# Algebraic Topology I

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## 1 Spectral Sequences

**Definition 1.1.** A **homologically** or **Serre-graded spectral sequence** is a triple $(E^\bullet, d^\bullet, h^\bullet)$ such that

- $(E^r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bi-graded abelian groups $E^r = \bigoplus_{p,q\in\mathbb{Z}^2} E^r_{p,q}$, where $E^r$ is called the **page** of the spectral sequence.
- $(d^r : E^r \to E^r)_{r\geq 2}$ is a sequence of morphism of bi-degree $(-r, r-1)$ satisfying $d^r \cdot d^r = 0$.
- $(h^r : H_*(E^r, d^r) \to E^{r+1})_{r\geq 2}$ is a sequence of bi-grading preserving isomorphism.

**Definition 1.2.** We say that a spectral sequence is **first quadrant** if for $p < 0$ or $q < 0$ we have $E^2_{p,q} = 0$.

**Lemma 1.1.** *For a first quadrant spectral sequence we have $\forall r \geq 0$ and whenever $p < 0$ or $q < 0$, $E^r_{p,q} = 0$. Moreover, for a given $p, q$ the map $h$ induces an isomorphism $E^r_{p,q} \xrightarrow{\sim} E^{r+1}_{p,q}$ for all $r > r_0 = \max(p, q+1)$.*

**Definition 1.3.** For a first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ we define the $E^\infty$-page as the bi-graded abelian group $E^\infty_{p,q} = E^{r_0+1}_{p,q}$ with $r_0 = \max(p, q+1)$. By lemma 1.1, $E^\infty_{p,q} \cong E^r_{p,q}$.

By a filtered object in an abelian Category $\mathcal{A}$ we mean an object $H \in \mathcal{A}$ with a sequence of inclusions $0 = F^{-1} \subset F^0 \subset F^1 \subset \ldots \subset F^n \subset \ldots \subset H$.

**Definition 1.4.** A first quadrant spectral sequence $(E^\bullet, d^\bullet, h^\bullet)$ is said to **converge** to a filtered object in a graded abelian group $(H, F)$ if there is a chosen isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$, $\forall p, q$ and $F^p_n = H_n$ if $n \geq p$.
In this case we write $E^2_{p,q} \implies H$.

*Remark.*

- Convergence is really just a datum of the isomorphism $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-1}_{p+q}$
- Convergence spectral sequences are often simply encoded as $E^2_{p,q} \implies H$, however this suppresses not only this data but also the higher numbered pages, differentials, and the filtration on H.

We now introduce the Serre spectral sequence for the homology of fibre sequences.

**Definition 1.5.** Let $f : Y \to X$ be a continuous map of topological spaces and $x \in X$. The homotopy fibre $hofib_x(f)$ of $f$ at $x$ is defined to be the space $hofib_x(f) = P_xX \times_X Y$ where $P_xX = \{\gamma : [0,1] \to X \mid \gamma(1) = x\}$.

Indeed

$$\begin{tikzcd} hofib_x(f) \arrow[r] \arrow[d] & P_xX \arrow[d, "ev_0"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

In other words, $hofib_x(f)$ is the space of pairs $(\gamma, y)$ where $y \in Y$ and $\gamma$ is a path from $f(y)$ to x.

*Remark.* Note that $P_xX$ is contractible by the homotopy $H : P_xX \times [0,1] \to P_xX$, $(\gamma, y) \mapsto (s \mapsto \gamma((1-t)s + t))$

*Example.* Take $f : * \to X$ then $hofib_x(f) = \Omega_x X$.

**Definition 1.6.** A **fibre sequence of topological spaces** is a sequence $F \xrightarrow{\iota} Y \xrightarrow{X}$, a base-point $x \in X$, a homotopy $h : F \to X^{[0,1]}$ from the composite $f \circ \iota$ to the constant map $C_X : F \to X$ and such that the induced map $F \to hofib_x(f)$, $z \mapsto (h(z), c(z))$.

*Remark.* Recall that weak homotopy equivalence = isomorphism on $\pi_n(,*)$ for all $n$, $*$.

*Example.*

1. Let $f : Y \to X$ be any continuous map $x \in X$. Then the pair $hofib_x(f) \xrightarrow{\iota} Y \xrightarrow{f} X, h)$ is a fibre sequence since by the construction the map $hofib_x(f) \to hofib_x(f)$ is the identity map.
   Every fibre sequence is equivalent to this example int he following sense: given $F \xrightarrow{\iota} Y \xrightarrow{X}$, there is a commutative diagram

$$\begin{tikzcd} F \arrow[r, "\sim"] \arrow[d] & hofib_x(f) \arrow[d] \\ Y \arrow[r] \arrow[d, "f"] & Y \arrow[d, "f"] \\ X \arrow[r] & X \end{tikzcd}$$

   an equivalence of fibre sequences.
   In particular $\Omega X \to * \to X$ is a fibre sequence where $h : \Omega X \times [0,1] \to X$ is the eval map.
   Note that if one instead chooses $h$ to be the constant homotopy, one does not obtain a fibre sequence (unless $X$ is contractible). This is because the induced map $\Omega X \to hofib_x(f) = \Omega X$ is the constant map, which is not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces $F$ and $X$, $x \in X$ the pair $(F \to F \times X \xrightarrow{pr_x} X$, $h$=const) is a fibre sequence, called the trivial fibre space. To see this, note that $hofib_x(pr_x) = F \times P_xX$, with induced map $F \to F \times P_xX$, $y \mapsto (y, const)$, which is a homotopy equivalence as $P_xX$ is contractible.

3. Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}$ for some $b \in B$. Then the sequence $F \to E \xrightarrow{p} B$ with constant homotopy is a fibre sequence. This is a special case of example 4.

4. The map $p : E \to B$ is a Serre fibration if every commutative diagram of the form

$$\begin{tikzcd} D^n \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

there exists a lift $g$ making both triangles commute.
Given a Serre fibration $p : E \to B$, $b \in B$ the sequence $F = p^{-1}(b) \to E \to B$ with the constant homotopy is a fibre sequence. (Proof as exercise).
Note that every fibre sequence is also equivalent to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle $S^1 \to S^3 \xrightarrow{\eta} S^2$ works by letting $S^1 = U(1)$ act on $S^3 \subset \mathbb{C}^2$ by $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$ with quotient $S^2 \cong \mathbb{C}P^1$.

6. Example 5 generalises to fibre bundles $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ with limit case

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\cong"] & S^\infty \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d, "\cong"] \\ \Omega\mathbb{C}P^1 \arrow[r] & * \arrow[r] & \mathbb{C}P^\infty \end{tikzcd}$$