Algebra II

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Information

Time & Room	MWF 13:30-14:20, Lockett 232
Exam	Monday May 5, 10:00-12:00
Textbook	Dummit & Foote, Abstract Algebra
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Office Hours	Monday 12:30-13:20, Tuesday 13:30-1420 (Tentative)

Here's what we're going to cover:

- Chapter 13-14, Field and Galois theory;
- Chapter 15, Commutative algebra and algebras over a field;
- Chapter 10, Basics for modules and their tensor products;
- Chapter 18, Wedderburn's theorem, Maschke's theorem and linear representations of finite groups;
- POSSIBLY Chapter 17, homological algebra.

Grade Distribution

Homework	60%
Midterm	20%
Final	20%

Chapter 13

Field Theory

13.1 Extensions

Definition 13.1.1 (Field). *A field is a commutative ring in which every nonzero element is invertible.*

We denote by $F^{\times} = F \setminus \{0\}$ *the set of all invertible elements of the field* F*.*

In general, we denote $R^{\times}$ *as the set of all units of the ring* R*.*

Definition 13.1.2 (Characteristic). *Let* F *be a field with identity* 1*. The characteristic of* F *is the order of* 1 *in the group* $(F, +)$*. If the order of* 1 *is not finite, we define the characteristic of* F *to be* 0*.*

We denote the characteristic as $\mathrm{ch}(F)$*.*

We know that $\mathbb{Z}/p\mathbb{Z}$ is a field of order p if p is a prime.

Because $\mathbb{Q}$ has $n1 \neq 0$ for $n \neq 0$, $\mathrm{ch}(\mathbb{Q}) = 0$. Some other fields with characteristic zero are $\mathbb{R}, \mathbb{C}, \mathbb{C}(x) \ldots$

We denote by $\mathbb{Z}_p(x)$ as the field of rational functions over $\mathbb{Z}_p$. That is, we're adjoining the element x. It is an infinite field with finite characteristic.

Let's say that G is an abelian group. If we write it multiplicatively, $g^n = g \cdots g$ n-many times. If we write it additively, we write $g = ng$.

Proposition 13.1.3 (Characteristic of a Field)**.** *The characteristic of a field is* 0 *or a prime number.*

Proof. Towards a contradiction, let F be a field with $\mathrm{ch}(F) \neq 0$. Then $\mathrm{ch}(F) = nm$ for $1 < n, m < \mathrm{ch}(F)$. Because $n \cdot 1_F := n, m \cdot 1_F := m \in F$, we have that $n \cdot m = nm \cdot 1_F = 0$.

Definition 13.1.4 (Prime Subfield)**.** *Let F be a field. The prime subfield of F is the subfield generated by* 1*.*

We follow with examples.

(a) $\mathbb{Z}_p(x) \geq \mathbb{Z}_p$.

(b) $\mathbb{R} > \mathbb{Q} \geq \mathbb{Q}$.

Proposition 13.1.5 (Prime Subfield)**.** *Let F be a field and K the prime subfield of F. If* $\mathrm{ch}(F) = p \neq 0$*, then $K \cong \mathbb{Z}_p$. If* $\mathrm{ch}(F) = 0$*, $K \cong \mathbb{Q}$.*

Proof. Define $\varphi : \mathbb{Z} \to F$; $\varphi(n) = n1$. We know that φ is a ring homomorphism; I omit the proof for being rather repetetive. Such a proof is necessary to remember, however.

The kernel of φ is an ideal. In particular, because $\mathbb{Z}$ is a principal ideal domain, $\ker(\varphi) = (a)$ for some nonnegative integer a. If $a = 1$, φ is the zero map. If $a = 0$, φ is injective. In this case, $\mathbb{Z} \cong \varphi(\mathbb{Z}) \subset F$. Hence, $\mathbb{Q} \xrightarrow{\tilde{\varphi}} F$ (this is to be read $\mathbb{Q}$ extends to F), so $K \cong \mathbb{Q}$.

In the case that $a \neq 0$, then $a1 = 0$. This implies that $\mathrm{ch}(F) = p|a$. Hence $(p) \subset \ker(\varphi) = (a) \subset (p)$; Thus $(p) = (a)$. Thus, $a = p$. Hence $\varphi(\mathbb{Z}) \cong \mathbb{Z}/(p) = \mathbb{Z}_p$ by the first isomorphism theorem, so $K \cong \mathbb{Z}_p$.

Definition 13.1.6 (Extension)**.** *If F contains a subfield K, we call F an extension of K, written as F/K.*

In this case, we have the diagram

$$\begin{array}{c} F \\ | \\ K \end{array}$$

Additionally, F is a $K-$vector space.

We denote the dimension or index of the extension $\dim_K F = [F : K]$.

For example, $\mathbb{C} \geq \mathbb{Q}$, so $\mathbb{C}$ is a $\mathbb{Q}-$linear space of uncountably infinite dimension: $[\mathbb{C} : \mathbb{Q}] = \infty$.

In a particularly obvious case, we have $[\mathbb{C} : \mathbb{R}] = 2$ for the extension $\mathbb{C}/\mathbb{R}$.

Let's take $\mathbb{Q}(i) = \{a + bi | a, b \in \mathbb{Q}\}$; $\mathbb{Q}(i)/\mathbb{Q}$ is an extension of degree 2.

Theorem 13.1.7 (Extension Index)**.** *Let $L/K, K/F$ be finite extensions. Then L/F is finite and $[L : F] = [L : K][K : F]$.*

$$\begin{tikzcd} L \arrow[d, dash, "{[L:K]<\infty}"] \\ K \arrow[d, dash, "{[K:F]<\infty}"] \\ F \end{tikzcd}$$

In time, we learn these are tensor products.

Proof. Let $A = \{\alpha_1, \dots, \alpha_n\}$ be a basis for K over F and $B = \{\beta_1, \dots, \beta_m\}$ be a basis for L over K. Let $C = \left\{\alpha_i \beta_j \middle| i \in [n], j \in m\right\}$. Naturally, $|C| = mn$. We seek to show that C is a basis. Let $x \in L$; then $x = \sum_{i=1}^n k_i \beta_i$ for some $k_i \in K$. But each k_i can be written as $k_i = \sum_{j=1}^m f_{ij} \alpha_j$. Hence, $x = \sum_{i=1}^n \sum_{j=1} f_{ij} \alpha_j \beta_i$. Hence, C spans L over F.

Suppose now that $\sum f_{ij} \alpha_j b_i = 0$. Then $\sum_i \left(\sum_j f_{ij} \alpha_j\right) \beta_i = 0$. By the independence of B over K, and A over F, $f_{ij} = 0$. Thus C is independent and C is a basis.

Theorem 13.1.8 (Finite Subextension)**.** *If L/F is finite and K/F is a subextension, that is,*

$$\begin{tikzcd} L \arrow[d, dash] \\ K \arrow[d, dash] \\ F \end{tikzcd}$$

then K/F is finite and $[K : F] | [L : F]$.

A neat consequence of this is: If L/F is finite and $[L : F]$ is prime, L/F has no nontrivial subextensions.

Proposition 13.1.9 (Field to Ring Homomorphism). *If $\varphi : F \to R$ is a ring homomorphism with $\varphi(1_F) \to 1_R$ where F is a field and R is a ring. Then φ is injective.*

Proof. The only ideals of F are (0) and F. As $\varphi(1) = 1$, $\ker(\varphi) =\neq F$ so $\ker \varphi = 0$ and φ is injective.

Let F be a field and $F[x]$ be a polynomial ring over F. Then, $F[x]^\times = F^\times$. Let p be irreducible in $F[x]$ and let $K = \frac{F[x]}{(p(x))}$, a field as $(p(x))$ is maximal.

Then, every element is of the form $\overline{f(x)} = f(x) + (p(x))$. Now, define $\varphi : F \to K$ via $\varphi(\alpha) = \alpha + (p(x))$. Naturally, $\varphi(1) = 1 + (p(x))$, $\varphi(\alpha + \beta) = \varphi(a) + \varphi(b)$, and $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ so it is a ring homomorpism. Therefore, φ is an embedding and $F \cong \varphi(F)$. Identify $F \equiv \overline{F} \subset K$.

Theorem 13.1.10 (Polynomial Extension). *Let $p(x) \in F[x]$ be an irreducible polynomial. Then $p(x)$ has a root $\theta = x + (p(x)) \in K$ and $[K : F] = \deg p(x)$. Moreover, the set $\{1, \theta, \dots, \theta^{\deg p(x) - 1}\}$ is a basis of K over F.*

Proof. $p(\theta) = p(x) + (p(x)) = (p(x)) = 0$ in K. We want to show that $\{1, \theta, \dots, \theta^{\deg p(x)-1}\}$ is a basis. For any $\overline{f(x)} \in K$, $\overline{f(x)} = f(x) + (p(x))$. By the division algorithm, $f(x) // p(x) = r(x)$ where $\deg r(x) < \deg p(x)$ or $r(x) = 0$. Let $r(x) = \sum_{i=0}^{n-1} a_i x^i$ for some $a_i \in F$. Moreover, $\overline{f(x)} = \overline{r(x)} = a_0\overline{1} + a_1\overline{x} + \dots a_{n-1}\overline{x}^{n-1} = \sum_{i=0}^{n-1} a_i\theta^i$ and hence $\{1, \theta, \dots, \theta^{\deg p(x)-1}\}$ spans.

Linear independence is left to the reader.

We were working on field extensions $K = \frac{F[x]}{(p(x))}$ where p is irreducible over F.

If $\theta = x + (p(x)) \in K$, then $p(\theta) = 0$. Moreover, $\{1, \theta, \dots, \theta^{n-1}\}$ spans K over F where $n = \deg p(x)$.

We want to show that $\{1, \theta, \dots, \theta^{n=1}\}$ is linearly independent over F.

Suppose $a_0 + a_1\theta + \dots + a_{n-a}\theta^{n-1} = 0$ in K.

Consider the polynomial $g(x) = a_0 + a_1x + \dots x_{n-1}^{n-1} \in F[x]$. This implies that $g(\theta) = 0$ in K. This means that $g(x) + (p(x)) = 0$. Hence $g(x) \in (p(x))$, so $p(x)|g(x)$. This is a contradiction as $\deg p > \deg g$, unless $g \equiv 0$ and $a_0, \dots, a_{n-1} = 0$.

Therefore, $K = F[x]/(p(x))$ is an extension in which $p(x)$ has a root.

Theorem 13.1.11 (Existence of Root Extensions)**.** *Let $f(x) \in F[x]$ be a nonconstant polynomial. There exists a field extension in which $f(x)$ has a root.*

Proof. Because $F[x]$ is a PID, we have the unique factorization $f(x) = p_1(x) \cdots p_n(x)$ for some irreducible p_i. By the preceding theorem, there exists a field K in which p_1 has a root. Therefore, $f(x)$ shares this root in K.

Theorem 13.1.12 (Existence of Root Extensions (again))**.** *Let $f(x) \in F[x]$ be a nonconstant polynomial. There exists a field extension in which $f(x)$ splits.*

Proof. Because $F[x]$ is a PID, we have the unique factorization $f(x) = p_1(x) \cdots p_n(x)$ for some irreducible p_i. By the preceding theorem, there exists a field K in which each p_i has a root. Iterate this process for all i to obtain the field.

For example, take the polynomial $x^2 + x + 1$ over $\mathbb{Q}$. If θ is a root, then it naturally has degree 2 so the extension field K has $[K : \mathbb{Q}] = 2$.

Let's do something concrete. We know that $\mathbb{C} \geq \mathbb{Q}$, so if we compute the roots of $p(x) = x^2 + x + 1$, we have the roots of unity of degree 3;

$$\theta = \frac{-1 \pm \sqrt{-3}}{2} = e^{\pm 2\pi i/3}.$$

It turns out that extending by either root induces isomorphic fields.

Definition 13.1.13 (Subfield Generation)**.** *Let K/F be an extension and let $\alpha_1, \alpha_2, \dots \in K$. $F(\alpha_1, \dots)$ denotes the smallest subfield of K containing F and each α_i. We call this construction the subfield of K generated by F and $a_1, \dots$*

We call $F(\alpha)$ a simple extension when we only extend via one element.

Moreover, note that $F(\alpha_1, \alpha_2) = F(\alpha_1)(\alpha_2)$.

However, we can very bad simple extensions. Take $\mathbb{Q} \leq \mathbb{Q}(x)$, rational functions over $\mathbb{Q}$.

Let's say that α is a root of $x^2 + x + 1$ in $\mathbb{C}$. Then $\mathbb{Q}(\alpha) = \{a + b\alpha | a, b \in \mathbb{Q}\}$.

Proposition 13.1.14 (Polynomial Extension Isomorphism)**.** *Let K be an extension of F and $\alpha \in K$ is a root of an irreducible polynomial $p(x) \in F[x]$. Then, as a field, $F[\alpha] \cong F[x]/(p(x))$.*

Proof. Define $\phi : F[x] \to K$, where $f(x) \mapsto f(\alpha)$. This is naturally a ring homomorphism; we omit the verification. Then, $\ker \phi = (g(x))$ for some $g(x) \in F[x]$. Because $p(\alpha) = 0$, $p(x) \in g(x)$. Hence, $g(x)|p(x)$. Because p is irreducible, $g(x), p(x)$ are associates and $(g(x)) = (p(x))$. Therefore, by the first isomorphism theorem, $F[x]/(p(x)) \cong \operatorname{im} \phi = F(\alpha)$ (If we were to do every single detail, we'd have to show two-way containment). Moreover, $x + (p(x)) \mapsto \alpha$.

This demonstrates that $F(\alpha)/F$ is a finite extension.

Theorem 13.1.15 (Polynomial Root Extension Isomorphism)**.** *Let K/F be an extension and $p(x)$ be irreducible in $F[x]$. If α_1, α_2 are two roots of $p(x)$ in K, $F(\alpha_1) \cong F(\alpha_2)$ via $\alpha_1 \mapsto \alpha_2$.*

13.2 Algebraic Extensions

Definition 13.2.1 (Algebraic Extension)**.** *K/F is called an algebraic extension if α is a root of a polynomial $f(x)$ for every $\alpha \in K$.*

If α is not algebraic, we say that α is transcendental.

For example, $\pi \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, but $\sqrt{2}$ is algebraic.

A few criteria for extensions to be algebraic:

Lemma 13.2.2 (Finite Extension is Algebraic)**.** *If K/F is finite, then K/F is algebraic.*

Proof. Interpret K as a finite dimensional vector space over F. Let $n = \deg_F K$. Let $\alpha \in K$. The set $1, \alpha, \ldots, \alpha^n$ is dependent, so there are coefficients a_k, not all 0, so that

$$a_0 + a_1\alpha + \ldots + a_k\alpha^n = 0.$$

Then α is a root of $a_0 + a_1x + \ldots + a_kx^n$.

Proposition 13.2.3 (Minimal Polynomial)**.** *Let $\alpha \in K$ be algebraic over F. Then there exists a unique monic irreducible polynomial $m_{\alpha,F}(x)$ such that α is a root.*

Moreover, for any polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$, $m_{\alpha,F}|f(x)$ in $F[x]$.

Proof. Consider the ring homomorphism $\varphi : F[x] \to K$ defined by $\varphi(f(x)) = f(\alpha)$. The kernel is a principal ideal and hence uniquely generated by a monic polynomial $m_{\alpha,F}$. Clearly, the second statement follows as $f \in (m_{\alpha,F})$. By the first isomorphism theorem, $F[x]/(m_{\alpha,F}) \cong \varphi(F[x] \subset K)$ so it is an integral domain and hence $(m_{\alpha,F})$ is prime. Hence, $m_{\alpha,F}$ is irreducible and we are done.

Theorem 13.2.4 (Intermediate Extension Polynomial Divisibility)**.** *Let L/F be an extension and $\alpha \in K$ be algebraic over F. Then $m_{\alpha,L}(x)|m_{\alpha,F}(x)$ in $L[x]$.*

Proof. Whence $m_{\alpha,F}(\alpha) \in L[x]$, so $m_{\alpha,L}|m_{\alpha,F}$.

Theorem 13.2.5 (Isomorphism of Extensions)**.** *Let $\alpha \in K$ be algebraic over F. Then $F(\alpha) \cong F[x]/(m_{\alpha,F})$ and $\{1, \alpha, \dots, \alpha^{n+1}\}$ is a basis of $F(\alpha)$ over F, where $n = \deg(m_{\alpha,F})$*

Theorem 13.2.6 (Algebraic Subfield)**.** *An element $\alpha \in K/F$ is algebraic over F if and only if $F(\alpha)/F$ is finite. The set $\overline{F}$ of all algebraic elements in K/F is a subfield of K.*

Proof. The first statement follows immediately. Let $\alpha, \beta \in K$ be algebraic over F. Then $F(\alpha, \beta)$ is finite. Thus, the sum and product of α, β are in $F(\alpha, \beta)$ and $\alpha^{-1} \in F(\alpha, \beta)$. Thus, $\overline{F}$ is a subfield.

Proposition 13.2.7 (Transitivity of Extensions)**.** *If $L/K, K/F$ are algebraic, then L/F is algebraic.*

Proof. Let $\alpha \in L$. Then α is algebraic over K, so α is a root of $a_0 + a_1x + \dots + a_nx^n \in K[x]$. Since $[F(a_0, \dots, a_n)(\alpha) : F(a_0, \dots, a_n)] < \infty$, and each a_k is algebraic over F, $[F(a_0, \dots, a_n, \alpha) : F] < \infty$ implying that α is algebraic over F.

$$\begin{tikzcd}
 & & L \arrow[dd, dash] \\
 & F(a_0,\dots a_n)(\alpha) \arrow[ur, dash] \arrow[ddl, dash] \arrow[dd, dash, "<\infty"] & \\
 & & K \arrow[dl, dash] \arrow[ddl, dash] \\
F(\alpha) \arrow[dr, dash] & F(a_0,\dots a_n) \arrow[d, dash, "<\infty"] & \\
 & F &
\end{tikzcd}$$

Definition 13.2.8 (Product Field)**.** *Let K_1, K_2 be subfields of K. We denote by K_1K_2 the subfield of K generated by K_1 and K_2.*

$$\begin{tikzcd}
 & K \arrow[ddl, dash] \arrow[ddr, dash] & \\
 & K_1K_2 \arrow[dl, dash] \arrow[dr, dash] & \\
K_1 \arrow[dr, dash] & & K_2 \arrow[dl, dash] \\
 & F &
\end{tikzcd}$$

Proposition 13.2.9 (Product Field Properties)**.** *Let $K_1, K_2)$ be subfields of K. Suppose K_1, K_2 are finite extensions over F.*

(a) K_1K_2/F *is finite.*

(b) $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$.

(c) $\operatorname{lcm}([K_1, F], [K_2 : F])|K_1K_2 : F$.

Proof. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis of K_1 over F and $\{\beta_1, \dots, \beta_m\}$ be a basis of K_2 over F. Then $K_1K_2 = F(\alpha_1, \dots, \alpha_n, \beta_1, \beta_n)$ and $K_1K_2 = K_1(\beta_1, \dots, \beta_m)$ implies $[K_1K_2 : K_1] \leq m$ so $[K_1K_2 : F] = [K_1K_2 : K_1][K_1][F] \leq mn$. Moreover, this final relationship implies that $[K_i : F]|[K_1K_2 : F]$ for each i, proving the final statement.

13.3 Constructibility

Definition 13.3.1 (Constructibility)**.** *An $\alpha \in \mathbb{R}$ is called constructible if $|\alpha|$ can be constructed by straightedge and compass. We call the set of all constructible numbers $\mathbb{F}$ and assume that $\mathbb{Z} \subset \mathbb{F}$.*

Proposition 13.3.2 (Subfield of Constructible Numbers)**.** *Let $\alpha, \beta \in \mathbb{F}$. Sums and differences are easy to construct by using a straight line through the origin, a segment of length β, and a compass of length α.*

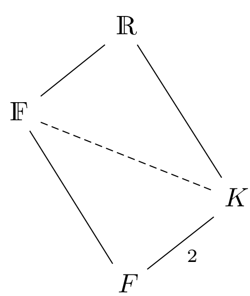
To construct products, take a triangle with one side β and foot 1, and construct a similar triangle with foot α. The corresponding side of β will have length $\alpha\beta$.

To show inverses, take a triangle of one side α and foot 1. Construct a similar triangle where the one side has length 1. The foot will have length α^{-1}.

Because $\operatorname{ch} \mathbb{F} = 0$, the prime subfield of $\mathbb{F}$ is $\mathbb{Q}$. However, as $\sqrt{2} \in \mathbb{F}$, $\mathbb{F} \supset \mathbb{Q}$.

Take $\alpha \in \mathbb{F}$. Draw a circle from $-\alpha$, 1, with centre on the x-axis. Construct a triangle to i by $-\alpha i 1$. Then, $x/1 = \alpha/x$ so $x^2 = \alpha$ and $\sqrt{\alpha} \in \mathbb{F}$.

Theorem 13.3.3 (Degree 2 Extension Closure)**.** *Let $F \subset \mathbb{F}$ and $K \subset \mathbb{R}$ with $[K : F] = 2$. Then, $K \subset \mathbb{F}$.*



Proof. Let $\alpha \in K \backslash F$. Then α is a root of a degree 2 polynomial $p(x) = x^2 + bx + c$. Therefore, $p(x) = m_{\alpha,\mathbb{F}}(x)$. Whence $\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ we have $F(\alpha) = F(\sqrt{b^2 - 4c})$. Because $\alpha \in \mathbb{R}$, $b^2 - 4c > 0$. However, we know already that $\sqrt{b^2 - 4c} \in \mathbb{F}$! Thus $K \subset \mathbb{F}$.

Definition 13.3.4 (Constructible Points ($\mathbb{F}^2$) in $\mathbb{R}^2$)**.** *We call $(x, y) \in \mathbb{R}^2$ constructible if $x, y \in \mathbb{F}$. Moreover, we could switch out for $x + iy \in \mathbb{C}$ if $x, y \in \mathbb{F}$. The constructible points of $\mathbb{C}$ make a field.*

We construct these points via intersections of the following objects.

(a) Straight lines through two points in $\mathbb{F}^2$;

(b) If $(h, k) \in \mathbb{F}^2$ and $r \in \mathbb{F}$, $(x-h)^2 + (y-k)^2 = r^2$.

We call $ax + by + c = 0$ an F-line if $a, b, c \in F \subset \mathbb{F}$; similarly, we infer the notion of an $\mathbb{F}$ circle.

Intersecting two F lines cannot "escape" F. A circle and line give quadratic extensions, and two circles, surprisingly, intersect on an $\mathbb{F}-$line and give also a quadratic extension. The proof requires solving the system and seeing that all of the quadratic terms cancel.

Theorem 13.3.5 (Power Two Necessity)**.** *If $\alpha \in \mathbb{R}$ is constructible, there exists an extension $\mathbb{Q} \subset K \subset \mathbb{R}$ such that $K = \mathbb{Q}(\alpha)$ $[K : \mathbb{Q}] = 2^k$ for some k.*

Proof. If α is constructible, there exists a finite sequence of constructible points (x_n, y_n) so that $\alpha \in \mathbb{Q}(x_1, y_1, \dots, x_n, y_n)$ and $[\mathbb{Q}(x_1, \dots, x_\ell) : \mathbb{Q}(\dots, y_{\ell-1})] \leq 2$. Therefore $[K : \mathbb{Q}]$ is a two-power.

Theorem 13.3.6 (Trisecting the Angle)**.** *Angles, in general, cannot be trisected by compass and straightedge.*

Proof. We consider the angle 60 deg $= 3\theta$. Then $(x, y) = e^{3\theta i} = \sqrt[3]{\cos\theta + i \sin\theta}^3$ is constructible. $\cos 3\theta$ is constructible, so we have that $(4\cos\theta)^3 - 3\cos\theta = 1$. Let $\alpha = \cos 3\theta$, yielding the equation $1 = 4\alpha^3 - 3\alpha$. Let $\beta = 2\alpha$. Then $0 = \beta^3 - 3\beta - 1$, an irreducible monic polynomial. Then $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$ would be a cubic extension, a contradiction by the previous, so the angle cannot be trisected.

Theorem 13.3.7 (Doubling the Cube)**.** *A cube may not be doubled.*

Proof. If it were so, doubling the unit cube would yield sides of length $\sqrt[3]{2}$, a root of a degree three irreducible polynomial. As three divides no power of two, such a cube is not constructible.

Theorem 13.3.8 (Square with Area π). *A swuare with area π cannot be constructed.*

Proof. This would mean that $\sqrt{\pi}$ would be constructible. However, $[\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}(\pi)] = [\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}(\pi)][\mathbb{Q}(\pi) : \mathbb{Q}] = 2\infty = \infty$ divides no power of two.

13.4 Splitting Fields

Definition 13.4.1 (Splitting Field). *Let $f(x) \in F[x]$ be a nonconstant polynomial. We call E/F a splitting field for f if f factors over E and f splits in no subfield of E.*

Theorem 13.4.2 (Roots to Splitting Field). *Let $\alpha_1, \dots, \alpha_n$ be the roots of $f(x) \in E$. Then, $F(\alpha_1, \dots, \alpha_n) \subset E$ and so we have equality.*

Example. Take $f(x) = x^3 + x + 1$ in $\mathbb{F}_2[x]$. Let α be a root of $f(x)$ in some extension $K/\mathbb{Z}_2$. Then, $f(\alpha) = 0$ and $f(\alpha^2) = (\alpha^2)^3 + \alpha^2 + 1 = (\alpha^3)^2 + \alpha^2 + 1 = (\alpha + 1)^2 + \alpha^2 + 1 = 0$, so α^2 is also a root. The other root is $\alpha^2 + \alpha$.

Splitting fields for degree n polynomials have, in general, degree $n!$.

Theorem 13.4.3 (Splitting Degree). *Every polynomial $f(x) \in F[x]$ of positive degree n splits in a field of degree at most $n!$.*

Proof. It suffices to show that there is an extension K/F such that $[K/F] \leq n!$ where $f(x)$ splits. Induction. Let degree $f(x) = 1$. Then $f(x)$ splits in F. Assume that the statement holds up to some $n - 1$. Let α be a root of f in some field E. We know α has degree at most n, so $f(x)/(x - \alpha)$ has degree $n - 1$ in $F(\alpha)$ where $[F(\alpha) : F] \leq n$. Then, f splits in $K/F(\alpha)$, so $[K : F] = [K : F(\alpha)][F(\alpha) : F] \leq (n - 1)!n = n!$.

13.4.1 Friday

Reminder: If we have a positive degree $f(x) \in F[x]$, we can always find a splitting field E/F of $f(x)$ such that $[E : F] \leq (\deg f)!$.

Lemma 13.4.4 (Isomorphism Extension)**.** *If $\alpha \in E/F$ a root of an irreducible $f(x) \in F[x]$, then there is an isomorphism ϕ from $F \to \overline{F}$. We define $\overline{f} = \phi(f)$. Then, we may extend to an isomorphism $\tilde{\phi} : F(\alpha) \to K$ such that $\tilde{\phi}|_F = \phi$; there are exactly k such extensions, where k is the number of distinct roots of $\overline{f}$ in $\overline{E}$.*

Proof. Let $\overline{\alpha} \in \overline{E}$ be a root of $\overline{f}(x)$. Then, because $\overline{F}(\overline{\alpha}) \cong \overline{F}[x]/((\overline{f})) \cong F[x]/(f(x)) \cong F(\alpha)$, we formally have the map

$$\tilde{\phi} : F(a)\overline{\eta^{-1}} \to F[x]/(f(x))\overline{\phi} \to \overline{F}[x]/(\overline{f}(x))\overline{\overline{\eta}} \to \overline{F}(\overline{\alpha}).$$

Hence, $\phi : \alpha \mapsto \overline{\alpha}$, so the number of such extensions is the number of distinct roots $\overline{\alpha}$ of $\overline{f}(x)$

Theorem 13.4.5. *Let $\phi : a \to \overline{a}$ be an isomorphism of a field F onto $\overline{F}$. Say that $f(x) \in F[x]$ has image $\overline{f}$, and let E and $\overline{E}$ be splitting fields for the two polynomials over their respective fields. Then, ϕ can be extended to an isomorphism $\tilde{E} \to \overline{E}$. The number of such extensions is less than or equal to $[E : F]$.*

The equality holds if $\bar{f}$ has no multiple roots in $\bar{E}$.

Theorem 13.4.6 (Uniqueness of Splitting Field)**.** *The splitting field of $f(x)$ over F is unique up to isomorphism.*

Proof. Induction on $[E : F]$. If $[E : F] = 1$, then $E = F$ and f splits completely. Thus, $\bar{f}$ also splits completely so $\bar{E} = \bar{F}$. Assume $[E : F] > 1$. Then f must have a monic irreducible factor of degree greater than 1. Let $\alpha \in E$ such that $g(\alpha) = 0$. Then, $\bar{g}$ is a monic irreducible factor of $\bar{f}$, and there exists $\bar{\alpha} \in \bar{E}$ such that $\bar{g}(\bar{\alpha}) = 0$. By the preceding lemma, ϕ can be extended to $\bar{\phi} : F(\alpha) \to \bar{E}$. Let $\bar{\alpha}_1, \dots, \bar{\alpha}_\ell$ be the distinct roots of $\bar{g}$ in $\bar{E}$. Hence, there are ℓ such extensions. Then, $[E : F(\alpha)] = [E : F]/[F(\alpha) : F]$ but $[F(\alpha) : F] = \deg g(x)$. By induction, there exist at most $[E : F(\alpha)]$ possible extensions of $\tilde{\phi}$ to $\bar{\phi}$. There are at most $\ell[E : F(\alpha)] \leq \deg g(x)[E : F(\alpha)] = [E : F]$ such extensions over ϕ.

If $\bar{f}(x)$ has no multiple roots in $\bar{E}$, then $\ell = \deg \bar{g}(x)$. Then, by induction, the number of extensions is equal to $[E : F(\alpha)] \deg g(x) = [E : F]$.

Proof. [Corollary] If $\mathrm{id} : F \to F$ and $f(x) \in F$ with $E, \overline{E}$ are splitting fields of $f(x)$ over F, then the number of such extensions ϕ of id is less than or equal to $[E : F]$.[1]

Example. For $f(x) = x^3 - 2 \in \mathbb{Q}[x]$, $E = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is the splitting field. Now, how many automorphisms do we have and what are they? Well, we have at most 6 automorphisms. We could take $\sqrt[3]{2} \mapsto \zeta_3^k \sqrt[3]{2}$, and $\zeta_3 \mapsto \bar{\zeta}_3, \zeta_3$.

13.4.2 Week 4

Last time, we learned that, if $f(x) \in F[x]$ has splitting field E and $\bar{f}(x) \in \bar{F}[x]$ has splitting field $\bar{E}$, then an isomorphism $\phi : F \to \bar{F}$ lifts to an isomorphism $\bar{\phi}$. There are at most $[E : F]$ such extensions.

$$\begin{tikzcd} E \arrow[r, "\bar{\phi}"] \arrow[d, dash] & \bar{E} \arrow[d, dash] \\ F \arrow[r, "\phi"] & \bar{F} \end{tikzcd}$$

Definition 13.4.7 (Algebraic Closure)**.** *A field $\overline{F}$ is called an algebraic closure of F if $\overline{F}/F$ is algebraic and every $f(x) \in F[x]$ splits completely.*

For an example, consider

$$\begin{tikzcd} \mathbb{C} \arrow[d, dash] \\ \bar{\mathbb{Q}} \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

However, even though $\bar{\mathbb{C}} = \mathbb{C}$, $\mathbb{C}/\mathbb{Q}$ is not algebraic.

Proposition 13.4.8 (Algebraic Closure is Closed)**.** *If $\bar{F}$ is an algebraic closure of F, then $\bar{\bar{F}} = \bar{F}$.*

[1] Is this really sufficient?

Proof. Let $f(x) \in \bar{F}[x]$ be irreducible. It suffices to show that $\deg f(x) = 1$. Let $E/\bar{F}$ be the splitting field of f.

We know that $E/\bar{F}$ is finite and hence algebraic. Since $\bar{F}/F$ is algebraic, E/F° is algebraic. For any $\alpha \in E$, α is not a root of some irreducible polynomial over F. Since $\bar{F}$ is the algebraic closure of F,

$$E \subset \bar{F} \implies E = \bar{F} \implies \deg f(x) = 1.$$

Definition 13.4.9 (Separable)**.** *We call $f(x) \in F[x]$ separable if $f(x)$ has no multiple root in a splitting field E of $f(x)$ over F.*

Moreover, we say E/F is called separable if every element of E is a root of a seperable polynomial over F.

For example, $\mathbb{F}_2(t)(\sqrt{t})/\mathbb{F}_2(t)$ is inseparable.

Theorem 13.4.10. *A polynomial $f(x) \in F[x]$ has a multiple root in its splitting field over F if and only if $f(x)$ and $f'(x)$ are not relatively prime.*

Proof. Let E be a splitting field of $f(x)$ over F and $\alpha \in E$ a multiple root of $f(x)$. Then $f(x) = (x-\alpha)^m h(x)$ for some $h(x) \in F[x]$ and $m > 1$. Then $f'(x) = m(x-\alpha)^{m-1}h(x) + (x-\alpha)^m h'(x) = (x-\alpha)^{m-1}$ in $E[x]$. Let $p(x)$ be the minimal polynomial of α over F. Then $p|f, f'$.

Assume $\gcd(f(x), f'(x)) \neq 1$. Then there exists an irreducible polynomial $p(x)|f(x), f'(x)$. Let α be a root of $p(x)$ in a splitting field E of $f(x)$ over F. Then, we have that $f(\alpha) = 0 = f'(\alpha)$, so $f(x) = (x-\alpha)g(x)$ for some $g(x) \in E[x]$. Hence,

$$f'(x) = g(x) + (x-\alpha)g'(x) \implies (x-\alpha)|g(x)$$

so $(x-\alpha)^2|f(x)$ in $E[x]$, so α is a multiple root of $f(x)$. In particular, $f(x)$ is nonseperable.

Theorem 13.4.11 (Algebraic Closure)**.** *Every field F is contained in an algebraically closed field K and the set of all elements of K algebraic over F is an algebraic closure of F. Moreover, the algebraic closure of F is unique.*

Hence, if $\mathbb{F}$ is another algebraically closed field and K is all of the elements of $\mathbb{F}$ algebraic over $\mathbb{Q}$,

$$\begin{tikzcd}
\mathbb{C} \arrow[d, dash] & \mathbb{F} \arrow[d, dash] \\
\bar{\mathbb{Q}} \arrow[r, "\sim"] \arrow[d, dash] & K \\
\mathbb{Q} \arrow[ur, dash] &
\end{tikzcd}$$

13.4.3 Wednesday

Let $\mathbb{F}$ be a finite field. Let $f(x) = x^{|\mathbb{F}|} - x$ in $\mathbb{Z}_p[x]$. So, $f'(x) = -1$. Thus, the greatest common divisor of the two is 1, and f is always seperable in $\mathbb{F}$.

Additionally, for $\alpha \in \mathbb{F}^\times$, $\alpha^{|F^\times|} = 1$ so α is a root of $x^{|\mathbb{F}|-1} - 1$ and $x^{|F|-1} - 1 = \prod_{\alpha \in \mathbb{F}^\times}(x - \alpha)$. However, $x^{|\mathbb{F}|} - x = \prod_{\alpha \in \mathbb{F}}(x - \alpha)$ so. Hm!

Definition 13.4.12 (Perfect Field)**.** *A field $\mathbb{F}$ is called perfect if every irreducible polynomial $f(x) \in F[x]$ is seperable.*

Theorem 13.4.13 (Characteristic 0 Perfection)**.** *Every field $\mathbb{F}$ of characteristic 0 is perfect.*

Proof. Let $f(x) \in \mathbb{F}[x]$ be irreducible. Then $f'(x) \neq 0$ and $\deg f'(x) < \deg f(x)$. In particular, $f(x) \nmid f'(x)$. Since $\gcd(f(x), f'(x)) = 1, f(x)$ by irreducibility, f and f' are coprime and f is separable.

Theorem 13.4.14 (Number Fields)**.** *Every algebraic extension over a field of characteristic 0 is separable.*

Theorem 13.4.15 (Finite Field Perfection)**.** *Every finite field is perfect.*

Proof. Let $f(x) \in F[x]$ be irreducible and E a splitting field of f over F. In particular, E/F is finite and E is finite itself. Therefore

$x^{|E|} - x = \prod_{\alpha \in E}(x - \alpha)$ and $f(x)|x^{|E|} - x$ in $F[x]$. Therefore, $f(x)$ is a factor of a seperable polynomial and is hence separable.

Definition 13.4.16 (Frobenius Automorphism)**.** *The function x^p is called the Frobenius automorphism for fields of characteristic p.*

13.5 Cyclotomic Extensions

Let's start with $\mu_n = \left\{e^{2\pi ik/n} = \zeta_n^k | k \in [n-1]\right\}$. This is a finite group under complex multiplication. Hence, $(\mu_n, \cdot) \cong (\mathbb{Z}_n, +)$.

Definition 13.5.1 (Cyclotomic Polynomial)**.**

$$\Phi_n(x) = \prod_{\alpha \in \mu_n \text{ that are generators}} (x - \alpha)$$

is the nth cyclotomic polynomial.

We often call the generators of μ_n the primitive nth roots of unity.

Chapter 14

Second Chapter

14.1 First Section

Appendix A

List Of Definitions

Appendix B

List Of Theorems