Alg Geo I PS 4

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1. Firstly we compute the Eilenberg-MacLane space $K(\mathbb{Z}_2, 3)$. To do that first we look at the path-loop fibration $K(\mathbb{Z}_2, 1) \to * \to K(\mathbb{Z}_2, 2)$. For simplicity we denote $K_n = K(\mathbb{Z}_2, n)$. As $K_1 = \mathbb{R}P^\infty$, the first few homology and cohomology groups are as follows,

$$
\begin{aligned}
H_*(K_1; \mathbb{Z}) &= \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2 \\
H^*(K_1; \mathbb{Z}) &= \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0 \\
H_*(K_1; \mathbb{Z}_2) &= \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2 \\
H^*(K_1; \mathbb{Z}_2) &= \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2 \\
H_*(K_2; \mathbb{Z}) &= \mathbb{Z}, 0, \mathbb{Z}_2, 0 \\
H^*(K_2; \mathbb{Z}) &= \mathbb{Z}, 0, 0, \mathbb{Z}_2 \\
H_*(K_2; \mathbb{Z}_2) &= \mathbb{Z}_2, 0 \\
H^*(K_2; \mathbb{Z}_2) &= \mathbb{Z}_2, 0
\end{aligned}
$$

We know that $H^*(K_1; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ the polynomial ring on one generator. Thus by UCT we can see that $H^p(K_2; H^q(K_1; \mathbb{Z}_2)) = H^p(K_2; \mathbb{Z}_2) \otimes H^q(K_1; \mathbb{Z}_2)$. The cohomological Serre spectral sequence on $\mathbb{Z}_2$ of the fibre sequence,

$$
\begin{tikzcd}
\mathbb{Z}_2 \arrow[drr] & 0 & \mathbb{Z}_2 & H^3 & H^4 & H^5 \\
\mathbb{Z}_2 & 0 & \mathbb{Z}_2 & H^3 & H^4 & H^5 \\
\mathbb{Z}_2 \arrow[drr] & 0 & \mathbb{Z}_2 \arrow[drr] & H^3 & H^4 & H^5 \\
\mathbb{Z}_2 & 0 & \mathbb{Z}_2 & H^3 & H^4 & H^5 \\
\mathbb{Z}_2 \arrow[drr] & 0 & \mathbb{Z}_2 \arrow[drr] & H^3 & H^4 & H^5 \\
\mathbb{Z}_2 & 0 & \mathbb{Z}_2 & H^3 & H^4 & H^5
\end{tikzcd}
$$

Take the generator $x \in E_2^{0,1}$ of $H^*(K_1; \mathbb{Z}_2)$, x is a transgressive element and by vanishing at E_∞ page, $H^2(K_2; \mathbb{Z}_2) = \mathbb{Z}_2$ with generator $y = d_2(x)$. Then $x^2 \in E_2^{0,2}$, $x^4 \in E_2^{0,4}$ are both transgressive hence $d_2(x^2) = d_2(x^4) = 0$ and $d_2(x^3) = yx^2$, $d_2(x^5) = yx^4$. Also the multiplicative structure $E_2^{2,q} = \mathbb{Z}_2$ is generated by yx^q. Hence $E_2^{2,2}$, $E_2^{2,4}$ vanish at the E_∞ page. By the transgression, $E_2^{2,1}$, $E_2^{2,3}$, $E_2^{2,5}$ do not vanish at the E^3 page, but $d_2(yx) = y^2$, $d_2(yx^3) = y^2x^2$, $d_2(yx^5) = y^2x^4$. In particular $E_2^{4,0}$ has a subgroup generated by y^2. Note that y has degree 2, so we take the 2nd order Steenrod square such that $\mathrm{Sq}^2(y) = y^2$. Similarly, x has degree one and so $\mathrm{Sq}^1(x) = x^2$ so $z = d_3(x^2) = \mathrm{Sq}^1(d_3(x)) = \mathrm{Sq}^1(y)$ and hence again $E_2^{3,0}$ contains $\mathbb{Z}_2$ generated by $\mathrm{Sq}^1(y)$. By a similar process, $E_2^{5,0}$ is $\mathbb{Z}_2$ generated by $\mathrm{Sq}^2\mathrm{Sq}^1(y)$.

For the integral co-homological sequence;

$$
\begin{tikzcd}
\mathbb{Z}_2 & 0 & 0 & & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}_2 \arrow[ddrrr, "d_3"'] & 0 & 0 & \mathbb{Z}_2 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}_2 & 0 & H^5 & H^6
\end{tikzcd}
$$

Which converges to the E^∞ page which vanishes everywhere except $E^\infty_{0,0} = \mathbb{Z}$. Note that $E^2_{4,0}$ vanishes as there is no-nontrival differential to it. Looking at the mulitplicative structure, the generator $x \in E^2_{0,2} = \mathbb{Z}_2$ maps to the generator $y = d_3(x) \in \mathbb{Z}$ and $d_3(x^2) = 2xy$ so $d_3 : E^2_{0,4} \to E^2_{3,2}$ is not an isomophism and so for vanishing at the E^∞ page H^6 must be non-empty. Note further that $d_3(xy) = d_3(x)y = y^2$ so it is clear that $H^6 = \mathbb{Z}_2$.

From UCT we have an exact sequence $0 \to \mathbb{Z}_2 \to H^3(K_2; H^2(K_1)) = H^3(K_2;\mathbb{Z}_2) \to \mathbb{Z}_2 \to 0$. Also there is a d_5 differential $d_5 : \mathbb{Z}_2 \to H^5(K_2)$ so $H^5(K_2) = \mathbb{Z}_4$. From this we can find the homology groups $H_*(K_2) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}_2$.
Similarly we do this for $K_2 \to * \to K_3$. We already have that $H_*(K_3) = \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0$, $H^*(K_3) = \mathbb{Z}, 0, 0, 0, \mathbb{Z}_2, 0$. Again as before $H^4(K_3; H^3(K_2)) = \mathbb{Z}_2 \otimes \mathbb{Z}_2$

$$
\begin{array}{|cccccc}
\mathbb{Z}_2 & 0 & 0 & 0 & & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 & \mathbb{Z}_2 \otimes \mathbb{Z}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}_2 & 0 \\
\hline
& & & & &
\end{array}
$$

Similarly to before we get that the only differential hitting $E_2^{0,5}$ is $d_6 : E_2^{0,5} = \mathbb{Z}_2 \to E_2^{6,0}$ and since both these vanish at the E_∞ page, it is an isomrphism, thus $H^6(K_3, \mathbb{Z}_2) = \mathbb{Z}_2$. Finally, we can apply UCT again to get homology groups $H_*(K_3) = \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2$.
Since S^3 is CW and 2-connected, there exists a Whitehead tower up to degree 4

$$
\begin{tikzcd}
K(\mathbb{Z}_2, 3) \arrow[d] & K(\mathbb{Z}, 2) \arrow[d] & \\
X_4 \arrow[r] & X_3 \arrow[r] & X_2 = S^3
\end{tikzcd}
$$

By long exact sequences on the fibrations $K(\mathbb{Z}_2, 3) \to X_4 \to X_3$ and $K(\mathbb{Z}, 2) \to X_3 \to S^3$, $\pi_5(S^3) = \pi_5(X_4)$ and by Hurewicz $\pi_5(X_4) = H_5(X_4)$ so it suffices to compute the fifth homology group of X_4. First we look at X_3, i.e the fibration $K(\mathbb{Z}, 2) \to X_3 \to S^3$. Note $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ so $H^q(K(\mathbb{Z}, 2)) = \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0 \ldots$. Then cohomological Serre spectral sequence has E_2-page $E_2^{p,q} = H^p(S^3, H^q(K(\mathbb{Z}, 2))) \implies H^{p+q}(X_3)$

$$
\begin{tikzcd}[row sep=small, column sep=small]
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0
\end{tikzcd}
$$

Non-trivial differentials are all d_3 then if $x \in E_2^{0,2} = H^2(K(\mathbb{Z}, 2)) = \mathbb{Z}$ is a generator then $y = d_3(x)$ is a generator for $E_2^{3,0} = H^3(S^3)$. By the multiplicative structure xy generates $E_2^{3,2}$ and as $d_3(x^2) = d_3(x)x + xd_3(x) = 2xy$ so x^2 is twice a generator of $E_2^{0,4} = H^4(K(\mathbb{Z}, 2))$. Similarly x^3 is thrice a generator of $H^6(K(\mathbb{Z}, 2))$ since $d_3(x^3) = 3x^2y$.
As there are no more larger degree differentials after this $E_4 = E_\infty$, and so the first four cohomology group are easily seen to be $H^n(X_3) = \mathbb{Z}, 0, 0, 0$ and due to the maps $\mathbb{Z} \xrightarrow{\times 2, \times 3} \mathbb{Z}$, $H^4(X_3) = 0$, $H^5(X_3) = \mathbb{Z}_2$, $H^6(X_3) = 0$ and $H^7(X_3) = \mathbb{Z}_3$. Then by UCT we see that the first few homology groups are $H_n(X_3) = \mathbb{Z}, 0, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}_3$.
Now looking at the homological Serre spectral sequence for $K(\mathbb{Z}_2, 3) \to X_4 \to X_3$.

$$
\begin{tikzcd}[row sep=small, column sep=small]
\mathbb{Z}_2 & 0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_3 \arrow[uuuuullllll, "d_6"']
\end{tikzcd}
$$

Since the only morphism $\mathbb{Z}_3 \to \mathbb{Z}_2$ is the trivial map, it follows that at the

E^∞-page $H_5(X_4) = E^\infty_{0,5} = \mathbb{Z}_2$. Hence $\pi_5(S^3) = \mathbb{Z}_2$.

2. Let $X = \mathrm{map}(S^1, S^3)$, take base-point $* \in S^1$ and consider the evaluation map $\mathrm{ev}_* : X \to S^3$, $f \mapsto f(*)$. Take base-point $b \in S^3$ and fibre sequence $F := \mathrm{ev}_*^{-1}(b) \to X \xrightarrow{\mathrm{ev}_*} S^3$. Note that F is just the loop space on S^3. Every map $S^1 \to S^3$ is null-homotopic, X is path connected. From the fibre sequence, there is a long exact sequence of homotopy groups. Since $\pi_0(S^3) = \pi_1(S^3) = \pi_2(S^3) = 0$ and the loop-space F is simply connected (in fact contractible), $\pi_1(X) = 0$. Looking at the long exact sequence

$$\ldots \to \pi_3(F) \to \pi_3(X) \to \pi_3(S^3) = \mathbb{Z} \to \pi_2(F) \to \pi_2(X) \to 0 \to \ldots$$

ev_0 has a right inverse $\sigma : S^3 \to X$ taking $x \in S^3$ to the constant map to x. Hence the map $\pi_3(X) \to \pi_3(S^3)$ is surjective, so $\pi_3(S^3) \to \pi_2(F)$ is the zero map. Therefore $\pi_2(X) = \pi_2(F)$ (also follows from splitting) and as $F \cong \Omega(S^3)$, $\pi_2(\Omega(S^3)) \cong \pi_3(S^3) = \mathbb{Z}$. By Hurewicz then $H_2(X) = H_2(F) = \mathbb{Z}$ and $H^2(X) = H^2(F) = \mathbb{Z}$ from UCT. We know that $H^*(F)$ is the divided power algebra on one generator $\Gamma(x)$ so looking at the cohomological Serre spectral sequence,

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[ddrrr] & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & 0
\end{tikzcd}$$

We see the only non-trival differentials are $d_3 : H^0(S^3; H^{2k}(F)) \to H^3(S^3; H^{2k-2}(F))$. The case $k = 1$ must be the zero map as $H^3(X) \cong E^{3,0}_\infty \cong H^3(S^3)/\mathrm{im}(d_3)$ so the map $H^3(S^3) \to H^3(X)$ induced by the right inverse from before is injective (since it has left inverse ev_*).
Then as $H^*(F)$ is the divided power algebra on one generator $\Gamma(x)$, the multiplicative structure shows inductively that all d_3 are zero maps. Hence we get

$$H^n(X) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbb{Z} & \text{else} \end{cases}$$