Math 307 E - Summer 2011
Mid-Term Exam
July 20, 2011

Name: ______________________ Student number: ______________

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	3*	
Total	60+	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 60 minutes to complete the exam.

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1. Solve for $y(t)$:

   (a) (5 points)
$$y' = 1 + t + y + ty, \qquad y(0) = 0.$$

   (b) (5 points)
$$ty' = 1 - y. \qquad y(1) = 0, \quad t > 0.$$

2. Find the general solution to:

(a) (5 points)

$$x^2\frac{dy}{dx} = (x-y)^2 + 3xy, \quad x, y > 0.$$

(b) (5 points)

$$e^{y-x}(\cos(x) - \sin(x)) - \sin(x) + \sin(x)e^{y-x}\frac{dy}{dx} = 0.$$

Hint: Integrate $N(x, y)$ with respect to y.

3. A tank initially contains 9 gal of water with 5 lb of salt in solution. A solution containing a constant concentration of γ lb salt per gallon runs into the tank at a rate of 1 gal/min; the well-mixed solution drains from the tank at the rate of γ^2 gal/min.

   (a) (5 points) Set up the initial value problem that models $Q(t)$, the amount of salt in pounds at time t for $0 \leq t < \frac{9}{\gamma^2}$.

   (b) (5 points) Solve this linear differential equation for $Q(t)$ using an integrating factor.

4. Jill has a saving's account with a balance of $S(t)$ dollars at time t. Assume that her credit is outstanding, so that any negative balance is acceptable to her bank and represents a loan. Assume that Jill's account earns interest at an annual rate of r compounded continuously. Furthermore, assume that Jill continuously withdraws money from her account at a rate proportional to the cube of her balance, with proportionality constant r^3.

   (a) (4 points) Write down a differential equation modeling $S(t)$.

   (b) (4 points) Find the equilibrium solutions, and classify as stable, unstable or semi-stable.

   (c) (2 points) If Jill has saved any money at $t = 0$, how much money do you expect her to have in her account for large values of t if $r = 1\%$?

5. Find the general solution to the following second-order differential equations:

(a) (3 points)

$$\frac{1}{2}y'' + 2y' + \frac{5}{2}y = 0.$$

(b) (3 points)

$$16y'' + 8y' + y = 0.$$

(c) (4 points)

$$y'' - y' - 2y = 0.$$

6. Consider the differential equation

$$y'' + 2y' + \left(1 - \frac{3}{4t^2}\right) y = 0 \qquad t > 0.$$

Following the reduction of order technique, assume that $y_1(t)$ is a solution and guess that $y_2(t) = w(t) \cdot y_1(t)$.

(a) (3 points) Show that if $y_2(t)$ is a solution, then $w(t)$ must satisfy

$$\frac{w''}{w'} = -2\left(\frac{y_1'}{y_1} + 1\right).$$

Hint: You do not need to know what $y_1(t)$ is, only that it is a solution.

(b) (7 points) It is true (and you may assume) that $y_1(t) = t^{3/2}e^{-t}$ satisfies the equation. Find $w(t)$ using the formula from part (a); write down the general solution as a linear combination of $y_1(t)$ and $y_2(t)$.

7. (3 bonus points) Assume that $y_1(t) = e^{t^2/2}\cos(t)$ is a solution to the differential equation

$$y'' - 2ty' + t^2 y = 0.$$

Find another independent solution. (Just guessing will not earn all points.)
Hint: Derive a similar formula as in problem 6 part (a), or use a trick.