

# Algebraic Topology I

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## 1 Spectral Sequences

**Definition 1.1.** A **homologically** or **Serre-graded spectral sequence** is a triple  $(E^\bullet, d^\bullet, h^\bullet)$  such that

- $(E^r)_{r \geq 2}$  is a sequence of  $\mathbb{Z}$ -bi-graded abelian groups  $E^r = \bigoplus_{p,q \in \mathbb{Z}^2} E_{p,q}^r$ , where  $E^r$  is called the **page** of the spectral sequence.
- $(d^r : E^r \rightarrow E^r)_{r \geq 2}$  is a sequence of morphism of bi-degree  $(-r, r-1)$  satisfying  $d^r \cdot d^r = 0$ .
- $(h^r : H_*(E^r, d^r) \rightarrow E^{r+1})_{r \geq 2}$  is a sequence of bi-grading preserving isomorphism.

**Definition 1.2.** We say that a spectral sequence is **first quadrant** if for  $p < 0$  or  $q < 0$  we have  $E_{p,q}^2 = 0$ .

**Lemma 1.1.** For a first quadrant spectral sequence we have  $\forall r \geq 0$  and whenever  $p < 0$  or  $q < 0$ ,  $E_{p,q}^r = 0$ . Moreover, for a given  $p, q$  the map  $h$  induces an isomorphism  $E_{p,q}^r \xrightarrow{\sim} E_{p,q}^{r+1}$  for all  $r > r_0 = \max(p, q+1)$ .

**Definition 1.3.** For a first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$  we define the  $E^\infty$ -page as the bi-graded abelian group  $E_{p,q}^\infty = E_{p,q}^{r_0+1}$  with  $r_0 = \max(p, q+1)$ . By lemma 1.1,  $E_{p,q}^\infty \cong E_{p,q}^r$ .

By a filtered object in an abelian Category  $\mathcal{A}$  we mean an object  $H \in \mathcal{A}$  with a sequence of inclusions  $0 = F^{-1} \subset F^0 \subset F^1 \subset \dots \subset F^n \subset \dots \subset H$ .

**Definition 1.4.** A first quadrant spectral sequence  $(E^\bullet, d^\bullet, h^\bullet)$  is said to **converge** to a filtered object in a graded abelian group  $(H, F)$  if there is a chosen isomorphism  $E_{p,q}^\infty \cong F_{p+q}^p / F_{p+q}^{p-1}$ ,  $\forall p, q$  and  $F_n^p = H_n$  if  $n \geq p$ . In this case we write  $E_{p,q}^2 \implies H$ .

*Remark.* • Convergence is really just a datum of the isomorphism  $E_{p,q}^\infty \cong F_{p+q}^p / F_{p+q}^{p-1}$

- Convergence spectral sequences are often simply encoded as  $E_{p,q}^2 \implies H$ , however this suppresses not only this data but also the higher numbered pages, differentials, and the filtration on  $H$ .

We now introduce the Serre spectral sequence for the homology of fibre sequences.

**Definition 1.5.** Let  $f : Y \rightarrow X$  be a continuous map of topological spaces and  $x \in X$ . The homotopy fibre  $hofib_x(f)$  of  $f$  at  $x$  is defined to be the space  $hofib_x(f) = P_x X \times_X Y$  where  $P_x X = \{\gamma : [0, 1] \rightarrow X \mid \gamma(1) = x\}$ .

Indeed

$$\begin{array}{ccc} hofib_x(f) & \longrightarrow & P_x X \\ \downarrow & & \downarrow ev_0 \\ Y & \xrightarrow{f} & X \end{array}$$

In other words,  $hofib_x(f)$  is the space of pairs  $(\gamma, y)$  where  $y \in Y$  and  $\gamma$  is a path from  $f(y)$  to  $x$ .

*Remark.* Note that  $P_x X$  is contractible by the homotopy  $H : P_x X \times [0, 1] \rightarrow P_x X$ ,  $(\gamma, t) \mapsto (s \mapsto \gamma((1-t)s + t))$

*Example.* Take  $f : * \rightarrow X$  then  $hofib_x(f) = \Omega_x X$ .

**Definition 1.6.** A **fibre sequence of topological spaces** is a sequence  $F \xrightarrow{\iota} Y \xrightarrow{X} X$ , a base-point  $x \in X$ , a homotopy  $h : F \rightarrow X^{[0,1]}$  from the composite  $f \circ \iota$  to the constant map  $C_X : F \rightarrow X$  and such that the induced map  $F \rightarrow hofib_x(f)$ ,  $z \mapsto (h(z), c(z))$ .

*Remark.* Recall that weak homotopy equivalence = isomorphism on  $\pi_n(*, *)$  for all  $n, *$ .

*Example.* 1. Let  $f : Y \rightarrow X$  be any continuous map  $x \in X$ . Then the pair  $(hofib_x(f) \xrightarrow{\iota} Y \xrightarrow{f} X, h)$  is a fibre sequence since by the construction the map  $hofib_x(f) \rightarrow hofib_x(f)$  is the identity map. Every fibre sequence is equivalent to this example in the following sense: given  $F \xrightarrow{\iota} Y \xrightarrow{X} X$ , there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\sim} & hofib_x(f) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \\ \downarrow f & & \downarrow f \\ X & \longrightarrow & X \end{array}$$

an equivalence of fibre sequences.

In particular  $\Omega X \rightarrow * \rightarrow X$  is a fibre sequence where  $h : \Omega X \times [0, 1] \rightarrow X$  is the eval map.

Note that if one instead chooses  $h$  to be the constant homotopy, one does not obtain a fibre sequence (unless  $X$  is contractible). This is because the induced map  $\Omega X \rightarrow hofib_x(f) = \Omega X$  is the constant map, which is not a weak homotopy equivalence. Hence, the choice of homotopy is important.

2. For every pair of spaces  $F$  and  $X$ ,  $x \in X$  the pair  $(F \rightarrow F \times X \xrightarrow{pr_x} X, h = \text{const})$  is a fibre sequence, called the trivial fibre space. To see this, note that  $hofib_x(pr_x) = F \times P_x X$ , with induced map  $F \rightarrow F \times P_x X$ ,  $y \mapsto (y, \text{const})$ , which is a homotopy equivalence as  $P_x X$  is contractible.

3. Let  $p : E \rightarrow B$  be a fibre bundle with fibre  $F = p^{-1}$  for some  $b \in B$ . Then the sequence  $F \rightarrow E \xrightarrow{p} B$  with constant homotopy is a fibre sequence. This is a special case of example 4.
4. The map  $p : E \rightarrow B$  is a Serre fibration if every commutative diagram of the form

$$\begin{array}{ccc} D^n \times 0 & \longrightarrow & E \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ D^n \times [0, 1] & \longrightarrow & B \end{array}$$

there exists a lift  $g$  making both triangles commute.

Given a Serre fibration  $p : E \rightarrow B$ ,  $b \in B$  the sequence  $F = p^{-1}(b) \rightarrow E \rightarrow B$  with the constant homotopy is a fibre sequence. (Proof as exercise).

Note that every fibre sequence is also equivalent to one of this form.

5. As a special case of example 3, the Hopf fibration is a fibre bundle  $S^1 \rightarrow S^3 \xrightarrow{\eta} S^2$  works by letting  $S^1 = U(1)$  act on  $S^3 \subset \mathbb{C}^2$  by  $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$  with quotient  $S^2 \cong \mathbb{C}P^1$ .
6. Example 5 generalises to fibre bundles  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$  with limit case

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^\infty & \longrightarrow & \mathbb{C}P^\infty \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \Omega \mathbb{C}P^1 & \longrightarrow & * & \longrightarrow & \mathbb{C}P^\infty \end{array}$$