# Lecture Notes

## MTH166
## Differential Equations
## and Vector Calculus

# MTH166

## Lecture-1
## Basics of Ordinary Differential Equations

**Unit 1: Ordinary Differential Equations**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-4**)

**Topic:**

Separable form (ODE)

**Learning Outcomes:**

**1.** Definition of Ordinary Differential Equation.

**2.** Solution of ODE by Variable separation method

**3**. First order Homogeneous Differential Equations.

**Ordinary Differential Equation (ODE):**

A *differential equation* can be defined as an equation containing derivatives of various orders and the variables.

A differential equation which involves only one independent variable is called an *ordinary differential equation* (ODE).

Let $y$ be the dependent variable and $x$ be an independent variable.

We denote the derivatives as:

$$\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'' \text{ , } etc.$$

**Ordinary Differential Equation (ODE):**

**Order:** The order of a differential equation is the order of the highest order derivative occurring in the equation.

**Degree:** The degree of a differential equation is the degree (power) of the highest order derivative occurring in the equation, after the equation has been made free of radicals and functions in its derivatives.

**Linear:** A differential equation is linear, when the dependent variable and its derivatives occur only in first degree and no products of the dependent variable and its derivatives or of various order derivatives occur.

**Non-Linear:** A differential equation which is not linear, is called non-linear.

**Ordinary Differential Equation (ODE):**

Some examples of ordinary differential equations are:

**1.** $y' = 6x^2$

Order = 1, Degree = 1, Linear.

**2.** $y'' + 16y = 2x$

Order = 2, Degree = 1, Linear.

**3.** $x^2y'' - xy' + 6y = \log x$

Order = 2, Degree = 1, Linear.

**4.** $y'y'' + y^2 = x^2$

Order = 2, Degree = 1, Non-Linear.

**Polling Quiz**

The equation: $yy' + y - x^2 = 0$ is a:

(A) A linear ODE with order 1 and degree 1

(B) A non-linear ODE with order 1 and degree 1

(C) A linear ODE with order 1 and degree 2

(D) A non-linear ODE with order 2 and degree 2

**ODE Solvable by Variable Separable form**

Let the given ODE be:

$y' = f(x, y)$ where $f(x, y) = g(x)h(y)$

$\Rightarrow \frac{dy}{dx} = g(x)h(y)$

$\Rightarrow \frac{dy}{h(y)} = g(x)dx$ ( Separating the variables)

$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)dx$ (Integrating both sides)

$\Rightarrow A(y) = B(x) + C$ which is the required solution.

**Solve the following differential equations:**

**Problem 1:** $(x\, log\, x)y' = y$

**Solution:** $(x\, log\, x)\frac{dy}{dx} = y$ (1)

$\Rightarrow \frac{dy}{y} = \frac{dx}{(x\, log\, x)}$ (Separating the variables)

$\Rightarrow \int \frac{dy}{y} = \int \frac{\frac{1}{x}dx}{\log x}$ (Integrating both sides)

$\Rightarrow \log y = \log(\log x) + \log c$ $\left(\int \frac{f'(x)}{f(x)}dx = \log f(x)\right)$

$\Rightarrow \log y = \log(c \log x)$

$\Rightarrow y = c \log x$ **Answer.**

**Problem 2:** $y' = e^{x+y} + x^2 e^y$

**Solution:** $\frac{dy}{dx} = e^{x+y} + x^2 e^y = e^y(e^x + x^2)$ (1)

$\Rightarrow \frac{dy}{e^y} = (e^x + x^2)dx$ (Separating the variables)

$\Rightarrow \int e^{-y}dy = \int (e^x + x^2)dx$ (Integrating both sides)

$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c$

$\Rightarrow e^x + e^{-y} + \frac{x^3}{3} + c = 0$ **Answer.**

**Problem 3:** $axy' = by, a \neq 0$

**Solution:** $ax\frac{dy}{dx} = by$ (1)

$\Rightarrow \frac{dy}{y} = \frac{b}{a}\frac{dx}{x}$ (Separating the variables)

$\Rightarrow \int \frac{dy}{y} = \frac{b}{a}\int \frac{dx}{x}$ (Integrating both sides)

$\Rightarrow \log y = \frac{b}{a}\log x + logc$

$\Rightarrow \log y = \log\left(cx^{b/a}\right)$

$\Rightarrow y = \left(cx^{b/a}\right)$ **Answer.**

**First order Homogeneous ODE**

A first order ODE: $y' = f(x, y)$ is said to be a homogeneous equation, if $f(x, y)$ is a homogeneous function, that is, ODE is of the form:

**1.** $y' = \frac{y}{x}$ (Homogeneous function)

**2.** $y' = \frac{x^2 - y^2}{xy}$ (Homogeneous function)

**3.** $y' = \frac{x^3 + y^3 + x^2 y}{x^3 - y^3}$ (Homogeneous function)

*If a first order ODE is homogeneous, then the substitution $y = vx$ or $x = uy$ reduces the equation to a separable form.*

The equation: $y' = \frac{x^3 - y^3 - y^2x}{x^3 + y^3}$ is Homogeneous:

(A) Yes

(B) No

**Problem 1.** Solve the ODE: $y' = \frac{x^2+y^2+xy}{x^2}$

**Solution:** $\frac{dy}{dx} = \frac{x^2+y^2+x}{x^2}$ (1)

Since it is a homogeneous ODE, so put $y = vx$ $\left(v = \frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$

Equation (1) becomes:

$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+(vx)^2+x(vx)}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2(1+v^2+v)}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = (1 + v^2 + v)$

$\Rightarrow x\frac{dv}{dx} = 1 + v^2$

$\Rightarrow \frac{dv}{1+v^2} = \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$

$\Rightarrow tan^{-1}v = \log x + c$

$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) = \log x + c$ **Answer.**

**Problem 2.** Solve the ODE: $(y + x)y' = (y - x)$

**Solution:** $\frac{dy}{dx} = \frac{y-x}{y+x}$ (1)

Since it is a homogeneous ODE, so put $y = vx$ $\left(v = \frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$

Equation (1) becomes:

$\Rightarrow v + x\frac{dv}{dx} = \frac{vx-x}{vx+x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$

$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v^2-v}{v+1}$

$\Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2+1}{v+1}\right)$

$\Rightarrow \frac{(1+v)dv}{1+v^2} = -\frac{dx}{x}$

$\Rightarrow \int \frac{(1+v)dv}{1+v^2} = -\int \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{1+v^2} + \frac{1}{2}\int \frac{2vdv}{1+v^2} = -\int \frac{dx}{x}$

$\Rightarrow tan^{-1}v + \frac{1}{2}\log v = \log x + c$

$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\log\left(\frac{y}{x}\right) = \log x + c$ **Answer.**

**Problem 3.** Solve the ODE: $xy' = y + x\, sec\left(\frac{y}{x}\right)$

**Solution:** $\frac{dy}{dx} = \frac{y}{x} + sec\left(\frac{y}{x}\right)$ (1)

Since it is a homogeneous ODE, so put $y = vx$ $\left(v = \frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$

Equation (1) becomes:

$\Rightarrow v + x\frac{dv}{dx} = v + \sec v$

$\Rightarrow x\frac{dv}{dx} = \sec v$

$\Rightarrow \frac{dv}{\sec v} = \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{\sec v} = \int \frac{dx}{x}$

$\Rightarrow \int \cos v\, dv = \int \frac{dx}{x}$

$\Rightarrow \sin v = \log x + c$

$\Rightarrow \sin\left(\frac{y}{x}\right) = \log x + c$ **Answer.**

# MTH166

## Lecture-2

## Exact Differential Equations (EDE)

**Unit 1: Ordinary Differential Equations**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-4**)

**Topic:**

Exact Differential Equations (EDE)

**Learning Outcomes:**

**1.** What is the form of Exact Differential Equations.

**2.** What is the Necessary and Sufficient condition for an equation to be called as Exact differential equations.

**3.** How to solve an Exact differential equation.

**Exact Differential Equation (EDE):**

An equation of the form:

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

is called an exact differential equation if and only if:

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ( Necessary and Sufficient Condition for EDE)

The solution of equation (1) is given by:

$\int_{y=con.} M(x,y)dx + \int (Terms\ of\ N\ free\ from\ x)dy = C$(Constant)

**Exact Differential Equation (EDE):**

EDE Form: $Mdx + Ndy = 0$

Necessary and Sufficient Condition: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution of EDE:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

**Solve the following differential equations:**

**Problem 1:** $(x^2 - ay)dx = (ax - y^2)dy$

**Solution:** $(x^2 - ay)dx + (y^2 - ax)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (x^2 - ay) \Rightarrow \frac{\partial M}{\partial y} = 0 - a(1) = -a$

And $N = (y^2 - ax) \Rightarrow \frac{\partial N}{\partial x} = 0 - a(1) = -a$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$\Rightarrow \int_{y=con.} (x^2 - ay)dx + \int (y^2)dy = C$

$\Rightarrow \frac{x^3}{3} - ay(x) + \frac{y^3}{3} = c$

$\Rightarrow x^3 - 3axy + y^3 = 3c$

$\Rightarrow x^3 - 3axy + y^3 = c_1$ **Answer.** $(3c = c_1)$

**Polling Quiz**

The differential equation: $M(x,y)dx + N(x,y)dy = 0$ will be an exact differential equation if:

(A) $\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} = 0$

(B) $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

(C) $\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 0$

(D) None of these

**Problem 2:** $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

**Solution:** $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (x^2 - 4xy - 2y^2) \Rightarrow \frac{\partial M}{\partial y} = -4x - 4y$

And $N = (y^2 - 4xy - 2x^2) \Rightarrow \frac{\partial N}{\partial x} = -4y - 4x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.}(x^2 - 4xy - 2y^2)dx + \int (y^2)dy = C$$

$$\Rightarrow \frac{x^3}{3} - 4y\left(\frac{x^2}{2}\right) - 2y^2(x) + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = 6c$$

$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = c_1$ **Answer.** $(6c = c_1)$

**<u>Problem 3:</u>** $\boldsymbol{(y \sin 2x)dx - \left(1 + y^2 + cos^2x\right)dy = 0}$

**<u>Solution:</u>** $(y \sin 2x)dx - (1 + y^2 + cos^2x)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (y \sin 2x) \Rightarrow \frac{\partial M}{\partial y} = \sin 2x$

And $N = -(1 + y^2 + cos^2x) \Rightarrow \frac{\partial N}{\partial x} = -(2\cos x(-\sin x)) = \sin 2x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.}(y \sin 2x)dx - \int (1 + y^2)dy = C$$

$$\Rightarrow y\left(\frac{-\cos 2}{2}\right) - y - \frac{y^3}{3} = c$$

$$\Rightarrow -3ycos2x - 6y - 2y^3 = 6c$$

$\Rightarrow 3ycos2x + 6y + 2y^3 + c_1 = 0$ **Answer.** $(6c = c_1)$

**<u>Problem 4:</u>** $\boldsymbol{(sec\ x\ tan\ x\ tan\ y - e^x)dx + \left(sec\ x\ sec^2y\right)dy = 0}$

**<u>Solution:</u>**$(sec\ x\ tan\ x\ tan\ y - e^x)dx - (sec\ x\ sec^2y)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (sec\ x\ tan\ x\ tan\ y - e^x) \Rightarrow \frac{\partial M}{\partial y} = secx\ tanx\ (sec^2y)$

And $N = (sec\ x\ sec^2y) \Rightarrow \frac{\partial N}{\partial x} = (secx\ tanx)sec^2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.} (secx\ tanx\ tany\ - e^{x})dx + \int (0)dy = C$$

$\Rightarrow tany(secx) - e^{x} = c$ **Answer.**

**Polling Quiz**

What is the relationship between $a$ and $b$, so that the given differential equ.: $(x^2 + ay)dx + (y^2 + bx)dy = 0$ is exact:

(A) $a = 2b$

(B) $a = b$

(C) $a \neq b$

(D) $a \neq 2b$

# MTH166

## Lecture-3

## Equations Reducible to Exact Form-I

**Topic:**

Equations Reducible to Exact Form

**Learning Outcomes:**

**1.** What is a Non-Exact Differential Equations.

**2.** What is an Integrating Factor (I.F.).

**3.** How to find an I.F. using different methods (Two methods).

**4.** How to convert a Non-exact equation into an Exact equation using I.F. and find its solution.

**Non-Exact Differential Equation:**

Let us consider:

$$\boldsymbol{xdy - ydx + a(x^2 + y^2)dx = 0} \qquad (1)$$

Equation (1) can be re-written as:

$(ax^2 + ay^2 - y)dx + xdy = 0$

Comparing it with: $Mdx + Ndy = 0$

Here $M = (ax^2 + ay^2 - y) \Rightarrow \frac{\partial M}{\partial y} = 2ay - 1$

And $N = x \Rightarrow \frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-Exact differential equation.

**Integrating Factor (I.F.):**

An Integrating Factor (I.F.) is a factor which when multiplied with a non-exact differential equation, makes it an exact differential equation.

Means: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

**There can be more than one I.F. for one non-exact diff. equ.

**Methods to find Integrating Factor (I.F.):**

**1. Inspection Method:**

In this method we look for a particular expression: $(xdy - ydx)$ which is non-exact.
There are many I.F. available to make this expression an exact one, like;
$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}, \frac{1}{x^2-y^2}$.
We know: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

1. $(xdy - ydx) \times \frac{1}{x^2} = \frac{(xdy-ydx)}{x^2} = d\left(\frac{y}{x}\right)$ (Quotient rule)
2. $(xdy - ydx) \times \frac{1}{y^2} = \frac{-(ydx-xdy)}{y^2} = -d\left(\frac{x}{y}\right)$ (Quotient rule)
3. $(xdy - ydx) \times \frac{1}{xy} = \frac{(xdy-ydx)}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$
4. $(xdy - ydx) \times \frac{1}{x^2+y^2} = \frac{(xdy-ydx)}{x^2+y^2} = d\left[tan^{-1}\left(\frac{y}{x}\right)\right]$
5. $(xdy - ydx) \times \frac{1}{x^2-y^2} = \frac{(xdy-ydx)}{x^2-y^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]$

**Solve the following differential equations:**

**Problem 1:** $xdy - ydx + a(x^2 + y^2)dx = 0$

**Solution:** $(xdy - ydx) + a(x^2 + y^2)dx = 0$ (1)

Multiplying both sides by I.F. $= \frac{1}{x^2+y^2}$

$[(xdy - ydx) + a(x^2 + y^2)dx] \times \frac{1}{x^2+y^2} = 0$

$\Rightarrow \frac{(xdy-ydx)}{x^2+y^2} + adx = 0$

$\Rightarrow d\left[tan^{-1}\left(\frac{y}{x}\right)\right] + adx = 0$

Integrating both sides:

$\int d\left[tan^{-1}\left(\frac{y}{x}\right)\right] + a\int dx = c$

$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) + ax = c$ **Answer.**

**Solve the following differential equations:**

**Problem 2:** $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

**Solution:** $(ydx - xdy) + 3x^2y^2e^{x^3}dx = 0$ (1)

Multiplying both sides by I.F. $= \frac{1}{y^2}$

$\left[(ydx - xdy) + 3x^2y^2e^{x^3}dx\right] \times \frac{1}{y^2} = 0$

$\Rightarrow \frac{(ydx-xdy)}{y^2} + 3x^2e^{x^3}dx = 0$

$\Rightarrow d\left(\frac{x}{y}\right) + 3x^2e^{x^3}dx = 0$

Integrating both sides:

$\int d\left(\frac{x}{y}\right) + \int e^{x^3}.3x^2dx = c$ $\left[\because \int e^{f(x)}.f'(x)dx = e^{f(x)} + c\right]$

$\Rightarrow \left(\frac{y}{x}\right) + e^{x^3} = c$ **Answer.**

**Polling Quiz**

Which is the solution of $xdy + ydx = 0$:

(A) $x + y = c$

(B) $x - y = c$

(C) $xy = c$

(D) $x/y = c$

**Methods to find Integrating Factor (I.F.):**

**2.** If $Mdx + Ndy = 0$ is a homogeneous differential equation in x and y, then:

I.F. $= \frac{1}{Mx+Ny}$

**Solve the following differential equations:**

**Problem 1.** $(x^3 + y^3)dx - xy^2dy = 0$

**Solution:** $(x^3 + y^3)dx - xy^2dy = 0$ (1)

Comparing it with: $Mdx + Ndy = 0$

Here $M = (x^3 + y^3) \Rightarrow \frac{\partial M}{\partial y} = 3y^2$

And$N = -xy^2 \Rightarrow \frac{\partial N}{\partial x} = -y^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

We need to make it exact using I.F.

Look at equation: $(x^3+y^3)dx - xy^2dy = 0$ (1)
Which is homogeneous in x and y

★ An equation is said to be homogeneous if each term of M and N is of same degree.
Here $M = (x^3+y^3)$. Each of these two terms x and y have same degree 3.
And $N = -x^1y^2$ is also of same degree 3 as sum of powers of x and y is 3
Thus, since equation (1) is homogeneous of degree 3.
So, I.F. $= \frac{1}{Mx+Ny}$
$\Rightarrow$I.F.$= \frac{1}{(x^3+y^3)x+(-xy^2)y} = \frac{1}{x^4}$
Multiplying equation (1) by I.F. to convert it to an exact equation:

$[(x^3+y^3)dx - xy^2dy] \times \frac{1}{x^4} = 0$

$\Rightarrow \left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$ (2)

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$\Rightarrow \int_{y=con.} \left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx + 0 = c$

$\Rightarrow \log x - \frac{y^3}{3x^3} = c$ **Answer.**

**Problem 2.** $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

**Solution:** $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ (1)

Comparing it with: $Mdx + Ndy = 0$
Here $M = (x^2y - 2xy^2) \Rightarrow \frac{\partial M}{\partial y} = x^2 - 4xy$
And $N = (3x^2y - x^3) \Rightarrow \frac{\partial N}{\partial x} = 6xy - 3x^2$
Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .
We need to make it exact using I.F.
since equation (1) is homogeneous of degree 3.
So, I.F. $= \frac{1}{Mx+Ny}$
$\Rightarrow$I.F.$= \frac{1}{(x^2y-2xy^2)x+(3x^2y-x^3)y} = \frac{1}{x^2y^2}$
Multiplying equation (1) by I.F. to convert it to an exact equation:

$[(x^2y - 2xy^2)dx + (3x^2y - x^3)dy] \times \frac{1}{x^2y^2} = 0$

$\Rightarrow \left(\frac{1}{y}-\frac{2}{x}\right)dx + \left(\frac{3}{y}-\frac{x}{y^2}\right)dy = 0$ (2)

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$\Rightarrow \int_{y=con.} \left(\frac{1}{y}-\frac{2}{x}\right)dx + \int \frac{3}{y}dy = c$

$\Rightarrow \frac{x}{y} - 2\log x + 3\log y = c$ **Answer.**

# MTH166

## Lecture-4

## Equations Reducible to Exact Form-II

**Topic:**

Equations of Reducible to Exact form

**Learning Outcomes:**

**1.** How to find I.F. using different methods (Next two methods)

**2.** How to convert a non-exact equation into an exact equation using I.F. and find its solution

**Methods to find I.F.**

**3.** If an equation: $Mdx + Ndy = 0$

can be re-written as:

$$f(xy)ydx + g(xy)xdy = 0$$

then, $I.F. = \frac{1}{Mx - Ny}$ provided $Mx - Ny \neq 0$

**Solve the following differential equations:**

**Problem 1.** $(\boldsymbol{y}+\boldsymbol{x}\boldsymbol{y}^2)\boldsymbol{dx}+(\boldsymbol{x}-\boldsymbol{x}^2\boldsymbol{y})\boldsymbol{dy}=\boldsymbol{0}$ **(1)**

**Solution:** $(y+xy^2)dx+(x-x^2y)dy=0$

Comparing it with: $Mdx+Ndy=0$

Here $M=(y+xy^2)\Rightarrow\frac{\partial M}{\partial y}=1+2xy$

And $N=(x-x^2y)\Rightarrow\frac{\partial N}{\partial x}=1-2xy$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Since equation (1) can be re-written as:
$(1+xy)ydx+(1-xy)xdy=0$
Which is of the form: $f(xy)ydx+g(xy)xdy=0$

$I.F.=\frac{1}{Mx-Ny}=\frac{1}{(y+xy^2)x-(x-x^2y)y}=\frac{1}{2x^2y^2}$

Multiplying equation (1) by I.F.

$[(y+xy^2)dx+(x-x^2y)dy]\times\frac{1}{2x^2y^2}=0$

$\Rightarrow\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0$

Which is an exact differential equation.

Solution: $\int_{y=con.}Mdx+\int(Terms\ of\ N\ free\ from\ x)dy=C$

$\Rightarrow\int_{y=con.}\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx-\int\left(\frac{1}{2y}\right)dy=C$

$\Rightarrow-\frac{1}{2x}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$ **Answer.**

**Solve the following differential equations:**

**Problem 2.** $(\boldsymbol{x}^2\boldsymbol{y}^3-\boldsymbol{y})\boldsymbol{dx}+(\boldsymbol{x}^3\boldsymbol{y}^2+\boldsymbol{x})\boldsymbol{dy}=\boldsymbol{0}$ **(1)**

**Solution:** $(x^2y^3-y)dx+(x^3y^2+x)dy=0$

Comparing it with: $Mdx+Ndy=0$

Here $M=(x^2y^3-y)\Rightarrow\frac{\partial M}{\partial y}=3x^2y^2-1$

And $N=(x^3y^2+x)\Rightarrow\frac{\partial N}{\partial x}=3x^2y^2+1$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Since equation (1) can be re-written as:
$(x^2y^2-1)ydx+(x^2y^2+1)xdy=0$
Which is of the form: $f(xy)ydx+g(xy)xdy=0$

$I.F.=\frac{1}{Mx-Ny}=\frac{1}{(x^2y^3-y)x-(x^3y^2+x)y}=\frac{-1}{2xy}$

Multiplying equation (1) by I.F.

$[(x^2y^3-y)dx+(x^3y^2+x)dy]\times\frac{-1}{2xy}=0$

$\Rightarrow\left(-\frac{xy^2}{2}+\frac{1}{2x}\right)dx+\left(-\frac{x^2y}{2}-\frac{1}{2y}\right)dy=0$

Which is an exact differential equation.

Solution: $\int_{y=con.}Mdx+\int(Terms\ of\ N\ free\ from\ x)dy=C$

$\Rightarrow\int_{y=con.}\left(-\frac{xy^2}{2}+\frac{1}{2x}\right)dx-\int\left(\frac{1}{2y}\right)dy=C$

$\Rightarrow-\frac{x^2y^2}{4}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$ **Answer.**

**Polling Quiz**

**For what values of $a$, given differential equ.: $(x^2 - ay)dx + (y^2 - ax)dy = 0$ is exact:**

(A)For all values of $a$

(B) There does not exist any value of $a$ for which given diff. equ. is exact.

(C) Only for $a = 1$

(D) None of these.

**Methods to find I.F.**

**4.** For an equation: $Mdx + Ndy = 0$ (1)

(I) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then $I.F. = e^{\int f(x)dx}$

(II) If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then $I.F. = e^{\int f(y)dy}$

**Problem 1.** $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$ (1)

**Solution:** $(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy = 0$

Comparing it with: $Mdx + Ndy = 0$

Here $M = (4xy + 3y^2 - x) \Rightarrow \frac{\partial M}{\partial y} = 4x + 6y$

and $N = (x^2 + 2xy) \Rightarrow \frac{\partial N}{\partial x} = 2x + 2y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Here $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(4x+6y)\ \ (2x+2\ \ )}{(x^2+2xy)} = \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$

So, $I.F. = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

Multiplying equation (1) by I.F., we get:

$[(4xy + 3y^2 - x)dx + (x^2 + 2xy)dy] \times x^2 = 0$

$\Rightarrow (4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$\Rightarrow \int_{y=con.} (4x^3y + 3x^2y^2 - x^3)dx + \int (0)dy = c$

$\Rightarrow \frac{4x^4y}{4} + 3\frac{x^3}{3}y^2 - \frac{x^4}{4} = c$

$\Rightarrow 4x^4y + 4x^3y^2 - x^4 = 4c$

$\Rightarrow 4x^4y + 4x^3y^2 - x^4 = c_1$ **Answer.** $(4c = c_1)$

**Problem 2.** $(xy^3+y)dx+(2x^2y^2+2x+2y^4)dy=0$ (1)

**Solution:** $(xy^3+y)dx+(2x^2y^2+2x+2y^4)dy=0$

Comparing it with: $Mdx+Ndy=0$

Here $M=(xy^3+y) \Rightarrow \frac{\partial M}{\partial y}=3xy^2+1$

and $N=(2x^2y^2+2x+2y^4) \Rightarrow \frac{\partial N}{\partial x}=4xy^2+2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

Here $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{(4x\ ^2+2)-(3x\ ^2+1)}{(xy^3+y)}=\frac{(xy^2+1)}{y(xy^2+1)}=\frac{1}{y}=f(y)$

So, $I.F.=e^{\int f(y)d}\ =e^{\int \frac{1}{y}dy}=e^{\log y}=y$

Multiplying equation (1) by I.F., we get:

$[(xy^3+y)dx+(2x^2y^2+2x+2y^4)dy]\times y=0$

$\Rightarrow (xy^4+y^2)dx+(2x^2y^3+2xy+2y^5)dy$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx+\int(Terms\ of\ N\ free\ from\ x)dy=C$

$\Rightarrow \int_{y=con.}(xy^4+y^2)dx+\int(2y^5)dy=c$

$\Rightarrow \frac{x^2y^4}{2}+xy^2+\frac{y^6}{3}=c$

$\Rightarrow 3x^2y^4+6xy^2+2y^6=6c$

$\Rightarrow 3x^2y^4+6xy^2+2y^6=c_1$ **Answer.** $(6c=c_1)$

**Polling Quiz**

**Solution of differential equation: $xdy+ydx=0$ is given by:**

(A) $xy=c$

(B) $\frac{x}{y}=c$

(C) $x+y=c$

(D) $x-y=c$

# MTH166

## Lecture-5

## Equs. of First Order and Higher Degree

**Topic:**

Equations of First Order and Higher Degree

**Learning Outcomes:**

**1.** What are the Equations of First Order and Higher Degree.

**2.** How to solve these equations first order and higher degree.

**Equations of First Order and Higher Degree:**

Let $\frac{dy}{dx} = p$ (A standard notation)

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = p^2$ $\left(Not\ \frac{d^2y}{dx^2}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^3 = p^3$ $\left(Not\ \frac{d^3y}{dx^3}\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)^4 = p^4$ and so on…

These types of equations can also be written as: $f(x, y, p) = 0$

**Eqations Solvable for p:**

**Find the general solution of following differential equations:**

**Problem 1.** $y\left(\frac{dy}{dx}\right)^2 + (x-y)\left(\frac{dy}{dx}\right) - x = 0$ **(1)**

**Solution:** Let $\frac{dy}{dx} = p$

Equation (1) can be re-written as:

$yp^2 + (x-y)p - x = 0$

$\Rightarrow yp^2 + px - py - x = 0$

$\Rightarrow p(yp + x) - 1(yp + x) = 0$

$\Rightarrow (yp + x)(p - 1) = 0$

Here $(yp + x) = 0$
$\Rightarrow y\frac{dy}{dx} + x = 0$
$\Rightarrow ydy + xdx = 0$
$\Rightarrow \int ydy + \int xdx = c_1$
$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c_1$
$\Rightarrow (y^2 + x^2 - 2c_1) = 0$

Also $(p - 1) = 0$
$\Rightarrow \frac{dy}{dx} - 1 = 0$
$\Rightarrow dy - dx = 0$
$\Rightarrow \int dy - \int dx = 0$
$\Rightarrow y - x = c_2$
$\Rightarrow (y - x - c_2) = 0$
So, the general solution of equation (1) is given by:
$(y^2 + x^2 - 2c_1)\,(y - x - c_2) = 0$ **Answer.**

**Problem 2.** $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\left(\frac{dy}{dx}\right) + xy = 0$ **(1)**

**Solution:** Let $\frac{dy}{dx} = p$
Equation (1) can be re-written as:
$xyp^2 - (x^2 + y^2)p + xy = 0$
$\Rightarrow xyp^2 - x^2p - y^2p + xy = 0$
$\Rightarrow xp(yp - x) - y(yp - x) = 0$
$\Rightarrow (yp - x)(xp - y) = 0$
Here $(yp - x) = 0$
$\Rightarrow y\frac{dy}{dx} - x = 0$
$\Rightarrow ydy - xdx = 0$
$\Rightarrow \int ydy - \int xdx = c_1$
$\Rightarrow \frac{y^2}{2} - \frac{x^2}{2} = c_1$
$\Rightarrow (y^2 - x^2 - 2c_1) = 0$

Here $(xp - y) = 0$
$\Rightarrow x\frac{dy}{dx} - y = 0$
$\Rightarrow xdy - ydx = 0$
$\Rightarrow xdy = ydx$
$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$
$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$
$\Rightarrow \log y = \log x + \log c_2$
$\Rightarrow \log y = \log xc_2$
$\Rightarrow y = xc_2 \quad \Rightarrow (y - xc_2) = 0$
So, the general solution of equation (1) is given by:
$(y^2 - x^2 - 2c_1)\,(y - xc_2) = 0$ **Answer.**

**Polling Quiz**

**What is the relationship between $a$ and $b$, so that the given equation:**

$(x^8 + x + ay^2)dx + (y^8 - y + bxy)dy = 0$ **is exact:**

(A) $b = 2a$

(B) $a = b$

(C) $a \neq b$

(D) $a = 1, b = 3$

**Problem 3.** $\boldsymbol{p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0}$ (1)

**Solution:** $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ (1)

$\Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0$

$\Rightarrow p[p(p + 2x) - y^2(p + 2x)] = 0$

$\Rightarrow p(p + 2x)(p - y^2) = 0$

Here $p = 0$ $\Rightarrow \frac{dy}{dx} = 0$ $\Rightarrow y = c_1$ $\Rightarrow (y - c_1) = 0$

Also $(p + 2x) = 0$

$\Rightarrow \frac{dy}{dx} + 2x = 0$

$\Rightarrow dy + 2xdx = 0$ $\Rightarrow \int dy + 2\int xdx = c_2$ $\Rightarrow y + 2\left(\frac{x^2}{2}\right) = c_2$

$\Rightarrow (y + x^2 - c_2) = 0$

Also $(p - y^2) = 0$

$\Rightarrow \frac{dy}{dx} - y^2 = 0$

$\Rightarrow \frac{dy}{dx} = y^2$

$\Rightarrow \frac{dy}{y^2} = dx$

$\Rightarrow \int y^{-2}\, dy = \int dx$

$\Rightarrow \frac{-1}{y} = x + c_3$

$\Rightarrow (xy + 1 + yc_3) = 0$

So, the general solution of equation (1) is given by:

$(y - c_1)(y + x^2 - c_2)(xy + 1 + yc_3) = 0$ **Answer.**

**Problem 4.** $\boldsymbol{p(p + y) = x(x + y)}$ **(1)**

**Solution:** $p(p + y) = x(x + y)$ (1)

$\Rightarrow p^2 + yp - x^2 - xy = 0$

$\Rightarrow (p^2 - x^2) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x) + y(p - x) = 0$

$\Rightarrow (p - x)(p + x + y) = 0$

Here $(p - x) = 0$

$\Rightarrow \frac{dy}{dx} - x = 0$

$\Rightarrow dy - xdx = 0$ $\Rightarrow \int dy - \int xdx = c_1$ $\Rightarrow y - \left(\frac{x^2}{2}\right) = c_1$

$\Rightarrow (2y - x^2 - 2c_1) = 0$

Also $(p + x + y) = 0$

$\Rightarrow \frac{dy}{dx} + x + y = 0$

$\Rightarrow (x + y)dx + dy = 0$ (2)

Comparing it with: $Mdx + Ndy = 0$

Here $M = (x + y)$ $\Rightarrow \frac{\partial M}{\partial y} = 1$

and $N = 1$ $\Rightarrow \frac{\partial N}{\partial x} = 0$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, equation (2) in non-exact.

Here $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-0}{1} = 1 = x^0 = f(x)$ which can be considered as a function of x.

So, $I.F. = e^{\int f(x)dx} = e^{\int 1dx} = e^x$

Multiplying equation (2) by I.F.

$[(x+y)dx + dy] \times e^x = 0$

$\Rightarrow (xe^x + ye^x)dx + e^x dy = 0$

Which is an exact differential equation.

Solution: $\int_{y=co\ .} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = c_2$

$\Rightarrow \int_{y=con.} (xe^x + ye^x)dx + 0 = c_2$

$\Rightarrow e^x(x-1) + ye^x = c_2$

$\Rightarrow (e^x(x-1) + ye^x - c_2) = 0$

So, the general solution of equation (1) is given by:

$(2y - x^2 - 2c_1)(e^x(x-1) + ye^x - c_2) = 0$ **Answer.**

**Polling Quiz**

**In the non-exact differential equation: $\boldsymbol{M(x,y)dx + N(x,y)dy = 0}$ if:**

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ **then Integrating factor is given by:**

(A) $f(x)$

(B) $\log f(x)$

(C) $e^{f(x)}$

(D) $e^{\int f(x)d}$

**Problem 5.** $y = x\left[p + \sqrt{1+p^2}\right]$

*Try it yourself.*

**Problem 6.** $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

*Try it yourself.*

# MTH166

## Lecture-6

## Clairaut's Equation

**Topic:**

Clairaut's Equation: A special case of equations solvable for y

**Learning Outcomes:**

**1.** How to find general solution (Not singular solution) of Clairaut's equation.

**2.** How to solve equations solvable for $y$ i.e. $y = f(x, p)$

**Clairaut's Equation:**

An equation of the form: $y = px \pm f(p)$ (1)
is called Clairaut's equation where $p = \frac{dy}{dx}$
General solution of equation (1) is given by:
$y = cx \pm f(c)$ (Just replace $p$ by $c$ in the question, we get the answer)

* It is also called as a special case of equations solvable for y: $y = f(x, p)$

** It is a very good concept for asking MCQ

**Find the general solution of following differential equations:**

**Problem 1.** $\boldsymbol{xp^2 - yp + a = 0}$ **(1)**

**Solution:** $xp^2 - yp + a = 0$

$\Rightarrow yp = xp^2 + a$

$\Rightarrow y = px + \frac{a}{p}$ (2)

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + \frac{a}{c}$ **Answer.**

**Problem 2.** $\boldsymbol{p = \log(px - y)}$ **(1)**

**Solution:** $p = \log(px - y)$

$\Rightarrow e^p = e^{\log(px-y)}$

$\Rightarrow e^p = (px - y)$

$\Rightarrow y = px - e^p$ (2)

Which is of Clairaut's form: $y = px \pm f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx - e^c$ **Answer.**

**Problem 3.** $\boldsymbol{y = px + \sqrt{a^2p^2 + b^2}}$ **(1)**

**Solution:** $y = px + \sqrt{a^2p^2 + b^2}$

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (1)

i.e. $y = cx + \sqrt{a^2c^2 + b^2}$ **Answer.**

**Problem 4.** $\boldsymbol{\sin px \cos y = \cos px \sin y + p}$ **(1)**

**Solution:** $\sin px \cos y = \cos px \sin y + p$

$\Rightarrow \sin px \cos y - \cos px \sin y = p$

$\Rightarrow \sin(px - y) = p$

$\Rightarrow (px - y) = sin^{-1}p$

$\Rightarrow y = px - sin^{-1}p$ (2)

Which is of Clairaut's form: $y = px \pm f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx - sin^{-1}c$ **Answer.**

**Problem 5.** $\boldsymbol{y + 2\left(\frac{dy}{dx}\right)^2 = (x + 1)\left(\frac{dy}{dx}\right)}$ **(1)**

**Solution:** $y + 2p^2 = (x + 1)p$

$\Rightarrow y = px + p - 2p^2$

$\Rightarrow y = px + p(1 - 2p)$ (2)

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + c(1 - 2c)$ **Answer.**

**Problem 6.** $(\boldsymbol{y} - \boldsymbol{px})(\boldsymbol{p} - \mathbf{1}) = \boldsymbol{p}$ **(1)**

**Solution:** $(y - px)(p - 1) = p$

$\Rightarrow y - px = \frac{p}{p-1}$

$\Rightarrow y = px + \frac{p}{p-1}$ (2)

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + \frac{c}{c-1}$ **Answer.**

**Polling Quiz**

**Which of the following is I.F. for equation:** $(\boldsymbol{xdy} - \boldsymbol{ydx}) + \boldsymbol{a}(\boldsymbol{x}^2 + \boldsymbol{y}^2)\boldsymbol{dx} = \mathbf{0}$

(A) $\frac{1}{x^2}$

(B) $\frac{1}{y^2}$

(C) $\frac{1}{xy}$

(D) $\frac{1}{(x^2+y^2)}$

**Equations solvable for y:** $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p})$

**Problem.** $\boldsymbol{y} = \mathbf{2}\boldsymbol{px} + \boldsymbol{p}^{\boldsymbol{n}}$ **(1)**

**Solution:** $y = 2px + p^n$ (1)

Equation (1) is neither of Clairaut's form: $y = px + f(p)$ nor it can be solved for p by factorisation.

But equation (1) is of the form $y = f(x, p)$ i.e. solvable for y. So, Differentiating equation (1) w.r.t x, we get:

$y = 2px + p^n$ (1)

$\Rightarrow \frac{dy}{dx} = 2\left[p\frac{d}{dx}(x) + x\frac{d}{dx}(p)\right] + np^{n-1}\frac{d}{dx}(p)$

$\Rightarrow p = 2\left[p + x\frac{dp}{dx}\right] + np^{n-1}\frac{dp}{dx}$

$\Rightarrow p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} = 0$

$\Rightarrow p + (2x + np^{n-1})\frac{dp}{dx} = 0$

$\Rightarrow pdx + (2x + np^{n-1})dp = 0$ (2)

Comparing it with: $Mdx + Ndp = 0$

Here $M = p \Rightarrow \frac{\partial M}{\partial p} = 1$

and $N = (2x + np^{n-1}) \Rightarrow \frac{\partial N}{\partial x} = 2$

Since, $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial x}$, so, equation (2) is non-exact.

Here $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial p}}{M} = \frac{2-1}{p} = \frac{1}{p} = f(p)$

So, $I.F. = e^{\int f(p)dp} = e^{\int \frac{1}{p}dp} = e^{\log p} = p$

Multiplying equation (2) by I.F.

$[pdx + (2x + np^{n-1})dp] \times p = 0$

$\Rightarrow p^2 dx + (2xp + np^n)dp = 0$

Which is an exact differential equation.

Solution: $\int_{p=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dp = c$

$\Rightarrow \int_{p=con.} p^2 dx + \int (np^n)dp = c$

$\Rightarrow p^2 x + n\frac{p^{n+1}}{n+1} = c$

$\Rightarrow x = \frac{c}{p^2} - n\frac{p^{n-1}}{n+1}$ (3)

$\Rightarrow x = \frac{c}{p^2} - n\frac{p^{n-1}}{n+1}$ (3)

Substituting this value of x from equation (3) in equation (1), we get:

$y = 2px + p^n$

$\Rightarrow y = 2p\left(\frac{c}{p^2} - n\frac{p^{n-1}}{n+1}\right) + p^n$ (4)

Thus, equations (3) and (4) taken together, in parameter p, give the general solution of equation (1).

**Polling Quiz**

**Solution of differential equation $xdy - ydx - 2y^3dy = 0$ is:**

(A) $x + y^3 = c$

(B) $x + y^3 - cy = 0$

(C) $x + y^3 + cy = 0$

(D) None of these

**Equations solvable for y:** $x = f(y, p)$

**Problem.** $x = \frac{y - y^2p^3}{2p}$ **(1)**

*Try it yourself.*

# MTH166

## Lecture-7

## Linear Differential Equations (LDE)

**Unit 2: Differential Equations of Higher Order**

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-5**)

**Topic:**

Linear Differential Equations (LDE)

**Learning Outcomes:**

**1.** Identification of Linear Differential Equations (LDE).

**2.** Necessary and Sufficient condition for LDE to be Normal on an interval

**Linear Differential Equations (LDE):**

A linear differential equation of order $n$ is written as:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + --- + a_{n-1} \frac{dy}{dx} + a_n y = r(x) \qquad (1)$$

or

$$a_0 y^n + a_1 y^{n-1} + ----- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

For example, a second order LDE is written as:

$$a_0 y'' + a_1 y' + a_2 y = r(x) \qquad (2)$$

* If $r(x) = 0$, then LDE is called Homogeneous LDE.

* If $r(x) \neq 0$, then LDE is called Non-Homogeneous LDE.

* If $a_0, a_1, ---, a_n$ are all constants, then LDE is called LDE with constant coefficients.

* If $a_0, a_1, ---, a_n$ are not all constants, then it is called LDE with variable coefficients.

**Classify the following LDE:**

**1.** $y'' + 4y' + 3y = x^2 e^x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

**2.** $y'' + 2y' + y = \sin x$

It is a 2nd order Non-homogeneous LDE with constant coefficients.

**3.** $x^2 y'' + xy' + (x^2 - 4)y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

**4.** $(1 - x^2)y'' - 2xy' + 20y = 0$

It is a 2nd order Homogeneous LDE with variable coefficients.

**Polling Question**

**The equation: $y'' + 4y' + xy = x^2 e^x$ is:**

(A) 1st order Homogeneous LDE with variable coefficients.

(B) 2nd order Homogeneous LDE with constant coefficients.

(C) 2nd order Non-homogeneous LDE with variable coefficients.

(D) 2nd order Non-homogeneous LDE with constant coefficients.

**Necessary and Sufficient condition for LDE to be Normal on an interval:**

A linear differential equation of order $n$:

$$a_0 y^n + a_1 y^{n-1} + ---- + a_{n-1} y' + a_n y = r(x) \qquad (1)$$

is said to be normal on an interval $I$ if:

**1.** $a_0, a_1, ---, a_n$ and $r(x)$ are all continuous on an interval $I$

**2.** $a_0 \neq 0$

* In homogeneous LDE ($r(x)$=0), the problem arises only because of $a_0 \neq 0$.

* In non-homogeneous LDE ($r(x) \neq 0$), the problem arises due to $a_0 \neq 0$ and also due to domain of $r(x)$.

* In almost all numerical $a_0, a_1, ---, a_n$ are all continuous, so first condition automatically holds. We are focused on second condition only.

**Find the intervals on which the following differential equations are normal.**

**Problem 1.** $\mathbf{(1 - x^2)y'' - 2xy' + 3y = 0}$

**Solution:** $(1 - x^2)y'' - 2xy' + 3y = 0 \qquad (1)$

Comparing with: $a_0 y'' + a_1 y' + a_2 y = 0$

**1.** $\left.\begin{array}{l} a_0 = (1 - x^2) \\ a_1 = -2x \\ a_2 = 3 \end{array}\right\}$ Being polynomials, $a_0$, $a_1$, $a_2$ are all continuous on $(-\infty, \infty)$

**2.** $a_0 \neq 0 \quad \Rightarrow (1 - x^2) \neq 0 \quad \Rightarrow x^2 \neq 1 \quad \Rightarrow x \neq \pm 1$


Thus, LDE (1) is normal on subintervals: $(-\infty, -1), (-1,1), (1, \infty)$.

**Problem 2.** $\boldsymbol{x^2y'' + xy' + (n^2 - x^2)y = 0;\ n}$ **is real.**

**Solution:** $x^2y'' + xy' + (n^2 - x^2)y = 0$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = 0$

**1.** $\left.\begin{aligned} a_0 &= x^2 \\ a_1 &= x \\ a_2 &= (n^2 - x^2) \end{aligned}\right\}$ Being polynomials, $a_0$, $a_1$, $a_2$ are all continuous on $(-\infty, \infty)$

**2.** $a_0 \neq 0 \quad \Rightarrow x^2 \neq 0 \quad \Rightarrow x \neq 0$


Thus, LDE (1) is normal on subintervals: $(-\infty, 0), (0, \infty)$.

**Problem 3.** $\boldsymbol{x(1 - x)y'' - 3xy' - y = 0}$

**Solution:** $x(1 - x)y'' - 3xy' - y = 0$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = 0$

**1.** $\left.\begin{aligned} a_0 &= x(1 - x) \\ a_1 &= -3x \\ a_2 &= -1 \end{aligned}\right\}$ Being polynomials, $a_0$, $a_1$, $a_2$ are all continuous on $(-\infty, \infty)$

**2.** $a_0 \neq 0 \quad \Rightarrow x(1 - x) \neq 0 \quad \Rightarrow x \neq 0,\ \ x \neq 1$


Thus, LDE (1) is normal on subintervals: $(-\infty, 0), (0,1), (1, \infty)$.

**Problem 4.** $\boldsymbol{y'' + 9y' + y = \log(x^2 - 9)}$

**Solution:** $y'' + 9y' + y = \log(x^2 - 9)$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = r(x)$

**1.** $\left.\begin{aligned} a_0 &= 1 \\ a_1 &= 9 \\ a_2 &= 1 \end{aligned}\right\}$ Being constants, $a_0$, $a_1$, $a_2$ are all continuous on $(-\infty, \infty)$

Also $r(x) = \log(x^2 - 9)$ will be defined if $(x^2 - 9) > 0$ i.e. $x^2 > 9$

$\Rightarrow |x| > 3 \quad \Rightarrow\ -\infty < x < -3$ and $3 < x < \infty$

**2.** $a_0 \neq 0 \quad \Rightarrow 1 \neq 0$ which is true.


Thus, LDE (1) is normal on subintervals: $(-\infty, -3), (3, \infty)$.

**Problem 5.** $\boldsymbol{\sqrt{x}y'' + 6xy' + 15y = \log(x^4 - 256)}$

**Solution:** $\sqrt{x}y'' + 6xy' + 15y = \log(x^4 - 256)$ (1)

Comparing with: $a_0y'' + a_1y' + a_2y = r(x)$

**1.** $\left.\begin{aligned} a_0 &= \sqrt{x} \\ a_1 &= 6x \\ a_2 &= 15 \end{aligned}\right\}$ $a_1$, $a_2$ are all continuous on $(-\infty, \infty)$, but $a_0$ is continuous on $(0, \infty)$

Also $r(x) = \log(x^4 - 256)$ will be defined if $(x^4 - 256) > 0$ i.e. $x^4 > 256$

$\Rightarrow |x| > 4 \quad \Rightarrow\ -\infty < x < -4$ and $4 < x < \infty$

**2.** $a_0 \neq 0 \quad \Rightarrow \sqrt{x} \neq 0 \Rightarrow x > 0$ (Square root of a negative number is not defined)


Thus, LDE (1) is normal on subintervals: $(4, \infty)$.

**Polling Question**

**A linear differential equation: $a_0 y''' + a_1 y'' + a_2 y' + a_3 y = 0$ is said to be normal on an interval $I$ if:**

(A) $a_0, a_1, a_2, a_3$ and $r(x)$ are all continuous on an interval $I$ and $a_0 \neq 0$

(B) $a_0, a_1, a_2, a_3$ are all continuous on an interval $I$ and $a_0 \neq 0$

(C) $a_0 \neq 0$

**Problem 6.** $y''+3y' + \sqrt{x}y = \sin x$

**Solution:** $y'' + 3y' + \sqrt{x}y = \sin x$ (1)

Comparing with: $a_0 y'' + a_1 y' + a_2 y = r(x)$

**1.** $\left.\begin{array}{l} a_0 = 1 \\ a_1 = 3 \\ a_2 = \sqrt{x} \end{array}\right\}$ $a_0, a_1$ are all continuous on $(-\infty, \infty)$, but $a_2$ is continuous on $(0, \infty)$

Also $r(x) = \sin x$ is continuous on $(-\infty, \infty)$.

**2.** $a_0 \neq 0 \quad \Rightarrow 1 \neq 0$ which is true.


Thus, LDE (1) is normal on subintervals: $(0, \infty)$.

# MTH166

## Lecture-8

## Linear Differential Equations (LDE)-II

**Topic:**

Linear Differential Equations (LDE)

**Learning Outcomes:**

**1.** Linear Dependence and Independence of functions (Solutions) via Wronskian.

**2.** Abel's Formula to find Wronskian.

**3.** Principle of Superposition.

**Linear Dependence and Independence of Functions or Solutions:**

Let $(y_1, y_2, y_3)$ be the given set of functions or solutions.

**Wronskian**, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

(I) If $\boldsymbol{W = 0}$, then functions $(y_1, y_2, y_3)$ are said to be **Linearly Dependent**.

(II) If $\boldsymbol{W \neq 0}$, then functions $(y_1, y_2, y_3)$ are said to be **Linearly Independent**.

**Fundamental Solutions or Basis:**

The solutions which are linearly independent are called as Fundamental solutions or Basis.

**Problem 1:** Show that the functions: $(1, \sin x, \cos x)$ are linearly independent.

**Solution:** Let $(y_1, y_2, y_3) = (1, \sin x, \cos x)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$

Expanding by first column:

$W = 1(-cos^2x - sin^2x) - 0 + 0 = -1(cos^2x + sin^2x) = -1$

Since $W \neq 0$, so the given functions are linearly independent.

**Problem 2:** Show that the functions: $(x, x^2, x^3)$ are linearly independent on an interval which does not contain zero.

**Solution:** Let $(y_1, y_2, y_3) = (x, x^2, x^3)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

Expanding by first column:

$W = x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) + 0 = 2x^3 \neq 0$ (because $x \neq 0$)

Since $W \neq 0$, so the given functions are linearly independent.

**Problem 3:** Show that the functions: $(2x, 6x+3, 3x+2)$ are linearly dependent.

**Solution:** Let $(y_1, y_2, y_3) = (2x, 6x+3, 3x+2)$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 2x & 6x+3 & 3x+2 \\ 2 & 6 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(If one row of a determinant is zero, then value of determinant is always zero.)

Since $W = 0$, so the given functions are linearly dependent.

**Polling Question**

**The functions $(y_1, y_2, y_3)$ are said to be Linearly Dependent if:**

(A)Wronskian $W \neq 0$

(B)Wronskian $W = 0$

**Abel's Formula to find Wronskian:**

Let us consider a 2nd order homogeneous LDE:

$a_0 y'' + a_1 y' + a_2 y = 0$ (1)

Where $a_0 \neq 0, a_1, a_2$ are continuous on an interval I and $y_1, y_2$ be its linearly independent solutions, then Wronskian is given as:

Wronskian, $\boldsymbol{W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}}$ where $c$ is a constant.

**Problem 1.** Using Abel's formula, find Wronskian for: $y'' - 4y' + 4y = 0$

**Solution:** The given equation is: $y'' - 4y' + 4y = 0$ (1)

Comparing it with: $a_0 y'' + a_1 y' + a_2 y = 0$

Here $a_0 = 1, \ a_1 = -4, \ a_2 = 4$

By Abel's formula, Wronskian is given by:

$W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}$ where $c$ is a constant

$\Rightarrow W = ce^{-\int\left(\frac{-4}{1}\right)dx}$

$\Rightarrow W = ce^{4\int dx}$

$\Rightarrow W = ce^{4x}$ **Answer.**

**Problem 2.** Using Abel's formula, find Wronskian for: $y'' + a^2y = 0, a \neq 0$.

**Solution:** The given equation is: $y'' + a^2y = 0, a \neq 0$ (1)

Comparing it with: $a_0y'' + a_1y' + a_2y = 0$

Here $a_0 = 1,\ \ a_1 = 0,\ \ a_2 = a^2$

By Abel's formula, Wronskian is given by:

$W = ce^{-\int\left(\frac{a_1}{a_0}\right)dx}$ where $c$ is a constant

$\Rightarrow W = ce^{-\int\left(\frac{0}{1}\right)dx}$

$\Rightarrow W = ce^0 = c(1)$

$\Rightarrow W = c$ **Answer.**

**Polling Question**

**Using Abel's formula, find Wronskian for: $y'' + y' + 4y = 0$**

(A) $W = ce^{4x}$

(B) $W = ce^{-x}$

(C) $W = ce^{x}$

(D) $W = ce^{2x}$

**Principle of Superposition:**

If functions $(y_1, y_2, --- , y_n)$ are the solutions of homogeneous LDE:

$a_0y^n + a_1y^{n-1} + ----- + a_{n-1}y' + a_ny = 0$ (1)

Then, their linear combination: $(c_1y_1 + c_2y_2 + --- + c_ny_n)$ is also a solution of LDE (1).

**Note:** Principle of superposition is not applicable to non-homogeneous LDE.

**Problem 1:** Show that $(e^x, e^{-x})$ and their linear combination $(c_1e^x + c_2e^{-x})$ are the solutions of homogeneous equation: $y'' - y = 0$. Also show that $(e^x, e^{-x})$ form basis or Fundamental solution.

**Solution:** The given homogeneous LDE: $y'' - y = 0$ (1)

Let $(y_1, y_2) = (e^x, e^{-x})$ be the given set of functions.

**Part 1:** Now they will be solutions of equation (1) if they satisfy equation (1).

i.e. $y_1$ will be solution of equation (1) if $y_1'' - y_1 = 0$ (2)

Here $y_1 = e^x$ $\Rightarrow y_1' = e^x$ $\Rightarrow y_1'' = e^x$

Substitute these values of $y_1$ and $y_1''$ in equation (2), we get:

$e^x - e^x = 0$ which is true.

So, $y_1$ is a solution of equation (1)

Now $y_2$ will be solution of equation (1) if $y_2'' - y_2 = 0$ (3)

Here $y_2 = e^{-x}$ $\Rightarrow y_2' = -e^{-x}$ $\Rightarrow y_2'' = e^{-x}$

Substitute these values of $y_2$ and $y_2''$ in equation (3), we get:

$e^{-x} - e^{-x} = 0$ which is true.

So, $y_2$ is a solution of equation (1)

**Part 2:** By principle of superposition, if $y_1$ and $y_2$ are solutions, then their linear combination $(c_1e^x + c_2e^{-x})$ will also be a solution.

Let us verify it:

Let $y_3 = (c_1e^x + c_2e^{-x})$ $\Rightarrow y_3' = (c_1e^x - c_2e^{-x}) \Rightarrow y_3'' = (c_1e^x + c_2e^{-x})$

Now $y_3$ will be solution of equation (1) if $y_3'' - y_3 = 0$

i.e. $(c_1e^x + c_2e^{-x}) - (c_1e^x + c_2e^{-x}) = 0$ which is true.

**Part 3:**

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$= -e^x . e^{-x} - e^x . e^{-x}$

$= -2e^x e^{-x} = -2e^{x-x}$

$= -2e^0 = -2(1)$

$= -2 \neq 0$

Since W $\neq 0$, So, Solutions $(y_1, y_2)$ are linearly independent.

Hence $(e^x, e^{-x})$ form basis or fundamental solutions of equation (1).

**<u>Problem 2:</u>** Show that $(1, x^2)$ form a set of fundamental solutions (basis) of homogeneous equation: $x^2y'' - xy' = 0$.

**<u>Solution:</u>** The given homogeneous LDE: $x^2y'' - xy' = 0$ (1)

Let $(y_1, y_2) = (1, x^2)$ be the given set of functions.

**Part 1:** Now they will be solutions of equation (1) if they satisfy equation (1).

i.e. $y_1$ will be solution of equation (1) if $x^2y_1'' - xy_1' = 0$ (2)

Here $y_1 = 1$ $\Rightarrow y_1' = 0$ $\Rightarrow y_1'' = 0$

Substitute these values of $y_1$ and $y_1''$ in equation (2), we get:

$x^2(0) - x(0) = 0$ which is true.

So, $y_1$ is a solution of equation (1)

Now $y_2$ will be solution of equation (1) if $x^2y_2'' - xy_2' = 0$ (3)

Here $y_2 = x^2$ $\Rightarrow y_2' = 2x$ $\Rightarrow y_2'' = 2$

Substitute these values of $y_2$ and $y_2''$ in equation (3), we get:

$x^2(2) - x(2x) = 0$ which is true.

So, $y_2$ is a solution of equation (1)

**Part 2:** Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x \neq 0$

Since W $\neq 0$, So, Solutions $(y_1, y_2)$ are linearly independent.

Hence $(1, x^2)$ form basis or fundamental solutions of equation (1).

# MTH166

## Lecture-9

## Solution of 2nd Order Homogeneous LDE with Constant Coefficients-I

**Topic:**

Solution of 2nd order Homogeneous LDE with Constant coefficients-I

**Learning Outcomes:**

**1.** How to apply differential operator for solving 2nd order Homogeneous LDE.

**2.** How to write solution when roots are real and unequal.

**3.** How to write solution when roots are real and equal.

**4.** How to write solution when roots are complex conjugates.

**Solution of 2nd order homogeneous LDE with constant coefficients:**

Let us consider 2nd order homogeneous LDE with constant coefficients as:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

or

$$ay'' + by' + cy = 0 \qquad (1)$$

Let $\frac{d}{dx} \equiv D$ be Differential operator ( An algebraic operator like $+, -, \times, \div$)

Equation (1) becomes:

$aD^2y + bDy + cy = 0$

Symbolic Form (S.F.): $(aD^2 + bD + c)y = 0$

Auxiliary Equ. (A.E.): $(aD^2 + bD + c) = 0$

$(aD^2 + bD + c) = 0$

$\Rightarrow D = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = m_1, m_2$ (Say)

**Case 1:** When roots are real and unequal (distinct) i.e. $m_1 \neq m_2$

Solution: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

**Case 2:** When roots are real and equal i.e. $m_1 = m_2$

Solution: $y = (c_1 + c_2 x)e^{m_1 x}$

**Case 3:** When roots are complex conjugates i.e. $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$

Solution: $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

## Polling Question

**Let roots of the equation: $(aD^2 + bD + c) = 0$ be given by:**

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**The roots are real and equal if:**

(A) $\sqrt{b^2 - 4ac} > 0$

(B) $\sqrt{b^2 - 4ac} < 0$

(C) $\sqrt{b^2 - 4ac} = 0$

**Find the general solution of the following differential equations:**

**Problem 1.** $y''-4y = 0$

**Solution:** The given equation is:

$y'' - 4y = 0$ (1)

**S.F. :** $(D^2 - 4)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 - 4) = 0$ $\Rightarrow D^2 = 4$ $\Rightarrow D = \pm 2$ (real and unequal roots)

Let $m_1 = 2$ and $m_2 = -2$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x}$ **Answer.**

**Problem 2.** $y'' - 4y' - 12y = 0$

**Solution:** The given equation is:

$y'' - 4y' - 12y = 0$ (1)

**S.F. :** $(D^2 - 4D - 12)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 - 4D - 12) = 0$ $\Rightarrow (D - 6)(D + 2) = 0$

$\Rightarrow D = 6, -2$ (real and unequal roots)

Let $m_1 = 6$ and $m_2 = -2$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{6x} + c_2 e^{-2x}$ **Answer.**

**Problem 3.** $y'' + 4y' + y = 0$

**Solution:** The given equation is:

$y'' + 4y' + y = 0$ (1)

**S.F. :** $(D^2 + 4D + 1)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 + 4D + 1) = 0$ $\Rightarrow D = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = \frac{2(-2 \pm \sqrt{3})}{2}$

$\Rightarrow D = -2 + \sqrt{3}, -2 - \sqrt{3}$ (real and unequal roots)

Let $m_1 = -2 + \sqrt{3}$ and $m_2 = -2 - \sqrt{3}$

∴ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}$ **Answer.**

**Problem 4.** $y''+2y' + y = 0$

**Solution:** The given equation is:

$y'' + 2y' + y = 0$ (1)

**S.F. :** $(D^2 + 2D + 1)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 + 2D + 1) = 0$ $\Rightarrow (D + 1)^2 = 0$

$\Rightarrow D = -1, -1$ (real and equal roots)

Let $m_1 = -1$ and $m_2 = -1$

∴ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x) e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x) e^{-1x}$ **Answer.**

**Problem 5.** $9y'' - 12y' + 4y = 0$

**Solution:** The given equation is:

$9y'' - 12y' + 4y = 0$ (1)

**S.F. :** $(9D^2 - 12D + 4)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(9D^2 - 12D + 4) = 0$ $\Rightarrow (3D - 2)^2 = 0$

$\Rightarrow D = \frac{2}{3}, \frac{2}{3}$ (real and equal roots)

Let $m_1 = \frac{2}{3}$ and $m_2 = \frac{2}{3}$

∴ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x) e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x) e^{\frac{2}{3}x}$ **Answer.**

**Problem 6.** $4y'' + 4y' + 1y = 0$

**Solution:** The given equation is:

$4y'' + 4y' + 1y = 0$ (1)

**S.F. :** $(4D^2 + 4D + 1)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(4D^2 + 4D + 1) = 0$ $\Rightarrow (2D + 1)^2 = 0$

$\Rightarrow D = -\frac{1}{2}, -\frac{1}{2}$ (real and equal roots)

Let $m_1 = -\frac{1}{2}$ and $m_2 = -\frac{1}{2}$

∴ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x) e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x) e^{-\frac{1}{2}x}$ **Answer.**

**Problem 7.** $y'' + 25y = 0$

**Solution:** The given equation is:

$y'' + 25y = 0$ (1)

**S.F. :** $(D^2 + 25)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 + 25) = 0$ $\Rightarrow D^2 = -25$ $\Rightarrow D = \pm 5i$ (Complex conjugate roots)

Let $m_1 = 0 + 5i$ and $m_2 = 0 - 5i$ $(\alpha \pm i\beta)$

∴ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow y = e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$

$\Rightarrow y = (c_1 \cos 5x + c_2 \sin 5x)$ **Answer.**

**Polling Question**

**The roots of equation: $y'' + 9y = 0$ are:**

(A) $3, -3$

(B) $0,9$

(C) $3i, -3i$

**Problem 8.** $y'' + 4y' + 5y = 0$

**Solution:** The given equation is:

$y'' + 4y' + 5y = 0$ (1)

**S.F. :** $(D^2 + 4D + 5)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 + 4D + 5) = 0$ $\Rightarrow D = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

Let $m_1 = -2 + 1i$ and $m_2 = -2 - 1i$ $(Complex\ roots\colon\ \alpha \pm i\beta)$

∴ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow y = e^{-2x}(c_1 \cos 1x + c_2 \sin 1x)$

$\Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$ **Answer.**

**Problem 9.** $y'' - 2y' + 2y = 0$

**Solution:** The given equation is:

$y'' - 2y' + 2y = 0$ (1)

**S.F. :** $(D^2 - 2D + 2)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 - 2D + 2) = 0$ $\Rightarrow D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Let $m_1 = 1 + 1i$ and $m_2 = 1 - 1i$ $(Complex\ roots\colon\ \alpha \pm i\beta)$

∴ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow y = e^{1x}(c_1 \cos 1x + c_2 \sin 1x)$

$\Rightarrow y = e^{x}(c_1 \cos x + c_2 \sin x)$ **Answer.**

# MTH166

## Lecture-10

## Solution of 2nd Order Homogeneous LDE with Constant Coefficients-II

**Topic:**

Solution of 2nd order Homogeneous LDE with Constant coefficients

**Learning Outcomes:**

**1.** Formulation of 2nd order homogeneous LDE when roots are given.

**2.** Solution of Initial Value Problems (IVP) and Boundary Value Problems (BVP).

**Formulation of LDE of the form: $ay'' + by' + cy = 0$ when Roots are given:**

Let the two given roots be: $m_1$ and $m_2$.

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

or

Then required 2nd order homogeneous LDE is:

$(D - m_1)(D - m_2)y = 0$ where $D \equiv \frac{d}{dx}$

**Find a LDE of the form: $ay'' + by' + cy = 0$ for which the following functions are solutions:**

**Problem 1.** $(e^{3x}, e^{-2}\ )$

**Solution:** Comparing with: $(e^{m_1x}, e^{m_2x})$

We have: $m_1 = 3, m_2 = -2$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$

$\Rightarrow y'' - (3 - 2)y' + (3)(-2)y = 0$

$\Rightarrow y'' - y' - 6y = 0$ **Answer.**

**Problem 2.** $(1, e^{-2x})$

**Solution:** Here $(1, e^{-2x}) = (e^{0x}, e^{-2}\ )$

Comparing with: $(e^{m_1x}, e^{m_2x})$

We have: $m_1 = 0, m_2 = -2$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$

$\Rightarrow y'' - (0 - 2)y' + (0)(-2)y = 0$

$\Rightarrow y'' + 2y' = 0$ **Answer.**

**Problem 3.** $(e^{2x}, xe^{2x})$

**Solution:** Comparing with: $(e^{m_1x}, xe^{m_2x})$

We have: $m_1 = 2, m_2 = 2$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$

$\Rightarrow y'' - (2 + 2)y' + (2)(2)y = 0$

$\Rightarrow y'' - 4y' + 4y = 0$ **Answer.**

**Problem 4.** $\left(e^{-3ix}, e^{3ix}\right)$

**Solution:** Comparing with: $(e^{m_1x}, e^{m_2x})$

We have: $m_1 = -3i, m_2 = 3i$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$

$\Rightarrow y'' - (-3i + 3i)y' + (-3i)(3i)y = 0$

$\Rightarrow y'' + 9y = 0$ **Answer.** $(i^2 = -1)$

**Problem 5.** $\left(e^{(5+3i)x}, e^{(5-3i)x}\right)$

**Solution:** Comparing with: $(e^{m_1 x}, e^{m_2 x})$

We have: $m_1 = 5 + 3i, m_2 = 5 - 3i$

Then required 2nd order homogeneous LDE is:

$y'' -$ (sum of roots) $y' +$ (Product of roots)$y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$

$\Rightarrow y'' - [(5 + 3i) + (5 - 3i)]y' + (5 + 3i)(5 - 3i)y = 0$

$\Rightarrow y'' - 10y' + 34y = 0$ **Answer.** $(i^2 = -1)$

**Polling Question**

If $(e^{-3}\ , e^{2x})$ are the roots, then the corresponding LDE is:

(A) $y'' + y' + 6y = 0$

(B) $y'' + y' - 6y = 0$

(C) $y'' - y' - 6y = 0$

**Problem:** Solve the Initial value problem: $y'' - y = 0,\ \ y(0) = 0, y'(0) = 2$.

**Solution:** The given equation is:

$y'' - y = 0$ (1)

Such that: $y(0) = 0, y'(0) = 2$

**S.F. :** $(D^2 - 1)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 - 1) = 0$ $\Rightarrow D^2 = 1$ $\Rightarrow D = \pm 1$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = -1$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{1x} + c_2 e^{-1x}$

$\Rightarrow y(x) = c_1 e^{x} + c_2 e^{-x}$ (2)

$\Rightarrow y'(x) = c_1 e^{x} - c_2 e^{-x}$ (3)

Using $y(0) = 0$ in equation (2), we get:

$y(0) = c_1 e^{0} + c_2 e^{-0}$ $\Rightarrow 0 = c_1 + c_2$ (4)

Using $y'(0) = 2$ in equation (3), we get:

$y'(0) = c_1 e^{0} - c_2 e^{-0}$ $\Rightarrow 2 = c_1 - c_2$ (5)

Solving equations (4) and (5), we get: $c_1 = 1, c_2 = -1$

Putting these values of $c_1$ and $c_2$ in equation (2), we get:

$y(x) = e^{x} - e^{-x}$ **Answer.**

**Problem:** Solve the Boundary value problem: $y'' - 4y' + 3y = 0$ such that $y(0) = 1, y(1) = 0$.

**Solution:** The given equation is:

$y'' - 4y' + 3y = 0$ (1)

Such that: $y(0) = 1, y(1) = 0$

**S.F. :** $(D^2 - 4D + 3)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^2 - 4D + 3) = 0$ $\Rightarrow (D-1)(D-3) = 0$

$\Rightarrow D = 1, 3$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = 3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y = c_1 e^{1x} + c_2 e^{3x}$

$y(x) = c_1 e^{1x} + c_2 e^{3x}$ (2)

Using $y(0) = 1$, we get:

$y(0) = c_1 e^0 + c_2 e^0$ $\Rightarrow 1 = c_1 + c_2$ (3)

Using $y(1) = 0$, we get:

$y(1) = c_1 e^1 + c_2 e^3$ $\Rightarrow 0 = c_1 e^1 + c_2 e^3$ (4)

Solving equations (3) and (4), we get: $c_1 = \frac{e^2}{e^2-1}$ and $c_2 = \frac{1}{e^2-1}$

Putting these values of $c_1$ and $c_2$ in equation (2), we get:

$y(x) = \frac{e^2}{e^2-1} e^x + \frac{1}{e^2-1} e^{3x}$ **Answer.**

## Polling Question

A Linear differential equation with conditions given as:

$y(a) = 0$ and $y(b) = 1$ (say) is called:

(A) Initial value problem

(B) Boundary value problem

# MTH166

## Lecture-11

## Solution of Higher Order Homogeneous LDE with Constant Coefficients-I

**Topic:**

Solution of Higher order Homogeneous LDE with Constant coefficients

**Learning Outcomes:**

**1.** Solution of 3rd order (Cubic) homogeneous LDE with constant coefficients.

**2.** Solution of 4th order (Biquadratic) homogeneous LDE with constant coefficients.

**Find the general solution of the following differential equations:**

**Problem 1.** $y''' - 9y' = 0$

**Solution:** The given equation is:

$y''' - 9y' = 0$ (1)

**S.F. :** $(D^3 - 9D)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(D^3 - 9D) = 0$ $\Rightarrow D(D^2 - 9) = 0 \Rightarrow D(D-3)(D+3) = 0$

$\Rightarrow D = 0, 3, -3$ (Real and distinct roots)

Let $m_1 = 0, m_2 = 3, m_3 = -3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$ $\Rightarrow y = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-3x}$

$\Rightarrow y = c_1 + c_2 e^{3x} + c_3 e^{-3}$ **Answer.**

**Problem 2.** $3y''' - 2y'' - 3y' + 2y = 0$

**Solution:** The given equation is:

$3y''' - 2y'' - 3y' + 2y = 0$ (1)

**S.F. :** $(3D^3 - 2D^2 - 3D + 2)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :** $(3D^3 - 2D^2 - 3D + 2) = 0 \Rightarrow D^2(3D-2) - 1(3D-2) = 0$

$\Rightarrow (3D-2)(D^2-1) = 0$ $\Rightarrow D = 1, -1, {}^2/_3$ (Real and distinct roots)

Let $m_1 = 1, m_2 = -1, m_3 = {}^2/_3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$ $\Rightarrow y = c_1 e^{1x} + c_2 e^{-1x} + c_3 e^{{}^2/_3 x}$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{{}^2/_3 x}$ **Answer.**

**Problem 3.** $y''' - 2y'' + y' = 0$

**Solution:** The given equation is:

$y''' - 2y'' + y' = 0 \qquad (1)$

**S.F. :** $(D^3 - 2D^2 + D)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(D^3 - 2D^2 + D) = 0 \Rightarrow D(D^2 - 2D + 1) = 0$

$\Rightarrow D(D-1)^2 = 0 \;\; \Rightarrow D = 0,1,1$ (Two equal and one distinct real roots)

Let $m_1 = 0, m_2 = 1, m_3 = 1$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + (c_2 + c_3 x)e^{m_2 x} \qquad \Rightarrow y = c_1 e^{0x} + (c_2 + c_3 x)e^{1x}$

$\Rightarrow y = c_1 + (c_2 + c_3 x)e^x$ **Answer.**

**Polling Question**

The roots of the equation: $y''' - 16y' = 0$ are:

(A)4,4

(B)4,-4

(C)0,4,-4

(D)0,16

**Problem 4.** $27y''' - 27y'' + 9y' - y = 0$

**Solution:** The given equation is:

$27y''' - 27y'' + 9y' - y = 0 \qquad (1)$

**S.F. :** $(27D^3 - 27D^2 + 9D - 1)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(27D^3 - 27D^2 + 9D - 1) = 0 \qquad [a^3 - b^3 - 3ab(a-b) = (a-b)^3]$

$\Rightarrow (3D-1)^3 = 0 \;\; \Rightarrow D = {}^1/_3, {}^1/_3, {}^1/_3$ (All real and equal roots)

Let $m_1 = {}^1/_3, m_2 = {}^1/_3, m_3 = {}^1/_3$

$\therefore$ General Solution of equation (1) is given by:

$y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x}$

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2)e^{{}^1/_3 x}$ **Answer.**

**Problem 5.** $y''' - 2y'' + 4y' - 8y = 0$

**Solution:** The given equation is:

$y''' - 2y'' + 4y' - 8y = 0 \qquad (1)$

**S.F. :** $(D^3 - 2D^2 + 4D - 8)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(D^3 - 2D^2 + 4D - 8) = 0 \Rightarrow D^2(D-2) + 4(D-2) = 0$

$\Rightarrow (D-2)(D^2+4) = 0 \;\; \Rightarrow D = 2, \pm 2i$ (one real and two complex roots)

Let $m_1 = 2, m_2 = 0 + 2i, m_3 = 0 - 2i$ [Complex roots:$(\alpha \pm i\beta)$]

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + e^{\alpha x}(c_2 \cos \beta x + c_3 \sin \beta x)$

$\Rightarrow y = c_1 e^{2x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

$\Rightarrow y = c_1 e^{2x} + (c_2 \cos 2x + c_3 \sin 2x)$ **Answer.**

**Problem 6.** $y^{IV} - 13y'' + 36y = 0$

**Solution:** The given equation is:

$y^{IV} - 13y'' + 36y = 0$ (1)

**S.F. :** $(D^4 - 13D^2 + 36)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(D^4 - 13D^2 + 36) = 0$

$\Rightarrow (D^2 - 4)(D^2 - 9) = 0 \;\Rightarrow D = 2, -2, 3, -3$ (real and distinct roots)

Let $m_1 = 2, m_2 = -2, m_3 = 3, m_4 = -3$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} + c_4 e^{-3x}$ **Answer.**

**Problem 7.** $y^{IV} + 8y'' - 9y = 0$

**Solution:** The given equation is:

$y^{IV} + 8y'' - 9y = 0$ (1)

**S.F. :** $(D^4 + 8D^2 - 9)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(D^4 + 8D^2 - 9) = 0$

$\Rightarrow (D^2 - 1)(D^2 + 9) = 0 \;\Rightarrow D = 1, -1, 3i, -3i$ (Mix of real and complex roots )

Let $m_1 = 1, m_2 = -1, m_3 = 0 + 3i, m_4 = 0 - 3i$

$\therefore$ General Solution of equation (1) is given by:

$y = c_1 e^{1x} + c_2 e^{-1x} + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} + (c_3 \cos 3x + c_4 \sin 3x)$ **Answer.**

**Problem 8.** $4y^{IV} + 101y'' + 25y = 0$

**Solution:** The given equation is:

$4y^{IV} + 101y'' + 25y = 0$ (1)

**S.F. :** $(4D^4 + 101D^2 + 25)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(4D^4 + 101D^2 + 25) = 0$

$\Rightarrow (4D^2 + 1)(D^2 + 25) = 0 \;\Rightarrow D = \frac{1}{2}i, -\frac{1}{2}i, 5i, -5i$ (Two sets of complex roots)

Let $m_1 = 0 + \frac{1}{2}i, m_2 = 0 - \frac{1}{2}i, m_3 = 0 + 5i, m_4 = 0 - 5i$ $[(\alpha \pm i\beta), (\gamma \pm i\delta)]$

$\therefore$ General Solution of equation (1) is given by:

$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + e^{\gamma x}(c_1 \cos \delta x + c_2 \sin \delta x)$

$\Rightarrow \; y = e^{0x}\left(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x\right) + e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$

$\Rightarrow y = \left(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x\right) + (c_1 \cos 5x + c_2 \sin 5x)$ **Answer.**

**Polling Question**

The roots of the equation: $y^{iv} - 25y'' + 144y = 0$ are:

(A) 3,3,4,4

(B) 3,-3,4,-4

(C) 0,3,4,6

(D) 5,5,12,12

**Problem 9.** $y^{IV} + 50y'' + 625y = 0$

**Solution:** The given equation is:

$y^{IV} + 50y'' + 625y = 0$ (1)

**S.F. :** $(D^4 + 50D^2 + 625)y = 0$ where $D \equiv \frac{d}{dx}$

**A.E. :**$(D^4 + 50D^2 + 625) = 0$ $\Rightarrow (D^2 + 25)^2 = 0$

$\Rightarrow (D^2 + 25)(D^2 + 25) = 0$ $\Rightarrow D = 5i, -5i, 5i, -5i$ (Repeated complex roots)

Let $m_1 = 0 + 5i, m_2 = 0 - 5i, m_3 = 0 + 5i, m_4 = 0 - 5i$ $[(\alpha \pm i\beta), (\alpha \pm i\beta)]$

$\therefore$ General Solution of equation (1) is given by:

$y = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x)]$

$\Rightarrow$ $y = e^{0x}[(c_1 + c_2x)\cos 5x + (c_3+c_4x)\sin 5x)]$

$\Rightarrow y = [(c_1+c_2x)\cos 5x + (c_3+c_4x)\sin 5x)]$**Answer.**

# MTH166

## Lecture-12

## Solution of Higher Order Homogeneous LDE with Constant Coefficients-II

**Topic:**

Solution of Higher order Homogeneous LDE with Constant coefficients

**Learning Outcomes:**

**1.** Formulation of 3rd order and 4th order homogeneous LDE when roots are given.

**Formulation of LDE: $ay''' + by'' + cy' + dy = 0$ when Roots are given:**

Let the three given roots be: $m_1, m_2$ and $m_3$.

Then required 3rd order homogeneous LDE is:

$y'''$ − (sum of roots taken one at a time ) $y''$ + (sum of roots taken one at a time ) $y'$ − (Product of roots)$y = 0$

i.e. $y''' - (m_1 + m_2 + m_3)y'' + (m_1m_2 + m_2m_3 + m_3m_1)y' - (m_1m_2m_3)y = 0$

or

Then required 3rd order homogeneous LDE is:

$(D - m_1)(D - m_2)(D - m_3)y = 0$ where $D \equiv \frac{d}{dx}$

**Formulation of LDE: $ay^{iv} + by''' + cy'' + dy' + ey = 0$ when Roots are given:**

Let the four given roots be: $m_1, m_2, m_3$ and $m_4$.

Then required 4th order homogeneous LDE is:

$(D - m_1)(D - m_2)(D - m_3)(D - m_4)y = 0$ where $D \equiv \frac{d}{dx}$

**Find a homogeneous LDE with constant coefficients of lowest order which has the following particular solution:**

**Q 1.** $5 + e^x + 2e^{3x}$

**Sol.** Here: $5 + e^x + 2e^{3x} = 5e^{0x} + e^{1x} + 2e^{3x}$

So, $m_1 = 0, m_2 = 1, m_3 = 3$

Then required 3rd order homogeneous LDE is:

$y''' - (m_1 + m_2 + m_3)y'' + (m_1m_2 + m_2m_3 + m_3m_1)y' - (m_1m_2m_3)y = 0$

$\Rightarrow y''' - (0 + 1 + 3)y'' + (0 + 3 + 0)y' - (0)y = 0$

$\Rightarrow y''' - 4y'' + 3y' = 0$ **Answer.**

**Q 2.** $xe^{-x} + e^{2x}$

**Sol.** Here: $xe^{-x} + e^{2x} = (0 + 1x)e^{-x} + e^{2x}$ $[(c_1 + c_2x)e^{m_1x} + c_3e^{m_2x}]$

So, $m_1 = -1, m_2 = -1, m_3 = 2$

Then required 3rd order homogeneous LDE is:

$y''' - (m_1 + m_2 + m_3)y'' + (m_1m_2 + m_2m_3 + m_3m_1)y' - (m_1m_2m_3)y = 0$

$\Rightarrow y''' - (-1 - 1 + 2)y'' + (1 - 2 - 2)y' - (2)y = 0$

$\Rightarrow y''' - 3y' - 2y = 0$ **Answer.**

**Q 3.** $e^{-x} + \cos 5x + 3\sin 5x$

**Sol.** Here: $e^{-x} + \cos 5x + 3\sin 5x = c_1 e^{m_1 x} + e^{\alpha x}[c_2 \cos \beta x + c_3 \sin \beta x]$

So, $m_1 = -1, m_2 = 0 + 5i, m_3 = 0 - 5i$

Then required 3rd order homogeneous LDE is:

$y''' - (m_1 + m_2 + m_3)y'' + (m_1 m_2 + m_2 m_3 + m_3 m_1)y' - (m_1 m_2 m_3)y = 0$

$\Rightarrow y''' - (-1 + 5i - 5i)y'' + (-5i + 25 + 5i)y' - (25i^2)y = 0$

$\Rightarrow y''' + y'' + 25y' + 25y = 0$ **Answer.**

**Q 4.** $1 + x + e^x - 3e^{3x}$

**Sol.** Here: $1 + x + e^x - 3e^{3x} = (1 + x)e^{0x} + e^x - 3e^{3x}$

So, $m_1 = 0, m_2 = 0, m_3 = 1, m_4 = 3$

Then required 4th order homogeneous LDE is:

$(D - m_1)(D - m_2)(D - m_3)(D - m_4)y = 0$ where $D \equiv \frac{d}{dx}$

$\Rightarrow (D - 0)(D - 0)(D - 1)(D - 3)y = 0$

$\Rightarrow D^2(D^2 - 4D + 3)y = 0$

$\Rightarrow (D^4 - 4D^3 + 3D^2)y = 0$

$\Rightarrow y^{iv} - 4y''' + 3y'' = 0$ **Answer.**

**Polling Questions:**

**Q1.** If $e^x, e^{4x}$ are solutions of differential equation $y'' + a(x)y' + b(x)y = 0$, then the values of $a(x)$ and $b(x)$ are:

**(A)** $a(x) = -5, b(x) = 4$ **(B)** $a(x) = 5, b(x) = 4$

**(C)** $a(x) = -5, b(x) = -4$ **(D)** $a(x) = 5, b(x) = -4$

**Q2.** The intervals on which the differential equation $y' = 3\frac{y}{x}$ is normal are:

**(A)** $(-\infty, 0), (0, \infty)$ **(B)** $(-\infty, \infty)$

**(C)** $(-\infty, 1), (1, \infty)$ **(D)** None of these.

**Q3.** If $y = e^{at}$ is solution of $y'' - 5y' + 4y = 0$, then possible value of $a$ is:

**(A)** $a = 2$ **(B)** $a = 3$

**(C)** $a = 4$ **(D)** $a = 5$

**Q4.** The general solution of $y'' - 9y = 0$ is:

**(A)** $Ae^{-3} \ + Be^{3x}$ **(B)** $Ae^{3x} + Be^{3x}$

**(C)** $Ae^{-3x} + Be^{-3}$ **(D)** None of these

**Q5.** The general solution of $y'' + 4y = 0$ is:

**(A)** $(Acos2x + Bsin\ 2x)$ **(B)** $Ae^{2x} + Be^{-2x}$

**(C)** $(A + Bx)e^{-2x}$ **(D)** None of these

**Q6.** The Wronskian of functions: $(1, \sin x, \cos x)$is:

**(A)** 0 **(B)** 1

**(C)** $-1$ **(D)** None of these

**Q7.** The general solution of $y'' - 10y' + 25y = 0$ is:

**(A)** $Ae^{4x} + Be^{5x}$ **(B)** $Ae^{5x} + Be^{5x}$

**(C)** $Ae^{4x} + Be^{7x}$ **(D)** $Ae^{5x} + Bxe^{5x}$

**Q15.** The second order linear homogeneous differential equation with variable coefficients is

(a) $y'' + 9y' + y = \log(x^2 - 9)$ (b) $(1+x^2)y'' + 2xy' + y = 0$

(c) $(1+x^2)y'' + 2xy' + y = x^2$ (d) $(1+x^2)y'' + 2yy' + 3y = 0$

**Q20.** The differential equation $yy'' + 2y' + y = sinx$ is...

(a)linear (b) homogeneous (c)non-linear (d)of order 3

**Q21.** The complementary function differential of the equation $(D^2 - 2D + 5)^2 y = 0$ *is*

(a)$e^x(Acos2x + Bsin2x)$ (b)$e^x\{(A + Bx)cos2x + (C + Dx)sin2x\}$

(c)$e^x(Acos2x + Bsin2x)$+$e^x(Ccos2x + Dsin2x)$ (d) None of these

# MTH166

## Lecture-13

## Solution of Non-Homogeneous LDE with Constant Coefficients Using Operator Method-I

**Topic:**

Solution of Non-Homogeneous LDE with Constant coefficients Using Operator Method-I

**Learning Outcomes:**

Solving Non-Homogeneous LDE Using operator method when:

**1.** Function is of the form: $r(x) = e^{\alpha x}$

**2.** Function is of the form: $r(x) = \cos \alpha x$ or $r(x) = \sin \alpha x$

**Solution of Non-homogeneous LDE with constant coefficients Using Operator Method:**

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x) \qquad (1)$$

or

$$ay'' + by' + cy = r(x) \qquad (1)$$

Let $\frac{d}{dx} \equiv \boldsymbol{D}$ be Differential operator ( An algebraic operator like $+, -, \times, \div$)

Equation (1) becomes:

$aD^2y + bDy + cy = r(x)$

Symbolic Form (S.F.): $(aD^2 + bD + c)y = r(x)$

$\Rightarrow f(D)y = r(x)$

To find Complimentary Function (C.F.):

A.E.: $f(D) = 0$

$\Rightarrow\ (aD^2 + bD + c) = 0$

$\Rightarrow D = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = m_1, m_2$ (Say) (Suppose here $m_1 \neq m_2$ are real roots)

Complementary function C.F. is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

To find Particular Integral (P.I.):

P.I.: $y_p = \frac{1}{f(D)} r(x)$ ( There are different methods to evaluate it)

General Solution: $y =$ C.F. +P.I.

i.e. $\boldsymbol{y = y_c + y_p}$

**Operator Method to find Particular Integral (P.I.):**

**Case 1:** If $r(x) = e^{\alpha x}$, then P.I. $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x}$ , (i.e. Put $D = \alpha$), provided $f(\alpha) \neq 0$

If $f(\alpha) = 0$, then:

$y_p = x \frac{1}{f'(D)} e^{\alpha x} = x \frac{1}{f'(\alpha)} e^{\alpha x}$, provided $f'(\alpha) \neq 0$

If $f'(\alpha) = 0$ , then:

$y_p = x^2 \frac{1}{f''(D)} e^{\alpha x} = x^2 \frac{1}{f''(\alpha)} e^{\alpha x}$, provided $f''(\alpha) \neq 0$

and so on…

**Problem 1.** Find the general solution of: $y'' + 5y' + 4y = 18e^{2x}$

**Solution:** The given equation is:

$y'' + 5y' + 4y = 18e^{2x}$ (1)

S.F. : $(D^2 + 5D + 4)y = 18e^{2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 5D + 4)$ and $r(x) = 18e^{2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 5D + 4) = 0$ $\Rightarrow (D + 1)(D + 4) = 0$

$\Rightarrow D = -1, -4$ (real and unequal roots)

Let $m_1 = -1$ and $m_2 = -4$

∴ Complimentary function is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{-1x} + c_2 e^{-4x}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+5D+4)} (18e^{2x})$

$\Rightarrow y_p = 18\left[\frac{1}{(D^2+5D+4)} e^{2x}\right]$

$\Rightarrow y_p = 18\left[\frac{1}{((2)^2+5(2)+4)} e^{2x}\right]$ $(Put\ D = 2)$

$\Rightarrow y_p = 18\left[\frac{1}{18} e^{2x}\right]$ $\Rightarrow y_p = e^{2x}$

∴ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{-1} + c_2 e^{-4x}) + e^{2x}$ **Answer.**

**Polling Question:**

**Q1.** The general solution of $y'' - 9y = 0$ is:

**(A)** $Ae^{-3x} + Be^{3x}$

**(B)** $Ae^{3x} + Be^{3x}$

**(C)** $Ae^{-3} + Be^{-3}$

**(D)** None of these

**Problem 2.** Find the general solution of: $y'' + y' - 6y = e^{2x}$

**Solution:** The given equation is:

$y'' + y' - 6y = e^{2x}$ (1)

S.F. : $(D^2 + D - 6)y = e^{2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + D - 6)$ and $r(x) = e^{2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + D - 6) = 0$ $\Rightarrow (D-2)(D+3) = 0$

$\Rightarrow D = 2, -3$ (real and unequal roots)

Let $m_1 = 2$ and $m_2 = -3$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-3x}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+D-6)}(e^{2x})$

$\Rightarrow y_p = \left[\frac{1}{((2)^2+(2)-6)} e^{2x}\right]$ $(Put\ D = 2)$ $\Rightarrow y_p = \left[\frac{1}{(6-6)} e^{2x}\right]$ (Case of failure)

$\therefore y_p = x \frac{1}{f'(D)} r(x) = x \frac{1}{(2D+1)}(e^{2x})$

$\Rightarrow y_p = x\left[\frac{1}{(2(2)+1)} e^{2x}\right]$ $(Put\ D = 2)$ $\Rightarrow y_p = \frac{x}{5} e^{2x}$

General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{2x} + c_2 e^{-3x}) + \frac{x}{5} e^{2x}$ **Answer.**

**Problem 3.** Find the general solution of: $y'' - 6y' + 9y = 14e^{3x}$

**Solution:** The given equation is:

$y'' - 6y' + 9y = 14e^{3x}$ (1)

S.F. : $(D^2 - 6D + 9)y = 14e^{3x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 6D + 9)$ and $r(x) = 14e^{3x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 6D + 9) = 0$ $\Rightarrow (D-3)(D-3) = 0$

$\Rightarrow D = 3,3$ (real and equal roots)

Let $m_1 = 3$ and $m_2 = 3$

$\therefore$ Complimentary function is given by:

$y_c = (c_1 + c_2 x) e^{m_2 x}$

$\Rightarrow y_c = (c_1 + c_2 x) e^{3x}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-6D+9)}(14e^{3x})$

$\Rightarrow y_p = 14\left[\frac{1}{((3)^2-6(3)+9)} e^{2x}\right]$ $(Put\ D = 3)$ $\Rightarrow y_p = 14\left[\frac{1}{(18-18)} e^{3x}\right]$ (Case of failure)

$\therefore y_p = 14x \frac{1}{f'(D)} r(x) = 14x \frac{1}{(2D-6)}(e^{3x}) = 14x\left[\frac{1}{(6-6)} e^{3x}\right]$ $(Put\ D = 3)$ (Case of failure)

$\therefore y_p = 14x^2\left[\frac{1}{f''(D)} r(x)\right] = 14x^2\left[\frac{1}{2} e^{3x}\right] = 7x^2 e^{3x}$

General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 + c_2 x)e^{3x} + 7x^2 e^{3x}$ **Answer.**

**Polling Question:**

The general solution of $y'' + 4y = 0$ is:

**(A)** $(Acos2x + Bsin\ 2x)$

**(B)** $Ae^{2x} + Be^{-2x}$

**(C)** $(A + Bx)e^{-2x}$

**(D)** None of these

**Operator Method to find Particular Integral (P.I.):**

**Case 2:** If $r(x) = \cos \alpha x$ or $r(x) = \sin \alpha x$

Then P.I is: $y_p = \frac{1}{f(D)} r(x) = \frac{1}{f(D)} \cos \alpha x$

$y_p = \frac{1}{f(D^2=-\alpha^2)} \cos \alpha x$, provided $f(D^2 = -\alpha^2) \neq 0$

If $f(D^2 = -\alpha^2) = 0$, then:

$y_p = x \frac{1}{f'(D)} \cos \alpha x = x \frac{1}{f'(D^2=-\alpha^2)} \cos \alpha x$, provided $f'(D^2 = -\alpha^2) \neq 0$

**Note:** **1.** $D[r(x)] = \frac{d}{dx}[r(x)]$

**2.** $\frac{1}{D}[r(x)] = \int r(x)dx$

**3.** $\frac{1}{D-a}[r(x)] = \frac{1}{D-a} \times \frac{D+a}{D+a}[r(x)]$ (Rationalize to create $D^2$in denominator)

**Problem 1.** Find the general solution of: $y'' - 16y = \cos 2x$

**Solution:** The given equation is:

$y'' - 16y = \cos 2x$ (1)

S.F. : $(D^2 - 16)y = \cos 2x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 16)$ and $r(x) = \cos 2x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 16) = 0$ $\Rightarrow (D - 4)(D + 4) = 0$

$\Rightarrow D = 4, -4$ (real and unequal roots)

Let $m_1 = 4$ and $m_2 = -4$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{4x} + c_2 e^{-4x}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-16)}(\cos 2x\ )$

$\Rightarrow y_p = \left[\frac{1}{(-(2)^2-16)} \cos 2x\right]$ $\quad (Put\ D^2 = -(2)^2)$

$\Rightarrow y_p = \left[\frac{1}{(-4-16)} \cos 2x\right]$

$\Rightarrow y_p = -\frac{1}{20} \cos 2x$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{4x} + c_2 e^{-4x}) - \frac{1}{20} \cos 2x$ **Answer.**

**Problem 2.** Find the general solution of: $y'' + 9y = \sin 3x$

**Solution:** The given equation is:

$y'' + 9y = \sin 3x$ (1)

S.F. : $(D^2 + 9)y = \sin 3x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 9)$ and $r(x) = \sin 3x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 9) = 0$ $\Rightarrow D^2 = -9$

$\Rightarrow D = 3i, -3i$ (complex roots)

Let $m_1 = 0 + 3i$ and $m_2 = 0 - 3i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$\Rightarrow y_c = (c_1 \cos 3x + c_2 \sin 3x)$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+9)}(\sin 3x\ )$

$\Rightarrow y_p = \left[\frac{1}{(-(3)^2+9)} \sin 3x\right]$ $\quad (Put\ D^2 = -(3)^2)$

$\Rightarrow y_p = \left[\frac{1}{(-9+9)} \sin 3x\right]$ (Case of Failure)

$\therefore y_p = x\left[\frac{1}{f'(D)} r(x)\right] = x\left[\frac{1}{2D}(\sin 3x\ )\right] = \frac{x}{2}\int \sin 3x\, dx = \frac{x}{2}\left[\frac{-\cos 3x}{3}\right]$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos 3x + c_2 \sin 3x) - \frac{x}{6} \cos 3x$ **Answer.**

**Problem 3.** Find the general solution of: $2y'' - 5y' + 3y = \sin x$

**Solution:** The given equation is:

$2y'' - 5y' + 3y = \sin x$ (1)

S.F. : $(2D^2 - 5D + 3)y = \sin x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (2D^2 - 5D + 3)$ and $r(x) = \sin x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (2D^2 - 5D + 3) = 0$ $\Rightarrow (2D - 3)(D - 1) = 0$

$\Rightarrow D = 1, {}^3/_2$ (real and unequal roots)

Let $m_1 = 1$ and $m_2 = {}^3/_2$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow y_c = c_1 e^{1x} + c_2 e^{{}^3/_2 x}$

<u>To find Particular Integral (P.I.):</u>

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(2D^2-5D+3)} (\sin x\ )$

$\Rightarrow y_p = \left[\frac{1}{(2(-(1)^2)-5D+3)} \sin x\right] \qquad (Put\ D^2 = -(1)^2)$

$\Rightarrow y_p = \left[\frac{1}{(1-5D)} \sin x\right] = \left[\frac{1}{(1-5D)} \times \frac{(1+5\ \ )}{(1+5D)} \sin x\right] = \left[\frac{(1+5D)}{(1-25D^2)} \sin x\right]$

$\Rightarrow y_p = \frac{1}{26}[(1+5D)\sin x] = \frac{1}{26}\left[\sin x + 5\frac{d}{dx}(\sin x)\right] = \frac{1}{26}(\sin x + 5\cos x)$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = \left(c_1 e^{1x} + c_2 e^{3/2 x}\right) + \frac{1}{26}(\sin x + 5\cos x)$ **Answer.**

# MTH166

## Lecture-14

## Solution of Non-Homogeneous LDE with Constant Coefficients Using Operator Method-II

**Topic:**

Solution of Non-Homogeneous LDE with Constant coefficients Using Operator Method-II

**Learning Outcomes:**

Solving Non-Homogeneous LDE Using operator method when:

**1.** Function is of the form: $r(x) = x^m$

**2.** Function is of the form: $r(x) = e^{\alpha x} g(x)$

**Operator Method to find Particular Integral (P.I.):**

**Case 3:** If $r(x) = x^m$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} x^m = \frac{1}{[1 \pm h(D)]} x^m$ ( By taking least degree term common from $f(D)$)

$\Rightarrow y_p = [1 \pm h(D)]^{-1} x^m$ and we expand this expression by Binomial expansion.

**Note:**

**1.** $[1 + h(D)]^{-1} = 1 - h(D) + (h(D))^2 - (h(D))^3 + ---$

**2.** $[1 - h(D)]^{-1} = 1 + h(D) + (h(D))^2 + (h(D))^3 + ---$

**Problem 1.** Find the general solution of: $y'' + 25y = 4x^2$

**Solution:** The given equation is:

$y'' + 25y = 4x^2$ (1)

S.F. : $(D^2 + 25)y = 4x^2$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 25)$ and $r(x) = 4x^2$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 25) = 0$ $\Rightarrow D^2 = -25$

$\Rightarrow D = 5i, -5i$ (Complex roots)

Let $m_1 = 5i$ and $m_2 = -5i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$

$\Rightarrow y_c = (c_1 \cos 5x + c_2 \sin 5x)$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+25)}(4x^2)$

$\Rightarrow y_p = 4\left[\frac{1}{25\left(1+\frac{D^2}{25}\right)} x^2\right] = \frac{4}{25}\left[\left(1 + \frac{D^2}{25}\right)^{-1} x^2\right]$

$\Rightarrow y_p = \frac{4}{25}\left[\left(1 - \left(\frac{D^2}{25}\right)^1 + \left(\frac{D^2}{25}\right)^2 ----\right) x^2\right] = \frac{4}{25}\left[x^2 - \frac{2}{25} + 0\right]$ $(D^2(x^2) = 2)$

$\Rightarrow y_p = \frac{4}{625}(25x^2 - 2)$

$\therefore$ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos 5x + c_2 \sin 5x) + \frac{4}{625}(25x^2 - 2)$ **Answer.**

**Polling Question:**

The Particular Integral of $y'' + 5y' + 4y = 18e^{2x}$ is:

**(A)** $e^{2x}$

**(B)** $xe^{2x}$

**(C)** $e^{3x}$

**(D)** None of these

**Problem 2.** Find the general solution of: $y'' - 6y' + 9y = 4x^2 - 1$

**Solution:** The given equation is:

$y'' - 6y' + 9y = 4x^2 - 1$ (1)

S.F. : $(D^2 - 6D + 9)y = 4x^2 - 1$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 6D + 9)$ and $r(x) = 4x^2 - 1$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 6D + 9) = 0$ $\Rightarrow (D - 3)(D - 3) = 0$

$\Rightarrow D = 3, 3$ (real and equal roots)

Let $m_1 = 3$ and $m_2 = 3$

∴ Complimentary function is given by:

$y_c = (c_1 + c_2 x)e^{m_1 x}$

$\Rightarrow y_c = (c_1 + c_2 x)e^{3x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2 - 6D + 9)}(4x^2 - 1)$$

$$\Rightarrow y_p = \left[\frac{1}{9\left(1 - \frac{(6D - D^2)}{9}\right)}(4x^2 - 1)\right] = \frac{1}{9}\left[\left(1 - \frac{(6D - D^2)}{9}\right)^{-1}(4x^2 - 1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[\left(1 + \left(\frac{(6D - D^2)}{9}\right)^1 + \left(\frac{(6D - D^2)}{9}\right)^2 + - - - -\right)(4x^2 - 1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2 - 1) + \left(\frac{(6D - D^2)}{9}\right)(4x^2 - 1) + \frac{36}{81}D^2(4x^2 - 1) + 0\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2 - 1) + \frac{6}{9}(8x) - \frac{1}{9}(8) + \frac{36}{81}(8)\right]$$

∴ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 + c_2 x)e^{3x} + \frac{1}{9}\left[(4x^2 - 1) + \frac{6}{9}(8x) - \frac{1}{9}(8) + \frac{36}{81}(8)\right]$$ **Answer.**

**Operator Method to find Particular Integral (P.I.):**

**Case 4:** If $r(x) = e^{\alpha x} g(x)$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} e^{\alpha x} g(x)$

$\Rightarrow y_p = e^{\alpha x}\left[\frac{1}{f(D + \alpha)} g(x)\right]$

Either $g(x) = x^m$ or $g(x) = \cos \alpha x$

Then we proceed with the rules that we already know

**Problem 1.** Find the general solution of: $y'' - 4y' + 5y = 24e^{2x} \sin x$

**Solution:** The given equation is:

$y'' - 4y' + 5y = 24e^{2x} \sin x$ (1)

S.F. : $(D^2 - 4D + 5)y = 24e^{3x} \sin x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 4D + 5)$ and $r(x) = 24e^{3x} \sin x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 4D + 5) = 0$

$\Rightarrow D = 2 \pm i$ (Complex roots)

Let $m_1 = 2 + i$ and $m_2 = 2 - i$

∴ Complimentary function is given by:

$y_c = e^{2x}(c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-4D+5)}(24e^{2x} \sin x\ )$

$\Rightarrow y_p = 24e^{2x}\left[\frac{1}{((D+2)^2-4(D+2)+5)} \sin x\right] \Rightarrow y_p = 24e^{2x}\left[\frac{1}{(D^2+1)} \sin x\right]$

$\Rightarrow y_p = 24e^{2x}\left[\frac{1}{(-1+1)} \sin x\right]$ (Put $D^2 = -(1)^2$) (Case of failure)

$\therefore y_p = 24e^{2x} x\left[\frac{1}{f'(D)} \sin x\right] = 24e^{2x} x\left[\frac{1}{2D} \sin x\right] = 12e^{2x} x \int \sin x dx = -12xe^{2x} \cos x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = e^{2x}(c_1 \cos x + c_2 \sin x) - 12xe^{2x} \cos x$ **Answer.**

**Polling Question:**

The Particular Integral of $y'' + 9y = \sin 3x$ is:

**(A)** $-\frac{x}{6} \sin 3x$

**(B)**$-\frac{x}{6} \cos 3x$

**(C)** $\frac{x}{6} \cos 3x$

**(D)** None of these

**Problem 2.** Find the general solution of: $y'' - y' - 6y = xe^{-2x}$

**Solution:** The given equation is:

$y'' - y' - 6y = xe^{-2x}$ (1)

S.F. : $(D^2 - D - 6)y = xe^{-2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - D - 6)$ and $r(x) = xe^{-2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - D - 6) = 0$ $\Rightarrow (D-3)(D+2) = 0$

$\Rightarrow D = 3, -2$ (real and distinct roots)

Let $m_1 = 3$ and $m_2 = -2$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{3x} + c_2 e^{-2x}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-D-6)}(xe^{-2x}\ )$

$\Rightarrow y_p = e^{-2}\ \left[\frac{1}{((D-2)^2-(D-2)-6)} x\right] \Rightarrow y_p = e^{-2x}\left[\frac{1}{(D^2-5\ \ )} x\right]$

$\Rightarrow y_p = e^{-2}\ \left[\frac{1}{-5\ \left(1-\frac{D^2}{5D}\right)} x\right] = -\frac{e^{-2}}{5}\left[\frac{1}{D}\left(1-\frac{D}{5}\right)^{-1} x\right] = -\frac{e^{-2x}}{5}\left[\frac{1}{D}\left(1+\frac{D}{5}+\left(\frac{D}{5}\right)^2+\cdots\right) x\right]$

$\therefore y_p = -\frac{e^{-2}}{5}\int\left(x+\frac{1}{5}\right) dx = -\frac{e^{-2x}}{5}\left(\frac{x^2}{2}+\frac{x}{5}\right) = -\frac{e^{-2}}{50}(5x^2+2x)$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 e^{3x} + c_2 e^{-2}\ \ - \frac{e^{-2x}}{50}(5x^2+2x)$ **Answer.**

# MTH166

## Lecture-15

## Method of Variation of Parameters

**Topic:**

Solution of Non-Homogeneous LDE with Constant coefficients.

**Learning Outcomes:**

To use Method of Variation of Parameters to solve Non-Homogeneous LDE with constant coefficients.

**Method of Variation of Parameters:**

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$ay'' + by' + cy = r(x)$ (1)

Let two solutions of C.F. $(aD^2 + bD + c) = 0$ be $y_1$ and $y_2$

i.e. $y_c = c_1y_1 + c_2y_2$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

P.I. $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

General solution is: $y = C.F. + P.I.$

i.e. $y = y_c + y_p$

**Problem 1.** Find the general solution of: $y'' + y = \sec x$

**Solution:** The given equation is:

$y'' + y = \sec x$ (1)

S.F. : $(D^2 + 1)y = \sec x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \sec x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 1) = 0$ $\Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing $y_c$ with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2 x + sin^2 x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\sec x)}{1} dx + \sin x \int \frac{\cos x(\sec x)}{1} dx$

$\Rightarrow y_p = -\cos x \int \tan x\, dx + \sin x \int dx$ $\left(sec\ x = \frac{1}{\cos x}\right)$

$\Rightarrow y_p = -\cos x(-\log|\cos x|) + \sin x(x)$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \cos x(\log|\cos x|) + x \sin x$ **Answer.**

**Polling Question:**

The Wronskian of $y'' + 5y' + 4y = 18e^{2x}$ is:

(a) $3e^{5x}$ (b) $5e^{-5x}$

(c) $3e^{-5x}$ (d) $3e^{5x}$

**Problem 2.** Find the general solution of: $y'' + y = \operatorname{cosec} x$

**Solution:** The given equation is:

$y'' + y = \operatorname{cosec} x$ (1)

S.F. : $(D^2 + 1)y = \operatorname{cosec} x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \operatorname{cosec} x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 1) = 0$ $\Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

Comparing $y_c$ with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x$, $y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2x + sin^2x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\cos \quad)}{1} dx + \sin x \int \frac{\cos x(\text{cose} \quad)}{1} dx$

$\Rightarrow y_p = -\cos x \int dx + \sin x \int \cot x\, dx$ $\qquad \left(cosec\ x = \frac{1}{\sin x}\right)$

$\Rightarrow y_p = -\cos x (x) + \sin x\ (\log|\sin x|)$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \sin x(\log|\sin x|) - x \cos x$ **Answer.**

**<u>Problem 3.</u>** Find the general solution of: $y'' + y = \tan x$

**<u>Solution:</u>** The given equation is:

$y'' + y = \tan x \qquad (1)$

<u>S.F. :</u> $(D^2 + 1)y = \tan x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \tan x$

<u>To find Complimentary Function (C.F.):</u>

<u>A.E. :</u> $f(D) = 0 \quad \Rightarrow (D^2 + 1) = 0 \quad \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

<u>To find Particular Integral (P.I.):</u>

Comparing $y_c$ with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x,\ y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2x + sin^2x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\tan x)}{1} dx + \sin x \int \frac{\cos x(\tan x)}{1} dx$

$\Rightarrow y_p = -\cos x \int \frac{sin^2 x}{\cos x} dx + \sin x \int \sin x\, dx$ $\qquad \left(tan\ x = \frac{\sin x}{\cos x}\right)$

$\Rightarrow y_p = -\cos x \int \frac{1 - cos^2 x}{\cos x} dx + \sin x(-\cos x)$

$\Rightarrow y_p = -\cos x \int (\sec x - \cos x) dx - \sin x \cos x$

$\Rightarrow y_p = -\cos x \log|\sec x + \tan x| + \cos x \sin x - \sin x \cos x$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \cos x(\log|\sec x + \tan x|)$ **Answer.**

**<u>Problem 4.</u>** Find the general solution of: $y'' + y = \cot x$

**<u>Solution:</u>** The given equation is:

$y'' + y = \cot x \qquad (1)$

<u>S.F. :</u> $(D^2 + 1)y = \cot x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 1)$ and $r(x) = \cot x$

<u>To find Complimentary Function (C.F.):</u>

<u>A.E. :</u> $f(D) = 0 \quad \Rightarrow (D^2 + 1) = 0 \quad \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$ Complimentary function is given by:

$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$

<u>To find Particular Integral (P.I.):</u>

Comparing $y_c$ with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = \cos x,\ y_2 = \sin x$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (cos^2x + sin^2x) = 1$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -\cos x \int \frac{\sin x(\cot x)}{1} dx + \sin x \int \frac{\cos x(\cot x)}{1} dx$

$\Rightarrow y_p = -\cos x \int cos\, x\, dx + \sin x \int \frac{cos^2x}{\sin x} dx \qquad \left(cot\ x = \frac{\cos x}{\sin x}\right)$

$\Rightarrow y_p = -\cos x\,(sin\ x) + \sin x \int \frac{1-sin^2x}{\sin x} dx$

$\Rightarrow y_p = -\cos x\,(sin\ x) + \sin x \int (cosecx - \sin x) dx$

$\Rightarrow y_p = -\cos x(\sin x) + \sin x\,(-\log|cosecx + \cot x|) + \sin x(\cos x)$

$\therefore$ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \sin x(\log|\operatorname{cosec} x + \cot x|)$ **Answer.**

**Polling Question:**

The Wronskian of $y'' + 9y = \cos 3x$ is:

**(A)** 1

**(B)** 3

**(C)** -3

**Problem 5.** Find the general solution of: $y'' - 2y' - 3y = e^x$

**Solution:** The given equation is:

$y'' - 2y' - 3y = e^x \qquad (1)$

S.F. : $(D^2 - 2D - 3)y = e^x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 2D - 3)$ and $r(x) = e^x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 2D - 3) = 0 \quad \Rightarrow (D-3)(D+1) = 0$

$\Rightarrow D = 3, -1$ (real and distinct roots)

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{3x} + c_2 e^{-1x}$

To find Particular Integral (P.I.):

Comparing $y_c$ with $y_c = c_1\, y_1 + c_2\, y_2$

Here $y_1 = e^{3x}$, $y_2 = e^{-x}$

Wronskian, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = (-e^{3x}e^{-x} - 3e^{3x}e^{-x}) = -4e^{2x}$

P.I. is given by: $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$\Rightarrow y_p = -e^{3x} \int \frac{e^{-x}(e^x)}{-4e^{2x}} dx + e^{-x} \int \frac{e^{3x}(e^x)}{-4e^{2x}} dx$

$\Rightarrow y_p = \frac{e^{3x}}{4} \int e^{-2x} dx - \frac{e^{-x}}{4} \int e^{2x}\, dx$

$\Rightarrow y_p = \frac{e^{3x}}{4}\left(\frac{e^{-2x}}{-2}\right) - \frac{e^{-x}}{4}\left(\frac{e^{2x}}{2}\right) = -\frac{1}{4}e^x$

$\therefore$ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{3x} + c_2 e^{-1x}) - \frac{1}{4}e^x$ **Answer.**

# MTH166

## Lecture-16

## Method of Undeterminant Coefficients

**Topic:**

Solution of Non-Homogeneous LDE with Constant coefficients.

**Learning Outcomes:**

To use Method of Undeterminant coefficients to solve Non-Homogeneous LDE with constant coefficients.

**Method of Undeterminant Coefficients:**

Let us consider 2nd order Non-homogeneous LDE with constant coefficients as:

$ay'' + by' + cy = r(x)$ (1)

In this method, we guess the trial solution (P.I.) $y_p$ by looking at the type of $r(x)$.

As $y_p$ is solution of equation (1), so it satisfies (1) from where we calculate the Undeterminant coefficients by comparing like coefficients on both sides.

For example:

**1.** If $r(x) = e^{\alpha x}$, then assumed trial solution: $y_p = A\, e^{\alpha x}$

**2.** If $r(x) = x^2$, then assumed trial solution: $y_p = A\, x^2 + Bx + C$

**3.** If $r(x) = \cos ax$, then assumed trial solution: $y_p = A \cos x + B \sin x$

where $A, B, C$ are Undeterminant coefficients to be evaluated.

**Problem 1.** Find the general solution of: $4y'' - y = e^{3x}$

**Solution:** The given equation is:

$4y'' - y = e^{3x}$ (1)

S.F. : $(4D^2 - 1)y = e^{3x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (4D^2 - 1)$ and $r(x) = e^{3x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (4D^2 - 1) = 0$ $\Rightarrow D^2 = \frac{1}{4}$

$\Rightarrow D = \frac{1}{2}, -\frac{1}{2}$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$

To find Particular Integral (P.I.):

Since $r(x) = e^{3x}$, So, let trial solution be: $y_p = ae^{3x}$

$\Rightarrow y_p' = 3ae^{3x}$ $\Rightarrow y_p'' = 9ae^{3x}$

Since $y_p$ is a solution of equation (1)

So, $4y_p'' - y_p = e^{3x}$

$\Rightarrow 4(9ae^{3x}) - ae^{3x} = e^{3x}$ $\Rightarrow 35ae^{3x} = e^{3x}$ $\Rightarrow 35a = 1$ $\Rightarrow a = \frac{1}{35}$

$\therefore y_p = \frac{1}{35} e^{3x}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = \left(c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}\right) + \frac{1}{35} e^{3x}$ **Answer.**

**Problem 2.** Find the general solution of: $y'' - 3y' - 10y = x^2 + 1$

**Solution:** The given equation is:

$y'' - 3y' - 10y = x^2 + 1$ (1)

S.F. : $(D^2 - 3D - 10)y = x^2 + 1$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 3D - 10)$ and $r(x) = x^2 + 1$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 3D - 10) = 0$ $\Rightarrow (D - 5)(D + 2) = 0$

$\Rightarrow D = 5, -2$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 e^{5x} + c_2 e^{-2x}$

To find Particular Integral (P.I.):

Since $r(x) = x^2 + 1$, So, let trial solution be: $y_p = ax^2 + bx + c$

$\Rightarrow y_p' = 2ax + b$ $\Rightarrow y_p'' = 2a$

Since $y_p$ is a solution of equation (1)

So, $y_p'' - 3y_p' - 10y_p = x^2 + 1$

$\Rightarrow 2a - 3(2ax + b) - 10(ax^2 + bx + c) = x^2 + 1$

$\Rightarrow -10ax^2 - (6a + 10b)x + (2a - 3b - 10c) = x^2 + 1$

Comparing the like coefficients:

Coeff. of $x^2$: $-10a = 1$ $\Rightarrow a = -\frac{1}{10}$

Coeff. of $x$ : $-(6a + 10b) = 0$ $\Rightarrow 10b = -6a$ $\Rightarrow b = \frac{6}{100}$

Constant terms: $(2a - 3b - 10c) = 1 \quad \Rightarrow 10c = 2a - 3b - 1$

$\Rightarrow c = \frac{1}{10}(2a - 3b - 1) = \frac{1}{10}\left(2\left(-\frac{1}{10}\right) - 3\left(\frac{6}{100}\right) - 1\right) = -\frac{138}{1000}$

Putting back values of $a, b, c$ in $y_p = ax^2 + bx + c$

$y_p = -\frac{1}{10}x^2 + \frac{6}{100}x - \frac{138}{1000}$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1e^{5x} + c_2e^{-2x}) - \frac{1}{10}x^2 + \frac{6}{100}x - \frac{138}{1000}$ **Answer.**

**Polling Quiz**

For $y'' - 3y' - 10y = x^3 + 1$, the assumed trial solution will be:

(A) $y_p = ax^3 + bx^2 + cx + d$

(B) $y_p = a + bx + cx^2 + dx^3$

(C) Both A and B

(D) None of these.

**Problem 3.** Find the general solution of: $y'' + y' - 6y = 39\cos 3x$

**Solution:** The given equation is:

$y'' + y' - 6y = 39\cos 3x$ (1)

S.F. : $(D^2 + D - 6)y = 39\cos 3x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + D - 6)$ and $r(x) = 39\cos 3x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 + D - 6) = 0 \quad \Rightarrow (D - 2)(D + 3) = 0$

$\Rightarrow D = 2, -3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{2x} + c_2e^{-3x}$

To find Particular Integral (P.I.):

Since $r(x) = 39\cos 3x$, So, let trial solution be: $y_p = a\cos 3x + b\sin 3x$

$\Rightarrow y_p{}' = -3a\sin 3x + 3b\cos 3x \quad \Rightarrow y_p{}'' = -9a\cos 3x - 9b\sin 3x$

Since $y_p$ is a solution of equation (1)

So, $y_p{}'' + y_p{}' - 6y_p = 39\cos 3x$

$\Rightarrow (-9a\cos 3x - 9b\sin 3x) + (-3a\sin 3x + 3b\cos 3x) - 6(a\cos 3x + b\sin 3x) = 39\cos 3x$

$\Rightarrow (-15a + 3b)\cos 3x + (-3a - 15b)\sin 3x = 39\cos 3x + 0\sin 3x$

Comparing the like coefficients:

Coeff. of $\cos 3x$: $-15a + 3b = 39$ (2)

Coeff. of $\sin 3x$ : $-3a - 15b = 0$ (3)

Solving equations (2) and (3):

$-15a + 3b = 39$ (2) $\times 3$

$-3a - 15b = 0$ (3) $\times 15$ and subtracting, we get:

$234b = 127 \Rightarrow b = \frac{1}{2}$

Put value of $b$ in equation (3): $3a = -15b = -\frac{15}{2} \quad \Rightarrow a = -\frac{5}{2}$

Putting back values of $a, b$ in $y_p = a\cos 3x + b\sin 3x$

$y_p = -\frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{2x} + c_2 e^{-3x}) - \frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$ **Answer.**

**Polling Quiz**

For $y'' + y' - 6y = 16\sin 2x$, the assumed trial solution will be:

(A) $y_p = a\cos 2x$

(B) $y_p = a\sin 2x$

(C) $y_p = a\cos 2x + b\sin 2x$

# MTH166

## Lecture-17

## Method of Undeterminant Coeff.-II

**Problem 4.** Find the general solution of: $y'' + y' - 6y = 39\cos 3x$

**Solution:** The given equation is:

$y'' + y' - 6y = 39\cos 3x$ (1)

S.F. : $(D^2 + D - 6)y = 39\cos 3x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + D - 6)$ and $r(x) = 39\cos 3x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + D - 6) = 0$ $\Rightarrow (D-2)(D+3) = 0$

$\Rightarrow D = 2, -3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1 e^{2x} + c_2 e^{-3x}$

To find Particular Integral (P.I.):

Since $r(x) = 39\cos 3x$, So, let trial solution be: $y_p = a\cos 3x + b\sin 3x$

$\Rightarrow y_p{}' = -3a\sin 3x + 3b\cos 3x$ $\Rightarrow y_p{}'' = -9a\cos 3x - 9b\sin 3x$

Since $y_p$ is a solution of equation (1)

So, $y_p{}'' + y_p{}' - 6y_p = 39\cos 3x$

$\Rightarrow (-9a\cos 3x - 9b\sin 3x) + (-3a\sin 3x + 3b\cos 3x) - 6(a\cos 3x +$
$b\sin 3x) = 39\cos 3x$

$\Rightarrow (-15a + 3b)\cos 3x + (-3a - 15b)\sin 3x = 39\cos 3x + 0\sin 3x$

Comparing the like coefficients:

Coeff. of $\cos 3x$: $-15a + 3b = 39$ (2)

Coeff. of $\sin 3x$ : $-3a - 15b = 0$ (3)

Solving equations (2) and (3):

$-15a + 3b = 39$ (2) $\times 3$

$-3a - 15b = 0$ (3) $\times 15$ and subtracting, we get:

$234b = 127 \Rightarrow b = \frac{1}{2}$

Put value of $b$ in equation (3): $3a = -15b = -\frac{15}{2}$ $\Rightarrow a = -\frac{5}{2}$

Putting back values of $a, b$ in $y_p = a\cos 3x + b\sin 3x$

$y_p = -\frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1 e^{2x} + c_2 e^{-3x}) - \frac{5}{2}\cos 3x + \frac{1}{2}\sin 3x$ **Answer.**

**Problem 5.** Find the general solution of: $y'' + 4y = \cos x$

**Solution:** The given equation is:

$y'' + 4y = \cos x$ (1)

S.F. : $(D^2 + 4)y = \cos x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 4)$ and $r(x) = \cos x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 4) = 0$

$\Rightarrow D = 2i, -2i$ (Complex roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1\cos 2x + c_2\sin 2x$

To find Particular Integral (P.I.):

Since $r(x) = \cos x$, So, let trial solution be: $y_p = a\cos x + b\sin x$

$\Rightarrow y_p{}' = -a\sin x + b\cos x \quad \Rightarrow y_p{}'' = -a\cos x - b\sin x$

Since $y_p$ is a solution of equation (1)

So, $y_p{}'' + 4y_p = \cos x$

$\Rightarrow (-a\cos x - b\sin x) + 4(a\cos x + b\sin x) = \cos x$

$\Rightarrow 3a\cos x + 3b\sin x = \cos x + 0\sin x$

Comparing the like coefficients:

Coeff. of $\cos x$: $3a = 1 \quad \Rightarrow a = \frac{1}{3}$

Coeff. of $\sin x$ : $3b = 0 \quad \Rightarrow b = 0$

Putting back values of $a, b$ in $y_p = \frac{1}{3}\cos x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\cos x$ **Answer.**

**Problem 6.** Find the general solution of: $y'' + 4y = \cos 2x$

**Solution:** The given equation is:

$y'' + 4y = \cos x \qquad (1)$

S.F. : $(D^2+4)y = \cos x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2+4)$ and $r(x) = \cos x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2+4) = 0$

$\Rightarrow D = 2i, -2i$ (Complex roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1\cos 2x + c_2\sin 2x$

To find Particular Integral (P.I.):

Since $r(x) = \cos 2x$, So, let trial solution be: $y_p = x(a\cos 2x + b\sin 2x)$

$\Rightarrow y_p{}' = x(-2a\sin 2x + 2b\cos 2x) + 1(a\cos 2x + b\sin 2x)$

$\Rightarrow y_p{}' = (a+2bx)\cos 2x + (b-2ax)\sin 2x$

$\Rightarrow y_p{}'' = [(a+2bx)(-2\sin 2x) + \cos 2x\,(2b)] + [(b-2ax)(2\cos 2x) + \sin 2x(-2a)]$

$\Rightarrow y_p{}'' = (4b-4ax)\cos 2x - (4a+4bx)\sin 2x$

Since $y_p$ is a solution of equation (1)

So, $y_p{}'' + 4y_p = \cos 2x$

$\Rightarrow \big((4b-4ax)\cos 2x - (4a+4bx)\sin 2x\big) + 4(x(a\cos 2x + b\sin 2x)) = \cos 2x$

$\Rightarrow 4b\cos 2x - 4a\sin 2x = \cos 2x + 0\sin 2x$

Comparing the like coefficients:

Coeff. of $\cos 2x$: $4b = 1$ $\Rightarrow b = \frac{1}{4}$

Coeff. of $\sin 2x$ : $-4a = 0$ $\Rightarrow a = 0$

Putting back values of $a, b$ in $y_p = x(0\cos 2x + \frac{1}{4}\sin 2x) = \frac{1}{4}x\sin 2x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x\sin 2x$ **Answer.**

**Polling Quiz**

For $y'' + y = \cos x$, the assumed trial solution will be:

(A) $y_p = x(a\cos x + b\sin x)$

(B) $y_p = (a\cos x + b\sin x)$

(C) $y_p = x(a\cos 2x + b\sin 2x)$

**Problem 7.** Find the general solution of: $y'' - 4y' + 13y = 12e^{2x}\sin 3x$

**Solution:** The given equation is:

$y'' - 4y' + 13y = 12e^{2x}\sin 3x$ (1)

S.F. : $(D^2 - 4D + 13)y = 12e^{2x}\sin 3x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 4D + 13)$ and $r(x) = 12e^{2x}\sin 3x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 - 4D + 13) = 0$

$\Rightarrow D = 2 + 3i, 2 - 3i$ (Complex roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = e^{2x}(c_1\cos 3x + c_2\sin 3x)$

To find Particular Integral (P.I.):

Since $r(x) = 12e^{2x}\sin 3x$, So, let trial solution be: $y_p = xe^{2x}(a\cos 3x + b\sin 3x)$

$\Rightarrow$ *calculate* $y_p'$ *and* $y_p''$

Since $y_p$ is a solution of equation (1)

So, $y_p'' - 4y_p' + 13y_p = 12e^{2x}\sin 3x$

Comparing the like coefficients:

$\Rightarrow a = -2$ and $b = 0$

Putting back values of $a, b$ in $y_p = -2xe^{2x}\cos 3x$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = e^{2x}(c_1\cos 3x + c_2\sin 3x) - 2xe^{2x}\cos 3x$ **Answer.**

# MTH166

## Lecture-18

## Euler-Cauchy Equation

**Topic:**

Solution of Non-Homogeneous LDE with Variable coefficients.

**Learning Outcomes:**

To convert equation with variable coefficients (Euler-Cauchy Equation) to an equation with constant coefficients and then solve it with standard known methods.

**Euler-Cauchy Equation:**

Let us consider 2nd order Non-homogeneous LDE with variable coefficients as:

$x^2y'' + xy' + y = r(x)$ (1)

Equation of type (1) called Euler-Cauchy Equation.

S.F. : $(x^2D^2 + xD + 1)y = r(x)$ (2) where $D \equiv \frac{d}{dx}$

The very first job is to convert this equation with variable coefficients to an equation with constant coefficients using an appropriate transformation.

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

Let $\theta \equiv \frac{d}{dt}$ ( Another differential operator)

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1),\ x^3D^3 = \theta(\theta - 1)(\theta - 2)$ and so on…

Equation (2) becomes: $[\theta(\theta - 1) + \theta + 1]y = r(t)$ (3)

which is an equation with constant coefficients and we know the methods to solve equ.(3)

**Problem 1.** Find the general solution of: $x^2y'' + xy' - 4y = 0$

**Solution:** The given equation is:

$x^2y'' + xy' - 4y = 0$ (1)

S.F. : $(x^2D^2 + xD - 4)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + \theta - 4]y = 0$

A.E. : $[\theta(\theta - 1) + \theta - 4] = 0 \quad \Rightarrow [\theta^2 - \theta + \theta - 4] = 0$

$\Rightarrow (\theta^2 - 4) = 0 \quad \Rightarrow \theta = 2, -2$ (real and unequal roots)

$\therefore$ General solution is given by:

$\Rightarrow y = c_1e^{2t} + c_2e^{-2} \quad \Rightarrow y = c_1x^2 + c_2x^{-2}$ **Answer.**

**Problem 2.** Find the general solution of: $9x^2y'' + 15xy' + y = 0$

**Solution:** The given equation is:

$9x^2y'' + 15xy' + y = 0$ (1)

S.F. : $(9x^2D^2 + 15xD + 1)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[9\theta(\theta - 1) + 15\theta + 1]y = 0$

A.E. : $[9\theta(\theta - 1) + 15\theta + 1] = 0 \quad \Rightarrow [9\theta^2 - 9\theta + 15\theta + 1] = 0$

$\Rightarrow (9\theta^2 + 6\theta + 1) = 0 \quad \Rightarrow (3\theta + 1)(3\theta + 1) = 0 \quad \Rightarrow \theta = -\frac{1}{3}, -\frac{1}{3}$ (equal roots)

$\therefore$ General solution is given by:

$\Rightarrow y = (c_1 + c_2t)e^{-\frac{1}{3}t} \quad \Rightarrow y = (c_1 + c_2 \log x)\, x^{-\frac{1}{3}}$ **Answer.**

**Polling Quiz**

The transformation used to convert a LDE with variable coefficients to LDE with constant coefficients is:

(A) $t = e^x$

(B) $x = e^t$

(C) None of these.

**Problem 3.** Find the general solution of: $2x^2y'' + 2xy' + y = 0$

**Solution:** The given equation is:

$2x^2y'' + 2xy' + y = 0$ (1)

S.F. : $(2x^2D^2 + 2xD + 1)y = 0$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t \quad \Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[2\theta(\theta - 1) + 2\theta + 1]y = 0$

A.E. : $[2\theta(\theta - 1) + 2\theta + 1] = 0 \quad \Rightarrow [2\theta^2 - 2\theta + 2\theta + 1] = 0$

$\Rightarrow (2\theta^2 + 1) = 0 \quad \Rightarrow \theta^2 = -\frac{1}{2} \quad \Rightarrow \theta = \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}}i$ (complex roots)

$\therefore$ General solution is given by:

$\Rightarrow y = e^{0t}(c_1 \cos\frac{1}{\sqrt{2}}t + c_2 \sin\frac{1}{\sqrt{2}}t) \quad \Rightarrow y = (c_1 \cos\frac{1}{\sqrt{2}}\log x + c_2 \sin\frac{1}{\sqrt{2}}\log x)$ **Ans.**

**Problem 4.** Find the general solution of: $x^2y'' - 2y = 2x$

**Solution:** The given equation is:

$x^2y'' - 2y = 2x$ (1)

S.F. : $(x^2D^2 - 2)y = 2x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) - 2]y = 2e^t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - \theta - 2)$ and $r(t) = 2e^t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 - \theta - 2) = 0$ $\Rightarrow (\theta - 2)(\theta + 1) = 0$

$\Rightarrow \theta = 2, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{2t} + c_2e^{-t}$ $\Rightarrow y_c = c_1x^2 + c_2x^{-1}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(\theta)}r(t) = \frac{1}{(\theta^2-\theta-2)}(2e^t)$

$\Rightarrow y_p = 2\left[\frac{1}{(\theta^2-\theta-2)}e^t\right]$ $\Rightarrow y_p = 2\left[\frac{1}{((1)^2-(1)-2)}e^t\right]$ $(Put\ \theta = 1)$

$\Rightarrow y_p = 2\left[\frac{1}{-2}e^t\right]$ $\Rightarrow y_p = -e^t = -x$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1x^2 + c_2x^{-1}) - x$ **Answer.**

**Problem 5.** Find the general solution of: $x^2y'' + 2xy' = \cos(\log x)$

**Solution:** The given equation is:

$x^2y'' + 2xy' = \cos(\log x)$ (1)

S.F. : $(x^2D^2 + 2xD)y = \cos(\log x)$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta - 1) + 2\theta]y = \cos t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 + \theta)$ and $r(t) = \cos t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 + \theta) = 0$ $\Rightarrow \theta(\theta + 1) = 0$

$\Rightarrow \theta = 0, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{0t} + c_2e^{-t}$ $\Rightarrow y_c = c_1 + c_2x^{-1}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(\theta)}r(t) = \frac{1}{(\theta^2+\theta)}(\cos t)$

$\Rightarrow y_p = \left[\frac{1}{(-(1)^2+\theta)}(\cos t)\right]$ $(Put\ \theta^2 = -(1)^2)$

$\Rightarrow y_p = \left[\frac{1}{(\theta-1)}(\cos t)\right] = \left[\frac{1}{(\theta-1)} \times \frac{(\theta+1)}{(\theta+1)}(\cos t)\right] = \left[\frac{(\theta+1)}{(\theta^2-1)}(\cos t)\right]$

$\Rightarrow y_p = \left[\frac{(\theta+1)}{(-1-1)}(\cos t)\right] = -\frac{1}{2}\left[\frac{d}{dt}(\cos t) + (\cos t)\right] = -\frac{1}{2}(-\sin t + \cos t)$

$\Rightarrow y_p = -\frac{1}{2}[-\sin(\log x) + \cos(\log x)]$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 + c_2x^{-1} - \frac{1}{2}[-\sin(\log x) + \cos(\log x)]$ **Answer.**

**Polling Quiz**

Which of the following is **not** an Euler-Cauchy differential equation:

(a) $x^2y'' + xy' - 4y = 0$

(b) $x^2y'' - 2xy = 2x$

(c) $x^2y'' - 3xy' + 3y = 2 + 3\log x$

(d) $x^2y'' - 2y = 2x$

**Problem 6.** Find the general solution of: $x^2y'' - 3xy' + 3y = 2 + 3\log x$

**Solution:** The given equation is:

$x^2y'' - 3xy' + 3y = 2 + 3\log x$ (1)

S.F. : $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta-1) - 3\theta + 3]y = 2 + 3t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - 4\theta + 3)$ and $r(t) = 2 + 3t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 - 4\theta + 3) = 0$ $\Rightarrow (\theta-1)(\theta-3) = 0$

$\Rightarrow \theta = 1, 3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^t + c_2e^{3t}$ $\Rightarrow y_c = c_1x + c_2x^3$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(\theta)}r(t) = \frac{1}{(\theta^2-4\theta+3)}(2+3t)$

$\Rightarrow y_p = \left[\frac{1}{3\left[1+\left(\frac{\theta^2-4\theta}{3}\right)\right]}(2+3t)\right] = \frac{1}{3}\left[\left(1+\left(\frac{\theta^2-4\theta}{3}\right)\right)^{-1}(2+3t)\right]$

$\Rightarrow y_p = \frac{1}{3}\left[\left(1 - \left(\frac{\theta^2-4\theta}{3}\right) + \left(\frac{\theta^2-4\theta}{3}\right)^2 - - - \right)(2+3t)\right]$

$\Rightarrow y_p = \frac{1}{3}\left[(2+3t) + \frac{4}{3}\frac{d}{dt}(2+3t) + 0\right] = \frac{1}{3}[(2+3t)+4] = 2 + t$

$\Rightarrow y_p = 2 + \log x$

$\therefore$ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1x + c_2x^3 + 2 + \log x$ **Answer.**

# MTH166

## Lecture-19

## Euler-Cauchy Equation-II

**Topic:**

Solution of Homogeneous and Non-Homogeneous LDE with Variable coefficients.

**Learning Outcomes:**

To convert equation with variable coefficients (Euler-Cauchy Equation) to an equation with constant coefficients and then solve it with standard known methods.

**Problem 1.** Find the general solution of: $x^2y'' - 2y = 2x^2$

**Solution:** The given equation is:

$x^2y'' - 2y = 2x^2$ (1)

S.F. : $(x^2D^2 - 2)y = 2x^2$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta-1) - 2]y = 2e^{2t}$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - \theta - 2)$ and $r(t) = 2e^{2t}$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 - \theta - 2) = 0$ $\Rightarrow (\theta - 2)(\theta + 1) = 0$

$\Rightarrow \theta = 2, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{2t} + c_2e^{-t}$ $\Rightarrow y_c = c_1x^2 + c_2x^{-1}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(\theta)}r(t) = \frac{1}{(\theta^2-\theta-2)}(2e^{2t})$

$\Rightarrow y_p = 2\left[\frac{1}{(\theta^2-\theta-2)}e^t\right]$ $\Rightarrow y_p = 2\left[\frac{1}{((2)^2-(2)-2)}e^t\right]$ $(Put\ \theta = 2)$

$\Rightarrow y_p = 2\left[\frac{1}{4-4}e^t\right]$ Case of failure

$y_p = 2t\left[\frac{1}{2\theta-1}e^{2t}\right] = 2t\left(\frac{1}{3}e^{2t}\right) = 2\log x\left(\frac{1}{3}x^2\right)$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = (c_1x^2 + c_2x^{-1}) + 2\log x\left(\frac{1}{3}x^2\right)$ **Answer.**

**Problem 2.** Find the general solution of: $x^2y'' + 2xy' = \cos(\log x)$

**Solution:** The given equation is:

$x^2y'' + 2xy' = \cos(\log x)$ (1)

S.F. : $(x^2D^2 + 2xD)y = \cos(\log x)$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta-1) + 2\theta]y = \cos t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 + \theta)$ and $r(t) = \cos t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 + \theta) = 0$ $\Rightarrow \theta(\theta + 1) = 0$

$\Rightarrow \theta = 0, -1$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^{0t} + c_2e^{-t}$ $\Rightarrow y_c = c_1 + c_2x^{-1}$

To find Particular Integral (P.I.):

$y_p = \frac{1}{f(\theta)}r(t) = \frac{1}{(\theta^2+\theta)}(\cos t)$

$\Rightarrow y_p = \left[\frac{1}{(-(1)^2+\theta)}(\cos t)\right]$ $(Put\ \theta^2 = -(1)^2)$

$\Rightarrow y_p = \left[\frac{1}{(\theta-1)}(\cos t)\right] = \left[\frac{1}{(\theta-1)} \times \frac{(\theta+1)}{(\theta+1)}(\cos t)\right] = \left[\frac{(\theta+1)}{(\theta^2-1)}(\cos t)\right]$

$\Rightarrow y_p = \left[\frac{(\theta+1)}{(-1-1)}(\cos t)\right] = -\frac{1}{2}\left[\frac{d}{dt}(\cos t) + (\cos t)\right] = -\frac{1}{2}(-\sin t + \cos t)$

$\Rightarrow y_p = -\frac{1}{2}[-\sin(\log x) + \cos(\log x)]$

$\therefore$ General solution is given by: $y$ = C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 + c_2x^{-1} - \frac{1}{2}[-\sin(\log x) + \cos(\log x)]$ **Answer.**

**Polling Quiz**

Which of the following is not an Euler-Cauchy differential equation:

(A) $x^2y'' + xy' - 4y = 0$

(B) $x^2y'' - 2xy = 2x$

(C) $x^2y'' - 3xy' + 3y = 2 + 3\log x$

**Problem 3.** Find the general solution of: $x^2y'' - 3xy' + 3y = 2 + 3\log x$

**Solution:** The given equation is:

$x^2y'' - 3xy' + 3y = 2 + 3\log x$ (1)

S.F. : $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$ (2) where $D \equiv \frac{d}{dx}$

Let transformation be: $x = e^t$ $\Rightarrow t = \log x$

then $xD = \theta,\ x^2D^2 = \theta(\theta - 1)$ where $\theta \equiv \frac{d}{dt}$

Equation (2) becomes: $[\theta(\theta-1) - 3\theta + 3]y = 2 + 3t$

$\Rightarrow f(\theta)y = r(t)$ where $f(\theta) = (\theta^2 - 4\theta + 3)$ and $r(t) = 2 + 3t$

To find Complimentary Function (C.F.):

A.E. : $f(\theta) = 0$ $\Rightarrow (\theta^2 - 4\theta + 3) = 0$ $\Rightarrow (\theta - 1)(\theta - 3) = 0$

$\Rightarrow \theta = 1, 3$ (real and unequal roots)

$\therefore$ Complimentary function is given by:

$\Rightarrow y_c = c_1e^t + c_2e^{3t}$ $\Rightarrow y_c = c_1x + c_2x^3$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(\theta)} r(t) = \frac{1}{(\theta^2-4\theta+3)}(2+3t)$$

$$\Rightarrow y_p = \left[\frac{1}{3\left[1+\left(\frac{\theta^2-4\theta}{3}\right)\right]}(2+3t)\right] = \frac{1}{3}\left[\left(1+\left(\frac{\theta^2-4\theta}{3}\right)\right)^{-1}(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[\left(1-\left(\frac{\theta^2-4\theta}{3}\right)+\left(\frac{\theta^2-4\theta}{3}\right)^2 ---\right)(2+3t)\right]$$

$$\Rightarrow y_p = \frac{1}{3}\left[(2+3t)+\frac{4}{3}\frac{d}{dt}(2+3t)+0\right] = \frac{1}{3}\left[(2+3t)+4\right] = 2+t$$

$$\Rightarrow y_p = 2+\log x$$

$\therefore$ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = c_1 x + c_2 x^3 + 2 + \log x$ **Answer.**

**Q27.** The Particular Integral (P.I.) for the differential equation $y'' + y' - 6y = 5e^{-3x}$ is:

(a) $-e^{-3x}$ (b) $-xe^{-3x}$ (c) $xe^{-3x}$ (d) $e^{-3x}$

**Q28.** By method of undetermined coefficient , the choice of particular integral for $y''-4y=5e^{-2x}$ is

(a) $Ce^{-2x}$ (b) $Cxe^{-2x}$

(c) $Cx^2e^{-2x}$ (d) $Cx^3e^{-2x}$

**Q29.** To transform $xy'' + y' = \frac{1}{x}$ into a linear differential equations with constant coefficient, suitable transformation of $x$ is

(a) $x = \sin t$ (b) $x = \log t$ (c) $x = e^t$ (d) none of these

**Q30.** Which of the following is Euler Cauchy equation?

(a) $x^2y''$-5$xy'$+13y=30$xy^2$
(b) $x^2y''$-5$xy'$+13$xy$=30$x^2$
(c) $x^2y''$-5$xy'$+13y=30$xy^2$
(d) None of these

**Q32.** The P.I. of the differential equation $(D^3+6D+9)y=5e^{-x}$ is

(a) $\frac{e^{-x}}{2}$
(b) $\frac{5e^{-x}}{2}$
(c) $-\frac{e^{-x}}{2}$
(d) $-\frac{5e^{-x}}{2}$

**Q33.** The trial solution for finding the P.I. of the differential equation $(D^2+3D+1)y=x^2+1$

(a) $C_0+C_1x$
(b) $C_0+C_1x+C_3x$
(c) $C_0+C_1x+C_3x^2$
(d) $C_0x+C_1x^2+Cx^3$

**Q35.** Particular integral solution of $(2D^3-3D^2+5)y=6\,e^{3x}$ will be

(a)$e^x/6$ (b)$3e^x/16$ (c)$6e^x$ (d)$e^x/32$

**Q36.** Solution of differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$ is
(a) $c_1x+c_2$ (b) $c_1\log x+c_2$ (c) $c_1x^2+c_2$ (d) $c_1+\frac{c_2}{x}$

**Q37.** The roots of the Auxilary equation of the differential equation $2x^2y''+3xy'-3y=0$, are
(a) 1,-3/2 (b) -1,3/2 (c) -1,-3/2 (d) 1,3/2

**Q38.** The Particular Integral of $\frac{d^2y}{dx^2}+y=\sin x, is$
(a) $-\frac{x}{2}\sin x$ (b) $\frac{x}{2}\sin x$ (c) $\frac{x}{2}\cos x$ (d) $-\frac{x}{2}\cos x$

**Q39.** The general solution of the differential equation $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=0$, is
(a) $y=c_1x+c_2x^3$ (b) $y=c_1x-c_2x^3$ (c) $y=c_1x+c_2x^2$ (d) $y=c_1+c_2x^3$

**Q40.** The operator form of the differential equation $\frac{dy_1}{dt} + \frac{dy_2}{dt} - 2y_1 + 3y_2 = t$ is

(a) $Dy_1 + Dy_2 + 2y_1 + 3y_2 = t$
(b) $Dy_1 + Dy_2 - 2y_1 + 3y_2 = t$
(c) $Dy_1 + Dy_2 - 2y_1 - 3y_2 = t$
(d) None of these

# MTH166

## Lecture-20

## Simultaneous Differential Equations

**Learning Outcomes:**

To write operator form of simultaneous system of LDE and solving these.

**Simultaneous Linear Differential Equations:**

The system involving two first order linear differential equations in two dependent variables $y_1$ and $y_2$ and one independent variable $x$ is called system of simultaneous linear differential equations.

**Write the operator form of following system of LDE:**

**Problem1.** $6\frac{dy_1}{dx} + 5\frac{dy_2}{dx} + 3y_1 + y_2 = 0, \frac{dy_2}{dx} - 5y_1 + 3y_2 = e^x$

**Solution.** The given system of simultaneous equations is:

$$6\frac{dy_1}{dx} + 5\frac{dy_2}{dx} + 3y_1 + y_2 = 0 \qquad (1)$$

$$\frac{dy_2}{dx} - 5y_1 + 3y_2 = e^x \qquad (2)$$

Let $D \equiv \frac{d}{dx}$, then operator form of given system can be written as:

$$(6D+3)y_1 + (5D+1)y_2 = 0 \qquad (3) \times 5$$

$$-5y_1 + (D+3)y_2 = e^x \qquad (4) \times (6D+3)$$

Solving these equations:

$$5(6D+3)y_1 - 5(5D+1)y_2 = 0$$

$$-5(6D+3)y_1 + (6D+3)(D+3)y_2 = (6D+3)e^x$$

On adding these equations, we get:

$$[(6D+3)(D+3) - 5(5D+1)]y_2 = (6D+3)e^x$$

$$\Longrightarrow (6D^2 - 4D + 4)y_2 = 6e^x + 3e^x = 9e^x \qquad (5)$$

To find Complimentary Function (C.F.):

$$(6D^2 - 4D + 4) = 0 \qquad \Longrightarrow D = \frac{1 \pm \sqrt{5}i}{3}$$

$$\Rightarrow y_c = e^{\frac{x}{3}}\left(c_1 \cos\frac{\sqrt{5}}{3}x + c_2 \sin\frac{\sqrt{5}}{3}x\right)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)}r(x) = \frac{1}{(6D^2-4D+4)}(9e^x)$$

$$\Rightarrow y_p = 9\left[\frac{1}{(6D^2-4D+4)}e^x\right]$$

$$\Rightarrow y_p = 9\left[\frac{1}{(6(1)^2-4(1)+4)}e^x\right] \qquad (Put\ D = 1)$$

$$\Rightarrow y_p = 9\left[\frac{1}{6}e^{2x}\right] \qquad \Rightarrow y_p = \frac{3}{2}e^{2x}$$

$\therefore$ General solution is given by: $y_2$ = C.F. +P.I.

i.e. $y_2 = y_c + y_p$

$$\Rightarrow y_2 = e^{\frac{x}{3}}\left(c_1 \cos\frac{\sqrt{5}}{3}x + c_2 \sin\frac{\sqrt{5}}{3}x\right) + \frac{3}{2}e^{2x}$$

Again from equations (3) and (4):

$(6D+3)y_1 + (5D+1)y_2 = 0$ (3) $\times (D+3)$

$-5y_1 + (D+3)y_2 = e^x$ (4) $\times (5D+1)$

Solving these equations:

$(6D+3)(D+3)y_1 + (5D+1)(D+3)y_2 = 0$

$-5(5D+1)y_1 + (5D+1)(D+3)y_2 = (5D+1)e^x$

Subtracting these equations, we get:

$[(6D+3)(D+3)+5(5D+1)]y_1 = (5D+1)e^x$

$\Rightarrow (6D^2+46D+14)y_1 = 6\,e^x$

***Try it yourself.***

**Polling Question:**

The operator form of $3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x}, \frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x$ is:

(a) $(3D+2)y_1 + y_2 = 0,$ $(D-2)y_1 + (D+3)y_2 = 0$

(b) $(3D+2)y_1 + y_2 = e^{-x},$ $(D-2)y_2 + (D+3)y_1 = x$

(c) $(3D+2)y_1 + y_2 = e^{-x},$ $(D-2)y_1 + (D+3)y_2 = x$

(d) None of these

**Problem2.** $3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x}, \frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x$

**Solution.** The given system of simultaneous equations is:

$3\frac{dy_1}{dx} + 2y_1 + y_2 = e^{-x}$ (1)

$\frac{dy_1}{dx} + \frac{dy_2}{dx} - 2y_1 + 3y_2 = x$ (2)

Let $D \equiv \frac{d}{dx}$, then operator form of given system can be written as:

$(3D+2)y_1 + y_2 = e^{-x}$ (3) $\times (D+3)$

$(D-2)y_1 + (D+3)y_2 = x$ (4) $\times 1$

Solving these equations:

$(3D+2)(D+3)y_1 + (D+3)y_2 = (D+3)e^{-x}$

$(D-2)y_1 + (D+3)y_2 = x$

Subtracting these equations, we get:

$[(3D+2)(D+3) - (D-2)]y_1 = (D+3)e^{-x} - x$

$\Rightarrow (3D^2+10D+8)y_1 = 2e^{-x} - x$ (5)

To find Complimentary Function (C.F.):

$(3D^2+10D+8) = 0$ $\Longrightarrow D = -2, -\frac{4}{3}$

$\Rightarrow y_c = \left(c_1e^{-2x} + c_2e^{-\frac{4}{3}x}\right)$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(3D^2+10D+\ )}(2e^{-x} - x)$$

$$\Rightarrow y_p = 2\left[\frac{1}{(3D^2+10D+8)} e^{-x}\right] - \left[\frac{1}{(3D^2+10D+\ )} x\right]$$

$$\Rightarrow y_p = 2\left[\frac{1}{(3(-1)^2+10(-1)+8)} e^{-x}\right] - \left[\frac{1}{8\left(1+\left(\frac{10D+3\ \ ^2}{8}\right)\right)} x\right]$$

$$\Rightarrow y_p = 2\left[\frac{1}{1} e^{-x}\right] - \frac{1}{8}\left[\left(1+\left(\frac{10D+3\ \ ^2}{8}\right)\right)^{-1} x\right]$$

$$\Rightarrow y_p = 2e^{-x} - \frac{1}{8}\left(x - \frac{10}{8}\right)$$

$\therefore$ General solution is given by: $y_1$ = C.F. +P.I.

i.e. $y_1 = y_c + y_p$

$$\Rightarrow y_1 = \left(c_1 e^{-2x} + c_2 e^{-\frac{4}{3}x}\right) + 2e^{-x} + \frac{1}{8}\left(x - \frac{10}{8}\right)$$

Again from equations (3) and (4):

$(3D+2)y_1 + y_2 = e^{-x}$ (3)$\times (D-2)$

$(D-2)y_1 + (D+3)y_2 = x$ (4)$\times (3D+2)$

Solving these equations:

$$(3D+2)(D-2)y_1 + (D-2)y_2 = (D-2)e^{-x}$$

$$(3D+2)(D-2)y_1 + (3D+2)(D+3)y_2 = (3D+2)x$$

Subtracting these equations, we get:

$$[(3D+2)(D+3) - (D-2)]y_2 = (3D+2)x - (D-2)e^{-x}$$

$$\Rightarrow (3D^2+10D+8)y_2 = (3+2x) + 3e^{-x} \qquad (6)$$

To find Complimentary Function (C.F.):

$(3D^2+10D+8) = 0$ $\Longrightarrow D = -2, -\frac{4}{3}$

$$\Rightarrow y_c = \left(c_3 e^{-2x} + c_4 e^{-\frac{4}{3}x}\right)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(3D^2+10D+\ )}(3e^{-x} + (2x+3))$$

$$\Rightarrow y_p = 3\left[\frac{1}{(3D^2+10D+\ )} e^{-x}\right] + \left[\frac{1}{(3D^2+10D+8)}(2x+3)\right]$$

$$\Rightarrow y_p = 3\left[\frac{1}{(3(-1)^2+1\ \ (-1)+8)} e^{-x}\right] + \left[\frac{1}{8\left(1+\left(\frac{10D+3D^2}{8}\right)\right)}(2x+3)\right]$$

$$\Rightarrow y_p = 3\left[\frac{1}{1} e^{-x}\right] + \frac{1}{8}\left[\left(1+\left(\frac{10D+3D^2}{8}\right)\right)^{-1}(2x+3)\right]$$

$$\Rightarrow y_p = 3e^{-x} + \frac{1}{8}\left((2x+3) - \frac{10}{4}\right)$$

$\therefore$ General solution is given by: $y_2$ = C.F. +P.I.

i.e. $y_2 = y_c + y_p$

$$\Rightarrow y_2 = \left(c_3 e^{-2x} + c_4 e^{-\frac{4}{3}x}\right) + 2e^{-x} + \frac{1}{8}\left((2x+3) - \frac{10}{4}\right)$$

# MTH166

## Lecture-21

## Partial Differential Equations (PDE)

**Unit 4: Partial Differential Equations**

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-9**)

**Topic:**

Partial Differential Equations (PDE)

**Learning Outcomes:**

**1.** Formation of PDE by elimination of arbitrary constants

**2.** Formation of PDE by elimination of arbitrary functions

**Partial Derivatives:**

Earlier: If $\boldsymbol{u = f(x)}$, it means there is dependent variable $u$ and one independent variable $x$.

So, we differentiate $u$ with respect to $x$ and denote it as: $\frac{du}{dx}$

Now: If $\boldsymbol{u = f(x, y)}$, it means there is dependent variable $u$ and two independent variable $x$ and $y$. So, we can differentiate $u$ with respect to $x$ or y, denoted as: $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial y}$ respectively.

**Standard Notations:**

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= u_x = p \\ \frac{\partial u}{\partial y} &= u_y = q \end{aligned}\right\} \text{These are called first order partial derivatives.}$$

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= u_{xx} = r \\ \frac{\partial^2 u}{\partial x \partial y} \ or \ \frac{\partial^2 u}{\partial y \partial x} &= u_{xy} \ or \ \ u_{yx} = s \\ \frac{\partial^2 u}{\partial y^2} &= u_{yy} = t \end{aligned}\right\} \text{These are called second order partial derivatives.}$$

**Partial Differential Equation:**

An equation of the form: $f(x, y, u, p, q) = 0$ is called first order partial differential equation.

An equation of the form: $g(x, y, u, p, q, r, s, t) = 0$ is called second order partial differential equation.

**Methods of Formation of PDE:**

1. **By elimination of arbitrary constants**

**Problem 1. Form the PDE for: $\boldsymbol{u = ax + by}$, *a* and *b* are constants**

**Solution.** Given $u = ax + by$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = a(1) + b(0) = a \quad \Rightarrow p = a$

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = a(0) + b(1) = b \quad \Rightarrow q = b$

Putting values of $a$ and $b$ in equation (1):

$u = px + qy$ **or** $u = \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y$ which is the required PDE.

**Problem 2. Form the PDE for: $\boldsymbol{u = ax + by + a^4 + b^4}$, *a* and *b* are constants**

**Solution.** Given $u = ax + by + a^4 + b^4$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = a(1) + b(0) = a \quad \Rightarrow p = a$

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = a(0) + b(1) = b \quad \Rightarrow q = b$

Putting values of $a$ and $b$ in equation (1):

$u = px + qy + p^4 + q^4$ **or** $u = \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y + \left(\frac{\partial u}{\partial x}\right)^4 + \left(\frac{\partial u}{\partial y}\right)^4$

which is required PDE.

**Problem 3.** **Form the PDE for**: $\boldsymbol{u = (x-\alpha)^2 + (y-\beta)^2}$, $\boldsymbol{\alpha}$ **and** $\boldsymbol{\beta}$ **are constants**

**Solution.** Given $u = (x-\alpha)^2 + (y-\beta)^2$ (1) $[u = f(x,y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = 2(x-\alpha) \quad \Rightarrow p = 2(x-\alpha) \quad \Rightarrow \frac{p}{2} = (x-\alpha)$

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = 2(y-\beta) \quad \Rightarrow q = 2(y-\beta) \quad \Rightarrow \frac{q}{2} = (y-\beta)$

Putting values of $(x-\alpha)$ and $(y-\beta)$ in equation (1):

$u = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$ **or** $4u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

which is required PDE.

**Polling Quiz**

**The PDE for function:** $\boldsymbol{u = ax + by + a^2b^2}$, ***a* and *b* are constants, is:**

**(A)** $u = aq + bp + q^2p^2$

**(B)** $u = ap + bq + a^2b^2$

**(C)** $u = px + qy + p^2q^2$

**Methods of Formation of PDE:**

**2. By elimination of arbitrary functions**

**Problem 1.** **Form the PDE for**: $\boldsymbol{u = f(x^2 + y^2)}$

**Solution.** Given $u = f(x^2+y^2)$ (1) $[u = f(x,y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = f'(x^2+y^2)(2x) \Rightarrow p = f'(x^2+y^2)(2x) \Rightarrow \frac{p}{2x} = f'(x^2+y^2)$ (2)

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = f'(x^2+y^2)(2y) \Rightarrow q = f'(x^2+y^2)(2y) \Rightarrow \frac{q}{2y} = f'(x^2+y^2)$ (3)

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$\Rightarrow py = qx$ **or** $\left(\frac{\partial u}{\partial x}\right)y = \left(\frac{\partial u}{\partial y}\right)x$ which is the required PDE.

**Problem 2.** **Form the PDE for**: $\boldsymbol{u = f\left(\frac{x}{y}\right)}$

**Solution.** Given $u = f\left(\frac{x}{y}\right)$ (1) $[u = f(x,y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow p = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow py = f'\left(\frac{x}{y}\right)$ (2)

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow -q\frac{y^2}{x} = f'\left(\frac{x}{y}\right)$ (3)

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$\Rightarrow py = -q\frac{y^2}{x} \quad \Rightarrow px + qy = 0$ **or** $\left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y = 0$

which is the required PDE.

**Problem 3.** **Form the PDE for**: $\boldsymbol{u = f(ax + by)}$

**Solution.** Given $u = f(ax + by)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t $x$

$\frac{\partial u}{\partial x} = f'(ax + by)(a) \quad \Rightarrow \frac{p}{a} = f'(ax + by) \quad (2)$

Differentiating (1) partially w.r.t $y$

$\frac{\partial u}{\partial y} = f'(ax + by)(b) \quad \Rightarrow \frac{q}{b} = f'(ax + by) \quad (3)$

Comparing (2) and (3), we get: $\frac{p}{a} = \frac{q}{b}$

$\Rightarrow pb = qa$ **or** $\left(\frac{\partial u}{\partial x}\right) b = \left(\frac{\partial u}{\partial y}\right) a$

which is the required PDE.

**Polling Quiz**

**The PDE for function**: $\boldsymbol{u = f\left(\frac{ax}{by}\right)}$, ***a* and *b* are constants, is:**

**(A)** $px = qy$

**(B)** $px + qy = 0$

**(C)** $py = qx$

# MTH166

## Lecture-22

## Classification of PDE

**Unit 4: Partial Differential Equations**

(**Book:** Advanced Engineering Mathematics by R.K.Jain and S.R.K Iyengar,**Chapter-9**)

**Topic:**

Partial Differential Equations (PDE)

**Learning Outcomes:**

**1.** Classification of PDE: Hyperbolic, Parabolic and Elliptic

**Classification/Nature of Partial Differential Equations:**

Let us consider a general second order homogeneous PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \quad \text{where} \quad u = f(x,y)$$

or

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ (1)

or $Ar + Bs + Ct + f(x,y,u,p,q) = 0$ (1)

Equation (1) will be:

**1.** Hyperbolic iff $(B^2 - 4AC) > 0$

**2.** Parabolic iff $(B^2 - 4AC) = 0$

**3.** Elliptic iff $(B^2 - 4AC) < 0$

Note: Coefficients of second order partial derivatives only decide nature of PDE.

**Classify the following PDE as Hyperbolic, Parabolic or Elliptic:**

**Problem 1.** $\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial y}$

**Solution.** Given equ. is: $u_{xy} - 3u_y = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 0, B = 1, C = 0$

Here $(B^2 - 4AC) = (1)^2 - 4(0)(0) = 1 > 0$

So, equation (1) is Hyperbolic.

**Problem 2.** $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

**Solution.** Given equ. is: $u_{xx} + 2u_{xy} + u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 2, C = 1$

Here $(B^2 - 4AC) = (2)^2 - 4(1)(1) = 4 - 4 = 0$

So, equation (1) is Parabolic.

**Problem 3.** $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$

**Solution.** Given equ. is: $u_{xx} + 3u_{yy} - u_x = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 0, C = 3$

Here $(B^2 - 4AC) = (0)^2 - 4(1)(3) = 0 - 12 < 0$

So, equation (1) is Elliptic.

**Polling Quiz**

Let us consider a general second order homogeneous PDE:

$Ar + Bs + Ct + f(x, y, u, p, q) = 0$ (1)

Equation (1) will be Hyperbolic iff :

(A) $(B^2 - 4AC) > 0$

(B) $(B^2 - 4AC) = 0$

(C) $(B^2 - 4AC) < 0$

**Problem 4.** $y\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$

**Solution.** Given equ. is: $yu_{xx} + 2xyu_{xy} + yu_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = y, B = 2x, C = y$

Here $(B^2 - 4AC) = (2x)^2 - 4(y)(y) = 4(x^2 - y^2)$

So, equation (1) is:

Hyperbolic iff $(x^2 - y^2) > 0$

Parabolic iff $(x^2 - y^2) = 0$

Elliptic iff $(x^2 - y^2) < 0$

**Problem 5.** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **[2D-Laplace Equation]**

**Solution.** Given equ. is: $u_{xx} + u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 0, C = 1$

Here $(B^2 - 4AC) = (0)^2 - 4(1)(1) = -4 < 0$

So, **Laplace** equation (1) is **Elliptic** in nature.

**Problem 6.** $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Wave Equation]; $c^2$ is Diffusivity constant.**

**Solution.** Given equ. is: $c^2 u_{xx} - u_{tt} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$

We have $A = c^2, B = 0, C = -1$

Here $(B^2 - 4AC) = (0)^2 - 4(c^2)(-1) = 4c^2 > 0$

So, **Wave** equation (1) is **Hyperbolic** in nature.

**Problem 7.** $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Heat Equation]; $c^2$ is Diffusivity constant.**

**Solution.** Given equ. is: $c^2 u_{xx} - u_t = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$

We have $A = c^2, B = 0, C = 0$

Here $(B^2 - 4AC) = (0)^2 - 4(c^2)(0) = 0$

So, **Heat** equation (1) is **Parabolic** in nature.

**Polling Quiz**

The nature of PDE: $(1-y)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (1+y)\frac{\partial^2 u}{\partial y^2} = 0$ is:

(A) Hyperbolic

(B) Parabolic

(C) Elliptic

(D) All of above is possible

**Problem 8.** $\mathbf{(1-y)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (1+y)\frac{\partial^2 u}{\partial y^2} = 0}$

**Solution.** Given equ. is: $(1-y)u_{xx} + 2xu_{xy} + (1+y)u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = (1-y), B = 2x, C = (1+y)$

Here $(B^2 - 4AC) = (2x)^2 - 4(1-y)(1+y) = 4(x^2 + y^2 - 1)$

So, equation (1) is:

Hyperbolic iff $(x^2 + y^2 - 1) > 0$

Parabolic iff $(x^2 + y^2 - 1) = 0$

Elliptic iff $(x^2 + y^2 - 1) < 0$

**Polling Quiz**

The nature of one dimensional Heat Equation is:

(A) Hyperbolic

(B) Parabolic

(C) Elliptic

**Polling Quiz**

The nature of one dimensional Wave Equation is:

(A) Hyperbolic

(B) Parabolic

(C) Elliptic

**Polling Quiz**

The nature of two dimensional Laplace Equation is:

(A) Hyperbolic

(B) Parabolic

(C) Elliptic

# MTH166

## Lecture-23
## Solution of Partial Differential Equations
## And
## Solution of 1D-Heat Equation

**Topic:**

Solution of Partial Differential Equations

**Learning Outcomes:**

To solve first order PDE using separation of variables solution.

To solve one dimensional Heat Equation.

**Find the separation of variables solution of following PDE:**

**Problem 1.** $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

**Solution.** The given equation is: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ (1)

Let solution be: $u(x, y) = XY$ (2) where $X = f(x), Y = g(y)$

$\Rightarrow \frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Equation (1) becomes: $X'Y = XY'$

$\Rightarrow \frac{X'}{X} = \frac{Y'}{Y} = k$ (Say)

Taking these pairs one by one

$\Rightarrow \frac{X'}{X} = k$

$\Rightarrow \int \frac{X'}{X} dx = k \int dx$

$\Rightarrow \log X = kx + c_1$

$\Rightarrow X = e^{(kx+c_1)}$

also $\frac{Y'}{Y} = k$

$\Rightarrow \int \frac{Y'}{Y} dy = k \int dy$

$\Rightarrow \log Y = ky + c_2$

$\Rightarrow Y = e^{(ky+c_2)}$

Required solution is: $u(x, y) = XY = e^{(kx+c_1)} e^{(ky+c_2)} = e^{(c_1+c_2)} e^{k(x+y)}$

$\Rightarrow u(x, y) = Ae^{k(x+y)}$ **Answer.** $(A = e^{(c_1+c_2)})$

**Find the separation of variables solution of following PDE:**

**Problem 2.** $4\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=0$

**Solution.** The given equation is: $4\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=0$ (1)

Let solution be: $u(x,y)=XY$ (2) where $X=f(x), Y=g(y)$

$\Rightarrow \frac{\partial u}{\partial x}=X'Y$ and $\frac{\partial u}{\partial y}=XY'$

Equation (1) becomes: $4X'Y+3XY'=0 \quad \Rightarrow 4X'Y=-3XY'$

$\Rightarrow 4\frac{X'}{X}=-3\frac{Y'}{Y}=k$ (Say)

Taking these pairs one by one

$\Rightarrow 4\frac{X'}{X}=k$

$\Rightarrow \int \frac{X'}{X}dx=\frac{k}{4}\int dx$

$\Rightarrow \log X=\frac{k}{4}x+c_1$

$\Rightarrow X=e^{\left(\frac{k}{4}x+c_1\right)}$

also $-3\frac{Y'}{Y}=k$

$\Rightarrow \int \frac{Y'}{Y}dy=-\frac{k}{3}\int dy$

$\Rightarrow \log Y=-\frac{k}{3}y+c_2$

$\Rightarrow Y=e^{\left(-\frac{k}{3}y+c_2\right)}$

Required solution is: $u(x,y)=XY=e^{\left(\frac{k}{4}x+c_1\right)}e^{\left(-\frac{k}{3}y+c_2\right)}=e^{(c_1+c_2)}e^{k(\frac{1}{4}x-\frac{1}{3}y)}$

$\Rightarrow u(x,y)=Ae^{\frac{k}{12}(3x-4y)}$ **Answer.** $(A=e^{(c_1+c_2)})$

**Find the separation of variables solution of following PDE:**

**Problem 3.** $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0$

**Solution.** The given equation is: $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0$ (1)

Let solution be: $u(x,y)=XY$ (2) where $X=f(x), Y=g(y)$

$\Rightarrow \frac{\partial u}{\partial x}=X'Y$ and $\frac{\partial u}{\partial y}=XY'$

Equation (1) becomes: $yX'Y+xXY'=0 \quad \Rightarrow yX'Y=-xXY'$

$\Rightarrow \frac{1}{x}\frac{X'}{X}=-\frac{1}{y}\frac{Y'}{Y}=k$ (Say)

Taking these pairs one by one

$\Rightarrow \frac{1}{x}\frac{X'}{X}=k$

$\Rightarrow \int \frac{X'}{X}dx=k\int x dx$

$\Rightarrow \log X=k\frac{x^2}{2}+c_1$

$\Rightarrow X=e^{\left(k\frac{x^2}{2}+c_1\right)}$

also $-\frac{1}{y}\frac{Y'}{Y}=k$

$\Rightarrow \int \frac{Y'}{Y}dy=-k\int y dy$

$\Rightarrow \log Y=-k\frac{y^2}{2}+c_2$

$\Rightarrow Y=e^{\left(-k\frac{y^2}{2}+c_2\right)}$

Required solution is: $u(x,y)=XY=e^{\left(k\frac{x^2}{2}+c_1\right)}e^{\left(-k\frac{y^2}{2}+c_2\right)}=e^{(c_1+c_2)}e^{k(\frac{x^2}{2}-\frac{y^2}{2})}$

$\Rightarrow u(x,y)=Ae^{\frac{k}{2}(x^2-y^2)}$ **Answer.** $(A=e^{(c_1+c_2)})$

**Polling Quiz**

If $u = f(x,t)$, then the solution will be of the form:

(A) $u(x,y) = X(x)Y(y)$

(B) $u(x,t) = X(x)T(t)$

(C) $u(x,y) = X(t)Y(x)$

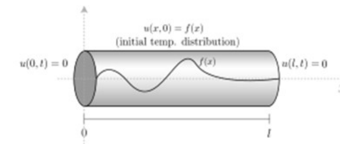
**Problem.** **Solve** $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Heat Equation];** $c^2$ **is Diffusivity constant.**

**Solution.** The given one dimensional Heat equation is:

$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ (1) $0 \le x \le l\ (length), t > 0\ (time)$

Let solution be: $u(x,t) = XT$ (2) where $X = f(x), T = g(t)$

$\Rightarrow \frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = X''T$




Equation (1) becomes: $XT' = C^2 X''T$

$\Rightarrow \frac{X''}{X} = \frac{1}{C^2}\frac{T'}{T} = k$ (Say)

As $k$ can take three values: zero, positive or negative, so we have following three cases.

**Case 1.** When $k = 0$

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = 0 \quad \Rightarrow X'' = 0 \quad \Rightarrow X = ax + b$

Also $\frac{1}{C^2}\frac{T'}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = 0 \quad \Rightarrow T' = 0 \quad \Rightarrow T = c$

Required solution of equation (1) is:

$u(x,t) = XT = (ax + b)c = Ax + B$

**Case 2.** When $k = p^2$ (Positive)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = p^2 \quad \Rightarrow X'' - p^2 X = 0$

S.F. $(D^2 - p^2)X = 0$

A.E. $(D^2 - p^2) = 0 \quad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $\frac{1}{C^2}\frac{T'}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = p^2 \quad \Rightarrow T' - C^2p^2T = 0$

S.F. $(D - C^2p^2)T = 0$

A.E. $(D - C^2p^2) = 0 \quad \Rightarrow D = C^2p^2$

$\therefore T = ce^{C^2p^2t}$

Required solution of equation (1) is:

$$u(x,t) = XT = (ae^{px} + be^{-p}\ )\left(ce^{C^2p^2t}\right) = (Ae^{px} + Be^{-p}\ )e^{C^2p^2t}$$

**Case 3.** When $k = -p^2$ (Negative)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2 X = 0$

S.F. $(D^2 + p^2)X = 0$

A.E. $(D^2 + p^2) = 0 \qquad \Rightarrow D = \pm ip$

$\therefore X = e^{0x}(a\cos px + b\sin px)$

Also $\frac{1}{C^2}\frac{T'}{T} = k \qquad \Rightarrow \frac{1}{C^2}\frac{T'}{T} = -p^2 \qquad \Rightarrow T' + C^2p^2T = 0$

S.F. $(D + C^2p^2)T = 0$

A.E. $(D + C^2p^2) = 0 \qquad \Rightarrow D = -C^2p^2t$

$\therefore T = ce^{-C^2p^2t}$

Required solution of equation (1) is:

$u(x,t) = XT = (a\cos px + b\sin px)\left(ce^{-C^2p^2t}\right) = (A\cos px + B\sin px)e^{-C^2p^2t}$

This is the most suitable and practically feasible solution of heat equation.

**Note: Heat Equation**

Equation: $\frac{\partial u}{\partial t} = C^2\frac{\partial^2 u}{\partial x^2}$

Nature: Parabolic

Solution: **1.** $u(x,t) = Ax + B$

**2.** $u(x,t) = (Ae^{px} + Be^{-p}\ )\,e^{C^2p^2t}$

or

$u(x,t) = (A\cosh px + B\sinh px)e^{C^2p^2t}$

**3.** $u(x,t) = (A\cos px + B\sin px)e^{-C^2p^2t}$ **(Most suitable one)**

# MTH166

## Lecture-24

## Solution of Wave Equation

**Topic:**

Solution of Partial Differential Equations

**Learning Outcomes:**

**1.** To solve one dimensional Wave Equation

**2.** D' Alembert's solution of infinitely long wave

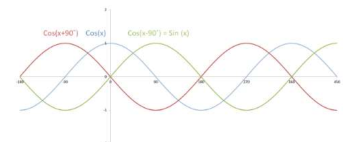
**Problem.** **Solve** $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Wave Equation];** $c^2$ **is Diffusivity constant.**

**Solution.** The given one dimensional wave equation is:

$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ (1) $0 \leq x \leq l\ (length), t > 0\ (time)$

Let solution be: $u(x,t) = XT$ (2) where $X = f(x), T = g(t)$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} = XT''$ and $\frac{\partial^2 u}{\partial x^2} = X''T$




Equation (1) becomes: $XT'' = C^2 X''T$

$\Rightarrow \frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = k$ (Say)

As $k$ can take three values: zero, positive or negative, so we have following three cases.

**Case 1.** When $k = 0$

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = 0 \quad \Rightarrow X'' = 0 \quad \Rightarrow X = ax + b$

Also $\frac{1}{C^2}\frac{T''}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = 0 \quad \Rightarrow T'' = 0 \quad \Rightarrow T = ct + d$

Required solution of equation (1) is:

$u(x,t) = XT = (ax + b)(ct + d)$

**Case 2.** When $k = p^2$ (Positive)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = p^2 \quad \Rightarrow X'' - p^2 X = 0$

S.F. $(D^2 - p^2)X = 0$

A.E. $(D^2 - p^2) = 0 \quad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $\frac{1}{C^2}\frac{T''}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = p^2 \quad \Rightarrow T'' - C^2 p^2 T = 0$

S.F. $(D^2 - C^2 p^2)T = 0$

A.E. $(D^2 - C^2 p^2) = 0 \quad \Rightarrow D = \pm Cp$

$\therefore T = ce^{Cpt} + de^{-Cpt}$

Required solution of equation (1) is:

$u(x,t) = XT = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = XT = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

**Case 3.** When $k = -p^2$ (Negative)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2 X = 0$

S.F. $(D^2 + p^2)X = 0$

A.E. $(D^2 + p^2) = 0 \quad \Rightarrow D = \pm ip$

$\therefore X = e^{0x}(a\cos px + b\sin px)$

Also $\frac{1}{C^2}\frac{T''}{T} = k \quad \Rightarrow \frac{1}{C^2}\frac{T''}{T} = -p^2 \quad \Rightarrow T'' + C^2 p^2 T = 0$

S.F. $(D^2 + C^2p^2)T = 0$

A.E. $(D^2 + C^2p^2) = 0 \quad \Rightarrow D = \pm iCp$

$\therefore T = e^{0t}(c\cos Cpt + d\sin Cpt)$

Required solution of equation (1) is:

$u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$

This is the most suitable and practically feasible solution of wave equation.

**Note: Wave Equation**

Equation: $\frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2}$

Nature: Hyperbolic

Solution: **1.** $u(x,t) = (ax+b)(ct+d)$

**2.** $u(x,t) = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

**3.** $u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$ **(Most suitable one)**

**Polling Quiz**

Which of the following is a one-dimensional wave equation?

(A) $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

(B) $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

(C) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(D) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

**D'Alembert's Solution of Infinitely long wave (string)**

Let given wave equation be: $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ (1)

such that $-\infty < x < \infty, t > 0$

with initial displacement $= f(x)$ and initial velocity $= g(x)$

Then. D' Alembert's solution of equation (1) is given by:

$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$

**Find D' Alembert's solution of following:**

**Problem 1.** $f(x) = \sin x, g(x) = a$

**Solution.** $f(x+ct) = \sin(x+ct),\ f(x-ct) = \sin(x-ct), g(s) = a$

D' Alembert's solution is given by:

$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$

$\Rightarrow u(x,t) = \frac{1}{2}[sin(x+ct) + sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} a\,ds$

$\Rightarrow u(x,t) = \frac{1}{2}[sinxcosct + cosxsinct + sinxcosct - cosxsinct] + \frac{a}{2c}[x+ct-x+c\,t]$

$\Rightarrow u(x,t) = \frac{1}{2}[2sinxcosct] + \frac{a}{2c}[2ct]$

$\Rightarrow u(x,t) = \sin x \cos ct + at$ **Answer.**

**Find D' Alembert's solution of following:**

**Problem 2.** $f(x) = 0, g(x) = \cos x$

**Solution.** $f(x+ct) = 0,\ f(x-ct) = 0,\quad g(s) = \cos s$

D' Alembert's solution is given by:

$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2C}\int_{x-ct}^{x+ct} g(s)ds$

$\Rightarrow u(x,t) = \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s\ ds$

$\Rightarrow u(x,t) = \frac{1}{2c}[\sin s]_{x-ct}^{x+ct} = \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)]$

$\Rightarrow u(x,t) = \frac{1}{2c}[sinxcosct + cosxsinct - sinxcosct + cosxsinct]$

$\Rightarrow u(x,t) = \frac{1}{c}[cosxsinct]$ **Answer.**

## Polling Quiz

D'Alembert's solution of infinitely long wave with initial displacement $f(x)$ and initial velocity $g(x)$ is given by:

(A) $u(x,t) = [f(x+Ct) + f(x-Ct)] + \int_{-x}^{x} g(s)ds$

(B) $u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$

(C) $u(x,t) = [f(x+Ct) - f(x-Ct)] + \int_{x-Ct}^{x+Ct} g(s)ds$

(D) $u(x,t) = \frac{1}{2}[f(x+Ct) + g(x-Ct)]$

# MTH166

## Lecture-25

## Solution of Laplace Equation

**Topic:**

Solution of Partial Differential Equations

**Learning Outcomes:**

To solve two dimensional Laplace Equation

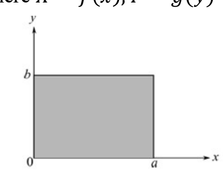
**Problem.** **Solve** $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ **[2D-Laplace Equation]**

**Solution.** The given two dimensional Laplace equation is:

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (1) $0 \le x \le a, 0 \le y \le b$

Let solution be: $u(x,y)=XY$ (2) where $X=f(x), Y=g(y)$

$\Rightarrow \frac{\partial^2 u}{\partial x^2}=X''Y$ and $\frac{\partial^2 u}{\partial y^2}=XY''$




Equation (1) becomes: $X''Y+XY''=0$

$\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=k$ (Say)

As $k$ can take three values: zero, positive or negative, so we have following three cases.

**Case 1.** When $k = 0$

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = 0 \quad \Rightarrow X'' = 0 \quad \Rightarrow X = ax + b$

Also $-\frac{Y''}{Y} = k \quad \Rightarrow -\frac{Y''}{Y} = 0 \quad \Rightarrow Y'' = 0 \quad \Rightarrow Y = cy + d$

Required solution of equation (1) is:

$u(x, y) = XY = (ax + b)(cy + d)$

**Case 2.** When $k = p^2$ (Positive)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = p^2 \quad \Rightarrow X'' - p^2X = 0$

S.F. $(D^2 - p^2)X = 0$

A.E. $(D^2 - p^2) = 0 \quad \Rightarrow D = \pm p$

$\therefore X = ae^{px} + be^{-px}$

Also $-\frac{Y''}{Y} = k \quad \Rightarrow -\frac{Y''}{Y} = p^2 \quad \Rightarrow Y'' + p^2Y = 0$

S.F. $(D^2 + p^2)Y = 0$

A.E. $(D^2 + p^2) = 0 \quad \Rightarrow D = \pm ip$

$\therefore Y = c\cos py + d\sin py$

Required solution of equation (1) is:

$u(x, y) = XY = (ae^{px} + be^{-p}\ )(c\cos py + d\sin py)$

**Case 3.** When $k = -p^2$ (Negative)

$\frac{X''}{X} = k \quad \Rightarrow \frac{X''}{X} = -p^2 \quad \Rightarrow X'' + p^2X = 0$

S.F. $(D^2 + p^2)X = 0$

A.E. $(D^2 + p^2) = 0 \quad \Rightarrow D = \pm ip$

$\therefore X = a\cos px + b\sin px$

Also $-\frac{Y''}{Y} = k \quad \Rightarrow -\frac{Y''}{Y} = -p^2 \quad \Rightarrow Y'' - p^2Y = 0$

S.F. $(D^2 - p^2)Y = 0$

A.E. $(D^2 - p^2) = 0$ Type equation here. $\Rightarrow D = \pm p$

$\therefore Y = ce^{py} + de^{-p}$

Required solution of equation (1) is:

$u(x, y) = XY = (a\cos px + b\sin px)\,(ce^{py} + de^{-py})$

**Note: Laplace Equation**

Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Nature: Elliptic

Solution: **1.** $u(x, y) = (ax + b)(cy + d)$

**2.** $u(x, y) = (ae^{px} + be^{-px})(c\cos py + d\sin py)$

or

$u(x, y) = (a\cosh px + b\sinh px)(c\cos py + d\sin py)$

**3.** $u(x, y) = (a\cos px + b\sin px)\,(ce^{py} + de^{-p}\ )$

or

$u(x, y) = (a\cos px + b\sin px)\,(c\cosh py + d\sinh py)$

**Problem.** Prove that a two dimensional heat equation becomes Laplace equation in steady state.

**Solution.** The two dimensional heat equation is:

$$\frac{\partial u}{\partial t} = C^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \qquad (1)$$

In steady-state: $\frac{\partial u}{\partial t} = 0$

So, in steady-state equation (1) becomes:

$$0 = C^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \Rightarrow \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \qquad (\text{As } C^2 \neq 0)$$

which is a two dimensional Laplace Equation.

# MTH166

## Lecture-26

## Boundary Value Problems of Heat Equation

**Topic:**

Solution of Partial Differential Equations

**Learning Outcomes:**

To solve Boundary value problems of one dimensional Heat Equation.

**<u>Recall:</u> <u>Heat Equation</u>**

<u>Equation:</u> $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

<u>Nature:</u> Parabolic

<u>Solution:</u> **1.** $u(x,t) = Ax + B$

**2.** $u(x,t) = (Ae^{px} + Be^{-p}\ )\, e^{C^2p^2t}$

or

$u(x,t) = (A\cosh px + B\sinh px)e^{C^2p^2t}$

**3.**$u(x,t) = (A\cos px + B\sin px)e^{-C^2p^2t}$ **(Most suitable one)**

**<u>Problem.</u>** Solve $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ , $0 \le x \le l, t > 0$ under boundary conditions:

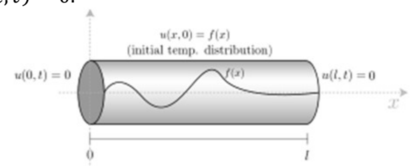
$u(0,t) = 0, u(l,t) = 0.$

**<u>Solution.</u>** The given one dimensional Heat equation is:

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$0 \le x \le l\ (length), t > 0\ (time)$

Such that :$u(0,t) = 0, u(l,t) = 0.$




We know the most appropriate solution of equation (1) is:

$$u(x,t) = (A\cos px + B\sin px)e^{-C^2p^2t} \qquad (2)$$

Using the condition $u(0,t) = 0$, we get:

$u(0,t) = (A\cos 0 + B\sin 0)e^{-C^2p^2t}$

$\Rightarrow 0 = A.e^{-C^2p^2t}$ $\quad \Rightarrow \boldsymbol{A = 0}$

Equation (2) becomes:

$$u(x,t) = (B\sin px)e^{-C^2p^2t} \qquad (3)$$

Using the condition $u(l,t) = 0$, we get:

$u(l,t) = (B\sin pl)e^{-C^2p^2t}$

$\Rightarrow 0 = (B\sin pl)e^{-C^2p^2t}$

$\Rightarrow 0 = (B\sin pl)e^{-C^2p^2t}$

$\Rightarrow \sin pl = 0$ (As $B \neq 0$)

$\Rightarrow \sin pl = \sin n\pi$

$\Rightarrow pl = n\pi$ $\quad \Rightarrow \boldsymbol{p = \frac{n\pi}{l}}$

Equation (3) becomes:

$u(x,t) = (B\sin px)e^{-C^2p^2t}$

$$\Rightarrow u(x,t) = (B\sin\frac{n\pi}{l})e^{-\frac{C^2n^2\pi^2}{l^2}t} \qquad (4)$$

This gives temperature distribution only at one point. There are inifinite number of points on the rod.

By principle of superposition:

$u(x,t)=\sum_{n=1}^{\infty}(B_n \sin\frac{n\pi x}{l})e^{-\frac{c^2n^2\pi^2}{l^2}t}$ (5)

Which is the required solution of Heat equation (1).

Where $B_n=\frac{1}{2l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx$

Here $u(x,0)$ is the initial temperature.

**Polling Quiz**

If $u(x,t)=(A\cos px+B\sin px)e^{-C^2p^2t}$ is the solution of one-dimensional heat equation, then which of the following is true when initial condition $u(0,t)=0$ is applied to the solution:

(A) $A=0$

(B) $p=n\pi/l$

(C) $B=0$

(D) $C=0$

# MTH166

## Lecture-27

## Boundary Value Problems -Wave Equation

**Topic:**

Solution of Partial Differential Equations

**Learning Outcomes:**

To solve Boundary value problems of one dimensional Wave Equation.

**Recall:Wave Equation**

Equation: $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$

Nature: Hyperbolic

Solution: **1.** $u(x,t) = (ax+b)(ct+d)$

**2.** $u(x,t) = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

**3.** $u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$ **(Most suitable one)**

**Problem.** Solve $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ , $0 \le x \le l, t > 0$ under boundary conditions:

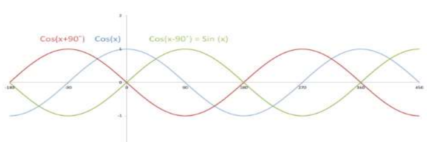
$u(0,t) = 0, u(l,t) = 0.$

**Solution.** The given one dimensional wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$0 \le x \le l\ (length), t > 0\ (time)$

Such that :$u(0,t) = 0, u(l,t) = 0.$




We know the most appropriate solution of equation (1) is:

$$u(x,t) = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt) \qquad (2)$$

Using the condition $u(0,t) = 0$, we get:

$u(0,t) = (a\cos 0 + b\sin 0)\,(c\cos Cpt + d\sin Cpt)$

$\Rightarrow 0 = a(c\cos Cpt + d\sin Cpt) \qquad \Rightarrow \boldsymbol{a = 0}$

Equation (2) becomes:

$$u(x,t) = (b\sin px)(c\cos Cpt + d\sin Cpt) \qquad (3)$$

Using the condition $u(l,t) = 0$, we get:

$u(l,t) = (b\sin pl)(c\cos Cpt + d\sin Cpt)$

$\Rightarrow 0 = (b\sin pl)(c\cos Cpt + d\sin Cpt)$

$\Rightarrow 0 = (b \sin pl)(c \cos Cpt + d \sin Cpt)$

$\Rightarrow \sin pl = 0$ (As $b \neq 0$)

$\Rightarrow \sin pl = \sin n\pi$

$\Rightarrow pl = n\pi$ $\Rightarrow \boldsymbol{p = \frac{n\pi}{l}}$

Equation (3) becomes:

$u(x,t) = (b \sin px)(c \cos Cpt + d \sin Cpt)$

$\Rightarrow u(x,t) = (b \sin \frac{n\pi}{l})(c \cos \frac{n\pi C}{l} t + d \sin \frac{n\pi}{l} t)$

$$\Rightarrow u(x,t) = \left(A \cos \frac{n\pi C}{l} t + B \sin \frac{n\pi C}{l} t\right) \sin \frac{n\pi x}{l} \quad (4)$$

By principle of superposition:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi C}{l} t + B_n \sin \frac{n\pi C}{l} t\right) \sin \frac{n\pi x}{l} \quad (5)$$

Which is the required solution of Wave equation (1).

Where $A_n = \frac{1}{2l} \int_0^l u(x,0) \sin \frac{n\pi x}{l} dx;$ $u(x,0)$ is the initial displacement.

And $B_n = \frac{1}{2\pi C} \int_0^l \frac{\partial u}{\partial t}(x,0) \sin \frac{n\pi x}{l} dx;$ $\frac{\partial u}{\partial t}(x,0)$ is the initial velocity.

**Polling Quiz**

If $u(x,t) = (a\, cos\, px + b\, sin\, px)\,(c\, cos\, Cpt + d\, sin\, Cpt)$is the solution of one-dimensional wave equation, then which of the following is true when initial condition $u(l,t) = 0$ is applied to the solution:

(A) $a = 0$

(B) $p = n\pi/l$

(C) $b = 0$

(D) $C = 0$

# MTH 166

## Lecture-28

## Level Surfaces and Parametric Equation of a Straight Line

**Unit 5:** **Vector Calculus-I**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-15**)

**Topic:**

Level Surfaces, Parametric equation of a Straight Line

**Learning Outcomes:**

**1.** To find Level surfaces of a given scalar function

**2.** To find parametric equation of a straight line

**Level Surfaces:**

Let $f(x,y,z)$ be a given scalar surface.

Level surfaces corresponding to this $f(x,y,z)$ are given by:

$f(x,y,z) = c$ (1)

Infact, this equation (1) gives family of surfaces that never intersect with each other

For different values of constant *c*, we get different members of this family of level surfaces.

**Find the Level surfaces of the scalar fields defined by following functions:**

**Problem 1.** $\boldsymbol{f = x + y + z}$

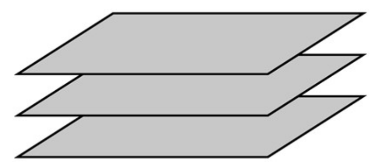
**Solution.** The given scalar function is:

$f = x + y + z$ (1)

Level surfaces are given by: $f = c$

$\Rightarrow x + y + z = c$

Which is a family of Parallel planes.



**Find the Level surfaces of the scalar fields defined by following functions:**

**Problem 2.** $\boldsymbol{f = x^2 + y^2 + z^2}$

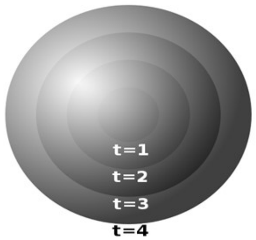
**Solution.** The given scalar function is:

$f = x^2 + y^2 + z^2$ (1)

Level surfaces are given by: $f = t$

$\Rightarrow x^2 + y^2 + z^2 = t$

Which is a family of concentric Spheres.




**Find the Level surfaces of the scalar fields defined by following functions:**

**Problem 3.** $\boldsymbol{f = x^2 + y^2 - z}$

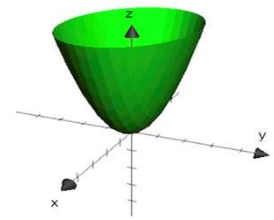
**Solution.** The given scalar function is:

$f = x^2 + y^2 - z$ (1)

Level surfaces are given by: $f = c$

$\Rightarrow x^2 + y^2 - z = c$

Which is a family of Paraboloids.




**Find the Level surfaces of the scalar fields defined by following functions:**

**Problem 4.** $\boldsymbol{f = x^2 + 9y^2 + 16z^2}$

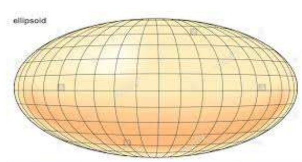
**Solution.** The given scalar function is:

$f = x^2 + 9y^2 + 16z^2$ (1)

Level surfaces are given by: $f = c$

$\Rightarrow x^2 + 9y^2 + 16z^2 = c$

Which is a family of Ellipsoids.




**Find the Level surfaces of the scalar fields defined by following functions:**

**Problem 5.** $\boldsymbol{f = z - \sqrt{x^2 + y^2}}$

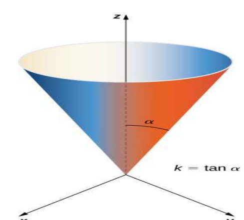
**Solution.** The given scalar function is:

$f = z - \sqrt{x^2 + y^2}$ (1)

Level surfaces are given by: $f = c$

$\Rightarrow z - \sqrt{x^2 + y^2} = c$

Which is a family of Cones.



## Polling Quiz

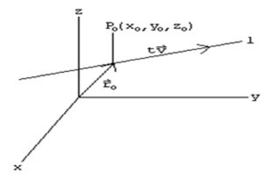
The level surfaces of the scalar field defined by the function f=x+y+z are:

(a) Parallel Planes
(b) Spheres
(c) Paraboloids
(d) Ellipsoids

**Parametric Equation of a Straight line passing through a point and having a given direction:**

Let given point is $P_0(x_0, y_0, z_0)$ and the given direction be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Then Parametric Equation of a Straight line passing through point $P_0$ and having given direction $\vec{v}$ is given by:




$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{v}$ where $t$ is parameter

$\Rightarrow \overrightarrow{r(t)} = (x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}) + t(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (x_0 + tv_1)\hat{\imath} + (y_0 + tv_2)\hat{\jmath} + (z_0 + tv_3)\hat{k}$

**Find the Parametric Equation of a Straight line passing through point $P_0$ and having direction $\vec{b}$ :**

**Problem 1.** $P_0(1,2,3)$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

**Solution.** Parametric Equation of a Straight line passing through point $P_0$ and having given direction $\vec{b}$ is given by:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{b}$ where $t$ is parameter

$\Rightarrow \overrightarrow{r(t)} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$\Rightarrow \overrightarrow{r(t)} = (1+t)\hat{\imath} + (2+2t)\hat{\jmath} + (3+3t)\hat{k}$ **Answer.**

**Find the Parametric Equation of a Straight line passing through point $P_0$ and having direction $\vec{b}$ :**

**Problem 2.** $P_0(1,-1,1)$, $\vec{b} = \hat{\imath} - \hat{\jmath}$

**Solution.** Parametric Equation of a Straight line passing through point $P_0$ and having given direction $\vec{b}$ is given by:

$\overrightarrow{r(t)} = \overrightarrow{P_0} + t\,\vec{b}$ where $t$ is parameter

$\Rightarrow \overrightarrow{r(t)} = (\hat{\imath} - \hat{\jmath} + \hat{k}) + t(\hat{\imath} - \hat{\jmath})$

$\Rightarrow \overrightarrow{r(t)} = (1+t)\hat{\imath} - (1+t)\hat{\jmath} + \hat{k}$ **Answer.**

**Polling Quiz**

The parametric representation of the straight line through the point P(1,2,3) and having direction b ⃗=i ̂+2j ̂+2k ̂is:

(a) (r(t)) ⃗=(1+t) i ̂+2(1+t)j ̂+(3+2t)k ̂

(b) (r(t)) ⃗=(1+t) i ̂-(1+t)j ̂+k ̂

(c) (r(t)) ⃗=(1+t) i ̂-2(1+t)j ̂+(3-2t)k ̂

(d) (r(t)) ⃗=(1+t) i ̂+(1+t) j ̂-k ̂

**Find the parametric representation of the following straight lines/curves. Use the indicated representation wherever given:**

**Problem 1.** $x = y, y = z$

**Solution.** Let $z = t$ where $t$ is a parameter

$\Rightarrow y = t \qquad (\because y = z)$

also $x = t \qquad (\because x = y)$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$ **Answer.**

**Problem 2.** $x + y + z = 3, y - z = 0.$

**Solution.** Let $z = t$ where $t$ is a parameter

$\Rightarrow y = t \qquad (\because y = z)$

also $x + y + z = 3$

$\Rightarrow x = 3 - y - z = 3 - t - t$

$\Rightarrow x = 3 - 2t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (3 - 2t)\hat{\imath} + t\hat{\jmath} + t\hat{k}$ **Answer.**

**Problem 3.** $y^2 + z^2 = 9, x = 9 - y^2,\ y = 3\sin t$.

**Solution.** Since $y = 3\sin t$

$\Rightarrow x = 9 - (3\sin t)^2$

$\Rightarrow x = 9(1 - sin^2 t) = 9cos^2 t$

also $z^2 = 9 - y^2 \quad = 9 - (3\sin t)^2 \quad = 9(1 - sin^2 t) \quad = 9cos^2 t$

$\Rightarrow z = \pm 3\cos t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (9cos^2 t)\hat{\imath} + (3\sin t)\hat{\jmath} \pm (3\cos t)\hat{k}$ **Answer.**

**Problem 4.** $y^2 = x^2 + z^2, y = 2,\ x = 2\sin t$.

**Solution.** Since $x = 2\sin t$ and $y = 2$

$z^2 = y^2 - x^2$

$\Rightarrow z^2 = 4 - (2\sin t)^2 \quad = 4(1 - sin^2 t) \quad = 4cos^2 t$

$\Rightarrow z = \pm 2\cos t$

So, Parametric representation is given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\Rightarrow \overrightarrow{r(t)} = (2\sin t)\hat{\imath} + 2\hat{\jmath} \pm (2\cos t)\hat{k}$ **Answer.**

# MTH 166

## Lecture-29

## Parametric Equation of a Tangent Line, Length of Space Curve, Motion of a Particle

**Topic:**

Vector Differential Calculus

**Learning Outcomes:**

**1.** To find parametric equation of tangent line

**2.** To find Length of Space Curve

**3.** Motion of a body or a particle

**Parametric Equation of Tangent Line:**

**Problem 1.** Find the parametric equation of tangent line to the curve:

$x = t, y = 2t^2, z = 3t^3$ at $t = 2$.

**Solution.** Let $P_0 = (x, y, z)_{t=2} = (t, 2t^2, 3t^3)_{t=2} = (2,8,24)$

Let $\vec{v} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)_{t=2} = \left(\hat{\imath} + 4t\hat{\jmath} + 9t^2\hat{k}\right)_{t=2} = (\hat{\imath} + 8\hat{\jmath} + 36\hat{k})$

Parametric equation of tangent line to the given curve is:

$\overrightarrow{r(\mu)} = \overrightarrow{P_0} + \mu\vec{v}$ where μ is parameter

$\Rightarrow \overrightarrow{r(\mu)} = (2\hat{\imath} + 8\hat{\jmath} + 24\hat{k}) + \mu(\hat{\imath} + 8\hat{\jmath} + 36\hat{k})$

$\Rightarrow \overrightarrow{r(\mu)} = (2 + \mu)\hat{\imath} + (8 + 8\mu)\hat{\jmath} + (24 + 36\mu)\hat{k}$ **Answer.**

**Problem 2.** Find the parametric equation of tangent line to the curve:

$x = t, y = e^t, z = 1$ at $t = 1$.

**Solution.** Let $P_0 = (x, y, z)_{t=1} = (t, e^t, 1)_{t=1} = (1, e, 1)$

Let $\vec{v} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)_{t=1} = \left(\hat{\imath} + e^t\hat{\jmath} + 0\hat{k}\right)_{t=1} = (\hat{\imath} + e\hat{\jmath})$

Parametric equation of tangent line to the given curve is:

$\overrightarrow{r(\mu)} = \overrightarrow{P_0} + \mu\vec{v}$ where μ is parameter

$\Rightarrow \overrightarrow{r(\mu)} = (\hat{\imath} + e\hat{\jmath} + \hat{k}) + \mu(\hat{\imath} + e\hat{\jmath})$

$\Rightarrow \overrightarrow{r(\mu)} = (1 + \mu)\hat{\imath} + (e + e\mu)\hat{\jmath} + \hat{k}$ **Answer.**

**Polling quiz**

The parametric equation of tangent line to the curve:

$x = t, y = t, z = t$ at $t = 1$ is:

(A) $\overrightarrow{r(\mu)} = (1 + \mu)\hat{\imath} + (1 + 2\mu)\hat{\jmath} + 3\hat{k}$

(B) $\overrightarrow{r(\mu)} = (1 + \mu)\hat{\imath} + (1 + \mu)\hat{\jmath} + (1 + \mu)\hat{k}$

(C) $\overrightarrow{r(\mu)} = (1 + \mu)\hat{\imath} + (1 + 2\mu)\hat{\jmath} + (1 + 3\mu)\hat{k}$

(D) None of these.

**Length of Space Curve:**

Let the curve $C$ be represented in the parametric form as:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b$

Then, length of curve $C$ is given by:

$$l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

**Problem 1.** Find the length of the following curve:

$\overrightarrow{r(t)} = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath}, \quad 0 \le t \le 2\pi$

**Solution.** Comparing with: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad a \le t \le b$

$x(t) = a\cos t \quad \Rightarrow \frac{dx}{dt} = -a\sin t$ and $y(t) = a\sin t \quad \Rightarrow \frac{dy}{dt} = a\cos t$

Length of curve $C$ is given by: $l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\Rightarrow l = \int_{t=0}^{t=2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}dt = a\int_{t=0}^{t=2\pi} dt = a(2\pi - 0)$

$\Rightarrow l = 2\pi a$ units **Answer.**

**Problem 2.** Find the length of the following curve:

$\overrightarrow{r(t)} = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + (3t)\hat{k}, \quad -2\pi \le t \le 2\pi$

**Solution.** Comparing with: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad a \le t \le b$

$x(t) = \cos t \Rightarrow \frac{dx}{dt} = -\sin t,\ y(t) = \sin t \Rightarrow \frac{dy}{dt} = \cos t, z(t) = 3t \Rightarrow \frac{dz}{dt} = 3$

Length of curve $C$ is given by: $l = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$\Rightarrow l = \int_{t=-2\pi}^{t=2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (3)^2}dt = \int_{t=-2\pi}^{t=2\pi} \sqrt{1+9}dt$

$\Rightarrow l = \sqrt{10}(2\pi + 2\pi) = 4\pi\sqrt{10}$ units **Answer.**

**Polling quiz**

The length of the following curve:

$\overrightarrow{r(t)} = (2\cos t)\hat{\imath} + (2\sin t)\hat{\jmath}, \quad 0 \le t \le \pi$ is:

(A) $2\pi$ units

(B) $4\pi$ units

(C) $\pi/2$ units

(D) None of these.

**Motion of a body or a particle**

Let position vector of a particle be given by:

$\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Then, we can calculate following quantities:

**1.** Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right)$

**2.** Speed, $S = \left|\overrightarrow{V(t)}\right|$ ( Magnitude of vector $\overrightarrow{V(t)}$ )

**3.** Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right)$

**Problem 1.** The position vector of a moving particle is given by:

$\overrightarrow{r(t)} = (\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.

**Solution.** $\overrightarrow{r(t)} = (\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}$

Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right) = \frac{d}{dt}\left[(\cos t + \sin t)\hat{\imath} + (\sin t - \cos t)\hat{\jmath} + t\hat{k}\right]$

$\Rightarrow \overrightarrow{V(t)} = (-\sin t + \cos t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + 1\hat{k}$

Speed, $S = \left|\overrightarrow{V(t)}\right| = \sqrt{(-\sin t + \cos t)^2 + (\cos t + \sin t)^2 + (1)^2}$

$\Rightarrow S = \sqrt{2(sin^2 t + cos^2 t) + 1} = \sqrt{3}$

Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right) = \frac{d}{dt}\left[(-\sin t + \cos t)\hat{\imath} + (\cos t + \sin t)\hat{\jmath} + 1\hat{k}\right]$

$\Rightarrow \overrightarrow{a(t)} = (-\cos t - \sin t)\hat{\imath} + (-\sin t + \cos t)\hat{\jmath} + 0\hat{k}$

**Problem 2.** The position vector of a moving particle is given by:

$\overrightarrow{r(t)} = t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}$. Determine the velocity, speed and acceleration of the particle in the direction of the motion.

**Solution.** $\overrightarrow{r(t)} = t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}$

Velocity, $\overrightarrow{V(t)} = \frac{d}{dt}\left(\overrightarrow{r(t)}\right) = \frac{d}{dt}\left[t^3\hat{\imath} + t\hat{\jmath} + t^2\hat{k}\right]$

$\Rightarrow \overrightarrow{V(t)} = 3t^2\hat{\imath} + \hat{\jmath} + 2t\hat{k}$

Speed, $S = \left|\overrightarrow{V(t)}\right| = \sqrt{(3t^2)^2 + (1)^2 + (2t)^2}$

$\Rightarrow S = \sqrt{9t^4 + 4t^2 + 1}$

Acceleration, $\overrightarrow{a(t)} = \frac{d}{dt}\left(\overrightarrow{V(t)}\right) = \frac{d}{dt}\left[3t^2\hat{\imath} + \hat{\jmath} + 2t\hat{k}\right]$

$\Rightarrow \overrightarrow{a(t)} = 6t\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$

## Polling quiz

The position vector of a moving particle is given by: $\overrightarrow{r(t)} = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$. The speed of the particle is:

(A) 3 units

(B) $\sqrt{3}$ units

(C) 9 units

(D) None of these.

## Rules of Vector Differentiation

If $\overrightarrow{u(t)}, \overrightarrow{v(t)}$ are vector functions and $\alpha, \beta$ are constants, then following rules of differentiation hold for vector functions in same way as that for scalar function.

**1.** $\frac{d}{dt}[\alpha\vec{u} \pm \beta\vec{v}] = \alpha\frac{d}{dt}(\vec{u}) \pm \beta\frac{d}{dt}(\vec{v})$

**2.** $\frac{d}{dt}[\vec{u}.\vec{v}] = \vec{u}\frac{d}{dt}(\vec{v}) + \vec{v}\frac{d}{dt}(\vec{u})$ $\left(\frac{d}{dt}(Dot\ product\ between\ \vec{u}\ and \vec{v})\right)$

**3.** $\frac{d}{dt}[\vec{u} \times \vec{v}] = \vec{u} \times \frac{d}{dt}(\vec{v}) + \vec{v} \times \frac{d}{dt}(\vec{u})$ $\left(\frac{d}{dt}(Cross\ product\ between\ \vec{u}\ and \vec{v})\right)$

**4.** $\frac{d}{dt}\left[\frac{\vec{u}}{\vec{v}}\right] = \frac{\vec{v}\frac{d}{dt}(\vec{u}) - \vec{u}\frac{d}{dt}(\vec{v})}{(\vec{v})^2}$, provided $\vec{v} \neq 0$

**Problem 1.** If $\overrightarrow{u(t)} = (\sin 2t)\hat{\imath} - (\cos 2t)\hat{\jmath} + t\hat{k}$, $\overrightarrow{v(t)} = (\cos 2t)\hat{\imath} - (\sin 2t)\hat{\jmath} + t^2\hat{k}$

Find $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right]'$

**Solution.** Here we will first find dot product between $\overrightarrow{u(t)}$ and $\overrightarrow{v(t)}$

i.e. $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right] = \left[(\sin 2t)\hat{\imath} - (\cos 2t)\hat{\jmath} + t\hat{k}\right].\left[(\cos 2t)\hat{\imath} - (\sin 2t)\hat{\jmath} + t^2\hat{k}\right]$

$= (\sin 2t)(\cos 2t) + (-\cos 2t)(-\sin 2t) + (t)(t^2)$

$= \sin 4t + t^3$

So, $\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{u(t)}.\overrightarrow{v(t)}\right] = \frac{d}{dt}[\sin 4t + t^3]$

$= 4\cos 4t + 3t^2$ **Answer.**

**Problem 2.** If $\overrightarrow{u(t)} = (6t^2)\hat{\imath} - (t)\hat{\jmath} + (3t^2)\hat{k}$, $\overrightarrow{v(t)} = (t)\hat{\imath} + (t^2)\hat{\jmath} + (2t)\hat{k}$

Find $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right]'$

**Solution.** Here we will first find cross product between $\overrightarrow{u(t)}$ and $\overrightarrow{v(t)}$

i.e. $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6t^2 & -t & 3t^2 \\ t & t^2 & 2t \end{vmatrix} = -(2t^2 + 3t^4)\hat{\imath} - 9t^3\hat{\jmath} + (6t^4 + t^2)\hat{k}$

So, $\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{u(t)} \times \overrightarrow{v(t)}\right] = \frac{d}{dt}\left[-(2t^2 + 3t^4)\hat{\imath} - 9t^3\hat{\jmath} + (6t^4 + t^2)\hat{k}\right]$

$= -(4t + 12t^3)\hat{\imath} - 27t^2\hat{\jmath} + (24t^3 + 2t)\hat{k}$**Answer.**

**Problem 3.** Prove that $\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]' = \overrightarrow{v(t)} \times \overrightarrow{v''(t)}$

**Solution.** Here $\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]' = \frac{d}{dt}\left[\overrightarrow{v(t)} \times \overrightarrow{v'(t)}\right]$

$= \overrightarrow{v(t)} \times \frac{d}{dt}\left(\overrightarrow{v'(t)}\right) + \overrightarrow{v'(t)} \times \frac{d}{dt}\left(\overrightarrow{v(t)}\right)$

$= \left[\overrightarrow{v(t)} \times \overrightarrow{v''(t)}\right] + \left[\overrightarrow{v'(t)} \times \overrightarrow{v'(t)}\right]$

$= \left[\overrightarrow{v(t)} \times \overrightarrow{v''(t)}\right] + 0$ (Cross product of a vector with itself is always zero)

$= \overrightarrow{v(t)} \times \overrightarrow{v''(t)}$

Hence proved.

**<u>Polling quiz</u>**

The position vector of a moving particle is given by: $\overrightarrow{r(t)} = t^2\hat{\imath} + t^2\hat{\jmath} + t^2\hat{k}$. The velocity and speed of the particle at $t = 2$ is:

(A) $\overrightarrow{V(t)} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}, s = 2\sqrt{3}$

(B) $\overrightarrow{V(t)} = 4\hat{\imath} + 4\hat{\jmath} + 4\hat{k}, s = 4\sqrt{3}$

(C) $\overrightarrow{V(t)} = \hat{\imath} + \hat{\jmath} + \hat{k}, s = \sqrt{3}$

(D) None of these.

# MTH 166

## Lecture-30

## Gradient of a Scalar field

**Topic:**

Vector Differential Calculus

**Learning Outcomes:**

**1.** To calculate gradient of a scalar field

**2.** To find normal vector and unit normal vector to a surface

3. To find angle between two given surfaces

**Gradient of a scalar field**

Let $f(x, y, z)$ be a given scalar surface.

Let us consider a vector differential operator called as **Del/Nabla** defined as:

$\vec{\nabla}=\left(\hat{\imath}\,\frac{\partial}{\partial x}+\hat{\jmath}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)$

Then gradient of scalar field $f$ is given by:

$$\text{grad}(f)=\vec{\nabla} f=\left(\hat{\imath}\,\frac{\partial}{\partial x}+\hat{\jmath}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right) f(x, y, z)$$

$$=\left(\hat{\imath}\,\frac{\partial f}{\partial x}+\hat{\jmath}\,\frac{\partial f}{\partial y}+\hat{k}\,\frac{\partial f}{\partial z}\right)$$

$$=\left(f_x\hat{\imath}+f_y\hat{\jmath}+f_z\hat{k}\right).$$

**Problem 1.** Compute the gradient of function $(x^3 - 3x^2y^2 + y^3)$ at point $(1,2)$.

**Solution.** Let $f(x,y,z) = (x^3 - 3x^2y^2 + y^3)$

$\Rightarrow f_x = (3x^2 - 6xy^2), f_y = (3y^2 - 6x^2y)$

Then gradient of scalar field $f$ is given by:

$\text{grad}(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath}) = (3x^2 - 6xy^2)\hat{\imath} + (3y^2 - 6x^2y)\hat{\jmath}$

At point $(1,2)$:

$\text{grad}(f) = \vec{\nabla} f = (3 - 24)\hat{\imath} + (12 - 12)\hat{\jmath} = -21\hat{\imath}$ **Answer.**

**Problem 2.** Compute the gradient of function $\log(x + y + z)$ at point $(1,2,-1)$.

**Solution.** Let $f(x,y,z) = \log(x + y + z)$

$\Rightarrow f_x = f_y = f_z = \frac{1}{(x+y+z)}$

Then gradient of scalar field $f$ is given by:

$\text{grad}(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = \frac{1}{(x+y+z)}(\hat{\imath} + \hat{\jmath} + \hat{k})$

At point $(1,2.-1)$:

$\text{grad}(f) = \vec{\nabla} f = \frac{1}{(1+2-1)}(\hat{\imath} + \hat{\jmath} + \hat{k}) = \frac{1}{2}(\hat{\imath} + \hat{\jmath} + \hat{k})$ **Answer.**

## <u>Polling quiz</u>

The gradient of function $f = x + y + z$ at point $(1,1,1)$

(A) $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

(B) $x\hat{\imath} - y\hat{\jmath} - z\hat{k}$

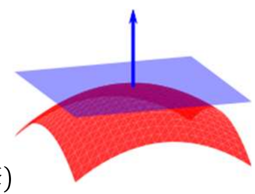
(C) $\hat{\imath} + \hat{\jmath} + \hat{k}$

(D) None of these.

## <u>Normal Vector to a Scalar Field Surface</u>

Geometrically, Gradient of a scalar field represents a vector normal to the surface,

Let $f(x,y,z)$ be a given scalar surface.



Let $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

Then vector normal to the surface $f$ is given by:

Normal vector, $\vec{n} = \text{grad}(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k})$

Unit normal vector, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$

**Problem 1.** Find the vector normal and unit normal to the surface $y^2 = 16x$ at (4,8).

**Solution.** Let $f(x, y, z) = 16x - y^2$

$\Rightarrow f_x = 16, \quad f_y = -2y$

Then gradient of scalar field $f$ is given by:

grad$(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath}) = 16\hat{\imath} - 2y\hat{\jmath}$

At point (4,8): $\vec{\nabla} f = 16\hat{\imath} - 16\hat{\jmath}$

Normal vector, $\vec{n} = \vec{\nabla} f = 16\hat{\imath} - 16\hat{\jmath}$

Unit normal vector, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{16\hat{\ } - 16\hat{\jmath}}{\sqrt{(16)^2 + (16)^2}} = \frac{16(\hat{\imath} - \hat{\jmath})}{16\sqrt{1+1}} = \frac{(\hat{\imath} - \hat{\jmath})}{\sqrt{2}}$ **Answer.**

**Problem 2.** Find vector normal and unit normal to the surface $x^2 + y^2 + z^2 = 4$ at (1,1,1).

**Solution.** Let $f(x, y, z) = x^2 + y^2 + z^2 - 4$

$\Rightarrow f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$

Then gradient of scalar field $f$ is given by:

grad$(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

At point (1,1,1): $\vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Normal vector, $\vec{n} = \vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Unit normal vector, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}}{\sqrt{4+4+4}} = \frac{2(\hat{\imath} + \hat{\jmath} + \hat{k})}{2\sqrt{3}} = \frac{(\hat{\imath} + \hat{\jmath} + \hat{k})}{\sqrt{3}}$ **Answer.**

## Polling quiz

The gradient of a scalar field function is:

(A) A scalar quantity

(B) A vector quantity

(C) None of these.

## Angle between two Scalar Surface

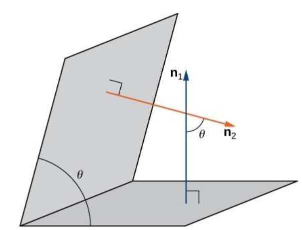
Angle between two surfaces is equal to the angle between their normal.

Let $f$ and $g$ be two given scalar surfaces. Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be the vectors normal to the surfaces $f$ and $g$ respectively.

Then angle between surfaces $f$ and $g$ is given by:

$$\cos\theta = [\widehat{n_1}.\widehat{n_2}] = \left[\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|} \cdot \frac{\overrightarrow{n_2}}{|\overrightarrow{n_2}|}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla} f}{|\vec{\nabla} f|} \cdot \frac{\vec{\nabla} g}{|\vec{\nabla} g|}\right]$$



**Problem.** Find the angle between the surfaces $z = x^2 + y^2, z = 2x^2 - 3y^2$ at (2,1,5).

**Solution.** Let $f(x, y, z) = x^2 + y^2 - z$

$\Rightarrow f_x = 2x, \quad f_y = 2y, \quad f_z = -1$

Then gradient of scalar field $f$ is given by:

grad$(f) = \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$

At point (2,1,5): $\vec{\nabla} f = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Normal vector, $\vec{n_1} = \vec{\nabla} f = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Let $g(x, y, z) = 2x^2 - 3y^2 - z$

$\Rightarrow g_x = 4x, \quad g_y = -6y, \quad g_z = -1$

Then gradient of scalar field $g$ is given by:

grad$(g) = \vec{\nabla} g = (g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k}) = 4x\hat{\imath} - 6y\hat{\jmath} - \hat{k}$

At point (2,1,5): $\vec{\nabla} g = 8\hat{\imath} - 6\hat{\jmath} - \hat{k}$

Normal vector, $\vec{n_2} = \vec{\nabla} g = 8\hat{\imath} - 6\hat{\jmath} - \hat{k}$

Then angle between surfaces $f$ and $g$ is given by:

$\cos\theta = [\widehat{n_1}.\widehat{n_2}] = \left[\frac{\vec{n_1}}{|\vec{n_1}|}.\frac{\vec{n_2}}{|\vec{n_2}|}\right]$

$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla} f}{|\vec{\nabla} f|}.\frac{\vec{\nabla} g}{|\vec{\nabla} g|}\right] \qquad \Rightarrow \theta = cos^{-1}\left[\frac{(4\hat{\imath}+2\hat{\jmath}-\hat{k})}{\sqrt{21}}.\frac{(8\hat{\imath}-6\hat{\jmath}-\hat{k})}{\sqrt{101}}\right]$

$\Rightarrow \theta = cos^{-1}\left[\frac{32-12+1}{\sqrt{21}\sqrt{101}}\right] = cos^{-1}\left[\sqrt{\frac{21}{101}}\right]$ **Answer.**

# MTH 166

## Lecture-31

## Directional Derivatives

**Topic:**

Vector Differential Calculus

**Learning Outcomes:**

**1.** To calculate directional derivatives

**2.** To find tangent plane to a surface

## Directional derivative of a scalar field

Let $f(x, y, z)$ be a given scalar surface.

Let the given direction be $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

Then directional derivative of $f$ in the direction of vector $\vec{b}$ is given by:

$$D_{\vec{b}}(f) = \vec{\nabla}f.\,\hat{b} = \vec{\nabla}f.\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right).\left(\frac{b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}\right)$$

$$= \frac{f_x b_1 + f_y b_2 + f_z b_3}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}$$

**Note:**

**1.** Maximum rate of increase (Minimum rate of decrease) $= |\vec{\nabla}f|$
It occurs in its own direction.

**2.** Minimum rate of increase (Maximum rate of decrease) $= -|\vec{\nabla}f|$
It occurs in its opposite direction.

**Problem 1.** Find the directional derivative of the scalar function $(x^2y - y^2z - xyz)$at the point $(1, -1, 0)$ in the direction $(\hat{\imath} - \hat{\jmath} + 2\hat{k})$.

**Solution.** Let $f = (x^2y - y^2z - xyz)$

$\Rightarrow f_x = (2xy - yz), f_y = (x^2 - 2yz - xz), f_z = (-y^2 - xy)$

$\vec{\nabla}f = \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right) = (2xy - yz)\hat{\imath} + (x^2 - 2yz - xz)\hat{\jmath} + (-y^2 - xy)\hat{k}$

At $(1, -1, 0)$: $\vec{\nabla}f = -2\hat{\imath} + \hat{\jmath} + 0\hat{k}$

Let $\vec{b} = (\hat{\imath} - \hat{\jmath} + 2\hat{k})$

$$D_{\vec{b}}(f) = \vec{\nabla}f.\,\hat{b} = \vec{\nabla}f.\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$\Rightarrow D_{\vec{b}}(f) = \left(-2\hat{\imath} + \hat{\jmath} + 0\hat{k}\right).\frac{(\hat{\imath} - \hat{\jmath} + 2\hat{k})}{\sqrt{1+1+4}} = \frac{-2-1+0}{\sqrt{6}} = \frac{-3}{\sqrt{6}}$ **Answer.**

**Problem 2.** Find the directional derivative of the scalar function $(xyz)$ at the point $(1,4,3)$ in the direction of line from $(1,2,3)$ to $(1,-1,-3)$.

**Solution.** Let $f = (xyz)$

$\Rightarrow f_x = (yz), f_y = (xz), f_z = (xy)$

$\vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$

At $(1,4,3)$: $\vec{\nabla} f = 12\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

Let $\vec{b} = \left((1-1)\hat{\imath} + (-1-2)\hat{\jmath} + (-3-3)\hat{k}\right) = 0\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$

$D_{\vec{b}}(f) = \vec{\nabla} f.\, \hat{b} = \vec{\nabla} f.\, \left(\frac{\vec{b}}{|\vec{b}|}\right)$

$\Rightarrow D_{\vec{b}}(f) = (12\hat{\imath} + 3\hat{\jmath} + 4\hat{k}).\frac{(0\hat{\imath}-3\hat{\jmath}-6\hat{k})}{\sqrt{0+9+36}} = \frac{0-9-24}{\sqrt{45}} = \frac{-33}{3\sqrt{5}} = -\frac{11}{\sqrt{5}}$ **Answer.**

## Polling quiz

The del operator, denoted as $\vec{\nabla}$, is defined as:

(a) $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

(b) $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} - \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

(c) $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} - \hat{k}\frac{\partial}{\partial z}$

(d) $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} - \hat{\jmath}\frac{\partial}{\partial y} - \hat{k}\frac{\partial}{\partial z}$

**Problem 3.** Find a direction that gives the direction of maximum rate of increase of scalar function $(3x^2 + y^2 + 2z^2)$ at $(0,1,2)$. Find the maximum rate too.

**Solution.** Let $f = (3x^2 + y^2 + 2z^2)$

$\Rightarrow f_x = 6x, \quad f_y = 2y, \quad f_z = 4z$

$\vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = 6x\hat{\imath} + 2y\hat{\jmath} + 4z\hat{k}$

At $(0,1,2)$: $\vec{\nabla} f = 0\hat{\imath} + 2\hat{\jmath} + 8\hat{k}$

which is the direction of maximum rate of increase.

Also, Maximum rate = $|\vec{\nabla} f| = \sqrt{0+4+64} = \sqrt{68} = 2\sqrt{17}$ **Answer.**

**Problem 4.** Find a direction that gives the direction of minimum rate of increase of scalar function $(x^3 - xy^2 + y^3)$ at $(-2,1)$. Find the minimum rate too.

**Solution.** Let $f = (x^3 - xy^2 + y^3)$

$\Rightarrow f_x = 3x^2 - y^2, \quad f_y = 3y^2 - 2xy$

$\vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath}) = (3x^2 - y^2)\hat{\imath} + (3y^2 - 2xy)\hat{\jmath}$

At $(-2,1)$: $\vec{\nabla} f = 11\hat{\imath} + 7\hat{\jmath}$

Direction of minimum rate of increase= $-\vec{\nabla} f = -(11\hat{\imath} + 7\hat{\jmath})$

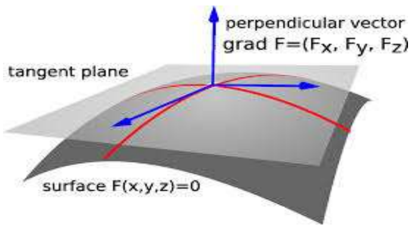
Also, Minimum rate = $-|\vec{\nabla} f| = -\sqrt{121+49} = -\sqrt{170}$ **Answer.**

**Tangent Plane to a Scalar Surface**

Let $f(x, y, z)$ be a given scalar surface.

Then equation of tangent plane to the surface $f(x, y, z)$ at the point $P_0(x_0, y_0, z_0)$ is:

$(x - x_0)f_x(P_0) + (y - y_0)f_y(P_0) + (z - z_0)f_z(P_0) = 0$




**Problem.** Find equation of tangent plane to surface $(x^2 - 3y^2 - z^2 = 2)$ at (3,1,2).

**Solution.** Let $f = x^2 - 3y^2 - z^2 = 2$

$\Rightarrow f_x = 2x, \quad f_y = -6y, \quad f_z = -2z$

At (3,1,2): $f_x = 6, \quad f_y = -6, \quad f_z = -4$

Equation of tangent plane is given by:

$(x - x_0)f_x(P_0) + (y - y_0)f_y(P_0) + (z - z_0)f_z(P_0) = 0$

$\Rightarrow (x - 3)(6) + (y - 1)(-6) + (z - 2)(-4) = 0$

$\Rightarrow 6x - 18 - 6y + 6 - 4z + 8 = 0$

$\Rightarrow 6x - 6y - 4z = 4$

$\Rightarrow 3x - 3y - 2z = 2$ **Answer.**

**Polling quiz**

The gradient of the scalar field function $f(x, y) = y^2 - 4xy$ at point (1,2) is:

(a) $8\hat{\imath}$ (b) $-8\hat{\imath}$ (c) $8\hat{\jmath}$ (d) $-8\hat{\jmath}$

**Polling quiz**

The directional derivative of a scalar point function $f$ in the direction $\hat{b}$ is given by:

(a) $f.\hat{b}$ (b) $curl(f).\hat{b}$

(c) $diverence(f).\hat{b}$ (d) $gradient(f).\hat{b}$

# MTH 166

## Lecture-32

## Divergence and Curl of a Vector Field

**Topic:**

Vector Differential Calculus

**Learning Outcomes:**

**1.** To calculate divergence of a vector field

**2.** To calculate curl of a vector field

Let the given vector field be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Let us consider a vector differential operator $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

**Divergence:** It is dot product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad (\because \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1)$$

Divergence is always a scalar quantity (Having no direction).

Note: If $\mathbf{div}(\vec{v}) = \mathbf{0}$, then vector $\vec{v}$ is called **Solenoidal** or **Incompressible**.

That means the field has no sources and no sinks.

**Curl:** It is cross product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Curl is always a vector quantity.

Note: If $\textbf{curl}(\vec{v}) = \vec{\mathbf{0}}$, then vector $\vec{v}$ is called **Irrotational** or **Conservative**.

Two Important Properties:

1. div(curl($\vec{v}$)) = 0 (Always)
2. curl(grad ($f$)) = $\vec{0}$ (Always)

**Problem 1.** Compute div($\vec{v}$) and curl($\vec{v}$) and verify that div(curl($\vec{v}$)) = 0 where $\vec{v} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

**Solution.** Here $\vec{v} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$v_1 = x, \quad v_2 = 2y, \quad v_3 = z$

$\Rightarrow \frac{\partial v_1}{\partial x} = 1, \quad \frac{\partial v_2}{\partial y} = 2, \quad \frac{\partial v_3}{\partial z} = 1$

$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$= 1 + 2 + 1 = 4$

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(z)}{\partial y} - \frac{\partial(2y)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z}\right) + \hat{k}\left(\frac{\partial(2y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)$$

$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0}$ (Irrotational)

div(curl($\vec{v}$)) = div($\vec{0}$) = 0

**Problem 2.** Compute div($\vec{v}$) and curl($\vec{v}$) and verify that div(curl($\vec{v}$)) = 0 where $\vec{v} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$

**Solution.** Here $\vec{v} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$v_1 = xy, \quad v_2 = yz, \quad v_3 = zx$

$\Rightarrow \frac{\partial v_1}{\partial x} = y, \quad \frac{\partial v_2}{\partial y} = z, \quad \frac{\partial v_3}{\partial z} = x$

$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$= y + z + x$

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(zx)}{\partial y} - \frac{\partial(yz)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(zx)}{\partial x} - \frac{\partial(xy)}{\partial z}\right) + \hat{k}\left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)$$

$$= -y\hat{\imath} - z\hat{\jmath} - x\hat{k}$$

$$\text{div}(\text{curl}(\vec{v})) = \text{div}(-y\hat{\imath} - z\hat{\jmath} - x\hat{k}) = \frac{\partial(-y)}{\partial x} + \frac{\partial(-z)}{\partial y} + \frac{\partial(-x)}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

**Polling quiz**

Let $\vec{v}$ be a differentiable vector field. Then:

(a) $div(curl\ \vec{v}) = \vec{v}$ (b) $div(curl\ \vec{v}) \neq 0$

(c) $div(curl\ \vec{v}) = 0$ (d) $div(curl\ \vec{v}) = \hat{\imath} + \hat{\jmath} + \hat{k}$

**Problem 3.** Compute grad ($f$) and verify that curl(grad ($f$)) = $\vec{0}$ where $f = x + y - 2z^2$

**Solution.** Let $f(x, y, z) = x + y - 2z^2$

$\Rightarrow f_x = 1, f_y = 1, f_z = -4z$

Then gradient of scalar field $f$ is given by:

grad($f$) $= \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = \hat{\imath} + \hat{\jmath} - 4z\hat{k}$

$$\text{curl(grad}\ (f)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & -4z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

**Problem 4.** Compute grad ($f$) and verify that curl(grad ($f$)) = $\vec{0}$ where $f = 16xy^3z^2$

**Solution.** Let $f(x, y, z) = 16xy^3z^2$

$\Rightarrow f_x = 16y^3z^2, f_y = 48xy^2z^2, f_z = 32xy^3z$

Then gradient of scalar field $f$ is given by:

grad($f$) $= \vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = 16y^3z^2\hat{\imath} + 48xy^2z^2\hat{\jmath} + 32xy^3z\hat{k}$

$$\text{curl(grad}\ (f)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16y^3z^2 & 48xy^2z^2 & 32xy^3z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

**Polling quiz**

Let $f$ be a differentiable scalar field. Then:

(a) $curl(grad\ f) = \vec{0}$ (b) $curl(grad\ f) \neq \vec{0}$

(c) $curl(grad\ f) = \vec{f}$ (d) $curl(grad\ f) = \hat{\imath} + \hat{\jmath} + \hat{k}$

**1.** If $\mathbf{div}(\vec{v}) = \mathbf{0}$, then vector $\vec{v}$ is called **Solenoidal** or **Incompressible**.

That means the field has no sources and no sinks.

**2.** If $\mathbf{curl}(\vec{v}) = \vec{\mathbf{0}}$, then vector $\vec{v}$ is called **Irrotational** or **Conservative**.

**Problem 1.** Show that the vector $\vec{v} = (2x + 3y)\hat{\imath} + (x - y)\hat{\jmath} - (x + y + z)\hat{k}$ is solenoidal.

**Solution.** Here $\vec{v} = (2x + 3y)\hat{\imath} + (x - y)\hat{\jmath} - (x + y + z)\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$v_1 = (2x + 3y),\quad v_2 = (x - y),\quad v_3 = -(x + y + z)$

$\Rightarrow \frac{\partial v_1}{\partial x} = 2,\quad \frac{\partial v_2}{\partial y} = -1,\quad \frac{\partial v_3}{\partial z} = -1$

$\operatorname{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$= 2 - 1 - 1 = 0$

So, the given vector $\vec{v}$ is Solenoidal.

**Problem 2.** Show that the vector $\vec{v} = 3x^2y^2z^4\hat{\imath} + 2x^3yz^4\hat{\jmath} + 4x^3y^2z^3\hat{k}$ is Irrotational.

**Solution.** Here $\vec{v} = 3x^2y^2z^4\hat{\imath} + 2x^3yz^4\hat{\jmath} + 4x^3y^2z^3\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$$\operatorname{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2z^4 & 2x^3yz^4 & 4x^3y^2z^3 \end{vmatrix}$$

$= \hat{\imath}(8x^3yz^3 - 8x^3yz^3) - \hat{\jmath}(12x^2y^2z^3 - 12x^2y^2z^3) + \hat{k}(6x^2yz^4 - 6x^2yz^4)$

$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0}$

So, the given vector $\vec{v}$ is Irrotational.

# MTH 166

## Lecture-33

## Line Integral

**Unit 6: Vector Calculus-II**

(**Book:** Advanced Engineering Mathematics by Jain and Iyengar,**Chapter-15**)

**Topic:**

Vector Integral Calculus

**Learning Outcomes:**

**1.** To calculate Line integral w.r.t to an arc length

**2.** To calculate Line integral of a vector field (Work Done)

**Line Integral w.r.t an Arc Length**

Let the parametric representation of curve $C$ be:

$C: x = x(t), y = y(t), z = z(t); \quad a \le t \le b.$

Let $f(x, y, z)$ be a scalar field function.

Then line integral of scalar function $f$ over curve $C$ w.r.t arc length $s$ is given by:

$$I = \int_C f(x, y, z) ds$$

$$= \int_{t=a}^{t=b} \left[ f(x(t), y(t), z(t)) \frac{ds}{dt} \right] dt$$

$$= \int_{t=a}^{t=b} \left[ f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \right] dt$$

**Problem 1.** Evaluate $\int_C f(x,y,z)ds$ where $f(x,y,z) = 2x + 3y$ and curve C is:

$C: x = t, y = 2t, z = 3t, 0 \le t \le 3.$

**Solution.** $C: x = t,\ y = 2t,\ z = 3t$

$$\Rightarrow \frac{dx}{dt} = 1,\quad \frac{dy}{dt} = 2,\quad \frac{dz}{dt} = 3$$

$f(x,y,z) = 2x + 3y \qquad \Rightarrow f(t) = 2(t) + 3(2t) = 8t$

Then line integral of scalar function $f$ over curve $C$ w.r.t arc length $s$ is given by:

$$I = \int_C f(x,y,z)ds = \int_{t=a}^{t=b} \left[f(x(t),y(t),z(t))\frac{ds}{dt}\right] dt$$

$$= \int_{t=a}^{t=b} \left[f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right] dt$$

$$I = \int_{t=0}^{t=3} \left[8t\sqrt{(1)^2 + (2)^2 + (3)^2}\right] dt$$

$$= 8\sqrt{14}\int_{t=0}^{t=3} t\,dt$$

$$= 8\sqrt{14}\left[\frac{t^2}{2}\right]_0^3$$

$$= 8\sqrt{14}\left[\frac{9}{2} - 0\right]$$

$= 36\sqrt{14}$ **Answer.**

**Problem 2.** Evaluate $\int_C f(x,y)ds$ where $f(x,y) = x^2y$ and curve C is:

$C: x = 3\cos t, y = 3\sin t, 0 \le t \le \frac{\pi}{2}.$

**Solution.** $C: x = 3\cos t,\ y = 3\sin t$

$$\Rightarrow \frac{dx}{dt} = -3\sin t,\quad \frac{dy}{dt} = 3\cos t$$

$f(x,y) = x^2y \quad \Rightarrow f(t) = (3\cos t)^2(3\sin t) = 27cos^2t\sin t$

Then line integral of scalar function $f$ over curve $C$ w.r.t arc length $s$ is given by:

$$I = \int_C f(x,y)ds = \int_{t=a}^{t=b} \left[f(x(t),y(t))\frac{ds}{dt}\right] dt$$

$$= \int_{t=a}^{t=b} \left[f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right] dt$$

$$I = \int_{t=0}^{t=\frac{\pi}{2}} 27cos^2t\sin t\left[\sqrt{(-3\sin t)^2 + (3\cos t)^2}\right] dt$$

$$= 27 \times 3\int_{t=0}^{t=\frac{\pi}{2}} cos^2t\sin t\,dt$$

$$= -81\int_{t=0}^{t=\frac{\pi}{2}} cos^2t\,(-\sin t)\,dt$$

$$= -81\left[\frac{(\cos t)^3}{3}\right]_0^{\frac{\pi}{2}}$$

$$= -27\left[\left(\cos\frac{\pi}{2}\right)^3 - (\cos 0)^3\right]$$

$= -27(0 - 1) = 27$ **Answer.**

## Polling quiz

The line integral $\int_C f(x,y,z)ds$ of function $f(x,y,z)$ which is continuous over the simple smooth curve $C(x(t),y(t),z(t)), a \le t \le b$, with respect to the arc length $s$ is given by:

(a) $\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

(b) $\int_a^b f(x(t),y(t),z(t))dt$

(c) $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

(d) None of these.

## Line Integral of a Vector Field (Work Done)

Let us consider a vector field $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

We write $\overrightarrow{v(t)} = v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}$; $\quad a \le t \le b$

Let curve $C$ is represented by: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

We write $C$: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Then line integral of vector $\vec{v}$ over curve $C$ given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right]dt$$

$$= \int_{t=a}^{t=b}\left[\left(v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}\right).\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)\right]dt$$

$$= \int_{t=a}^{t=b}\left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dy}{dt} + v_3(t)\frac{dz}{dt}\right)\right]dt$$ This also called as **Work Done.**

**Problem 1.** Evaluate $\int_C \vec{v}.\overrightarrow{dr}$ where $\vec{v} = xy\hat{\imath} + y^2\hat{\jmath} + e^z\hat{k}$ and curve c is given by:

$C: x = t^2, y = 2t, z = t, 0 \le t \le 1.$

**Solution.** We write $C$: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \quad \Rightarrow \overrightarrow{r(t)} = t^2\hat{\imath} + 2t\hat{\jmath} + t\hat{k}$

$\Rightarrow \frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \frac{dz}{dt} = 1$

$\vec{v} = xy\hat{\imath} + y^2\hat{\jmath} + e^z\hat{k} \quad \Rightarrow \overrightarrow{v(t)} = 2t^3\hat{\imath} + 4t^2\hat{\jmath} + e^t\hat{k}$

Then line integral of vector $\vec{v}$ over curve $C$ given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right]dt$$

$$= \int_{t=a}^{t=b}\left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dy}{dt} + v_3(t)\frac{dz}{dt}\right)\right]dt$$

$$I = \int_{t=a}^{t=b}\left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dy}{dt} + v_3(t)\frac{dz}{dt}\right)\right]dt$$

$$= \int_{t=0}^{t=1}\left[\left((2t^3)(2t) + (4t^2)(2) + (e^t)(1)\right)\right]dt$$

$$= \int_{t=0}^{t=1}[4t^4 + 8t^2 + e^t]dt$$

$$= \left[\frac{4}{5}t^5 + \frac{8}{3}t^3 + e^t\right]_{t=0}^{t=1}$$

$$= \left[\frac{4}{5} + \frac{8}{3} + e\right] - [0 + 0 + 1]$$

$$= \frac{37}{15} + e$$ **Answer.**

**Problem 2.** Evaluate $\int_C \vec{v}.\overrightarrow{dr}$ where $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $C$ is the line segment from (1,2,2) to (3,6,6). **Or.**

Find work done by force $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ in moving a particle from (1,2,2) to (3,6,6).

**Solution.** Two point form of line $C$ is given by:

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$ (Say)

$\Rightarrow \frac{x-1}{3-1} = \frac{y-2}{6-2} = \frac{z-2}{6-2} = t$

$\Rightarrow \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} = t$

$\Rightarrow x = 2t+1,\ y = 4t+2,\ z = 4t+2, (when\ x = 1, t = 0\ and\ x = 3, t = 1)$

So, $C$: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\Rightarrow \overrightarrow{r(t)} = (2t+1)\hat{\imath} + (4t+2)\hat{\jmath} + (4t+2)\hat{k}$

$\Rightarrow \frac{dx}{dt} = 2,\ \frac{dy}{dt} = 4,\ \frac{dz}{dt} = 4$

$\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \Rightarrow \overrightarrow{r(t)} = (2t+1)\hat{\imath} + (4t+2)\hat{\jmath} + (4t+2)\hat{k}$

Then line integral of vector $\vec{v}$ over curve $C$ given by:

$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right] dt$

$I = \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dy}{dt} + v_3(t)\frac{dz}{dt}\right)\right] dt$

$= \int_{t=0}^{t=1} [((2t+1)(2) + (4t+2)(4) + (4t+2)(4))] dt$

$= \int_{t=0}^{t=1} [36t+18] dt = \left[36\frac{t^2}{2} + 18t\right]_{t=0}^{t=1} = (18+18) - 0 = 36$ **Answer.**

# MTH 166

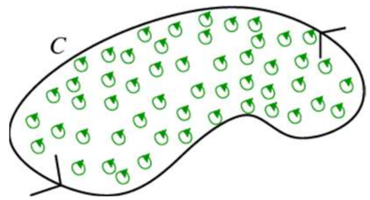
## Lecture-34

## Greens' Theorem

**Statement:**

Let C be a piecewise smooth simple closed curve bounding a region R traced in anticlockwise direction. If f and g are two scalar functions which are continuous and have continuous first order partial derivatives on R, then:

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$




**Important Results for MCQ Practice:**

**1.** By Greens' theorem: $\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$

**2.** Greens' theorem is a relationship between double integral and line integral.

**3.** Greens' theorem is also called as First fundamental theorem of integral vector calculus.

**4.** Area of region R = $\oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C (xdy - ydx)$

**5.** If $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$, then by Greens' theorem $\oint_C fdx + gdy = 0$

**Problem:** Use Greens' theorem to evaluate: $\oint_C (x+y)dx + x^2dy$, where C is a triangle with the vertices (0,0),(2,0) and (2,4) taken in the order.

**Solution:** Here the given integral is: $\oint_C (x+y)dx + x^2dy$

Comparing it with: $\oint_C fdx + gdy$

$f = (x+y)$ implies $\frac{\partial f}{\partial y} = 1$ , $g = x^2$ implies $\frac{\partial g}{\partial x} = 2x$

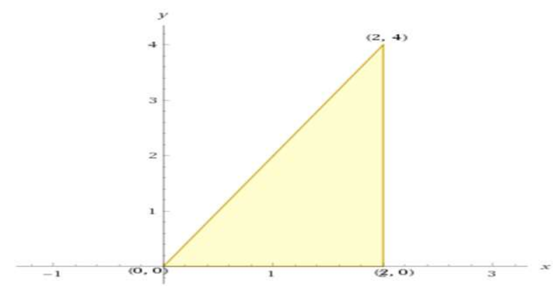
By Greens' theorem:

$$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$\oint_C (x+y)dx + x^2dy = \iint_R (2x - 1)dxdy$$

Now we are to get the limits of x and y for the evaluation of double integral.

Let us first draw the figure and find limits




In the given figure: $R: \begin{Bmatrix} 0 \le x \le 2 \\ 0 \le y \le 2x \end{Bmatrix}$

Now, Let us evaluate the integral:

$\oint_C (x+y)dx + x^2dy = \iint_R (2x-1)dxdy$

$= \int_{x=0}^{2}\int_{y=0}^{2x}(2x-1)dydx$

$= \int_{x=0}^{2}(2xy-y)_{y=0}^{y=2x}dx$

$= \int_{x=0}^{x=2}(4x^2-2x)dx$

$= \left[4\left(\frac{8}{3}\right)-4\right]$

$= \frac{20}{3}$ **Answer**

**Polling quiz**

Let $C$ be a piecewise smooth simple closed curve bounding a region $R$. If $f, g, \frac{\partial f}{\partial y}$ *and* $\frac{\partial g}{\partial x}$ are continuous on $R$. If the integration is carried in the anti-clockwise direction of $C$, then by **Green's theorem:**

(a) $\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\right)dxdy$

(b) $\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial y}+\frac{\partial f}{\partial x}\right)dxdy$

(c) $\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy$

(d) $\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial y}-\frac{\partial f}{\partial x}\right)dxdy$

**Problem:** Use Greens' theorem to evaluate: $\oint_C x^3dy - y^3dx$, where C is a the circle: $x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le 2\pi$.

**Solution:** Here the given integral is: $\oint_C x^3dy - y^3dx$

Comparing it with: $\oint_C fdx + gdy$

$f = -y^3$implies $\frac{\partial f}{\partial y} = -3y^2$, $g = x^3$ implies $\frac{\partial g}{\partial x} = 3x^2$

By Greens' theorem:

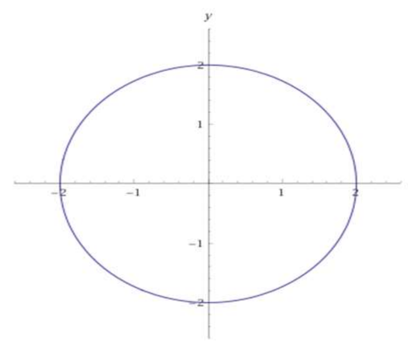
$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy$

$\oint_C x^3dy - y^3dx = 3\iint_R (x^2+y^2)dxdy$

Here limits of x and y are in polar coordinates:

$x = 2\cos\theta, y = 2\sin\theta, 0 \le \theta \le 2\pi$

In the given region R: $0 \le r \le 2, 0 \le \theta \le 2\pi$



$$\oint_C x^3 dy - y^3 dx = 3\iint_R (x^2+y^2)dxdy$$
$$= 3\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r^2)rdrd\theta$$
$$= 3\int_{r=0}^{2} r^3 dr \int_{\theta=0}^{2\pi} d\theta$$
$$= 3\left[\frac{r^4}{4}\right]_{r=0}^{2}[\theta]_{\theta=0}^{2\pi}$$
$$= 3(4)(2\pi)$$
$= 24\pi$ **Answer**

## Line Integral Independent of Path of Integration

An integral of the form: $\int_P^Q f(x,y)dx + g(x,y)dy$ is independent of path of integration if and only if: $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

**Problem:** Show that the integral $\int_P^Q 2xy^2dx + (2x^2y+1)dy$ is independent of path of integration.

**Solution:** Compare the given integral with: $\int_P^Q f(x,y)dx + g(x,y)dy$

Here $f = 2xy^2$ and $g = (2x^2y+1)$

This implies: $\frac{\partial f}{\partial y} = 4xy$ and $\frac{\partial g}{\partial x} = 4xy$

Since, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 4xy$

So, the given integral is independent of path of integration.

## Polling quiz

Let $C$ be a positively oriented simple closed path enclosing a simply connected region $R$. Using Green's theorem, area of the region $R$ is given by:

(a) $\oint_C xdx + ydy$ (b) $\oint_C xdy - ydx$

(c) $\frac{1}{2}\oint_C xdx + ydy$ (d) $\frac{1}{2}\oint_C xdy - ydx$

# MTH 166

## Lecture-35

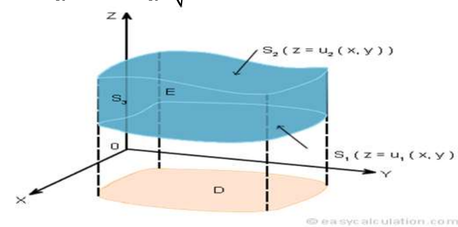
## Surface Area and Surface Integral

**Area Element and Surface Area**

**1.** If the projection of the surface $z = f(x, y)$ is taken on $xy$ −plane.

Area element, $dA = \sqrt{1 + {f_x}^2 + {f_y}^2}$

Surface Area, $A = \iint_R \, dA = \iint_R \, \sqrt{1 + {f_x}^2 + {f_y}^2} dxdy$




**2.** If the projection of the surface $x = g(y, z)$ is taken on $yz$ −plane.

Area element, $dA = \sqrt{1 + {g_y}^2 + {g_z}^2}$

Surface Area, $A = \iint_R \, dA = \iint_R \, \sqrt{1 + {g_y}^2 + {g_z}^2} dydz$

**3.** If the projection of the surface $y = h(z, x)$ is taken on $zx$ −plane.

Area element, $dA = \sqrt{1 + {h_z}^2 + {h_x}^2}$

Surface Area, $A = \iint_R \, dA = \iint_R \, \sqrt{1 + {h_z}^2 + {h_x}^2} dzdx$

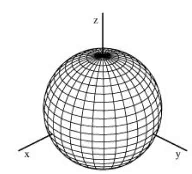
**Important Questions from MCQ Point of View:**

**Problem 1.** Find the surface area of $x^2 + y^2 + z^2 = a^2$

**Solution:** Since it is sphere of radius $a$

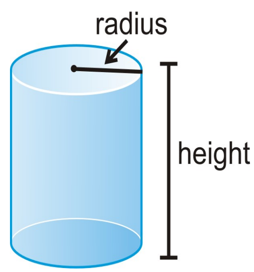
So, Surface area of sphere, $S = 4\pi r^2$

$\Rightarrow S = 4\pi(a)^2$ **Answer.**



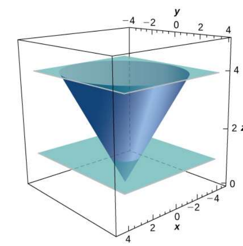
**Problem 2.** Find the surface area of $x^2 + y^2 = 16, 0 \le z \le 2$

**Solution:** Since it is cylinder of radius 4 and height 2.
So, Surface area of cylinder, $S = 2\pi rh$
$\Rightarrow S = 2\pi(4)(2) = 16\pi$ **Answer.**




**Problem 3.** Find the surface area of $z^2 = x^2 + y^2, 0 \le z \le 4$

**Solution:** Since it is a cone of height h = 4 and radius of topmost part r = 4.
Here slanting height, $l = \sqrt{h^2 + r^2}$
$\Rightarrow l = \sqrt{(4)^2+(4)^2} = 4\sqrt{2}$
So, Surface area of cone, $S = \pi rl$
$\Rightarrow S = \pi(4)(4\sqrt{2})$
$\Rightarrow S = 16\sqrt{2}\pi$ **Answer.**



## Polling quiz

If the orthogonal projection of the surface $z = f(x, y)$ is taken on the $xy$ plane, the element surface area $dA$ is given by:

(a) $\sqrt{f_x^{\,2} + f_y^{\,2}}$ (b) $\sqrt{1 + f_x^{\,2} + f_y^{\,2}}$

(c) $\sqrt{1 + f_y^{\,2} + f_z^{\,2}}$ (d) $\sqrt{1 + f_z^{\,2} + f_x^{\,2}}$

## Surface Integral

The expression for surface integral is: $I = \iint_S f(x, y, z)dA$

Where $dA$ is the area element and it can be determined from one of the following ways:

If projection is taken on xy-plane (z=0 plane), then: $dA = \frac{dxdy}{\hat{n}.\hat{k}}$
If projection is taken on yz-plane (x=0 plane), then: $dA = \frac{dydz}{\hat{n}.\hat{\imath}}$
If projection is taken on zx-plane (y=0 plane), then: $dA = \frac{dzdx}{\hat{n}.\hat{\jmath}}$

**Problem 4.** Evaluate the surface integral $\iint_S f(x,y,z)dA$ where $f = 6xyz$ and S is the portion of the plane $x+y+z=1$ in the first octant.

**Solution:** Let $g = x+y+z=1$

$\Rightarrow g_x = 1, g_y = 1, g_z = 1$

Now, $\vec{\nabla} g = g_x\hat{\imath} + g_y\hat{\jmath} + g_z\hat{k} = \hat{\imath}+\hat{\jmath}+\hat{k}$

$\hat{n} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{1+1+1}}$

$\Rightarrow \hat{n} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}$

Let projection of Surface S be taken on xy-plane:

$z = 1 - x - y$

$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}.\hat{k}}$

$\Rightarrow dA = \frac{dxdy}{\frac{1}{\sqrt{3}}}$

For limits: On xy-plane( z = 0)

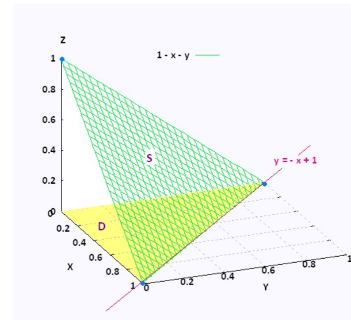
So, x + y + 0 = 1 $\Rightarrow y = (1-x)$

$0 \le y \le (1-x)$

On x-axis (y = 0, z = 0)

So, x + 0 + 0 = 1 $\Rightarrow x = 1$

$0 \le x \le 1$




So, required surface integral is:$I = \iint_S f(x,y,z)dA$

$\Rightarrow I = \iint_S (6xyz)\frac{dxdy}{\frac{1}{\sqrt{3}}}$

$= 6\sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x} xy(1-x-y)dxdy$

$= 6\sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x}(xy - x^2y - xy^2)dydx$

On simplification, we get:

$I = \frac{\sqrt{3}}{20}$ **Answer.**

## Polling quiz

Which of the following options is **Not** true for element surface area $dA$:

(a) $dA = \frac{dydz}{\hat{n}.\hat{n}}$ (b) $dA = \frac{dydz}{\hat{n}.\hat{\imath}}$

(c) $dA = \frac{dzdx}{\hat{n}.\hat{\jmath}}$ (d) $dA = \frac{dxdy}{\hat{n}.\hat{k}}$

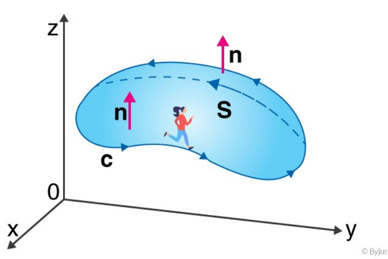
# MTH 166

## Lecture-36

## Stokes' Theorem

**Statement:**
Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve c traced in a positive direction. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = \int_S \int (\vec{\nabla} \times \vec{V}).\hat{n}dA$$




**Calculations required to Apply Stokes' Theorem**

**1.** Calculate $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

**2.** Calculate $\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{grad\, f}{|grad\, f|}$

**3.** Calculate $dA$ : By taking projection of given surface S

If projection is taken on xy-plane (z=0 plane), then: $dA = \frac{dxdy}{\hat{n}.\hat{k}}$

If projection is taken on yz-plane (x=0 plane), then: $dA = \frac{dydz}{\hat{n}.\hat{\imath}}$

If projection is taken on zx-plane (y=0 plane), then: $dA = \frac{dzdx}{\hat{n}.\hat{\jmath}}$

** Stokes' theorem gives a relationship between line integral and double integral like that of Green's Theorem

**Problem 1:** Evaluate $\oint_C \vec{V}.\vec{dr}$ using Stokes' theorem where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and S is the surface bounded by sphere $x^2 + y^2 + z^2 = 16$.

**Solution:** Here $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

**1.** $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$

By Stokes' theorem:

$\oint_C \vec{V}.\vec{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = 0$

* * This is how an MCQ can be framed on Stokes' theorem.

**Problem 2:** Show that $\oint_C \vec{V}.\vec{dr}$ is always zero ,using Stokes' theorem where $\vec{V} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$ and S is bounded by any closed curve c.

**Solution:** Here $\vec{V} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$

**1.** $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$

$= (x - x)\hat{\imath} - (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{0}$

By Stokes' theorem:

$\oint_C \vec{V}.\vec{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = 0$

* * This is how an MCQ can be framed on Stokes' theorem.

**Problem 3:** Evaluate $\oint_C \vec{V}.\vec{dr}$ by Stokes' theorem where $\vec{V} = (3x - y)\hat{\imath} - 2yz^2\hat{\jmath} - 2y^2z\hat{k}$ and S is the surface bounded by sphere $x^2 + y^2 + z^2 = 16, z > 0$.

**Solution:** Here $\vec{V} = (3x - y)\hat{\imath} - 2yz^2\hat{\jmath} - 2y^2z\hat{k}$

**1.** $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x - y) & -2yz^2 & -2y^2z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$

$Curl\,\vec{V} = 0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$

**2.** $\hat{n} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{grad\ f}{|grad\ f|}$

Let $f = x^2 + y^2 + z^2 = 16\ where\ z > 0$

$\Rightarrow f_x = 2x, f_y = 2y, f_z = 2z$

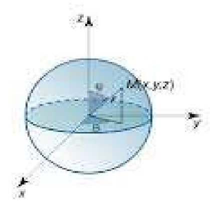
Now, $\vec{\nabla}f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

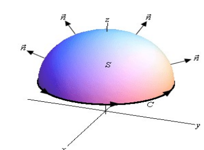
$\hat{n} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{(2x\hat{} + 2y\hat{\jmath} + 2z\hat{k})}{\sqrt{4x^2 + 4y^2 + 4\ ^2}}$

$\hat{n} = \frac{2(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{\sqrt{16}}$

$\hat{n} = \frac{(x\hat{\imath} + y\hat{\jmath} + z\hat{k})}{4}$

**3.** To find dA: Let the projection of surface S be taken on xy-plane.(z=0) The sphere becomes a circle of radius 4, $x^2 + y^2 + 0^2 = 16$ that is $x^2 + y^2 = 16$



$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4}.\hat{k}}$$

$$dA = \frac{dxdy}{\frac{(z)}{4}}$$

By Stokes' Theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA$$

$$= \int_S \int [(0\hat{\imath} + 0\hat{\jmath} + 1\hat{k}).\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{4}] \frac{dxdy}{\frac{(z)}{4}}$$

$= \int_S \int \frac{(z)}{4} \frac{dxdy}{\frac{(z)}{4}} = \int_S \int dxdy =$ Area of a circle $x^2 + y^2 = 16$

$= \pi r^2 = \pi(4^2) = 16\pi$ **Answer.**

## Polling quiz

Let $S$ be a piecewise smooth orientable surface bounded by piecewise smooth simple closed curve $C$. Let $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$ be a vector field on $C$. If $\hat{n}$ is the outer unit normal vector on $S$, then by **Stokes's theorem:**

(a) $\oint_C \vec{v}.\overrightarrow{dr} = \iint_S (div\,\vec{v}).\hat{n}dA$

(b) $\oint_C \vec{v}.\overrightarrow{dr} = \iint_S (grad\,\vec{v}).\hat{n}dA$

(c) $\oint_C \vec{v}.\overrightarrow{dr} = \iint_S (curl\,\vec{v}).\hat{n}dA$

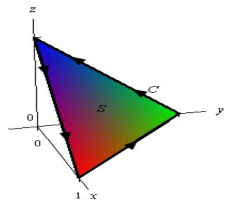
(d) None of these

**Problem 4:** Evaluate $\oint_C \vec{V}.\overrightarrow{dr}$ by Stokes' theorem where $\vec{V} = z^2\hat{\imath} + y^2\hat{\jmath} + x\,\hat{k}$ and S is the surface bounded by triangle with the vertices (1,0,0), (0,1,0), (0,0,1).

**Solution:** Here $\vec{V} = z^2\hat{\imath} + y^2\hat{\jmath} + x\,\hat{k}$

**1.** $Curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 0\hat{\imath} + (2z-1)\hat{\jmath} + 0\hat{k}$

$Curl\,\vec{V} = 0\hat{\imath} + (2z-1)\hat{\jmath} + 0\hat{k}$




**2.** $\hat{n} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{grad\,f}{|grad\,f|}$

Let f = x + y + z = 1

$\Rightarrow f_x = 1, f_y = 1, f_z = 1$

Now, $\vec{\nabla}f = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$

$$\hat{n} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{1+1+1}}$$

$$\hat{n} = \frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}$$

**3.** To find dA: Let the projection of surface S be taken on xy-plane.(z=0)

$z = 1 - x - y$

$$dA = \frac{dxdy}{\hat{n}.\hat{k}} = \frac{dxdy}{\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}.\hat{k}}$$

$$dA = \frac{dxdy}{\frac{1}{\sqrt{3}}}$$

By Stokes' Theorem:

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA$$

$$= \int_S \int [(0\hat{\imath} + (2z-1)\hat{\jmath} + 0\hat{k}).\frac{(\hat{\imath}+\hat{\jmath}+\hat{k})}{\sqrt{3}}] \frac{dxdy}{\frac{(1)}{\sqrt{3}}}$$

$$= \int_S \int [2(1-x-y)-1]\, dxdy$$
$$= \int_{x=0}^{1} \int_{y=0}^{1-x} [2(1-x-y)-1] dydx$$
$$= \int_{x=0}^{1} [y-2xy-y^2]_{y=0}^{y=1-x} dx$$
$$= \int_{x=0}^{1} (x^2-x) dx$$
$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=0}^{1}$$
$$= \frac{1}{3} - \frac{1}{2}$$
$= \frac{1}{6}$ **Answer**

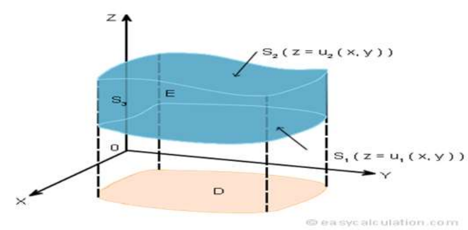
# MTH 166

## Lecture-37

## Gauss's Divergence Theorem

**Statement:** Let D be a closed and bounded region in 3-dimensional space whose boundary is a piecewise smooth surface S oriented outwards. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

$$\iint_S (\vec{V}.\hat{n}) dA = \iiint_D (div\, \vec{V}) dv = \iiint_D (\vec{\nabla}.\vec{V}) dv$$



**Important Results from MCQ point of view:**

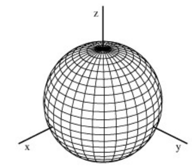
1. By gauss divergence theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv = \iiint_D (\nabla.\vec{V})dv$
2. Gauss Divergence theorem gives a relationship between double integral and triple integral, unlike Green's theorem and Stokes' theorem.
3. Gauss Divergence theorem is applicable to a closed region bounded by a surface, where as Stokes' theorem is applicable to an open surface bounded by a closed curve.
4. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, \vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$(a constant vector) and $V$is the volume,

(I). $\iint_S (\vec{a}.\hat{n})dA = 0$

(II). $\iint_S (curl\,\vec{r}.\hat{n})dA = 0$

(III). $\iint_S (\vec{r}.\hat{n})dA = 3V = 3(Volume\ of\ the\ given\, region)$

**Problem 1:** Evaluate $\iint_S (\vec{V}.\hat{n})dA$ using Gauss Divergence Theorem, where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and D is the region bounded by the sphere $x^2 + y^2 + z^2 = 16$.




**Solution:** Here: $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$

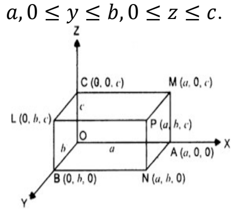
$\Rightarrow div\vec{V} = 3$

By Gauss Divergence Theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv$

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D 3dv = 3\iiint_D dv = 3(Volume\ of\ sphere x^2 + y^2 + z^2 = 16.)$

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = 3\left(\frac{4}{3}\pi r^3\right) = 4\pi(4^3) = 256\pi$ **Answer.**

**Problem 2:** Evaluate $\iint_S (\vec{V}.\hat{n})dA$ using Gauss Divergence Theorem, where $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and D is bounded by the edges $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.




**Solution:** Here: $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$

$\Rightarrow div\vec{V} = 3$

By Gauss Divergence Theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv$

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D 3dv = 3\iiint_D dv = 3(Volume\ of$ cuboid.)

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = 3(lbh) = 3(abc)$ **Answer.**

**Problem 3:** Evaluate $\iint_S (\vec{V}.\hat{n})dA$ using Gauss Divergence Theorem, where $\vec{V} = 3x^2\hat{\imath} + 6y^2\hat{\jmath} + z\hat{k}$ and D is the region bounded by the closed cylinder $x^2 + y^2 = 16$, z = 0 and z = 4.

**Solution:** Here: $\vec{V} = 3x^2\hat{\imath} + 6y^2\hat{\jmath} + z\hat{k}$

$\Rightarrow div\vec{V} = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(6y^2) + \frac{\partial}{\partial z}(z)$

$\Rightarrow div\vec{V} = 6x + 12y + 1$

By Gauss Divergence Theorem: $\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv$

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D (6x + 12y + 1)dv$

Let us calculate the limits of x, y, z

$0 \le z \le 4$ (Given height of cylinder)

Also $x^2 + y^2 = 16$

$\Rightarrow y = \pm\sqrt{16 - x^2}$

$\Rightarrow -\sqrt{16 - x^2} \le y \le \sqrt{16 - x^2}$

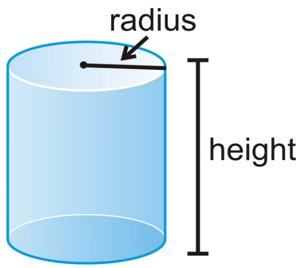
On x-axis (y=0)

$\Rightarrow x^2 + 0^2 = 16$

$\Rightarrow x = \pm 4$

$\Rightarrow -4 \le x \le 4$

Let us put these limits in the given integral




$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \iiint_D (6x + 12y + 1)dv$

$\Rightarrow \iint_S (\vec{V}.\hat{n})dA = \int_{z=0}^{z=4} \int_{x=-4}^{x=4} \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x + 12y + 1)dydxdz$

$= \int_{z=0}^{z=4} dz \int_{x=-4}^{x=4} \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x + 12y + 1)dydx$

$= 4 \int_{x=-4}^{x=4} \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x + 12y + 1)dydx$

$= 4 \int_{x=-4}^{x=4} \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (1)dydx$ (Because x and y are odd functions)

$= 4(2)(2) \int_{x=0}^{x=4} \int_{y=0}^{\sqrt{16-x^2}} (1)dydx$ (Because 1 is an even function)

$= 16 \int_{x=0}^{4} \sqrt{16 - x^2}dx = 16\left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2} sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{4}$

$= 64\pi$ **Answer.**