INDEFINITE INTEGRATION

Integration is the inverse process of differentiation. That is, the process of finding a function, whose differential coefficient is known, is called integration.

If the differential coefficient of F(x) is f (x),

i.e. $\frac{d}{dx}[F(x)] = f(x)$, then we say that the **antiderivative** or **integral** of f (x) is F(x), written as $\int f(x)dx = F(x)$,

Here $\int dx$ is the notation of integration f (x) is the integrand, x is the variable of integration and dx denotes the integration with respect to x.

1. INDEFINITE INTEGRAL

We know that if $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)dx = F(x)$.

Also, for any arbitrary constant C, $\frac{d}{dx}[F(x)+C] = \frac{d}{dx}[F(x)] + 0 = f(x)$.

$\therefore \int f(x)dx = F(x) + C,$

This shows that F(x) and F(x) + C are both integrals of the same function f(x). Thus, for different values of C, we obtain different integrals of f(x). This implies that the integral of f(x) is not definite. By virtue of this property F(x) is called the indefinite integral of f(x).

1.1 Properties of Indefinite Integration

1. $\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$

2. $\int f'(x)dx = \int \frac{d}{dx}[f(x)]dx = f(x) + c$

3. $\int k\, f(x)dx = k\int f(x)dx$, where k is any constant

4. If $f_1(x), f_2(x), f_3(x)$, ... (finite in number) are functions of x, then

$$\int [f_1(x) \pm f_2(x) \pm f_3(x)...]dx$$

$$= \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm ...$$

5. If $\int f(x)dx = F(x) + c$

then $\int f(ax \pm b)dx = \frac{1}{a}F(ax \pm b) + c$

1.2 Standard Formulae of Integration

The following results are a direct consequence of the definition of an integral.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$

2. $\int \frac{1}{x}dx = \log|x| + C$

3. $\int e^x\, dx = e^x + C$

4. $\int a^x\, dx = \frac{a^x}{\log_e a} + C.$

5. $\int \sin x\, dx = -\cos x + C$

6. $\int \cos x\, dx = \sin x + C$

7. $\int \sec^2 x\, dx = \tan x + C$

8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C.$

11. $\int \tan x \, dx = -\log|\cos x| + C = \log|\sec x| + C.$

12. $\int \cot x \, dx = \log|\sin x| + C$

13. $\int \sec x \, dx = \log|\sec x + \tan x| + C$

14. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \; ; |x| < 1$

16. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

17. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C \; ; |x| > 1$

2. METHODS OF INTEGRATION

2.1 Method of Substitution

By suitable substitution, the variable x in $\int f(x)\,dx$ is changed into another variable t so that the integrand $f(x)$ is changed into F(t) which is some standard integral or algebraic sum of standard integrals.

There is no general rule for finding a proper substitution and the best guide in this matter is experience.

However, the following suggestions will prove useful.

(i) If the integrand is of the form $f'(ax + b)$, then we put $ax + b = t$ and $dx = \frac{1}{a} dt$.

Thus, $\int f'(ax+b)\,dx = \int f'(t)\frac{dt}{a}$

$$= \frac{1}{a}\int f'(t)\,dt = \frac{f(t)}{a} = \frac{f(ax+b)}{a} + c$$

(ii) When the integrand is of the form $x^{n-1} f'(x^n)$, we put $x^n = t$ and $nx^{n-1}\,dx = dt$.

Thus, $\int x^{n-1} f'(x^n)\,dx = \int f'(t)\frac{dt}{n} = \frac{1}{n}$

$$\int f'(t)\,dt = \frac{1}{n} f(t) = \frac{1}{n} f(x^n) + c$$

(iii) When the integrand is of the form $[f(x)]^n \cdot f'(x)$, we put $f(x) = t$ and $f'(x)\,dx = dt$.

Thus, $\int [f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1} = \frac{[f(x)]^{n+1}}{n+1} + c$

(iv) When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put $f(x) = t$ and $f'(x)\,dx = dt$.

Thus, $\int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log|t| = \log|f(x)| + c$

2.1.1 Some Special Integrals

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

4. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

5. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

6. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

7. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

8. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

9. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

2.1.2 Integrals of the Form

(a) $\int f\left(a^2-x^2\right)dx,$

(b) $\int f\left(a^2+x^2\right)dx,$

(c) $\int f\left(x^2-a^2\right)dx,$

(d) $\int f\left(\frac{a-x}{a+x}\right)dx,$

Working Rule

Integral	**Substitution**
$\int f\left(a^2-x^2\right)dx,$	$x = a\sin\theta$ or $x = a\cos\theta$
$\int f\left(a^2+x^2\right)dx,$	$x = a\tan\theta$ or $x = a\cot\theta$
$\int f\left(x^2-a^2\right)dx,$	$x = a\sec\theta$ or $x = a\,\mathrm{cosec}\,\theta$
$\int f\left(\frac{a-x}{a+x}\right)dx$ or $\int f\left(\frac{a+x}{a-x}\right)dx$	$x = a\cos 2\theta$

2.1.3 Integrals of the Form

(a) $\int \frac{dx}{ax^2+bx+c}$, (b) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$,

(c) $\int \sqrt{ax^2+bx+c}\,dx$ a,

Working Role

(i) Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.

(ii) Complete the square in the terms involving x, i.e. write $ax^2 + bx + c$ in the form

$$a\left[\left(x+\frac{b}{2a}\right)^2\right] - \frac{\left(b^2-4ac\right)}{4a}.$$

(iii) The integrand is converted to one of the nine special integrals.

(iv) Integrate the function.

2.1.4 Integrals of the Form

(a) $\int \frac{px+q}{ax^2+bx+c}dx,$ (b) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,$

(c) $\int (px+q)\sqrt{ax^2+bx+c}\;dx$

Integral Working Rule

$\int \frac{px+q}{ax^2+bx+c}dx$ Put $px + q = \lambda\,(2ax + b) + \mu$ or $px + q = \lambda$ (derivative of quadratic) $+ \mu$.

Comparing the coefficient of x and constant term on both sides, we get

$p = 2a\lambda$ and $q = b\lambda + \mu \Rightarrow \lambda = \frac{}{2a}$ and $\mu = \left(q - \frac{bp}{2a}\right)$. Then integral becomes

$$\int \frac{px+q}{ax^2+bx+c}dx$$

$$= \frac{}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

$$= \frac{}{2a}\log|ax^2+bx+c| + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ In this case the integral becomes

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx =$$

$$\frac{p}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{p}{a}\sqrt{ax^2+bx+c} + \left(q-\frac{bp}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int (px+q)\sqrt{ax^2+bx+c}\;dx$$

The integral in this case is converted to

$$\int (px+q)\sqrt{ax^2+bx+c}\;dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\;dx$$

$$+\left(q-\frac{bp}{2a}\right)\int \sqrt{ax^2+bx+c}\;dx$$

$$= \frac{p}{3a}(ax^2+bx+c)^{3/2} + \left(q-\frac{bp}{2a}\right)\int \sqrt{ax^2+bx+c}\;dx$$

2.1.5 Integrals of the Form

$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx$, where $P(x)$ is a polynomial in x of degree $n \geq 2$.

Working Rule: Write

$$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx =$$

$$= (a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1})\;\sqrt{ax^2+bx+c} + k\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

where k, a_0, a_1, ... a_{n-1} are constants to be determined by differentiating the above relation and equating the coefficients of various powers of x on both sides.

2.1.6 Integrals of the Form

$$\int \frac{x^2+1}{x^4+kx^2+1}dx \quad \text{or} \quad \int \frac{x^2-1}{x^4+kx^2+1}dx,$$

where k is a constant positive, negative or zero.

Working Rule

(i) Divide the numerator and denominator by x^2.

(ii) Put $x-\frac{1}{x}=z$ or $x+\frac{1}{x}=z$, whichever subsitution, on differentiation gives, the numerator of the resulting integrand.

(iii) Evaluate the resulting integral in z

(iv) Express the result in terms of x.

2.1.7 Integrals of the Form

$\int \frac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral	**Substitution**
$\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$	$cx + d = z^2$
$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	$px + q = z^2$
$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px + q = \frac{1}{z}$
$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$	$x = \frac{1}{z}$

3. METHOD OF PARTIAL FRACTIONS FOR RATIONAL FUNCTIONS

Integrals of the type $\int \frac{p(x)}{g(x)}$ can be integrated by resolving the integrand into partial fractions. We proceed as follows:

Check degree of p (x) and g (x).

If degree of p (x) $\geq$ degree of g (x), then divide p (x) by g (x) till its degree is less, i.e. put in the form $\frac{p(x)}{g(x)} = r(x) + \frac{f(x)}{g(x)}$ where degree of f (x) < degree of g (x).

CASE 1: When the denominator contains non-repeated linear factors. That is

$$g(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

In such a case write f (x) and g (x) as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x-\alpha_1)} + \frac{A_2}{(x-\alpha_2)} + \ldots + \frac{A_n}{(x-\alpha_n)}$$

where A_1, A_2, ... A_n are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 2 : When the denominator contains repeated as well as non-repeated linear factor. That is

$$g(x) = (x - \alpha_1)^2 (x - \alpha_3) \ldots (x - \alpha_n).$$

In such a case write f (x) and g (x) as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \frac{A_3}{x-\alpha_3} + \ldots + \frac{A_n}{(x-\alpha_n)}$$

where A_1, A_2, ... A_n are constants to determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

Note : Corresponding to repeated linear factor $(x - a)^r$ in the denominator, a sum of r partial fractions of the type $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_r}{(x-a)^r}$ is taken.

CASE 3 : When the denominator contains a non repeated quadratic factor which cannot be factorised further:

$$g(x) = (ax^2 + bx + c)(x - \alpha_3)(x - \alpha_4) \ldots (x - \alpha_n).$$

In such a case express f (x) and g (x) as:

$$\frac{f(x)}{g(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3}{x-\alpha_3} + \ldots + \frac{A_n}{x-\alpha_n}$$

where A_1, A_2, ... A_n are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 4 : When the denominator contains a repeated quadratic factor which cannot be factorised further: That is

$$g(x) = (ax^2 + bx + c)^2 (x - \alpha_5)(x - \alpha_6) \ldots (x - \alpha_n)$$

In such a case write f (x) and g (x) as

$$\frac{f(x)}{g(x)} = \frac{A_1x + A_2}{ax^2 + bx + c} + \frac{A_3x + A_4}{(ax^2 + bx + c)^2} + \frac{A_5}{x-\alpha_5} + \ldots + \frac{A_n}{(x-\alpha_n)}$$

where A_1, A_2, ... A_n are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 5 : If the integrand contains only even powers of x

(i) Put $x^2 = z$ in the integrand.

(ii) Resolve the resulting rational expression in z into partial fractions

(iii) Put $z = x^2$ again in the partial fractions and then integrate both sides.

4. METHOD OF INTEGRATION BY PARTS

The process of integration of the product of two functions is known as integration by parts.

For example, if u and v are two functions of x, then $\int (uv)\,dx = u.\int v\,dx - \int \left(\frac{du}{dx}.\int v\,dx\right) dx.$

In words, integral of the product of two functions = first function × integral of the second – integral of (differential of first × integral of the second function).

Working Hints

(i) Choose the first and second function in such a way that derivative of the first function and the integral of the second function can be easily found.

(ii) In case of integrals of the form $\int f(x).x^n$ dx, take x^n as the first function and f (x) as the second function.

(iii) In case of integrals of the form $\int (\log x)^n \cdot 1 dx$, take 1 as the second function and $(\log x)^n$ as the first function.

(iv) Rule of integration by parts may be used repeatedly, if required.

(v) If the two functions are of different type, we can choose the first function as the one whose initial comes first in the word "ILATE", where

I — Inverse Trigonometric function

L — Logarithmic function

A — Algebraic function

T — Trigonometric function

E — Exponential function.

(vi) In case, both the functions are trigonometric, take that function as second function whose integral is simple. If both the functions are algebraic, take that function as first function whose derivative is simpler.

(vii) If the integral consists of an inverse trigonometric function of an algebraic expression in x, first simplify the integrand by a suitable trigonometric substitution and then integrate the new integrand.

4.1 Integrals of the Form

$$\int e^x \left[f(x) + f'(x) \right] dx$$

Working Rule

(i) Split the integral into two integrals.

(ii) Integrate only the first integral by parts, i.e.

$$\int e^x \left[f(x) + f'(x) \right] \ x = \int e^x f(x) \ x + \int e^x f'(x)\, dx$$

$$= \left[f(x).e^x - \int f'(x).e^x \ x \right] + \int e^x f'(x)\, dx$$

$$= e^x f(x) + C.$$

4.2 Integrals of the Form:

Where the initial integrand reappears after integrating by parts.

Working Rule

(i) Apply the method of integration by parts twice.

(ii) On integrating by parts second time, we will obtain the given integrand again, put it equal to I.

(iii) Transpose and collect terms involving I on one side and evaluate I.

5. INTEGRAL OF THE FORM (TRIGONOMETRIC FORMATS)

5.1 (a) $\int \frac{dx}{a + b\cos x}$ (b) $\int \frac{dx}{a + b\sin x}$

(c) $\int \frac{dx}{a + b\cos x + c\sin x}$

Working Rule

(i) Put $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\sin x = \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ so that the given integrand becomes a function of $\tan \frac{x}{2}$.

(ii) Put $\tan \frac{x}{2} = z \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2} dx = dz$

(iii) Integrate the resulting rational algebraic function of z

(iv) In the answer, put $z = \tan \frac{x}{2}$.

5.2 Integrals of the Form

(a) $\int \frac{dx}{a + b\cos^2 x}$ (b) $\int \frac{dx}{a + b\sin^2 x}$

(c) $\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$

Working Rule

(i) Divide the numerator and denominator by $\cos^2 x$.

(ii) In the denominator, replace $\sec^2 x$, if any, by $1 + \tan^2 x$.

(iii) Put $\tan x = z \Rightarrow \sec^2 x\, dx = dz$.

(iv) Integrate the resulting rational algebraic function of z.

(v) In the answer, put $z = \tan x$.

5.3 Integrals of the Form

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx$$

Working Role

(i) Put Numerator = λ (denominator) + μ (derivative of denominator)

$a\cos x + b\sin x = \lambda(c\cos x + d\sin x) + \mu(-c\sin x + d\cos x)$.

(ii) Equate coefficients of sin x and cos x on both sides and find the values of λ and μ.

(iii) Split the given integral into two integrals and evaluate each integral separately, i.e.

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx =$$

$$\lambda \int 1dx + \mu \int \frac{-c\sin x + d\cos x}{a\cos x + b\sin x}\, dx = \lambda x + \mu \log|a\cos x + b\sin x|.$$

(iv) Substitute the values of λ and μ found in step 2.

5.4 Integrals of the Form

$$\int \frac{a + b\cos x + c\sin x}{e + f\cos x + g\sin x}\, dx$$

Working Rule

(i) Put Numerator = l (denominator) + m

(derivative of denominator) + n

$a + b\cos x + c\sin x = l(e + f\cos x + g\sin x) + m(-f\sin x + g\cos x) + n$

(ii) Equate coefficients of sin x, cos x and constant term on both sides and find the values of l, m, n.

(iii) Split the given integral into three integrals and evaluate each integral separately, i.e.

$$\int \frac{a + b\cos x + c\sin x}{e + f\cos x + g\sin x}\, dx$$

$$l \int 1dx + m \int \frac{-f\sin x + g\cos x}{e + f\cos x + g\sin x}\, dx +$$

$$n \int \frac{dx}{e + f\cos x + g\sin x}$$

$$= lx + m\log|e + f\cos x + g\sin x| + n\int \frac{dx}{e + f\cos x + g\sin x}\, dx$$

(iv) Substitute the values of l, m, n found in Step (ii).

5.5 Integrals of the Form

$$\int \sin^m x \cos^n x\, dx$$

Working Rule

(i) If the power of sin x is an odd positive integer, put cos x = t.

(ii) If the power of cos x is an odd positive integer, put sin x = t.

(iii) If the power of sin x and cos x are both odd positive integers, put sin x = t or cos x = t.

(iv) If the power of sin x and cos x are both even positive integers, use De' Moivre's theorem as follows:

Let, $\cos x + i\sin x = z$. Then $\cos x - i\sin x = z^{-1}$

Adding these, we get $z + \frac{1}{z} = 2\cos x$ and $z - \frac{1}{z} = 2i\sin x$

By De'Moivre's theorem, we have

$$z^n + \frac{1}{z^n} = 2\cos nx \text{ and } z^n - \frac{1}{z^n} = 2i\sin^n x \quad ...(1)$$

$$\therefore \quad \sin^m x\cos^n x = \frac{1}{(2i)^m}\cdot\frac{1}{2^n}\left(z + \frac{1}{z}\right)^n\left(z - \frac{1}{z}\right)^m$$

$$= \frac{1}{2^{m+n}}\cdot\frac{1}{i^m}\left(z + \frac{1}{z}\right)^n\left(z - \frac{1}{z}\right)^m.$$

Now expand each of the factors on the R.H.S. using Binomial theorm. Then group the terms equidistant from the beginning and the end. Thus express all such pairs as the sines or cosines of multiple angles. Further integrate term by term.

(v) If the sum of powers of sin x and cos x is an even negative integer, put tan x = z.

SOLVED EXAMPLES

Example – 1

Evaluate : $\int\left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right)dx$

Sol. $\int\left(x^3+5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}\right)dx$

$=\int x^3\,dx+\int 5x^2\,dx-\int 4dx+\int\frac{7}{x}dx+\int\frac{2}{\sqrt{x}}dx$

$=\int x^3\,dx+5\,.\int x^2\,dx - 4\,.\int 1\,.dx+\ 7\,.\int\frac{1}{x}dx+2\,.\int x^{-1/2}dx$

$=\frac{x^4}{4}+5.\frac{x^3}{3}-4x+7\log|x|+2\left(\frac{x^{1/2}}{1/2}\right)+C$

$=\frac{x^4}{4}+\frac{5}{3}x^3\text{-}4x+7\log|x|+4\sqrt{x}+C$

Example – 2

Evaluate : $\int e^{x\log a}+e^{a\log x}+e^{a\log a}\,dx$

Sol. We have,

$\int e^{x\log a}+e^{a\log x}+e^{a\log a}\,dx$

$=\int e^{\log a^x}+e^{\log x^a}+e^{\log a^a}\,dx$

$=\int(a^x+x^a+a^a)\,dx$

$=\int a^x\,dx+\int x^a\,dx+\int a^a\,dx$

$=\frac{a^x}{\log a}+\frac{x^{a+1}}{a+1}+a^a\,.\,x+C.$

Example – 3

Evaluate : $\int\frac{x^4}{x^2+1}\,dx$

Sol. $\int\frac{x^4}{x^2+1}\,dx$

$=\int\frac{x^4-1+1}{x^2+1}\,dx=\int\frac{x^4-1}{x^2+1}+\frac{1}{x^2+1}\,dx$

$=\int(x^2-1)\,dx+\int\frac{1}{x^2+1}\,dx=\frac{x^3}{3}-x+\tan^{-1}x+C$

Example – 4

Evaluate : $\int\frac{2^x+3^x}{5^x}\,dx$

Sol. $\int\frac{2^x+3^x}{5^x}\,dx$

$=\int\left(\frac{2^x}{5^x}+\frac{3^x}{5^x}\right)dx$

$=\int\left[\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x\right]dx=\frac{(2/5)^x}{\log_e 2/5}+\frac{(3/5)^x}{\log_e 3/5}+C$

Example – 5

Evaluate : $\int x^3\sin\ \ ^4\,dx$

Sol. We have

$I=\int x^3\sin\ \ ^4\,dx$

Let $x^4=t\ \Rightarrow\ d(x^4)=dt$

$\Rightarrow\ 4x^3\,dx=dt\Rightarrow\ dx=\frac{1}{4x^3}\,dt$

Implicit Differentiation

mc-TY-implicit-2009-1

Sometimes functions are given not in the form $y = f(x)$ but in a more complicated form in which it is difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions. In this unit we explain how these can be differentiated using implicit differentiation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate functions defined implicitly

Contents

1. Introduction

In this unit we look at how we might differentiate functions of y with respect to x.

Consider an expression such as

$$x^2 + y^2 - 4x + 5y - 8 = 0$$

It would be quite difficult to re-arrange this so y was given explicitly as a function of x. We could perhaps, given values of x, use the expression to work out the values of y and thereby draw a graph. In general even if this is possible, it will be difficult.

A function given in this way is said to be defined **implicitly**. In this unit we study how to differentiate a function given in this form.

It will be necessary to use a rule known as the *the chain rule* or the rule for differentiating a *function of a function*. In this unit we will refer to it as the chain rule. There is a separate unit which covers this particular rule thoroughly, although we will revise it briefly here.

2. Revision of the chain rule

We revise the chain rule by means of an example.

Example

Suppose we wish to differentiate $y = (5 + 2x)^{10}$ in order to calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$.

We make a substitution and let $u = 5 + 2x$ so that $y = u^{10}$.
The chain rule states

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Now

$$\text{if } y = u^{10} \quad \text{then} \quad \frac{dy}{du} = 10u^9$$

and

$$\text{if } u = 5 + 2x \quad \text{then} \quad \frac{du}{dx} = 2$$

hence

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 10u^9 \times 2 \\ &= 20u^9 \\ &= 20(5 + 2x)^9 \end{aligned}$$

So we have used the chain rule in order to differentiate the function $y = (5 + 2x)^{10}$.

In quoting the chain rule in the form $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ note that we write y in terms of u, and u in terms of x. i.e.

$$y = y(u) \qquad \text{and} \qquad u = u(x)$$

We will need to work with different variables. Suppose we have z in terms of y, and y in terms of x, i.e.

$$z = z(y) \qquad \text{and} \qquad y = y(x)$$

The chain rule would then state:

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

Example

Suppose $z = y^2$. It follows that $\frac{dz}{dy} = 2y$. Then using the chain rule

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \times \frac{dy}{dx} \\ &= 2y \times \frac{dy}{dx} \\ &= 2y\frac{dy}{dx} \end{aligned}$$

Notice what we have just done. In order to differentiate y^2 with respect to x we have differentiated y^2 with respect to y, and then multiplied by $\frac{dy}{dx}$, i.e.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx}$$

We can generalise this as follows:

to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$.

Key Point

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right) \times \frac{dy}{dx}$$

We are now ready to do some implicit differentiation. Remember, every time we want to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$.

3. Implicit differentiation

Example

Suppose we want to differentiate the implicit function

$$y^2 + x^3 - y^3 + 6 = 3y$$

with respect x.

We differentiate each term with respect to x:

$$\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(y^3\right) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

Differentiating functions of x with respect to x is straightforward. But when differentiating a function of y with respect to x we must remember the rule given in the previous keypoint. We find

$$\frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx} \; + \; 3x^2 - \frac{d}{dy}\left(y^3\right) \times \frac{dy}{dx} \; + \; 0 = \frac{d}{dy}(3y) \times \frac{dy}{dx}$$

that is

$$2y\frac{dy}{dx} \; + \; 3x^2 - 3y^2\frac{dy}{dx} = 3\frac{dy}{dx}$$

We rearrange this to collect all terms involving $\frac{dy}{dx}$ together.

$$3x^2 = 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

then

$$3x^2 = \left(3 - 2y + 3y^2\right)\frac{dy}{dx}$$

so that, finally,

$$\frac{dy}{dx} = \frac{3x^2}{3 - 2y + 3y^2}$$

This is our expression for $\dfrac{dy}{dx}$.

Example

Suppose we want to differentiate, with respect to x, the implicit function

$$\sin y + x^2y^3 - \cos x = 2y$$

As before, we differentiate each term with respect to x.

$$\frac{d}{dx}(\sin y) + \frac{d}{dx}\left(x^2y^3\right) - \frac{d}{dx}(\cos x) = \frac{d}{dx}(2y)$$

Recognise that the second term is a product and we will need the product rule. We will also use the chain rule to differentiate the functions of y. We find

$$\frac{d}{dy}(\sin y) \times \frac{dy}{dx} + \left\{x^2\frac{d}{dx}\left(y^3\right) + y^3\frac{d}{dx}\left(x^2\right)\right\} + \sin x = \frac{d}{dy}(2y) \times \frac{dy}{dx}$$

so that

$$\cos y\frac{dy}{dx} + \left\{x^2.\frac{d}{dy}\left(y^3\right)\frac{dy}{dx} + y^3.\,2x\right\} + \sin x = 2\frac{dy}{dx}$$

Tidying this up gives

$$\cos y\frac{dy}{dx} + x^2\,3y^2\frac{dy}{dx} + 2xy^3 + \sin x = 2\frac{dy}{dx}$$

We now start to collect together terms involving $\frac{dy}{dx}$.

$$2xy^3 + \sin x = 2\frac{dy}{dx} - \cos y\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx}$$

$$2xy^3 + \sin x = (2 - \cos y - 3x^2y^2)\frac{dy}{dx}$$

so that, finally

$$\frac{dy}{dx} = \frac{2xy^3 + \sin x}{2 - \cos y - 3x^2y^2}$$

We have deliberately included plenty of detail in this calculation. With practice you will be able to omit many of the intermediate stages. The following two examples show how you should aim to condense the solution.

Example

Suppose we want to differentiate $y^2 + x^3 - xy + \cos y = 0$ to find $\frac{dy}{dx}$. The condensed solution may take the form:

$$\begin{aligned}
2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} &= 0 \\
(2y - \sin y)\frac{dy}{dx} + 3x^2 - \left\{x\frac{dy}{dx} + y.1\right\} &= 0 \\
(2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y &= 0 \\
(2y - \sin y - x)\frac{dy}{dx} &= y - 3x^2
\end{aligned}$$

so that

$$\frac{dy}{dx} = \frac{y - 3x^2}{2y - \sin y - x}$$

Example

Suppose we want to differentiate

$$y^3 - x\sin y + \frac{y^2}{x} = 8$$

The solution is as follows:

$$3y^2\frac{dy}{dx} - \left\{x\cos y\frac{dy}{dx} + \sin y.1\right\} + \frac{x\;2y\frac{dy}{dx} - y^2.1}{x^2} \quad = \quad 0$$

Multiplying through by x^2 gives:

$$\begin{aligned} 3x^2y^2\frac{dy}{dx} - x^3\cos y\frac{dy}{dx} - x^2\sin y + 2xy\frac{dy}{dx} - y^2 &= 0 \\ \frac{dy}{dx}\left(3x^2y^2 - x^3\cos y + 2xy\right) &= x^2\sin y + y^2 \end{aligned}$$

so that

$$\frac{dy}{dx} = \frac{x^2\sin y + y^2}{3x^2y^2 - x^3\cos y + 2xy}$$

Exercises

1. Find the derivative, with respect to x, of each of the following functions (in each case y depends on x).

a) y b) y^2 c) $\sin y$ d) e^{2y} e) $x+y$
f) xy g) $y\sin x$ h) $y\sin y$ i) $\cos(y^2+1)$ j) $\cos(y^2+x)$

2. Differentiate each of the following with respect to x and find $\frac{dy}{dx}$.
a) $\sin y + x^2 + 4y = \cos x$.
b) $3xy^2 + \cos y^2 = 2x^3 + 5$.
c) $5x^2 - x^3\sin y + 5xy = 10$.
d) $x - \cos x^2 + \frac{y^2}{x} + 3x^5 = 4x^3$.
e) $\tan 5y - y\sin x + 3xy^2 = 9$.

Answers to Exercises on Implicit Differentiation

1.
a) $\frac{dy}{dx}$ b) $2y\frac{dy}{dx}$ c) $\cos y\frac{dy}{dx}$
d) $2\mathrm{e}^{2y}\frac{dy}{dx}$ e) $1+\frac{dy}{dx}$ f) $x\frac{dy}{dx}+y$
g) $y\cos x + \sin x\frac{dy}{dx}$ h) $(\sin y + y\cos y)\frac{dy}{dx}$ i) $-2y\sin(y^2+1)\frac{dy}{dx}$
j) $-\left(2y\frac{dy}{dx}+1\right)\sin(y^2+x)$

2.
a) $\frac{dy}{dx} = \frac{-\sin x - 2x}{4+\cos y}$ b) $\frac{dy}{dx} = \frac{6x^2-3y^2}{6xy-2y\sin y^2}$
c) $\frac{dy}{dx} = \frac{10x - 3x^2\sin y + 5y}{x^3\cos y - 5x}$ d) $\frac{dy}{dx} = \frac{12x^4 - 15x^6 + y^2 - 2x^3\sin x^2 - x^2}{2xy}$
e) $\frac{dy}{dx} = \frac{y\cos x - 3y^2}{5\sec^2 5y - \sin x + 6xy}$

Integration by parts

mc-TY-parts-2009-1

A special rule, **integration by parts**, is available for integrating products of two functions. This unit derives and illustrates this rule with a number of examples.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state the formula for integration by parts
- integrate products of functions using integration by parts

Contents

1. Introduction

Functions often arise as products of other functions, and we may be required to integrate these products. For example, we may be asked to determine

$$\int x\cos x\,\mathrm{d}x\,.$$

Here, the integrand is the product of the functions x and $\cos x$. A rule exists for integrating products of functions and in the following section we will derive it.

2. Derivation of the formula for integration by parts

We already know how to differentiate a product: if

$$y = u\,v$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}\,.$$

Rearranging this rule:

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\,.$$

Now integrate both sides:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}(uv)}{\mathrm{d}x}\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\,.$$

The first term on the right simplifies since we are simply integrating what has been differentiated.

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = u\,v - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\,.$$

This is the formula known as **integration by parts**.

Key Point

Integration by parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = u\,v - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

The formula replaces one integral (that on the left) with another (that on the right); the intention is that the one on the right is a simpler integral to evaluate, as we shall see in the following examples.

3. Using the formula for integration by parts

Example

Find $\displaystyle\int x\,\cos x\,\mathrm{d}x$.

Solution

Here, we are trying to integrate the product of the functions x and $\cos x$. To use the integration by parts formula we let one of the terms be $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ and the other be u. Notice from the formula that whichever term we let equal u we need to differentiate it in order to find $\dfrac{\mathrm{d}u}{\mathrm{d}x}$. So in this case, if we let u equal x, when we differentiate it we will find $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$, simply a constant. Notice that the formula replaces one integral, the one on the left, by another, the one on the right. Careful choice of u will produce an integral which is less complicated than the original.
Choose

$$u = x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x\,.$$

With this choice, by differentiating we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1\,.$$

Also from $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, by integrating we find

$$v = \int \cos x\,\mathrm{d}x = \sin x\,.$$

(At this stage do not concern yourself with the constant of integration). Then use the formula

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = u\,v - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\ :$$

$$\begin{aligned}\int x\,\cos x\,\mathrm{d}x &= x\,\sin x - \int (\sin x)\cdot 1\,\mathrm{d}x \\ &= x\sin x + \cos x + c\end{aligned}$$

where c is the constant of integration.

In the next Example we will see that it is sometimes necessary to apply the formula for integration by parts more than once.

Example

Find $\displaystyle\int x^2 \mathrm{e}^{3x}\,\mathrm{d}x$.

Solution

We have to make a choice and let one of the functions in the product equal u and one equal $\frac{\mathrm{d}v}{\mathrm{d}x}$. As a general rule we let u be the function which will become simpler when we differentiate it. In this case it makes sense to let

$$u = x^2 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x}\,.$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \text{and} \qquad v = \int \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x}\,.$$

Then, using the formula for integration by parts,

$$\begin{aligned} \int x^2\mathrm{e}^{3x}\,\mathrm{d}x &= \frac{1}{3}\mathrm{e}^{3x}\cdot x^2 - \int \frac{1}{3}\mathrm{e}^{3x}\cdot 2x\,\mathrm{d}x \\ &= \frac{1}{3}x^2\mathrm{e}^{3x} - \int \frac{2}{3}x\mathrm{e}^{3x}\,\mathrm{d}x\,. \end{aligned}$$

The resulting integral is still a product. It is a product of the functions $\frac{2}{3}x$ and e^{3x}. We can use the formula again. This time we choose

$$u = \frac{2}{3}x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x}\,.$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{3} \qquad \text{and} \qquad v = \int \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x}\,.$$

So

$$\begin{aligned} \int x^2\mathrm{e}^{3x}\,\mathrm{d}x &= \frac{1}{3}x^2\mathrm{e}^{3x} - \int \frac{2}{3}x\mathrm{e}^{3x}\,\mathrm{d}x \\ &= \frac{1}{3}x^2\mathrm{e}^{3x} - \left\{\frac{2}{3}x\cdot\frac{1}{3}\mathrm{e}^{3x} - \int \frac{1}{3}\mathrm{e}^{3x}\cdot\frac{2}{3}\,\mathrm{d}x\right\} \\ &= \frac{1}{3}x^2\mathrm{e}^{3x} - \frac{2}{9}x\mathrm{e}^{3x} + \frac{2}{27}\mathrm{e}^{3x} + c \end{aligned}$$

where c is the constant of integration. So we have done integration by parts twice to arrive at our final answer.

Remember that to apply the formula you have to be able to integrate the function you call $\frac{\mathrm{d}v}{\mathrm{d}x}$. This can cause problems — consider the next Example.

Example

Find $\int x\ln|x|\,\mathrm{d}x$.

Solution

Remember the formula:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = u\,v - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\,.$$

It would be natural to choose $u = x$ so that when we differentiate it we get $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$. However this choice would mean choosing $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln|x|$ and we would need to be able to integrate this. This integral is not a known standard form. So, in this Example we will choose

$$u = \ln|x| \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = x$$

from which

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \qquad \text{and} \qquad v = \int x\,\mathrm{d}x = \frac{x^2}{2}\,.$$

Then, applying the formula

$$\begin{aligned}\int x\ln|x|\,\mathrm{d}x &= \frac{x^2}{2}\ln|x| - \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln|x| - \int \frac{x}{2}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln|x| - \frac{x^2}{4} + c\end{aligned}$$

where c is the constant of integration.

Example

Find $\int \ln|x|\mathrm{d}x$.

Solution

We can use the formula for integration by parts to find this integral if we note that we can write $\ln|x|$ as $1\cdot\ln|x|$, a product. We choose

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \qquad \text{and} \qquad u = \ln|x|$$

so that

$$v = \int 1\,\mathrm{d}x = x \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}\,.$$

Then,

$$\begin{aligned}\int 1\cdot\ln|x|\mathrm{d}x &= x\ln|x| - \int x\cdot\frac{1}{x}\mathrm{d}x \\ &= x\ln|x| - \int 1\,\mathrm{d}x \\ &= x\ln|x| - x + c\end{aligned}$$

where c is a constant of integration.

Example

Find $\displaystyle\int \mathrm{e}^x \sin x \,\mathrm{d}x$.

Solution

Whichever terms we choose for u and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ it may not appear that integration by parts is going to produce a simpler integral. Nevertheless, let us make a choice:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \qquad \text{and} \qquad u = \mathrm{e}^x$$

so that

$$v = \int \sin x \,\mathrm{d}x = -\cos x \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x\,.$$

Then,

$$\begin{aligned}\int \mathrm{e}^x \sin x \,\mathrm{d}x &= \mathrm{e}^x \cdot -\cos x - \int -\cos x \cdot \mathrm{e}^x \mathrm{d}x \\ &= -\cos x \cdot \mathrm{e}^x + \int \mathrm{e}^x \cos x \mathrm{d}x\,.\end{aligned}$$

We now integrate by parts again choosing

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad u = \mathrm{e}^x$$

so that

$$v = \int \cos x \,\mathrm{d}x = \sin x \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x\,.$$

So

$$\begin{aligned}\int \mathrm{e}^x \sin x \,\mathrm{d}x &= -\cos x \cdot \mathrm{e}^x + \left\{\mathrm{e}^x \sin x - \int \sin x \cdot \mathrm{e}^x \mathrm{d}x\right\} \\ &= -\mathrm{e}^x \cos x + \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x \mathrm{d}x\,.\end{aligned}$$

Notice that the integral we have ended up with is exactly the same as the one we started with. Let us call this I. That is $I = \displaystyle\int \mathrm{e}^x \sin x \,\mathrm{d}x$.
So

$$I = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - I$$

from which

$$2I = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x$$

and

$$I = \frac{1}{2}\left(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x\right)\,.$$

So

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \frac{1}{2}\left(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x\right) + c$$

where c is the constant of integration.

Exercises

1. Evaluate the following integrals:

(a) $\displaystyle\int x\sin x\,dx$ (b) $\displaystyle\int x\cos 4x\,dx$ (c) $\displaystyle\int xe^{-x}dx$ (d) $\displaystyle\int x^2\cos x\,dx$

(e) $\displaystyle\int 2x^2e^x dx$ (f) $\displaystyle\int x^2\ln|x|\,dx$ (g) $\displaystyle\int \tan^{-1}x\,dx$ (h) $\displaystyle\int \sin^{-1}x\,dx$

(i) $\displaystyle\int e^x\cos x\,dx$ (j) $\displaystyle\int \sin^3 x\,dx$ (Hint: write $\sin^3 x$ as $\sin^2 x\,\sin x$.)

2. Calculate the value of each of the following:

(a) $\displaystyle\int_0^{\pi} x\cos\tfrac{1}{2}x\,dx$ (b) $\displaystyle\int_0^1 x^2e^x dx$ (c) $\displaystyle\int_1^2 x^3\ln|x|\,dx$ (d) $\displaystyle\int_0^{\pi/4} x^2\sin 2x\,dx$

(e) $\displaystyle\int_0^1 x\tan^{-1}x\,dx$

Answers

1.

(a) $-x\cos x+\sin x+C$ (b) $\frac{1}{4}x\sin 4x+\frac{1}{16}\cos 4x+C$

(c) $-xe^{-x}-e^{-x}+C$ (d) $x^2\sin x+2x\cos x-2\sin x+C$

(e) $2x^2e^x-4xe^x+4e^x+C$ (f) $\frac{1}{3}x^3\ln|x|-\frac{1}{9}x^3+C$

(g) $x\tan^{-1}x-\frac{1}{2}\ln|1+x^2|+C$ (h) $x\sin^{-1}x+\sqrt{1-x^2}+C$

(i) $\frac{1}{2}e^x(\cos x+\sin x)+C$ (j) $-\frac{1}{3}(\cos x\sin^2 x+2\cos x)+C$

2.

(a) $2\pi-4$ (b) $e-2$ (c) $4\ln 2-\dfrac{15}{16}$ (d) $\dfrac{\pi}{8}-\dfrac{1}{4}$ (e) $\dfrac{\pi}{4}-\dfrac{1}{2}$

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Parametric Differentiation

mc-TY-parametric-2009-1

Instead of a function $y(x)$ being defined explicitly in terms of the independent variable x, it is sometimes useful to define both x and y in terms of a third variable, t say, known as a parameter. In this unit we explain how such functions can be differentiated using a process known as parametric differentiation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate a function defined parametrically
- find the second derivative of such a function

Contents

1. Introduction

Some relationships between two quantities or variables are so complicated that we sometimes introduce a third quantity or variable in order to make things easier to handle. In mathematics this third quantity is called a **parameter**. Instead of one equation relating say, x and y, we have two equations, one relating x with the parameter, and one relating y with the parameter. In this unit we will give examples of curves which are defined in this way, and explain how their rates of change can be found using parametric differentiation.

2. The parametric definition of a curve

In the first example below we shall show how the x and y coordinates of points on a curve can be defined in terms of a third variable, t, the **parameter**.

Example

Consider the parametric equations

$$x = \cos t \qquad y = \sin t \qquad \text{for } 0 \le t \le 2\pi \tag{1}$$

Note how both x and y are given in terms of the third variable t.

To assist us in plotting a graph of this curve we have also plotted graphs of $\cos t$ and $\sin t$ in Figure 1. Clearly,

when $t = 0$, $x = \cos 0 = 1$; $y = \sin 0 = 0$

when $t = \frac{\pi}{2}$, $x = \cos\frac{\pi}{2}$=0; $y = \sin\frac{\pi}{2} = 1$.

In this way we can obtain the x and y coordinates of lots of points given by Equations (1). Some of these are given in Table 1.

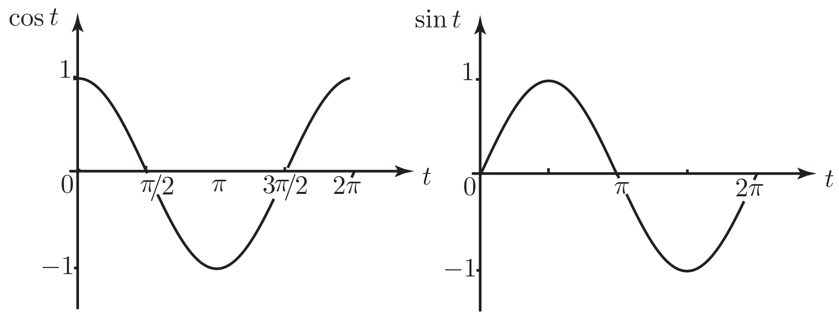

Figure 1. Graphs of $\sin t$ and $\cos t$.

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

Table 1. Values of x and y given by Equations (1).

Plotting the points given by the x and y coordinates in Table 1, and joining them with a smooth curve we can obtain the graph. In practice you may need to plot several more points before you can be confident of the shape of the curve. We have done this and the result is shown in Figure 2.

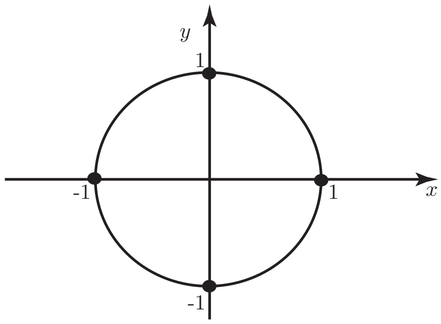

Figure 2. The parametric equations define a circle centered at the origin and having radius 1.

So $x = \cos t$, $y = \sin t$, for t lying between 0 and 2π, are the parametric equations which describe a circle, centre $(0, 0)$ and radius 1.

3. Differentiation of a function defined parametrically

It is often necessary to find the rate of change of a function defined parametrically; that is, we want to calculate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. The following example will show how this is achieved.

Example

Suppose we wish to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = \cos t$ and $y = \sin t$.

We differentiate both x and y with respect to the parameter, t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$$

From the chain rule we know that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

so that, by rearrangement

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \text{ is not equal to } 0$$

So, in this case

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\cos t}{-\sin t} = -\cot t$$

Key Point

parametric differentiation: if $x = x(t)$ and $y = y(t)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided } \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

Example

Suppose we wish to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = t^3 - t$ and $y = 4 - t^2$.

$$x = t^3 - t \qquad\qquad y = 4 - t^2$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 1 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -2t$$

From the chain rule we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-2t}{3t^2 - 1} \end{aligned}$$

So, we have found the gradient function, or derivative, of the curve using parametric differentiation.

For completeness, a graph of this curve is shown in Figure 3.

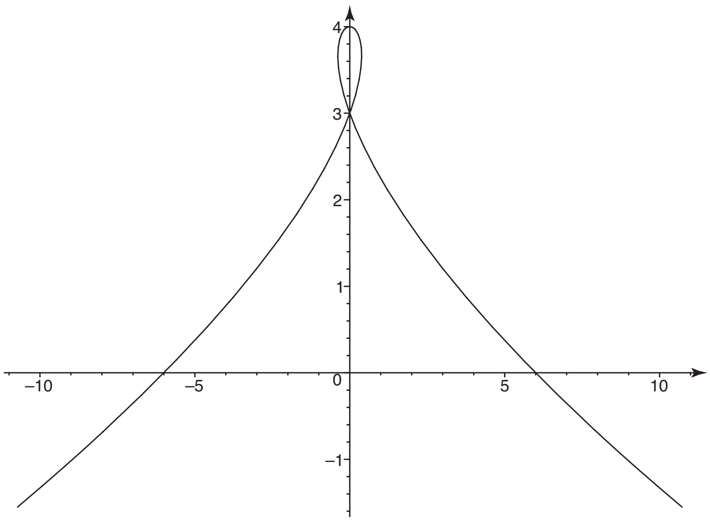

Figure 3

Example

Suppose we wish to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x = t^3$ and $y = t^2 - t$.

In this Example we shall plot a graph of the curve for values of t between -2 and 2 by first producing a table of values (Table 2).

t	-2	-1	0	1	2
x	-8	-1	0	1	8
y	6	2	0	0	2

Table 2

Part of the curve is shown in Figure 4. It looks as though there may be a turning point between 0 and 1. We can explore this further using parametric differentiation.

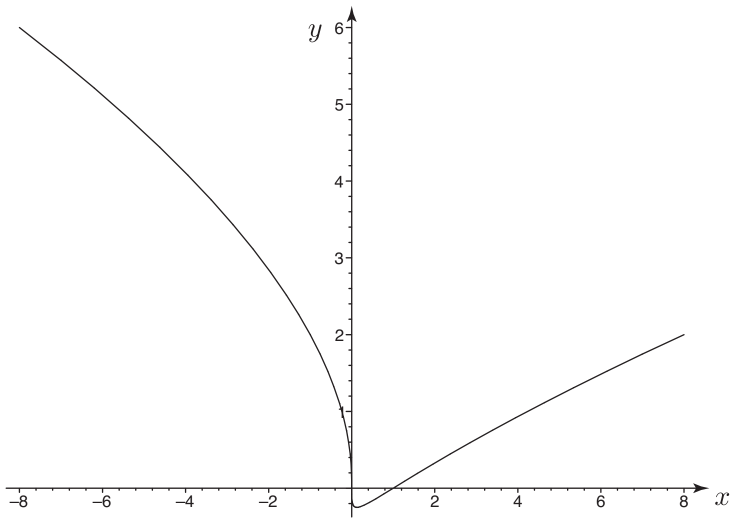

Figure 4.

From

$$x = t^3 \qquad\qquad y = t^2 - t$$

we differentiate with respect to t to produce

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t - 1$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided} \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-1}{3t^2}$$

From this we can see that when $t = \dfrac{1}{2}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and so $t = \dfrac{1}{2}$ is a stationary value. When $t = \dfrac{1}{2}$, $x = \dfrac{1}{8}$ and $y = -\dfrac{1}{4}$ and these are the coordinates of the stationary point.

We also note that when $t = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is infinite and so the y axis is tangent to the curve at the point $(0, 0)$.

Exercises 1

1. For each of the following functions determine $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

   (a) $x = t^2 + 1, y = t^3 - 1$
   (b) $x = 3\cos t, y = 3\sin t$
   (c) $x = t + \sqrt{t}, y = t - \sqrt{t}$
   (d) $x = 2t^3 + 1, y = t^2 \cos t$
   (e) $x = t\mathrm{e}^{-t}, y = 2t^2 + 1$

2. Determine the co-ordinates of the stationary points of each of the following functions

   (a) $x = 2t^3 + 1, y = t\mathrm{e}^{-2t}$
   (b) $x = \sqrt{t} + 1, y = t^3 - 12t$ for $t > 0$
   (c) $x = 5t^4, y = 5t^6 - t^5$ for $t > 0$
   (d) $x = t + t^2, y = \sin t$ for $0 < t < \pi$
   (e) $x = t\mathrm{e}^{2t}, y = t^2\mathrm{e}^{-t}$ for $t > 0$

4. Second derivatives

Example

Suppose we wish to find the second derivative $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when

$$x = t^2 \qquad y = t^3$$

Differentiating we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \qquad \text{provided} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2}{2t} = \frac{3t}{2}$$

We can apply the chain rule a second time in order to find the second derivative, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\
&= \frac{\frac{3}{2}}{2t} \\
&= \frac{3}{4t}
\end{aligned}$$

Key Point

if $x = x(t)$ and $y = y(t)$ then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example

Suppose we wish to find $\dfrac{d^2y}{dx^2}$ when

$$x = t^3 + 3t^2 \qquad y = t^4 - 8t^2$$

Differentiating

$$\frac{dx}{dt} = 3t^2 + 6t \qquad \frac{dy}{dt} = 4t^3 - 16t$$

Then, using the chain rule,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{provided} \quad \frac{dx}{dt} \neq 0$$

so that

$$\frac{dy}{dx} = \frac{4t^3 - 16t}{3t^2 + 6t}$$

This can be simplified as follows

$$\begin{aligned}\frac{dy}{dx} &= \frac{4t(t^2-4)}{3t(t+2)} \\ &= \frac{4t(t+2)(t-2)}{3t(t+2)} \\ &= \frac{4(t-2)}{3}\end{aligned}$$

We can apply the chain rule a second time in order to find the second derivative, $\dfrac{d^2y}{dx^2}$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\ &= \frac{\frac{4}{3}}{3t^2+6t} \\ &= \frac{4}{9t(t+2)}\end{aligned}$$

Exercises 2

For each of the following functions determine $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t

1. $x = \sin t, y = \cos t$
2. $x = 3t^2 + 1, y = t^3 - 2t^2$
3. $x = \frac{1}{2}t^2 + 2, y = \sin(t+1)$
4. $x = \mathrm{e}^{-t}, y = t^3 + t + 1$
5. $x = 3t^2 + 4t, y = \sin 2t$

Answers
Exercise 1

1. a) $\frac{3t}{2}$ b) $-\cot t$ c) $\frac{2\sqrt{t}-1}{2\sqrt{t}+1}$ d) $\frac{2\cos t - t\sin t}{6t}$ e) $\frac{4t\mathrm{e}^t}{1-t}$

2. a) $\left(\frac{5}{4}, \frac{1}{2\mathrm{e}}\right)$ b) $(1+\sqrt{2}, -16)$ c) $\left(\frac{5}{1296}, \frac{-1}{46656}\right)$ d) $\left(\frac{\pi}{2} + \frac{\pi^2}{4}, 1\right)$ e) $(2\mathrm{e}^4, 4\mathrm{e}^{-2})$

Exercise 2

1. $-\sec^3 t$ 2. $\frac{1}{12t}$ 3. $\frac{-t\sin(t+1) - \cos(t+1)}{t^3}$

4. $(3t^2 + 6t + 1)\mathrm{e}^{2t}$ 5. $\frac{-2(3t+2)\sin 2t - 3\cos 3t}{2(3t+2)^3}$

