Topology II

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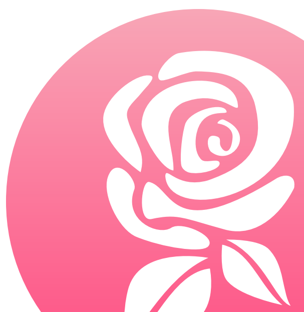

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Information

Time & Room	TTth 12:00-13:20, Lockett 232
Exam	?
Textbook	Hatcher, *Algebraic Topology*
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First assignment: Get your hand on the textbook! Additionally, read the introduction to Chapter 2.

Chapter 1

Weeks 1-3

1.1 Week 1

1.1.1 Tuesday

Theorem 1.1.1 (Brower Fixed Point for $n = 2$). *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Towards a contradiction, suppose f is a fixed point free map. Then $r(x) = \frac{x-f(x)}{\|x-f(x)\|}$ is a continuous map from $D^2 \to S^1$. Because r is a retract, we have the composition $D^2 \xrightarrow{r} S^1 \xrightarrow{i} D^2$ yielding the maps

$$\pi_1(D^2) \xrightarrow{r_*} \pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2).$$

However, as retracts should induce a surjection, this is a contradiction!

What about $n \geq 3$?

We want to study the relationship between topological spaces and algebraic objects. Maybe we associate groups, maybe rings...We want this association to be functorial. Roughly speaking, this means that spaces that are the same should get sent to the same objects! Another property of functorality is that continuous maps f should be sent to homomorphisms f_*.

In topology 1, we take a space X and assign to it its fundamental group $\pi_1(X)$. Roughly speaking, $\pi_1(X)$ is the set of homotopy classes of maps $S^1 \to X$. To generalize, we could look at the homotopy classes of maps $S^k \to X$. We denote this set $\pi_k(X)$ and we call them the higher homotopy groups. Additionally, they turn out to be abelian groups for $k \geq 2$. These are hard to determine!

A guess: Are these groups given by generators of faces and relations of 4-cells with a simplicial complex?

In this course, we're going to take X and take it to $\{H_k X\}$, homology, and $\{H^k X\}$, cohomology groups.

Let's denote pseudomanifolds $\mathcal{X}$.

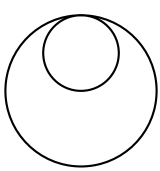
Definition 1.1.2 (Informal k–dimensional Manifold). *A space that looks locally like $\mathbb{R}^k$.*

Definition 1.1.3 (Informal k–dimensional $\mathcal{X}$-Manifold). *Can have singularities; points of being noneuclidean.*

Some properties:

1. *The part of P where it is a k–manifold is open, dense, and oriented.*

2. *The set of singularities has dimension $\leq k - 2$*



Definition 1.1.4 (k–Simplex). *A k-simplex is the convex hull of points $p_0 \dots p_k$ in general position in some Euclidean space. Its faces are $k-1$simplicies.*

Definition 1.1.5 (Simplicial Complex). *A simplicial complex is the set S of simplicies in some $\mathbb{R}^N$ satisfying*

- *Any face of a simplex in S is in S.*

- *Any two simplices in S are either disjoint or intersect in a set that is a face of both of them.*

Simplicial complexes are not pseudomanifolds; the edges are too big!

Definition 1.1.6 (Orientation on a Simplex)**.** *An orientation $\mathcal{O}$ in $\Delta^{k>0}$ $k-$simplex is an ordering of the vertices of each simplex. Two orderings are the same or equivalent if they differ by an even permutation.*

An orientation on a point is either a $+$ or $-$.

Hence, there are two orientations on any simplex; we call them "opposites."

The orientation has a concept of induced orientations; just negotiate the missing vertex to the last position via a permutation and delete it.

Definition 1.1.7 (k-dimensional Pseudomanifold)**.** *A $k-$dimensional pseudomanifold is a simplicial complex with a $\mathcal{O}(\Delta)$ orientation of each $k-$simplex such that*

1. *Every simplex is a face of a $k-$simplex.*
2. *Every $(k-1)$-simplex is a face of exactly two $k-$simplices.*
3. *Continuity of orientation: If Δ' is a $(k-1)$-simplex, face of $\Delta, \tilde{\Delta}$, $\mathcal{O}(\Delta)$ and $\mathcal{O}(\tilde{\Delta})$ must induce opposite orientations on Δ'.*

Definition 1.1.8 (i-cycle)**.** *Let X be a topological space. An i-cycle of X is a pseudomanifold of dimension i, P, and a map $\sigma : P \to X$ that "captures" a hole.*

For example, take P, an oriented triangle, and X an annulus. Map the triangle around the hole.

Let $\sigma_1 : P_1 \to X$, $\sigma_2 : P_2 \to X$ be i-cycles. Let $P_1 + P_2 = P_1 \sqcup P_2$, and σ be defined as you would think. We can also define $-P$; take the same psuedomanifold and same map, but put the opposite orientation on P.

Definition 1.1.9 (Pseudomanifold with Boundary)**.** *A simplicial complex Q and an orientation $\mathcal{O}(\Delta)$ on each $k-$simplex, along with a subsimplex B that satisfies:*

1. *Each simplex in Q is a face of a $k-$simplex*

2. *B a $k-1$ dimensional pseudomanifold;*
3. *Each $k-1$-dim Δ' not in B is a face of 2 k-simplices;*
4. *Each $k-1$-dim Δ' in B is a face of 1 $k-$simplex;*
5. *Orientation is inherited.*

Definition 1.1.10 (Cobordism of i-cycles). *A cobordism of i-cycles P_1, P_2 in X is an $(i+1)$-dimensional psuedomanifold with boundary C and a map $\sigma : C \to X$ So that $\partial B = P_1 - P_2$ and $\sigma|_B$ coincides with σ_1, σ_2.*

Two i-cycles that have a cobordism between them are called cobordant.

Homework problem:

Proposition 1.1.11 (Cobordism relation). *Cobordism of i-cycles in X is an equivalence relation.*

Proof.

Reflexive Let $P = \{x_1, \dots, x_k\}$. Let $P \cong Q = \{y_1, \dots, y_k\}$. Construct Δ as follows. The $k+1$ cells are of the form $\Delta_n = p_n \sqcup q_{k+1-n}$ where $p_n = \{x_i \mid i \in [n]\}$ and $q_m = \{y_i \mid i \in [m]\}$ for $1 \leq n \leq k$. Now, the k cells are of the form $p_n \sqcup q_{k-n}$ or $p_{n-1} \sqcup q_{k+1-n}$. Thus, Δ_n shares a face with Δ_{n+1} and Δ_{n-1}. The only faces not shared are P, from $n = 1$, and Q, from $n = k$. Hence, $P \sim Q = P$ so $P \sim P$.

Symmetric Given $P \sim Q$ via Δ, σ, we know $\partial\Delta = P - Q$ and hence $\partial\overline{\Delta} = (-P) - (-Q) = Q - P$ so $Q \sim P$ via $\overline{\sigma}$.

Transitive Let $P \sim Q$ and $Q \sim R$ via Δ, σ and Δ', σ', construct Δ'', σ'' by identifying Q in Δ with Q in Δ'. Then, $\overline{Q}$ is a face of some $X \in \Delta$, Q a face of some $\Upsilon \in \Delta''$, so Q is a face of both X and Y and $\partial\Delta'' = P - R$. Let $\sigma''|_{\Delta} = \sigma$ $\sigma''|_{\Delta'} = \sigma'$. Hence, $P \sim R$.

Definition 1.1.12 (Homology). *$H_i(X)$ is the set of equivalence classes of $+, -$ of i-cycles giving an abelian group structure. The identity element of this group is represented by the empty $i-$dimensional psuedomanifold.*

1.1.2 Thursday

Types of topological spaces:

$$\begin{tikzcd}
 & \text{Metric spaces} & \\
\text{Manifolds} \arrow[ur, "\subset"] \arrow[d, "\subset"] & \text{Simplicial complexes} \arrow[u, "\subset"] \arrow[d, "\subset"] & \\
\text{Pseudomanifold} & \Delta\text{-Complex} \arrow[r, "\subset"] & \text{CW-Complex}
\end{tikzcd}$$

What's a manifold? Well, a manifold is a 2nd-countable Hausdorff space that is locally euclidean.

Examples:

- $\mathbb{R}^n$;
- S^n;
- Products of manifolds: $T = T^2 = S^1 \times S^1$.
- $\square$-Complex: Combinatorial model for space X, built from "cells" or related objects.

Let's mess with the torus $T = S^1 \times S^1$ for a bit. Whenever we're constructing it from a gluing diagram, we have one zero cell, the point, two 1-cells, the edges, and one 2-cell, the square itself. We can think of $n-$cells as n-disks. Hence,

$$T = X \supset X^1 \supset X^0$$

where X^0 is the $0 - cell$, X^1 is $X_0 \cup \{\text{1-cells}\}$. We could say, if we're so inclined,

$$X^1 = \left(X^0 \sqcup e_1^1 \sqcup e_2^1\right) / \sim_1$$

and

$$X = \left(X^1 \sqcup e^2\right) / \sim_2 .$$

Definition 1.1.13 (CW-Complex)**.**

1. *A discrete set of points X^0;*
2. *Inductively form the n-skeleton X^n from X^{n-1} by attaching some collection of n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} = \partial D^n \to X^{n-1}$, and setting $X^n = \left(X^{n-1} \sqcup_\alpha D_\alpha^n\right) / \sim$ where $x \sim \varphi_\alpha(x)$ for $x \in S^{n-1} = \partial D^n$.*

3. *Either $X = X^n$ for some n and we say that n is the dimension of X, or $X = \bigcup_n X^n$ with the weak topology.*

Definition 1.1.14 ((aside) Weak Topology)**.** *Let $X = \bigcup_n X_n$. Then $A \subset X$ is open if $A \cap X_n$ is open in X_n for all n.*

Let's do some examples; $X = X^1$ is a graph, S^1 is a point and a quotient-ed one cell, or two points and two one cells. More interestingly, we could have $S^2 = x_0 \cup D^2/\sim$ where $x_0 \sim y$ if $y \in \partial D^2$...And we could generalize this to all $n-$spheres.

Another really interesting example is T^2, formed by the CW-Complex by an octagon!

We can do similar things with non-orientable surfaces. Take $\mathbb{R}P^2$ and the standard gluing diagram. In general,

$$\mathbb{R}P^n = \{\text{lines through the origin in } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1} \backslash 0/\sim$$

where $v \sim \lambda\omega$ for $\lambda \neq 0$. It could also be written $S^n/\sim$, $x \sim -x$, and $D^n/\sim$ for $y = -y$ for $y \in \partial D^n = S^{n-1}$. Yet another description is $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$.

For $m < n$, $\mathbb{R}P^m \subset \mathbb{R}P^n$ is a subcomplex, with $\mathbb{R}P^\infty$ are lines through the origin in $\mathbb{R}^\infty$ (weak topology).

Now here's where the fun begins: Klein bottles with their standard gluing map!

This leads into the observation that closed surfaces can be realized as the quotient spaces of polygons. Moreover, a polygon can be cut into triangles! Thus, any surface can be built out of triangles. This is where the notion of a Δ-complex (or more stringently, a simplicial complex) comes from.

Let's get some standard notation. A standard n-simplex

$$\Delta^n = \{(t_0, \dots, t_j) \in \mathbb{R}^{n+1} \mid \textstyle\sum_{i=1}^n t_i = 1, t_i \geq 0\}$$

. Hence, Δ^0 is a point, Δ^1 is the line segment from $(1,0)$ to $(0,1)$, so on, so forth.

The $n-$simplex on $(p_0, \dots, p_n)$ in general position (the set

$$\{v_1 - v_0, \dots, v_n - v_0\}$$

is linearly independent) is the convex hull of $(p_0, \dots, p_n)$. We call it $\Delta = [p_0 p_1 \dots p_n]$. In this position, $\Delta^n = [e_0 e_1 \dots e_n]$. Any two n-simplicies are homeomorphic. Eg, $\Delta^n \to \Delta = [v_0 \dots v_n]$ via $(t_0, \dots, t_n) \mapsto (\sum t_i v_i)$.

Some terminology that we'll use: The boundary of $[v_0 \dots v_n] = \cup\{\text{all (n-1)-dim faces}\}$, the interior of Δ is $\Delta-$boundary, so $]v_0 \dots v_n[= [v_0 \dots v_n] - \partial[v_0 \dots v_n]$.

Definition 1.1.15 (Δ-Complex)**.** *A Δ-complex is a quotient of a disjoint union of simplicies obtained by identifying certain faces using orientation preserving homeomorphisms.*

To be more refined, a Δ-complex on X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ where

- $\sigma_\alpha|_{\overset{o}{\Delta^n}}$ *is a homeomorphism onto* $\sigma_\alpha(\overset{o}{\Delta^n}) = e^n_\alpha$.
- *Each* $x \in X$ *is in exactly one* e^n_α.
- $\sigma_\alpha|_{\text{(n-1)-face of } \Delta^n} = \sigma_\beta : \Delta^{n-1} \to X$.
- $U \subset X$ *is open if and only if* $\sigma_\alpha^{-1}(U)$ *is open in* Δ^n *for all* n.

A homework problem:

A Δ-Complex is a CW-Complex.

1.2 Week 3

1.2.1 Tuesday

Definition 1.2.1 (Free Abelian Group)**.** *Let $G_1, G_2, \dots, G_n$ be groups. Then, we have the direct product $\prod_1^n G_i$ with operations defined pointwise. If each G_i is abelian, we speak instead of their direct sum $\bigoplus_1^n G_i$. If each and every G_i is infinite cyclic, we call their direct sum* free abelian*. If $G_i = \langle a_i \rangle$, then elements of $\bigoplus G_i$ look like $(m_1 a_1, \dots, m_n a_n)$ for $m_i \in \mathbb{Z}$. We call $\{a_1, \dots, a_n\}$ a basis for $\bigoplus G_i \cong \bigoplus \mathbb{Z} = \mathbb{Z}^n$.*

We can generalize this; for an infinite list $S = \{a_1, a_2, ...\}$, the free abelian group with basis S is $G = G(S)$ with elements $\sum m_i a_i$ for finitely many $m_i \neq 0$.

Recall that a Δ complex is a topological space X with characteristic maps $\sigma_\alpha : \Delta^n \to X$ with each $x \in X$ in a unique open $n - cell$ $\sigma_\alpha(\Delta^n)$ and compatibility, openness conditions.

For example, take $X = T^2$. Then we have one 0-cell $\sigma_v : \Delta^0 \to X$, three 1-cells $\sigma_a, \sigma_b, \sigma_c : \Delta^1 \to X$, and two 2-cells $\sigma_U, \sigma_L : \Delta^2 \to T$ with some identifications.

Definition 1.2.2 (Simplicial Homology of Δ-Complexes)**.** *Let $\Delta_n(X)$ be the free abelian group with basis n-simplicies in X (or the maps $\sigma_\alpha : \Delta^n \to X$).*

We also inflict the following boundary homomorphism: Let $\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ *via*

$$\partial[v_0, v_2] = +[v_1] - [v_0],$$

$$\partial[v_0, v_1, v_2] = +[v_0, v_1] + [v_1, v_2] - [v_0, v_2],$$

In general...literally just take the boundary.

$$\partial[v_0, \dots, n_n] = \sum (-1)^i [v_0 \dots \hat{v}_i \dots v_n].$$

Example: Using the torus again, we have $\Delta_0(T) = \mathbb{Z}$, $\Delta_1(T) = \mathbb{Z}^3$, $\Delta_2(T) = \mathbb{Z}^2$, and otherwise 0.

For general Δ complexes, we say

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 v_1 \dots \hat{v}_i \dots v_n]}.$$

Lemma 1.2.3. *We have that* $\partial_n \circ \partial_{n+1} = 0$*. Tersely, we can say*

$$\Delta_{n+1}(x) \overset{\partial_n+1}{\to} \Delta_n(x) \overset{\partial_n}{\to} \Delta_{n-1}(x)$$

to mean that $\operatorname{im}\partial_{n+1} \subset \ker \partial_n$*. We then define the* n*th simplicial homology group to be* $H_n^\Delta(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

Proof. For $\sigma : \Delta^{n+1} \to X$,

$$\partial\sigma = \sum_{i=0}^{n+1} (-1)^i \sigma|_{\hat{v}_i}$$

so

$$\begin{aligned}
\partial\partial\sigma &= \sum_{i=0}^{n+1} (-1^i) \partial\sigma_{\hat{v}_i} \\
&= \sum_{i=0}^{n+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j \sigma|_{\hat{v}_i, \hat{v}_j} + \sum_{j=i+1}^{n+1} (-1)^{j-1} \sigma|_{\hat{v}_i, \hat{v}_j} \right) \\
&= \sum_{0 \le p < q \le n+1} (-1)^{p+q} \sigma|_{\hat{v}_q, \hat{v}_p} \\
&\qquad \sum_{0 \le q < p \le n+1} (-1)^{p+q-1} \sigma|_{\hat{v}_p, \hat{v}_q} \\
&= 0
\end{aligned}$$

For example, take $X = [v_0 v_1 v_2]$ with bases $\sigma_0, \sigma_1, \sigma_2, \sigma_{01}, \sigma_{02}, \sigma_{12}$ and σ_{012}. We have that $\partial\sigma_{12} = \sigma_2 - \sigma_1$, so on, so forth. Hence,

$$
\begin{tikzcd}
0 \arrow[r] & \Delta_2(X) \cong \mathbb{Z} \arrow[r] & \Delta_1(X) \cong \mathbb{Z}^2 \arrow[r] & \Delta_0(X) \cong \mathbb{Z}^3 \arrow[r] & 0 \\
0,0 \arrow[r, mapsto, "\partial"] & 0,0 & & & \\
& \mathbb{Z}^2, 0 \arrow[r, mapsto, "\partial"] & \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle, 0 & & \\
& & \mathbb{Z}^2, \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle \arrow[r, mapsto, "\partial"] & \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle, 0 & \\
& & & \mathbb{Z}^3, \mathbb{Z}^3 \arrow[r, mapsto, "\partial"] & 0,0 \\
0 \arrow[r, dash] & H_2 \cong 0/0 = 0 \arrow[r, dash] & H_1 \cong \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle / \langle \sigma_{12} - \sigma_{02} + \sigma_{01} \rangle = 0 \arrow[r, dash] & H_0 \cong \langle \sigma_2 - \sigma_1, \sigma_1 - \sigma_0 \rangle \cong \mathbb{Z} \arrow[r, dash] & \bullet
\end{tikzcd}
$$

Now, take $X = S^1$, with one 0 cell v and one 1 cell a.

$$\begin{tikzcd}
0 \arrow[r, "\partial"] & \Delta_1 = \langle a \rangle \arrow[r, "\partial"] & \Delta_0 = \langle v \rangle \arrow[r, "\partial"] & \bullet \\
0,0 \arrow[r] & 0,0 & & \\
& \mathbb{Z},\mathbb{Z} \arrow[r] & 0,0 & \\
& & \mathbb{Z},\mathbb{Z} \arrow[r] & 0,0 \\
0 \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & 0
\end{tikzcd}$$

What about a different Δ complex on the circle? Say, two points g, f with edges p, q ending in f.

$$\begin{tikzcd}
0 \arrow[r, "\partial"] & \Delta_1 = \langle p, q \rangle \arrow[r, "\partial"] & \Delta_0 = \langle g, f \rangle \arrow[r, "0"] & \bullet \\
0,0 \arrow[r] & 0,0 & & \\
& \mathbb{Z}^2, \langle p - q \rangle \arrow[r] & \mathbb{Z},0 & \\
& & \mathbb{Z}^2,\mathbb{Z}^2 \arrow[r] & 0,0 \\
0 \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & 0
\end{tikzcd}$$

1.2.2 Thursday

Definition 1.2.4 (Chain Complex). *A chain complex* $\{C_n, \partial_n\}$ *is a sequence of abelian groups* C_n *with homomorphisms* $\partial_n = \partial C_n \to C_{n-1}$ *with* $\partial_n \partial_{n-1} = 0$. *As abelian groups are* $\mathbb{Z}$ *modules, we could replace* $\mathbb{Z}$ *with any ring.*

Elements of C_n *are called chains, elements of* $\ker \partial$ *are called cycles, and elements in* $\operatorname{im} \partial$ *are called boundaries.*

If one has a chain complex, one can take its homology:

Definition 1.2.5 (Homology). *The homology of a chain complex is*

$$H_c(C_\bullet) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

We say that $z, w \in \ker \partial$ *are homologous if* $z - w \in \operatorname{im} \partial$.

For example, we return to $C_n = \Delta_n(X)$ with ∂ being the simplicial boundary.

Let's take the torus.

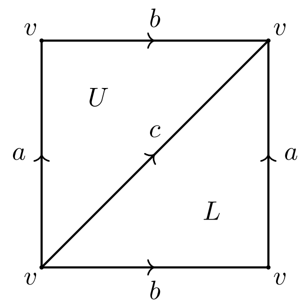


Figure 1.1: A Δ-complex on the torus.

Then

$$\Delta_2(T) = \langle U, L\rangle \to \Delta_1(T) = \langle a, b, c\rangle \to \Delta_0 = \langle v\rangle.$$

See that $\ker \partial_2 = \langle U, L\rangle$ and $\ker \partial_1 = \langle a, b, c\rangle = \Delta_1$. Thus, $\operatorname{im} \partial_2 = \langle a + b\rangle$. Hence,

$$H_2 = \langle U + L\rangle \cong \mathbb{Z}$$

$$H_1 = \langle a, b, c\rangle / \langle a + b - c\rangle = \langle a, b\rangle \cong \mathbb{Z}^2$$

$$H_0 = \langle v\rangle \cong \mathbb{Z}.$$

Recall that a simplicial complex is a Δ complex for which each σ_α is injective on the vertices of the standard $n-$simplex and furthermore no other $n-$simplex has the exact same set of vertices.

For example, to get a simplicial complex on a circle, you require at least three vertices.

This gives rise to some fun problems: What's the minimal simplicial complex structure on S^2? How about T? $\mathbb{R}P^2$? What does it mean to be minimal here, minimize the number of triangles?

	Simplex	Subdivision
$n = 0$	$[v_0]$	$[v_0]$
$n = 1$	$[v_0 v_1]$	$[bv_0], [bv_1]$
$n = 2$	$[xyz]$	

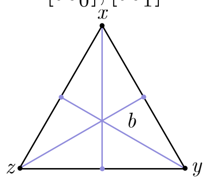


Figure 1.2: Barycentric subdivision.

Given a Δ-complex, we can produce a simplicial complex structure by using barycentric subdivision.

Tale a closed $n-$simplex $[v_0, v_1, \dots, v_n]$. Recall that these points are of the form $\sum_{i=0}^{n} t_i v_i$ where the sum of the t_i's is one and each is non zero. A barycenter is $\sum \frac{1}{n+1} v_i$. For barycentric subdivision, we take a simplex $[v_0 \dots v_n]$ and decompose it into $n-$simplices by taking $[v_0 \dots \hat{v}_j \dots v_n b]$ where b is the barycenter where inductively $[v_0 \dots \hat{v}_j \dots v_n]$ is a face in the barycentric subdivision of a face.

Given a Δ-complex structure on X with $\sigma_\alpha^n : \Delta^n \to X$, the barycentric subdivision is the Δ complex with characteristic maps $\sigma_\alpha^n \tau_\beta^m : \Delta^m \to X$.

Claim: Barycentric subdivision twice gives a simplicial complex. See H§2.3#23.

Now, we move to a special little thing called singular homology. It's defined for any topological space and will agree with simplicial homology for Δ complexes.

Definition 1.2.6 (Singular n-simplex)**.** *A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$.*

We define the singular n-chains to be $C_n(X)$, the free abelian group generated by singular n-simplices.

Elements of $C_n(X)$ look like $m_1\sigma_1 + \dots + m_k\sigma_k$ for $m_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$. We develop a map ∂ given by $\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0 \dots \hat{v}_j \dots v_n]}$. We still have $\partial \circ \partial = 0$.

Definition 1.2.7 (Singular Homology)**.** *With the previous definitions, $\{C_n, \partial_n\}$ is the singular chain complex of X giving rise to singular homology.*

Let's do an example with $X = \{*\}$. Then, $\sigma^n : \Delta^n \to X = *$

is the unique continuous map into X for every $0 < n$. Hence, $C_n(X) = \langle \sigma_n \rangle \cong \mathbb{Z}$ for each $n \geq 0$.

$$\begin{array}{ccccccc} & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & \\ \dashrightarrow & C_2(X) & \xrightarrow{\partial_2, \cong} & C_1(X) & \xrightarrow{\partial_1, 0} & C_0(X) & \longrightarrow 0 \end{array}$$

Concretely, we can determine that $\partial \sigma^n = \sigma^{n-1}$ for n even and 0 for n odd.

So, $H_{2k} = \ker \partial_{2k} / \operatorname{im} \partial_{2k-1} = 0/0 = 0$ and $H_{2k+1} = \ker \partial_{2k+1} / \operatorname{im} 2k = \mathbb{Z}/\mathbb{Z} = 0$. Hence $H_n(*) = 0$ except at $k = 0$. $H_0(*) = \ker \partial_0 / \operatorname{im} \partial_{-1} = \mathbb{Z}/0 = \mathbb{Z}$.

Let's go back and do one more simplicial calculation, H_*^Δ, for $X = \mathbb{R}P^2$.

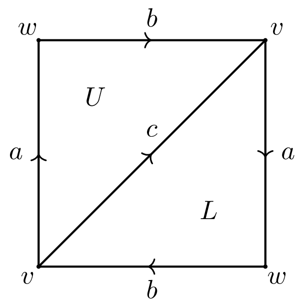


Figure 1.3: A Δ-complex on $\mathbb{R}P^2$.

We have $\Delta_0(X) = \langle \sigma_v, \sigma_w \rangle \cong \mathbb{Z}^2$, $\Delta_1(X) = \langle \sigma_a, \sigma_b, \sigma_c \rangle \cong \mathbb{Z}^3$ and $\Delta_2(X) = \langle \sigma_U, \sigma_L \rangle \cong \mathbb{Z}^2$. We see that

$$\begin{aligned} \sigma_a &\mapsto \sigma_w - \sigma \\ \sigma_b &\mapsto \sigma_v - \sigma_w \\ \sigma_c &\mapsto 0 \end{aligned}$$

$$\begin{aligned} \sigma_U &\mapsto \sigma_a + \sigma_b - \sigma_c \\ \sigma_L &\mapsto -\sigma_a - \sigma_b - \sigma_c \end{aligned}$$

Now, we compute the homology groups.

$$H_0^\Delta(X) = \Delta_0(x)/\operatorname{im} \partial_1 = \langle \sigma_v, \sigma_w \rangle / \langle \sigma_v - \sigma_w \rangle \cong \mathbb{Z}$$

For at home: Check that ∂_2 is injective so that $H_2^\Delta = \ker \partial_2 = 0$, and that

$$H_1^\Delta(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \langle \sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle / \langle 2\sigma_c, \sigma_a + \sigma_b - \sigma_c \rangle \cong \mathbb{F}_2.$$

Chapter 2

Weeks 4-6

2.1 Week 4

2.1.1 Tuesday

Last time, we were looking at the singular homology. Take the singular chain complex $\{C_n, \partial_n(x)\}$, where C_n are the free abelian groups generated by $\sigma : \Delta^n \to X$, defining

$$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

So far we've done a grand total of one example, that of the point. We calculated

$$H_n(*) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \end{cases}$$

Proposition 2.1.1. *If $X = \bigsqcup X_\alpha$ is a disjoint union of path components, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.*

Proof. The image of $\sigma : \Delta^n \to X$ is path connected, implying $C_n(X) = \oplus_\alpha C_n(X_\alpha)$. Hence, ∂ respects the decomposition:

$$\ker \partial_n(X) = \oplus_\alpha \ker \partial_n(X_\alpha)$$

with the same relation for the images, so

$$H_n(x) = \oplus H - n(X_\alpha).$$

Proposition 2.1.2. *If $X \neq ;$ is path connected then $H_0(x) \cong \mathbb{Z}$.*

Proof. We have $H_0(x) = C_0(x)/\operatorname{im}\partial_1$. Define $\epsilon : C_0(x) \to \mathbb{Z}$ by $\epsilon(\sum m_i\sigma_i^0) = \sum m_i$. We claim that $\operatorname{im}\partial_1 = \ker\epsilon$. If this is true, noting that ϵ is surjective, we get $H_0(x) = C_0(X)/\operatorname{im}\partial_1 = C_0(x)/\ker\epsilon \cong \operatorname{im}\epsilon = \mathbb{Z}$.

If $\sigma : \Delta^1 \to X$ is a 1-simplex, $\partial\sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$ so $\epsilon(\partial_1\sigma) = \epsilon(\sigma|_{[v_1]}) - \epsilon(\sigma|_{[v_2]}) = 1 - 1 = 0$. This implies $\operatorname{im}\partial_1 \subset \ker\epsilon$. Now, if $\sum m_i\sigma_i^0 \in C_0(X)$ is in $\ker\epsilon$, then $\sum m_i = 0$. Note that σ_i^0 is a 0-simplex, that is, a point $\sigma_i^0(v_0) \in X$.

Fix $x_0 \in X$ and choose paths $\gamma_i : I \to X$ with $\gamma_i(0) = x_0$, $\gamma_i(1) = \sigma_i^0(v_0)$. As the interval is the 1-simplex, each path is a singular 1−simplex in X. Moreover, $\partial_1\gamma_i = \sigma_i^0(v_0) - x_0$. Computing $\partial_1(\sum m_i\gamma_i) = \sum m_i\sigma_i^0(v_0) - (\sum m_i)\,\sigma_0^0(v_0) = \sum m_i\sigma_i^0(v_0)$ as the second sum goes to 0. This implies that $\ker\epsilon \subset \operatorname{im}\partial_1$ so we are done.

Theorem 2.1.3. *If $X = \bigsqcup X_\alpha$ is a decomposition of X into path components, then $H_0(X) \cong \oplus_\alpha\mathbb{Z}$.*

We now discuss reduced homology.

Definition 2.1.4 (Reduced Homology)**.** *The reduced homology of X is the homology of*

$$\ldots \longrightarrow C_n(X) \longrightarrow \ldots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

denoted $\tilde{H}_n(X)$.

The fact that $\epsilon\partial_1 = 0$ implies that ϵ induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\tilde{H}_0(\mathbb{Z})$ implies that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$; that is, we have the exact sequence

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow 0$$

If X is path connected, then $\tilde{H}_0(x) = 0$. Otherwise, the homology groups agree.

$H_*(X)$ is a homotopy type invariant.

Definition 2.1.5 (Homotopic). *We say that maps $f, g : X \to Y$ are homotopic, $f \simeq g$, if there exists continuous $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We call H a homotopy. Homotopy is an equivalence relation on $C[x, y]$.*

Definition 2.1.6 (Homotopy Equivalence). *We say X and Y are homotopy equivalent, or have the same homotopy type, if there are maps $f : X \to Y$, $g : Y \to X$ with $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. once again, this is an equivalence relation on topological spaces. We communicate this briefly as $X \simeq Y$.*

It's worth noting that if $X \cong Y$, $X \simeq Y$.

Let's do an example. Say that X is convex in $\mathbb{R}^n$, and that $*$ is a point. They are homotopic and are called contractible.

Another example. An annulus is homotopic to S^1! We might then try our hand at computing the simplicial homology of the annulus.

We'd like to show that $f : X \to Y$ induces $f_* : H_n(X) \to H_n(Y)$ for all n, and that homotopy equivalences induce isomorphisms.

We first need to work on the chain level, that $f : X \to Y$ induces $f_\sharp : C_n(X) \to C_n(Y)$. If $\sigma : \Delta^n \to X$ is an n-simplex in X, then $f\sigma : \Delta^n \to Y$ is an $n-$simplex in Y. Therefore, we define $f_\sharp : C_n(X) \to C_n(Y)$ by $f_\sharp(\sigma m_i \sigma_i) = \sum m_i(f\sigma_i)$.

Lemma 2.1.7.

$$f_\sharp \circ \partial_n(X) = \partial_n(Y) \circ f_\sharp.$$

That is,

$$\begin{tikzcd}
C_n(X) \arrow[r, "\partial_n(X)"] \arrow[d, "f_\sharp"'] & C_{n-1}(X) \arrow[d, "f_\sharp"] \\
C_n(Y) \arrow[r, "\partial_n(Y)"'] & C_{n-1}Y
\end{tikzcd}$$

Proof. For $\sigma : \Delta^n \to X$,

$$\partial_n(X)(\sigma) = \sum_i (-1)^i \sigma|_{\hat{v}_i}$$

so

$$\begin{aligned}
f_\sharp(\partial_n(X)(\sigma)) &= \sum_i (-1)^i f \circ \sigma|_{\hat{v}_i} \\
&= \partial_n(Y)(f_\sharp(\sigma))
\end{aligned}$$

To be even more illustrative,

$$\begin{array}{ccccccc} \longrightarrow & C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) & \longrightarrow \\ & \Big\downarrow{\scriptstyle f_\sharp} & & \Big\downarrow{\scriptstyle f_\sharp} & & \Big\downarrow{\scriptstyle f_\sharp} & \\ \longrightarrow & C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) & \longrightarrow \end{array}$$

Definition 2.1.8 (Chain Map). *Let $\{C_n, \partial\}, \{C'_n, \partial'\}$ be two chains. A chain map $f : C_n \to C'_n$ for all n has $f \circ \partial = \partial' \circ f$ for all n.*

Hence, $f_\sharp$ takes cycles to cycles (elements of the kernel to elements in the kernel, as it commutes with ∂). Additionally, $f_\sharp$ takes boundaries to boundaries! If $x = \partial_n(X)(y)$, then

$$\begin{aligned} f_\sharp(x) &= f_\sharp \circ \partial_n(X)(y) \\ &= \partial_n(Y) \circ f_\sharp(y) \\ &= \partial_n(Y)(f_\sharp(y)). \end{aligned}$$

Consequently, $f_\sharp$ induces a homomorphism $f_\star : H_n(X) \to H_n(Y)$.

We remark that if $x \in C_n(X)$ satisfies $\partial_n(x) = 0$, $[x] \in H_n(X)$, $f_\star([x]) = [f_\sharp(x)]$.

2.1.2 Thursday

Last time, we saw that a continuous map of spaces $f : X \to Y$ induces a chain map $f_\sharp : C_*(X) \to C_*(Y)$. We further saw that any chain map induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for all n.

There are some desirable properties of f_*. For

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

$(g \circ f)_* = g_* \circ f_*$, and ditto for $f_\sharp, g_\sharp$. The underlying workings of this are

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y \xrightarrow{g} Z \implies g \circ f \circ \sigma : \Delta^n \to Z$$

with associative composition.

Additionally, $\mathrm{id}_X : X \to X$ induces $\mathrm{id} = (\mathrm{id}_X)_* : H_n(x) \to H_n(x)$. A consequence of this is that $H_*(x)$ is a topological invariant. If $X \cong Y$, then $H_n(X) \cong H_n(Y)$. The proof is easy. If $f : X \to Y$ is a homeomorphism, let $g = f^{-1}$. Then $g \circ f = \mathrm{id}_X$. Hence, $(g \circ f)_* = (\mathrm{id}_X)_*$ implying $f_* \circ g_* = \mathrm{id}_{H_*(X)}$ and similarly $g_* \circ f_* = \mathrm{id}_{H_*(Y)}$. More generally,

Theorem 2.1.9. *If $f \simeq g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.*

Theorem 2.1.10. *If $X \to Y$ is a homotopy equivalence, then $H_n(X) \cong H_n(Y)$.*

For example, if $X \simeq *$ is contractible, then $\tilde{H}_n(X) = \tilde{H}_n(*) = 0$. We use homological algebra to pursue this proof.

Definition 2.1.11 (Chain Homotopy)**.** *We say that $\psi, \phi : C_* \to C'_*$ are chain homotopic if there is a chain homotopy $P : C_* \to C'_{*+1}$ such that*

$$\partial' P + P\partial = \psi - \phi.$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_n \arrow[dl, "P"'] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] \\
C'_{n+1} \arrow[r, "\partial'"] & C'_n
\end{tikzcd}$$

Lemma 2.1.12. *If $\phi, \psi : C_* \to C'_*$ are chain homotopic, then $\phi_* = \psi_*$ for all n.*

Proof. Given a chain homotopy $P : C_* \to C'_*$, let $x \in C_n$ be a cycle. Then,

$$\partial' P(x) + P\partial x = \psi(x) - \phi(x) \implies \partial'(P(x)) = \psi(x) - \phi(x).$$

So, $[\phi(x)] = [\psi(x)]$ in $H_*(C'_*)$, that is,

$$\phi_*([x]) = \psi_*([x]).$$

Proof. Of the previous theorem. We will prove by building a chain homotopy between $f_\sharp$ and $g_\sharp$. We already have a homotopy $H : X \times I \to Y$ between f and g.

If $\sigma : \Delta^n \to X$ is a generator of C_n, we want P such that $P\sigma \in C_{n+1}(Y)$.

We almost have exactly what we want. We already have

$$\Delta^n \times I \overset{(\sigma,\mathrm{id})}{\to} X \times I \overset{H}{\to} Y.$$

The problem we face is that $\Delta^n \times I$ is not an $n+1$ simplex. The intuition here is that the square is the union of triangles[1].

So, given $H : X \times I \to Y$ and $\sigma : \Delta^n \to X$, we define $P : C_n(x) \to C_{n+1}(Y)$ by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i H \circ (\sigma, \mathrm{id})|_{[v_0 \dots v_i w_i \dots w_n]}.$$

We want to see $g_\sharp - f_\sharp - P\delta = \delta P$.

$$\begin{aligned}
\partial P(\sigma) &= \partial(\sum_i (-1)^i H \circ (\sigma, \mathrm{id})|_{v_0 \dots v_i w_i \dots w_n}) \\
&= \sum_{j \le i} (-1)^{i+j} (\sigma, \mathrm{id})|_{v_0 \dots \hat{v}_j \dots v_i w_i \dots w_n} \\
&\quad \sum_{j \ge i} (-1)^{i+j+1} (\sigma, \mathrm{id})|_{v_0 \dots v_i w_i \dots w_j \dots w_n}.
\end{aligned}$$

For the $i = j$ case, all terms cancel except for the $i = 1, j = n$ terms. What's left is

$$(H \circ (\sigma, \mathrm{id}))_{[w_0 \dots w_n]} - H \circ (\sigma, \mathrm{id}))_{[v_0 \dots v_n]} = g \circ \sigma - f \circ \sigma = g_\sharp(\sigma) - f_\sharp(\sigma).$$

We hope that the rest of the sum, $j \ne i$, yields $-P\partial(\sigma)$.

We know that

$$\begin{aligned}
P(\partial \sigma) &= P(\sum_j (-1)^j \sigma|_{v_0 \dots \hat{v}_j \dots v_n}) \\
&= \sum_{i<j} (-1)^i (-1)^j H \circ (\sigma, \mathrm{id})|_{[v_0 \dots v_i w_i \dots \hat{w}_j \dots w_n]} + \\
&\quad \sum_{i>j} (-1)^{i-1} (-1)^j H \circ (\sigma, \mathrm{id})|_{[v_0 \dots \hat{v}_j \dots v_i w_i \dots w_n]}.
\end{aligned}$$

This is left as an exercise.

[1] I used this idea on the homework problem that wasn't to turn in!

Theorem 2.1.13. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an induced isomorphism for all n.*

What we're building up to is that if we have X, a topological space with a Δ−complex, remarking about the difference between $\{\Delta_n(X), \partial_n\}$ and $\{C_n(X), \partial_n\}$. To be blunt: the difference is vast, as we have $\Delta_n(X) \hookrightarrow C_n(X)$. This is transparently a chain map, inducing a homomorphism $H_n^{\Delta}(X) \to H_*(X)$. Our fervent wish, which will eventually come true, is that this homomorphism is also an isomorphism.

For the moment, we think about the inclusion structure. This is what we'll refer to as a subcomplex.

Here's another example where this structure rears its head. If $A \subset X$ is a subspace, then it is natural to ask whether there's a relation between $H_*(A)$ and $H_*(X)$. We can view $\sigma : \Delta^n \to A$ as $i \circ \sigma : \Delta^n \to A \overset{i}{\hookrightarrow} X$. This realizes $i_\sharp(C_n(A))$ as a subgroup of $C_n(X)$. What we're saying here is that the boundary of X, when restricted to chains on A, remains in chains on A. Hence, $(C_*(A), \partial_*(A))$ is a subcomplex of $(C_*(X), \partial_*(X))$.

2.2 Week 5

2.2.1 Tuesday

Let's start with an algebraic definition.

Definition 2.2.1 (Subcomplex, Quotient Complex)**.** *Suppose we have a chain complex $\mathcal{B} = (B_n, \partial_n^B)$. A subcomplex of $\mathcal{B}$ is another chain complex $\mathcal{A} = (A_n, \partial_n^A)$ with an injective chain map $i : \mathcal{A} \hookrightarrow \mathcal{B}$.*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial^A"] & A_{n+1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_n \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_{n-1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & \cdots \\
\cdots \arrow[r, "\partial^B"'] & B_{n+1} \arrow[r, "\partial^B"'] & B_n \arrow[r, "\partial^B"'] & B_{n-1} \arrow[r, "\partial^B"'] & \cdots
\end{tikzcd}$$

This setup yields a quotient complex $\mathcal{C} = (C_n, \partial_n^C)$ $C_n = B_n/i(A_n)$. We write $j : B_n \twoheadrightarrow C_n$. If $c = j(b)$, then $\partial_n^C(c) = \partial_n^B(b + iA_n) = \partial_n^B(b) + iA_{n-1}$.

$$\begin{tikzcd}
\dots \arrow[r, "\partial^A"] & A_{n+1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_n \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & A_{n-1} \arrow[r, "\partial^A"] \arrow[d, hook, "i"'] & \dots \\
\dots \arrow[r, "\partial^B"'] & B_{n+1} \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"'] & B_n \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"'] & B_{n-1} \arrow[r, "\partial^B"'] \arrow[d, two heads, "j"'] & \dots \\
\dots \arrow[r, "\partial^C"'] & C_{n+1} \arrow[r, "\partial^C"'] & C_n \arrow[r, "\partial^C"'] & C_{n-1} \arrow[r, "\partial^C"'] & \dots
\end{tikzcd}$$

Or, tersely, a short exact sequence $0 \to \mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{C} \to 0$.

Example. Let $i : A \to X$ be an inclusion of a subspace A into a space X. Now, $i_\sharp : C_*(A) \to C_*(X)$. This realizes $(C_*(A), \partial)$ as a subcomplex of $(C_*(X), \partial)$.

Now, we write $C_*(X, A) = C_*(X)/i_\sharp C_*(A)$. Evidently, ∂^C induces $\partial^{C,A}$. This means we have a short exact sequence $0 \to C_*(A) \to C_n(X) \to C_n(X, A) \to 0$. For the sake of consistency, name $j_\sharp$ as the onto map.

Definition 2.2.2 (Relative Homology)**.** *The homology of the complex $(C_*(X, A), \partial)$ is the relative homology an is denoted $H_n(X, A)$.*

Elements in $H_n(X, A)$ are represented by relative cycles: α an $n-$chain in X with $\partial\alpha$ in A. To be precise, $\alpha \in C_n(X)$ with $\partial\alpha \in i_\sharp C_n(A)$. A relative cycle α is trivial in $H_n(X, A)$ if $\alpha = \partial\beta + i_\sharp\gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Theorem 2.2.3. *If $A \hookrightarrow X$ by the inclusion map i is a subspace, there is a long exact sequence in singular homology*

$$\dots \longrightarrow H_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_n$$

We call δ a connecting homomorphism.

If $[\alpha] \in H_{n+1}(X, A)$ is represented by the relative cycle $\alpha \in C_n(X)$, then $\partial([\alpha]) = [\partial\alpha] \in H_n(A)$.

Proposition 2.2.4. *If $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$ is a short exact sequence of chain complexes of abelian groups, there is an associated long exact sequence in homology*

$$\dots \longrightarrow H_{n+1}(\mathcal{C}) \xrightarrow{\delta} H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{j_*} H_n(\mathcal{C}) \xrightarrow{\delta} H_{n-1}(\mathcal{A})$$

Proof. We define $\delta : H_n(\mathcal{C}) \to H_{n-1}(\mathcal{A})$. Take $[c] \in H_n(\mathcal{C})$ arising from a cycle $c \in C_n$, $\partial c = 0$. Hence, $c = j(b)$ for some $b \in B_n$. If we first take the boundary and apply j, $j\partial b = 0$. This says $\partial b \in \ker j$. As $\ker j = \operatorname{im} i$, there is an $a \in A_{n-1}$ so that $ia = \partial b$. We then define $\delta([c]) = [a]$. We should note that a is a cycle, as $i\partial a = \partial ia = \partial\partial b = 0$ and as i is injective, $\partial a = 0$.

We need to check well-definition of δ. The first thing we checked is, as i is injective, a is uniquely determined by ∂b. Suppose $c = j(b) = j(b')$. If we take their difference $b' - b$ and we apply j, we get 0. Hence, their difference is in the kernel of j. By exactness, we have a preimage $ia' = b' - b$. Hence, $b' = b + ia'$ so $\partial b' = \partial b + \partial ia' = \partial b + i\partial a' = ia + i\partial a' = i(a + \partial a')$.

Now, say $\delta[c] = [a + \partial a'] = [a]$.

We need to check if we took a diffent representative cycle for c, $c + \partial c'$. This c' is jb' for some b', so $c = \partial c' = c + \partial jb' = c + j\partial b = jb + j\partial b' = j(b + \partial b')$. But $\partial(b + \partial b') = \partial b = ia$ is not altered.

Time to check if δ is a homomorphism. If $\delta[c_1] = [a_1], \delta[c_2] = [a_2]$, we have $c_1 = jb_1, \partial b_1 = ia_1$ and so on. Hence, $c_1 + c_2 = j_1 b_1 + jb_2 = j(b_1 + b_2)$. Also, $\partial(b_1 + b_2) = \partial b_1 + \partial b_2 = ia_1 + ia_2 = i(a_1 + a_2)$ so $\delta[c_1 + c_2] = [a_1 + a_2] = [a_1] + [a_2] = \partial[c_1] + \partial[c_2]$.

We have six chores to prove this proposition, two for every map we have. Some are easy, some are involved.

Here's an easy one: $\operatorname{im} i_* \subset \ker j_*$ by short exactness.

Next: $\operatorname{im} j_* \subset \ker \delta$. $[b] \in H_n(\mathcal{B})$ represented by $b \in B_n$ with $\partial b = 0$. Now, $j_*([b]) = [j(b)]$. Then $\delta([j(b)]) = 0$ as $\partial b = 0$.

Let's go for $\operatorname{im} \delta \subset i_*$. $\partial[c] = [a]$ where $c = j(b)$, $\partial b = i(a)$. Then,

$$i_*(\partial c) = i_*([a]) = [i(a)] = [\partial b] = 0.$$

The next one–a new hard one–is $\ker j_* \subset \operatorname{im} i_*$. Suppose $j_*([b]) = 0$. Thus, $j_*[b] = [j(b)] = 0$. Therefore $j(b) = \partial c'$ for some c'. Well, $c' = j(b')$ for some b'. Thus, $\partial c' = \partial jb' = j\partial b'$. Compare $j(b - \partial b') = jb - j\partial b'$. Now, this all equals $\partial c' - \partial c' = 0$. This means that $b - \partial b' \in \ker k = \operatorname{im}_i$. $b - \partial b' = ia$. Hence, $[b - \partial b'] = [ia] = i_*[a] = [b]$.

Lastly, $\ker i_* \subset \operatorname{im} \delta$ and $\ker \delta \subset \operatorname{im} j_*$ are...homework!

2.2.2 Thursday

Last time, we talked about what happens if we have an inclusion map from a subspace into a space. We then have the exact sequence of chain complexes

$$0 \longrightarrow C_n(A) \xhookrightarrow{i_\sharp} C_n \xrightarrow{j_\sharp} C_n(X,A) \longrightarrow 0$$

Yielding the long exact homology sequence of the pair (X,A),

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \longrightarrow \ldots$$

We can then talk about the reduced relative homology,

$$\begin{tikzcd}
 & \ldots \arrow[d] & \ldots \arrow[d] & \ldots \arrow[d] & \\
0 \arrow[r] & C(A) \arrow[r] \arrow[d, "\epsilon"] & C_0(X) \arrow[r] \arrow[d, "\epsilon"] & C_0(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

If $A \neq ;$, the $\tilde{H}(X,A) = H(X,A)$.

If $A = \{*\}$, a point, then $\tilde{H}_n(A) = 0$ so $\tilde{H}_n(X) \cong \tilde{H}_n(X,A)$.

Another example. Let $(X,A) = (D^n, \partial D^n)$. Because $D^n \simeq *$, $\tilde{H}_n(D^n) = 0$. Moreover, $\tilde{H}_n(D^n, S^{i-1}) \cong \tilde{H}_{i-1}(S^{i-1})$ for all n, by long exactness. We will start seeing reasons why this group is isomorphic to $\mathbb{Z}$ for $i = n$ and 0 otherwise.

Let's make a comment about functorality of relative homology. A map of pairs $f : (X,A) \to (Y,B)$ induces a homomorphism $f_* : H_n(X,A) \to H_n(Y,B)$ for all n. Viewing $f : X \to Y$, we get $f_\sharp : C_n(X) \to C_n(Y)$ taking $C_n(A)$ to $C_n(B)$. So we get an induced $f_\sharp C_n(X,A) \to C_n(Y,B)$. Hence, we get an f_* on the relative homology.

We say that $f \simeq g : (X,A) \to (Y,B)$ if there exists a homotopy where $H(a,t) \in B$ for all $a \in A$ and $t \in I$. This induces the same map $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

A couple of comments on how one might proceed: Take the induced chain homotopy P from $f_\sharp$ and $g_\sharp$.

Going back to some old stuff. We can view $j_\sharp : C_n(X) \to C_n(X,A)$ as induced by $j : (X,*) \to (X,A)$ where j is the identity map on X and $j(*) \in A$.

For triples (X,A,B) that are stacked $B \subset A \subset X$, there is a long exact sequence in relative homology

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta} H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow$$

In the special case $B = *$, we get the long exact sequence of X, A.

Two fundamental properties of singular homology:

Definition 2.2.5 (Excision). *If $Z \subset A \subset X$ with $\bar{Z} \in \mathring{A}$, then the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \cong H_n(X, A)$ for all n.*

Closely related is the

Definition 2.2.6 (Mayer-Vietoris Sequence). *If $A, B \subset X$ with $X = \mathring{A} \cup \mathring{B}$, there is a long exact sequence*

$$\dots \longrightarrow H_{n+1}(X) \xrightarrow{\delta} H_n(A \cap B) \xrightarrow{i_*, j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_* - \ell_*} H_n($$

$$i : A \cap B \hookrightarrow A \qquad k : A \hookrightarrow X$$

$$j : A \cap B \hookrightarrow B \qquad \ell : B \hookrightarrow X$$

In both of these contexts we can work with reduced homology.

Example. Let $X = S^n$ with A the "extended northern hemisphere" and B the "extended southern hemisphere".

Interestingly! $A, B \cong \mathring{D}^n \simeq *$. Additionally, $A \cap B \cong S^{n-1} \times \mathring{I} \simeq S^{n-1}$. Plugging into Mayer-Vietoris thus gives

$$\dots \longrightarrow 0 \longrightarrow \tilde{H}_{i+1}(S^n) \longrightarrow \tilde{H}_i(S^{n-1}) \longrightarrow 0 \longrightarrow \dots$$

Or $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$. What is $\tilde{H}_j(S^0)$? Two points? It's 0 for $j \neq 0$ and $\mathbb{Z}$ for $j = 0$. Hence, this yields $\tilde{H}_j(S^n) \cong \mathbb{Z}$ for $j = n$ and 0 otherwise.

Let's revisit the Brower Fixed Point Theorem, saying that a continuous map $f : D^n \to D^n$ must have a fixed point. *Proof.* Let i be the map $S^{n-1} = \partial D^n \hookrightarrow D^n$. Suppose $f : D^n \to D^n$ has no fixed point. Then we can use f to define $r : D^n \to S^{n-1}$. Make a vector from $f(x)$ to x and let $r(x)$ be where it hits the boundary of the disk. Then $r : D^n \to S^{n-1}$ with $r|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. The proof concludes by the diagrams

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[dr, "\mathrm{id}"'] & D^n \arrow[d, two heads, "r"] \\ & S^{n-1} \end{tikzcd} \qquad \begin{tikzcd} \tilde{H}_{n-1}(S^{n-1}) \arrow[r, hook, "i_*"] \arrow[dr, "\mathrm{id}"'] & \tilde{H}_{n-1}(D^n) \arrow[d, two heads, "j_*"] \\ & \tilde{H}_{n-1}(S^{n-1}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr, "\mathrm{id}"'] & 0 \arrow[d] \\ & \mathbb{Z} \end{tikzcd}$$

So then f having no fixed point must have been a lie.

2.3 Week 6

2.3.1 Tuesday

Recall that last class we started talking about excision and Mayer-Vietoris.

If $Z \subset A \subset X$ with $\bar{\mathbb{Z}} \subset \mathring{A}$, then $(X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X - Z, A - Z) \to H_n(X, A)$.

Equivalently, if $A, B \subset X$ with $X = \mathring{A} \cup \mathring{B}$, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

It's easy to see that these two statements are essentially the same; take $B = X - Z$ to traverse between the two frameworks.

In the second setting, we also have the Mayer-Vietoris sequence.

$$\begin{tikzcd} \dots \arrow[r] & H_{n+1}(X) \arrow[dl] \\ H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[dl] \\ H_n(X) \arrow[r] & \dots \end{tikzcd}$$

and an analogous sequence in reduced homology.

We're going to fairly methodically work on proving these two tools. But first, let's illustrate some utility of excision.

Let's make a non-manifold. Take the standard $\Delta-$complex on the torus T. Let's think about $X = CT$, the cone on T. Claim:

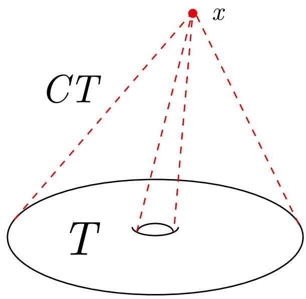


Figure 2.1: Cone on the Torus.

No neighbourhood of x, the point at the tip of the cone, is homeomorphic to a disc D^3. This says X is NOT a manifold!

Let's pursue this using local homology. Say we have a space X and a point $x \in X$. The local homology of X at x is $H_*(X, X - x)$.

Why this name? If U is a neighbourhood of $x \in X$, then $H_n(X, X - x) \cong H_n(U, U - x)$ by excising $X \backslash U$. If $f : X \to Y$ is a homeomorphism, then $f_* : H_n(X, X - x) \to H_n(Y, Y - f(x))$ is an isomorphism for all n.

Theorem 2.3.1 (Invariance of Domain). *If $; \neq U \subset \mathbb{R}^m$ and $; \neq V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.*

Proof. To prove invariance of domain, compute the local homology of $\mathbb{R}^n$ at $x \in \mathbb{R}^n$. Then we look at the long exact sequence

$$\dots \longrightarrow \tilde{H}_i(\mathbb{R}^n) \longrightarrow \tilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - x) \longrightarrow \tilde{H}_{i-1}(\mathbb{R}^n - x) \longrightarrow \tilde{H}_{i-1}(\mathbb{R}^n$$

$$\dots \longrightarrow 0 \longrightarrow \tilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - x) \xrightarrow{\sim} \tilde{H}_{i-1}(S^{n-1}) \longrightarrow 0 \longrightarrow$$

So $\tilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - x) \cong \mathbb{Z}$ for $i = n$, and otherwise, 0.

Using excision, we can then show that all manifolds have this local homology at every point.

Exercise: For $X = CT = T \times I / T \times \{1\}$ and x the cone point, compute $H_*(X, X - x)$. Conclude that X is not a $3 - manifold$.

Let's move on. We'll pursue Mayer-Vietoris. Let $X = \mathring{A} \cup \mathring{B}$ and let $C_*(A + B)$ be the chain complex generated by chains in X, $\sigma : \Delta^n \to X$, such that $\operatorname{im} \sigma \subset A$ or $\operatorname{im} \sigma \subset B$.

Naturally, $C_*(A+B) \hookrightarrow C_*(X)$ is a subcomplex by †.

We will eventually show that † induces an isomorphism in homology.

We have

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A+B) \to 0.$$

If $(y,z) \in C_*(A) \oplus C_*(B)$, then $(y,z) \mapsto y+z$. If we take $x \mapsto (x,-x)$, we can make this into a short exact sequence of chain complexes. Let's real quick name the chain maps so we can come back to them. Name them ϕ, ψ in order. Then ϕ is injective, ψ is surjective, $\operatorname{im}\phi \subset \ker\phi$ fairly transparently. The only thing we need left is that $\operatorname{im}\phi \supset \ker\psi$ for short exactness. If $y+z=0$, then $z=-y$. Because $y \in C_n(A)$ and $z \in C_n(B)$, $z,y \in C_n(A \cap B)$. Hence, $(y,z) = \phi(y)$.

Now it's time to zig zag. The resulting long exact homology sequence is

$$\ldots \longrightarrow H_{i+1}(C(A+B)) \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(\cdots$$

with induced homomorphisms $\Phi = \phi_*$ and $\Psi = \psi_*$.

Our goal now, to obtain Mayer-Vietoris, is that $H_n(C(A+B)) \cong H_n(X)$. We'll eventually prove this, but not today.

Let's use Mayer-Vietoris for a couple of examples.

Take $X = S^1 \vee S^1$. Let A be the left circle and a little more and let B be the right circle and a little more. Their intersection is then a little X shape, homotopic to a point. Both A and B defret to circles. Now,

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{H}_2(A) \oplus \tilde{H}_2(B) \arrow[r] & \tilde{H}_2(X) \arrow[r] & \tilde{H}_1(A \cap B) \arrow[dl] & \\
 & & \tilde{H}_1(A) \oplus \tilde{H}_1(B) \arrow[r] & \tilde{H}_1(X) \arrow[r] & \tilde{H}_0(\cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & \tilde{H}_2(X) \arrow[r] & 0 \arrow[dl] & \\
 & & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \tilde{H}_1(X) \arrow[r] & \cdots
\end{tikzcd}$$

So $H_i(X) \cong \mathbb{Z}$ for $i=0$, $\mathbb{Z} \oplus \mathbb{Z}$ for $i=1$, and 0 otherwise.

Recall that $H_i^{\Delta}(T) \cong \mathbb{Z}$ for $i=0$, $\mathbb{Z} \oplus \mathbb{Z}$ for $i=1$, $\mathbb{Z}$ for $i=2$, and 0 otherwise. We pinky promise that $H_i^{\Delta}(T) \cong H_i(T)$. Can we get $H_i(T)$ using Mayer-Vietoris? Or maybe $H_i(\Sigma)$, for Σ any orientable surface? Maybe nonorientable surfaces?

We can decompose the torus into two elbow noodles with intersection ditalini. The elbows defret to S^1, and so do each of the ditalini components of the intersection.

We could also decompose the torus into a small disc around a point and basically everything else. The disc is homotopic to a point, and T minus a point defrets to $S^1 \vee S^1$. If we now run Mayer-Vietoris,

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{H}_2(A) \oplus \tilde{H}_2(B) \arrow[r] & \tilde{H}_2(T) \arrow[r] & \tilde{H}_1(A \cap B) \arrow[dl] & \\
& & \tilde{H}_1(A) \oplus \tilde{H}_1(B) \arrow[r] & \tilde{H}_1(T) \arrow[r] & \tilde{H}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & \tilde{H}_2(T) \arrow[r] & \mathbb{Z} \arrow[dl, "\Phi"'] & \\
& & \mathbb{Z}^2 \arrow[r, "\Psi"] & \tilde{H}_1(X) \arrow[r] & {}
\end{tikzcd}$$

We claim that $\Psi = 0$. If c is a cycle in $A \cap B$ representing a generator of $\tilde{H}_1(A \cap B)$, then c is a boundary in A. We also claim that c is a boundary in B, which requires a little more pictoral reasoning but with Δ complexes it doesn't take too much work.

This means that we get $\tilde{H}_2 T \cong \mathbb{Z}$ and $\tilde{H}_1 T \cong \mathbb{Z}^2$.

2.3.2 Thursday

Given X, let $\mathcal{U} = \{U_i\}$ with $X = \bigcup \mathring{U}_i$. Then $C_n^{\mathcal{U}}(X)$ is a subroup of $C_n(X)$ generated by $\sigma : \Delta^n \to X$ with $\operatorname{im} \sigma \subset U_j$ for some j. Let $\partial : C_n(X) \to C_{n-1}(X)$ take $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$ So $C^{\mathcal{U}}(X) \hookrightarrow C(X)$ is a subcomplex. If $\mathcal{U} = \{A, B\}$, then this is the familiar Mayer-Vietoris.

Proposition 2.3.2. *$i : C^{\mathcal{U}} \hookrightarrow C(X)$ is a chain homotopy equivalence.*

Let's first look at convex $Y \subset \mathbb{R}^n$ and let $LC_n(T)$ be the subgroup of linear $\lambda : \Delta^n \to Y$ in $C_n(Y)$. We know that λ is determined by $w_i = \lambda(\text{vtx of } \Delta^n)$, so we write it in direct correspondence.

The collection of these guys gives us a subcomplex. We augment: $LC_{-1}(Y) = \mathbb{Z}$ generated by [;].

For $b \in Y$ define $b : LC_n(Y) \to LC_{n+1}(Y)$ by $b([w_0 \dots w_n]) = [bw_0 \dots w_n]$ which is a coning operation.

We check that $\partial(b([w_{-} \dots w_n])) = [w_0 \dots w_n] - b[\partial[w_0 \dots w_n]]$. By linearity, $\partial b\alpha = \alpha - b\partial\alpha$. Hence, in homology, $\partial b + b\partial = \mathrm{id}$, saying that $\mathrm{id}_* = 0_*$. Hence, the groups all have to be 0, $H_i(LC_*(Y)) = 0$ for all i.

Now let $S : LC_n(Y) \to LC_n(Y)$ be a subdivision homomorphism, where $S([;]) = (;)$ and $S([w_0]) = [w_0]$. For $\lambda : \Delta^n \to Y$ a generator of $LC_n(Y)$, let b_λ be the image of the barycenter of Δ^n, $b_\lambda = \sum_{i=0}^{n} \frac{1}{n+1} w_i$. For $n \geq 1$ define $S(\lambda) = b_\lambda(S(\partial\lambda))$. We claim that $\partial S = S\partial$ is a chain map.

Proof. $S = \mathrm{id}$ on LC_{-1} and LC_0 implies that $\partial S = S\partial$ on LC_0. Other,

$$
\begin{aligned}
\partial S\lambda &= \partial(b_\lambda(S(\partial\lambda))) \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) \\
&= S\partial\lambda.
\end{aligned}
$$

Hence, S is a chain map.

Now, we make $T : LC_*(Y) \to LC_{*+1}(Y)$ to be a chain homotopy between S and id on $LC_*(Y)$. Specifically, we want $\partial T + T\partial = \mathrm{id} - S$.

Define $T([;]) = 0$. Let $T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$ for $n \geq 0$.

On $LC_{-1}(Y)$, $\partial T + T\partial = \mathrm{id} - S$ just says $0 = 0$.

For general n, observe that $\partial\lambda = \partial T\partial\lambda = (\mathrm{id} - \partial T)(\partial\lambda) = (S + T\partial)(\partial\lambda)$ for induction.

$$
\begin{aligned}
\partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda)) \\
&\lambda - T\partial\lambda - b_\lambda\partial(\lambda - T\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda((S + T\partial)(\partial\lambda)) \\
&= \lambda - T\partial\lambda - b_\lambda S\partial\lambda - b_\lambda T\partial\partial \\
&= \lambda - T\partial\lambda - S\lambda
\end{aligned}
$$

This shows that $\partial T\lambda + T\partial\lambda = \lambda - S(\lambda) = (\mathrm{id} - S)(\lambda)$.

Next week, we're going to extend this to $C(X)$ generally.

For future use:

Definition 2.3.3 (Diameter of Simplex). *Suppose we have a simplex $[v_0 \dots v_n]$ in some $\mathbb{R}^N$. The diameter is the maximum distance between two points in the simplex.*

Our first claim is htat this is the maximum distance between any two of its vertices.

Suppose we have v and $\sum t_i v_i$. Then $|v - \sum t_i v_i| = |\sum t_i (v - v_i)| \leq \sum t_i |v - v_i|$. Lastly, that is at most $\sum t_i \max(v - v_j) = \max(v - v_j)$ as desired.

Fact: $\operatorname{diam}\{\text{simplex in barycentric subdivision of } [v_0 \dots v_n]\} / \operatorname{diam}[v_0 \dots v_n]$ $n/(n+1)$.

Chapter 3

Weeks 7-9

3.1 Week 7

3.1.1 Tuesday

For $Y \subset \mathbb{R}^n$ convex, we defined $S : LC_*(Y) \to LC_*(Y)$ for $\lambda : \Delta^n \to Y$ linear given by $S(\lambda) = b_\lambda(S\partial\lambda)$.

Now, we make a subdivision operator on $C_*(X)$ generally. For $\sigma : \Delta^n \to X$, define $S(\sigma) = \sigma_\sharp(S(\Delta^n))$.

S is a chain map.

$$\begin{aligned}
\partial S(\sigma) &= \partial\sigma_\sharp(S(\Delta^n)) \\
&= \sigma_\sharp \partial S(\Delta^n) \\
&= \sigma_\sharp S\partial\Delta^n \\
&= \sigma_\sharp S(\sum_{i=0}^{n}(-1)^i\Delta_i^n) \\
&- \sum_0^n(-1)^i\sigma_\sharp S(\Delta_i^n) \\
&= \sum_0^n(-1)^i S(\sigma|_{\Delta_i^n}) \\
&= S(\sum_0^n(-1)^i\sigma|_{\Delta_i^n}) \\
&= S\partial\sigma
\end{aligned}$$

Define $T : C_*(X) \to C_{*+1}(X)$ by $T(\sigma) = \sigma_\sharp T(\Delta^n)$. This is a chain homotopy between id and S. We can iterate subdivision, S^m, which is still chain homotopic to the identity via $D_m = \sum_{i=0}^m TS^i$.

$$\begin{aligned}
\partial D_m + D_m\partial &= \sum_0^{m=1}\partial TS^i + TS^i\partial \\
&= \sum \partial TS^i + T\partial S^i \\
&= \sum(\partial T + T\partial)S^i \\
&=\sim (\mathrm{id} - S)S^i \\
&= \sum_0^{n-1} S^i - S^{i+1} \\
&= \mathrm{id} - S^m.
\end{aligned}$$

Now suppose $X = \bigcup_{U_j\in\mathcal{U}}\mathring{U}_j$. Given $\sigma : \Delta^n \to X$, I claim there exists $m = m(\sigma)$ so that $S^m(\delta) \in C_*^{\mathcal{U}}(X)$.

Why: Well, $\{\sigma^{-1}(\mathring{U}_j)\}$ covers Δ^n which has the compact metric. This implies that there exists $\epsilon > 0$ such that for all $x \in \Delta^n$, $B_\epsilon(x) \subset \sigma^{-1}(\mathring{U}_j)$ for some j. Take m big so that $(\frac{n}{n+1})^m < \epsilon$. Let $m(\sigma)$ be the smallest m for which this holds.

Define $D : C_*(X) \to C_{*+1}(X)$ by $D(\sigma) = D_{m(\sigma)}(\sigma)$.

Then, define $\rho(\sigma) : S^{m(\sigma)}(\sigma) + D_{m(\sigma)}\partial\sigma - D\partial\sigma$, then we can substitute to $\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma)$. This says D is a chain homotopy between id and ρ.

Our chores are as follows: $\rho(\sigma) \in C_*^{\mathcal{U}}(X)$, $\rho : C_*(X) \to C_*^{\mathcal{U}}(X)$ is a chain map, and $i\rho, \rho i$ are chain homotopic to the identity. This says that i is a chain homotopy equivalence which induces an isomorphism.

If we take $\mathcal{U} = \{A, B\}$, we get $H_*(\mathcal{C}_*(A + B)) \cong H_*(X)$ which yields Mayer-Vietoris via

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A + B) \to 0.$$

We also get excision. If $Z \subset A \subset X$ with $\overline{Z} \subset \mathring{A}$, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomohprism in homology $H_n(X - Z, A - Z) \to H(X, A)$.

Equivalently, if $X = \mathring{A} \cup \mathring{B}$, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism in homology. Set $B = X - Z$. We can prove this for $\mathcal{U} = \{A, B\}$. We have that $C_*^{\mathcal{U}}(X) \xrightarrow{i} C_*(X)$ with $\partial D + D\partial = \mathrm{id} - i\rho$, and $\rho i = \mathrm{id}$. The upshot; all of these maps preserve chains on A. From this, $C_*^{\mathcal{U}}/C_*(A) \hookrightarrow C_*(X)/C_*(A)$ induces an isomorphism in homology and $C_*(B)/C_*(A \cap B) \to C_*^{\mathcal{U}}(X)/C_*(A)$ is just plainly an isomorphism. Hence, $H_*(B, A \cap B) \cong H_*(X, A)$.

Let's do a sample chore. $\partial\rho\sigma = \partial\sigma - \partial\partial D\sigma - \partial D\partial\sigma$ and $\rho\partial\sigma = \partial\sigma - \partial D\partial\rho - D\partial\partial\sigma$. Hence, $\partial\rho = \rho\partial$.

We call X, A a good pair if A is a subspace of X which is a deformation retract of a neighbourhood in X.

For example, take X a cw complex and A a subcomplex.

Lemma 3.1.1. *If $y_0 \in Y$ path connected, then $H_n(Y, y_0) \cong \tilde{H}_n(Y)$.*

Proposition 3.1.2. *If (X, A) is a good pair, the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism in homology $H_n(X, A) \cong \tilde{H}_n(X/A)$.*

For an example, if $X = S^1 \vee S^1$, then $(X, *)$ is a good pair.

3.1.2 Thursday

The above means that for (X, A) good, we have

$$\ldots \to H_n(A) \to H_n(X) \to \tilde{H}_n(X/A) \to H_{n-1}(A) \to \ldots$$

Proof. For $A \subset V \subset X$, we have the commutative diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "f"] \arrow[d, "q_*"'] & H_n(X, V) \arrow[d, "q_*"] & H_n(X - A, V - A) \arrow[l, "\sim"'] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r, "g"'] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\sim"']
\end{tikzcd}$$

with g and f induced by inclusions. Moreover, this q_* on the right hand side is an isomorphism, as q is the identity on $X - A$, $V - A$. We need to see that f, g are also isomorphisms.

Let's use the long exact sequences of triples for this, e.g., (X, V, A). On the level of chains we have

$$0 \to C_*(V, A) \to C_*(X, A) \to C_*(X, V) \to 0.$$

This yields the long exact sequence

$$... \to H_n(V, A) \to H_n(X, A) \to H_n(X, V) \to ...$$

Up to homotopy, $(V, A) \simeq (A, A)$ for A a deformation retract of V. This implies $H_n(V, A) = 0$ for all n. Hence, $H_n(X, A) \to H_n(X, V)$ is an isomorphism. Then, do it with the triple $(X/A, V/A, A/A)$.

For example. $\tilde{H}_k(S^k) = \mathbb{Z}$ and trivial elsewhere, generated by e.g. $\Delta^k_N - \Delta^k_S$. We can think about this as the identity.

Time to talk about the 0 sphere. $S^0 = N \cup S$. On the level of chains,

$$... \to C_1(S^0) \to C_0(S^0) \xrightarrow{\epsilon} \mathbb{Z}$$

Where $C_0(S^0)$ has things like $aN + bS$ and $\mathbb{Z}$ has things like $a + b$. With a tiny bit of work, $\partial_1 = 0$ so $\tilde{H}_0(S^0) = \ker \epsilon$ which is generated by $N - S$.

Let's see if we can't soup this up now for general k. We once again go to the chain level.

$$\begin{tikzcd}
0 \arrow[r] & C_k(A \cap B) \arrow[r] \arrow[d, dashed, "\partial"] & C_k(A) \oplus C_k(B) \arrow[r] \arrow[d, dashed, "\partial"] & C_k(A + B) \arrow[r] \arrow[d, dashed, "\partial"] & 0 \\
0 \arrow[r] & C_k(A \cap B) \arrow[r] & C_k(A) \oplus C_k(B) \arrow[r] & C_k(A + B) \arrow[r] & 0
\end{tikzcd}$$

We want a cycle $x + y$ with $\partial[x + y] = [g]$ and $g \mapsto (g, -g)$. We need

$$\begin{tikzcd}
 & x, y \arrow[d, dashed, "\partial"] \\
g \arrow[r] & (g, -g) = (\partial x, \partial y)
\end{tikzcd}$$

So we oughta check that $x = \Delta^k_N, y = \Delta^k_S$ do the job.

Let's look at

$$H_i(D^k, \partial D^k) = H_i(\Delta^k, \partial\Delta^k) \cong \tilde{H}_i(\Delta^k/\partial\Delta^k) \overset{\text{good pair}}{\cong} \tilde{H}_i(S^k)$$

Thus, $\mathrm{id} : \Delta^k \to \Delta^k$ represents a generator of $H_k(D^k, \partial D^k)$.

Now we pursue $H_*^\Delta(X) \cong H_*(X)$.

We need a little preamble. Let X be a Δ complex and A a subcomplex. Then $\Delta_*(A) \hookrightarrow \Delta_*(X)$ is a subchain complex. We can build $\Delta_*(X, A) := \Delta_*(X)/\Delta_*(A)$ and declare that $H_*^\Delta(X, A) := H_k(\Delta_*(X, A))$.

We certainly have $\Delta_*(X) \hookrightarrow C_*(A)$ and $\Delta_*(X) \hookrightarrow C_*(X)$. There is also a relative map $\Delta_*(X, A) \to C_*(X, A)$ by a simplex mapping to its characteristic map. These induce homomorphisms in homology that we want to show are isomorphisms.

First consider X a finite dimensional Δ complex. Let X^k be the k skeleton of X, all simplicies of dimension less than or equal to k.

We now see that the following diagram commutes:

$$\begin{tikzcd}
H_{i+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "1"] & H_i^\Delta(X^{k-1}) \arrow[r] \arrow[d, dashed, "2"] & H_i^\Delta(X^k) \arrow[r] \arrow[d, dashed, "3"] & H_i^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, dashed, "4"] & {} \\
H_{i+1}(X^k, X^{k-1}) \arrow[r] & H_i(X^{k-1}) \arrow[r] & H_i^\Delta(X^k) \arrow[r] & H_i(X^k, X^{k-1}) \arrow[r] & {}
\end{tikzcd}$$

Read page 127. The rows are exact. By induction, 2 and 5 are isomorphisms.

We claim that 1 and 4 are isomorphisms.

$\Delta_i(X^k, X^{k-1}) = \Delta_i(X^k)/\Delta_i(X^{k-1}) = 0$ if $i = k$ and 0 otherwise. Therefore $H_k^\Delta(X^k, X^{k-1}) = \Delta_k(X^k, X^{k-1}) \cong \mathbb{Z}^{\beta k}$ and 0 otherwise.

Yow about $H_i(X^k, X^{k-1})$? It's a good pair! Hence

$$H_i(X^k, X^{k-1}) \cong \tilde{H}_i(X^k/X^{k-1})$$

What's X^k/X^{k-1}? A wedge of k-spheres, one for each simplex in X, β_k!

Lemma 3.1.3. *For good pairs $(X, x_0), (Y, y_0)$, and $X \vee Y = X \sqcup Y/x_0\tilde{y}_0$, we have $\tilde{H}_i X \vee Y \cong \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$.*

Thus, $H_k(X^k, X^{k-1}) = \mathbb{Z}^{\beta_k}$ and 0 otherwise.

We get a basis for this homology group corresponding to the characteristic maps of the simplicies in $(\Delta^k, \partial\Delta^k)$. He then wrote $H_k(\Delta^k, \partial\Delta^k) \cong \mathbb{Z}$ generated by the id on Δ^k imply the isomorphism.

3.2 Week 8

3.2.1 Thursday

Let X be a finite Δ-complex and X^k be its finite $k-$skeleton. Then

$$\begin{tikzcd}
H_{i+1}^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "1"'] & H_i^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "2"] & H_i^{\Delta}(X^k) \arrow[r] \arrow[d, "3"] & H_i^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "4"] & {} \\
H_{i+1}(X^k, X^{k-1}) \arrow[r] & H_i(X^{k-1}) \arrow[r] & H_i^{\Delta}(X^k) \arrow[r] & H_i(X^k, X^{k-1}) \arrow[r] & {}
\end{tikzcd}$$

commutes with exact rows, where each of the labelled maps are isomorphisms.

Now we continue to infinite dimensional Δ-complexes.

Lemma 3.2.1. *If $C \subset X$ is compact, then C meets only finitely many of the simplicies of X.*

Proof. Suppose C meets infinitely many. Then we can find $\{x_i\}_{i=1}^{\infty}$ points in C, all in different open simplicies. Hence, we build $U_i = X - \bigcup_{j \neq i}$. The U_i cover X but there is no finite subcover of C. ❀

We claim that $H_i^{\Delta}(X) \to H_i(X)$ is onto. Let $[z] \in H_i(X)$ be represented by a cycle $z \in C_i(X)$. Then $z = \sum c_j \sigma_j$ where $\sigma_j : \Delta^i \to X$ is continuous with $c_j \in \mathbb{Z}$. Then $\sigma_j(\Delta^i)$ is compact in X since Δ^i is.

That means that this cycle z can meet only finitely many cells in X, so $z \in C_i(X^k)$ for some $k < \infty$. From before, $H_i^{\Delta}(X^k) \cong H_i(X^k)$ so is certainly onto. What this says is that this cycle z is homologous to some simplicial cycle $z' \in \Delta_i(X^k) \subset \Delta_i(X)$. Hence, $z = z' + \partial y$.

Now we want to show that $H_i^{\Delta}(X) \to H_i(X)$ is $1-1$. If $z \in \Delta_i(X)$ is a boundary in $C_i(X)$, $[z] \mapsto 0$ in $H_i(X)$. This says that $z = \partial x$ for some $x \in C_{i+1}(X)$. Now $x = y + \partial w$ for some $y \in \Delta_{i+1}(X)$ so $\partial x = z = \partial y$ in $H_i^{\Delta}(X)$, so $[z] = 0$ in $H_i^{\Delta}(X)$.

For a general $\Delta-$complex pair (X, A), we can compare and contrast sequences to again show an isomorphism between the relative homologies.

Concept time. We're going to go over the Eilenberg-Steenrod Axioms. Let $\mathcal{C}$ be a class of pairs of topological spaces with the property that (X, A) in this class implies $(X, ;), (A, ;), (X, X), (A, A), (X \times I, A \times I) \in \mathcal{C}$. Moreover, $(*, ;) \in \mathcal{C}$ where $*$ is a point space.

A homology theory on $\mathcal{C}$ consists of:

1. An abelian group $H_n(X, A)$ for each pair $(X, A) \in \mathcal{C}$ and $n \in \mathbb{Z}$;
2. A homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$ for each map of pairs $f : (X, A) \to (Y, B)$ and each integer $n \in \mathbb{Z}$;
3. A connecting homomorphism $H_n(X, A) \to H_{n-1}(A, ;)$ for each (X, A) in $\mathcal{C}$ and $n \in \mathbb{Z}$

satisfying the axioms

1. $\mathrm{id}_* = \mathrm{id}$
2. $(f \circ g)_* = f_* \circ g_*$ (that is, $H_n(\bullet, \bullet)$ is a functor from pairs of spaces with morphisms continuous maps to the category of abelian groups with morphisms homomorphisms of groups)
3. δ must be natural:

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\delta"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "(f|_A)_*"] \\
H_n(Y, B) \arrow[r, "\delta"'] & H_{n-1}(B)
\end{tikzcd}$$

4. There is a long exact sequence of the pair;
5. $f \simeq g$ implies $f_* = g_*$;
6. Excision;
7. Dimension; $H_n(*, ;) = \mathbb{Z}$ for $n = 0$ and 0 otherwise.

,

Let $f : S^n \to S^n$ continuous. Then it induces $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ by $\mathbb{Z} \to \mathbb{Z}$, where $f(z) = \alpha z$ for some $\alpha \in \mathbb{Z}$. We say $\deg f = \alpha$ is the degree of f.

Let's go through some properties: $\deg \mathrm{id} = 1$. Moreover, $f \simeq g$ if and only if $\deg f = \deg g$—one direction of this is significantly harder than the other.

If $f : S^n \to S^n$ is not onto, then $\deg f = 0$.

Suppose $y \in S^n$ is not in the image of f. Then we have maps $S^n \to S^n - y \to S^n$. This composite is f, and if we look at the homology, f_* has to factor through the 0 map and so is 0.

We also, by this same motion, show that $\deg f \circ g = \deg f \deg g$.

So if $f : S^n \to S^n$ is a homotopy equivalence, then $\deg f = \pm 1$.

That begs the question: can we exhibit a map that has $\deg f = -1$? Yes! It'll end up being the antipodal map.

If $f : S^n \to S^n$ is the restriction to S^n of a reflection in $\mathbb{R}^{n+1}$ across a hyperplane through the origin, then $\deg f = -1$.

3.3 Week 9

3.3.1 Tuesday

Last time, we started talking about degree, $f : S^n \to S^n$ inducing a map $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n); \alpha \mapsto \deg(f)\alpha$.

The observation about the hyperplane reflections can be used to get our hands on the antipodal map. We can realise $a(x) = -x$ as a composition of $n+1$ reflections, reflecting once over each coordinate. Now, via the composition rule, the degree of a must be $(-1)^{n+1}$.

A quick and dirty consequence: if n is even, $\mathrm{id} \not\simeq a$ and if n odd, $\mathrm{id} \simeq a$.

If $f : S^n \to S^n$ has no fixed point, $\deg f = (-1)^{n+1}$.

Suppose we have a map f with no fixed points. Define $(1-t)f(x) - tx$ as the line segment from $f(x)$ to $-x$. This is nowhere 0, so we may divide by its length to obtain a function $F_t(x)$ that is a homotopy from $f(x)$ to $a(x)$.

Theorem 3.3.1. *S^n has a nowhere zero tangent vector field if and only if n is odd.*

Proof. For n odd, $S^n \subset \mathbb{R}^{2k}$. For $x = (x_1, x_2, x_3, x_4, \dots, x_{2k-1}, x_{2k}) \in S^n$, we define the nowhere zero vector field as $x = (-x_2, x_1, -x_4, x_3, \dots, -x_{2k}, x_2$

Suppose we have such a vector field $V : S^n \to \mathbb{R}^{n+1}$. Then we can normalize the field as $U(x) = V(x)/|V(x)| \in S^n$ that has $U(x) \perp x$. We can use this to define a homotopy

$$F_t(x) = \cos(\pi t)x + \sin(\pi t)u(x)$$

from $S^n \times I \to S^n$. This gives a homotopy $F_0(x) = x$ and $F_1(x) = -x = a(x)$. Hence, $\deg \mathrm{id} = \deg a$ so we must be in odd dimension.

Now we discuss computing degrees from local degrees. If $f : S^n \to S^n$ for $n > 0$, suppose $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$ for a finite list of points in the source sphere. Pick neighbourhoods U_i of x_i and V of Y with $f(U_i, x_i) \to (V, y)$. Now,

$$\begin{tikzcd}
 & H_n(U_i, U_i - x) \arrow[dl, "\cong(\text{excision})"'] \arrow[d, "k_i"'] \arrow[r, "f_*"] & H_n(V, V - y) \arrow[d, "\cong(\text{excision})"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "\rho_i"] \arrow[r, "f_*"'] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[ul, "\cong(\text{pair})"] \arrow[u, "j"] \arrow[r, "f_*"'] & H_n(S^n) \arrow[u, "\cong(\text{pair})"']
\end{tikzcd}$$

Along the top, $f_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by $\deg f|_{x_i} \in \mathbb{Z}$, the local degree of f at x_i.

Proposition 3.3.2. $\deg f = \sum_{i=1}^{n} \deg(f|_{x_i})$.

I'm leaving the proof out because I first don't want to prove it and second I want to read along in class.

What integers can be realised as degrees? Well, if $f : S^1 \to S^1$, we can view $S^1 \subset \mathbb{C}$ to get the map $z \mapsto z^k$. If $k = 0, 1$, $\deg f = k$. If $k = -1, z \mapsto z^{-1}$ is a reflection so has $\deg = -1 = k$. For $k \in \mathbb{Z}^+$, f is covering map so is a local homeomorphism, that is, stretch and rotation. As rotations are homeomorphic to id, $\deg f|_{x_i} = 1$ so we get $\deg(f) = 1 + 1 + \cdots + 1 = k$. For $k < 0$, just compose with z^{-1} and we are done.

Proposition 3.3.3. *For any* $n \geq 1$ *and any* $k \in \mathbb{Z}$, *there exists a map* $f : S^n \to S^n$ *with* $\deg(f) = k$.

Recall the suspension of X, $SX = X/I/\sim$ where $\sim$ identifies the cone points. For $X = S^n, SS^n = S^{n+1}$. For $f : S^n \to S^n$, we can suspend the map $Sf : S^{n+1} \to S^{n+1}$ induced by $f \times \text{id}$ and quotient maps. Claim: $\deg(Sf) = \deg f$

This map f also induces $Cf : (CS^n, S^n) \to (CS^n, S^n)$. If we take the quotient of this, we get the suspension Sf.

$$\begin{tikzcd}
 & & \tilde{H}_{n+1}(CS^n/S^n = SS^n = S^{n+1}) & \\
{} \arrow[r] & H_{n+1}(CS^n) = 0 \arrow[r] \arrow[d] & H_{n+1}(CS^n, S^n) \arrow[r] \arrow[d] \arrow[u, dashed, "\cong"] & H_n(S^n) \arrow[d] \\
{} \arrow[r] & H_{n+1}(CS^n) = 0 \arrow[r] & H_{n+1}(CS^n, S^n) \arrow[r] \arrow[d, dashed, "\cong"] & H_n(S^n) \\
 & & \tilde{H}_{n+1}(CS^n/S^n = SS^n = S^{n+1}) &
\end{tikzcd}$$

3.3.2 Thursday

Time to move on to something new: Cellular homology of CW-complexes X.

A brief recollection: A CW-complex is built from cells; X^0, the zero cells, X^1, the attaching of the one cells, all the way to X^n, the n-skeleton, the attaching of n-cells e^n_α to X^{n-1} via maps $\phi_\alpha S^{n-1} = \partial D^n_\alpha \to S^{n-1}$. We build the n-skeleton by

$$X^n = X^{n-1} \sqcup \{n \text{ disks}\}/(x \sim \phi_\alpha(x) \text{ for } x \in \partial D^n_\alpha)$$

For example, take $X = T = S^1 \times S^1$. X^0 is a point, $X^1 = S^1 \vee S^1$, and $X^2 = X =$ the torus.

What we're going to do is define $H^{CW}_*(X)$ using C^{CW}_*, the cellular chain complex.

We define the chain groups $C^{CW}_n(X)$ as the free abelian groups generated with basis the $n-$cells of X. Moreover, we define the boundary maps. For $n = 1$, $\partial = d$ with the usual simplicial boundary. For larger n, $d(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ for $d_{\alpha\beta} \in \mathbb{Z}$. These integers are the reason we spent so long on degrees.

$$d_{\alpha\beta} = \deg(S^{n-1} \overset{\phi_\alpha}{\to} X^{n-1} \to X^{n-1}/(X^{n-1} \backslash e^{n-1}\beta) = S^{n-1})$$

To continue the torus example, we have that $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ generated by the face, both edges, and the point respectively. $d_1 = 0$ transparently. $d_2(e) = d_{ea} a + d_{eb} b$. First we traverse b in the direction it likes, then in the way it dislikes. So we claim that $d_{eb} = 0$ because the face traverses b both ways and so is nulhomotopic. Similarly, $d_{ea} = 0$. Hence, $d_2 = 0$ also. This is as nice a homology as you'd ever hope for. Therefore, $H^{CW}_i(T)$ is $\mathbb{Z}$ for $i = 0, 2$, $\mathbb{Z} \oplus \mathbb{Z}$ for $i = 1$, and 0 otherwise.

Another example. $X = \Sigma_2$. This has the stop sign identification face, and $C_n^{CW}(X)$ by $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. Once again, the boundary maps vanish and we are left with that chain as our homology groups.

Let's go forward with a bit more formalism in order to show the equivalence of this to singular homology.

Lemma 3.3.4. *If (X_α, x_α) are good pairs, then the inclusions $X_\alpha \hookrightarrow {}_\alpha X_\alpha$ induce an isomorphism $\oplus_\alpha \tilde{H}_*(X_\alpha) \to \tilde{H}_*({}_\alpha X_\alpha)$.*

Lemma 3.3.5. *Suppose we have a CW-complex X. Then*

1. *$H_k(X^n, X^{n-1}) = 0$ for $k \neq n$ and free abelian generated by a basis corresponding to the $n-$cells of X.*
2. *$H_k(X^n) = 0$ for $k > n$.*
3. *$X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n) \xrightarrow{\cong} H_k(X)$ for $k < n$.*

Proof.

1. $H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) = \tilde{H}_k({}_\alpha S_\alpha^n) = \begin{cases} \text{free abelian} & k = n \\ 0 & k \neq n \end{cases}.$
2. Let's look at the long exact sequence of (X^n, X^{n-1}). The homology of the pairs vanishes, and induces an isomorphism $H_k(X^{n-1}) \cong H_k(X^{n-1})$. Chain this down to X^0 to finally get 0.
3. Now, look at $(X^{n+\ell+1}, X^{n+\ell})$ for $\ell = 0, 1, 2, ...$ The long exact sequence of this induces $H_k(X^n) \to H_k(X^{n+1})$. For $k < n$, this is an isomorphism. Now we can bootstrap to see $H_k(X^n) \cong ... \cong H_k(X^{n+m})$. If X is finite dimensional, we are done. Else, use the image of singular chain is compact and so meets finitely many cells. Mimic part of $H_*^\Delta \cong H_*$ proof.

So define d_n as

$$\begin{tikzcd}
 & & & & & & 0 \\
 & & 0 \arrow[dr] & & H_n(X^{n+1}) \arrow[urr] & & \\
 & & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & \\
 {} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\delta_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\delta_n"] & & {} \\
 & & & & & H_{n-1}(X & \\
 {} \arrow[uur] & & & & 0 \arrow[ur] & &
\end{tikzcd}$$

First claim: This is a chain complex with $C_n^{CW}(X) := H_n(X^n, X^{n-1})$. This is due to the fact that $\delta_n \circ j_n = 0$.

Theorem 3.3.6. $H_*(C_*^{CW}(X), d_*) =: H_*^{CW}(X) \cong H_*(X)$.

Proof.

$$H_n(X) \cong H_n(X^{n+1}) \cong H_n(X^n)/\operatorname{im}\delta_{n+1}$$

We can apply j_n, as it is injective, to get $\cong j_n(H_n(X^n))/\operatorname{im}(j_n \circ \delta_{n+1})$. The bottom is $\operatorname{im} d_{n+1}$, and we can then rewrite this as $\cong \ker \delta_n / \operatorname{im} d_{n+1}$. But the kernel of δ^n is the same thing as $\ker j_{n-1}\delta_n = \ker d_n$. Hence, we are done.

An immediate consequence of this is that if X has no n-cells, then $H_n(X) = 0$. if X has k $n-$cells, then $H_n(X)$ has at most k generators.

This makes computing the homology of the sphere almost trivial. Try it on your own! The circle requires the extra acknowledgement that $d_1 = 0$.

Time to talk about projective spaces. First, $\mathbb{R}P^n$; the space of lines through the origin in $\mathbb{R}^{n+1}$. We can communicate this by $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{0\}/\sim$ where $x \sim \lambda x$ for $\lambda \in \mathbb{R} - \{0\}$. We could also construct it via $S^n/\sim$ by essentially the same quotient, $x \sim -x$. We could also construct it by $D^n/\sim$ with $x \sim -x$ for $x \in \partial D^n$. What this last thing is saying is that $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$.

Chapter 4

Weeks 10-12

4.1 Week 10

4.1.1 Tuesday

Today we discuss euler characteristic. Let X be a finite CW-complex and $c_n = \#$ of $n-$cells of X. Then, $\chi(X) = \sum_{n\geq 0}(-1)^n c_n$.

For example, take the sphere. It's two if n is even and 0 if n is odd.

Theorem 4.1.1.

$$\chi(X) = \sum_{n\geq 0}(-1)^n \beta_n$$

where β_n is the nth betti number of X, the rank of $H_n(X)$. As $H_n(X) = H_n^{CW}(X)$, this is a topological invariant.

We need the following fact: If $0 \to A \to B \to C \to 0$ is short exact with finitely generated abelian groups, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

This is an exercise.

Proof. Let

$$0 \to ... \to C_k \to C_{k-1} \to ... \to C_1 \to C_0 \to 0$$

be a chain comples of finitely generated abelian groups. Let $Z_n = \ker(d_n)$ be the cycles and $B_n = \mathrm{im}(d_{n+1})$ be the boundaries. Then $H_n = Z_n/B_n$. We have the following sequences:

$$0 \to B_n \to Z_n \to H_n \to 0$$

and

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

We can now use the fact twice and we get that $\mathrm{rank}\,(C_n) = \mathrm{rank}\,(Z_n) + \mathrm{rank}\,(B_{n-1})$ and $\mathrm{rank}\,(Z_n) = \mathrm{rank}\,(B_n) + \mathrm{rank}\,(H_n)$. Then we define

$$\chi(C_*) = \sum_{n\geq 0} (-1)^n \mathrm{rank}\,(C_n)$$

which we can rewrite as

$$\chi(C_*) = \sum_{n\geq 0} (-1)^n \left(\mathrm{rank}\, B_n + \mathrm{rank}\, H_n + \mathrm{rank}\, B_{n-1}\right).$$

After the telescoping, $\chi(X) = \sum_{n\geq 0} (-1)^n \mathrm{rank}\, H_n$.

The theorem is the case $C_* = C_*^{CW}(X)$. ❀

This allows us to talk about the classicication of surfaces. A compact surface without boundary is homeomorphic to one of

- S^2
- Σ_g
- $N_h = \#^h \mathbb{R}P^2$

A big point of this theorem is that all are distinct spaces.

We've already computed that $\chi(S^2) = 2$; and we compute Σ_g to be $1 - 2g + 1 = 2 - 2g$. For the nonorientable case, $\chi(N_h) = 1 - h + 1 = 2 - h$.

As we can see, the only thing we have left is to compute orientability. But orientability is hard. Let's think about how the homology relates to it.

It's a fact that if we have a nonorientable n-manifold, then the nth homology vanishes.

So let's calculate the cellular homology of these surfaces. We called the cellular boundary $d : C_n^{CW}(X) = H_n(X^n, X^{n-1}) \to H_n(X^{n-1}, X^{n-2})$. For one of the n-cells of X, we have these two maps:

$$\Phi_\alpha(D^n_\alpha, \partial D^n_\alpha) \to (X^n, X^{n-1})$$

and

$$\phi_\alpha = \Phi_\alpha \mid_{\partial D^n_\alpha}$$

We call Φ the characteristic map and ϕ the attaching map.

We may have already ruined the surprise: $d_n(e^n_\alpha) = \sum_{n-1 \text{ cells}} d_{\alpha\beta} e^{n-1}_\beta$ where $d_{\alpha\beta}$ os the degree of

$$S^{n-1}_\alpha \to X^{n-1} \to X^n/(X^{n-1} - e^{n-1}_\beta) = S^{n-1}_\beta.$$

For $n = 1$ this is just the simplicial boundary! Moreover, if X is connected with only one zero cell, then $d_1 = 0$.

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\delta"'] \arrow[d, "\Phi_{\alpha_*}"] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "(\Delta_{\alpha\beta})_*"] \arrow[d, "\phi_{\alpha_*}"'] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"'] \arrow[d, "j_{n-1}"'] & \tilde{H}_{n-1}(X^{n-1}/X^{n-1}) \arrow[u, dashed, "(q_\beta)_*"'] \arrow[d, dashed, "\mu"] \\
 & H_{n-1}(X^{n-1}, X^{n-1}) \arrow[r, "\lambda"'] \arrow[ur, "\psi"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^n
\end{tikzcd}$$

Where q is a quotient and q_β is as outlined before. This $\Delta_{\alpha\beta} = q_\beta q \phi_\alpha$, and it's the map we want the degree of.

This e^n_α, one of the generators of the free abelian group, is $\Phi_{\alpha*}([D^n_\alpha])$. By definition, $d_n(e^n_\alpha) = j_{n-1}\delta_n(e^n_\alpha)$. Then $(q_\beta)_*$ projects $\tilde{H}_{n-1}(X^{n-1}/X^{n-2})$ into $\mathbb{Z} = \tilde{H}_{n-1}(S^{n-1}_\beta)$ generated by e^{n-1}_β.

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\delta{,}\cong"'] \arrow[d, "\Phi_{\alpha_*}"] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "(\Delta_{\alpha\beta})_*"] \arrow[d, "\phi_{\alpha_*}"'] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"'] \arrow[d, "j_{n-1}"'] & \tilde{H}_{n-1}(X^{n-1}/X^{n-1}) \arrow[u, dashed, "(q_\beta)_*"'] \arrow[d, dashed, "\mu{,}\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\lambda{,}\cong"'] \arrow[ur, "\psi{,}\cong"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^n
\end{tikzcd}$$

Now, $\psi d_n e^n_\alpha = \psi j_{n-1}\delta_n(e^n_\alpha) = \psi j_{n-1}\delta_n \Phi_{\alpha*}([D^n_\alpha])$. Working on the commutativity, we have that this whole thing is $\psi j_{n-1}\phi_{\alpha*}(\delta[D^{n-1}_\alpha])$. Hence, this is $q_*\phi_{\alpha*}(\delta[D^{n-1}_\alpha])$. So $q_{\beta*}\psi d_n(e^n_\alpha) = q_{\beta*}q_*\phi_{\alpha*}\delta[D^n_\alpha]$, and the first three functions give our friend. Hence

$$q_{\beta*}\psi d_n(e^n_\alpha) = \deg \Delta_{\alpha\beta} \cdot \delta[D^n_\alpha]$$

We should go back and revisit the surfaces a little more carefully now.

We've done the homology of the sphere calculation many times, but let's think of $C_*^{CW}(\Sigma_g)$. Well, we can read things off as

$$\ldots 0 \to \mathbb{Z} \to \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \to 0.$$

The orientation goes... a_1 with, stay, a_1 against, so $S^1_e \to S^1_{a_1}$ has degree 0. In fact, the degree of the map itself is 0! So $d_2 = 0$. The cellular homology then stares at one in the face.

Now, on $C_*^{CW}(N_h)$,

$$\ldots 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}^h \xrightarrow{0} \mathbb{Z} \to 0.$$

The only sort of interesting thing happens for $H_1 \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}_2$. Now we're done with the classification of surfaces, for good.

Let's now take a closer look at real projective space $\mathbb{R}P^n$. We saw $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$, and the quotient from the sphere. We saw that we can then build $\mathbb{R}P^n = e^0 \cup \cdots \cup e^n$. This is just a bunch of copies of $\mathbb{Z}$, so as long as we can get our hands on the degree of each map, we're golden!

For d_{n+1}, we need $\deg(S^n \xrightarrow{\phi} \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} = S^n)$. We compute via local degrees. One point in the target has two preimages (one under q, two under ϕ), so we can write y has preimage $x_1, -x_1$ with neighbourhoods V, U_1, U_2. We can assume $f : U_1 \to V$ is a local homeomorphism with local degree 1. Now, $f : -U_1 \to V$ is just fa so has degree 1 in dimension odd and -1 in dimension even. Hence, $\deg f = 1 + (-1)^{n+1}$.

4.1.2 Thursday

$$C_*^{CW}(\mathbb{R}P^n): \qquad \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z} \longrightarrow 0$$

Where $d_k(e^k) = 2e^{k-1}$ for k even and 0 for k odd.

So for $n = 2$, $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ yields $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$, and $H_2 = 0$.

What about $n = 3$? $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0}$ where $H_0 = \mathbb{Z}$, and $H_1(\mathbb{Z}_2)$, $H_2 = 0$, and $H_3 = \mathbb{Z}$.

Blah blah blah,

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i < n \text{ odd} \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Now let's think about two maps $\mathbb{R}P^2 \to S^2$. Then $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^2$ and $S^2 = \hat{e}^0 \cup \hat{e}^2$.

This first one is stupid. $c : \mathbb{R}P^2 \to S^2; e^i \mapsto \hat{e}^0$. The other, $f : \mathbb{R}P^2 \to S^2; e^2 \mapsto \hat{e}^2, e^i \mapsto \hat{e}^0$. This means that f is homeomorphic on open 2-cells. We get

$$c_* : H_*(\mathbb{R}P^2) \to H_*(S^2), f_* : H_*(\mathbb{R}P^2) \to H_*(S^2)$$

and isomorphisms on H_0.

What about $c_* : H_1(\mathbb{R}P^2) \to H_1(S^2), \mathbb{Z}_2 \to 0$? 0. Similar for c_* on H_2. f_* has to be the same; so $f \simeq c$? Doesn't feel like it. Let's now talk about $\mathbb{C}P^n$, the space of complex lines thou the origin in $\mathbb{C}^{n+1}$. The list of inclusions $C_1 \subset C_2 \dots$ gives rise to the inclusions $\mathbb{C}P^0 \subset \mathbb{C}P^1 \dots$. The first two are a point and S^2, up to homeomorphism. What about $\mathbb{C}P^n$ more generally? Then $\mathbb{C}P^n = \mathbb{C}^{n+1} \backslash \{0\} / \sim$, where $v \sim \lambda v$ for $\lambda \in \mathbb{C}^*$. We can then, without loss of information, $S^{2n+1}/\sim$ for $v \sim \lambda v$ for $|v| = 1$. Let's name the quotient map q. It's frequently referred to as a Hopf map. What are the fibres of this map? They are circles! Thus, $\mathbb{C}P^n$ can be obtained by attaching $2n$-cell to $\mathbb{C}P^{n-1}$. If $\Phi : (D^{2n}, \partial D^{2n} = S^{2n-1}) \to (\mathbb{C}P^n, \mathbb{C}P^{n-1})$ is the characteristic map of the $2n$-cell $\phi = \Phi|_{\partial D^{2n}}$ attaching map is just $S^{2n-1} \to \mathbb{C}P^{n-1}$. We can write down explicit formulas:

$$\Phi(z_1, \dots, z_n) = (z_1 : z_2 : \dots : z_n : \sqrt{1 - |z|^2})$$

This implies $\mathbb{C}P^k$ can be viewed as the $2k$-skeleton of $\mathbb{C}P^n$. So $\mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$. Homework: What's $H_*(\mathbb{C}P^n)$?

Now it's time to talk about homology with coefficients. Let X be a topological space, and G a fixed abelian group. An n-chain with coefficients in G is a member of the set $C_n(X : G)$. We take the same boundary as usual:

$$\partial_n(\sum_i m_i \sigma_i^n) = \sum_i m_i \sum_j (-1)^j \sigma_i^n|_{v_0 \dots \hat{v}_j \dots v_n}.$$

We still have $\partial\partial = 0$, so... homology?

We have that $C_*(X:G), \partial_*$ is a new chain complex. Its homology is $H_*(X:G)$, the singular homology of X with coefficients in G.

For $A \subset X$, $C_*(A:G) \hookrightarrow C_*(X:G)$ is a subcomplex. $C_*(X,A:G)$ is the quotient complex, which has the relative homology with $G-$coefficients.

Additionally, we discuss the reduced homology $\tilde{H}_*(X:G) = H_*(C_*(X:G) \xrightarrow{\epsilon} G)$ where $\sum m_i\sigma_i^0 \mapsto \sum m_i$.

What are $\tilde{H}_*(\_ : G)$ properties?

- $\tilde{H}_i(\text{point} : G) = 0$;
- homotopy invariant,
- long exact sequence of the pair,
- excision.
- When relevant, $H_*(X:G) \cong H_*^{\Delta}(X:G) \cong H_*^{CW}(X:G)$.

Let's look at $H_*(\_ : \mathbb{Z}_2)$. Recall our map f from way earlier today. How can we apply this to cellular homology?

Call $\Phi : (D^2, \partial D^2) \to (\mathbb{R}P^2, e^0 \cup e^1)$ the characteristic map of e^2, the two cell of $\mathbb{R}P^2$.

$$
\begin{tikzcd}
(D^2, \partial D^2) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & (\mathbb{R}P^2, e^0 \cup e^1) \arrow[d, "f"] \\
& (S^2, \hat{e}^0)
\end{tikzcd}
$$

$$
\begin{array}{cccccc}
 & 2 & & 1 & & 0 \\
C_*^{CW}(\mathbb{R}P^2) & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & \downarrow & & \downarrow{\scriptstyle f_*=0} & & \downarrow{\scriptstyle f_*=\mathrm{id}} \\
C_*^{CW}(S^2) & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}
\end{array}
$$

$$
\begin{tikzcd}
H_2(D^2, \partial D^2) \arrow[r, "\Phi_*"] \arrow[dr, "\Psi_*"'] & H_2(\mathbb{R}P^2, e^0 \cup e^1) \arrow[d, "f_*"] \\
& H_2(S^2, \hat{e}^0)
\end{tikzcd}
$$

Now think about $C_*^{CW}(\_ : \mathbb{Z}_2)$.

$$\begin{array}{lccc} & 2 & 1 & 0 \\ C_*^{CW}(\mathbb{R}P^2 : \mathbb{Z}_2) & \mathbb{Z}_2 \xrightarrow{0} & \mathbb{Z}_2 \xrightarrow{0} & \mathbb{Z}_2 \\ & \downarrow \cong & \downarrow 0 & \downarrow \cong \\ C_*^{CW}(S^2 : \mathbb{Z}_2) & \mathbb{Z}_2 \xrightarrow{0} & 0 \xrightarrow{0} & \mathbb{Z}_2 \end{array}$$

In the last two minues, what happens here when you pass to homology?

We get $H_2(\mathbb{R}P^2 : \mathbb{Z}_2) = \mathbb{Z}_2$, $H_1(\mathbb{R}P^2 : \mathbb{Z}_2) = \mathbb{Z}_2$, $H_2(S^2 : \mathbb{Z}_2) = \mathbb{Z}_2$, $H_1(S^2 : \mathbb{Z}_2) = 0$. We still have $f_* = 0$ on dimension 1, but now we have an isomorphism in dimension 2. Here, we see that f_* is nonzero but c_* IS zero.

4.2 Week 11

4.2.1 Tuesday

Lemma 4.2.1 (Hatcher p. 262)**.** *If $A \xrightarrow{i} \xrightarrow{j} C \to 0$ is exact of abelian groups, then $A \otimes G \xrightarrow{i \otimes \mathrm{id}_G} B \otimes G \xrightarrow{j \otimes \mathrm{id}_G} C \otimes G \to 0$ is exact.*

If we have a chain complex, we can form a new chain complex $C_* \otimes G = (C_n \otimes G, \partial_n \otimes \mathrm{id}_G)$. This gives rise to

$$H_*(C_* : G)$$

our current object of study.

Lemma 4.2.2 (Hatcher p. 147)**.** *If*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is short exact of abelian groups, the following are equivalent:

1. *There exists an isomorphism $B \to A \oplus C$ so that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus C & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

commutes.

2. *There exists a homomorphism* $\rho : B \to A$ *so that* $\rho i = \mathrm{id}_A$.
3. *There exists a homomorphism* $s : C \to B$ *so that* $js = \mathrm{id}_C$.

How do we relate $H_*(C_*; G)$ and $H_*(C_*)$? As we frequently do, we call $Z_n = \ker \partial_n$ and $B_n = \operatorname{im} \partial_{n+1}$.

Then we have a beautiful little ladder:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Because B_n is free, the exact rows split.

The diagram is a split short exact sequence of chain complexes. Now, if we tensor with G, our rows still split. See the calculation $(A \oplus B) \otimes G \cong (A \otimes G) \oplus (B \otimes G)$. This means we get a long exact sequence in homology.

$$\cdots \longrightarrow Z_n \otimes G \longrightarrow H_n(C_* : G) \longrightarrow B_{n-1} \otimes G \longrightarrow Z_{n-1} \otimes G \longrightarrow \cdots$$

Check that the connecting homomorphism is just $i_n \otimes \mathrm{id}_G$. This gives

$$0 \to \operatorname{coker}(i_n \otimes \mathrm{id}_G) \to H_n(C; G) \to \ker i_{n-1} \otimes \mathrm{id}_G \to 0$$

Tensoring the right exact squence, we get $\operatorname{coker}(i_n \otimes \mathrm{id}_G) = H_n(C_n) \otimes G$.

Theorem 4.2.3 (Universal Coefficient Theorem)**.** *Let* C_* *be a chain complex of free abelian groups and* G *also abelian. For each* n*, there is a natural short exact sequence*

$$0 \to H_n(C_*) \otimes G \to H_n(C_*(G)) \to \operatorname{Tor}(H_{n-1}(C_*), G) \to 0.$$

The sequence splits but not naturally.

Here's what natural would mean: If $\phi : C_* \to D_*$ is a chain map, we get a commuting diagram:

$$\begin{tikzcd} 0 \arrow[r] & H_n(C_*) \otimes G \arrow[r] \arrow[d, dashed, "\phi_* \otimes \mathrm{id}_G"] & H_n(C_*; G) \arrow[r] \arrow[d, dashed, "(\phi \otimes \mathrm{id}_G)_*"] & \mathrm{Tor}(H_{n-1}(C_*), G) \arrow[r] \arrow[d, dashed, "\phi_*"] & 0 \\ 0 \arrow[r] & H_n(D_*) \otimes G \arrow[r] & H_n(D_*; G) \arrow[r] & \mathrm{Tor}(H_{n-1}(D_*), G) \arrow[r] & 0 \end{tikzcd}$$

A couple comments about the splitting: The group in the middle, $H_n(C_*; G) \cong (H_n(C_*) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(C_*), G)$ but the decomposition does not, in general, commute with chain maps.

This all begs the question: What is Tor?

Let H and G be abelian groups. We define $\mathrm{Tor}(H, G) = \mathrm{Tor}_1^{\mathbb{Z}}(H, G)$ using resolutions. H has a free resolution of the form

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where F_0, F_1 are free abelian. E.g., F_0 is free abelian on generators of H and f_0 is the natural map. Now, $F_1 = \ker f_0$, $f_1 : F_1 \hookrightarrow F_0$ is the inclusion. F_1 is then a subgroup of F_0 and hence free abelian.

Then form

$$F_1 \otimes G \to F_0 \otimes G \to H \otimes G \to 0$$

and define $\mathrm{Tor}(H, G) = \ker(f_1 \otimes \mathrm{id}_G)$.

Before, we had

$$0 \to \mathrm{coker}(i_n \otimes \mathrm{id}_G) \to H_n(C_*; G) \to \ker(i_{n-1} \otimes \mathrm{id}_G) \to 0$$

where $i_n : B_n \hookrightarrow Z_n$. We also saw that $\mathrm{coker}(i_n \otimes \mathrm{id}_G) = H_n(C_*) \otimes G$.

But, the exact sequence from earlier is a free resolution of $H_{n-1}(C_*)$, so $\ker(i_{n-1} \otimes \mathrm{id}_G) = \mathrm{Tor}(H_{n-1}(C_*), G)$.

Now let's restate the theorem, but for topological spaces.

Theorem 4.2.4 (Universal Coefficients for Spaces). *Let X be a topological space and G be an abelian group. Then*

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0$$

that is exact for each n and is natural with respect to continuous maps. The sequence splits but the splitting is not natural.

4.2.2 Thursday

Recall that $\mathrm{Tor}(H,G) = \ker[f_1 \otimes \mathrm{id}_G : F_1 \otimes G \to F_0 \otimes G]$ where

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to)$$

is a free resolution of H.

Let's talk about some Tor properties.

1. Independent of choice of resolution;
2. $\mathrm{Tor}(H,G) \cong \mathrm{Tor}(G,H)$;
3. If A or B is free abelian, then $\mathrm{Tor}(A,B) = 0$. A toy illustration. If $A = \mathbb{Z}$, we have the resolution $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$
4. $\mathrm{Tor}(A \oplus B, G) = \mathrm{Tor}(A,G) \oplus \mathrm{Tor}(B,G)$.
5. $\mathrm{Tor}(\mathbb{Z}_k, G) = \{g \in G \mid kg = 0\}$. The reason; we have the resolution $\mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \to \mathbb{Z}_k$ so $\mathrm{Tor}(\mathbb{Z}_k, G) = \ker[\mathbb{Z} \otimes G \to \mathbb{Z} \otimes G] = \{g \in G \mid kg = 0\}$.
6. $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}$.

On the splitting...

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

is short exact of free abelian groups. Hence, this sequence splits, implying that there exists $\rho : C_n \to Z_n$ that is a left splitting. This gives us a map $C_n \xrightarrow{\rho} Z_n \xrightarrow{\text{project}} H_n(C_*) = Z_n/B_n$. Let's view this as a chain map:

$$\begin{tikzcd}
C_n \arrow[r] \arrow[d, "\partial"'] & Z_n \arrow[r] & H_n(C_*) \arrow[d] \\
C_{n-1} \arrow[rr] & & H_{n-1}(C_*)
\end{tikzcd}$$

from $(C_*, \partial_*) \to (H_*(C_*), 0)$, we obtain a chain map $(C_* \otimes G, \partial_* \otimes \mathrm{id}_G) \to (H_*(C_*) \otimes G, 0)$ inducing $H_n(C_*; G) \to H_n(C_*) \otimes G$. This gives a left splitting

$$0 \to H_n(C_*) \otimes G \to H_n(C_*; G) \to \mathrm{Tor}(H_{n-1}(C_*), G) \to 0.$$

For example, take $H_*(\mathbb{R}P^2; \mathbb{Z}_2)$ for $n = 0$.

$$H_0(\mathbb{R}P^2) \otimes \mathbb{Z}_2 \to H_0(\mathbb{R}P^2) \to 0$$

so transparently, $H_0(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$. For $n = 1$,

$$H_1(\mathbb{R}P^2) \otimes \mathbb{Z}_2 \to H_1(\mathbb{R}P^2;\mathbb{Z}_2) \to \mathrm{Tor}(H_0(\mathbb{R}P^2),\mathbb{Z}_2)$$

and as $H_0(\mathbb{R}P^2)$ is free, $H_1(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$. For $n = 2$, as $H_2(\mathbb{R}P^2)$ is trivial and $\mathrm{Tor}(H_1(\mathbb{R}P^2),\mathbb{Z}_2) \cong \mathbb{Z}_2$, our result is also simple. Hence,

$$H_1(\mathbb{R}P^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & i = 0,1,2 \\ 0 & \text{else} \end{cases}$$

Let's do an even easier one. What about $H_*(\mathbb{R}P^2,\mathbb{Q})$? Well, $\mathrm{Tor}(H,\mathbb{Q}) = 0$ for any abelian group H. On the other side, $\mathbb{Z}_k \otimes \mathbb{Q} = 0$ so

$$H_n(X) \otimes \mathbb{Q} \cong \mathbb{Q}^{b_n}$$

where b_n is the rank of $H_n(X)$.

This leads us to cohomology, but first, the Hom tensor.

Let A, B be abelian groups. Then $\mathrm{Hom}(A,B)$ is the group homomorphisms from A to B. This naturally has an abelian group structure. If you have a homomorphism $f : A \to B$, then you get a homomorphism $\mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ $\phi : B \to G$ maps to $f^*\phi = \phi f$.

What if we have two maps $A \xrightarrow{f} B \xrightarrow{g} C$, and $\phi \in \mathrm{Hom}(C,G)$? Then $(gf)^* : \mathrm{Hom}(C,G) \to \mathrm{Hom}(A,G)$ is produced by $(gf)^*\phi = \phi gf = (\phi g)f = f^*(\phi g) = f^*g^*\phi$. Thus, we have a functor $\mathrm{Hom}(\_,G)$. However, it's not exact. The standard example is:

$$0 \to \mathbb{Z} \xrightarrow{\times 2}\to \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

when we apply $\mathrm{Hom}(\_,\mathbb{Z})$. We get

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z})$$

which is

$$\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0$$

so fails exactness on the left.

Lemma 4.2.5. *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact sequence of abelian groups, then $\mathrm{Hom}(A,G) \xleftarrow{f^*} \mathrm{Hom}(B,G) \overset{g^*}{\mathrm{Hom}}(C,G) \leftarrow 0$ *is exact.*

Proof. g^* injects: If $g^*(\phi) = 0$, then $\phi(g(b)) = 0$ for all $b \in B$. This implies $\phi(c) = 0$ for all $c \in C$ since g is onto, and hence $\phi = 0$.

$f^*g^* = 0$: $f^*g^* = (gf)^* = 0^* = 0$ so $\operatorname{im} g^* \subseteq \ker f^*$. The other containment is as follows: If $f^*(\psi) = 0$, then $\psi(f(a)) = 0$ for all $a \in A$.

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "0"] & \\
B \arrow[r, "\psi"] \arrow[d, "g"'] & G \\
C \arrow[ur, "\bar{\psi}"'] &
\end{tikzcd}$$

This induces a map $\bar{\psi}(g(b)) = \psi(b)$, so $\psi = \bar{\psi}g = g^*(\bar{\psi})$, so $\psi \in \Im g^*$.

Remark: Hom is exact when the sequence is split.

We can now start talking about cohomology.

Let (C_*, ∂_*) be a chain complex of abelian groups and some abelian group G. We form a cochain complex (C^*, δ^*) where $C^n = \operatorname{Hom}(C_n, G)$ and $\delta^n = \partial^*_{n+1}$. We have $\delta\delta = 0$, because $\delta\delta = \partial^*\partial^* = (\partial\partial)^* = 0$. We have that the homology of this hain complex is called cohomology, where

$$H^n(C_*; G) = \frac{\ker(\delta^n)}{\operatorname{im}(\delta^{n-1})}.$$

Singular cohomology time.

Take $C_* = C_*(X)$. Then $H^*(X; G)$ and when $G = \mathbb{Z}$ we write $H^*(X)$

Appendix A

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