Algebraic Topology I PS5

Mathieu Wydra

1. Consider the fibration on the two-sheeted cover $S^0 \to S^1 \xrightarrow{p} S^1$

2. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with B path connected. Let $\{c_j\} \in H^*(E;\mathbb{Z})$ with only finitely many in any degree, such that $\{i^*c_j\}$ form a $\mathbb{Z}$ basis for the cohomology of $H^*(F;\mathbb{Z})$. Note first that this condition implies that the induced map is $i^* : H^q(E;\mathbb{Z}) \to H^q(F;\mathbb{Z})$ is a surjection. We have the natural Serre spectral sequence $E_2^{p,q} = H^p(B; H^q(F,\mathbb{Z})) \implies H^{p+q}(E;\mathbb{Z})$, and also a Serre sequence arising from the fibration $F \to F \to *$, $\tilde{E}_2^{p,q} = H^p(*; H^q(F;\mathbb{Z})) \implies H^{p+q}(F;\mathbb{Z})$. The following diagram commutes:

$$\begin{tikzcd}
F \arrow[r, "="] \arrow[d, "="] & F \arrow[r] \arrow[d, "i"] & * \arrow[d] \\
F \arrow[r, "i"] \arrow[d] & E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, "="] \\
* \arrow[r] & B \arrow[r, "="] & B
\end{tikzcd}$$

Then i induces a map on spectral sequences $E_2^{p,q} \xrightarrow{i^*} \tilde{E}_2^{p,q}$ which collapses on the E_∞ page so the map

$$H^q(E;\mathbb{Z}) \twoheadrightarrow E_\infty^{p,0} = E_{q+1}^{0,q} \hookrightarrow E_q^{0,q} \hookrightarrow \dots \hookrightarrow E_2^{0,q} = H^q(F;\mathbb{Z})$$

is just i^*. Hence as i^* is surjective, each inclusion map must also be a bijection, so the $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ vanish for all r. Also on the E_2-page, $H^n(F;\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module. So by UCT $H^p(B; H^q(F;\mathbb{Z})) = H^p(B;\mathbb{Z}) \otimes H^q(F;\mathbb{Z})$. Since d_r is zero on both the p and q axes, by multiplicative structure (ADD DETAIL) it is zero everywhere to the sequence collapses on the E_2 page. Then $H^p(B;\mathbb{Z}) \otimes H^q(B;\mathbb{Z}) \to H^{p+q})(E;\mathbb{Z})$, $x \otimes y \mapsto s(x) \cap \pi^*(y)$ is an isomorphism.