t\right)$$

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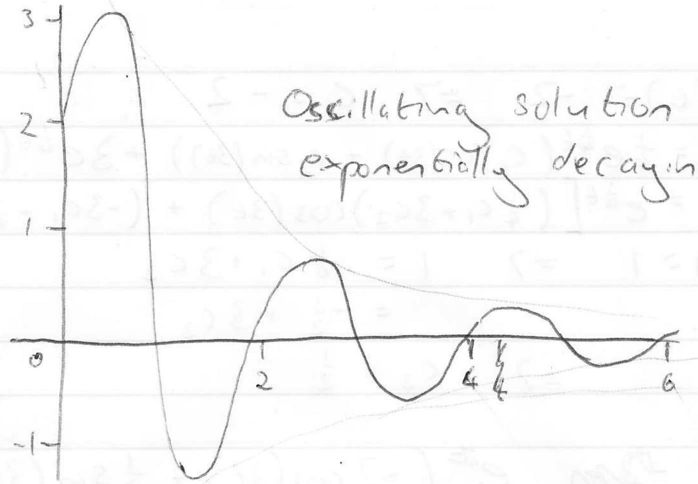
Plot:



General Case 2.2.5: $ay'' + by' + cy = 0$

has CE $ar^2 + br + c = 0$

with roots $r = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$

If $b^2 - 4a < 0$, then the equation has complex roots.

Note that we may always the two complex roots as

$r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$

where $\lambda, \omega \in \mathbb{R}$.

The general solution is

$$y = c_1 e^{(\lambda + i\omega)t} + c_2 e^{(\lambda + i\omega)t}$$

which we can always write as (Different c_1, c_2)

$$y = e^{\lambda t}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right)$$

Definition: λ is called the amplitude or growth, ω the angular frequency of the system.

Example 2.2.6: Solve the IVP

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2$$
$$y'(0) = 1$$

CE: $16r^2 - 8r + 145 = 0 \Rightarrow r = \frac{1}{4} \pm \frac{1}{32}\sqrt{64 - 9280}$

$= \frac{1}{4} \pm \frac{1}{32} \cdot 96i$

$= \frac{1}{4} \pm i \cdot 3$

So here $\lambda = \frac{1}{4}$, $\omega = 3$, & $r_1, r_2 = \lambda \pm i\omega$.

Hence General solution is $y = e^{\frac{1}{4}t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$

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ICs: $y(0) = -2 \Rightarrow c_1 = -2$

$$y' = \tfrac{1}{4}e^{\frac{1}{4}t}\left(c_1\cos(3t) + c_2\sin(3t)\right) + 3e^{\frac{1}{4}t}\left(-c_1\sin(3t) + c_2\cos(3t)\right)$$

$$= e^{\frac{1}{4}t}\left[\left(\tfrac{1}{4}c_1 + 3c_2\right)\cos(3t) + \left(-3c_1 + \tfrac{1}{4}c_2\right)\sin(3t)\right)$$

$$y'(0) = 1 \Rightarrow 1 = \tfrac{1}{4}c_1 + 3c_2$$

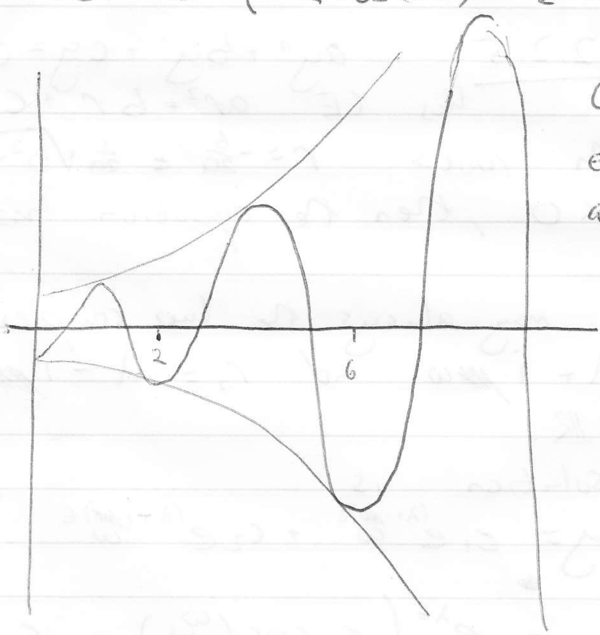
$$= -\tfrac{1}{2} + 3c_2$$

$$\Rightarrow c_2 = \tfrac{1}{2}.$$

angular frequency

So $y = e^{\frac{1}{4}t}\left(-2\cos(3t) + \tfrac{1}{2}\sin(3t)\right)$.

<u>Plot</u>:



Oscillating solution with exponentially growing amplitude.

<u>General Behaviour 2.2.6</u>: Any solution to $ay'' + by' + cy = 0$, where $ar^2 + by + c = 0$ has complex roots, will

- Oscillate with angular frequency $\omega = \frac{1}{2a}\sqrt{4ac - b^2}$
- Grow or decay exponentially according to amplitude $\lambda = -\frac{b}{2a}$:
  - if $\lambda < 0$, Solutions decay $\to 0$
  - if $\lambda > 0$, solutions grow in amplitude exponentially
  - if $\lambda = 0$, solutions have constant amplitude.

c.f. $y = e^{\lambda t}\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right)$