Topology II

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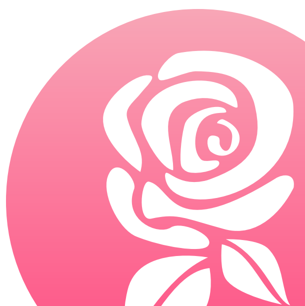

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Information

Time & Room	TTth 12:00-13:20, Lockett 232
Exam	?
Textbook	Hatcher, *Algebraic Topology*
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Office Hours	Tuesday, Thursday 13:30-14:20

First assignment: Get your hand on the textbook! Additionally, read the introduction to Chapter 2.

Chapter 1

Weeks 1-3

1.1 Week 1

1.1.1 Tuesday

Theorem 1.1.1 (Brower Fixed Point for $n = 2$)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Towards a contradiction, suppose f is a fixed point free map. Then $r(x) = \frac{x - f(x)}{\|x - f(x)\|}$ is a continuous map from $D^2 \to S^1$. Because r is a retract, we have the composition $D^2 \xrightarrow{r} S^1 \xrightarrow{i} D^2$ yielding the maps

$$\pi_1(D^2) \xrightarrow{r_*} \pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2).$$

However, as retracts should induce a surjection, this is a contradiction!

What about $n \geq 3$?

We want to study the relationship between topological spaces and algebraic objects. Maybe we associate groups, maybe rings...We want this association to be functorial. Roughly speaking, this means that spaces that are the same should get sent to the same objects! Another property of functorality is that continuous maps f should be sent to homomorphisms f_*.

In topology 1, we take a space X and assign to it its fundamental group $\pi_1(X)$. Roughly speaking, $\pi_1(X)$ is the set of homotopy classes of maps $S^1 \to X$. To generalize, we could look at the homotopy classes of maps $S^k \to X$. We denote this set $\pi_k(X)$ and we call them the higher homotopy groups. Additionally, they turn out to be abelian groups for $k \geq 2$. These are hard to determine!

A guess: Are these groups given by generators of faces and relations of 4-cells with a simplicial complex?

In this course, we're going to take X and take it to $\{H_k X\}$, homology, and $\{H^k X\}$, cohomology groups.

Let's denote pseudomanifolds $\mathcal{X}$.

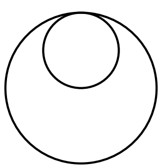
Definition 1.1.2 (Informal $k-$dimensional Manifold)**.** *A space that looks locally like* $\mathbb{R}^k$*.*

Definition 1.1.3 (Informal $k-$dimensional $\mathcal{X}$-Manifold)**.** *Can have singularities; points of being noneuclidean.*

Some properties:

(1) The part of P where it is a $k-$manifold is open, dense, and oriented.

(2) The set of singularities has dimension $\leq k-2$



Definition 1.1.4 ($k-$Simplex)**.** *A k-simplex is the convex hull of points $p_0 \ldots p_k$ in general position in some Euclidean space. Its faces are $k-1$simplicies.*

Definition 1.1.5 (Simplicial Complex)**.** *A simplicial complex is the set S of simplicies in some $\mathbb{R}^N$ satisfying*

- *Any face of a simplex in S is in S.*
- *Any two simplices in S are either disjoint or intersect in a set that is a face of both of them.*

Simplicial complexes are not pseudomanifolds; the edges are too big!

Definition 1.1.6 (Orientation on a Simplex). *An orientation $\mathcal{O}$ in $\Delta^{k>0}$ $k-$simplex is an ordering of the vertices of each simplex. Two orderings are the same or equivalent if they differ by an even permutation.*

An orientation on a point is either a $+$ or $-$.

Hence, there are two orientations on any simplex; we call them "opposites."

The orientation has a concept of induced orientations; just negotiate the missing vertex to the last position via a permutation and delete it.

Definition 1.1.7 (k-dimensional Pseudomanifold). *A $k-$dimensional pseudomanifold is a simplicial complex with a $\mathcal{O}(\Delta)$ orientation of each $k-$simplex such that*

(1) Every simplex is a face of a $k-$simplex.

(2) Every $(k-1)$-simplex is a face of exactly two $k-$simplices.

(3) Continuity of orientation: If Δ' is a $(k-1)$-simplex, face of $\Delta, \tilde{\Delta}$, $\mathcal{O}(\Delta)$ and $\mathcal{O}(\tilde{\Delta})$ must induce opposite orientations on Δ'.

Definition 1.1.8 (i-cycle). *Let X be a topological space. An i-cycle of X is a pseudomanifold of dimension i, P, and a map $\sigma : P \to X$ that "captures" a hole.*

For example, take P, an oriented triangle, and X an annulus. Map the triangle around the hole.

Let $\sigma_1 : P_1 \to X$, $\sigma_2 : P_2 \to X$ be i-cycles. Let $P_1 + P_2 = P_1 \sqcup P_2$, and σ be defined as you would think. We can also define $-P$; take the same psuedomanifold and same map, but put the opposite orientation on P.

Definition 1.1.9 (Pseudomanifold with Boundary). *A simplicial complex Q and an orientation $\mathcal{O}(\Delta)$ on each $k-$simplex, along with a subsimplex B that satisfies:*

(1) Each simplex in Q is a face of a $k-$simplex

(2) B a $k-1$ dimensional pseudomanifold;

(3) Each $k-1$-dim Δ' not in B is a face of 2 k-simplices;

(4) Each $k-1$-dim Δ' in B is a face of 1 $k-$simplex;

(5) Orientation is inherited.

Definition 1.1.10 (Cobordism of i-cycles). *A cobordism of i-cycles P_1, P_2 in X is an $(i+1)$-dimensional psuedomanifold with boundary C and a map $\sigma : C \to X$ So that $\partial B = P_1 - P_2$ and $\sigma|_B$ coincides with σ_1, σ_2.*

Two i-cycles that have a cobordism between them are called cobordant.

Homework problem:

Proposition 1.1.11 (Cobordism relation). *Cobordism of i-cycles in X is an equivalence relation.*

Proof.

Reflexive Let $P = \{x_1, \dots, x_k\}$. Let $P \cong Q = \{y_1, \dots, y_k\}$. Construct Δ as follows. The $k+1$ cells are of the form $\Delta_n = p_n \sqcup q_{k+1-n}$ where $p_n = \{x_i \mid i \in [n]\}$ and $q_m = \{y_i \mid i \in [m]\}$ for $1 \leq n \leq k$. Now, the k cells are of the form $p_n \sqcup q_{k-n}$ or $p_{n-1} \sqcup q_{k+1-n}$. Thus, Δ_n shares a face with Δ_{n+1} and Δ_{n-1}. The only faces not shared are P, from $n = 1$, and Q, from $n = k$. Hence, $P \sim Q = P$ so $P \sim P$.

Symmetric Given $P \sim Q$ via Δ, σ, we know $\partial\Delta = P - Q$ and hence $\partial\overline{\Delta} = (-P) - (-Q) = Q - P$ so $Q \sim P$ via $\overline{\sigma}$.

Transitive Let $P \sim Q$ and $Q \sim R$ via Δ, σ and Δ', σ', construct Δ'', σ'' by identifying Q in Δ with Q in Δ'. Then, $\overline{Q}$ is a face of some $X \in \Delta$, Q a face of some $\Upsilon \in \Delta''$, so Q is a face of both X and Y and $\partial\Delta'' = P - R$. Let $\sigma''|_\Delta = \sigma$ $\sigma''|_{\Delta'} = \sigma'$. Hence, $P \sim R$.

Definition 1.1.12 (Homology). *$H_i(X)$ is the set of equivalence classes of $+, -$ of i-cycles giving an abelian group structure. The identity element of this group is represented by the empty $i-$dimensional psuedomanifold.*

1.1.2 Thursday

Types of topological spaces:

$$\begin{tikzcd}
& \text{Metric spaces} & \\
\text{Manifolds} \arrow[ur, "\subset"] \arrow[d, "\subset"] & \text{Simplicial complexes} \arrow[u, "\subset"'] \arrow[d, "\subset"] & \\
\text{Pseudomanifold} & \Delta\text{-Complex} \arrow[r, "\subset"] & \text{CW-Complex}
\end{tikzcd}$$

What's a manifold? Well, a manifold is a 2nd-countable Hausdorff space that is locally euclidean.

Examples:

- $\mathbb{R}^n$;
- S^n;
- Products of manifolds: $T = T^2 = S^1 \times S^1$.
- $\square$-Complex: Combinatorial model for space X, built from "cells" or related objects.

Let's mess with the torus $T = S^1 \times S^1$ for a bit. Whenever we're constructing it from a gluing diagram, we have one zero cell, the point, two 1-cells, the edges, and one 2-cell, the square itself. We can think of $n-$cells as n-disks. Hence,

$$T = X \supset X^1 \supset X^0$$

where X^0 is the $0 - \textit{cell}$, X^1 is $X_0 \cup \{\text{1-cells}\}$. We could say, if we're so inclined,

$$X^1 = \left(X^0 \sqcup e_1^1 \sqcup e_2^1\right) / \sim_1$$

and

$$X = \left(X^1 \sqcup e^2\right) / \sim_2 .$$

Definition 1.1.13 (CW-Complex)**.** *(1) A discrete set of points* X^0*;*

(2) Inductively form the n-skeleton X^n from X^{n-1} by attaching some collection of n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} = \partial D^n \to X^{n-1}$, and setting $X^n = \left(X^{n-1} \sqcup_\alpha D_\alpha^n\right) / \sim$ where $x \sim \varphi_\alpha(x)$ for $x \in S^{n-1} = \partial D^n$.

(3) Either $X = X^n$ for some n and we say that n is the dimension of X, or $X = \bigcup_n X^n$ with the weak topology.

Definition 1.1.14 ((aside) Weak Topology)**.** *Let $X = \bigcup_n X_n$. Then $A \subset X$ is open if $A \cap X_n$ is open in X_n for all n.*

Let's do some examples; $X = X^1$ is a graph, S^1 is a point and a quotient-ed one cell, or two points and two one cells. More interestingly, we could have $S^2 = x_0 \cup D^2/\sim$ where $x_0 \sim y$ if $y \in \partial D^2$...And we could generalize this to all $n-$spheres.

Another really interesting example is T^2, formed by the CW-Complex by an octagon!

We can do similar things with non-orientable surfaces. Take $\mathbb{R}P^2$ and the standard gluing diagram. In general,

$$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1}\backslash 0/\sim$$

where $v \sim \lambda\omega$ for $\lambda \neq 0$. It could also be written $S^n/\sim$, $x \sim -x$, and $D^n/\sim$ for $y = -y$ for $y \in \partial D^n = S^{n-1}$. Yet another description is $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$.

For $m < n$, $\mathbb{R}P^m \subset \mathbb{R}P^n$ is a subcomplex, with $\mathbb{R}P^\infty$ are lines through the origin in $\mathbb{R}^\infty$ (weak topology).

Now here's where the fun begins: Klein bottles with their standard gluing map!

This leads into the observation that closed surfaces can be realized as the quotient spaces of polygons. Moreover, a polygon can be cut into triangles! Thus, any surface can be built out of triangles. This is where the notion of a Δ-complex (or more stringently, a simplicial complex) comes from.

Let's get some standard notation. A standard n-simplex

$$\Delta^n = \{(t_0, \dots, t_j) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} t_i = 1, t_i \geq 0\}$$

. Hence, Δ^0 is a point, Δ^1 is the line segment from $(1,0)$ to $(0,1)$, so on, so forth.

The $n-$simplex on $(p_0, \dots, p_n)$ in general position (the set

$$\{v_1 - v_0, \dots, v_n - v_0\}$$

is linearly independent) is the convex hull of $(p_0, \dots, p_n)$. We call it $\Delta = [p_0 p_1 \dots p_n]$. In this position, $\Delta^n = [e_0 e_1 \dots e_n]$. Any two n-simplicies are homeomorphic. Eg, $\Delta^n \to \Delta = [v_0 \dots v_n]$ via $(t_0, \dots, t_n) \mapsto (\sum t_i v_i)$.

Some terminology that we'll use: The boundary of $[v_0 \dots v_n] = \cup\{\text{all (n-1)-dim faces}\}$, the interior of Δ is $\Delta-$boundary, so $]v_0 \dots v_n[= [v_0 \dots v_n] - \partial[v_0 \dots v_n]$.

Definition 1.1.15 (Δ-Complex)**.** *A Δ-complex is a quotient of a disjoint union of simplicies obtained by identifying certain faces using orientation preserving homeomorphisms.*

To be more refined, a Δ-complex on X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ where

- $\sigma_\alpha|_{\overset{o}{\Delta^n}}$ *is a homeomorphism onto* $\sigma_\alpha(\overset{o}{\Delta^n}) = e^n_\alpha$.
- *Each $x \in X$ is in exactly one e^n_α.*
- $\sigma_\alpha|_{\textit{(n-1)-face of } \Delta^n} = \sigma_\beta : \Delta^{n-1} \to X$.
- *$U \subset X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in Δ^n for all n.*

A homework problem:

A Δ-Complex is a CW-Complex.

1.2 Week 3

1.2.1 Tuesday

Definition 1.2.1 (Free Abelian Group)**.** *Let $G_1, G_2, \ldots, G_n$ be groups. Then, we have the direct product $\prod_1^n G_i$ with operations defined pointwise. If each G_i is abelian, we speak instead of their direct sum $\bigoplus_1^n G_i$. If each and every G_i is infinite cyclic, we call their direct sum* free abelian. *If $G_i = \langle a_i \rangle$, then elements of $\bigoplus G_i$ look like $(m_1 a_1, \ldots, m_n a_n)$ for $m_i \in \mathbb{Z}$. We call $\{a_1, \ldots, a_n\}$ a basis for $\bigoplus G_i \cong \bigoplus \mathbb{Z} = \mathbb{Z}^n$.*

We can generalize this; for an infinite list $S = \{a_1, a_2, \ldots\}$, the free abelian group with basis S is $G = G(S)$ with elements $\sum m_i a_i$ for finitely many $m_i \neq 0$.

Recall that a Δ complex is a topological space X with characteristic maps $\sigma_\alpha : \Delta^n \to X$ with each $x \in X$ in a unique open $n - cell$ $\sigma_\alpha(\Delta^n)$ and compatibility, openness conditions.

For example, take $X = T^2$. Then we have one 0-cell $\sigma_v : \Delta^0 \to X$, three 1-cells $\sigma_a, \sigma_b, \sigma_c : \Delta^1 \to X$, and two 2-cells $\sigma_U, \sigma_L : \Delta^2 \to T$ with some identifications.

Definition 1.2.2 (Simplicial Homology of Δ-Complexes)**.** *Let $\Delta_n(X)$ be the free abelian group with basis n-simplicies in X (or the maps $\sigma_\alpha : \Delta^n \to X$).*

We also inflict the following boundary homomorphism: Let $\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ via

$$\partial[v_0, v_2] = +[v_1] - [v_0],$$

$$\partial[v_0, v_1, v_2] = +[v_0, v_1] + [v_1, v_2] - [v_0, v_2],$$

In general...literally just take the boundary.

$$\partial[v_0, \dots, n_n] = \sum(-1)^i[v_0 \dots \hat{v}_i \dots v_n].$$

Example: Using the torus again, we have $\Delta_0(T) = \mathbb{Z}$, $\Delta_1(T) = \mathbb{Z}^3$, $\Delta_2(T) = \mathbb{Z}^2$, and otherwise 0.

For general Δ complexes, we say

$$\partial\sigma = \sum_{i=0}^{n}(-1)^i\sigma|_{[v_0 v_1 \dots \hat{v}_i \dots v_n]}.$$

Lemma 1.2.3. *We have that $\partial_n \circ \partial_{n+1} = 0$. Tersely, we can say*

$$\Delta_{n+1}(x) \overset{\partial_n+1}{\to} \Delta_n(x) \overset{\partial_n}{\to} \Delta_{n-1}(x)$$

to mean that $\operatorname{im}\partial_{n+1} \subset \ker\partial_n$. We then define the nth simplicial homology group to be $H_n^{\Delta}(X) = \ker\partial_n / \operatorname{im}\partial_{n+1}$.

Proof. For $\sigma : \Delta^{n+1} \to X$,

$$\partial\sigma = \sum_{i=0}^{n+1}(-1)^i\sigma|_{\hat{v}_i}$$

so

$$\begin{aligned}
\partial\partial\sigma &= \sum_{i=0}^{n+1}(-1^i)\partial\sigma_{\hat{v}_i} \\
&= \sum_{i=0}^{n+1}(-1)^i\left(\sum_{j=0}^{i-1}(-1)^j\sigma|_{\hat{v}_i,\hat{v}_j} + \sum_{j=i+1}^{n+1}(-1)^{j-1}\sigma|_{\hat{v}_i,\hat{v}_j}\right) \\
&= \sum_{0\le p<q\le n+1}(-1)^{p+q}\sigma|_{\hat{v}_q,\hat{v}_p} \\
&\qquad \sum_{0\le q<p\le n+1}(-1)^{p+q-1}\sigma|_{\hat{v}_p,\hat{v}_q} \\
&= 0
\end{aligned}$$

For example, take $X = [v_0 v_1 v_2]$ with bases $\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}$ and σ_{012}. We have that $\partial\sigma_{12} = \sigma_2 - \sigma_1$, so on, so forth. Hence,

$$
\begin{tikzcd}
0 \arrow[r] & \Delta_2(X) \cong \mathbb{Z} \arrow[r] & \Delta_1(X) \cong \mathbb{Z}^2 \arrow[r] & \Delta_0(X) \cong \mathbb{Z}^3 \arrow[r] & 0 \\
0, 0 \arrow[r, mapsto, "\partial"] & 0, 0 & & & \\
& \mathbb{Z}^2, 0 \arrow[r, mapsto, "\partial"] & \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle, 0 & & \\
& & \mathbb{Z}^2, \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle \arrow[r, mapsto, "\partial"] & \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle, 0 & \\
& & & \mathbb{Z}^3, \mathbb{Z}^3 \arrow[r, mapsto, "\partial"] & 0, 0 \\
0 \arrow[r, dash] & H_2 \cong 0/0 = 0 \arrow[r, dash] & H_1 \cong \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle / \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle = 0 \arrow[r, dash] & H_0 \cong \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle \cong \mathbb{Z} \arrow[r, dash] & \bullet
\end{tikzcd}
$$

Now, take $X = S^1$, with one 0 cell v and one 1 cell a.

$$0 \xrightarrow{\partial} \Delta_1 = \langle a \rangle \xrightarrow{\partial} \Delta_0 = \langle v \rangle \xrightarrow{\partial} \bullet$$

$$\begin{array}{ccccccc}
0,0 & \longrightarrow & 0,0 & & & & \\
 & & \mathbb{Z},\mathbb{Z} & \longrightarrow & 0,0 & & \\
 & & & & \mathbb{Z},\mathbb{Z} & \longrightarrow & 0,0 \\
0 & \text{———} & \mathbb{Z} & \text{———} & \mathbb{Z} & \text{———} & 0
\end{array}$$

What about a different Δ complex on the circle? Say, two points g, f with edges p, q ending in f.

$$0 \xrightarrow{\partial} \Delta_1 = \langle p, q \rangle \xrightarrow{\partial} \Delta_0 = \langle g, f \rangle \xrightarrow{0} \bullet$$

$$\begin{array}{ccccccc}
0,0 & \longrightarrow & 0,0 & & & & \\
 & & \mathbb{Z}^2, \langle p - q \rangle & \longrightarrow & \mathbb{Z}, 0 & & \\
 & & & & \mathbb{Z}^2, \mathbb{Z}^2 & \longrightarrow & 0,0 \\
0 & \text{———} & \mathbb{Z} & \text{———} & \mathbb{Z} & \text{———} & 0
\end{array}$$

1.2.2 Thursday

Definition 1.2.4 (Chain Complex)**.** *A chain complex $\{C_n, \partial_n\}$ is a sequence of abelian groups C_n with homomorphisms $\partial_n = \partial C_n \to C_{n-1}$ with $\partial_n \partial_{n-1} = 0$. As abelian groups are $\mathbb{Z}$ modules, we could replace $\mathbb{Z}$ with any ring.*

Elements of C_n are called chains, elements of $\ker \partial$ *are called cycles, and elements in* $\operatorname{im} \partial$ *are called boundaries.*

If one has a chain complex, one can take its homology:

Definition 1.2.5 (Homology)**.** *The homology of a chain complex is*

$$H_c(C_\bullet) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

We say that $z, w \in \ker \partial$ *are homologous if* $z - w \in \operatorname{im} \partial$.

For example, we return to $C_n = \Delta_n(X)$ with ∂ being the simplicial boundary.

Let's take the torus.

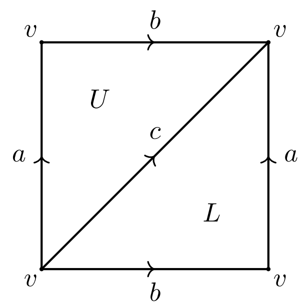


Figure 1.1: A Δ-complex on the torus.

Then

$$\Delta_2(T) = \langle U, L\rangle \to \Delta_1(T) = \langle a, b, c\rangle \to \Delta_0 = \langle v\rangle.$$

See that $\ker \partial_2 = \langle U, L\rangle$ and $\ker \partial_1 = \langle a, b, c\rangle = \Delta_1$. Thus, $\operatorname{im} \partial_2 = \langle a + b\rangle$. Hence,

$$H_2 = \langle U + L\rangle \cong \mathbb{Z}$$

$$H_1 = \langle a, b, c\rangle / \langle a + b - c\rangle = \langle a, b\rangle \cong \mathbb{Z}^2$$

$$H_0 = \langle v\rangle \cong \mathbb{Z}.$$

Recall that a simplicial complex is a Δ complex for which each σ_α is injective on the vertices of the standard $n-$simplex and furthermore no other $n-$simplex has the exact same set of vertices.

For example, to get a simplicial complex on a circle, you require at least three vertices.

This gives rise to some fun problems: What's the minimal simplicial complex structure on S^2? How about T? $\mathbb{R}P^2$? What does it mean to be minimal here, minimize the number of triangles?

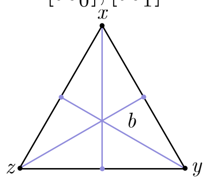
	Simplex	Subdivision
$n=0$	$[v_0]$	$[v_0]$
$n=1$	$[v_0v_1]$	$[bv_0],[bv_1]$
$n=2$	$[xyz]$	

Figure 1.2: Barycentric subdivision.

Given a Δ-complex, we can produce a simplicial complex structure by using barycentric subdivision.

Tale a closed $n-$simplex $[v_0, v_1, \ldots, v_n]$. Recall that these points are of the form $\sum_{i=0}^{n} t_i v_i$ where the sum of the t_i's is one and each is non zero. A barycenter is $\sum \frac{1}{n+1} v_i$. For barycentric subdivision, we take a simplex $[v_0 \ldots v_n]$ and decompose it into $n-$simplices by taking $[v_0 \ldots \hat{v}_j \ldots v_n b]$ where b is the barycenter where inductively $[v_0 \ldots \hat{v}_j \ldots v_n]$ is a face in the barycentric subdivision of a face.

Given a Δ-complex structure on X with $\sigma_\alpha^n : \Delta^n \to X$, the barycentric subdivision is the Δ complex with characteristic maps $\sigma_\alpha^n \tau_\beta^m : \Delta^m \to X$.

Claim: Barycentric subdivision twice gives a simplicial complex. See H§2.3#23.

Now, we move to a special little thing called singular homology. It's defined for any topological space and will agree with simplicial homology for Δ complexes.

Definition 1.2.6 (Singular n-simplex)**.** *A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$.*

We define the singular n-chains to be $C_n(X)$, the free abelian group generated by singular n-simplices.

Elements of $C_n(X)$ look like $m_1\sigma_1 + \ldots + m_k\sigma_k$ for $m_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. We develop a map ∂ given by $\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|_{[v_0 \ldots \hat{v}_j \ldots v_n]}$. We still have $\partial \circ \partial = 0$.

Definition 1.2.7 (Singular Homology)**.** *With the previous definitions, $\{C_n, \partial_n\}$ is the singular chain complex of X giving rise to singular homology.*

Let's do an example with $X = \{*\}$. Then, $\sigma^n : \Delta^n \to X = *$ is the unique continuous map into X for every $0 < n$. Hence, $C_n(X) = \langle \sigma_n \rangle \cong \mathbb{Z}$ for each $n \geq 0$.

$$\begin{array}{ccccccc} & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & \\ & & & & & & \\ \dashrightarrow & C_2(X) & \xrightarrow{\partial_2, \cong} & C_1(X) & \xrightarrow{\partial_1, 0} & C_0(X) & \longrightarrow 0 \end{array}$$

Concretely, we can determine that $\partial \sigma^n = \sigma^{n-1}$ for n even and 0 for n odd.

So, $H_{2k} = \ker \partial_{2k} / \operatorname{im} \partial_{2k-1} = 0/0 = 0$ and $H_{2k+1} = \ker \partial_{2k+1} / \operatorname{im} 2k = \mathbb{Z}/\mathbb{Z} = 0$. Hence $H_n(*) = 0$ except at $k = 0$. $H_0(*) = \ker \partial_0 / \operatorname{im} \partial_{-1} = \mathbb{Z}/0 = \mathbb{Z}$.

Let's go back and do one more simplicial calculation, H_*^Δ, for $X = \mathbb{R}P^2$.

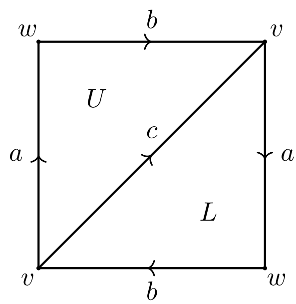


Figure 1.3: A Δ-complex on $\mathbb{R}P^2$.

We have $\Delta_0(X) = \langle \sigma_v, \sigma_w \rangle \cong \mathbb{Z}^2$, $\Delta_1(X) = \langle \sigma_a, \sigma_b, \sigma_c \rangle \cong \mathbb{Z}^3$ and $\Delta_2(X) = \langle \sigma_U, \sigma_L \rangle \cong \mathbb{Z}^2$. We see that

$$\begin{aligned} \sigma_a &\mapsto \sigma_w - \sigma \\ \sigma_b &\mapsto \sigma_v - \sigma_w \\ \sigma_c &\mapsto 0 \end{aligned}$$

$$\begin{aligned} \sigma_U &\mapsto \sigma_a + \sigma_b - \sigma_c \\ \sigma_L &\mapsto -\sigma_a - \sigma_b - \sigma_c \end{aligned}$$

Now, we compute the homology groups.

$$H_0^\Delta(X) = \Delta_0(x) / \operatorname{im} \partial_1 = \langle \sigma_v, \sigma_w \rangle / \langle \sigma_v - \sigma_w \rangle \cong \mathbb{Z}$$

For at home: Check that ∂_2 is injective so that $H_2^\Delta = \ker \partial_2 = 0$, and that

$$H_1^\Delta(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \langle \sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle / \langle 2\sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle \cong \mathbb{F}_2.$$

Chapter 2

Weeks 4-6

2.1 Week 4

2.1.1 Tuesday

Last time, we were looking at the singular homology. Take the singular chain complex $\{C_n, \partial_n(x)\}$, where C_n are the free abelian groups generated by $\sigma : \Delta^n \to X$, defining

$$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

So far we've done a grand total of one example, that of the point. We calculated

$$H_n(*) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 \end{cases}$$

Proposition 2.1.1. *If $X = \bigsqcup X_\alpha$ is a disjoint union of path components, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.*

Proof. The image of $\sigma : \Delta^n \to X$ is path connected, implying $C_n(X) = \oplus_\alpha C_n(X_\alpha)$. Hence, ∂ respects the decomposition:

$$\ker \partial_n(X) = \oplus_\alpha \ker \partial_n(X_\alpha)$$

with the same relation for the images, so

$$H_n(x) = \oplus H - n(X_\alpha).$$

Proposition 2.1.2. *If $X \neq ;$ is path connected then $H_0(x) \cong \mathbb{Z}$.*

Proof. We have $H_0(x) = C_0(x)/\operatorname{im}\partial_1$. Define $\epsilon : C_0(x) \to \mathbb{Z}$ by $\epsilon(\sum m_i \sigma_i^0) = \sum m_i$. We claim that $\operatorname{im}\partial_1 = \ker\epsilon$. If this is true, noting that ϵ is surjective, we get $H_0(x) = C_0(X)/\operatorname{im}\partial_1 = C_0(x)/\ker\epsilon \cong \operatorname{im}\epsilon = \mathbb{Z}$.

If $\sigma : \Delta^1 \to X$ is a 1-simplex, $\partial\sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$ so $\epsilon(\partial_1\sigma) = \epsilon(\sigma|_{[v_1]}) - \epsilon(\sigma|_{[v_2]}) = 1 - 1 = 0$. This implies $\operatorname{im}\partial_1 \subset \ker\epsilon$. Now, if $\sum m_i\sigma_i^0 \in C_0(X)$ is in $\ker\epsilon$, then $\sum m_i = 0$. Note that σ_i^0 is a 0-simplex, that is, a point $\sigma_i^0(v_0) \in X$.

Fix $x_0 \in X$ and choose paths $\gamma_i : I \to X$ with $\gamma_i(0) = x_0$, $\gamma_i(1) = \sigma_i^0(v_0)$. As the interval is the 1-simplex, each path is a singular 1$-$simplex in X. Moreover, $\partial_1\gamma_i = \sigma_i^0(v_0) - x_0$. Computing $\partial_1(\sum m_i\gamma_i) = \sum m_i\sigma_i^0(v_0) - (\sum m_i)\,\sigma_0^0(v_0) = \sum m_i\sigma_i^0(v_0)$ as the second sum goes to 0. This implies that $\ker\epsilon \subset \operatorname{im}\partial_1$ so we are done.

Theorem 2.1.3. *If $X = \bigsqcup X_\alpha$ is a decomposition of X into path components, then $H_0(X) \cong \oplus_\alpha \mathbb{Z}$.*

We now discuss reduced homology.

Definition 2.1.4 (Reduced Homology)**.** *The reduced homology of X is the homology of*

$$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

denoted $\tilde{H}_n(X)$.

The fact that $\epsilon\partial_1 = 0$ implies that ϵ induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(\mathbb{Z})$ implies that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$; that is, we have the exact sequence

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow 0$$

If X is path connected, then $\tilde{H}_0(x) = 0$. Otherwise, the homology groups agree.

$H_*(X)$ is a homotopy type invariant.

Definition 2.1.5 (Homotopic). *We say that maps $f, g : X \to Y$ are homotopic, $f \simeq g$, if there exists continuous $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We call H a homotopy. Homotopy is an equivalence relation on $C[x, y]$.*

Definition 2.1.6 (Homotopy Equivalence). *We say X and Y are homotopy equivalent, or have the same homotopy type, if there are maps $f : X \to Y$, $g : Y \to X$ with $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. once again, this is an equivalence relation on topological spaces. We communicate this briefly as $X \simeq Y$.*

It's worth noting that if $X \cong Y$, $X \simeq Y$.

Let's do an example. Say that X is convex in $\mathbb{R}^n$, and that $*$ is a point. They are homotopic and are called contractible.

Another example. An annulus is homotopic to S^1! We might then try our hand at computing the simplicial homology of the annulus.

We'd like to show that $f : X \to Y$ induces $f_* : H_n(X) \to H_n(Y)$ for all n, and that homotopy equivalences induce isomorphisms.

We first need to work on the chain level, that $f : X \to Y$ induces $f_\sharp : C_n(X) \to C_n(Y)$. If $\sigma : \Delta^n \to X$ is an n-simplex in X, then $f\sigma : \Delta^n \to Y$ is an $n-$simplex in Y. Therefore, we define $f_\sharp : C_n(X) \to C_n(Y)$ by $f_\sharp(\sigma m_i \sigma_i) = \sum m_i(f\sigma_i)$.

Lemma 2.1.7.

$$f_\sharp \circ \partial_n(X) = \partial_n(Y) \circ f_\sharp.$$

That is,

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n(X)"] \arrow[d, "f_\sharp"'] & C_{n-1}(X) \arrow[d, "f_\sharp"] \\ C_n(Y) \arrow[r, "\partial_n(Y)"'] & C_{n-1}Y \end{tikzcd}$$

Proof. For $\sigma : \Delta^n \to X$,

$$\partial_n(X)(\sigma) = \sum_i (-1)^i \sigma|_{\hat{v}_i}$$

so

$$\begin{aligned} f_\sharp(\partial_n(X)(\sigma)) &= \sum_i (-1)^i f \circ \sigma|_{\hat{v}_i} \\ &= \partial_n(Y)(f_\sharp(\sigma)) \end{aligned}$$

To be even more illustrative,

$$
\begin{array}{ccccccc}
\longrightarrow & C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) & \longrightarrow \\
 & \Big\downarrow{\scriptstyle f_\sharp} & & \Big\downarrow{\scriptstyle f_\sharp} & & \Big\downarrow{\scriptstyle f_\sharp} & \\
\longrightarrow & C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) & \longrightarrow
\end{array}
$$

Definition 2.1.8 (Chain Map)**.** *Let $\{C_n, \partial\}, \{C'_n, \partial'\}$ be two chains. A chain map $f : C_n \to C'_n$ for all n has $f \circ \partial = \partial' \circ f$ for all n.*

Hence, $f_\sharp$ takes cycles to cycles (elements of the kernel to elements in the kernel, as it commutes with ∂). Additionally, $f_\sharp$ takes boundaries to boundaries! If $x = \partial_n(X)(y)$, then

$$
\begin{aligned}
f_\sharp(x) &= f_\sharp \circ \partial_n(X)(y) \\
&= \partial_n(Y) \circ f_\sharp(y) \\
&= \partial_n(Y)(f_\sharp(y)).
\end{aligned}
$$

Consequently, $f_\sharp$ induces a homomorphism $f_\star : H_n(X) \to H_n(Y)$.

We remark that if $x \in C_n(X)$ satisfies $\partial_n(x) = 0$, $[x] \in H_n(X)$, $f_\star([x]) = [f_\sharp(x)]$.

2.1.2 Thursday

Last time, we saw that a continuous map of spaces $f : X \to Y$ induces a chain map $f_\sharp : C_*(X) \to C_*(Y)$. We further saw that any chain map induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for all n.

There are some desirable properties of f_*. For

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

$(g \circ f)_* = g_* \circ f_*$, and ditto for $f_\sharp, g_\sharp$. The underlying workings of this are

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y \xrightarrow{g} Z \implies g \circ f \circ \sigma : \Delta^n \to Z$$

with associative composition.

Additionally, $\mathrm{id}_X : X \to X$ induces $\mathrm{id} = (\mathrm{id}_X)_* : H_n(x) \to H_n(x)$. A consequence of this is that $H_*(x)$ is a topological invariant. If $X \cong Y$, then $H_n(X) \cong H_n(Y)$. The proof is easy. If

$f : X \to Y$ is a homeomorphism, let $g = f^{-1}$. Then $g \circ f = \mathrm{id}_X$. Hence, $(g \circ f)_* = (\mathrm{id}_X)_*$ implying $f_* \circ g_* = \mathrm{id}_{H_*(X)}$ and similarly $g_* \circ f_* = \mathrm{id}_{H_*(Y)}$. More generally,

Theorem 2.1.9. *If $f \simeq g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.*

Theorem 2.1.10. *If $X \to Y$ is a homotopy equivalence, then $H_n(X) \cong H_n(Y)$.*

For example, if $X \simeq *$ is contractible, then $\tilde{H}_n(X) = \tilde{H}_n(*) = 0$. We use homological algebra to pursue this proof.

Definition 2.1.11 (Chain Homotopy)**.** *We say that $\psi, \phi : C_* \to C'_*$ are chain homotopic if there is a chain homotopy $P : C_* \to C'_{*+1}$ such that*

$$\partial' P + P\partial = \psi - \phi.$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_n \arrow[dl, "P"'] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] \\
C'_{n+1} \arrow[r, "\partial'"] & C'_n
\end{tikzcd}$$

Lemma 2.1.12. *If $\phi, \psi : C_* \to C'_*$ are chain homotopic, then $\phi_* = \psi_*$ for all n.*

Proof. Given a chain homotopy $P : C_* \to C'_*$, let $x \in C_n$ be a cycle. Then,

$$\partial' P(x) + P\partial x = \psi(x) - \phi(x) \implies \partial'(P(x)) = \psi(x) - \phi(x).$$

So, $[\phi(x)] = [\psi(x)]$ in $H_*(C'_*)$, that is,

$$\phi_*([x]) = \psi_*([x]).$$

Proof. Of the previous theorem. We will prove by building a chain homotopy between $f_\sharp$ and $g_\sharp$. We already have a homotopy $H : X \times I \to Y$ between f and g.

If $\sigma : \Delta^n \to X$ is a generator of C_n, we want P such that $P\sigma \in C_{n+1}(Y)$.

We almost have exactly what we want. We already have

$$\Delta^n \times I \overset{(\sigma,\mathrm{id})}{\to} X \times I \overset{H}{\to} Y.$$

The problem we face is that $\Delta^n \times I$ is not an $n+1$ simplex. The intuition here is that the square is the union of triangles[1].

So, given $H : X \times I \to Y$ and $\sigma : \Delta^n \to X$, we define $P : C_n(x) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma, \mathrm{id})|_{[v_0 \dots v_i w_i \dots w_n]}.$$

We want to see $g_\sharp - f_\sharp - P\delta = \delta P$.

$$\begin{aligned}\partial P(\sigma) &= \partial(\sum_i (-1)^i H \circ (\sigma, \mathrm{id})|_{v_0 \dots v_i w_i \dots w_n}) \\ &= \sum_{j \le i} (-1)^{i+j} (\sigma, \mathrm{id})|_{v_0 \dots \hat{v}_j \dots v_i w_i \dots w_n} \\ &\quad \sum_{j \ge i} (-1)^{i+j+1} (\sigma, \mathrm{id})|_{v_0 \dots v_i w_i \dots w_j \dots w_n}.\end{aligned}$$

For the $i = j$ case, all terms cancel except for the $i = 1, j = n$ terms. What's left is

$$(H \circ (\sigma, \mathrm{id}))_{[w_0 \dots w_n]} - H \circ (\sigma, \mathrm{id}))_{[v_0 \dots v_n]} = g \circ \sigma - f \circ \sigma = g_\sharp(\sigma) - f_\sharp(\sigma).$$

We hope that the rest of the sum, $j \ne i$, yields $-P\partial(\sigma)$.

We know that

$$\begin{aligned}P(\partial\sigma) &= P(\sum_j (-1)^j \sigma|_{v_0 \dots \hat{v}_j \dots v_n}) \\ &= \sum_{i<j} (-1)^i (-1)^j H \circ (\sigma, \mathrm{id})|_{[v_0 \dots v_i w_i \dots \hat{w}_j \dots w_n]} + \\ &\quad \sum_{i>j} (-1)^{i-1} (-1)^j H \circ (\sigma, \mathrm{id})|_{[v_0 \dots \hat{v}_j \dots v_i w_i \dots w_n]}.\end{aligned}$$

This is left as an exercise.

Theorem 2.1.13. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an induced isomorphism for all n.*

[1] I used this idea on the homework problem that wasn't to turn in!

What we're building up to is that if we have X, a topological space with a Δ−complex, remarking about the difference between $\{\Delta_n(X), \partial_n\}$ and $\{C_n(X), \partial_n\}$. To be blunt: the difference is vast, as we have $\Delta_n(X) \hookrightarrow C_n(X)$. This is transparently a chain map, inducing a homomorphism $H_n^{\Delta}(X) \to H_*(X)$. Our fervent wish, which will eventually come true, is that this homomorphism is also an isomorphism.

For the moment, we think about the inclusion structure. This is what we'll refer to as a subcomplex.

Here's another example where this structure rears its head. If $A \subset X$ is a subspace, then it is natural to ask whether there's a relation between $H_*(A)$ and $H_*(X)$. We can view $\sigma : \Delta^n \to A$ as $i \circ \sigma : \Delta^n \to A \overset{i}{\hookrightarrow} X$. This realizes $i_\sharp(C_n(A))$ as a subgroup of $C_n(X)$. What we're saying here is that the boundary of X, when restricted to chains on A, remains in chains on A. Hence, $(C_*(A), \partial_*(A))$ is a subcomplex of $(C_*(X), \partial_*(X))$.

2.2 Week 5

2.2.1 Tuesday

Let's start with an algebraic definition.

Definition 2.2.1 (Subcomplex, Quotient Complex)**.** *Suppose we have a chain complex $\mathcal{B} = (B_n, \partial_n^B)$. A subcomplex of $\mathcal{B}$ is another chain complex $\mathcal{A} = (A_n, \partial_n^A)$ with an injective chain map $i : \mathcal{A} \hookrightarrow \mathcal{B}$.*

$$
\begin{tikzcd}
\cdots \arrow[r, "\partial^A"] & A_{n+1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_n \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_{n-1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & \cdots \\
\cdots \arrow[r, "\partial^B"'] & B_{n+1} \arrow[r, "\partial^B"'] & B_n \arrow[r, "\partial^B"'] & B_{n-1} \arrow[r, "\partial^B"'] & \cdots
\end{tikzcd}
$$

This setup yields a quotient complex $\mathcal{C} = (C_n, \partial_n^C)$ $C_n = B_n/i(A_n)$. We write $j : B_n \twoheadrightarrow C_n$. If $c = j(b)$, then $\partial_n^C(c) = \partial_n^B(b + iA_n) = \partial_n^B(b) + iA_{n-1}$.

$$
\begin{tikzcd}
\dots \arrow[r, "\partial^A"] & A_{n+1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"] & A_n \arrow[r, "\partial^A"] \arrow[d, hook, "i"] & A_{n-1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"] & \dots \\
\dots \arrow[r, "\partial^B"'] & B_{n+1} \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"] & B_n \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"] & B_{n-1} \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"] & \dots \\
\dots \arrow[r, "\partial^C"'] & C_{n+1} \arrow[r, "\partial^C"'] & C_n \arrow[r, "\partial^C"'] & C_{n-1} \arrow[r, "\partial^C"'] & \dots
\end{tikzcd}
$$

Or, tersely, a short exact sequence $0 \to \mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C} \to 0$.

Example. Let $i : A \to X$ be an inclusion of a subspace A into a space X. Now, $i_\sharp : C_*(A) \to C_*(X)$. This realizes $(C_*(A), \partial)$ as a subcomplex of $(C_*(X), \partial)$.

Now, we write $C_*(X, A) = C_*(X)/i_\sharp C_*(A)$. Evidently, ∂^C induces $\partial^{C,A}$. This means we have a short exact sequence $0 \to C_*(A) \to C_n(X) \to C_n(X, A) \to 0$. For the sake of consistency, name $j_\sharp$ as the onto map.

Definition 2.2.2 (Relative Homology)**.** *The homology of the complex* $(C_*(X, A), \partial)$ *is the relative homology an is denoted* $H_n(X, A)$.

Elements in $H_n(X, A)$ are represented by relative cycles: α an $n-$chain in X with $\partial\alpha$ in A. To be precise, $\alpha \in C_n(X)$ with $\partial\alpha \in i_\sharp C_n(A)$. A relative cycle α is trivial in $H_n(X, A)$ if $\alpha = \partial\beta + i_\sharp\gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Theorem 2.2.3. *If* $A \hookrightarrow X$ *by the inclusion map* i *is a subspace, there is a long exact sequence in singular homology*

$$\dots \longrightarrow H_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_n$$

We call δ *a connecting homomorphism.*

If $[\alpha] \in H_{n+1}(X, A)$ is represented by the relative cycle $\alpha \in C_n(X)$, then $\partial([\alpha]) = [\partial\alpha] \in H_n(A)$.

Proposition 2.2.4. *If* $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$ *is a short exact sequence of chain complexes of abelian groups, there is an associated long exact sequence in homology*

$$\dots \longrightarrow H_{n+1}(\mathcal{C}) \xrightarrow{\delta} H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{j_*} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A})$$

Proof. We define $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$. Take $[c] \in H_n(\mathcal{C})$ arising from a cycle $c \in C_n$, $\partial c = 0$. Hence, $c = j(b)$ for some $b \in B_n$. If we first take the boundary and apply j, $j\partial b = 0$. This says $\partial b \in \ker j$. As $\ker j = \operatorname{im} i$, there is an $a \in A_{n-1}$ so that $ia = \partial b$. We then define $\delta([c]) = [a]$. We should note that a is a cycle, as $i\partial a = \partial ia = \partial\partial b = 0$ and as i is injective, $\partial a = 0$.

We need to check well-definition of δ. The first thing we checked is, as i is injective, a is uniquely determined by ∂b. Suppose $c = j(b) = j(b')$. If we take their difference $b' - b$ and we apply j, we get 0. Hence, their difference is in the kernel of j. By exactness, we have a preimage $ia' = b' - b$. Hence, $b' = b + ia'$ so $\partial b' = \partial b + \partial ia' = \partial b + i\partial a' = ia + i\partial a' = i(a + \partial a')$.

Now, say $\delta[c] = [a + \partial a'] = [a]$.

We need to check if we took a diffent representative cycle for c, $c + \partial c'$. This c' is jb' for some b', so $c = \partial c' = c + \partial jb' = c + j\partial b = jb + j\partial b' = j(b + \partial b')$. But $\partial(b + \partial b') = \partial b = ia$ is not altered.

Time to check if δ is a homomorphism. If $\delta[c_1] = [a_1], \delta[c_2] = [a_2]$, we have $c_1 = jb_1, \partial b_1 = ia_1$ and so on. Hence, $c_1 + c_2 = j_1 b_1 + jb_2 = j(b_1 + b_2)$. Also, $\partial(b_1 + b_2) = \partial b_1 + \partial b_2 = ia_1 + ia_2 = i(a_1 + a_2)$ so $\delta[c_1 + c_2] = [a_1 + a_2] = [a_1] + [a_2] = \partial[c_1] + \partial[c_2]$.

We have six chores to prove this proposition, two for every map we have. Some are easy, some are involved.

Here's an easy one: $\operatorname{im} i_* \subset \ker j_*$ by short exactness.

Next: $\operatorname{im} j_* \subset \ker \delta$. $[b] \in H_n(\mathcal{B})$ represented by $b \in B_n$ with $\partial b = 0$. Now, $j_*([b]) = [j(b)]$. Then $\delta([j(b)]) = 0$ as $\partial b = 0$.

Let's go for $\operatorname{im} \delta \subset i_*$. $\partial[c] = [a]$ where $c = j(b)$, $\partial b = i(a)$. Then,

$$i_*(\partial c) = i_*([a]) = [i(a)] = [\partial b] = 0.$$

The next one–a new hard one–is $\ker j_* \subset \operatorname{im} i_*$. Suppose $j_*([b]) = 0$. Thus, $j_*[b] = [j(b)] = 0$. Therefore $j(b) = \partial c'$ for some c'. Well, $c' = j(b')$ for some b'. Thus, $\partial c' = \partial jb' = j\partial b'$. Compare $j(b - \partial b') = jb - j\partial b'$. Now, this all equals $\partial c' - \partial c' = 0$. This means that $b - \partial b' \in \ker k = \operatorname{im}_i$. $b - \partial b' = ia$. Hence, $[b - \partial b'] = [ia] = i_*[a] = [b]$.

Lastly, $\ker i_* \subset \operatorname{im} \delta$ and $\ker \delta \subset \operatorname{im} j_*$ are...homework! ❀

2.2.2 Thursday

Last time, we talked about what happens if we have an inclusion map from a subspace into a space. We then have the exact sequence of chain complexes

$$0 \longrightarrow C_n(A) \xhookrightarrow{i_\sharp} C_n \xrightarrow{j_\sharp} C_n(X,A) \longrightarrow 0$$

Yielding the long exact homology sequence of the pair (X,A),

$$... \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \longrightarrow ...$$

We can then talk about the reduced relative homology,

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & C(A) \arrow[r] \arrow[d, "\epsilon"] & C_0(X) \arrow[r] \arrow[d, "\epsilon"] & C_0(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

If $A \neq$;, the $\tilde{H}(X,A) = H(X,A)$.

If $A = \{*\}$, a point, then $\tilde{H}_n(A) = 0$ so $\tilde{H}_n(X) \cong \tilde{H}_n(X,A)$.

Another example. Let $(X,A) = (D^n, \partial D^n)$. Because $D^n \simeq *$, $\tilde{H}_n(D^n) = 0$. Moreover, $\tilde{H}_n(D^n, S^{i-1}) \cong \tilde{H}_{i-1}(S^{i-1})$ for all n, by long exactness. We will start seeing reasons why this group is isomorphic to $\mathbb{Z}$ for $i = n$ and 0 otherwise.

Let's make a comment about functorality of relative homology. A map of pairs $f : (X,A) \to (Y,B)$ induces a homomorphism $f_* : H_n(X,A) \to H_n(Y,B)$ for all n. Viewing $f : X \to Y$, we get $f_\sharp : C_n(X) \to C_n(Y)$ taking $C_n(A)$ to $C_n(B)$. So we get an induced $f_\sharp C_n(X,A) \to C_n(Y,B)$. Hence, we get an f_* on the relative homology.

We say that $f \simeq g : (X,A) \to (Y,B)$ if there exists a homotopy where $H(a,t) \in B$ for all $a \in A$ and $t \in I$. This induces the same map $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

A couple of comments on how one might proceed: Take the induced chain homotopy P from $f_\sharp$ and $g_\sharp$.

Going back to some old stuff. We can view $j_\sharp : C_n(X) \to C_n(X,A)$ as induced by $j : (X,*) \to (X,A)$ where j is the identity map on X and $j(*) \in A$.

For triples (X,A,B) that are stacked $B \subset A \subset X$, there is a long exact sequence in relative homology

$$... \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta} H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow$$

In the special case $B = *$, we get the long exact sequence of X, A.

Two fundamental properties of singular homology:

Definition 2.2.5 (Excision)**.** *If $Z \subset A \subset X$ with $\bar{Z} \in A$, then the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \cong H_n(X, A)$ for all n.*

Closely related is the

Definition 2.2.6 (Mayer-Vietoris Sequence)**.** *If $A, B \subset X$ with $X = A \cup B$, there is a long exact sequence*

$$\ldots \longrightarrow H_{n+1}(X) \xrightarrow{\delta} H_n(A \cap B) \xrightarrow{i_*, j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_* - \ell_*} H_n($$

$$i : A \cap B \hookrightarrow A \qquad k : A \hookrightarrow X$$

$$j : A \cap B \hookrightarrow B \qquad \ell : B \hookrightarrow X$$

In both of these contexts we can work with reduced homology.

Example. Let $X = S^n$ with A the "extended northern hemisphere" and B the "extended southern hemisphere".

Interestingly! $A, B \cong \circ D^n \simeq *$. Additionally, $A \cap B \cong S^{n-1} \times$.

Appendix A

List Of Definitions

Appendix B

List Of Theorems