Topology II

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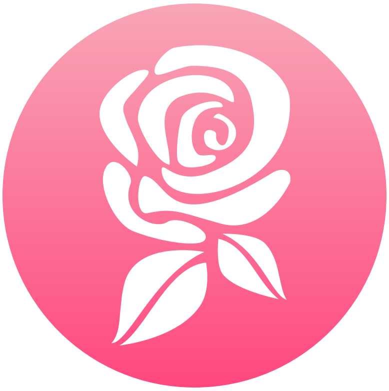

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Information

Time & Room	TTth 12:00-13:20, Lockett 232
Exam	?
Textbook	Hatcher, *Algebraic Topology*
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Office Hours	Tuesday, Thursday 13:30-14:20

First assignment: Get your hand on the textbook! Additionally, read the introduction to Chapter 2.

Chapter 1

Weeks 1-3

1.1 Week 1

1.1.1 Tuesday

Theorem 1.1.1 (Brower Fixed Point for $n = 2$)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Towards a contradiction, suppose f is a fixed point free map. Then $r(x) = \frac{x-f(x)}{\|x-f(x)\|}$ is a continuous map from $D^2 \to S^1$. Because r is a retract, we have the composition $D^2 \overset{r}{\to} S^1 \overset{i}{\to} D^2$ yielding the maps

$$\pi_1(D^2) \overset{r_*}{\to} \pi_1(S^1) \overset{i_*}{\to} \pi_1(D^2).$$

However, as retracts should induce a surjection, this is a contradiction!

What about $n \geq 3$?

We want to study the relationship between topological spaces and algebraic objects. Maybe we associate groups, maybe rings...We want this association to be functorial. Roughly speaking, this means that spaces that are the same should get sent to the same objects! Another property of functorality is that continuous maps f should be sent to homomorphisms f_*.

In topology 1, we take a space X and assign to it its fundamental group $\pi_1(X)$. Roughly speaking, $\pi_1(X)$ is the set of homotopy classes of maps $S^1 \to X$. To generalize, we could look at the homotopy classes of maps $S^k \to X$. We denote this set $\pi_k(X)$ and we call them the higher homotopy groups. Additionally, they turn out to be abelian groups for $k \geq 2$. These are hard to determine!

A guess: Are these groups given by generators of faces and relations of 4-cells with a simplicial complex?

In this course, we're going to take X and take it to $\{H_k X\}$, homology, and $\{H^k X\}$, cohomology groups.

1.1.2 Informal Homology via Pseudomanifolds

Let's denote pseudomanifolds $\mathscr{X}$.

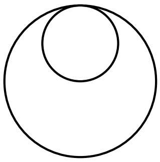
Definition 1.1.2 (Informal $k-$dimensional Manifold)**.** *A space that looks locally like* $\mathbb{R}^k$*.*

Definition 1.1.3 (Informal $k-$dimensional $\mathscr{X}$-Manifold)**.** *Can have singularities; points of being noneuclidean.*

Some properties:

(1) The part of P where it is a $k-$manifold is open, dense, and oriented.

(2) The set of singularities has dimension $\leq k-2$



Definition 1.1.4 ($k-$Simplex)**.** *A k-simplex is the convex hull of points $p_0 \dots p_k$ in general position in some Euclidean space. Its faces are $k-1$simplicies.*

Definition 1.1.5 (Simplicial Complex)**.** *A simplicial complex is the set S of simplicies in some* $\mathbb{R}^N$ *satisfying*

- *Any face of a simplex in S is in S.*
- *Any two simplices in S are either disjoint or intersect in a set that is a face of both of them.*

Simplicial complexes are not pseudomanifolds; the edges are too big!

Definition 1.1.6 (Orientation on a Simplex)**.** *An orientation $\mathscr{O}$ in $\Delta^{k>0}$ $k-$simplex is an ordering of the vertices of each simplex. Two orderings are the same or equivalent if they differ by an even permutation.*

An orientation on a point is either a $+$ or $-$.

Hence, there are two orientations on any simplex; we call them "opposites."

The orientation has a concept of induced orientations; just negotiate the missing vertex to the last position via a permutation and delete it.

Definition 1.1.7 (k-dimensional Pseudomanifold)**.** *A $k-$dimensional pseudomanifold is a simplicial complex with a $\mathcal{O}(\Delta)$ orientation of each $k-$simplex such that*

(1) Every simplex is a face of a $k-$simplex.

(2) Every $(k-1)$-simplex is a face of exactly two $k-$simplices.

(3) Continuity of orientation: If Δ' is a $(k-1)$-simplex, face of $\Delta, \tilde{\Delta}$, $\mathcal{O}(\Delta)$ and $\mathcal{O}(\tilde{\Delta})$ must induce opposite orientations on Δ'.

Definition 1.1.8 (i-cycle)**.** *Let X be a topological space. An i-cycle of X is a pseudomanifold of dimension i, P, and a map $\sigma : P \to X$ that "captures" a hole.*

For example, take P, an oriented triangle, and X an annulus. Map the triangle around the hole.

Let $\sigma_1 : P_1 \to X$, $\sigma_2 : P_2 \to X$ be i-cycles. Let $P_1 + P_2 = P_1 \sqcup P_2$, and σ be defined as you would think. We can also define $-P$; take the same psuedomanifold and same map, but put the opposite orientation on P.

Definition 1.1.9 (Pseudomanifold with Boundary)**.** *A simplicial complex Q and an orientation $\mathcal{O}(\Delta)$ on each $k-$simplex, along with a subsimplex B that satisfies:*

(1) Each simplex in Q is a face of a $k-$simplex

(2) B a $k-1$ dimensional pseudomanifold;

(3) Each $k-1$-dim Δ' not in B is a face of 2 k-simplices;

(4) Each $k-1$-dim Δ' in B is a face of 1 $k-$simplex;

(5) Orientation is inherited.

Definition 1.1.10 (Cobordism of i-cycles)**.** *A cobordism of i-cycles P_1, P_2 in X is an $(i+1)$-dimensional psuedomanifold with boundary C and a map $\sigma : C \to X$ So that $\partial B = P_1 - P_2$ and $\sigma|_B$ coincides with σ_1, σ_2.*

Two i-cycles that have a cobordism between them are called cobordant. Homework problem:

Proposition 1.1.11 (Cobordism relation)**.** *Cobordism of i-cycles in X is an equivalence relation.*

Proof.

Reflexive Let $P = \{x_1, \dots, x_k\}$. Let $P \cong Q = \{y_1, \dots, y_k\}$. Construct Δ as follows. The $k+1$ cells are of the form $\Delta_n = p_n \sqcup q_{k+1-n}$ where $p_n = \{x_i \,|\, i \in [n]\}$ and $q_m = \{y_i \,|\, i \in [m]\}$ for $1 \leq n \leq k$. Now, the k cells are of the form $p_n \sqcup q_{k-n}$ or $p_{n-1} \sqcup q_{k+1-n}$. Thus, Δ_n shares a face with Δ_{n+1} and Δ_{n-1}. The only faces not shared are P, from $n = 1$, and Q, from $n = k$. Hence, $P \sim Q = P$ so $P \sim P$.

Symmetric Given $P \sim Q$ via Δ, σ, we know $\partial\Delta = P - Q$ and hence $\partial\overline{\Delta} = (-P) - (-Q) = Q - P$ so $Q \sim P$ via $\overline{\sigma}$.

Transitive Let $P \sim Q$ and $Q \sim R$ via Δ, σ and Δ', σ', construct Δ'', σ'' by identifying Q in Δ with Q in Δ'. Then, $\overline{Q}$ is a face of some $X \in \Delta$, Q a face of some $\Upsilon \in \Delta''$, so Q is a face of both X and Y and $\partial\Delta'' = P - R$. Let $\sigma''|_{\Delta} = \sigma$ $\sigma''|_{\Delta'} = \sigma'$. Hence, $P \sim R$.

Definition 1.1.12 (Homology)**.** *$H_i(X)$ is the set of equivalence classes of $+, -$ of i-cycles giving an abelian group structure. The identity element of this group is represented by the empty $i-$dimensional psuedomanifold.*

1.1.3 Thursday

Types of topological spaces:

$$\begin{tikzcd}
 & \text{Metric spaces} & \\
\text{Manifolds} \arrow[ur, "\subset"] \arrow[d, "\subset"] & \text{Simplicial complexes} \arrow[u, "\subset"'] \arrow[d, "\subset"] & \\
\text{Pseudomanifold} & \Delta\text{-Complex} \arrow[r, "\subset"] & \text{CW-Complex}
\end{tikzcd}$$

What's a manifold? Well, a manifold is a 2nd-countable Hausdorff space that is locally euclidean.

Examples:

- $\mathbb{R}^n$;
- S^n;
- Products of manifolds: $T = T^2 = S^1 \times S^1$.
- $\square$-Complex: Combinatorial model for space X, built from "cells" or related objects.

Let's mess with the torus $T = S^1 \times S^1$ for a bit. Whenever we're constructing it from a gluing diagram, we have one zero cell, the point, two 1-cells, the edges, and one 2-cell, the square itself. We can think of $n-$cells as n-disks. Hence,

$$T = X \supset X^1 \supset X^0$$

where X^0 is the $0 - cell$, X^1 is $X_0 \cup \{\text{1-cells}\}$. We could say, if we're so inclined,

$$X^1 = \left(X^0 \sqcup e_1^1 \sqcup e_2^1\right) / \sim_1$$

and

$$X = (X^1 \sqcup e^2) / \sim_2 .$$

Definition 1.1.13 (CW-Complex)**.** *(1) A discrete set of points X^0;*

(2) Inductively form the n-skeleton X^n from X^{n-1} by attaching some collection of n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} = \partial D^n \to X^{n-1}$, and setting $X^n = (X^{n-1} \sqcup_\alpha D^n_\alpha) / \sim$ where $x \sim \varphi_\alpha(x)$ for $x \in S^{n-1} = \partial D^n$.

(3) Either $X = X^n$ for some n and we say that n is the dimension of X, or $X = \bigcup_n X^n$ with the weak topology.

Definition 1.1.14 ((aside) Weak Topology)**.** *Let $X = \bigcup_n X_n$. Then $A \subset X$ is open if $A \cap X_n$ is open in X_n for all n.*

Let's do some examples; $X = X^1$ is a graph, S^1 is a point and a quotiented one cell, or two points and two one cells. More interestingly, we could have $S^2 = x_0 \cup D^2/\sim$ where $x_0 \sim y$ if $y \in \partial D^2$...And we could generalize this to all $n-$spheres.

Another really interesting example is T^2, formed by the CW-Complex by an octagon!

We can do similar things with non-orientable surfaces. Take $\mathbb{R}P^2$ and the standard gluing diagram. In general,

$$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1}\backslash 0/\sim$$

where $v \sim \lambda\omega$ for $\lambda \neq 0$. It could also be written $S^n/\sim$, $x \sim -x$, and $D^n/\sim$ for $y = -y$ for $y \in \partial D^n = S^{n-1}$. Yet another description is $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$.

For $m < n$, $\mathbb{R}P^m \subset \mathbb{R}P^n$ is a subcomplex, with $\mathbb{R}P^\infty$ are lines through the origin in $\mathbb{R}^\infty$ (weak topology).

Now here's where the fun begins: Klein bottles with their standard gluing map!

This leads into the observation that closed surfaces can be realized as the quotient spaces of polygons. Moreover, a polygon can be cut into triangles! Thus, any surface can be built out of triangles. This is where the notion of a Δ-complex (or more stringently, a simplicial complex) comes from.

Let's get some standard notation. A standard n-simplex

$$\Delta^n = \{(t_0, \dots, t_j) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n t_i = 1, t_i \geq 0\}$$

. Hence, Δ^0 is a point, Δ^1 is the line segment from $(1, 0)$ to $(0, 1)$, so on, so forth.

The $n-$simplex on $(p_0, \dots, p_n)$ in general position (the set $\{v_1 - v_0, \dots, v_n - v_0\}$ is linearly independent) is the convex hull of $(p_0, \dots, p_n)$. We call it $\Delta = [p_0 p_1 \dots p_n]$. In this position, $\Delta^n = [e_0 e_1 \dots e_n]$. Any two n-simplicies are homeomorphic. Eg, $\Delta^n \to \Delta = [v_0 \dots v_n]$ via $(t_0, \dots, t_n) \mapsto (\sum t_i v_i)$.

Some terminology that we'll use: The boundary of $[v_0 \dots v_n] = \cup\{\text{all (n-1)-dim faces}\}$, the interior of Δ is $\Delta-$boundary, so $]v_0 \dots v_n[= [v_0 \dots v_n] - \partial[v_0 \dots v_n]$.

Definition 1.1.15 (Δ-Complex)**.** *A Δ-complex is a quotient of a disjoint union of simplicies obtained by identifying certain faces using orientation preserving homeomorphisms.*

To be more refined, a Δ-complex on X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ where

- $\sigma_\alpha|_{\overset{o}{\Delta^n}}$ *is a homeomorphism onto* $\sigma_\alpha(\overset{o}{\Delta^n}) = e^n_\alpha$.
- *Each $x \in X$ is in exactly one e^n_α.*
- $\sigma_\alpha|_{\text{(n-1)-face of }\Delta^n} = \sigma_\beta : \Delta^{n-1} \to X$.
- *$U \subset X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in Δ^n for all n.*

A homework problem:

A Δ-Complex is a CW-Complex.

1.2 Week 3

1.2.1 Free Abelian Groups

Definition 1.2.1 (Free Abelian Group)**.** *Let $G_1, G_2, \dots, G_n$ be groups. Then, we have the direct product $\prod_1^n G_i$ with operations defined pointwise. If each G_i is abelian, we speak instead of their direct sum $\bigoplus_1^n G_i$. If each and every G_i is infinite cyclic, we call their direct sum* free abelian. *If $G_i = \langle a_i \rangle$, then elements of $\bigoplus G_i$ look like $(m_1 a_1, \dots, m_n a_n)$ for $m_i \in \mathbb{Z}$. We call $\{a_1, \dots, a_n\}$ a basis for $\bigoplus G_i \cong \bigoplus \mathbb{Z} = \mathbb{Z}^n$.*

We can generalize this; for an infinite list $S = \{a_1, a_2, \dots\}$, the free abelian group with basis S is $G = G(S)$ with elements $\sum m_i a_i$ for finitely many $m_i \neq 0$.

Recall that a Δ complex is a topological space X with characteristic maps $\sigma_\alpha : \Delta^n \to X$ with each $x \in X$ in a unique open $n-cell$ $\sigma_\alpha(\Delta^n)$ and compatibility, openness conditions.

For example, take $X = T^2$. Then we have one 0-cell $\sigma_v : \Delta^0 \to X$, three 1-cells $\sigma_a, \sigma_b, \sigma_c : \Delta^1 \to X$, and two 2-cells $\sigma_U, \sigma_L : \Delta^2 \to T$ with some identifications.

Definition 1.2.2 (Simplicial Homology of ⊠-Complexes)**.** *Let $\Delta_n(X)$ be the free abelian group with basis n-simplicies in X (or the maps $\sigma_\alpha : \Delta^n \to X$).*

We also inflict the following boundary homomorphism: Let $\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ via

$$\partial[v_0, v_2] = +[v_1] - [v_0],$$

$$\partial[v_0, v_1, v_2] = +[v_0, v_1] + [v_1, v_2] - [v_0, v_2],$$

In general...literally just take the boundary. $\partial[v_0, \ldots, n_n] = \sum(-1)^i[v_0 \ldots \hat{v}_i \ldots v_n]$.

Example: Using the torus again, we have $\Delta_0(T) = \mathbb{Z}$, $\Delta_1(T) = \mathbb{Z}^3$, $\Delta_2(T) = \mathbb{Z}^2$, and otherwise 0.

For general Δ complexes, we say

$$\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|_{[v_0 v_1 \ldots \hat{v}_i \ldots v_n]}.$$

Lemma 1.2.3. *We have that* $\partial_n \circ \partial_{n+1} = 0$. *Tersely, we can say*

$$\Delta_{n+1}(x) \overset{\partial_n+1}{\to} \Delta_n(x) \overset{\partial_n}{\to} \Delta_{n-1}(x)$$

to mean that $\operatorname{im}\partial_{n+1} \subset \ker\partial_n$. *We then define the nth simplicial homology group to be* $H_n^\Delta(X) = \ker\partial_n / \operatorname{im}\partial_{n+1}$.

Proof. For $\sigma : \Delta^{n+1} \to X$,

$$\partial\sigma = \sum_{i=0}^{n+1}(-1)^i \sigma|_{\hat{v}_i}$$

so

$$\begin{aligned}
\partial\partial\sigma &= \sum_{i=0}^{n+1}(-1^i)\partial\sigma_{\hat{v}_i} \\
&= \sum_{i=0}^{n+1}(-1)^i \left(\sum_{j=0}^{i-1}(-1)^j \sigma|_{\hat{v}_i, \hat{v}_j} + \sum_{j=i+1}^{n+1}(-1)^{j-1}\sigma|_{\hat{v}_i, \hat{v}_j} \right) \\
&= \sum_{0 \le p < q \le n+1} (-1)^{p+q}\sigma|_{\hat{v}_q, \hat{v}_p} \\
&\quad \sum_{0 \le q < p \le n+1} (-1)^{p+q-1}\sigma|_{\hat{v}_p, \hat{v}_q} \\
&= 0
\end{aligned}$$

For example, take $X = [v_0 v_1 v_2]$ with bases $\sigma_0, \sigma_1, \sigma_2$, $\sigma_{01}, \sigma_{02}, \sigma_{12}$ and σ_{012}. We have that $\partial\sigma_{12} = \sigma_2 - \sigma_1$, so on, so forth. Hence,

$$
\begin{tikzcd}
0 \arrow[r] & \Delta_2(X) \cong \mathbb{Z} \arrow[r] & \Delta_1(X) \cong \mathbb{Z}^2 \arrow[r] & \Delta_0(X) \cong \mathbb{Z}^3 \arrow[r] & 0 \\
0, 0 \arrow[r, mapsto, "\partial"] & 0, 0 & & & \\
& \mathbb{Z}^2, 0 \arrow[r, mapsto, "\partial"] & \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle, 0 & & \\
& & \mathbb{Z}^2, \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle \arrow[r, mapsto, "\partial"] & \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle, 0 & \\
& & & \mathbb{Z}^3, \mathbb{Z}^3 \arrow[r, mapsto, "\partial"] & 0, 0 \\
0 \arrow[r, dash] & H_2 \cong 0/0 = 0 \arrow[r, dash] & H_1 \cong \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle / \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle = 0 \arrow[r, dash] & H_0 \cong \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle \cong \mathbb{Z} \arrow[r, dash] & \bullet
\end{tikzcd}
$$

Now, take $X = S^1$, with one 0 cell v and one 1 cell a.

$$\begin{tikzcd}
0 \arrow[r, "\partial"] & \Delta_1 = \langle a \rangle \arrow[r, "\partial"] & \Delta_0 = \langle v \rangle \arrow[r, "\partial"] & \bullet \\
0,0 \arrow[r] & 0,0 & & \\
& \mathbb{Z},\mathbb{Z} \arrow[r] & 0,0 & \\
& & \mathbb{Z},\mathbb{Z} \arrow[r] & 0,0 \\
0 \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & 0
\end{tikzcd}$$

What about a different Δ complex on the circle? Say, two points g, f with edges p, q ending in f.

$$\begin{tikzcd}
0 \arrow[r, "\partial"] & \Delta_1 = \langle p, q \rangle \arrow[r, "\partial"] & \Delta_0 = \langle g, f \rangle \arrow[r, "0"] & \bullet \\
0,0 \arrow[r] & 0,0 & & \\
& \mathbb{Z}^2, \langle p - q \rangle \arrow[r] & \mathbb{Z}, 0 & \\
& & \mathbb{Z}^2, \mathbb{Z}^2 \arrow[r] & 0,0 \\
0 \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & 0
\end{tikzcd}$$

1.2.2 Thursday

Definition 1.2.4 (Chain Complex). *A chain complex $\{C_n, \partial_n\}$ is a sequence of abelian groups C_n with homomorphisms $\partial_n = \partial C_n \to C_{n-1}$ with $\partial_n \partial_{n-1} = 0$. As abelian groups are $\mathbb{Z}$ modules, we could replace $\mathbb{Z}$ with any ring.*

Elements of C_n are called chains, elements of $\ker \partial$ are called cycles, and elements in $\operatorname{im} \partial$ are called boundaries.

If one has a chain complex, one can take its homology:

Definition 1.2.5 (Homology). *The homology of a chain complex is*

$$H_c(C_\bullet) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

We say that $z, w \in \ker \partial$ *are homologous if* $z - w \in \operatorname{im} \partial$.

For example, we return to $C_n = \Delta_n(X)$ with ∂ being the simplicial boundary.

Let's take the torus.

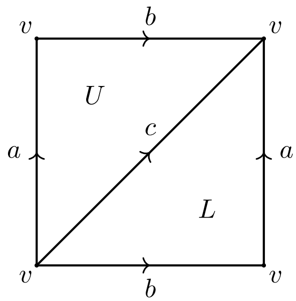


Figure 1.1: A Δ-complex on the torus.

Then

$$\Delta_2(T) = \langle U, L \rangle \to \Delta_1(T) = \langle a, b, c \rangle \to \Delta_0 = \langle v \rangle.$$

See that $\ker \partial_2 = \langle U, L \rangle$ and $\ker \partial_1 = \langle a, b, c \rangle = \Delta_1$. Thus, $\operatorname{im} \partial_2 = \langle a + b \rangle$. Hence,

$$H_2 = \langle U + L \rangle \cong \mathbb{Z}$$

$$H_1 = \langle a, b, c \rangle / \langle a + b - c \rangle = \langle a, b \rangle \cong \mathbb{Z}^2$$

$$H_0 = \langle v \rangle \cong \mathbb{Z}.$$

Recall that a simplicial complex is a Δ complex for which each σ_α is injective on the vertices of the standard $n-$simplex and furthermore no other $n-$simplex has the exact same set of vertices.

For example, to get a simplicial complex on a circle, you require at least three vertices.

This gives rise to some fun problems: What's the minimal simplicial complex structure on S^2? How about T? $\mathbb{R}P^2$? What does it mean to be minimal here, minimize the number of triangles?

Given a Δ-complex, we can produce a simplicial complex structure by using barycentric subdivision.

Tale a closed $n-$simplex $[v_0, v_1, \ldots, v_n]$. Recall that these points are of the form $\sum_{i=0}^n t_i v_i$ where the sum of the t_i's is one and each is non zero. A barycenter is $\sum \frac{1}{n+1} v_i$. For barycentric subdivision, we take a simplex $[v_0 \ldots v_n]$ and decompose it into $n-$simplices by taking $[v_0 \ldots \hat{v}_j \ldots v_n b]$ where b is the barycenter where inductively $[v_0 \ldots \hat{v}_j \ldots v_n]$ is a face in the barycentric subdivision of a face.

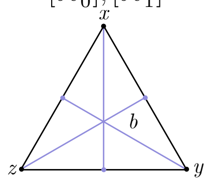
	Simplex	Subdivision
$n=0$	$[v_0]$	$[v_0]$
$n=1$	$[v_0v_1]$	$[bv_0], [bv_1]$
$n=2$	$[xyz]$	

Figure 1.2: Barycentric subdivision.

Given a Δ-complex structure on X with $\sigma_\alpha^n : \Delta^n \to X$, the barycentric subdivision is the Δ complex with characteristic maps $\sigma_\alpha^n \tau_\beta^m : \Delta^m \to X$.

Claim: Barycentric subdivision twice gives a simplicial complex. See H§2.3#23.

Now, we move to a special little thing called singular homology. It's defined for any topological space and will agree with simplicial homology for Δ complexes.

Definition 1.2.6 (Singular n-simplex)**.** *A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$.*

We define the singular n-chains to be $C_n(X)$, the free abelian group generated by singular n-simplices.

Elements of $C_n(X)$ look like $m_1\sigma_1 + ... + m_k\sigma_k$ for $m_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. We develop a map ∂ given by $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0 \dots \hat{v}_j \dots v_n]}$. We still have $\partial \circ \partial = 0$.

Definition 1.2.7 (Singular Homology)**.** *With the previous definitions, $\{C_n, \partial_n\}$ is the singular chain complex of X giving rise to singular homology.*

Let's do an example with $X = \{*\}$. Then, $\sigma^n : \Delta^n \to X = *$ is the unique continuous map into X for every $0 < n$. Hence, $C_n(X) = \langle \sigma_n \rangle \cong \mathbb{Z}$ for each $n \geq 0$.

$$\begin{tikzcd}
& \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \\
{} \arrow[r, dashed] & C_2(X) \arrow[r, "\partial_2{,}\cong"] & C_1(X) \arrow[r, "\partial_1{,}0"] & C_0(X) \arrow[r] & 0
\end{tikzcd}$$

Concretely, we can determine that $\partial\sigma^n = \sigma^{n-1}$ for n even and 0 for n odd.

So, $H_{2k} = \ker \partial_{2k} / \operatorname{im} \partial_{2k-1} = 0/0 = 0$ and $H_{2k+1} = \ker \partial_{2k+1} / \operatorname{im} 2k = \mathbb{Z}/\mathbb{Z} = 0$. Hence $H_n(*) = 0$ except at $k = 0$. $H_0(*) = \ker \partial_0 / \operatorname{im} \partial_{-1} = \mathbb{Z}/0 = \mathbb{Z}$.

Let's go back and do one more simplicial calculation, H_*^Δ, for $X = \mathbb{R}P^2$.

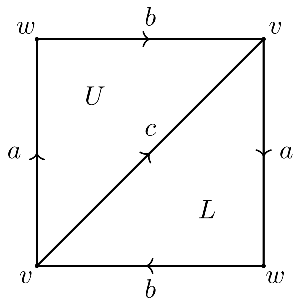


Figure 1.3: A Δ-complex on $\mathbb{R}P^2$.

We have $\Delta_0(X) = \langle \sigma_v, \sigma_w \rangle \cong \mathbb{Z}^2$, $\Delta_1(X) = \langle \sigma_a, \sigma_b, \sigma_c \rangle \cong \mathbb{Z}^3$ and $\Delta_2(X) = \langle \sigma_U, \sigma_L \rangle \cong \mathbb{Z}^2$. We see that

$$\begin{aligned}
\sigma_a &\mapsto \sigma_w - \sigma \\
\sigma_b &\mapsto \sigma_v - \sigma_w \\
\sigma_c &\mapsto 0
\end{aligned}$$

$$\begin{aligned}
\sigma_U &\mapsto \sigma_a + \sigma_b - \sigma_c \\
\sigma_L &\mapsto -\sigma_a - \sigma_b - \sigma_c
\end{aligned}$$

Now, we compute the homology groups.

$$H_0^\Delta(X) = \Delta_0(x)/\operatorname{im}\partial_1 = \langle \sigma_v, \sigma_w \rangle / \langle \sigma_v - \sigma_w \rangle \cong \mathbb{Z}$$

For at home: Check that ∂_2 is injective so that $H_2^\Delta = \ker\partial_2 = 0$, and that

$$H_1^\Delta(X) = \ker\partial_1/\operatorname{im}\partial_2 = \langle \sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle / \langle 2\sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle \cong \mathbb{F}_2.$$

Chapter 2

Second Chapter

2.1 First Section

Appendix A

List Of Definitions

Appendix B

List Of Theorems