Algebraic Topology I PS 8

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1. Denote $X = \mathbb{R}P^\infty$ and note that $\Sigma^n X$ is n-connected.

   First let us do the case $n = 0$, then $\pi_1(X) = \mathbb{F}_2$ and $\pi_1(A \vee B) \cong \pi_1(A) * \pi_1(B)$. So $\mathbb{F}_2$ must lie in one of them say A. The rest of the degrees follow by taking the suspension of X and considering the following cases, then using the stability of Sq^i.
   Now take some $n \geq 1$. Consider the weak homotopy equivalence $\Sigma^n X \simeq A \vee B$. These spaces are both 1-connected so this induces an isomorphism on cohomology $\mathbb{F}_2[\iota_1] \cong H^{*+n}(X; \mathbb{F}_2) \cong H^*(\Sigma^n X; \mathbb{F}_2) \cong H^*(A \vee B; \mathbb{F}_2)$. Moreover $\tilde{H}^*(A \vee B; \mathbb{F}_2) = \tilde{H}^*(A; \mathbb{F}_2) \oplus \tilde{H}^*(B; \mathbb{F}_2)$. Pick $x = \iota_1 \in H^1(X; \mathbb{F}_2)$, the generator for the cohomology ring on $\mathbb{R}P^\infty$. Then the generator on the ring $H^*(\Sigma^n X; \mathbb{F}_2)$ is x^n which will shall now consider as such. Without loss of generality let x be contained in $H^*(A; \mathbb{F}_2)$. Since we have Steenrod squares on X and so on $\Sigma^n X$ we also have it on $H^*(A \vee B; \mathbb{F}_2)$.
   We have $\mathrm{Sq}^1(x^k) = \begin{cases} x^{k+1} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$ and $\mathrm{Sq}^2(x^k) = \begin{cases} x^{k+2} & k \equiv 2,3 \bmod 4 \\ 0 & k \text{ even} \end{cases}$

   $\mathrm{Sq}^1(x) = x^2$, $\mathrm{Sq}^2(x^2) = x^4$ so x^2, x^4 is contained in $H^*(A; \mathbb{F}_2)$. Moreover $\mathrm{Sq}^1(x^3) = x^4$ so x^3 also in $H^*(A; \mathbb{F}_2)$. We can now proceed inductively: $x^{4j+2} \in H^*(A; \mathbb{F}_2) \implies x^{4j+4} \in H^*(A; \mathbb{F}_2)$, $x^{4j+3} \in H^*(A; \mathbb{F}_2) \implies x^{4j+1} \in H^*(A; \mathbb{F}_2)$ by the above relations. Moreover, $\mathrm{Sq}^1(x^{4j+1}) = x^{4j+2}$, $\mathrm{Sq}^1(x^{4j+3}) = x^{4j+4}$ so $x^{4j+1}, x^{4j+3} \in H^*(A; \mathbb{F}_2)$. Therefore if x lies in $H^*(A; \mathbb{F}_2)$ then all rest x^k lies in $H^*(A; \mathbb{F}_2)$ hence $H^*(B; \mathbb{F}_2)$ is trivial on cohomology and therefore also on homotopy so is weakly contractible.

2. Let X be $(n-1)$-connected for some $n \geq 2$. By Hurewicz $\pi_n(X) \cong H_n(X)$ and from the suspension isomorphism $s : H_k(X) \to H_{k+1}(\Sigma X)$, ΣX is n-connected. Consider the fibre sequence $\Omega\Sigma X \to P\Sigma X \xrightarrow{p} \Sigma X$, and take the maps $\phi : X \to \Omega\Sigma X$, $\phi(x)(t) = [x, t]$, $\varphi : CX \to P\Sigma X$, $\varphi([x,t])(s) = [x, st]$ we get diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\phi"] & CX \arrow[r] \arrow[d, "\varphi"] & \Sigma X \arrow[d, equal] \\ \Omega\Sigma X \arrow[r] & P\Sigma X \arrow[r, "p"] & \Sigma X \end{tikzcd}$$

   which commutes with the rest of the maps being the natural ones.
   Consider the Serre spectral sequence in homology. ΣX is n-connected and $\Omega\Sigma X$ is $(n-1)$-connected, so $H_p(\Sigma X) = 0$ for $0 < p \leq n$ and

$H_q(\Omega\Sigma X) = 0$ for $0 < q \leq n-1$.
(Local coefficient system is trivial as $\Omega\Sigma X$ is path-connected).

In the diagram $H_k(\Sigma X, b_0; \mathbb{Z}) \xleftarrow{p^*} H_k(P\Sigma X, \Omega\Sigma X; \mathbb{Z}) \xrightarrow{\delta} H_{k-1}(\Omega\Sigma X; \mathbb{Z})$ we get the transgression τ by restricting to the domain $E^k_{k,0}$ such that it maps to the range $H_{k-1}(\Omega\Sigma X)/\ker p^*$.

$$
\begin{tikzcd}
H_{n+1}(\Omega\Sigma X) & 0 & 0 & & \\
H_n(\Omega\Sigma X) & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & H_{n+1}(\Sigma X) \arrow[uulll] & H_{n+2}(\Sigma X)
\end{tikzcd}
$$

Consider the diagram induced by the previous one.

$$
\begin{tikzcd}
H_m(X) \arrow[d, "\phi_*"] & H_{m+1}(CX, X) \arrow[l, "\cong"', "\partial"] \arrow[r, "\cong"] \arrow[d, "\varphi_*"] & H_{m+1}(\Sigma X) \arrow[d, equal] \\
H_m(\Omega\Sigma X) & H_{m+1}(P\Sigma X, \Omega\Sigma X) \arrow[l, "\cong"', "\partial"] \arrow[r] & H_{m+1}(\Sigma X)
\end{tikzcd}
$$

Since $\rho := p \circ \varphi : CX \to \Sigma X$ is the natural quotient map and $\rho_* : H_m(CX, X) \to H_k(\Sigma X, b_0)$ is an isomorphism it follows that p_* is surjective so the trangression τ is defined on all of its domain $H_k(\Sigma X, b_0)$. So from the diagram we get $\tau \circ s = \partial \circ \varphi_* \circ (\rho_*)^{-1} \circ s = \phi_*$.

We have the Serre exact sequence

$$
H_{2n}(\Omega\Sigma X; \mathbb{Z}) \longrightarrow H_{2n}(P\Sigma X; \mathbb{Z}) \xrightarrow{p^*} H_{2n}(\Sigma X; \mathbb{Z}) \xrightarrow{\tau} H_{2n-1}(\Omega\Sigma X; \mathbb{Z}) \longrightarrow \dots
$$

Then as $P\Sigma X$ contractible, this gives us that $H_{k+1}(\Sigma X; \mathbb{Z}) \xrightarrow{\tau} H_k(\Omega\Sigma X; \mathbb{Z})$ is an isomorphism for all $k \leq 2n-1$. Hence as τ is an isomorphism, so is ϕ and thus $\varphi : H_k(CX, X) \to H_k(P\Sigma X, \Omega\Sigma X)$ for $k \leq 2n-1$. Then by relative Hurewicz theorem, as $H_k(\Omega\Sigma X, X) = 0$ for $k \leq 2n-1$ then $\pi_k(\Omega\Sigma X, X) = 0$ for $k \leq 2n-1$. Hence for the long exact sequence $\pi_k(X) \to \pi_k(\Omega\Sigma X)$ is an isomorphism for $k < 2n-1$ and an epimorphism for $k = 2n-1$. Then the result follows from the isomorphism $\pi_{k+1}(\Sigma X) \cong \pi_k(\Omega\Sigma X)$.