Symmetric Spaces

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Definition 1.1. A **Riemann symmetric space** is a Riemannian manifold (M, g) such that $\forall p \in M$ there exists a isometry $s_p : M \to M$ satisfying $s_p(p)$ and $ds_p|_p = -\mathrm{id}$.

Remark. A Riemannian manifold M does not have isometries in general.

A few examples:

Example.

- On $\mathbb{R}^n$, $s_p(p + v) = p - v$.
- $S^n \subset \mathbb{R}^n$ given w.l.o.g at $p = (0, 0, \ldots, 0, 1)$ as $s_p(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_n, x_{n+1})$.
- $\mathbb{H}^n$ as the Poincare disc model $(\mathbb{D}^n, g_{\text{hyp}})$ by $s_0(v) = -v$.

Definition 1.2. A Riemannian manifold (M, g) is called a **locally symmetric space** if $\forall p \in M$ there exists a neighbourhood of p, $p \in \mathcal{U}_p$ and an isometry $s_p : \mathcal{U}_p \to \mathcal{U}_p$ of $(\mathcal{U}_p, g|_{U_p})$ such that $ds_p|_p = -id$.