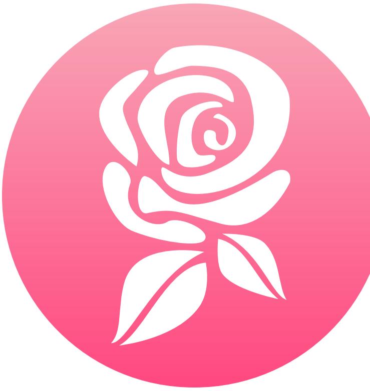
Diff. Geo.

Hailey Jay

February 13, 2025



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Information

Time & Room	TTh 09:30-, Lockett 232
Exam	May 7, 07:30
Textbook	Bredon, “Topology and Geometry”
Professor	Baldridge, Scott
Office	Lockett 380
Email	baldridge@math.lsu.edu
Homepage	?
Office Hours	?

A little bit about the professor: Mathematical physicist studying gague theory with spin-c bundles over 4-manifolds invariant under gague transforms.

Grade Distribution

Homework	50%
Midterm	30% (March 13)
Final	20%

Philosophy

Why am I here? Is it because it’s the next thing to do? Are you here because your mom and dad were very proud of you graduating from University Y?

I’m here because I love mathematics and I want to learn about geometry!

Why should you take good notes?

It’ll help structure your lectures in the future! Don’t take it on the basis of authority; take it for your own betterment.

Questions

(1) Can we find a mathematical model of objective reality? Moreover, what can this possibly mean?

With a TOE model, it should answer:

(2) What is time?

(3) Why does it have a direction?

(4) What is entropy?

(5) Why 3 spatial dimensions?

(6) Is the universe superdeterministic? Random? Other?

Math model

Let's build a mathematical model of an apple falling. Let's say that the mass of the apple is $136g$. But in reality...it's impossible to say. We can say $135.5g < m < 136.5g$, but not much more. Maybe we can tighten the bounds, but hell, it changes per atom.

Okay, let's look at gravity, $9.8m/s^2$. However, once again, this is a mere estimate!

Another pick, this being $4.9m$ above the ground. But hell, from where on the apple do we measure from? Apples aren't points! But I guess it's good enough for now.

Let's keep going. Let's set $v_0 = 0m/s$. But gosh, even this is a massive assumption.

We ignore air resistance, every dimension other than up and down, friction, the rotation of the earth, quantum fluctuations...it turns out that we're ignoring most factors in the universe.

Now, we can actually build the model, $s(t) = -4.9t^2 + 4.9$. Additionally, $t \geq 0$ and $s(t) \geq 0$. Finally, after all of this, we can write down a theorem:

The apple will hit the ground after one second.

In reality, though, the apple will hit the ground between .9 and 1.1 seconds.

Proof.

$$\begin{aligned} 0 &= -4.9t^2 + 4.9 \\ 0 &= -t^2 + 1 \\ \pm 1 &= t \\ \implies 1 &= t. \end{aligned}$$

This type of theorem is an absolute truth claim: "1 second *exactly*."

But in the objective reality, it's a tolerance truth claim: "between 0.9 seconds and 1.1 seconds."

We should not have the right to have this level of precision. We live in paradise.

Some people are excited about Excel spreadsheets! And that's fine! If that gets her up in the morning, that's fine!

Let's do the following.

Imagine: Suppose I have a math model that contains absolute truth claims, and that it is one to one with objective reality. Such a model should include...

- Classical mechanics;
- General relativity;
- Quantum mechanics;
- ??
- ??
- ??????????????

This would be, truly, a theory of everything.

Smooth manifolds are a great place to start!

Your professors are tying to take you to the abyss. Our entire life has been spent looking at shells on the seashore. The abyss is scary; but it's a lot of fun.

Chapter 1

Weeks 1-3

1.1 Week 1

1.1.1 Thursday

"Something is rotten in the state of teaching calc 1."

Class begins with a brief aside on space, belief, and mathematical philosophy. Math is plug and play like no other type of theory. Once you define what a circle is, with the definition Euclid gave us 2000 years ago, you can use that definition everywhere.

In this class, we're going to assume things are as nice as possible. The point of the reading is just a review; we're going to assume basically everything is C^∞. But even in the nicest case, we can still run into complications!

Traditional spaces include

(a) Euclidean space;

(b) $\mathbb{R}^n$;

(c) $\mathbb{C}^n$.

But now, we ask: What is Euclidean space?

Definition 1.1.1 (Affine Space)**.** *An affine space $A(E, V)$ is a set E together with a vector space V, and a transitive and free action of the additive group of V on E.*

We call $p \in E$ points and $v \in V$ vectors, translations, or free vectors.

The action is defined as

$$E \times V \to E; p, v \mapsto p + v$$

such that

(1) For all $p \in E$, $p + 0 = p$.

(2) For $v, w \in V$, and all $p \in E$, $(p + v) + w = p + (v + w)$.

(3) For every $p \in E$, $V \to E; v \mapsto p + v$ is bijective.

That together imply

(4) For all $v \in V$, there is a map $E \to E; p \mapsto p+v$ that is bijective.

(3) is usually stated as (5): For all $p, q \in E$, there exists a unique $v \in V$, denoted $p - q$, such that $q + (p - q) = q$.

Affine spaces are often characterized as $E \times E \to V; (p, q) \mapsto p - q$.

"You can only subtract points to free vectors."

Now, what are the basic objects of euclidean space? Well, points, lines, and planes. But these are undefined terms, so that we can end circular definitions. Axioms tell us basic relations between undefined terms. In ZFC, our single undefined term is a set.

Definition 1.1.2 (Euclidean Space)**.** *A Euclidean space is an affine space $\mathcal{A}(E, V)$ such that the associated vector space V is a finite dimensional vector space over the reals with*

- *An inner product $\langle , \rangle : V \times V \to \mathbb{R}$;*
- *A norm $\| : \|V \to \mathbb{R}$, $\|v\| = \sqrt{\langle v, v \rangle}$;*
- *A distance $d : E \to \mathbb{R}$, $d(p, q) = \|p - q\|$;*
- *Angles $m\angle v_1 v_2 = \arccos\left(\frac{\langle v_1, v_2 \rangle}{\|v_1\|\|v_2\|}\right)$;*
- *Dimension $E := \dim_{\mathbb{R}} V = n$.*

What about the topology of E? Well, because we have a distance, we can define $\overset{o}{B}_\varepsilon(p) = \{q \in E \,|\, d(p, q) < \varepsilon\}$. This means that $\mathcal{A}\left((E, d, \tau_d), (V, \langle , \rangle)\right)$

We can speak of continuous functions from E to other topological spaces. Firstly, limits.

Definition 1.1.3 (Limit). *Given $f : (E, d) \to (E', d')$ $\lim_{x\to c} f(x) = L$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x, c) < \delta$, $d(f(x), L) < \varepsilon$*

As soon as you have limits, you have derivatives.

Definition 1.1.4 (Derivative). *Let $f : E \to E'$ with $v \in V$. Then*

$$D_v f(p) = \lim_{t\to 0} \frac{f(p + tv) - f(p)}{t}.$$

Hence $D_v f : E \to V'$

What are the coordinates in (E^2, d)? Step one: pick a point, any point. Label it 0. We'll call it the origin. Step 2: construct two lines that intersect at the origin; make them perpendicular. Call them axes, label them x and y. Step 3: Use the distance function to put coordinates on the lines, one to one with the real numbers. Define coordinates $(a, b) \mapsto$the intersection of xa and yb.

Thus, E^2 is the underlying space $\mathbb{R}\times\mathbb{R}$ with no algebraic structure. We really have $\mathcal{A} = (E^n, V^n)$ which is the same as, after identifications have been made, $E = \mathbb{R}$ and $V = \mathbb{R}$.

1.2 Week 2

1.2.1 Tuesday

This is the snow week! Therefore, we're all on Zoom, but we're still taking notes. Last time, we started with an affine space $\mathcal{A}((E, d, e_d), (v, \langle\cdot,\cdot\rangle, \|\cdot\|))$ where E is a apsace and V is a vector space over the reals.

Moreover, E has a distance $d(p, q) = \|p - q\|$ that induces the topology τ_d.

With a topology, we can define a derivative in the v direction. Given $f : (E, d) \to (E', d')$ and $v \in V$, we define $D_v f : E \to V'$ by

$$D_v f(p) = \lim_{t\to 0} \frac{f(p + tv) - f(p)}{t}.$$

We can wrap this up as

$$TE := E \times V$$

Where TE is the tangent space. Differential geometry starts here!

We can more or less place a coordinate system on E, by $E \sim \mathbb{R}^n$ (roughly) and $V \sim \mathbb{R}^n$ (as a vector space over $\mathbb{R}$, with the standard orthonormal basis).

Definition 1.2.1 (Tangent Space)**.** *We define*

$$TE := E \times V$$

where $T_p = \{p\} \times V$*. We have the action* $E \times V \to E; (p, v) \mapsto p + v$*.*

All of this gives us one model for space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, \|\cdot\|, d, \tau_d)$. From this model, we can get the old model just by throwing away the topology and algebraic structure in turn to get the algebraic structure and the topology.

Now, this means that $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with a lot of assumptions.

Our choice of coordinate system can change! For example, (M, d_m) as a second countable Hausdorff space homeomorphic to $(\mathbb{R}^2, \tau_d)$ via φ.

We can use φ to put coords on M, via $(x, y) \mapsto \varphi(x, y) \in M$. Later, we show that M is not necessarily Euclidean space.

But we could've chosen a completely different coordinate system with a different function! If ψ is another such function, we've induced the change of coordinates $F = \psi\varphi^{-1} : E \to E'$.

Moreover, F induces a map $F_* : V \to V'$ by $(a, b) \mapsto a\varphi(1, 0) + b\varphi(0, 1)$.

Directional derivatives give us a way to calculate T_*. Because we have the action $((x, y), te_1) \mapsto (x + t, y)$, we have

$$F_{e_1}F(x, y) = \lim_{t\to 0} \frac{F(x + t, y) - F(x, y)}{t}.$$

Now, I didn't copy the specific example, the affine transformation F being $T_{10,10}\sqrt{2}R_{45}$, but calculating this out gives $F_* = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$.

In general, $D_{ae_1+be_2}F(p) = aD_{e_1}F(p) + bD_{e_2}F(p)$.

Let's fix p and let the vector vary, $D_\bullet F(p) : V_p \to V'_{F(p)}$. This is a linear transformation!

Furthermore, $F(x, y) = (x - y + 10, x + y + 10)$, and

$$F_*|_p = \begin{pmatrix} \frac{\partial F_1}{\partial x}\big|_p & \frac{\partial F_1}{\partial y}\big|_p \\ \frac{\partial F_2}{\partial x}\big|_p & \frac{\partial F_2}{\partial y}\big|_p \end{pmatrix}$$

.

This is basically going to hold in general; it's the Jacobian!

But what do we notice about F_*? It's 2, relating to our scale factor...in particular, it's not 0. Moreover, $D_\bullet F(p)$ is a matrix and $F \in C^\infty(\mathbb{R}^2)$. We should think that $F_*|_p = D_\bullet F(p)$.

However, these coordinate systems do not really need to make sense on M! Ultimately, we need to make sure that φ, ψ are local homeomorphisms. That's not a big deal. We'll call these maps "charts". Additionally, we need to make sure that $\varphi\psi^{-1}$ and $\omega\varphi^{-1}$ are C^∞ and we call them transition functions.

If $F = \psi\varphi^{-1}$, we say $F(x_1, x_2, \ldots, x_n) = (F_1(\ldots), \ldots, F_n(\ldots))$, and $F_*|_p$ is the jacobian.

The point: Charts that satisfy the first two conditions allow us to put coordinates on M that are compatible with eachother. In particular, charts help us work with functions of M by

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbb{R} \\ \mathbb{R}^2 \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

We say a smooth function of M is a function $f : M \to \mathbb{R}$ such that for all charts φ, $\tilde{f}$ is C^∞

1.2.2 Thursday

Last time, we saw that for M homeomorphic to $E = \mathbb{R}^n$, we can define coordinates on M via homeomorphisms from $\mathbb{R}^n$ to M. In particular, we needed that the maps $\varphi, \psi : M \to \mathbb{R}^n$ are homeomorphisms (charts) and that the transition functions $\psi\varphi^{-1}$ and $\varphi\psi^{-1}$ are C^∞.

We observed that for all $p \in E$, $F_*(p) : V_p \to V'_{F(p)}$ is a nonsingular linear isomorphism given by $D_\bullet F(p)$. If $F = \psi\varphi^{-1}$, and we say that F acts componentwise ($F = F_1 \times F_2 \times \ldots \times F_n$), then

$$F_*|_p = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}|_p & \cdots & \frac{\partial F_1}{\partial x_1}|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}|_p & \cdots & \frac{\partial F_n}{\partial x_n}|_p \end{pmatrix}$$

New stuff now!

How do we model an open cylinder? Well, we can use open sets U_i of $\mathbb{R}^n$ to model disc-like chunks with some overlap!

Definition 1.2.2 (Smooth Manifold)**.** *A smooth n-dimensional manifold is a 2nd countable Hausdorff topological space with a collection of maps, called charts, satisfying*

(a) *Each chart is a homeomorphism from an open set of M to an open set of $\mathbb{R}^n$.*

(b) *Each $x \in M$ is in the domain of some chart.*

(c) *For charts $\varphi_i : U_i \to \varphi_i(U_i)$ for $i = 1, 2$, then $\varphi_2\varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is C^∞.*

(d) *The collection of all charts for M is maximal with respect to the above criteria.*

A collection of charts was once called an atlas. Manifolds locally look like Euclidean space.

Some examples:

(a) S^n;

(b) T^2;

(c) Σ_n;

(d) If $g : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla g|_p \neq 0$ for all $p \in g^{-1}(0)$, then $M = g^{-1}(0)$ is an $n-1$ dimensional smooth manifold. For S^n, we may use $g(x) = x_1^2 + \ldots + x_n^2 - 1$.

Let's do an explicit construction, $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Let's take $U_1 = S^n - N$ (where N) is the north pole) and $U_2 = S^n - S$ (where S) is the south pole). Define $\varphi_1 : U_1 \to \mathbb{R}^n$ by $\varphi_1(x_1, x_2, \ldots, x_{n+1}) = \frac{2}{x_{n+1}+1}(x_1, \ldots, x_n)$. This is defined as $x_{n+1} = -1$ corresponds exactly to the south pole. We can construct an inverse, so it's a homeomorphism; however, the construction is mildly tedious so I omit it from the notes.

$$\varphi_1^{-1}(y_1, \ldots, y_n) = \frac{4}{\sum_i y^2 + 4}(y_1, \ldots, y_n, 1 - \frac{\sum y_i^2}{4}).$$

We also define $\varphi_1(x_1, x_2, \ldots, x_{n+1}) = \frac{-2}{1-x_{n+1}}(x_1, \ldots, x_n)$.

We need to check whether $\varphi_2\varphi_1^{-1}$ is C^∞. We can explicitly calculate out that it is given by

$$\frac{4}{\sum y_i^2}(y_1, \ldots, y_n)$$

which on every point of $\mathbb{R}^{n-1} - 0$ is C^∞.

Hence, $S^n/x \sim a(x)$ is a smooth manifold called $\mathbb{R}P^n$. Similarly, $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ is complex projective space.

1.3 Week 3

1.3.1 Tuesday

Last time we defined a smooth n-dimensional manifold as a 2nd countable Hausdorff topological space M with a collection of charts such that

(a) Each chart is a homomorphism from an open subset of M to an open subset of $\mathbb{R}^n$.

(b) Each $x \in M$ is in the domain of a chart.

(c) For charts $\phi_i : U_i \to \phi_i(U_i)$,

$$\phi_j \circ \phi_i \in C^\infty.$$

(d) The collection of all charts is maximal with respect to the above.

Definition 1.3.1 (Quotient Space)**.** *Consider X a topological space, $\sim$ as an equivalence relation on X. Define $Y = X/\sim = \{[x] \mid x \in X\}$ where $[x]$ is the equivalence class of x. The map π takes $\pi(x) = [x]$. We define a topology on Y wherein $V \in Y$ is open if and only if $\pi^{-1}(V)$ is open. It's the largest topology that makes π continuous!*

Theorem 1.3.2 (Quotient of Smooth Group)**.** *If M is a connected smooth manifold and Γ is a discrete group acting smoothly, freely, and properly on M, then the quotient M/Γ has a unique smooth structure such that $\pi : M \to M/\Gamma$ is smooth.*

We say that $\Gamma \times M \to M$ is proper if $\pi^{-1}(C)$ is compact for C compact.

We say that Γ acts freely if non-identity elements do not fix every point.

For example, let $\mathbb{Z}_2 \times S^n \to S^n$ given by $a(x) = -x$ acting freely, smoothly, and properly. Then, $S^n/\mathbb{Z}_2 \cong \mathbb{R}P^n$.

As another example, $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R} \times \mathbb{R}$ by $(x, y) \sim (n + x, m + y)$ for $(n, m) \in \mathbb{Z} \times \mathbb{Z}$. We can set $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mathbb{R}^2 \xrightarrow{\pi} T^2$.

Similarly, $\mathbb{R}/\mathbb{Z} = S^1$. We can take $f : \mathbb{R} \to \mathbb{C}$ via $t \mapsto e^{2\pi i t}$.

Definition 1.3.3 (Product Manifold)**.** *Let X be a smooth $n-$manifold and Y be a smooth m-manifold. Then $X \times Y$ is a smooth $n+m$-manifold. The charts of this space are $\varphi, \psi : U \times V \to \mathbb{R}^n \times \mathbb{R}^m$ where φ is a chart on U and ψ is a chart on V.*

For example, $S^1 \times S^1 = T^2$. Does the torus have a smooth map to the knotted torus? Well, yes, actually. If we have a chart of the torus $\varphi : U \to \mathbb{R}^2$ and a chart on the knotted torus $\psi : V \to \mathbb{R}^2$, then we can have a smooth map f from the image of φ to the image of ψ inducing $\psi f \varphi^{-1}$.

But now we have to ask the question: What is smooth? Well...

Definition 1.3.4 (Smooth). *A map $f : M^m \to N^n$ between two smooth manifolds is said to be smooth or differentiable or C^∞ if for any charts φ on M and ψ on N, then the function $\psi f \varphi^{-1}$ is C^∞ where it is defined.*

For $f : S^1 \hookrightarrow \mathbb{R}^3$ a smooth embedding, S^1 is diffeomorphic to $f(S^1)$. However, $\mathbb{R}^3 \backslash f(S^1)$ may not be diffeomorphic to $\mathbb{R}^3 \backslash S^1$.

Definition 1.3.5 (Diffeomorphism). *A diffeomorphism is a map $f : M \to N$ is called a diffeomorphism if it is a bijection, f is diffentiable, and $f^{-1} : M \to N$ is differentiable.*

Facts: A closed topological 2-manifold has exactly one smooth structure up to diffeomorphism.

This fact also holds for 3 dimensions; smooth closed 3 manifolds have the property that a C^1 map can extend to a C^∞ map. However, C^0 between S^3 is the Poincaré conjecture.

The smooth Poincaré conjecture, which is still open, is as follows: if $F \cong_{\text{homeo}} S^4$, is $X \cong_{\text{diffeo}} S^4$? We know that $X \cong \mathbb{R}^n$ implies X diffeomorphic to $\mathbb{R}^n$ is true for all $n \neq 4$. This statement is false for 4; such counterexamples are called exotic $\mathbb{R}^4$'s.

A statement that our professor proved back in $'07$: There exists a 4-manifold X such that $X \cong_{\text{homeo}} \mathbb{C}P^2 \# 3\mathbb{C}P^2$ but $X! \cong_{\text{diffeo}} \mathbb{C}P^2 \# 3\mathbb{C}P^2$.

Definition 1.3.6 (Connect Sum). *Suppose X, Y are smooth n-manifolds with $x_0, y_0 \in X, Y$. Choose charts ϕ_x, ϕ_y at x_0, y_0 and $\epsilon > 0$ small enough such that $B_\epsilon(\phi_x(x_0)) \subset \operatorname{im} \phi_x$ and $B_\epsilon(\phi_y(y_0)) \subset \operatorname{im} \phi_y$.*

Define

$$F : \phi_x^{-1}(B_\epsilon(\phi_x(x_0))) \backslash \{x_0\} \to \phi_y^{-1}(B_\epsilon(\phi_y(y_0))) \backslash \{y_0\}$$

by $F(\phi^{-1}(x)(u, r)) = \phi_y^{-1}(u, \epsilon - r)$.

Then $X \# Y = ((X \backslash \{x_0\}) \sqcup (Y \backslash \{y_0\})) / F$.

The connect sums have these properties:

(a) $X \# Y \cong Y \# X$

(b) $X\#(Y\#Z) \cong (X\#Y)\#Z$

(c) S^n *is the identity.*

The sum is orientable when both spaces are orientable.

Definition 1.3.7 (Orientation)**.** *A smooth manfold is orientable if there exist charts* $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ *such that* $X = \bigcup_\alpha U_\alpha$ *and for all* α, β*, the transition function* $\phi_\beta \circ \phi_\alpha^{-1}$ *has positive Jacobian determinant.*

For example, S^2 is orientable. We have S^2 with $U_1 = \mathbb{C}$ and $U_2 = \mathbb{C}$ with transition map $\tau : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ by $\tau(z) = 1/z$.

Let $z = x + iy$. Then,

$$\begin{aligned}
\tau(x+iy) &= \frac{1}{x+iy}\frac{(x-iy)}{(x-iy)} = \frac{\overline{z}}{|z|^2} \\
&\sim= \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right) \in C^\infty \\
\tau_*(x,y) &= \begin{pmatrix} \frac{\partial \tau_1}{\partial x} & \frac{\partial \tau_1}{\partial y} \\ \frac{\partial \tau_2}{\partial x} & \frac{\partial \tau_2}{\partial y} \end{pmatrix} \\
&= \frac{1}{(x^2+y^2)^4}\begin{pmatrix} y^2-x^2 & -2xy \\ 2xy & y^2-x^2 \end{pmatrix} \\
\implies \det \tau_*(x,y) &= \frac{(x^2+y^2)^2}{(x^2+y^2)^4} \\
&= \frac{1}{(x^2+y^2)^2} > 0.
\end{aligned}$$

If X is oriented, we denote $\overline{X}$ as X with opposite orientation.

1.3.2 Thursday

The need for quantity spaces or...

We need to build up to the notion of a tangent space.

First of all, "tangent space" should have "tangents," like the tangent line to the circle. The line at a point is literally the tangent space; we literally assign a space to each point. For $z = 1$, we'd denote it $T_1 S^1 \cong i\mathbb{R}$.

Where else do we talk about tangents? Calc 1!

Let's show how absurd this is to calc 1 standards. Going back to our original model, with the apple falling from the tree: $s(t) = 4.9 - 4.9t^2$, $0 \le t \le 1$.

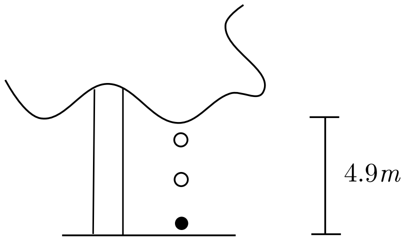

Figure 1.1: Our original model.

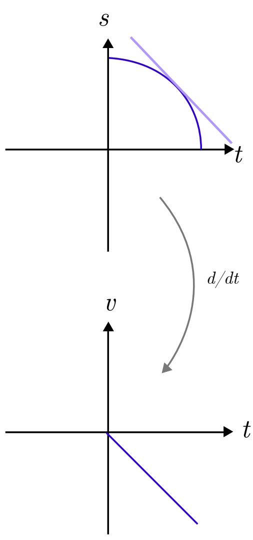

Figure 1.2: The derivative of s.

Recall that $v(t) = s'(t) = -9.8t$, which is our instantaneous velocity.

There are deeper issues still waiting to be exposed!

To see, we have to put units into the function $s(t) = 4.9 - 4.9t^2$.

$$s(t\,sec) = 4.9m - \frac{1}{2}(9.8m/sec^2)(t\,sec)^2$$

Physicists have a function denoted by brackets that pull out the unit.

Next, take the derivative with respect to $t\,sec$.

What problems do we see now?

(1) Why is $m/s^2 \cdot sec = m/sec$?

(2) $\frac{d}{d(t\,sec)} = \lim_{h\ \mathrm{sec} \to 0} \frac{s(t\,sec - h\,sec) - s(t\,sec)}{h\,sec}$

(3) What the hell is $5m/2\,sec$?

(4) What is $2.5m/sec$?

(5) What the hell is a quantity?

Context, and language, matter.

(1) Otto Hölder: "The axioms of Quantity and the theory of measurement." (1901) He was the first to come up with axioms of quantity. The main Axioms of Qualities are:

    (a) We may add two quantities;
    (b) We may compare two quantities;
    (c) We may multiply by a scalar in $\mathbb{R}^+$.

    The other axioms are to make everything nice.

(2) Hassler Whitney: "The mathematics of physical quantities." (1968) Quantity structure is modelled on an oriented, one dimensional vector space over $\mathbb{R}$. Now he can define square roots, subtraction, etc.

(3) Modern: Baldridge-Madden. This is what we'll do on Tuesday!

Chapter 2

Weeks 4-6

2.1 Week 4

2.1.1 Tuesday

Definition 2.1.1 (Quantity Space). *A quantity space with metric is modeled on an oriented, 1-dim, oriented, linear product space V over* $\mathbb{R}$*:*

$$(\mathbb{R},\ \langle \cdot, \cdot \rangle,\ \| \cdot \|,\ d, \tau_d,\ , \mathcal{O}_{std}).$$

Why $\mathbb{R}$? Well, we need order and metric completeness.

For example, let's look at the type of quantity known as length. To choose a 1, we take the vector v so that $2\langle v, v\rangle = 1$. We cal $v \in V^+$ such that $\|v\| = 1$ the unit. If we say our length is in meters, then the unit is one meter. Thus, $V_m = \{am \mid a \in \mathbb{R}\}$ and $\langle am, am = a^2\rangle$. We have another function, sending $V_m \to \mathbb{R}$: $\langle am, 1m\rangle = a$

Another examle, the quantity type of $V_m \otimes V_m = V_{m^2}$ is area in m^2. Think: if we have $\langle 3m \otimes 4m, 3m \otimes 4m\rangle$, it's equal to $\langle 3m, 3m\rangle\langle 4m, 4m\rangle = 9 \cdot 16 = 144$. So whatever that vector is, $\|3m \otimes 4m\| = 12$

Now we talk about product and sum spaces. What is $V \otimes W$ First, what is $V \oplus W$?

Let's put $V = \text{span}\ \{e_1, e_2\}$ and $W = \text{span}\ \{f_1, f_2, f_3\}$. Then,

$V \oplus W = \text{span}\,\{(e_1, 0), (e_2, 0), (0, f_1), (0, f_2), (0, f_2)\}$ so it has dimension 5. To formalize,

Definition 2.1.2 (Direct Sum)**.** *$V \oplus W$ is $V \times W$ with the following equivalence relations:*

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad a(v_1, w_1) = (av_1, aw_1).$$

The vector space is of dimension $\dim V + \dim W$.

Now we can really talk about the tensor product.

Definition 2.1.3 (Tensor Product)**.** *Given two vector spaces V, W, we define*

$$V \otimes W = F(V \times W)/\sim$$

where F is formally

$$\sum_i a_i(v_i, w_i)$$

and $\sim$ is defined by

- *(v,w)⊠(v, w)*
- *a(v, w)⊠(av,w)⊠(v,aw)*
- *(v,w)+(v',w)⊠(v+v',w)*

In $V \otimes W$, we write (v, w) as $v \otimes w$. We 1have the relations as defined above, just now with $\otimes$. We have that every tensor in $V \otimes w$ may be written as

$$\sum_i a^i(v^i \otimes w^i) = \sum_i a^i(v^i_1 e_1 + v^i_2 e_2) \oplus (w^i_1 f_1 + w^i_2 f_2 + w^i_3 f_3)$$

which equals ultimately a massive thing that looks like a matrix. Hmmmm...

In general, $V \otimes W = \text{span}\,\{v \otimes w \,|\, v, w \text{ are from selected bases for } V, W\}$, so $\dim V \otimes W = \dim V \cdot \dim W$.

The universal property of tensor spaces is this (copied mostly verbatim from wikipedia)

Definition 2.1.4 (By Universal Property)**.** *The tensor product of two vector spaces V and W is a vector space $V \otimes W$ along with a bilinear map $\otimes(v, w) \mapsto v \otimes w$ from $V \times W$ to $V \otimes W$, such that, for every bilinear map $h : V \times W \to Z$, we have a unique linear map $\tilde{h} : V \otimes W \to Z$ such that the following diagram commutes:*

$$\begin{tikzcd} V \times W \arrow[r, "\phi"] \arrow[dr, "h"'] & V \otimes W \arrow[d, dashed, "\exists!\,\tilde{h}"] \\ & Z \end{tikzcd}$$

Let's verify that this is a quantity space. Because V and W are both 1-dimensional, then $V \otimes W = 1$. Also if we have inner products on both spaces $\langle , \rangle_V$, $\langle , \rangle_W$, then we have an inner product on $V \otimes W$ by $\langle v \otimes w, u \otimes t \rangle_{V \otimes W} = \langle v, u \rangle_V \langle w, t \rangle_W$.

Example. Take V_N and V_m where $N = kg \cdot m/s^2$. Then $V_N \otimes V_m$ is V_J, measuring Newton-meters in Joules. Incidentally, it's also equal to

$$V_{1/s} \otimes V_{1/s} \otimes V_k \otimes V_m \otimes V_m.$$

We're almost there. We want to figure out why we want $V_s \otimes V_{1/s} = \mathbb{R}$.

First, we need to talk about $\mathrm{Hom}(V, W)$, the set of all linear transformations from V to W. It really ought to be a tensor, because... well, it's a matrix. First, we need the dual space $V' = \mathrm{Hom}(V, \mathbb{R})$. Because V is one dimensional, we can represent every v' as $\langle \cdot, v \rangle : V \to \mathbb{R}$. We could also represent, by this logic, anything in $\mathrm{Hom}(V, W)$ as $w \langle \cdot, v \rangle$. Hey, wait a minute... this means that $\mathrm{Hom}\, V, W = V' \otimes W$. This holds more generally; as the Riesz representation theorem.

Moreover, we can see that $\mathrm{Hom}(V_s, V_m)$ has elements like, well, I don't know, $2m/s$. Hence, $\mathrm{Hom}(V_s, \mathbb{R})$ is the quantity space for $1/s$.

This allows us to make sense of $(1/sec)(t\,sec) = t$.

2.1.2 Thursday

Now, a measure space has a preferred unit u wherein $\langle u, u \rangle = 1$. For example, in the space denoted V_g, for grams, the unit is one gram. What is $V_g \to V_{kg}$? It's measurement conversion or, more rigorously, change of basis.

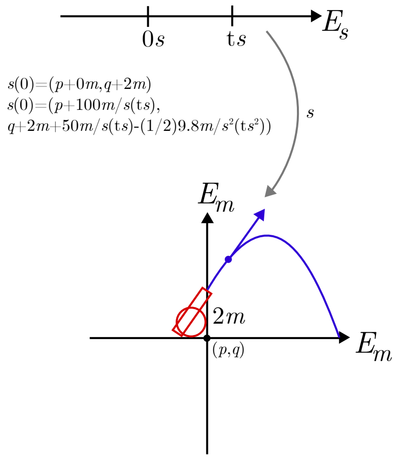

Figure 2.1: The cannon model.

We also talked abut hom-sets, and in particular, linear functionals. Let's say that

$$2.5m/s \in V_{m/s} = \mathrm{Hom}(V_s, V_m).$$

The dual space of V_s is V_s^*. Because $V_s^* = \mathrm{Hom}(V_s, \mathbb{R})$, we have a name for the unit. It's $V_s^* = V_{1/s}$!

We claim (towards the Riesz Representation Theorem) that

$$\mathrm{Hom}(V_s, V_m) = V_s^* \otimes V_m.$$

To check: $f \in \mathrm{Hom}(V_s, V_m)$, and $f(t\,s) = (2.5t)m$, so $f = 2.5(\frac{1}{s} \otimes m)$.

So now, finally, what is $5m/2s$? We can define division via $V_m \times V_s \backslash \{0\} \to V_s^* \otimes V_m$ wherein $(a\,m, b\,s) \mapsto \frac{a}{b}(1/s \otimes m)$.

What happens when we multiply like $V_{m/s^2} \otimes V_s$? Well,

$$V_{m/s^2} \otimes V_s = V_m \otimes V_s^* \otimes V_s^* \otimes V_s = V_m \otimes V_s^* = V_{m/s}.$$

Obviously, if we have something called V_{cm^2}, we'd very much like $\sqrt{cm^2} = cm$. Let's define *the* space $V_{\sqrt{u}}$ such that

$$V_{\sqrt{U}} \otimes V_{\sqrt{U}} = V_U.$$

We can now express what Calc 1 should look like.

Let's consider $V_s|_{0+t\,s} \xrightarrow{s_*} V_m \oplus V_m|_{s(ts)}$. We compute

$$\frac{d}{dt\,s}s(t\,s) = (100m/s, 50m/s + 9.8m/s^2(t\,s))$$

which is in the space

$$(V_m \otimes V_m) \otimes V_s^*.$$

But... it'd be really nice just to write $(100, 50-9.8t)$. So eventually we'll just forget all of this vector space stuff.

Even though the map itself isn't going to be linear, the induced map on the units WILL be linear when we restrict to a specific time input. Think derivatives. Neat!

One last thing (minute left in class).

The "wrapped up" object is the tangent space of E in meters. We said

$$TE = E \times V.$$

But it's actually...

$$TE = E \times (V_{m/s} \oplus ... \oplus V_{m/s})$$

where $\dim E = n$.

2.2 Week 5

2.2.1 Tuesday

Last time, we looked at the object $L_u = (\mathbb{R}, \langle,\rangle, \|\cdot\|, d, \tau_d, \mathcal{O}_{std})$ where u is chosen to be the unit. We call this Euclidean space. We could, if we wanted, split this into a vector space and an affine space.

If we make it one dimensional, we can think of it as a line cross a vector space.

Because we have an orientation, we know which direction is positive.

We also have unit conversions which are change of bases of our vector spaces.

At this point, he stressed the difference between projection and action. The object we're going to study is the projection, rather than the action.

The tangent space is a different space than the previous space. We take two units and we take a "per" unit. I.E., kg/hr. We're really going to restrict to m/s.

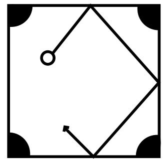

Figure 2.2: A unit pool table.

Our tangent space is the product of a bunch of copies of our euclidean space, product with the direct sum of $V_{m/s}$. We'd call $TE = \hat{E} \times \hat{V} = (E_m \times E_m \dots \times E_m) \times (V_{m/s} \oplus \dots \oplus V_{m/s})$.

The first component is the potential positions of an object, denoted by q. The secont part is all potential velocities of the object at a particular position, denoted q.

New stuff. This leads to the Lagrangian, $\mathcal{L} : T\hat{E} \times \mathbb{R} \to \mathbb{R}$. In order to capture all the physics we need this bigger space. We write this $(q, q, t) \mapsto \mathcal{L}(q, q, t)$. We know Lagrangians as $L = K - U$, kinetic energy minus potential energy, but we're ALSO familiar with Lagrangians as defined by Hamilton as $H = K + U$.

Let's do an example: A $[0, 1] \times [0, 1]$ pool table.

Th configuration space is, naturally, $[0, 1m] \times [0, 1m]$. The celocity phase space is

$$(I_m \times I_m) \times ((V_m \oplus V_m) \otimes V_s^*).$$

We've just written down all possible positions and all possible velocities.

This makes sense; if we wanted to talk about $3m$ as part of the vector space, we merely couldn't. It just doesn't work, it's out of the domain! But talking about $3m/s$ is a lot more natural. Here, we just parametrize the positions!

If we were to do another ball, the configuration space suddenly jumps to $(I_m \times I_m) \times (I_m \times I_m) = \hat{E}$. Now, the velocity phase space is

$$(I_m \times I_m) \times (I_m \times I_m) \times ((V_m \oplus V_m) \otimes V_s^*) \times ((V_m \oplus V_m) \otimes V_s^*)$$

Then, $\gamma : E_s \to \hat{E}$ is given by $\gamma(t\, s) = (\gamma_1^1, \gamma_2^1, \gamma_1^2, \gamma_2^2)$. When, if at all, do they touch? Well... The diagonal. The place where their positions equal. We point that out because this is used as a key moment in a famous proof.

If we return to the example from the first lesson, Figure 1.3.2, we can visualize $-4.9m/s$ as either a map or a vector. The map as a member of, say, $\mathrm{Hom}(V_s|_{1/2s}, V_m|_{3.675m})$, and the vector by $V_m \otimes V_s^*$.

Now we return with a vengance to the book. Here comes homeworks!

Let's put all of this on a manfold. Let's take a copy of $\mathbb{R}$, imagining our head that it's E_s, and pick a 0. Now we look at a map γ from that into an actual manifold. The velocity of γ at any particular point really is a vector! If $\gamma(0) = p$, we'd want a chart around p, and our velocity is $\gamma'(0)$. The vector, "overlaid" on the configuration space, actually lives in the tangent space. We have $(\gamma(0), \gamma'(0) \in T_pM)$. Locally, T_pM is $\mathbb{R}^2 \times \mathbb{R}^2$.

We want to capture all paths through a point with a given velocity. Suppose $\gamma_1, \gamma_2 : (-1,1) \to M$ such that $\gamma_1(0) = \gamma_2(0) = p$. Let ϕ be a chart at 0. Then $\phi \circ \gamma_1, \phi \circ \gamma_2 : (-1,1) \to \mathbb{R}^n$ are C^∞. We declare γ_1 equivalent to γ_2 at t_0 if and only if the derivatives of $\phi \circ \gamma_1, \phi \circ \gamma_2$ are the same at $t = 0$.

This defines an equivalence relation $\sim$ on the set of all C^∞ curves at p. This equivalence class is what we'll call the tangent vectors.

2.2.2 Thursday

Last time we looked at $\gamma_1, \gamma_2 : (-1,1) \to M$ with $\gamma_1(0) = \gamma_2(0) = p$ such that $\phi \circ \gamma_1, \phi \circ \gamma_2$ are C^∞. We say that $\gamma_1 \sim \gamma_2$ at p if their derivatives coincide at 0.

The equivalence class of all such curves is called a tangent vector. The class is called $\gamma'(0)$ for any such curve. We say that T_pM is the set of all tangent vectors at p, read as the tangent space of M at p.

It's easy to see how to put a vector space structure on T_pM, we say $T_pM \cong \mathbb{R}^n$ by thinking of it as a velocity phase space, $V_{m/s} \oplus \ldots \oplus V_{m/s}$.

Choose a chart $\phi : U \to \mathbb{R}^n$, $\phi(p) = (0, 0, \ldots, 0)$, and a path $\gamma : (-1,1) \to M$ such that $\gamma(p) = p$, $\gamma'(0) \in T_pM$. Define $d\phi_p : T_pM \to \mathbb{R}^n$ by $d\phi_p(\gamma'(0)) = \frac{d}{dt}(\phi \circ \gamma)(t)|_{t=0}$. This map $d\phi_p$ is onto and $1-1$. Does this depend on γ? Nope. It's an equivalence class of all such γs that has that property. Does it depend on ϕ? No again. Once again, choosing any chart with that property also puts us in an equivalence class.

We can think of $v \in T_pM$ as just an n-tuple after choosing a chart!

Let's go on. Another important way to think about tangent vectors is to think about them as directional derivatives.

Definition 2.2.1 (Derivation Operator)**.** *A derivation operator on C^∞ is a linear operator D such that $D(fg) = (Df)g + f(Dg)$.*

Functions in physics are called observables. Let's go to our cue ball example, where $M = I_m \times I_m$. Let $f : M \to V_m$, $f(xm, ym) = (xm - pm)$, that is, the distance from p in the $x-$direction. Given $v = (2m/s, 3m/s)$, let $\gamma(t\,s) = (p_1 m + 2(m/s)(t\,s), p_2 m + 3(m/s)(t\,s))$.

Define

$$\begin{aligned} D_v f|_p &= \frac{d}{dt\,s} f(p_1 m + 2m/s(t\,s)|_{t=0}, p_1 m + 3m/s(t\,s)) \\ &= \frac{d}{dt\,s}(2m/s(t\,s)) = 2m/s. \end{aligned}$$

This is the component velocity in the $x-$direction for a particle moving through p with velocity $(2, 3)$, speed $\sqrt{13}$.

Definition 2.2.2 (Directional Derivative)**.** *Let M be a smooth manifold and $\gamma : \mathbb{R} \to M$ a smooth curve with $\gamma(0) = p$. Let $f : U \to \mathbb{R}$ be a smooth function of an open set $p \in U \subset$, then the directional derivative of f along γ at p is*

$$D_\gamma(f)|_p := \frac{d}{dt} f(\gamma(t))|_{t=0}.$$

Relax the f component– $D_\gamma(\cdot)$–and this gives a tangent vector to γ at p, that is, $D_\gamma \in T_p M$.

When is $D_{\gamma_1} = D_{\gamma_2}$? When they are equivalent in the old way.

Let's do yet another view! Take the same γ as defined before and break it up into the components $\gamma_i(t\,s)$. Then take any f,

$$\begin{aligned} D_v f|_p &= \frac{d}{dt\,s} f(\gamma(t\,s))|_0 \\ &= \frac{\partial f}{\partial xm}|_p \cdot 2m/s + \frac{\partial f}{\partial ym}|_p \cdot 3m/s \\ &= ((2m/s) \cdot \frac{\partial}{\partial xm} + (3m/s) \cdot \frac{\partial}{\partial ym})f \\ &= X(f)|_p. \end{aligned}$$

Definition 2.2.3 (Germ)**.** *A germ of a smooth real valued function f at $p \in M$ on a smooth manifold M is the equivalence class of f under the relation $f_1 \sim_2$ if and only if $f_1(x) = f_2(x)$ for all $x \in U$ for some open U containing p.*

As an exercise, check that X is a derivation!

Also $X(p)$ is also a form of a tangent vector at p.

Let $TM = \bigcup_{p\in M} T_pM$, the tangent bundle of M. Then $X(p) \in T_pM$ varies smoothly over a manifold—a vector field.

Here's the point. Let's put a coordinate chart on it,

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[dr, dashed, "\tilde{\gamma}=\phi\circ\gamma"'] & M \arrow[r, "f"] \arrow[d, "\phi"] & \mathbb{R} \\
& \phi(U) \arrow[ur, dashed, "\tilde{f}=f\circ\phi^{-1}"'] &
\end{tikzcd}$$

Now we do a little slight of hand. With a coordinate chart, we can transfer γ to $\tilde{\gamma}$ and f to $\tilde{f}$. Then,

$$f(\gamma(t)) = f(\gamma_1(t), \dots, \gamma_n(t)).$$

Also,

$$\begin{aligned}
D_\gamma f|_p &= \frac{d}{dt} f(\gamma_1(t), \dots, \gamma_n(t)) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{d\gamma_i}{dt}|_{t=0} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot v_i \\
&= \left(\sum_{i=1}^n (v_i) \frac{\partial}{\partial x_i} \right) f.
\end{aligned}$$

Therefore, D_γ, in the presence of a chart, is $\sum_{i=1}^n (v_i) \frac{\partial}{\partial x_i}$ corresponding to $(v_1, \dots, v_n)$.

Appendix A

List Of Definitions

Appendix B

List Of Theorems