Theorem 3.1.6: for constant a & function $f(t)$,
$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f](s-a).$$

Proof: $\mathcal{L}[e^{at}f] = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt$
$= \mathcal{L}[f](s-a).$

Example 3.1.7 $f(t) = e^{at}\cos(bt)$
Then $\mathcal{L}[f] = \int_0^\infty e^{at}\cos(bt)e^{-st}\,dt = \int \cos(bt)e^{-(s-a)t}\,dt$

So $\mathcal{L}[e^{at}\cos(bt)] = \dfrac{s-a}{(s-a)^2+b^2}$

WED 26 FEB
→ MATH 307A LECTURE 19.

Definition 3.1.8 The Gamma function $\Gamma(a) : (-1,\infty) \to \mathbb{R}$ is defined by
$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt$$

We'll most often use it in the form $\Gamma(1+a) = \int_0^\infty t^a e^{-t}\,dt$

Observe: $\Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = 1$

And for positive integers n,
$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}$$
IBP $\quad = -t^n e^{-t}\Big|_0^\infty + n\int_0^\infty t^{n-1}e^{-t}\,dt$
$\quad = n\Gamma(n)$

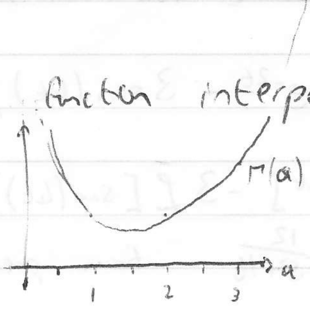
So $\Gamma(n+1) = n\Gamma(n)$.

Since $\Gamma(1) = 1 \Rightarrow \Gamma(2) = 1\cdot\Gamma(1) = 1$
$\Gamma(3) = 2\cdot\Gamma(2) = 2$
$\Gamma(4) = 3\cdot\Gamma(3) = 6$ etc.

So $\Gamma(n+1) = n!$

Thus the Gamma function interpolates the factorial function.

Looks like:



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WED 26

LAPLACE TRANSFORMS INTRO, PART 2 (BOYCE 6.1)

<u>Example 3.1.9</u>: $f(t) = t^a$ for some $a > -1$

Then $\mathcal{L}[f] = \int_0^\infty t^a e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u}\cdot\frac{1}{s}\,du$

$u = st$ $t=0$ $u=0$

$du = s\,dt$ $t=\infty$ $u=\infty$

or $\frac{1}{s}du = dt$

$= \frac{1}{s^{a+1}}\int_0^\infty u^a e^{-u}\,du$

$= \frac{\Gamma(1+a)}{s^{1+a}}$.

So $\mathcal{L}[t^a] = \frac{\Gamma(1+a)}{s^{1+a}}$

Suppose we know that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ (show if there's time)

<u>Example 3.1.10</u> i) $f(t) = \frac{1}{\sqrt{t}} = t^{-\frac{1}{2}}$.

Then $\mathcal{L}[f] = \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{1}{\sqrt{s}}\Gamma(\frac{1}{2})$ by example 3.1.9.

And $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du$

$u = \sqrt{t}$ $t=0$ $u=0$

$du = \frac{1}{2\sqrt{t}}dt$ $t=\infty$ $u=\infty$

$= 2\cdot\frac{\sqrt{\pi}}{2}$ by our supposition

$= \sqrt{\pi}$

Hence $\mathcal{L}[t^{-\frac{1}{2}}] = \sqrt{\frac{\pi}{s}}$.

2) $f(t) = \sqrt{t} = t^{\frac{1}{2}}$

Then $\mathcal{L}[f] = \frac{\Gamma(\frac{3}{2})}{s^{3/2}}$ by Ex. 3.1.9

But $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$ by properties of the Γ-function

$= \frac{1}{2}\cdot\sqrt{\pi}$ by i).

So $\mathcal{L}[\sqrt{t}] = \sqrt{\frac{\pi}{2s^3}}$ or $\frac{\sqrt{\pi}}{2s^{3/2}}$.

WED 20 Feb

MATH 307A LECTURE 19:

§3.2 SOLVING IVPs WITH LAPLACE TRANSFORMS (BOYCE 6.2)

How do we use Laplace transforms to solve initial value problems? The fundamental insight comes from the following theorem:

Theorem 3.2.1 Let $f(t)$, $t \geq 0$ be a continuous, differentiable function whose Laplace transform exists, s.t. f' is piecewise continuous. Then

$$\boxed{\mathcal{L}[f'] = s \cdot \mathcal{L}[f] - f(0)}$$

for $s >$ some a.

Proof sketch:
$$\mathcal{L}[f'] = \int_0^\infty f'(t)e^{-st}\,dt \overset{\text{IBP}}{=} f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}\,dt = -f(0) + s\mathcal{L}[f].$$

Corollary 3.2.2
$$\mathcal{L}[f''] = s \cdot \mathcal{L}[f'] - f'(0) = s\left(s\cdot\mathcal{L}[f] - f(0)\right) - f'(0)$$

So $$\boxed{\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)}$$

when f is doubly differentiable & well behaved (i.e. its derivatives are at most exponentially growing etc.)

We can of course use the above methodology to obtain formulae for the Laplace transform of higher derivatives of f (if they exist).

General approach 3.2.3 The idea to solving IVPs using Laplace transforms is therefore as follows: Given a constant coefficient linear ODE with known initial values,

- Suppose $\phi(t)$ solves the ~~equation~~ IVP. Let $\Phi(s)$ be the Laplace transform of $\phi(t)$, i.e. $\mathcal{L}[\phi] = \Phi(s)$.

PTO

- Take the Laplace transform of the DE, to obtain an equation in Φ, s, $\phi(0)$, $\phi'(0)$ etc.
- Solve for Φ as a function of s, all other quantities being known
- Look up in a table to see which $\phi(t)$ have Laplace transform $\Phi(s)$.

Example 3.2.4 We start with an easy homogeneous equation:

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

[We know how to solve this conventionally: CE is $r^2 - r - 2 = 0$, so solution has the form $y = c_1 e^{-t} + c_2 e^{2t}$

ICs: $y(0) = 1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = 0 \Rightarrow -c_1 + 2c_2 = 0$

Solving yields $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$, so $y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$.]

Laplace transform way: Suppose $y = \phi(t)$ is the solution to this IVP. Then $\phi'' - \phi' - 2\phi = 0$, $\phi(0) = 1$, $\phi'(0) = 0$.

- Hit the DE with $\mathcal{L}$:

$$\mathcal{L}[\phi'' - \phi' - 2\phi] = \mathcal{L}[0]$$

$$\Rightarrow \mathcal{L}[\phi''] - \mathcal{L}[\phi'] - 2\mathcal{L}[\phi] = 0, \text{ as } \mathcal{L}[0] = 0, \text{ \& linearity.}$$

$$\Rightarrow \left(s^2\mathcal{L}[\phi] - s\phi(0) - \phi'(0)\right) - \left(\mathcal{L}[\phi] - \phi(0)\right) - 2\mathcal{L}[\phi] = 0$$ using Theorems 3.2.1 & 3.2.2.

Suppose $\mathcal{L}[\phi] = \Phi$.

Then $(s^2 - s - 2)\Phi - s - 0 + 1 = 0$ using $\phi'(0) = 0$, $\phi(0) = 1$

So $$\Phi(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

Now we use partial fractions to get $\frac{s-1}{(s+1)(s-2)} = \frac{2/3}{s+1} + \frac{1/3}{s-2}$.

Hence

$$\Phi(s) = \frac{2}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2}.$$

Finally, we know $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ from previously, so we conclude that we must have

$$\phi(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$ $\square$.

PTO

Note that ~~what~~ in the last step, where we go from $\Phi(s)$ to obtaining the ~~Φ~~ $\phi(t)$ whose Laplace transform is $\Phi(s)$, we have made some implicit assumptions, which are laid out below.

Definition 3.2.5 The Inverse Laplace Operator $\mathcal{L}^{-1}$ is a linear (integral) operator that takes as input functions in s-space & returns functions in t-space. It is (for all practical purposes) the unique inverse operator of $\mathcal{L}$, i.e.

$$\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$$
$$\mathcal{L}[\mathcal{L}^{-1}[F(s)]] = F(s)$$

for all $f(t)$ & $F(s)$ suitably well-behaved.

The Inverse Laplace Operator can be defined using a complex integral, and hence is a bit ~~outside the~~ outside the scope of the course. Thus what we usually do is consult a lookup table to see which functions $f(t)$ have Laplace transform $F(s)$.

Note: $\mathcal{L}^{-1}$ is linear, so $\mathcal{L}^{-1}[c_1 F_1(s) + c_2 F_2(s)]$
$= c_1 \mathcal{L}^{-1}[F_1(s)] + c_2 \mathcal{L}^{-1}[F_2(s)]$.

The Laplace Transform is particularly effective when it comes to solving nonhomogeneous DEs.

Example 3.2.6: Find the solution to $y'' + y = \sin(2t)$, $y(0) = 2$, $y'(0) = 1$

So let $\phi(t)$ be the solution to the IVP, & let $\Phi(s) = \mathcal{L}[\phi(t)]$.

Then $\phi'' + \phi = \sin(2t)$

So $\mathcal{L}[\phi''] + \mathcal{L}[\phi] = \mathcal{L}[\sin(2t)]$

$\Rightarrow (s^2\Phi - s\phi(0) - \phi'(0)) + \Phi = \frac{2}{s^2+4}$ using $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$

$\Rightarrow (s^2+1)\Phi - 2s - 1 = \frac{2}{s^2+4}$

$\Rightarrow (s^2+1)\Phi = \frac{2}{s^2+4} + \frac{(2s+1)(s^2+4)}{s^2+4}$ PTO

$$\Rightarrow \quad \Phi(s) = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}$$

We can use partial fractions again to decompose the RHS:

$$\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

So
$$2s^3+s^2+8s+6 = (as+b)(s^2+4) + (cs+d)(s^2+1)$$
$$= (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d).$$

Hence we must have
$$a+c = 2, \quad b+d = 1$$
$$4a+c = 8, \quad 4b+d = 6$$

These are 2 independent systems of linear equations in 2 variables each, which we know how to solve. We get:
$a = 2$, $c = 0$, $b = \frac{5}{3}$ and $d = -\frac{2}{3}$,

so
$$\Phi(s) = \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4}.$$

Hence
$$\phi(t) = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \frac{5}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]$$
$$\Rightarrow \quad \underline{\phi(t) = 2\cos(t) + \tfrac{5}{3}\sin(t) - \tfrac{1}{3}\sin(2t).} \qquad \square$$

We see then that the Laplace Transform method has advantages & disadvantages associated with it:

<u>Advantages</u>:
- Converts differential problem into purely algebraic problem.
- Can be used to solve higher-order IVP's.
- works on NH equations with a large range of forcing functions

<u>Disadvantages</u>:
- Algebra can get messy
- Need a lookup table of Laplace (inverse) transforms
- Constant coefficient DEs only
  (can be extended to non-constant coeff. equation, but the algebra can become unsolvably messy).