Algebraic Topology I PS7

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1. Let $\tilde{x}, \tilde{y}$ be generators for $H^n(S^n;\mathbb{Z})$, $H^{2n}(S^{2n};\mathbb{Z})$, $f : S^{2n-1} \to S^n$ and induced generators $x \in H^n(C(f);\mathbb{Z})$, $y \in H^{2n}(C(f);\mathbb{Z})$.

   (a) Let n be odd. The cup product $x \smile x = (-1)^{n^2}(x \smile x)$ hence $x^2 = 0$ so $h(f) = 0$.

   (b) Define $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ as follows: pick representative of some homotopy class $f \in [f] \in \pi_{2n-1}(S^n)$, then $[f] \mapsto h(f)$. This is well defined as the mapping cone $C(f)$ is homotopic to the cone over any other choice of representatives. For homomorphism, pick two representatives $f \in [f], g \in [g]$ then sum of $[f], [g]$ in $\pi_{2n-1}(S^n)$ is represented by the composition

   $$f + g : S^{2n-1} \xrightarrow{\alpha} S^{2n-1} \vee S^{2n-1} \xrightarrow{f \wedge g} S^n \vee S^n \xrightarrow{\beta} S^n$$

   .

   Consider the map $\phi : C(f \vee g) \circ \alpha) \to C(f+g)$ defined by folding $S^n \vee S^n$ to S^n via β. Take generators $a \in H^n(C(f+g))$, $b \in H^{2n}(C(f+g))$, $c \in H^{2n}(C((f \vee g) \circ \alpha)))$, $d, d' \in H^n(S^n \vee S^n)$. Note that $\phi^*(a \smile a) = (\phi^* a)^2 = (d + d')^2$ via the action of the folding map on $C((f \vee g) \circ \alpha))$.
   Since d, d' corresponding to generators of $H^n(S^n)$, $d^2 = d'^2 = 0$ and the maps of $S^n \to C(f), C(g)$ induce $(d + d')^2 = h(f)y_f + h(g)y_g$ where y_f, y_g are generators of $H^{2n}(C(f)), H^{2n}(C(g))$ respectively. Moreover, y_f, y_g also generate $H^{2n}(C(f) \vee C(g))$ so by collapsing e^{2n} in $C((f \vee g) \circ \alpha)$ we get map $\varphi : C((f \vee g) \circ \alpha) \to C(f) \vee C(g)$ and so looking at d, d' as elements in $H^{2n}(C((f \vee g) \circ \alpha))$; $(d + d')^2 = h(f)\varphi^*(y_f) + h(g)\varphi^*(y_g)$.
   Thus

   $$h(f + g)\phi^* b = \phi^* a^2 = (d + d')^2 = h(f)\varphi^* y_f + h(g)\varphi^* y_g$$

   since $a^2 = h(f + g)b$. Since y_f, y_g, a all map to a generator of $H^{2n}(C((f \vee g) \circ \alpha)))$ we can chose them to map all to c (by sign). Then the result follows.

   (c) Let $g : S^n \to S^n$ be a map of degree d, then consider the mapping cone $C(g \circ f)$ and the natural map $\phi : C(f) \to C(g \circ f)$. Then ϕ induces a morphism $\phi^* : H^n(C(g \circ f)) \to H^n(C(f))$ which sends a generator $\sigma \in H^n(C(g \circ f))$ to $d\tau \in H^n(C(f))$ where τ is a generator of $H^n(C(f))$ such that the signs match up. Since

$H^{2n}(C(f)) \cong H^{2n}(C(g \circ f))$, then pick a generators $H^{2n}(C(f)) \ni y = y' \in H^{2n}(C(g \circ f))$. Thus in $C(g \circ f)$, $h(g \circ f)y' = \sigma^2 = d^2\tau^2 = d^2 h(f)y$ the result follows.

(d) Consider the composite map

$$\alpha : S^{2n-1} \to S^n \vee S^n \xrightarrow{f} S^n.$$

By attaching a e^{2n} cell to S^n along α we can get $\alpha' : S^{2n-1} \to S^n \hookrightarrow C(\alpha)$ which is null-homotopic by construction. Pick generators $z, z' \in H^{2n}(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ corresponding to the different $\mathbb{Z}$-parts. The folding map $f : S^n \vee S^n \to S^n$ is such that $f^*(\tilde{x}) = z + z'$. Moreover we can extend f to a map $F : S^n \times S^n \to S^n$. Note that F^* induces an isomorphism $H^{2n}(C(\alpha)) \cong H^{2n}(S^n \times S^n)$.
The cup square $z \smile z = z' \smile z' = 0$ in $S^n \times S^n$ and $z \smile z'$ is a generator of $H^{2n}(S^n \times S^n)$. For any generator $\sigma \in H^{2n}(C(\alpha); \mathbb{Z})$ we then have such that $F^*(\sigma) = \pm(z \smile z')$, then the Hopf invariant here is such that $\tilde{x} \smile \tilde{x} = h(\alpha)\sigma$, so $F^*(\tilde{x} \smile \tilde{x}) = h(\alpha)F^*(\sigma) = \pm h(\alpha)(z \smile z')$. Then as $F^*(\tilde{x}) = z + z'$,

$$F^*(\tilde{x}^2) = (F^*\tilde{x})^2 = (f^*\tilde{x})^2 = (z + z')^2 = z \smile z' + z' \smile z = (1 + (-1)^{n^2})z \smile z' = 2z \smile z'$$

Then $\pm h(a)z \smile z' = 2z \smile z'$ and the result follows.

(e) We have that $H^{kn}(\Omega S^{n+1}) = \mathbb{Z}$ for all k. Let $f : S^{2n-1} \to \Omega S^{n+1}$ be that attaching map of the e^2n-cell onto the e^n cell of ΩS^{n+1} (which under attaching with the 0-cell is S^n).

2. Consider based CW-pair (X, A) with inclusion map $i : A \to X$ and base-point $*$. i is an n-equivalence for some n, so (X, A) is an n-connected pair. By the 'correct' Blakers-Massey theorem, the pushout

$$\begin{tikzcd} A/X & X \arrow[l] \\ * \arrow[u] & A \arrow[l, "p"] \arrow[u, "i"'] \end{tikzcd}$$

where p is obviously a m-equivalence, induces an equivalence of $\pi(X/A) \cong \pi_k(X/A, *) \cong \pi_k(X, A) \cong \pi_k(X, A, *)$ for $k \leq m + n - 1$. Then from the exact sequence on relative homotopy

$$\ldots \longrightarrow \pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X, A) \longrightarrow \pi_{k-1}(A) \longrightarrow \ldots$$

we get

$$\ldots \longrightarrow \pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X/A) \longrightarrow \pi_{k-1}(A) \longrightarrow \ldots$$

By Freudenthal, $\pi_{k+\ell}(\Sigma^\ell X)$, $\pi_{k+\ell}(\Sigma^\ell A)$ stabilise for large enough ℓ (as we are doing this on CW-complexs) and then so does $\pi_k(X, A)$, hence passing to the colimit gives us the required exact sequence on stable homotopy groups.