Algebraic Geometry PS8

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4 (a) Let U_1, U_2 be open affine subschemes of X, say $U_1 \cong \text{Spec } A$ and $U_2 \cong \text{Spec } B$. Take $x \in U_1 \cap U_2$. We can assume wlog that we have $f \in \text{Spec } A$, $g \in \text{Spec } B$ such that $x \in D_{U_1}(f) \subset D_{U_2}(g) \subset U_1 \cap U_2$. Consider $F = f|_{D_{U_2}(g)} \in \Gamma(D_{U_2}(g), \mathcal{O}_X) = B_g$. So $F = h/g^n$ for some $b \in B$ and $n \geq 0$. As $\text{Spec } A_f = \text{Spec } B_g \backslash V(F) = \text{Spec } (B_g)_F$, then $\text{Spec } (B_g)_F = \text{Spec } (B_g h)$ so $D_{U_2}(gh) = D_{U_1}(f)$.

(b) Let $f : X \to S$ be a morphism of schemes such that for an open cover $\{\text{Spec } A_i\}_i$ of X and $\{\text{Spec } B_i\}_i$ of S with $f(\text{Spec } B_i) \subset \text{Spec } A_i$ with $A_i \to B_i$ being of finite presentation. Let affine opens be such that $f(\text{Spec } B) \subset \text{Spec } A$. Pick $p \in \text{Spec } B$, and p is contained in some $\text{Spec } B_i$. By (a) we can choose an affine open neighbourhood of $U = \text{Spec } C \subset \text{Spec } B \cap \text{Spec } B_i$ of p such that it is principal open in both subschemes. Then $\Gamma(\text{Spec } A_i, \mathcal{O}_S) \to \Gamma(\text{Spec } C, \mathcal{O}_X)$ is of finite presentation as $\text{Spec } C$ is principal open. Pick a neighbourhood $V = \text{Spec } D \subset \text{Spec } A_i \text{Spec } A$ of $f(p)$ again such that $\text{Spec } D$ is principal open in both subschemes. Then $f^{-1}(V) \cap U$ is a principal open in $\text{Spec } B_i$ so then also in $\text{Spec } B$ therefore $\Gamma(\text{Spec } A, \mathcal{O}_S) \to \Gamma(\text{Spec } B_i, \mathcal{O}_X)$ is of finite presentation. This can be done for all $p \in \text{Spec } B$, and $\{\text{Spec } B_i\}$ covers $\text{Spec } B$. Since $\text{Spec } B$ is quasi-compact, we can pick finitely many such affines to cover $\text{Spec } B$. By gluing argument since $A_i \to B_i$ is of finite presentation, then so is $A_i \to B$ then via similar argument on $\text{Spec } A$ we have that $A \to B$ is of finite presentation.