Homework 3

Total: 20 points
Due: Wed 22 Oct 2014 at the beginning of class

If a question is taken from the textbook, the refence is given on the right of the page.

1. **EXISTENCE & UNIQUENESS**

   (a) In each of the following problems, use the existence and uniqueness theorems to determine (without solving the DE) the largest interval in which the solution to the given initial value problem is guaranteed to exist. Write a sentence or two justifying your answer.

   i. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$ Boyce 2.4 Q1

   ii. $t(t-4)y' + y = 0, \quad y(2) = 1$ Q2

   iii. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$ Q3

   iv. $(\ln t)y' + y = \cot t, \quad y(2) = 3$ Q6

   (b) Find all solutions to the initial value problem

   $$(y')^2 - y = 4, \quad y(2) = 0.$$

   (c) Consider the initial value problem

   $$\frac{dy}{dt} = (y-1)^{\frac{1}{5}}, \quad y(0) = y_0$$

   i. For which value of y_0 does the IVP *not* have a unique solution in some interval about $x = 0$?

   ii. For this value of y_0, find all continuous differentiable functions which satisfy the differential equation.

2. **AUTONOMOUS EQUATIONS**

   (a) In the following two autonomous equations $\frac{dy}{dt} = f(y)$, sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each equilibrium solution as asymptotically stable, unstable or semistable. Then sketch a graph of several solutions on the ty-plane, including the equilibrium solutions and a few other solutions to indicate asymptotic behaviour.

   i. $\frac{dy}{dt} = y(y-1)(y-2)$ Boyce 2.5 Q3

   ii. $\frac{dy}{dt} = e^{-y} - 1$ Boyce 2.5 Q5

   (b) Boyce 2.5 Q18

   A pond forms as water collects in a conical depression of radius a and depth h. Suppose that the water flows in at a constant rate k, and is lost through evaporation at a rate proportional to the pond's surface area.

   i. Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

   $$\frac{dV}{dt} = k - \alpha\pi \left(\frac{3a}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}},$$

   where α is the coefficient of evaporation.

   ii. Find the equilibrium depth of the water in the pond. Is the equilibrium asymptotically stable or unstable?

   iii. Find a condition relating k and α that must be satisfied if the pond is not to overflow.

[More questions overleaf!]

3. **EULER'S METHOD** Consider the initial value problem

$$\frac{dy}{dt} = (t-1)(y+1), \quad y(1) = 1.$$

(a) Let $y = \phi(t)$ be the unique solution to this IVP. Estimate the value of $\phi(2)$ using Euler's method with a step size of $h = 1$. Then do the same for step sizes of $h = 0.5$ and $h = 0.2$.

(b) Solve the IVP and state the true value of $\phi(2)$. Do your estimates underpredict or overpredict $\phi(2)$? Do they get more accurate as h decreases?

4. **2ND ORDER LINEAR DIFFERENTIAL EQUATIONS**

(a) In each of the following problems, find the general solution to the given differential equation

i. $y'' + 2y' - 3y = 0$ — Boyce 3.1 Q1

ii. $y'' + 3y' + 2y = 0$ — Boyce 3.1 Q2

iii. $y'' + 5y' = 0$ — Boyce 3.1 Q5

iv. $y'' - 2y' - 2y = 0$ — Boyce 3.1 Q8

(b) In each of the following problems, find the solution to the given initial value problem, and sketch a graph of th solution, indicating the behaviour as t increases.

i. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$ — Boyce 3.1 Q9

ii. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$ — Boyce 3.1 Q10

iii. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$ — Boyce 3.1 Q12

(c) Boyce 3.1 Q23

Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0,$$

where α is a given constant. Determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.