
Algebra II



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Information

Time & Room	MWF 13:30-14:20, Lockett 232
Exam	Monday May 5, 10:00-12:00
Textbook	Dummit & Foote, Abstract Algebra
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Here's what we're going to cover:

- Chapter 13-14, Field and Galois theory;
- Chapter 15, Commutative algebra and algebras over a field;
- Chapter 10, Basics for modules and their tensor products;
- Chapter 18, Wedderburn's theorem, Maschke's theorem and linear representations of finite groups;
- POSSIBLY Chapter 17, homological algebra.

Grade Distribution

Homework	60%
Midterm	20%
Final	20%

Chapter 13

Field Theory

13.1 Extensions

Definition 13.1.1 (Field). *A field is a commutative ring in which every nonzero element is invertible.*

We denote by $F^\times = F \setminus \{0\}$ the set of all invertible elements of the field F .

In general, we denote R^\times as the set of all units of the ring R .

Definition 13.1.2 (Characteristic). *Let F be a field with identity 1. The characteristic of F is the order of 1 in the group $(F, +)$. If the order of 1 is not finite, we define the characteristic of F to be 0.*

We denote the characteristic as $\text{ch}(F)$.


We know that $\mathbb{Z}/p\mathbb{Z}$ is a field of order p if p is a prime.

Because \mathbb{Q} has $n1 \neq 0$ for $n \neq 0$, $\text{ch}(\mathbb{Q}) = 0$. Some other fields with characteristic zero are $\mathbb{R}, \mathbb{C}, \mathbb{C}(x) \dots$

We denote by $\mathbb{Z}_p(x)$ as the field of rational functions over \mathbb{Z}_p . That is, we're adjoining the element x . It is an infinite field with finite characteristic.

Let's say that G is an abelian group. If we write it multiplicatively, $g^n = g \cdots g$ n -many times. If we write it additively, we write $g = ng$.

Proposition 13.1.3 (Characteristic of a Field). *The characteristic of a field is 0 or a prime number.*

Proof. Towards a contradiction, let F be a field with $\text{ch}(F) \neq 0$. Then $\text{ch}(F) = nm$ for $1 < n, m < \text{ch}(F)$. Because $n \cdot 1_F := n, m \cdot 1_F := m \in F$, we have that $n \cdot m = nm \cdot 1_F = 0$. 

Definition 13.1.4 (Prime Subfield). *Let F be a field. The prime subfield of F is the subfield generated by 1.*


We follow with examples.

1. $\mathbb{Z}_p(x) \geq \mathbb{Z}_p$.
2. $\mathbb{R} > \mathbb{Q} \geq \mathbb{Q}$.

Proposition 13.1.5 (Prime Subfield). *Let F be a field and K the prime subfield of F . If $\text{ch}(F) = p \neq 0$, then $K \cong \mathbb{Z}_p$. If $\text{ch}(F) = 0$, $K \cong \mathbb{Q}$.*

Proof. Define $\varphi : \mathbb{Z} \rightarrow F$; $\varphi(n) = n1$. We know that φ is a ring homomorphism; I omit the proof for being rather repetitive. Such a proof is necessary to remember, however.

The kernel of φ is an ideal. In particular, because \mathbb{Z} is a principal ideal domain, $\ker(\varphi) = (a)$ for some nonnegative integer a . If $a = 1$, φ is the zero map. If $a = 0$, φ is injective. In this case, $\mathbb{Z} \cong \varphi(\mathbb{Z}) \subset F$. Hence, $\mathbb{Q} \xrightarrow{\varphi} F$ (this is to be read \mathbb{Q} extends to F), so $K \cong \mathbb{Q}$.

In the case that $a \neq 0$, then $a1 = 0$. This implies that $\text{ch}(F) = p|a$. Hence $(p) \subset \ker(\varphi) = (a) \subset (p)$; Thus $(p) = (a)$. Thus, $a = p$. Hence $\varphi(\mathbb{Z}) \cong \mathbb{Z}/(p) = \mathbb{Z}_p$ by the first isomorphism theorem, so $K \cong \mathbb{Z}_p$. 

Definition 13.1.6 (Extension). *If F contains a subfield K , we call F an extension of K , written as F/K .*

In this case, we have the diagram

$$\begin{array}{c} F \\ | \\ K \end{array}$$

Additionally, F is a K -vector space.

We denote the dimension or index of the extension $\dim_K F = [F : K]$.

For example, $\mathbb{C} \supseteq \mathbb{Q}$, so \mathbb{C} is a \mathbb{Q} -linear space of uncountably infinite dimension: $[\mathbb{C} : \mathbb{Q}] = \infty$.

In a particularly obvious case, we have $[\mathbb{C} : \mathbb{R}] = 2$ for the extension \mathbb{C}/\mathbb{R} .


Let's take $\mathbb{Q}(i) = \{a + bi | a, b \in \mathbb{Q}\}$; $\mathbb{Q}(i)/\mathbb{Q}$ is an extension of degree 2.

Theorem 13.1.7 (Extension Index). *Let $L/K, K/F$ be finite extensions. Then L/F is finite and $[L : F] = [L : K][K : F]$.*

$$\begin{array}{c} L \\ \Big|_{[L:K] < \infty} \\ K \\ \Big|_{[K:F] < \infty} \\ F \end{array}$$

In time, we learn these are tensor products.

Proof. Let $A = \{\alpha_1, \dots, \alpha_n\}$ be a basis for K over F and $B = \{\beta_1, \dots, \beta_m\}$ be a basis for L over K . Let $C = \{\alpha_i \beta_j | i \in [n], j \in m\}$. Naturally, $|C| = mn$. We seek to show that C is a basis. Let $x \in L$; then $x = \sum_{i=1}^n k_i \beta_i$ for some $k_i \in K$. But each k_i can be written as $k_i = \sum_{j=1}^m f_{ij} \alpha_j$. Hence, $x = \sum_{i=1}^n \sum_{j=1}^m f_{ij} \alpha_j \beta_i$. Hence, C spans L over F .

Suppose now that $\sum f_{ij} \alpha_j \beta_i = 0$. Then $\sum_i \left(\sum_j f_{ij} \alpha_j \right) \beta_i = 0$. By the independence of B over K , and A over F , $f_{ij} = 0$. Thus C is independent and C is a basis. 


Theorem 13.1.8 (Finite Subextension). *If L/F is finite and K/F is a subextension, that is,*

$$\begin{array}{c} L \\ \Big| \\ K \\ \Big| \\ F \end{array}$$

then K/F is finite and $[K : F][L : F]$.

A neat consequence of this is: If L/F is finite and $[L : F]$ is prime, L/F has no nontrivial subextensions.


Proposition 13.1.9 (Field to Ring Homomorphism). *If $\varphi : F \rightarrow R$ is a ring homomorphism with $\varphi(1_F) \rightarrow 1_R$ where F is a field and R is a ring. Then φ is injective.*

Proof. The only ideals of F are (0) and F . As $\varphi(1) = 1$, $\ker(\varphi) \neq F$ so $\ker \varphi = 0$ and φ is injective. 

Let F be a field and $F[x]$ be a polynomial ring over F . Then, $F[x]^\times = F^\times$. Let p be irreducible in $F[x]$ and let $K = \frac{F[x]}{(p(x))}$, a field as $(p(x))$ is maximal.

Then, every element is of the form $\overline{f(x)} = f(x) + (p(x))$. Now, define $\varphi : F \rightarrow K$ via $\varphi(\alpha) = \alpha + (p(x))$. Naturally, $\varphi(1) = 1 + (p(x))$, $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$, and $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ so it is a ring homomorphism. Therefore, φ is an embedding and $F \cong \varphi(F)$. Identify $F \equiv \overline{F} \subset K$.

Theorem 13.1.10 (Polynomial Extension). *Let $p(x) \in F[x]$ be an irreducible polynomial. Then $p(x)$ has a root $\theta = x + (p(x)) \in K$ and $[K : F] = \deg p(x)$. Moreover, the set $\{1, \theta, \dots, \theta^{\deg p(x)-1}\}$ is a basis of K over F .*

Proof. $p(\theta) = p(x) + (p(x)) = (p(x)) = 0$ in K . We want to show that $\{1, \theta, \dots, \theta^{\deg p(x)-1}\}$ is a basis. For any $\overline{f(x)} \in K$, $\overline{f(x)} = f(x) + (p(x))$. By the division algorithm, $f(x)/p(x) = r(x)$ where $\deg r(x) < \deg p(x)$ or $r(x) = 0$. Let $r(x) = \sum_{i=0}^{n-1} a_i x^i$ for some $a_i \in F$. Moreover, $\overline{f(x)} = \overline{r(x)} = a_0 \overline{1} + a_1 \overline{x} + \dots + a_{n-1} \overline{x}^{n-1} = \sum_{i=0}^{n-1} a_i \theta^i$ and hence $\{1, \theta, \dots, \theta^{\deg p(x)-1}\}$ spans. 

Linear independence is left to the reader.

We were working on field extensions $K = \frac{F[x]}{(p(x))}$ where p is irreducible over F .

If $\theta = x + (p(x)) \in K$, then $p(\theta) = 0$. Moreover, $\{1, \theta, \dots, \theta^{n-1}\}$ spans K over F where $n = \deg p(x)$.


We want to show that $\{1, \theta, \dots, \theta^{n-1}\}$ is linearly independent over F .

Suppose $a_0 + a_1 \theta + \dots + a_{n-a} \theta^{n-1} = 0$ in K .


Consider the polynomial $g(x) = a_0 + a_1x + \dots x_{n-1}^{n-1} \in F[x]$. This implies that $g(\theta) = 0$ in K . This means that $g(x) + (p(x)) = 0$. Hence $g(x) \in (p(x))$, so $p(x)|g(x)$. This is a contradiction as $\deg p > \deg g$, unless $g \equiv 0$ and $a_0, \dots, a_{n-1} = 0$.

Therefore, $K = F[x]/(p(x))$ is an extension in which $p(x)$ has a root.

Theorem 13.1.11 (Existence of Root Extensions). *Let $f(x) \in F[x]$ be a nonconstant polynomial. There exists a field extension in which $f(x)$ has a root.*

Proof. Because $F[x]$ is a PID, we have the unique factorization $f(x) = p_1(x) \cdots p_n(x)$ for some irreducible p_i . By the preceding theorem, there exists a field K in which p_1 has a root. Therefore, $f(x)$ shares this root in K . 

Theorem 13.1.12 (Existence of Root Extensions (again)). *Let $f(x) \in F[x]$ be a nonconstant polynomial. There exists a field extension in which $f(x)$ splits.*

Proof. Because $F[x]$ is a PID, we have the unique factorization $f(x) = p_1(x) \cdots p_n(x)$ for some irreducible p_i . By the preceding theorem, there exists a field K in which each p_i has a root. Iterate this process for all i to obtain the field. 

For example, take the polynomial $x^2 + x + 1$ over \mathbb{Q} . If θ is a root, then it naturally has degree 2 so the extension field K has $[K : \mathbb{Q}] = 2$.

Let's do something concrete. We know that $\mathbb{C} \geq \mathbb{Q}$, so if we compute the roots of $p(x) = x^2 + x + 1$, we have the roots of unity of degree 3;

$$\theta = \frac{-1 \pm \sqrt{-3}}{2} = e^{\pm 2\pi i/3}.$$

It turns out that extending by either root induces isomorphic fields.

Definition 13.1.13 (Subfield Generation). *Let K/F be an extension and let $\alpha_1, \alpha_2, \dots \in K$. $F(\alpha_1, \dots)$ denotes the smallest subfield of K containing F and each α_i . We call this construction the subfield of K generated by F and α_1, \dots*


We call $F(\alpha)$ a simple extension when we only extend via one element.

Moreover, note that $F(\alpha_1, \alpha_2) = F(\alpha_1)(\alpha_2)$.

However, we can very bad simple extensions. Take $\mathbb{Q} \leq \mathbb{Q}(x)$, rational functions over \mathbb{Q} .

Let's say that α is a root of $x^2 + x + 1$ in \mathbb{C} . Then $\mathbb{Q}(\alpha) = \{a + b\alpha \mid a, b \in \mathbb{Q}\}$.

Proposition 13.1.14 (Polynomial Extension Isomorphism). *Let K be an extension of F and $\alpha \in K$ is a root of an irreducible polynomial $p(x) \in F[x]$. Then, as a field, $F[\alpha] \cong F[x]/(p(x))$.*

Proof. Define $\phi : F[x] \rightarrow K$, where $f(x) \mapsto f(\alpha)$. This is naturally a ring homomorphism; we omit the verification. Then, $\ker \phi = (g(x))$ for some $g(x) \in F[x]$. Because $p(\alpha) = 0$, $p(x) \in g(x)$. Hence, $g(x) \mid p(x)$. Because p is irreducible, $g(x), p(x)$ are associates and $(g(x)) = (p(x))$. Therefore, by the first isomorphism theorem, $F[x]/(p(x)) \cong \text{im } \phi = F(\alpha)$ (If we were to do every single detail, we'd have to show two-way containment). Moreover, $x + (p(x)) \mapsto \alpha$. 

This demonstrates that $F(\alpha)/F$ is a finite extension.

Theorem 13.1.15 (Polynomial Root Extension Isomorphism). *Let K/F be an extension and $p(x)$ be irreducible in $F[x]$. If α_1, α_2 are two roots of $p(x)$ in K , $F(\alpha_1) \cong F(\alpha_2)$ via $\alpha_1 \mapsto \alpha_2$.*

13.2 Algebraic Extensions

Definition 13.2.1 (Algebraic Extension). *K/F is called an algebraic extension if α is a root of a polynomial $f(x)$ for every $\alpha \in K$.*

If α is not algebraic, we say that α is transcendental.

For example, $\pi \in \mathbb{C}$ is transcendental over \mathbb{Q} , but $\sqrt{2}$ is algebraic.

A few criteria for extensions to be algebraic:

Lemma 13.2.2 (Finite Extension is Algebraic). *If K/F is finite, then K/F is algebraic.*

Proof. Interpret K as a finite dimensional vector space over F . Let $n = \deg_F K$. Let $\alpha \in K$. The set $1, \alpha, \dots, \alpha^n$ is dependent, so there are coefficients a_k , not all 0, so that

$$a_0 + a_1\alpha + \dots + a_k\alpha^n = 0.$$

Then α is a root of $a_0 + a_1x + \dots + a_kx^n$.



Proposition 13.2.3 (Minimal Polynomial). *Let $\alpha \in K$ be algebraic over F . Then there exists a unique monic irreducible polynomial $m_{\alpha,F}(x)$ such that α is a root.*

Moreover, for any polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$, $m_{\alpha,F} | f(x)$ in $F[x]$.

Proof. Consider the ring homomorphism $\varphi : F[x] \rightarrow K$ defined by $\varphi(f(x)) = f(\alpha)$. The kernel is a principal ideal and hence uniquely generated by a monic polynomial $m_{\alpha,F}$. Clearly, the second statement follows as $f \in (m_{\alpha,F})$. By the first isomorphism theorem, $F[x]/(m_{\alpha,F}) \cong \varphi(F[x] \subset K)$ so it is an integral domain and hence $(m_{\alpha,F})$ is prime. Hence, $m_{\alpha,F}$ is irreducible and we are done.



Theorem 13.2.4 (Intermediate Extension Polynomial Divisibility). *Let L/F be an extension and $\alpha \in K$ be algebraic over F . Then $m_{\alpha,L}(x) | m_{\alpha,F}(x)$ in $L[x]$.*

Proof. Whence $m_{\alpha,F}(\alpha) \in L[x]$, so $m_{\alpha,L} | m_{\alpha,F}$.



Theorem 13.2.5 (Isomorphism of Extensions). *Let $\alpha \in K$ be algebraic over F . Then $F(\alpha) \cong F[x]/(m_{\alpha,F})$ and $\{1, \alpha, \dots, \alpha^{n+1}\}$ is a basis of $F(\alpha)$ over F , where $n = \deg(m_{\alpha,F})$.*

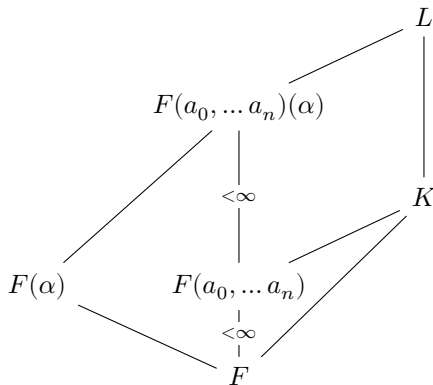
Theorem 13.2.6 (Algebraic Subfield). *An element $\alpha \in K/F$ is algebraic over F if and only if $F(\alpha)/F$ is finite. The set \overline{F} of all algebraic elements in K/F is a subfield of K .*

Proof. The first statement follows immediately. Let $\alpha, \beta \in K$ be algebraic over F . Then $F(\alpha, \beta)$ is finite. Thus, the sum and product of α, β are in $F(\alpha, \beta)$ and $\alpha^{-1} \in F(\alpha, \beta)$. Thus, \overline{F} is a subfield.

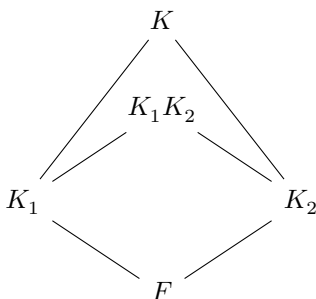


Proposition 13.2.7 (Transitivity of Extensions). *If $L/K, K/F$ are algebraic, then L/F is algebraic.*

Proof. Let $\alpha \in L$. Then α is algebraic over K , so α is a root of $a_0 + a_1x + \dots + a_nx^n \in K[x]$. Since $[F(a_0, \dots, a_n)(\alpha) : F(a_0, \dots, a_n)] < \infty$, and each a_k is algebraic over F , $[F(a_0, \dots, a_n, \alpha) : F] < \infty$ implying that α is algebraic over F .



Definition 13.2.8 (Product Field). Let K_1, K_2 be subfields of K . We denote by K_1K_2 the subfield of K generated by K_1 and K_2 .



Proposition 13.2.9 (Product Field Properties). Let K_1, K_2 be subfields of K . Suppose K_1, K_2 are finite extensions over F .

1. K_1K_2/F is finite.
2. $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$.
3. $\text{lcm}([K_1, F], [K_2 : F]) | [K_1K_2 : F]$.

Proof. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of K_1 over F and $\{\beta_1, \dots, \beta_m\}$ be a basis of K_2 over F . Then $K_1 K_2 = F(\alpha_1, \dots, \alpha_n, \beta_1, \beta_m)$ and $K_1 K_2 = K_1(\beta_1, \dots, \beta_m)$ implies $[K_1 K_2 : K_1] \leq m$ so $[K_1 K_2 : F] = [K_1 K_2 : K_1][K_1 : F] \leq mn$. Moreover, this final relationship implies that $[K_i : F] | [K_1 K_2 : F]$ for each i , proving the final statement. \square

13.3 Constructibility

Definition 13.3.1 (Constructibility). An $\alpha \in \mathbb{R}$ is called *constructible* if $|\alpha|$ can be constructed by straightedge and compass. We call the set of all constructible numbers \mathbb{F} and assume that $\mathbb{Z} \subset \mathbb{F}$.

Proposition 13.3.2 (Subfield of Constructible Numbers). Let $\alpha, \beta \in \mathbb{F}$. Sums and differences are easy to construct by using a straight line through the origin, a segment of length β , and a compass of length α .

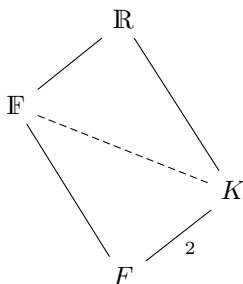
To construct products, take a triangle with one side β and foot 1, and construct a similar triangle with foot α . The corresponding side of β will have length $\alpha\beta$.


To show inverses, take a triangle of one side α and foot 1. Construct a similar triangle where the one side has length 1. The foot will have length α^{-1} .

Because $\text{ch } \mathbb{F} = 0$, the prime subfield of \mathbb{F} is \mathbb{Q} . However, as $\sqrt{2} \in \mathbb{F}$, $\mathbb{F} \supset \mathbb{Q}$.

Take $\alpha \in \mathbb{F}$. Draw a circle from $-\alpha, 1$, with centre on the x -axis. Construct a triangle to i by $-\alpha i 1$. Then, $x/1 = \alpha/x$ so $x^2 = \alpha$ and $\sqrt{\alpha} \in \mathbb{F}$.

Theorem 13.3.3 (Degree 2 Extension Closure). Let $F \subset \mathbb{F}$ and $K \subset \mathbb{R}$ with $[K : F] = 2$. Then, $K \subset \mathbb{F}$.



Proof. Let $\alpha \in K \setminus F$. Then α is a root of a degree 2 polynomial $p(x) = x^2 + bx + c$. Therefore, $p(x) = m_{\alpha, F}(x)$. Whence $\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ we have $F(\alpha) = F(\sqrt{b^2 - 4c})$. Because $\alpha \in \mathbb{R}$, $b^2 - 4c > 0$. However, we know already that $\sqrt{b^2 - 4c} \in F$! Thus $K \subset F$. 

Definition 13.3.4 (Constructible Points (\mathbb{F}^2) in \mathbb{R}^2). We call $(x, y) \in \mathbb{R}^2$ *constructible* if $x, y \in \mathbb{F}$. Moreover, we could switch out for $x + iy \in \mathbb{C}$ if $x, y \in \mathbb{F}$. The constructible points of \mathbb{C} make a field.

We construct these points via intersections of the following objects.

1. Straight lines through two points in \mathbb{F}^2 ;
2. If $(h, k) \in \mathbb{F}^2$ and $r \in \mathbb{F}$, $(x - h)^2 + (y - k)^2 = r^2$.

We call $ax + by + c = 0$ an F -line if $a, b, c \in F \subset \mathbb{F}$; similarly, we infer the notion of an \mathbb{F} circle.

Intersecting two F lines cannot “escape” F . A circle and line give quadratic extensions, and two circles, surprisingly, intersect on an \mathbb{F} -line and give also a quadratic extension. The proof requires solving the system and seeing that all of the quadratic terms cancel.

Theorem 13.3.5 (Power Two Necessity). If $\alpha \in \mathbb{R}$ is constructible, there exists an extension $\mathbb{Q} \subset K \subset \mathbb{R}$ such that $K = \mathbb{Q}(\alpha)$ $[K : \mathbb{Q}] = 2^k$ for some k .

Proof. If α is constructible, there exists a finite sequence of constructible points (x_n, y_n) so that $\alpha \in \mathbb{Q}(x_1, y_1, \dots, x_n, y_n)$ and $[\mathbb{Q}(x_1, \dots, x_\ell) : \mathbb{Q}(\dots, y_{\ell-1})] \leq 2$. Therefore $[K : \mathbb{Q}]$ is a two-power.




Theorem 13.3.6 (Trisecting the Angle). Angles, in general, cannot be trisected by compass and straightedge.

Proof. We consider the angle $60 \deg = 3\theta$. Then $(x, y) = e^{3\theta i} = \sqrt[3]{\cos \theta + i \sin \theta}^3$ is constructible. $\cos 3\theta$ is constructible, so we have that $(4 \cos \theta)^3 - 3 \cos \theta = 1$. Let $\alpha = \cos 3\theta$, yielding the equation $1 = 4\alpha^3 - 3\alpha$. Let $\beta = 2\alpha$. Then $0 = \beta^3 - 3\beta - 1$, an irreducible


monic polynomial. Then $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$ would be a cubic extension, a contradiction by the previous, so the angle cannot be trisected.



Theorem 13.3.7 (Doubling the Cube). *A cube may not be doubled.*

Proof. If it were so, doubling the unit cube would yield sides of length $\sqrt[3]{2}$, a root of a degree three irreducible polynomial. As three divides no power of two, such a cube is not constructible. 

Theorem 13.3.8 (Square with Area π). *A square with area π cannot be constructed.*

Proof. This would mean that $\sqrt{\pi}$ would be constructible. However, $[\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}(\pi)] = [\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}(\pi)][\mathbb{Q}(\pi) : \mathbb{Q}] = 2\infty = \infty$ divides no power of two. 

13.4 Splitting Fields


Definition 13.4.1 (Splitting Field). *Let $f(x) \in F[x]$ be a non-constant polynomial. We call E/F a splitting field for f if f factors over E and f splits in no subfield of E .*

Theorem 13.4.2 (Roots to Splitting Field). *Let $\alpha_1, \dots, \alpha_n$ be the roots of $f(x) \in E$. Then, $F(\alpha_1, \dots, \alpha_n) \subset E$ and so we have equality.*

Example. Take $f(x) = x^3 + x + 1$ in $\mathbb{F}_2[x]$. Let α be a root of $f(x)$ in some extension K/\mathbb{Z}_2 . Then, $f(\alpha) = 0$ and $f(\alpha^2) = (\alpha^2)^3 + \alpha^2 + 1 = (\alpha^3)^2 + \alpha^2 + 1 = (\alpha + 1)^2 + \alpha^2 + 1 = 0$, so α^2 is also a root. The other root is $\alpha^2 + \alpha$.

Splitting fields for degree n polynomials have, in general, degree $n!$.

Theorem 13.4.3 (Splitting Degree). *Every polynomial $f(x) \in F[x]$ of positive degree n splits in a field of degree at most $n!$.*

Proof. It suffices to show that there is an extension K/F such that $[K/F] \leq n!$ where $f(x)$ splits. Induction. Let $\deg f(x) = 1$. Then $f(x)$ splits in F . Assume that the statement holds up to some $n - 1$. Let α be a root of f in some field E . We know α has degree at most n , so $f(x)/(x - \alpha)$ has degree $n - 1$ in $F(\alpha)$ where $[F(\alpha) : F] \leq n$. Then, f splits in $K/F(\alpha)$, so $[K : F] = [K : F(\alpha)][F(\alpha) : F] \leq (n - 1)!n = n!$. 


13.4.1 Friday

Reminder: If we have a positive degree $f(x) \in F[x]$, we can always find a splitting field E/F of $f(x)$ such that $[E : F] \leq (\deg f)!$.

Lemma 13.4.4 (Isomorphism Extension). *If $\alpha \in E/F$ a root of an irreducible $f(x) \in F[x]$, then there is an isomorphism ϕ from $F \rightarrow \bar{F}$. We define $\tilde{f} = \phi(f)$. Then, we may extend to an isomorphism $\tilde{\phi} : F(\alpha) \rightarrow K$ such that $\tilde{\phi}|_F = \phi$; there are exactly k such extensions, where k is the number of distinct roots of \tilde{f} in \bar{E} .*

Proof. Let $\bar{\alpha} \in \bar{E}$ be a root of $\tilde{f}(x)$. Then, because $\bar{F}(\bar{\alpha}) \cong \bar{F}[x]/(\tilde{f}) \cong F[x]/(f(x)) \cong F(\alpha)$, we formally have the map

$$\tilde{\phi} : F(\alpha)\overline{\eta}^{-1} \rightarrow F[x]/(f(x))\overline{\phi} \rightarrow \bar{F}[x]/(\tilde{f}(x))\overline{\eta} \rightarrow \bar{F}(\bar{\alpha}).$$


Hence, $\phi : \alpha \mapsto \bar{\alpha}$, so the number of such extensions is the number of distinct roots $\bar{\alpha}$ of $\tilde{f}(x)$ 

Theorem 13.4.5. *Let $\phi : a \rightarrow \bar{a}$ be an isomorphism of a field F onto \bar{F} . Say that $f(x) \in F[x]$ has image \tilde{f} , and let E and \bar{E} be splitting fields for the two polynomials over their respective fields. Then, ϕ can be extended to an isomorphism $\tilde{E} \rightarrow \bar{E}$. The number of such extensions is less than or equal to $[E : F]$.*


The equality holds if \tilde{f} has no multiple roots in \bar{E} .

Theorem 13.4.6 (Uniqueness of Splitting Field). *The splitting field of $f(x)$ over F is unique up to isomorphism.*

Proof. Induction on $[E : F]$. If $[E : F] = 1$, then $E = F$ and f splits completely. Thus, \tilde{f} also splits completely so $\bar{E} = \bar{F}$. Assume $[E : F] > 1$. Then f must have a monic irreducible factor of degree

greater than 1. Let $\alpha \in E$ such that $g(\alpha) = 0$. Then, \bar{g} is a monic irreducible factor of \bar{f} , and there exists $\bar{\alpha} \in \bar{E}$ such that $\bar{g}(\bar{\alpha}) = 0$. By the preceding lemma, ϕ can be extended to $\bar{\phi} : F(\alpha) \rightarrow \bar{E}$. Let $\bar{\alpha}_1, \dots, \bar{\alpha}_\ell$ be the distinct roots of \bar{g} in \bar{E} . Hence, there are ℓ such extensions. Then, $[E : F(\alpha)] = [E : F]/[F(\alpha) : F]$ but $[F(\alpha) : F] = \deg g(x)$. By induction, there exist at most $[E : F(\alpha)]$ possible extensions of $\bar{\phi}$ to $\bar{\phi}$. There are at most $\ell[E : F(\alpha)] \leq \deg g(x)[E : F(\alpha)] = [E : F]$ such extensions over ϕ . 

If $\bar{f}(x)$ has no multiple roots in \bar{E} , then $\ell = \deg \bar{g}(x)$. Then, by induction, the number of extensions is equal to $[E : F(\alpha)] \deg g(x) = [E : F]$.

Proof. [Corollary] If $\text{id} : F \rightarrow F$ and $f(x) \in F$ with E, \bar{E} are splitting fields of $f(x)$ over F , then the number of such extensions ϕ of id is less than or equal to $[E : F]$.¹ 

Example. For $f(x) = x^3 - 2 \in \mathbb{Q}[x]$, $E = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is the splitting field. Now, how many automorphisms do we have and what are they? Well, we have at most 6 automorphisms. We could take $\sqrt[3]{2} \mapsto \zeta_3^k \sqrt[3]{2}$, and $\zeta_3 \mapsto \bar{\zeta}_3, \zeta_3$.

13.4.2 Week 4

Last time, we learned that, if $f(x) \in F[x]$ has splitting field E and $\bar{f}(x) \in \bar{F}[x]$ has splitting field \bar{E} , then an isomorphism $\phi : F \rightarrow \bar{F}$ lifts to an isomorphism $\bar{\phi}$. There are at most $[E : F]$ such extensions.

$$\begin{array}{ccc} E & \xrightarrow{\bar{\phi}} & \bar{E} \\ | & & | \\ F & \xrightarrow{\phi} & \bar{F} \end{array}$$

Definition 13.4.7 (Algebraic Closure). A field \bar{F} is called an algebraic closure of F if \bar{F}/F is algebraic and every $f(x) \in F[x]$ splits completely.

For an example, consider

¹Is this really sufficient?

$$\begin{array}{c} \mathbb{C} \\ | \\ \bar{\mathbb{Q}} \\ | \\ \mathbb{Q} \end{array}$$

However, even though $\bar{\mathbb{C}} = \mathbb{C}$, \mathbb{C}/\mathbb{Q} is not algebraic.

Proposition 13.4.8 (Algebraic Closure is Closed). *If \bar{F} is an algebraic closure of F , then $\bar{\bar{F}} = \bar{F}$.*

Proof. Let $f(x) \in \bar{F}[x]$ be irreducible. It suffices to show that $\deg f(x) = 1$. Let E/\bar{F} be the splitting field of f .

We know that E/\bar{F} is finite and hence algebraic. Since \bar{F}/F is algebraic, E/F° is algebraic. For any $\alpha \in E$, α is not a root of some irreducible polynomial over F . Since \bar{F} is the algebraic closure of F ,

$$E \subset \bar{F} \implies E = \bar{F} \implies \deg f(x) = 1.$$



Definition 13.4.9 (Separable). *We call $f(x) \in F[x]$ separable if $f(x)$ has no multiple root in a splitting field E of $f(x)$ over F .*

Moreover, we say E/F is called separable if every element of E is a root of a separable polynomial over F .


For example, $\mathbb{F}_2(t)(\sqrt{t})/\mathbb{F}_2(t)$ is inseparable.

Theorem 13.4.10. *A polynomial $f(x) \in F[x]$ has a multiple root in its splitting field over F if and only if $f(x)$ and $f'(x)$ are not relatively prime.*

Proof. Let E be a splitting field of $f(x)$ over F and $\alpha \in E$ a multiple root of $f(x)$. Then $f(x) = (x - \alpha)^m h(x)$ for some $h(x) \in F[x]$ and $m > 1$. Then $f'(x) = m(x - \alpha)^{m-1} h(x) + (x - \alpha)^m h'(x) = (x - \alpha)^{m-1}$ in $E[x]$. Let $p(x)$ be the minimal polynomial of α over F . Then $p|f, f'$.

Assume $\gcd(f(x), f'(x)) \neq 1$. Then there exists an irreducible polynomial $p(x) \mid f(x), f'(x)$. Let α be a root of $p(x)$ in a splitting field E of $f(x)$ over F . Then, we have that $f(\alpha) = 0 = f'(\alpha)$, so $f(x) = (x - \alpha)g(x)$ for some $g(x) \in E[x]$. Hence,

$$f'(x) = g(x) + (x - \alpha)g'(x) \implies (x - \alpha) \mid g(x)$$

so $(x - \alpha)^2 \mid f(x)$ in $E[x]$, so α is a multiple root of $f(x)$. In particular, $f(x)$ is nonseparable. 

Theorem 13.4.11 (Algebraic Closure). *Every field F is contained in an algebraically closed field K and the set of all elements of K algebraic over F is an algebraic closure of F . Moreover, the algebraic closure of F is unique.*

Hence, if \mathbb{F} is another algebraically closed field and K is all of the elements of \mathbb{F} algebraic over \mathbb{Q} ,

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{F} \\ \downarrow & & \downarrow \\ \bar{\mathbb{Q}} & \xrightarrow{\sim} & K \\ \downarrow & \nearrow & \\ \mathbb{Q} & & \end{array}$$

13.4.3 Wednesday

Let \mathbb{F} be a finite field. Let $f(x) = x^{|\mathbb{F}|} - x$ in $\mathbb{Z}_p[x]$. So, $f'(x) = -1$. Thus, the greatest common divisor of the two is 1, and f is always separable in \mathbb{F} .

Additionally, for $\alpha \in \mathbb{F}^\times$, $\alpha^{|\mathbb{F}^\times|} = 1$ so α is a root of $x^{|\mathbb{F}|-1} - 1$ and $x^{|\mathbb{F}|-1} - 1 = \prod_{\alpha \in \mathbb{F}^\times} (x - \alpha)$. However, $x^{|\mathbb{F}|} - x = \prod_{\alpha \in \mathbb{F}} (x - \alpha)$ so. Hm!

Definition 13.4.12 (Perfect Field). *A field \mathbb{F} is called perfect if every irreducible polynomial $f(x) \in F[x]$ is separable.*


Theorem 13.4.13 (Characteristic 0 Perfection). *Every field \mathbb{F} of characteristic 0 is perfect.*

Proof. Let $f(x) \in \mathbb{F}[x]$ be irreducible. Then $f'(x) \neq 0$ and $\deg f'(x) < \deg f(x)$. In particular, $f(x) \nmid f'(x)$. Since $\gcd(f(x), f'(x)) = 1$, $f(x)$ by irreducibility, f and f' are coprime and f is separable.



Theorem 13.4.14 (Number Fields). *Every algebraic extension over a field of characteristic 0 is separable.*

Theorem 13.4.15 (Finite Field Perfection). *Every finite field is perfect.*

Proof. Let $f(x) \in F[x]$ be irreducible and E a splitting field of f over F . In particular, E/F is finite and E is finite itself. Therefore $x^{|E|} - x = \prod_{\alpha \in E} (x - \alpha)$ and $f(x) \mid x^{|E|} - x$ in $F[x]$. Therefore, $f(x)$ is a factor of a separable polynomial and is hence separable. 

Definition 13.4.16 (Frobenius Automorphism). *The function x^p is called the Frobenius automorphism for fields of characteristic p .*

13.5 Cyclotomic Extensions

Let's start with $\mu_n = \{e^{2\pi i k/n} = \zeta_n^k \mid k \in [n-1]\}$. This is a finite group under complex multiplication. Hence, $(\mu_n, \cdot) \cong (\mathbb{Z}_n, +)$.

Definition 13.5.1 (Cyclotomic Polynomial).

$$\Phi_n(x) = \prod_{\alpha \in \mu_n \text{ that are generators}} (x - \alpha)$$

is the n th cyclotomic polynomial.

We often call the generators of μ_n the primitive n th roots of unity.

13.5.1 Friday


Proposition 13.5.2 (Cyclotomic Polynomial is Monic). $\Phi_n(x)$ is a monic polynomial in $\mathbb{Z}[x]$.

Proof. We have initially that $\Phi_n(x) \in \mathbb{C}[x]$ and $x^n - 1 \in \mathbb{Z}[x]$. We proceed by induction on n .

For our base step, $\Phi_1(x) = x - 1$. Assume now that the statement holds for any n .

Consider

$$f(x) = \prod_{d|n, n \neq d} \Phi_d(x).$$


Hence, $x^n - 1 = f(x)\Phi_n(x)$. By the division algorithm, $x^n - 1 = f(x)q(x) + r(x)$ in $\mathbb{Q}[x]$. We at least have that $f(x)|r(x) \in \mathbb{C}[x]$, meaning $r(x) = 0$. Hence, $q(x) = \Phi_n(x) \in \mathbb{Q}[x]$ so we are done by Gauss's lemma. 

Theorem 13.5.3 (Irreducibility of Cyclotomic Polynomial). $\Phi_n(x)$ is irreducible in $\mathbb{Q}[x]$.

Proof. Let α be a primitive n th root of unity. Let $g(x)$ be the minimal polynomial of α over \mathbb{Q} . Then $g(x)h(x) = \Phi_n(x)$ for some $h(x) \in \mathbb{Q}[x]$. By Gauss's lemma, $g(x), h(x)$ are monic in $\mathbb{Z}[x]$.

We claim that α^p is a root of $g(x)$ for any prime $p \nmid n$. Contradiction; there exists p so that $g(\alpha^p) \neq 0$. However, $(\alpha^p)^n - 1 = 0$. By Gauss's lemma, $x^n - 1 = g(x)f(x)$ for some $f(x) \in \mathbb{Z}[x]$. Therefore $f(\alpha^p) = 0$, $f(x^p) = g(x)k(x)$, and α is a root of $f(x^p) \in \mathbb{Z}[x]$. Consider $f(x^p) \in \mathbb{Z}_p[x]$. Then $f(x^p) = f(x)^p$ and $g(x)$ has an irreducible factor dividing $f(x)$. In mod p , $x^n - 1 = g(x)f(x)$, implying that $x^n - 1$ is not separable. But if we take the derivative nx^{n-1} , it shares no factor with $x^n - 1$, implying separability, and so we have a contradiction.

By induction, $\alpha^{p_1, \dots, p_\ell}$ is a root of $g(x)$ for any primes $p_1, p_2, \dots, p_\ell \nmid n$.

Now, let θ be a primitive n th root of 1. Then $\theta = \alpha^k$ for some $k > 1, (k, n) = 1$. Hence $g(x) = \prod_{\alpha \in \mu_n \text{ primitive}} = \Theta_n(x)$ 

Theorem 13.5.4.

$$[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$$

and $\mathbb{Q}(\zeta_n)$ is the splitting field of $x^n - 1$ over \mathbb{Q} .

We know $\Phi_p(x)$, but that is $\Phi_{p^n}(x)$? Well, it's quite simply $\Phi_p(x^{p^{n-1}})$!

What about $\Phi_{2n}(x)$ for n odd? We know that $\phi(2n) = \phi(n)$, and $-\zeta_n = \zeta_{2n}$. With a little bit of work we can show that $\Phi_{2n}(x) = \Phi_n(-x)$!

Chapter 14

Galois Theory

14.1 Automorphisms

Definition 14.1.1 ($\text{Aut}(K)$). *Let K be a field, and denote by $\text{Aut}(K)$ the set of all ring automorphisms of K .*

It is endowed with a natural group structure under composition. If K/F is a field extension, define $\text{Aut}(K/F) = \{\sigma \in \text{Aut}(K) \mid \sigma(\alpha) = \alpha, \alpha \in F\}$

Because we insist $\sigma(1) = 1$, $\sigma(\alpha) = \alpha$ for $\alpha \in F$, the prime subfield of K and $\text{Aut}(K) = \text{Aut}(K/F)$.

These automorphism groups play nicely with composite extensions.

$$\begin{array}{c} K \\ | \\ E \\ | \\ F \end{array}$$

implies $\text{Aut}(K/E) \leq \text{Aut}(K/F)$.

Definition 14.1.2 (Invariant Subfield). *Let $G \leq \text{Aut}(K/F)$. Define $\text{Inv}(G) = \{\alpha \in K \mid \sigma(\alpha) = \alpha \forall \sigma \in G\}$. This is a subfield of K .*

$$\begin{array}{c}
K \\
| \\
\text{Inv}(G) \\
| \\
F
\end{array}$$

The point of Galois theory is this: If K/F is good, there is a one to one correspondence between subextensions E/F of K/F and the subgroups of $\text{Aut}(K/F)$.

Proposition 14.1.3.

1. If $G_1 \leq G_2 \leq \text{Aut}(K/F)$, then $K \supset \text{Inv}(G_1) \supset \text{Inv}(G_2) \supset F$.
2. If $F \subset E_1 \subset E_2 \subset K$, then $\text{Aut}(K/E_1) \geq \text{Aut}(K/E_2)$.
3. Let $F \subset E \subset K$. Then $E \subset \text{Inv}(\text{Aut}(K/E))$.
4. Let $G \leq \text{Aut}(K/F)$. Then $G \leq \text{Aut}(K/\text{Inv}(G))$.

Proposition 14.1.4. If $\alpha \in K$ is a root of an irreducible polynomial $p(x) \in F[x]$, then $\sigma\alpha$ is also a root of $p(x)$ for any $\sigma \in \text{Aut}(K/F)$.


Proof. Let $p(x) = a_0 + a_1x + \dots + a_nx^n$ for some $a_0, \dots, a_n \in F$. Then,

$$\begin{aligned}
0 &= p(\alpha) = a_0 + \dots + a_n\alpha^n \\
\implies 0 &= \sigma p(\alpha) = \sigma a_0 + \dots + \sigma a_n \sigma \alpha^n \\
\implies 0 &= p(\sigma\alpha) = a_0 + \dots + a_n(\sigma\alpha)^n.
\end{aligned}$$



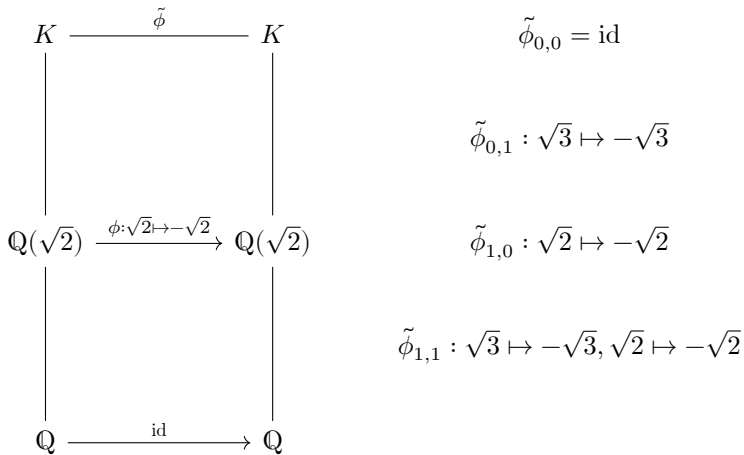
Theorem 14.1.5. $\text{Aut}(K/F)$ acts on the roots of any irreducible polynomial $p(x) \in F[x]$.

Proposition 14.1.6. If K is a splitting field for $f(x)$ over F , then $|\text{Aut}(K/F)| \leq [K : F]$. If $f(x)$ has no multiple root, then we have equality.

Proof. We have that $\sigma \in \text{Aut}(K/F)$ is an extension of the identity map on F by definition. However, we know already that the number of such extensions is less than or equal to $[K : F]$ with equality when $f(x)$ has no multiple roots. 

For example, let $K = \mathbb{Q}(\sqrt[3]{2})$. What is $\text{Aut}(K/\mathbb{Q})$? It consists of one element; Because the three roots of $x^3 - 2 \in \mathbb{Q}[x]$ are $\sqrt[3]{2}$, $\sqrt[3]{2}\zeta_3$, $\sqrt[3]{2}\zeta_3^2$, $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}$ must be the identity.

Another example. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. We already know $[K : \mathbb{Q}] = 4$. We show the data in the following diagram



to show that $\text{Aut}(K/\mathbb{Q}) = \{\phi_{i,j} | i, j \in \{0, 1\}\} \cong K_4$. The subfields are

$$\begin{array}{ccc}
 K & \xrightarrow{\tilde{\phi}} & K \\
 \downarrow & & \downarrow \\
 \mathbb{Q}(\sqrt{2}) & \xrightarrow{\phi: \sqrt{2} \mapsto -\sqrt{2}} & \mathbb{Q}(\sqrt{2}) \\
 \downarrow & & \downarrow \\
 \mathbb{Q} & \xrightarrow{\text{id}} & \mathbb{Q}
 \end{array}$$

$$\begin{aligned}
 \tilde{\phi}_{0,0} &= \text{id} \\
 \tilde{\phi}_{0,1} &: \sqrt{3} \mapsto -\sqrt{3} \\
 \tilde{\phi}_{1,0} &: \sqrt{2} \mapsto -\sqrt{2} \\
 \tilde{\phi}_{1,1} &: \sqrt{3} \mapsto -\sqrt{3}, \sqrt{2} \mapsto -\sqrt{2}
 \end{aligned}$$

Lemma 14.1.7. *Let K/F be a finite extension. Then $|\text{Aut}(K/F)| < \infty$.*

Proof. Recall that this is true if K is a splitting field over F .

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ form a basis for K over F and $p_i(x)$ a minimal polynomial for each. Let E be the splitting field of $f(x) = p_1(x)p_2(x) \dots$. Then E is the splitting field of $f(x)$ over F . Hence,

$$|\text{Aut}(K/F)| \leq |\text{Aut}(E/F)| \leq [E : F].$$



Definition 14.1.8 (Galois). *K/F is called Galois if $|\text{Aut}(K/F)| = [K : F]$. In this case we say $\text{Aut}(K/F) = \text{Gal}(K/F)$*

For example, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not Galois. The automorphism group is trivial but it's of degree 3. However, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois!

Lemma 14.1.9 (Artin). *Let $G \leq \text{Aut}(K)$ be finite. Then $[K : \text{Inv}(G)] \leq |G|$.*


Proof. Let $F = \text{Inv}(G)$. Let $u_1, \dots, u_n \in K$ where $n > |G|$. We want to show that this set is always dependent. Consider the system of linear equations $\sigma_i(u_1)x_1 + \dots + \sigma_i(u_n)x_n = 0$ for $i \in [n]$ and $G = \{\sigma_i\}$. More unknowns than the number of equations, we have nontrivial solutions in K ! That is, there exist $b_1, \dots, b_n \in K \setminus \{0\}$ which is a solution to the system.

One can assume such a solution with the least number of zero entries. Further assume that $b_1 = 1$. In particular, we have $\sigma_i(u_i) + \sigma_i(u_2)b_2 + \dots \sigma_i(u_m)b_m = 0$. Apply σ_j to get

$$\sigma_j\sigma_i(u_i) + \sigma_j(\sigma_i(u_2)b_2) + \dots \sigma_j(\sigma_i(u_m)b_m) = 0.$$

As i runs over all indices, we can rewrite

$$\sigma_i(u_i) + \sigma_i(u_2)\sigma_j(b_2) + \dots \sigma_i(u_m)\sigma_j(b_m) = 0.$$

Hence, we have another nontrivial minimal solution. Take the difference of the two solutions, and now the first coefficient is 0. This produces another solution with at least one more 0, implying that every coefficient is zero and that $b_k \in \text{Inv}(G)$. This all implies that $\{u_1, \dots, u_n\}$ is F -linearly dependent. 

Definition 14.1.10 (Character). *Let G be a group and L be a field. A group homomorphism $\chi : G \rightarrow L^\times$ is a character. Well, $\text{Hom}_{\text{Set}}(G, L)$ is an L -linear space.*

Let's do some examples of characters. Take $C_n = \langle g \rangle$. Let $\chi : C_n \rightarrow \mathbb{C}^\times$ by $\chi(g) = e^{2\pi k/n}$. Now let $G = GL_n(K)$. Then $\det : GL_n(K) \rightarrow K$ is a character. Last example. $\sigma \in \text{Aut}(K/F)$ is a character of K^\times !

Theorem 14.1.11. *Let χ_1, \dots, χ_n be distinct characters of a group G valued in L . Then χ_1, \dots, χ_n are L -linearly independent.*

Proof. Assume to the contrary that they are linearly dependent by the coefficients b_i ,

$$b_1\chi_1 + \dots + b_n\chi_n = 0.$$

If $n = 1$, we're automatically fine. There exists $g_0 \in G$ such that $\chi_1(g_0) \neq g_n(g_0)$. Then, plug in gg_0 for some $g \in G$ to get


$$b_1\chi_1gg_0 + \dots + b_n\chi_ngg_0 = 0.$$

This implies

$$b_1\chi_1(g)\chi_1(g_0) + \dots + b_n\chi_n(g)\chi_n(g_0) = 0.$$

On the other hand, $b_1\chi_1(g)\chi_n(g_0) + \dots + b_n\chi_n(g)\chi_n(g_0) = 0$. Then we subtract.

$$b_1\chi_1(g_0) - b_1\chi_n(g_0) + \dots + b_n\chi_n(g_0) - b_n\chi_n(g_0) = 0.$$

We could pick a maximal amount of zeros and nonzero leading coefficient for our original solution, which here would derive a contradiction. 

Here is how we can apply it. If we let $\sigma_1, \dots, \sigma_n \in \text{Aut}(K)$ be distinct, then they are K -linearly independent.

Theorem 14.1.12. *Let $G \leq \text{Aut}(K)$ be finite.*

Then $|G| = [K : \text{Inv}(G)]$, $K/\text{Inv}(G)$ is Galois, and $G = \text{Aut}(K/\text{Inv}(G))$.

Proof. We have already proved that $|G| \geq [K : \text{Inv}(G)]$. Assume that $n = |G| > [K : \text{Inv}(G)] = m$. Let α, \dots, α_m be a basis of $K/\text{Inv}(G)$ and $\{\sigma_1, \dots, \sigma_n\} = G$. Consider the system of equations


$$\sigma_1(\alpha_i)x_i + \dots + \sigma_n(\alpha_i)x_n = 0, i \in [m] \setminus \{0\}.$$

Then, there exists a nontrivial solution as there are more unknowns than equations. However, we already know the σ_i s are linearly independent, contradiction.


We have $G \subset \text{Aut}(K/\text{Inv}(G))$. This yields $\text{Inv}(G) \supset \text{Inv}(\text{Aut}(K/\text{Inv}(G))) \supset \text{Inv}(G)$ implying that $\text{Inv}(G) = \text{Inv}(\text{Aut}(K/\text{Inv}(G)))$. Now, we know

$$|G| = [K : \text{Inv}(G)] = [K : \text{Inv}(\text{Aut}(K/\text{Inv}(G)))] = |\text{Aut}(K/\text{Inv}(G))|$$

so we have equality.

Repeating the statement, $[K : \text{Inv}(G)] = |\text{Aut}(K/\text{Inv}(G))|$ implying that $K/\text{Inv}(G)$ is Galois. 

Theorem 14.1.13. *Let K/F be any finite extension. Then $|\text{Aut}(K/F)| = [K : F]$.*

Proof. Let $F_1 = \text{Inv}(\text{Aut}(K/F)) \supset F$. Then $[K : F_1] = |\text{Aut}(K/F)| = [K : F_1][F_1 : F]$, so we have our result. 

Definition 14.1.14 (Normal). *K/F is said to be normal if any irreducible polynomial $p(x) \in F[x]$ which has a root in K splits completely in $K[x]$.*

Theorem 14.1.15. *Let K/F be an extension. Then the following are equivalent:*

1. K is the splitting field of a separable polynomial in $F[x]$.
2. K/F is Galois.
3. K/F is finite and $F = \text{Inv}(\text{Aut}(K/F))$.
4. $F = \text{Inv}(G)$ for some finite subgroup of $\text{Aut}(K/F)$.
5. K/F is finite, normal, and separable.


Moreover, $G = \text{Aut}(K/F)$.

Proof. (1) \implies (2) We know that the number of extensions of $\text{id} : F \rightarrow F$ are equal to $[K : F]$ but also $|\text{Aut}(K/F)|$. This implies K/F is Galois.


(2) \implies (3) K/F is finite and $[K : F_1] = |\text{Aut}(K/F)| = [K : F]$ where $F_1 = \text{Inv}(\text{Aut}(K/F)) \supset F$. As $[F_1 : F] = [K : F]/[K : F_1] = 1$, we have equality.

(3) \implies (4) Take $G = \text{Aut}(K/F)$.

(4) \implies (5) We have $[K : F] = |G|$ as $F = \text{Inv } G$ and $G = \text{Aut}(K/F)$. Let $\alpha \in K$ and $\{\alpha = r_1, \dots, r_m\} = G\alpha$. Then let $p(x) = (x - r_1) \dots (x - r_m)$. Now apply $\sigma \in G$. This fixes the polynomial, so $p(x) \in F[x]$. Therefore K/F is separable. Let $q(x)$ be the minimal polynomial of α . Then $q(x)|p(x)$ and $q(\sigma(\alpha)) = 0$ for all $\sigma \in G$. Moreover, $q(x) = p(x)$. Also, $q(x)$ splits completely in $K[x]$.


(5) \implies (1). Let $\alpha_1, \dots, \alpha_m$ be a basis of K/F , and $p_i(x)$ be the minimal polynomial of α_i over F . Let $f(x) = p_1(x) \dots p_m(x)$. By removing the repeated prime factors, $f(x)$ is a product of distinct monic irreducible factors. Therefore $f(x)$ is separable and K is the splitting field of $f(x)$ over F . 

Theorem 14.1.16. *Let K/F be Galois and $F \subset E \subset K$. Then K/E is Galois.*

Proof. K is the splitting field of some separable polynomial $f(x)$ over F . Then K is the splitting field of $f(x)$ over E . 

However, K/E Galois, E/F Galois does NOT IMPLY that K/F is Galois! For an example, take $F = \mathbb{Q}$, $E = \sqrt{2}$, and $K = \mathbb{Q}(\sqrt[4]{2})$.

Theorem 14.1.17 (Fundamental Theorem of Galois Theory). *Let K/F be Galois. Then let \mathcal{F} be the set of all subfields E of K containing F . Let J be the set of all subgroups of $\text{Aut}(K/F)$. The assignment $\phi : G \rightarrow F$ given by $\phi(H) = \text{Inv}(H)$ is bijective whose inverse is ϕ^{-1} given by $\phi(E) = \text{Aut}(K/E)$.*

Proof. Apply that $H = \text{Aut}(K/\text{Inv}(H))$ and that if K/E is Galois (guaranteed by the corollary) then $E = \text{Inv}(\text{Aut}(K/E))$. 

Lemma 14.1.18. *$H \leq \text{Aut}(K/F)$ is normal if and only if $\text{Inv}(H)/F$ is normal. In this case, $\text{Inv}(H)/F$ is Galois and $\text{Gal}(\text{Inv}(H)/F) \cong G/H$ as a group.*

Proof. Let H be a normal subgroup of $G = \text{Aut}(K/F) = \text{Gal}(K/F)$. Hence, for any $\sigma \in G$, $\sigma H \sigma^{-1} = H$. Then $E = \text{Inv}(H) = \text{Inv}(\sigma H \sigma^{-1}) = \sigma E$. For all $\alpha \in E$, $\sigma h \sigma^{-1} \sigma \alpha = \sigma \alpha$.


For any monic irreducible polynomial $p(x) \in F[x]$ which has a root $\alpha \in E$, we can write $p(x) = \prod_{r \in \alpha G} (x - r)$, so $p(x)$ splits completely in $F[x]$.

Conversely, assume $\text{Inv}(H) = E$ is normal. Let $\alpha \in E$ and $p(x)$ the minimal polynomial for α over F . For $\sigma \in G$, $\sigma \alpha$ is also a root of $p(x)$. Therefore $\sigma(E) \subset E$ implies that $\sigma(E) = E$. Hence $\text{Inv}(H) = \text{Inv}(\sigma H \sigma^{-1})$ so therefore $H = \sigma H \sigma^{-1}$ by Galois correspondence. Therefore H is normal in G .

Hence, $G = \text{Gal}(K/F) \rightarrow \text{Aut}(E/F)$ by $\sigma \mapsto \sigma|_E$ defines an epimorphism, and by the first isomorphism theorem, $\text{Aut}(E/F) \cong G/\text{Gal}(K/E)$.

Hence,

$$\begin{aligned} |\text{Aut}(E/F)| &= \frac{|\text{Aut}(K/F)|}{|\text{Aut}(K/E)|} \\ &= \frac{[K : F]}{[K : E]} \\ &= [E : F] \end{aligned}$$

So E/F is Galois. 

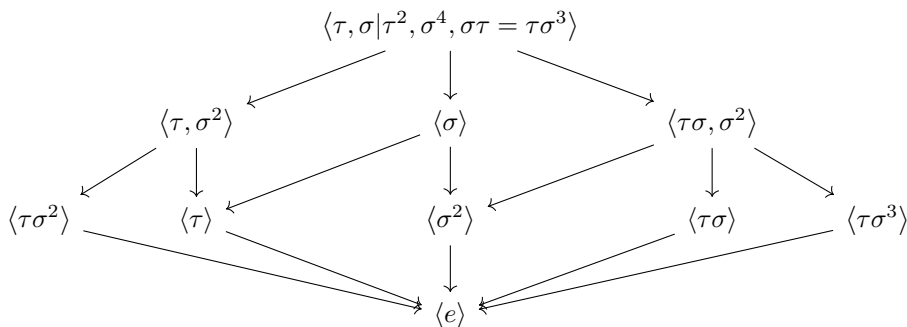
For posterity, we include that

$$1 \rightarrow \text{Gal}(K/E) \rightarrow \text{Gal}(K/F) \rightarrow \text{Aut}(E/F) \rightarrow 1$$

is exact.

Example. Consider the polynomial $x^4 + 2 \in \mathbb{Q}[x]$. The splitting field is $\mathbb{Q}(\sqrt[4]{2}, i)$. We can verify that the Galois group is D_8 with generators $\tau : i \mapsto -i$ and $\sigma : \sqrt[4]{2} \mapsto i\sqrt[4]{2}$.

The subgroup lattice is



so we have the table

H	$\text{Inv}(H)$	$[K : \text{Inv}(H)]$	$[\text{Inv}(H) : \mathbb{Q}]$
$\langle e \rangle$	K	1	8
$\langle \sigma \rangle$	$\mathbb{Q}(i)$	4	2
$\langle \tau, \sigma^2 \rangle$	$\mathbb{Q}(\sqrt{2})$	4	
$\langle \tau\sigma, \sigma^2 \rangle$	$\mathbb{Q}(i\sqrt{2})$	4	2
$\langle \sigma^2 \rangle$	$\mathbb{Q}(i, \sqrt{2})$	2	4
$\langle \tau \rangle$	$\mathbb{Q}(\sqrt[4]{2})$		
$\langle \tau\sigma^2 \rangle$	$\mathbb{Q}(i\sqrt[4]{2})$		
$\langle \tau\sigma \rangle$	$\mathbb{Q}(\zeta, \sqrt[4]{2})$		
$\langle \tau\sigma^3 \rangle$	$\mathbb{Q}(\zeta\sqrt[4]{2})$		

Next example, $\mathbb{Q}(\alpha) = K$ with $\alpha = \sqrt{2 + \sqrt{2}}$. The minimal polynomial of α is $p(x) = x^4 - 4x^2 + 2$. We can easily verify that it is irreducible over \mathbb{Q} using Eisenstein or the quadratic formula. The roots of this polynomial can be found in the homework associated with this class period.

14.2 Finite Fields

Let p be a prime and $n \in \mathbb{N}$. Then, \mathbb{F}_{p^n} exists and is unique up to isomorphism. Moreover, the extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is always simple with $\mathbb{F}_{p^n} \cong \mathbb{F}_p(\alpha)$, where the minimal polynomial of α over \mathbb{F}_p has

degree n . We define

$$P_n(x) = \prod \text{all monic irreducible polynomials of degree } n \text{ over } \mathbb{F}_p.$$


We have that $\deg P_n/n = \#$ of irreducible polynomials of degree n over \mathbb{F}_p . Moreover, we can figure out the degree of this polynomial. The degree of P_n is equal to the number of the roots of the polynomial is equal to the number of elements of \mathbb{F}_p^n which are not in any proper subfield.

For example, for $n = 6$, then $\mathbb{F}_p \subseteq \mathbb{F}_{p^2}, \mathbb{F}_{p^3}$ and

$$|\mathbb{F}_{p^6} \setminus \mathbb{F}_{p^2} \cup \mathbb{F}_{p^3}| = p^6 - p^2 - p^2 + p.$$

Therefore the number of monic irreducible polynomials of degree 6 over \mathbb{F} is $(p^6 - p^2 - p^2 + p)/6$.

As a brief remark, $x^4 + 1$ is reducible over \mathbb{F}_p for any prime p .

Proof. Quickly, $x^4 + 1 = \Phi_8(x)$ is irreducible over \mathbb{Q} . If $p = 2$, then $(x^4 + 1) = (x + 1)^4$ so we are done. Let p be an odd prime. Then, $8|p^2 - 1$. With an elementary calculation, $x^4 + 1 | x^8 - 1 | x^{p^2-1} - 1$. The polynomial on the right splits in \mathbb{F}_{p^2} , so $x^4 + 1$ splits completely in \mathbb{F}_{p^2} and $[\mathbb{F}_{p^2} : \mathbb{F}_p] = 2$. Hence, any root of $x^4 + 1$ has degree at most 2. There thus exists a polynomial $q(x)$ of degree 1 or 2 such that $q(x) | x^4 + 1$. Therefore, $x^4 + 1$ is reducible over \mathbb{F}_p . 


14.3 Cyclotomic Extensions

We study fields of the form $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ where ζ_p is a primitive n th root of unity in \mathbb{C} .

Firstly, $\mathbb{Q}(\zeta_n)$ is the splitting field of $\Phi_n(x)$. Therefore $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois. We also have that $|\text{Aut}(\mathbb{Q}(\zeta_n))/\mathbb{Q}| = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) = \deg \Phi_n(x)$.

Let's throw in another statement. For $\bar{\ell} \in \mathbb{Z}_n^\times$, $\sigma : \zeta_n \rightarrow \zeta_n^{\bar{\ell}}$ defines an automorphism of $\text{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. This induces a map $\bar{\psi} : \mathbb{Z}_n^\times \rightarrow \text{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ that we claim is a group isomorphism. *Proof.* Let j, k be integers coprime to n . Then,

$$\bar{\psi}(\bar{j}\bar{k}) = \bar{\psi}(\bar{j})\bar{\psi}(\bar{k}) = \sigma_{jk} = \sigma_j\sigma_k = \bar{\psi}(\bar{j})\bar{\psi}(\bar{k}).$$

Let $\bar{j} \in \mathbb{Z}_n^\times$ such that $\bar{\psi}(\bar{j}) = \text{id}$. Then that means that $\bar{j} = 1$ so $\bar{\psi}$ is injective. Hence, it is also bijective as it is between finite sets. 

Going back to highschool, let $n = p_1^{n_1} \dots p_k^{n_k}$ be a prime factorization. Then

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}}^\times \times \dots \times \mathbb{Z}_{p_k^{n_k}}^\times.$$

Additionally,

$$\mathbb{Z}_{2^n}^\times \cong \begin{cases} 1 & n = 1 \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}} & n \geq 2 \end{cases}$$

and

$$\mathbb{Z}_{p^n}^\times \cong \mathbb{Z}_{p^n - p^{n-1}}.$$

This leads to the theorem

Theorem 14.3.1.

$$\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}_n^\times.$$

And if p is prime

$$\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}.$$

For the prime case, every subgroup is cyclic and normal. Hence, if $H \leq \mathbb{Z}_{p-1}$, $\text{Inv}(H) = \mathbb{Q}(\alpha_H)$ where $\alpha_H = \sum_{\sigma \in H} \sigma \zeta_p$.

14.4 Constructibility, Revisited

Theorem 14.4.1. *A regular n -gon is constructible if and only if $n = 2^{n_0} p_1^{n_1} \dots p_\ell^{n_\ell}$ for p_i distinct fermat primes and $0 \leq n_1, \dots, n_\ell \leq 1$.*

14.5 Discriminant (Catch-up!)

Let $f(x)$ be a separable polynomial over F and K be a splitting field of $f(x)$ over F . Let $\alpha_1, \dots, \alpha_n \in K$ be the roots of $f(x)$. Then $\Delta = \prod_{i < j} (\alpha_i - \alpha_j)^2 \in F$ as $\sigma \in \text{Aut}(K/F)$ fixes Δ .

Now, consider $\sqrt{\Delta}$. We know that σ , interpreted as a member of a transitive subgroup of S_n , is a member of A_n if and only if $\sigma\sqrt{\Delta} = \sqrt{\Delta}$. That is,

$$\text{Aut}(K/F) \subseteq A_n$$

if and only if

$$\sqrt{\Delta} \in F.$$

Now, how do we construct discriminants?

- $f(x) = x^2 + bx + c$ has $\Delta = b^2 - 4c$.
- $f(x) = x^3 + px + q$ has $\Delta = -4p^3 - 27q^3$.

Example. Let $f(x) = x^3 - 3x + 1 \in \mathbb{Q}[x]$. In this case, $\Delta = -4(-3)^3 - 27 = 81$. In particular, this is a square, as $\sqrt{\Delta} = \pm 9 \in \mathbb{Q}$. Hence, $\text{Gal}(f) = A_3 \cong C_3$.

Now let $f(x) = x^3 - 2x + 2$. Then $\Delta = -76$ so $\text{Gal}(f) \cong S_3$.

Chapter 10

Modules

Let R be a ring; we take the convention that R has a 1. A *left R -module* is an additive abelian group with an R -action on the left. That is, an action

$$R \times M \rightarrow M; (r, m) \mapsto rm$$

satisfying the conditions:

1. $rs(m) = r(sm)$ for $r, s \in R$ and $m \in M$;
2. $(r + s)m = rm + sm$;
3. $r(m + n) = rm + rn$;
4. $1m = m$.

Example time! Let $\mathbb{F} = R$. For fields \mathbb{F} , we have \mathbb{F} -modules as vector spaces. Moreover, every \mathbb{Z} module is just an abelian group and vice-versa.

Now let $R = \mathbb{F}[x]$ be a polynomial ring over a field \mathbb{F} . Let V be a finite dimensional vector space over \mathbb{F} . Then if $T : V \rightarrow V$ is a linear transformation, then V admits an R -module structure by $xv = Tv$ for $v \in V$. Thus, V is an $\mathbb{F}[x]$ -module.

Another. Let $R = M_n(\mathbb{F})$. Let \mathbb{F}^n be the column space of dimension n . Then R acts on \mathbb{F}^n by matrix multiplication.

Definition 10.0.1. Let M, N be left modules. An R -module map $\phi : M \rightarrow N$ is an additive group homomorphism that commutes with the action of R .

Now we have the isomorphism theorem of R -modules.

1. Let $\phi : M \rightarrow N$. Then $\ker \phi$ is an R -submodule of M and $\text{im } \phi$ is as well. We have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \pi \downarrow & \nearrow \exists! \tilde{\phi} & \\ M/\ker \phi & & \end{array}$$

- 2.
- 3.
4. He neglected to go over the other ones.

10.1 Free R -modules

An R -module M is called free if there exists a basis X of M over R such that every element x can be uniquely written as an R -linear combination of X .

We may also define free modules via universal properties. Say M is free over X if for any R -module N with a map $f : X \rightarrow N$, then there exists a unique R -map called $\phi : M \rightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ i \uparrow & \nearrow f & \\ X & & \end{array}$$

One can consider X as a basis of the free R -module M .

For example, say $X = \{1, 2, 3\}$. Then $F_R(X) = R_1 \oplus R_2 \oplus R_3$.

10.2 Tensor Products

We now move on to tensor products. Let $M_R, {}_R N$ be R modules. Then $M \otimes_R N$ is the quotient of $F_{\mathbb{Z}}(M \times N)$ by the subgroup k generated by:

1. $(m + m', n) - (m, n) - (m', n)$
2. $(m, n + n') - (m, n) - (m, -n)$
3. $(mr, n) - (m, rn)$

For now, we only know that this is an abelian group.

This is actually a functor: $\mathbf{Mod}_R \times_R \mathbf{Mod} \rightarrow \mathbf{Ab}$.

We have to now talk about balanced maps. Let $M_R, {}_R N$ be R -modules. Let $f : M \times N \rightarrow G$, G an additive group, be called balanced if f is bilinear with $f(mr, n) = f(m, rn)$.

For example, $t : M \times N \rightarrow M \otimes_R N$ is balanced.

We now state the universal property.

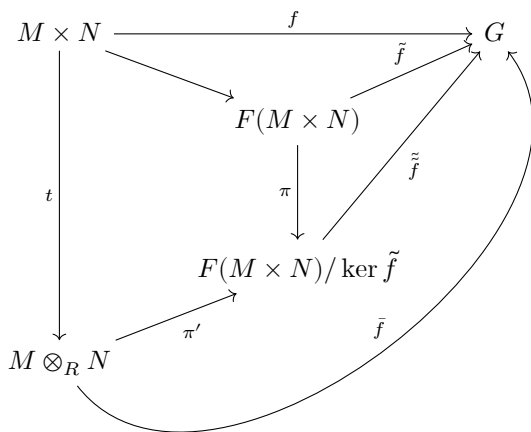
Let $M_R, {}_R N$ be R modules and $t : M \times N \rightarrow M \otimes_R N$ be the natural balanced map. Then t is universal, that is, if f is a balanced map, then there exists a unique homomorphism \tilde{f} such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & G \\ t \downarrow & \nearrow \exists! \tilde{f} & \\ M \otimes_R N & & \end{array}$$

Proof. First, we tackle uniqueness. If \tilde{f}_1, \tilde{f}_2 both satisfy $\tilde{f}_1 t(m, n) = \tilde{f}_2 t(m, n)$, then $\tilde{f}_1(m \otimes n) = \tilde{f}_2(m \otimes n)$. Since $\{m \otimes n\}$ generate the abelian group, $\tilde{f}_1 = \tilde{f}_2$.

Now existence. Let $f : M \times N \rightarrow G$ be a balanced map. Now, there exists a homomorphism $\tilde{f} : F(M \times N) \rightarrow G$ where $\tilde{f}(m, n) = f(m, n)$. Now, $\ker \tilde{f}$ is a subgroup of $F(M \times N)$, so we can take the quotient and apply the first isomorphism theorem to get a map $\tilde{\tilde{f}} : \frac{F(M \times N)}{\ker \tilde{f}} \rightarrow G$ so that the maps $\pi, \tilde{f}, \tilde{\tilde{f}}$ commute.

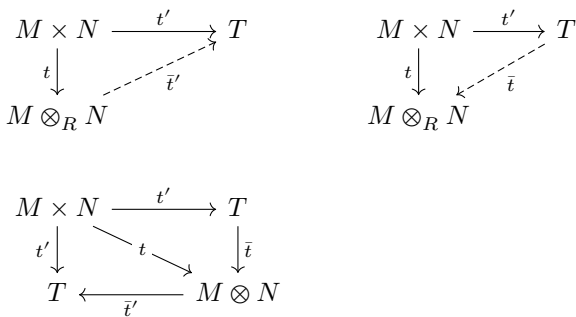
As f is balanced, $\ker \tilde{f}$ is contained in K . So there exists a map $\pi' : F(M \times N)/K \rightarrow F(M \times N)/\ker \tilde{f}$ by the first isomorphism theorem. So the following diagram commutes:



Theorem 10.2.1.


If $t' : M \times N \rightarrow T$ is a universal balanced map, $T \cong M \otimes_R N$ as an abelian group

Proof.



Let $f : M_R \rightarrow M'_R$ and $g : N_R \rightarrow N'_R$ be R -module maps. Then

$$\begin{array}{ccccc}
M & & N & & M \otimes N \\
\downarrow f & & \downarrow g & \Longrightarrow & \downarrow f \otimes g \\
M' & & N' & & M' \otimes N'
\end{array}$$

Proof. Define $\phi : M \times N \rightarrow M' \otimes N'$ by $\phi(m, n) = f(m) \otimes g(n)$. We need to first check ϕ is balanced; if it is, there exists a unique $\bar{\phi} : M \otimes N \rightarrow M' \otimes N'$ satisfying the universal property; we denote $\bar{\phi}f \otimes g$. 

Additionally,

$$\begin{array}{ccccc}
M & & N & & M \otimes N \\
\downarrow f & & \downarrow g & \Longrightarrow & \downarrow f \otimes g \\
M' & & N' & & M' \otimes N' \\
\downarrow f' & & \downarrow g' & \Longrightarrow & \downarrow f' \otimes g' \\
M'' & & N'' & & M'' \otimes N''
\end{array}$$

Proof. Write $\phi : M \times N \rightarrow M'' \otimes N''$ by $\phi(m, n) = (f' \circ f)(m) \otimes (g' \circ g)(n)$. Again, there exists a unique map, which we denote by the expected name. We then verify

$$\begin{aligned}
(f' \otimes g')(f \otimes g)(m \otimes n) &= (f' \otimes g')(f(m) \otimes g(n)) \\
&= (f' \circ f)(m) \otimes (g' \circ g)(n).
\end{aligned}$$



Lastly, we have that $\text{id}_M \otimes \text{id}_N = \text{id}_{M \otimes N}$ and if f, g are isomorphisms, $f \otimes g$ are also isomorphisms.


This shows that $_ \otimes _$ is a bifunctor.

10.3 Bimodules

Let's talk about S – R –bimodules. We write ${}_S M_R$ to indicate that M is a left S –module and right R –module, alongside the associative property $(sm)r = s(mr)$ for $s \in R, r \in R, m \in M$.

For an example. Every M_R is also ${}_Z M_R$. Moreover, if R is commutative, $M_R = {}_R M = {}_R M_R$.

Proposition 10.3.1. *If ${}_S M_R$ is an $S - R$ -module and ${}_R N_T$ is an $R - T$ -module, then $M \otimes_R N$ is an $S - T$ -bimodule.*

Proof. Let $s \in S$. Define $\phi_s(m, n) = sm \otimes n$. We need to check that this is balanced. Then there exists a unique $\bar{\phi}_s(m \otimes n) = (sm) \otimes n$. Now we define $s(m \otimes n) = \bar{\phi}_s(m \otimes n) = (sm) \otimes n$. Similarly define $\bar{\phi}_t(m \otimes n) = (m) \otimes (nt)$. We now need check the associative property of bimodules. 


Proposition 10.3.2. *If $f : {}_S M_R \rightarrow {}_S M'_R$ and $g : {}_R N_T \rightarrow {}_R N'_T$ are bimodule maps, $f \otimes g$ is also a $S - T$ -bimodule map.*

If $f \in \text{Hom}_R(M_R, M'_R)$ and $g \in \text{Hom}_R({}_R N, {}_R N')$, then $f \otimes g \in \text{Hom}_Z(M \otimes_R N, M' \otimes_R N')$. Moreover,

$$f \otimes _ : \text{Hom}_R({}_R N, {}_R N') \rightarrow \text{Hom}_Z(M \otimes_R N, M' \otimes_R N')$$

is a group homomorphism.

Proposition 10.3.3. *Let R be a ring and ${}_R N$ an R -module. Then, $R \otimes_R N \cong {}_R N$.*

Proof. Let $\alpha : R \times N \rightarrow N$ by $\alpha(r, n) \rightarrow rn$ be a balanced map. Then there exists a unique $\bar{\alpha} : R \otimes_R N \rightarrow {}_R N$ such that $\bar{\alpha}(r \otimes n) = rn$. Moreover, $\bar{\alpha}(sr \otimes n) = (sr)n = s(rn) = s(\bar{\alpha}(r \otimes n))$ implies $\bar{\alpha}$ is an R module map. Then let $\psi : N \rightarrow R \otimes_R N$ by $n \mapsto 1 \otimes n$. We can then see that ψ and $\bar{\alpha}$ are inverses so we are done. 

If we have ${}_S M_R, {}_R N_T, {}_T L_P$, then $(M \otimes N) \otimes L = M \otimes (N \otimes L)$ is a $S - P$ -bimodule.

Theorem 10.3.4. *Let $M_R, M'_R, {}_R N$ be R modules. Then,*

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N).$$


Let's talk about direct sums. $M_R \oplus M'_R = M_R \times M'_R$. But how do we do it in category theory?

If $N \cong M_R \oplus M'_R$, we have the projection and inclusion maps defined naturally such that $\pi i = \text{id}_M$ and $\pi' i' = \text{id}_{M'}$. What about

$i\pi$? Well, $i\pi(M) = (M, 0)$ and the other is more or less the same. Hence, $i\pi + i'\pi' = \text{id}_N$.

We say N is a direct sum of M, M' if we have a pair of maps $i_k : M \rightarrow N$ and $\pi_j : M \rightarrow N$, such that $\pi_j i_k = \delta_{jk} \text{id}_k$.

Theorem 10.3.5. *Let V, W be finite dimensional vector spaces over \mathbb{F} . Then $V \otimes_{\mathbb{F}} W \cong \mathbb{F}^{\dim(V) \dim(W)}$.*

Proof. $V \cong \mathbb{F}^{\dim V}$ and $W \cong \mathbb{F}^{\dim W}$. Then $V \otimes_{\mathbb{F}} W \cong \mathbb{F}^{\dim(V)} \otimes_{\mathbb{F}} \mathbb{F}^{\dim(W)} \cong (\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F})^{\dim(V) \dim(W)} \cong \mathbb{F}^{\dim(V) \dim(W)}$. 

If R is commutative, $M \otimes_R N \cong N \otimes_R M$ as R -modules by the flip map.

Let's say that we have M_R and ${}_R N$, so that we may write $M \otimes_R N$. Also, let's say we have a map ${}_R N \xrightarrow{f} {}_R N'$. Then, $\text{id} \otimes f : M \otimes_R N \rightarrow M \otimes_R N'$.

10.4 Algebra

Definition 10.4.1 (Algebra). *Let A be a ring, R a commutative ring, and A is an R -module such that the multiplication of A is a balanced bimodule map. Then A is said to be an algebra over R or an R -algebra.*

Every ring A may be viewed as a \mathbb{Z} -algebra. Another example we probably know is $A = M_n(R)$ over the ring R . A slightly more novel example is the group algebra RG .

Let G be a finite group. Then RG is a free R -module over G , $RG = \left\{ \sum_{g \in G} \alpha_g g \mid \alpha_g \in R \right\}$ with multiplication given by

$$\left(\sum_g \alpha_g g \right) \left(\sum_h \beta_h h \right) = \sum_{gh} \alpha_g \beta_h gh.$$

Another example. Let K/F be a field extension. Then K is an F -algebra.

We might then also write the multiplication of A as a bimodule map $m : A \otimes A \rightarrow A$ where

$$m(1 \otimes a) = m(a \otimes a) = a$$

and


$$m(a \otimes m(b \otimes c)) = m(m(a \otimes b) \otimes c).$$

Let's move on.

Lemma 10.4.2. *Let $M_R \rightarrow M'_R \rightarrow 0$ be an exact sequence of R -modules. Then for any left R -module ${}_R N$,*

$$M \otimes N \rightarrow M' \otimes N \rightarrow 0.$$

That is, ${}_{} \otimes N$ is right exact.

Proof. For any $m' \in M'$, there exists $m \in M$ such that $f(m) = m'$ since f is surjective. Therefore, $(f \otimes \text{id})(m \otimes n) = m' \otimes n$. As $m' \otimes n$ generate $M' \otimes N$ as an abelian group, $f \otimes \text{id}$ is surjective. 

What about injectivity? Take $\mathbb{Z}_2 \xrightarrow{\phi} \mathbb{C}^*$, $\phi(i) = (-1)^i$ as a group homomorphism (a \mathbb{Z} -module map). Now what about $\phi \otimes \text{id}$? Then $\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Z}_2 = 0$ so there is no way $\phi \otimes \text{id}$ is injective as $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is nontrivial. Hence, ${}_{} \otimes_R N$ does not generally preserve injective maps.

Definition 10.4.3. *An R -module N is called flat if ${}_{} \otimes N$ preserves injective maps.*

Which R -modules are flat? Well, free modules, for one, but that might be all we say.

Proposition 10.4.4. *Let ${}_R N \rightarrow {}_R N' \rightarrow {}_R N'' \rightarrow 0$ be right exact. Then, for any R -module M_R , we have*

$$M \otimes N \rightarrow M \otimes N' \rightarrow M \otimes N'' \rightarrow 0$$

is also right exact.

Let's try to finish the proof. We want to show that, if $g \circ f = 0$ then $M \otimes (g \circ f) = (M \otimes g) \circ M \otimes f$ is also 0. But as the first is $M \otimes 0$, $\text{im } M \otimes f \subset \ker M \otimes g$. Halfway done.

$$\begin{array}{ccc}
 M \otimes N' & \xrightarrow{M \otimes g} & M \otimes N'' \\
 \downarrow & \nearrow \phi & \uparrow \\
 M \otimes N' / \text{im}(M \otimes f) & & \\
 \downarrow & \nearrow \tilde{\phi} & \\
 M \otimes N' / \ker(M \otimes g) & &
 \end{array}$$

Because $\phi(m \otimes n' + \text{im}(M \otimes f)) = m \otimes g(n')$. Then $\ker(\phi) = \ker(M \otimes g) / \text{im}(M \otimes f)$.

We want now to show that ϕ is injective. Define $\psi : M \otimes N'' \rightarrow M \otimes N' / \text{im}(M \otimes f)$ by $\psi(m \otimes n'') = m \otimes n'$ where $g(n') = n''$.

We need check that ψ is well defined. Assume $n'_1, n'_2 \in N'$ such that $g(n'_1) = g(n'_2) = n''$. Then $n'_1 - n'_2 \in \ker g = \text{im } f$. Thus $n'_1 + \text{im } f = n'_2 + \text{im } f$. Thus $m \otimes n'_1 + \text{im } M \otimes f = m \otimes n'_2 + \text{im } M \otimes f$. Thus ψ is well defined. We need check that ψ is additive, but that's an exercise.

Then $\psi\phi(m \otimes n' + \text{im}(M \otimes f)) = \psi(m \otimes g(n')) = m \otimes n' + \text{im}(M \otimes f)$. Thus ϕ is injective.

Chapter 18

Finite Group Representations

Definition 18.0.1. Let G be a group and V a vector space over a field k . A representation of G on V is a group homomorphism $\rho : G \rightarrow GL(V)$.

Definition 18.0.2. A matrix representation of G is a group homomorphism $\rho : G \rightarrow GL(n, k)$.

When do we say two representations are the same? $\rho : G \rightarrow GL(V)$ and $\rho' : G \rightarrow GL(W)$. Let $T : V \rightarrow W$ is an isomorphism with $\hat{T} : GL(V) \rightarrow GL(W)$ by $\sigma \rightarrow T\sigma T^{-1}$. We say ρ and ρ' are equivalent if

$$\begin{array}{ccc}
 M \otimes N' & \xrightarrow{M \otimes g} & M \otimes N'' \\
 \downarrow & \nearrow \phi & \\
 M \otimes N' / \text{im}(M \otimes f) & & \\
 \downarrow & \nearrow \tilde{\phi} & \\
 M \otimes N' / \ker(M \otimes g) & &
 \end{array}$$

Similarly, if ρ and ρ' are matrix representations, then they are equivalent if there exists an invertible matrix $S \in GL(n, k)$ such that $S\rho(g)S^{-1} = \rho'(g)$.

Let $\rho : G \rightarrow GL(V)$ be a representation. Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of V . Then $[\rho(g)]_{\mathcal{B}}$ denotes the matrix of $\rho(g)$ with respect to \mathcal{B} . Then we can define a matrix representation of G .

Last time, we were working with KG -modules, that is, representations of G over the field K . Moreover, we were talking about KG -modules corresponding to the irreducible representations of G . Moreover, morphisms between representations of G correspond directly to morphisms between KG -modules. That is, if $\phi : M \rightarrow N$ is a KG -module map, and $\rho : G \rightarrow GL(M)$ and $\psi : G \rightarrow GL(N)$, then $\phi\rho(g) = \psi(g)\phi$. We can abbreviate that as $\phi(gm) = g\phi(m)$.

We now define faithful KG -module. A module M is faithful if $xM = 0$ implies $x = 0$. Faithfulness is not categorical; a faithful representation $\rho : G \rightarrow GL(M)$ is one such that ρ is injective.

Example, $\rho : D_8 \rightarrow GL_2(\mathbb{C})$ with the presentation $\langle x, y \mid x^4 = y^2 = yxyx = 1 \rangle$. Then

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\rho(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines a faithful representation. Because $\rho : G \rightarrow GL(\mathbb{C}^2)$ is faithful, but $\rho : \mathbb{C}D_* \rightarrow \text{End}(\mathbb{C}^2)$ maps a 8 dimensional space to a 4 dimensional one.


Another example. Let $\rho : D^8 \rightarrow GL_2(\mathbb{C})$. Then ρ is irreducible. Let's show it directly. Well, ρ defines a G -action on \mathbb{C}^2 . Let $z \in \mathbb{C}^2 \setminus \{0\}$. Then $z = (a, b)^T$, not both 0. We then see that $\rho(x)(z) = (b, -a)^T$ but $\rho(y)(z) = (b, a)^T$. That means $(2b, 0)^T = ((\lambda_1 + \lambda_2)a, (\lambda_1 + \lambda_2)b)^T$. Then either $b = 0$ or $\lambda_1 + \lambda_2 = 0$. The latter implies $b = 0$. If $b = 0$ but $\lambda_1 + \lambda_2 \neq 0$, then $a = 0$ and we are done. Thus $b = 0$ and $\lambda_1 + \lambda_2 = 0$, but again this implies $a = 0$ and we are done.

Moreover, 1-dimensional representations are always irreducible.

Definition 18.0.3. Let R be a ring. An R -module M is called completely reducible if for any submodule N of M , there exists a submodule N' of M such that $M = N \oplus N'$.

Theorem 18.0.4. Let M be completely reducible. Then every submodule of M is completely reducible.

Proof. Let N be a submodule of M and suppose N_1 is a submodule of N . Let N'_1 be the complement of N_1 in M . For $x \in N$, $x = x'_1 + x_1$ for some $x'_1 \in N'_1$ and $x_1 \in N_1$. Thus $x - x_1 \in N$.

Therefore $x'_1 \in N \cap N'_1$ and $x \in N_1 + (N'_1 \cap N)$ implying $N \subseteq N_1 + (C'_1 \cap N)$. We want to show $N_1 \oplus (N'_1 \cap N) = N$. Let $x \in N \cap (N'_1 \cap N)$ and $x \in N_1 \cap N' = 0$. 

Theorem 18.0.5 (Maschke). *Let G be a finite group and $\text{ch}(k) \nmid |G|$. Then every kG -module is completely reducible.*

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