Confidence Interval

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Confidence Interval

An interval estimator is a rule for calculating two numbers L and U that define an interval which you are fairly certain containing the parameter of interest.

The concept of fairly certain can be quantified using a statistical concept called the "Confidence coefficient" designated by $(1-\alpha)$ called the confidence level.

So, if θ be the parameter of interest based on the sample $X_1, X_2, ...X_n$ we are looking for L and U which are the functions of the sample such that we are $(1-\alpha)100\%$ confident that θ lies in the interval(L, U).

In summary, a 95% confidence interval can be interpreted to mean that if all possible samples of given size are taken from a population, 95% of the samples would produce intervals that captured the true population mean and 5% would not. So, the confidence interval is used to describe the method of construction rather than a particular interval.

for Loop

Often we need to repeat an action several times - sometimes over subjects in a data set.

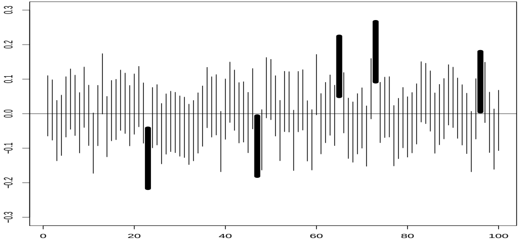
```
> for(i in 1:n){
> the action to be repeated
}
```

for indicates that we're going to loop from a start index to an end index.
i is the index we're looping over
1 is our start index
n is the end index
{ opens the loop
} closes the loop.
Example:

```
> index<-NULL # (Initiates the variable; assigns NULL value)
> for(i in 1:4){
+ index<-c(index,factorial(i))
+ }
> index
[1]  1  2  6 24
```

Confidence Interval

The function `interval.plot()` from **PASWR** library can be used to create the intervals.




```
set.seed(402)
m<-100 # Number of samples
n<-500 # Sample size
a<-array(0,m)
ll<-array(0,m)
ul<-array(0,m)
i<-0
while (i<m) {i<-i+1
a[i]<-mean(rnorm(n))
ll[i]<-a[i]+qnorm(0.025)*sqrt(1/n)
ul[i]<-a[i]+qnorm(0.975)*sqrt(1/n)}
interval.plot(ll,ul)
```

Central Limit Theorem

Generating from Uniform

```
plot(0,0,type="n",xlim=c(0,1),ylim=c(0,13.5),
xlab="Density", ylab="f(x)")
m=500;  a=0;  b=1
n=2
res<- mean(runif(n, a, b))
for ( i in 1:m) res[i]<- mean(runif(n, a, b))
lines( density(res), lwd=2)
```

Please repeat the above program by replacing n by 5, 10, 100

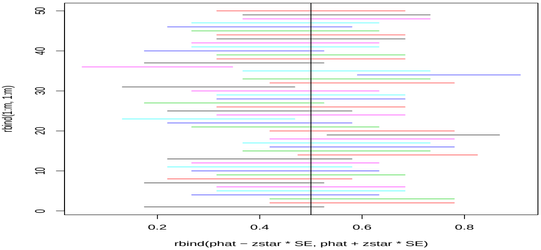
Generating from Gamma

```
plot(0,0,type="n", xlim=c(0,3), ylim=c(0,2),
 xlab=" Density", ylab="f(x)")
m=500; a=1; b=1
  n=2
  res<- mean(rgamma(n, a, b))
 for ( i in 1:m) res[i]<- mean(rgamma(n, a, b))
 lines( density(res), lwd=2)
```

Please repeat the above program by replacing n by 5, 10, 100

Confidence Interval

Each CI is a Bernoulli trial: It either contains the true parameter value or not. After the CI is constructed, we talk about how "confident" we are that it contains the true value.




```
> m = 50; n=20; p = .5 # toss 20 coins 50 times
> phat = rbinom(m,n,p)/n # divide by n for proportions
> SE = sqrt(phat*(1-phat)/n) # compute SE
> alpha = 0.10; zstar = qnorm(1-alpha/2)
> matplot(rbind(phat - zstar*SE, phat + zstar*SE),
+ rbind(1:m,1:m),type="l",lty=1)
> abline(v=p) # draw vertical line at p=0.5
```

Confidence Interval

Note that $(1-\alpha)100\%$ confidence interval for μ is given by

$$\overline{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}. \qquad \text{If } \sigma \text{ is known}$$

$$\overline{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{s}{\sqrt{n}}. \qquad \text{If } \sigma \text{ is unknown}$$

Confidence Interval- Example

The consumer expenditure survey , created by the U.S. Department of Labor, was administrated to 30 household. The amount of money each household spent per week is provided in the data frame `Grocery` in the PASWR package. Using the historical record of variance 900, Construct a 95% confidence interval for average expenditure.

```
library(PASWR)
attach(Grocery)
xbar <- mean(groceries)
Z <- qnorm(.975)
sigma <- 30
n <- 30
round(c(xbar-Z*sigma/sqrt(n),xbar+Z*sigma/sqrt(n)),2)
[1] 109.90 131.37
```

Assessing the Normality of the Data

```
library(PASWR)
attach(Grocery)
qqnorm(groceries)
qqline(groceries)
```

Confidence Interval- Example

The price for fourteen houses (three bedrooms/two-bathrooms) in Watauga County NC are provided in the data frame `House` in the PASWR package. Calculate a 95% confidence interval for the average price of a three bedroom/two bathroom house in this county.

```
library(PASWR)
attach(House)
MEAN<-mean(Price)
CT<-qt(.975,13)
ST<-sd(Price)
round(c(MEAN-CT*ST/sqrt(14), MEAN+CT*ST/sqrt(14)),2)
[1] 198.54 272.34
```

Assessing the Normality of the Data

```
library(PASWR)
attach(House)
qqnorm(House$Price)
qqline(House$Price)
```

Confidence Interval for the Difference in Population Means: Equal Variances

Suppose two normal populations $N(\mu_X, \sigma_X)$ and $N(\mu_Y, \sigma_Y)$, where $\sigma_X = \sigma_Y = \sigma$ is known. Take sample of sizes n_X and n_Y then $(1-\alpha)100\%$ confidence interval for $\mu_X - \mu_Y$ is given by

$$\overline{X} - \overline{Y} \pm Z_{\frac{\alpha}{2}} \sigma \sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}$$

$$\left[(\overline{X} - \overline{Y}) - Z_{\frac{\alpha}{2}} \sigma \sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}, (\overline{X} - \overline{Y}) + Z_{\frac{\alpha}{2}} \sigma \sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}\right]$$

Example

Example: Suppose independent random samples are taken from two normal distribution with the following summary

$$\begin{aligned}\bar{X} &= 4.00, \sigma_X = 3, n_X = 15 \\ \bar{Y} &= 4.41, \sigma_Y = 3, n_Y = 22\end{aligned}$$

Calculate a 95% confidence interval for the difference in population means $(\mu_X - \mu_Y)$.

```
round(c((4-4.41)-qnorm(0.975)*3*sqrt(1/15+1/22),
(4-4.41)+qnorm(0.975)*3*sqrt(1/15+1/22)),2)
[1] -2.38  1.56
```

Example

The hardness of a piece of fruit is a good indicator of the fruits's ripeness. The data frame `Apple` in the **PASWR** package contains the hardness of 17 fresh apples and 17 warehouse stored apples. Assuming that both group have equal variances of 2.25, construct a 95% confidence interval for the difference in the mean hardness for fresh and warehoused apples.

Assessing Normality: Q-Q plot

```
library(PASWR)
attach(Apple)
fresh <- qqnorm(Fresh)
par(new=T)
old <-qqnorm(Warehouse)
plot(fresh,type="n",ylab="Sample Quantiles",
xlab="Theoretical Quantiles")
qqline(Fresh, col = 1)
qqline(Warehouse, col = 2)
points(fresh, col = 1, pch = 16, cex = 1.2)
points(old, col = 2, pch = 17,cex=1.2)
legend(-2, 9, c("Fresh", "Warehouse"),pch=c(16,17),
 col = c(1, 2),title("Q-Q Normal Plots"))
```

Example- Apple hardness

```
> library(PASWR)
> attach(Apple)
> summary(Apple)
     Fresh          Warehouse
 Min.   :5.530   Min.   : 6.190
 1st Qu.:6.480   1st Qu.: 7.070
 Median :7.270   Median : 7.790
 Mean   :7.254   Mean   : 7.895
 3rd Qu.:8.090   3rd Qu.: 8.830
 Max.   :9.560   Max.   :10.500

> mean.old<-mean(Warehouse)
> mean.fresh<-mean(Fresh)
> round(c(mean.fresh - mean.old - qnorm(0.975)*1.5*sqrt(2/17),
+     mean.fresh - mean.old + qnorm(0.975)*1.5*sqrt(2/17)),2)
[1] -1.65  0.37
```

Therefore, we are 95% confident that the difference in the mean hardness for fresh and warehoused apples is (-1.65,0.37).

Confidence Interval for the Difference in Population Means: With Known and UnequalVariances

Suppose two normal populations $N(\mu_X, \sigma_X)$ and $N(\mu_Y, \sigma_Y)$, where $\sigma_X \neq \sigma_Y$ is known. Take sample of sizes n_X and n_Y then $(1-\alpha)100\%$ confidence interval for $\mu_X - \mu_Y$ is given by

$$\overline{X} - \overline{Y} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}$$

$$\left[(\overline{X} - \overline{Y}) - Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}, (\overline{X} - \overline{Y}) + Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}} \right]$$

Example

Example: Suppose independent random samples are taken from two normal distribution with the following summary

$$
\begin{aligned}
\bar{X} &= 8.4, \sigma_X = 4.5, n_X = 50 \\
\bar{Y} &= 8.8, \sigma_Y = 6, n_Y = 46
\end{aligned}
$$

Calculate a 95% confidence interval for the difference in population means $(\mu_X - \mu_Y)$.

```
round(c((8.4-8.8)-qnorm(0.975)*sqrt((4.5)^2/50+6^2/46),
(8.4-8.8)+qnorm(0.975)*3*sqrt((4.5)^2/50+6^2/46)),2)
```

```
[1] -2.54  6.01
```

Confidence Interval of means when the variances are unknown

Suppose two normal populations $N(\mu_X, \sigma_X)$ and $N(\mu_Y, \sigma_Y)$, where σ_X and $sigma_Y$ are unknown. The sampling distribution of $\bar{X} - \bar{Y}$ is students' t distribution with the standard statistic

$$t = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{SE(\bar{X} - \bar{Y})}$$

Note that

$$SE(\bar{X} - \bar{Y}) = \begin{cases} \sqrt{s_p^2(1/n_X + 1/n_Y)} & \text{if} \sigma_X = \sigma_Y \\ \sqrt{s_X^2/n_X + s_Y^2/n_Y} & \text{if} \sigma_X \neq \sigma_Y \end{cases}$$

where, $s_p^2 = \frac{(n_X-1)s_X^2 + (n_Y-1)s_Y^2}{n_X+n_Y-2}$ is called the pooled variance. Note that the degrees of freedom is given by

$$\text{degrees of freedom} = \begin{cases} n_X + n_Y - 2 & \text{if} \sigma_X = \sigma_Y \\ \left(\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}\right)^2 \times \left(\frac{(s_X^2/n_X)^2}{n_X-1} + \frac{(s_Y^2/n_Y)^2}{n_Y-1}\right)^{-1} & \text{if} \sigma_X \neq \sigma_Y \end{cases}$$

Confidence Interval

$(1-\alpha)100\%$ Confidence Interval for $\mu_X - \mu_y$ is
case 1: Unknown but equal variances so, $\sigma_X = \sigma_Y$

$$\left[(\bar{X}-\bar{Y}) - t_{\alpha/2,\nu} s_p \sqrt{\frac{1}{n_X}+\frac{1}{n_Y}}, (\bar{X}-\bar{Y}) + t_{\alpha/2,\nu} s_p \sqrt{\frac{1}{n_X}+\frac{1}{n_Y}}\right]$$

where, $s_p^2 = \frac{(n_X-1)s_X^2+(n_Y-1)s_Y^2}{n_X+n_Y-2}$ and $\nu = n_X + n_y - 2$.
case 2: Unknown and unequal variances so $\sigma_X \neq \sigma_Y$

$$\left[(\bar{X}-\bar{Y}) - t_{\alpha/2,\nu} \sqrt{\frac{s_X^2}{n_X}+\frac{s_Y^2}{n_Y}}, (\bar{X}-\bar{Y}) + t_{\alpha/2,\nu} \sqrt{\frac{s_X^2}{n_X}+\frac{s_Y^2}{n_Y}}\right]$$

where, $\nu = \left(\frac{s_X^2}{n_X}+\frac{s_Y^2}{n_Y}\right)^2 \times \left(\frac{(s_X^2/n_X)^2}{n_X-1}+\frac{(s_Y^2/n_Y)^2}{n_Y-1}\right)^{-1}$

Example

Suppose a random sample from a $N(\mu_X, \sigma)$ is taken where

$$n_X = 15, \sum_{i=1}^{15} x_i = 53, \text{and} \sum_{i=1}^{15} x_i^2 = 222.$$

Similarly another random sample is taken from a $N(\mu_Y, \sigma)$ population independent of the first sample such that

$$n_Y = 11, \sum_{i=1}^{11} y_i = 77, \text{and} \sum_{i=1}^{11} y_i^2 = 560$$

Obtain a 95% confidence interval for $\mu_X - \mu_Y$.
Solution: Note that

$$\bar{X} = 3.53, s_X^2 = 2.51, \bar{Y} = 7.00, s_Y^2 = 2.10$$

Now, using R we have

```
sp=sqrt((14*2.51+10*2.1)/24)
round(c((3.53-7)-qt(0.975,24)*sp*sqrt(1/15+1/11),
(3.53-7)+qt(0.975,24)*sp*sqrt(1/15+1/11)),2)
[1] -4.72 -2.22
```

Example

Suppose a random sample from a $N(\mu_X, \sigma_X)$ is taken where

$$n_X = 15, \sum_{i=1}^{15} x_i = 63, \text{and} \sum_{i=1}^{15} x_i^2 = 338.$$

Similarly another random sample is taken from a $N(\mu_Y, \sigma_Y)$ population independent of the first sample such that

$$n_Y = 11, \sum_{i=1}^{11} y_i = 66.4, \text{and} \sum_{i=1}^{11} y_i^2 = 486$$

Obtain a 95% confidence interval for $\mu_X - \mu_Y$.
Solution: Note that

$$\bar{X} = 4.2,\ s_X^2 = 5.24,\ \bar{Y} = 6.04,\ s_Y^2 = 8.47,\ \nu = \left(\frac{s_X^2}{n_X} + \frac{s_Y^2}{n_Y}\right)^2 \times \left(\frac{(s_X^2/n_X)^2}{n_X - 1} + \frac{(s_Y^2/n_Y)^2}{n_Y - 1}\right)^{-1} = 18.43 \approx 18$$

Now, using R we have

```
 round(c((4.2-6.04)-qt(0.975,18)*sqrt(5.24/15+8.47/11),
(4.2-6.04)+qt(0.975,18)*sqrt(5.24/15+8.47/11)),2)
[1] -4.06  0.38
```

Confidence Interval for Population Variance

Let $X_1, X_2, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$ and both are unknown. We know that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

We want χ^2_L and χ^2_U such that

$$P(\chi^2_L \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_U) = 1-\alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_U} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_L}\right) = 1-\alpha$$

Hence, the $(1-\alpha)100\%$ CI for σ^2 is

$$\left(\frac{(n-1)S^2}{\chi^2_U}, \frac{(n-1)S^2}{\chi^2_L}\right).$$

But if $\chi^2_U = \chi^2_{\alpha/2}$ then $\chi^2_L = \chi^2_{1-\alpha/2}$. Therefore, $(1-\alpha)100\%$ CI for σ^2 is

$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right)$$

Example

Suppose 15 random sample is taken from a $N(\mu_X, \sigma_X)$. Suppose the sample variance is 5.24. Construct a 80% confidence interval for σ^2.

```
round(c(14*5.24/qchisq(0.9, 14), 14*5.24/qchisq(0.1,14)),2)
[1] 3.48 9.42
```

Confidence Interval for Population Proportion

We know that the maximum likelihood estimate of the population proportion p is the sample proportion $\hat{p}$. The asymptomatic properties of MLE allows that

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

as $n \to \infty$
Hence, $(1-\alpha)100\%$ confidence interval for p is

$$\left[\hat{p} - Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

A more accurate confidence interval for p can be given as

$$\left[\frac{\hat{p} + \frac{Z^2_{\alpha/2}}{2n} - Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{Z^2_{\alpha/2}}{4n^2}}}{\left(1 + \frac{Z^2_{\alpha/2}}{n}\right)}, \frac{\hat{p} + \frac{Z^2_{\alpha/2}}{2n} + Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{Z^2_{\alpha/2}}{4n^2}}}{\left(1 + \frac{Z^2_{\alpha/2}}{n}\right)}\right]$$

Example

A researcher is interested in the the gender distribution of students in high school. He takes a random sample of 40 students and finds that only 26 are female. help him to construct 95% confidence intervals for the true proportion of female students.

```
> n<-40
> p=26/40
> z<-qnorm(0.975)
> round(c(p-z*sqrt(p*(1-p)/n), p+z*sqrt(p*(1-p)/n)),3)
[1] 0.502 0.798
```

The preferred method for smaller sample size yields

```
> n<-40
> p=26/40
> z<-qnorm(0.975)
> round(c((p+z^2/(2*n)-z*sqrt(p*(1-p)/n+z^2/(4*n^2)))/(1+z^2/n),
(p+z^2/(2*n)+z*sqrt(p*(1-p)/n+z^2/(4*n^2)))/(1+z^2/n)),3)
[1] 0.495 0.779
```