Algebraic Geometry I PS 10

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2 Let $f : Y \to X$ be a scheme morphism.

(a) Assume $|f|$ is homeomorphic to its closed image. Pick some $x \in X$. If x is not in the image of f then we can take an open affine neighbourhood $\mathcal{U}_x$ such that $f^{-1}(\mathcal{U}_x) = \emptyset$ since $f(Y)$ is closed. Assume then that $x \in f(Y)$ i.e $\exists! y \in Y$ mapping to x, and take affine neighbourhood $\mathcal{U}_x \ni x$. Pick some open affine neighbourhood V_y such that $f(V_y) \subset \mathcal{U}_x$. We take $g \in \Gamma(\mathcal{U}_x, \mathcal{O}_x)$ such that $x \in D(g)$ and $D(g) \cap \operatorname{im}(f) \subset f(V_y)$ (as $f(V_y)$ is open in $\mathcal{U}_x \cap \operatorname{im}(f)$) . Now let $h = f^*(g)|_{V_y}$, then $D(h) = f^{-1}(D(g))$. Thus every point has an open affine neighbourhood with affine preimage, and so f is affine.

(b) Assume that f is a universal homeomorphism. Then from $a)$ f is clearly affine. Since f is a homeomorphism under base change then it is also injective, closed under based change as well as surjective. Then affine and universally closed $\implies$ integral.
Conversely assume that f is surjective, universally injective and integral. It is then also affine and universally closed. Consider a base change $T \to S$.

$$\begin{tikzcd} Y_T \arrow[r, "p"] \arrow[d, "f_T"] & Y \arrow[d, "f"] \\ X_T \arrow[r, "q"] & X \end{tikzcd}$$

As f is integral, its base change is also integral. Note that $q^{-1}(f(Y)) = f_{S'}(p^{-1}(Y))$ (show by fibre product) so as f is surjective, as is $f_{S'}$. Thus it is universally bijective and as it is universally closed it is universally a homeomorphism.

3 (a) $f : X \to S$. Let $\mathcal{K} = \ker f^\flat$ and let $\{\mathcal{K}_i\}$ be the set of q-coh $\mathcal{O}_S$ submodules of $\mathcal{K}$. $\mathcal{K}$ in general is not q-coh. Then $\mathcal{K}' := \operatorname{im}(\oplus_i \mathcal{K}_i \to \mathcal{O}_S)$ is a q-coh $\mathcal{O}_S$ submodule. Moreover it is contained in $\mathcal{K}$ and by construction the largest q-coh submodule contained in $\mathcal{K}$. Since $\mathcal{K}$ is an ideal sheaf than so is $\mathcal{K}'$ and also q-coh so the subscheme Z associated to this ideal sheaf is the schematic image.

(b) Let f be q-com with kernel K. Wlog we can assume S is affine and take a finite open affine covering $\{\mathcal{U}_\alpha\}$ of X. Considering the map $\mathcal{O}_X \hookrightarrow \bigoplus_\alpha (\iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$ where ι_α is the inclusion of $\mathcal{U}_\alpha$, we can get an injection $f_*\mathcal{O}_X \to \bigoplus_\alpha (f \circ \iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$. Note that $(f \circ \iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$ and so $\bigoplus_\alpha (f \circ \iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$ are q-coh. Then we see that $\mathcal{K}$ is the kernel of the map $\mathcal{O}_S \to \bigoplus_\alpha (f \circ \iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$ and thus is q-coh. Hence $\mathcal{K} = \ker(\mathcal{O}_X \to \bigoplus_\alpha (f \circ \iota_\alpha)_* \mathcal{O}_{\mathcal{U}_\alpha}$.

(c) Let p be prime. Consider map $\bigsqcup_n \text{Spec}\,(\mathbb{Z}/p^n) \to \text{Spec}\,\mathbb{Z}$.