MATH EXERCISES

YOUR NAME$_\circ$

1. Exercise 4

1.1. **Roots and superscripts.** The two roots of $x^2 = 2$ are $\pm\sqrt{2}$. If $\sqrt{2} = p/q$ for some integers p and q, then we have $p^2 = 2q^2$, which is impossible. A similar argument holds for $\sqrt[n]{2}$ with $n \geq 3$.

1.2. **Pythagorean theorem.** The Pythagorean theorem says $\alpha^2+\beta^2 = \gamma^2$. It follows that $\gamma = \sqrt{\alpha^2 + \beta^2}$.

1.3. **Fibonacci numbers.** Fibonacci numbers satisfy

$$F_n = F_{n-1} + F_{n-2}$$

for $n = 1, 2, \ldots$.

1.4. **Inner products.** If $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$, then

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n.$$

1.5. **Repeating decimals.** If $0.\bar{9} < 1$, then $0 < 1 - 0.\bar{9} \leq \epsilon$ should hold for every $\epsilon > 0$.

1.6. **Approximating sine.** We have $\sin x \approx x$ for small x.

1.7. **Closures.** If $A_1, \ldots, A_n \subset X$, then $\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cup \cdots \cup \overline{A_n}$.

1.8. **Inverse function.** If $f \circ g$ and $g \circ f$ are both identities, then g is denoted by f^{-1}.

1.9. **Set builder notation.** If $R = \{x \mid x \notin x\}$, then $R \in R$ and $R \notin R$ are both true.

2. Exercise 5

2.1. **Quadratic formula.** If a, b, and c are complex numbers where $a \neq 0$, the roots of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

2.2. **Basel sum.** In 1734, Euler proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

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2.3. Euler's infinite product. Euler's infinite product formula says

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right).$$

2.4. Definition of derivative. If $y = f(x)$, we have

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2.5. Stokes' theorem. Let S be an oriented surface, and F be a C^1 vector field on S. Then,

$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds.$$

2.6. Integration by parts. The variance of the standard normal random variable can be computed by the following integration by parts:

$$\int_{-\infty}^{\infty} x^2 \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = \left[\frac{-xe^{-x^2/2}}{\sqrt{2\pi}}\right]_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx.$$

3. Exercise 7

3.1. A random set.

$$\left\{\frac{1}{p} + \frac{1}{q} : p \text{ and } q \text{ are prime}\right\}$$

3.2. Lévy equivalence theorem. If $X_1, X_2, \ldots$ are independent, then

$$\sum_{n=1}^{\infty} X_n \text{ converges a.s.} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} X_n \text{ converges in distribution.}$$

3.3. Completing the computation. We continue the computation given in 2.6:

$$\begin{aligned}\int_{-\infty}^{\infty} x^2 \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx &= \left[\frac{-xe^{-x^2/2}}{\sqrt{2\pi}}\right]_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx.\\ &= (0-0) + 1\\ &= 1.\end{aligned}$$

3.4. A long inequality.

$$\begin{aligned}|\mathbf{E}[f(Z)] - \mathbf{E}[f(S)] &- \mathbf{E}[f''(S)]\mathbf{E}[Y^2]/2|\\ &\le \frac{\epsilon}{2}\mathbf{E}[Y^2] + M\mathbf{E}[Y^2; |Y| > \delta].\end{aligned}$$

3.5. **Definition of cross product.** The cross product of $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ is given by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

3.6. **Large matrices.**

$$\begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & 0 & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & 0 \end{pmatrix}$$

4. Exercise 8

Theorem 4.1 (Baum–Katz). *If either*

- $t \geq 2$ *and* $r > t/2$*; or*
- $0 < t < 2$ *and* $r \geq 1$,

then

$$\mathbf{E}\, X = 0 \text{ (in case } t \geq 1 \text{ and } r/t \leq 1\text{)} \quad \text{and} \quad \mathbf{E}\,|X|^t < \infty \tag{4.1}$$

is equivalent to

$$\sum_{n=1}^{\infty} n^{r-2}\, \mathbf{P}(|S_n| > n^{r/t}\epsilon) < \infty \quad \text{for all } \epsilon > 0. \tag{4.2}$$

Proof. It is trival that (4.1) implies (4.2). We leave the other direction as an exercise. □

Theorem 4.2 (continuity). *For any function $f\colon \mathbb{R} \to \mathbb{R}$, the following are equivalent:*

1. f *is continuous;*
2. $f^{-1}(U)$ *is open for any open* $U \subset \mathbb{R}$*;*
3. $f^{-1}(C)$ *is closed for any closed* $C \subset \mathbb{R}$.

Corollary 4.3. *The composition of any two continuous functions from $\mathbb{R}$ to itself is continuous.*

Remark 4.1. The previous corollary holds for any continuous maps between arbitrary topological spaces.

Proof of Corollary 4.3. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous maps. For any open $U \subset \mathbb{R}$, the set $f^{-1}(U)$ is open by Theorem 4.2, and so is $g^{-1}(f^{-1}(U))$ by the same theorem. Since $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$, the continuity of $f \circ g$ follows. □