Symmetric Spaces

1 (Locally) Symmetric Spaces

Definition 1.1. A **Riemann symmetric space** is a Riemannian manifold (M, g) such that $\forall p \in M$ there exists a isometry $s_p : M \to M$ satisfying $s_p(p)$ and $ds_p|_p = -\mathrm{id}$.

Remark. A Riemannian manifold M does not have isometries in general.

A few examples:

Example.
- On $\mathbb{R}^n$, $s_p(p + v) = p - v$.
- $S^n \subset \mathbb{R}^n$ given w.l.o.g at $p = (0, 0, \ldots, 0, 1)$ as $s_p(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_n, x_{n+1})$.
- $\mathbb{H}^n$ as the Poincare disc model (D^n, g_{hyp}) by $s_0(v) = -v$.

Definition 1.2. A Riemannian manifold (M, g) is called a **locally symmetric space** if $\forall p \in M$ there exists a neighbourhood $\mathcal{U}_p \ni p$ and an isometry $s_p : \mathcal{U}_p \to \mathcal{U}_p$ of $(\mathcal{U}_p, g|_{\mathcal{U}_p})$ such that $ds_p|_p = -id$.

Note that a diffeomorphism with the same properties (no necessarily an isometry) always exists.

Theorem 1.1. *Let (M, g) be a Riemannian manifol, then the following are equal.*

1. *M is a locally symmetric space.*
2. *The curvature tensor is parallel, i.e. $\nabla_X R(Y, Z)T \equiv 0$, $\forall X, Y, Z, T \in V(M)$.*

Proof. Take some s_p as definite at $p \in M$. Given $X, Y, Z, T \in V(M)$ when we have

$$-\nabla_X R(Y,Z)T(p) = d(s_p)|_p(\nabla_X R(Y,Z)T)(p) = \nabla_{d(s_p)|_p}(R(Y,Z)T) = \\ = \nabla_{d(s_p)|_p} R(d(s_p)Y, d(s_p)Z)d(s_p)T = (-1)^4 \nabla_X R(Y,Z)T(p).$$

Conversely, follows from the following theorem as follows: Since M as parallel curvature tensor and $-\mathrm{Id} : T_pM \to T_pM$ is a linear isometry preserving the curvature tensor, then for $p \in M$ we can find neighbourhoods $\mathcal{U} \ni p$ and isometry $s_p : \mathcal{U} \to \mathcal{U}$ such that $d(s_p)|_p = -\mathrm{Id}$. $\square$

Theorem 1.2. *Let (M, g^M), (N, g^N) be Riemannian manifolds with parallel curvature tensor. Let $m \in M$, $n \in N$ then for any linear isometry $\phi : T_mM \to T_nN$ preserving curvature tensors ($\phi(R^M(X,Y), Z) = R^N(\phi(X), \phi(Y))\phi(Z)$, $\forall X, Y, Z \in T_mM$), and for any normal neighbourhood $\mathcal{U} = \exp_m(W)$ of m such that $\exp_m|_W$ is a diffeomorphism, there is a normal neighbourhood $\mathcal{V}$ of n and a local isometry $f : \mathcal{U} \to \mathcal{V}$ such that $f(m) = n$ and $df|_m = \phi$.*

Proof. We want to show $f := \exp_m \circ\phi \circ \exp_m^{-1} |\mathcal{U} : \mathcal{U} \to N$ is a local isometry. We already know that f is an isometry at m.
Given $x \in \mathcal{U}$, $w \in T_xM$ we need to show that $\|df_x w\|_{g^N} = \|w\|_{g^M}$.
Let $v \in T_mM$ such that $\exp_m(v) = x$ and denote $w' = d(\exp_m)^{-1}(w) \in T_mM$. Consider the Jacobi field J along $\exp_m(tv)$ with $J(0) = 0$, $J'(0) = w'$ (i.e. $J(t) = d(\exp_m)|_{tv}(tw')$). Let $e_1 = v$, $(e_2, \dots, e_n)$ an orthonormal basis of $e_1^\perp \subset T_mM$ and $e_i(t)$ the parallel transport of e_i along $c(t) := \exp_m(tv)$, hence $(e_1(t), \dots e_n(t))$ is orthogonal basis of $T_{\exp_m(tv)}M$.
Then there are $y_i(t) \in C^\infty([0,1])$ such that $J(t) = \sum_{i=1}^n y_i(t)e_i(t)$.
Let $(\epsilon_1(t), \dots, \epsilon_n(t))$ be the parallel transport of $(\phi(e_1), \dots, \phi(e_n))$ along $\gamma(t) := \exp_n(t\phi(e_1))$ (hence $\epsilon_1(t) = \gamma'(t)$). Define $I(t) := \sum_{i=1}^n y_i(t)\epsilon_i(t)$, and we claim that I is a Jacobi field along γ, with $I(0) = 0$ and $I'(0) = \phi(w')$. For any $i = 1, \dots, n$, $t \in [0,1]$,

$$\begin{aligned}
g^N_{\gamma(t)}\left(I'' + R^N(\gamma', I)\gamma', \epsilon_i\right) &= g^N_{\gamma(t)}\left(\sum y_i\epsilon_i, \epsilon_i\right) + g^N_{\gamma(t)}\left(R^N(\gamma', \sum y_i\epsilon_i)\gamma', \epsilon\right) \\
&= y_i'' + \sum y_i g^N_{\gamma(t)}(R^N(\epsilon_1(t), \epsilon_j(t))\epsilon_1(t), \epsilon_i(t)) \\
&= y_i'' + \sum y_i(t) g^N_{\gamma(0)}(R^N(\epsilon_1(0), \epsilon_j(0))\epsilon_1(0), \epsilon_i(0)) \\
&= y_i'' + \sum y_i(t) g^N_{c(0)}(R^N(e_1(0), e_j(0))e_1(0), e_i(0)) \\
&= y_i'' + \sum y_i g^N_{c(t)}(R^N(e_1(t), e_j(t))e_1(t), e_i(t)) \\
&= 0
\end{aligned}$$

So $I'' + R^N(\gamma', I)\gamma' = 0$ and I is a Jacobi field. Moreover $I'(0) = \sum y_i'(0)\epsilon_i(0) = \sum y_i'(0)\phi(e_1(0)) = \phi(J'(0)) = \phi(w')$.
By definition

$$df|_x(w) = df_{\exp_m(v)}(d\exp_m|_v(w') = d\exp_m|_{\phi(v)}(\phi(w')$$

so $df_x(J(1)) = I(1)$ but

$$\begin{aligned}
\|J(1)\|^2_{g^M} &= \|v\|^2_{g^M}|y_1(t)|^2 + \sum_{i=2}^n |y_i(1)|^2 \\
&= \|\phi(v)\|^2_{g^N}|y_1(1)|^2 + \sum_{i=2}^n |y_i(1)|^2 = \|I(1)\|^2_{g^N}
\end{aligned}$$

Thus f is a local isometry. □

Corollary 1.3. *Let M, N be complete locally symmetric spaces with M simply connected with non-positive sectional curvature. For any curvature tensor preserving linear isometry $\phi : T_mM \to T_nN$ that preserves the curvature tensor, then there exists a Riemannian covering $f : M \to N$ with $f(m) = n$ and $df|_m = \phi$.*

Proof. By Cartan-Hadamard, $\exp_m : T_mM \to M$ is a diffeomorphism. Thus we get a local isometry $f : M \to N$ with $f(m) = n$ and $df|_m = \phi$. Since M is complete, this guarantees that f is a Riemannian covering. □

Theorem 1.4. *Let M, N be complete locally symmetric spaces with M simply connected. For any curvature tensor preserving linear isometry $\phi : T_mM \to$*

T_nN that preserves the curvature tensor, then there exists a Riemannian covering $f : M \to N$ with $f(m) = n$ and $df|_m = \phi$.

Corollary 1.5. *Let M be a complete locally symmetric space. Then the Riemannian universal cover $\tilde{M}$ of M is a symmetric space.*

Proof. The universal cover of a complete Riemannian manifold is complete. We claim that $\tilde{M}$ is a locally symmetric space: $\pi : \tilde{M} \to M$ be the covering map, which is a local isometry, then

$$\begin{aligned}\pi(\nabla_X R^{\tilde{M}}(Y,Z),T) = \nabla_{d\pi(X)} d\pi(R^{\tilde{M}}(Y,Z),T) = \\ = \nabla_{d\pi(X)} R^M(d\pi(Y), d\pi(Z)) d\pi T = 0\end{aligned}$$

Since $d\pi|_p$ is an isomorphism, $\nabla_X R^{\tilde{M}}(Y,Z)T = 0$ if and only if $\tilde{M}$ is locally symmetric.
Given $p \in \tilde{M}$, $\mathrm{Id}_{T_p\tilde{M}}$ is a linear isometry and preserves the curvature tensor. Then by a previous corollary, there exists $s_p : \tilde{M} \to \tilde{M}$ a covering such that $s_p(p) = p$ and $ds_p|_p = -\mathrm{id}_{T_p\tilde{M}}$. As every smooth covering of a simply connected space is a diffeomorphism, this implies that s_p is a diffeomorphism (and a local isometry), so s_p is an isometry. □

Example. Let $M = S^n$, $N = \mathbb{R}P^n$ with natural metric (with sectional curvature $= 1$), then M, N are complete locally symmetric spaces. In this case there are many linear isometries that preserve the curvature tensor (in fact they identify with $O(n)$).

Theorem. (Idea). Such an f should be defined such that

$$\begin{tikzcd} T_mM \arrow[r, "\phi"] \arrow[d, "\exp_m"] & T_nN \arrow[d, "\exp_n"] \\ M \arrow[r, dotted, "f"] & N \end{tikzcd}$$

□

The issue is that in general this is not well defined i.e. there can exist $v, w \in T_mM$ such that $\exp_m v \exp_m w$.
We solve this by extending f along paths and show that the extension is independent of the chosen path.
Remark. Recall that $\mathcal{U} \subset M$ is a *normal neighbourhood* if $\mathcal{U} = \exp_p(W)$ for a star-shaped neighbourhood of 0, $W \subset T_pM$ and diffeomorphism $\exp_p|_W : W \to \mathcal{U}$.
If additionally $\mathcal{U} = B_r(p)$ ($W = B_r(0)$) then $\mathcal{U}$ is called a *normal ball.*

We also need the following two observations

1. Let M, N be complete locally symmetric spaces $B_r(p)$, $B_\ell(p) \subset M$ normal balls with $\ell > r$ and $f : B_r(p) \subset M \to V \subset N$ is a local isometry, then $\hat{f} := \exp^N_{f(p)} \circ df_p \circ (\exp^M_p)^{-1} : B_\ell(p) \to B_\ell(f(p))$ is a local isometry extending f.
   Note that it follows from a previous theorem that $\hat{f}$ is a local isometry with $\hat{f}(p) = f(p)$, $d\hat{f}|_p = df|_p$.
   Recall that two isometries are equal if they and their differentials are equal at p.

2. For any $p \in M$, $\exists r > 0$ such that $B_r(q)$ is a normal ball for any $q \in B_r(p)$.

Lemma 1.6. *Let M, N be complete locally symmetric spaces, $m \in M$, $m \in \mathcal{U}$ a normal neighbourhood, $f : \mathcal{U} \to N$ a local isometry and $\sigma : [0,1] \to M$ a smooth curve with $\sigma(0) = m$. Then f can be continued along σ i.e. $\forall t \in [0,1]$, there exist $\mathcal{U}_t$ neighbourhood of $\sigma(t)$ such that $f_t : \mathcal{U}_t \to N$ is an isometry, and $\exists \epsilon > 0$ such that $|t - s| < \epsilon \implies \mathcal{U}_t \cap \mathcal{U}_s \neq \emptyset$ and $f_t|_{\mathcal{U}_t \cap \mathcal{U}_s} = f_t s|_{\mathcal{U}_t \cap \mathcal{U}_s}$..*

Proof. Define $I = \{t \in [0,1] \mid f$ can be continued along $\sigma|_{[0,t)}\}$. I is non-empty (contains t such that $\sigma([0,t]) \subset \mathcal{U}$) and open (If $t_0 \in I$ choose ϵ small enough such that $\sigma(t_0 - \epsilon, t_0 + \epsilon) \subset \mathcal{U}_{t_0}$ and set $\mathcal{U}_t = \mathcal{U}_{t_0}$, $f_t = f_{t_0}$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$). We show that I is closed. Let q be an accumulation point of $\underline{f_t(\sigma(t))}$ for $t \to T :=$ $\sup I$. Such a point always exists as N is complete and $f_t(\sigma(t)) \subset \overline{B_{L(\sigma)}(f(\sigma(0)))}$ (since f_t is an isometry and $f_t = f_s$ on $\mathcal{U}_t \cap \mathcal{U}_s$).
Choose $r > 0$ such that it satisfied (2) for q, $\sigma(T)$. Then by construction there exists $t_0 < T$ such that $\sigma(t_0) \in B_r(\sigma(T))$ and $F_t(\sigma(t_0)) \in B_r(q)$. By (1) we can then extend f_{t_0} on $B_r(\sigma(t_0))$. Setting $f_t = f_{t_0}$ and $\mathcal{U}_t = B_r(\sigma(t_0))$ on $(t_0 - \epsilon, t_0 + \epsilon)$ for ϵ small enough, we can extend f around $\sigma(T)$ thus I is closed. Hence $I = [0,1]$.
The second part follows from the compactness of I - we can find finitely many t_i such that $\mathcal{U}_{t_i}$ cover I and chose $\mathcal{U}_t = \mathcal{U}_{t_i}$ for t_i minimal such that $t \in \mathcal{U}_{t_i}$ and $f_t = f_{t_i}$. ☐

Lemma 1.7. *The continuation of f along σ is unique in the sense that if $\{\mathcal{U}_t, f_t\}, \{V_t, \bar{f}_t\}$ are two different continuations and $A_t$is the connected component of $\mathcal{U}_t \cap V_t$ containing $\sigma(t)$ then $f_t|_{A_t} = \bar{f}_t|_{A_t}$.*

Proof. $I\{t \in [0,1] \mid f_t(\sigma(t)) = \bar{f}_t(\sigma(t)), df_t|_{\sigma(t)} = d\bar{f}_t|_{\sigma(t)}\}$, then I is open, closed and non-empty $\implies I = [0,1]$. This holds because isometries are uniquely defined by their value at a point and the differential at the point.
For the same reason $f_t|_{A_T} = \bar{f}_t|_{A_T}$ ☐

Lemma 1.8. *Let M, N be complete locally symmetric spaces $\mathcal{U}$ a normal neighbourhood of m, $f : \mathcal{U} \to N$ a local isometry. Let $\sigma, \tau : [0,1] \to M$ smooth curves with $\sigma(0) = \tau(0) = m$, $\tau(1) = \sigma(1)$, curves being homotopic rel ∂I.*

Proof. Let H be the homotopy between σ, τ rel ∂I. Fix s and let f^s be the continuation of f along the path $t \mapsto H(t,s)$. Let

$$I = \{s \in [0,1] \mid \forall r \leq s\, f^r(\sigma(1)) = f^\sigma(\sigma(1)) \text{ and } df^r|_{\sigma(1)} = df^\sigma|_{\sigma(1)}\}.$$

I is non-empty as$0 \in I$ and open as follows: given $s_0 \in I$ we find $\epsilon > 0$ such that for all $s' \in (s_0 - \epsilon, s_0 + \epsilon)$, $H(t,s') \in \mathcal{U}_t$. By (2) and compactness, there exists $r > 0$ such that $B_r(\sigma(t))$ is a normal ball. Assume then that $B_r(H(s_0,t)) \subset \mathcal{U}_t$ (as H is smooth). Setting $\mathcal{U}_t^s = \mathcal{U}_t^{s_0}$ and $f_t^{s'} = f_t^{s_0}$ gives continuation along $H(,s')$ thus $(s_0 - \epsilon, s_0 + \epsilon) \subset I$ (by construction) and I is open.
For I closed, let $A = \sup I$ and as before $\exists r > 0$ such that $B_r(H(t,A))$ is a normal ball for all $t \in [0,1]$ and $B_r(f^A(H(t,A))$ is a normal ball. As before $\exists \epsilon > 0$ such that $\forall s : |A - s| < \epsilon$, $H(t,s) \in B_r(H(t,A))$ so f^A is a continuation of f along $H(,s)$ for $s \in (A - \epsilon, A)$ where $f^A(\sigma(1)) = f^s(\sigma(1)) = f^\sigma(\sigma(1))$ and $df^A_{\sigma(1)} = df^s_{\sigma(1)} = df^\sigma_{\sigma(1)}$ thus $A \in I \implies I = [0,1]$. ☐

Back to the theorem we are trying to prove

Proof. Theorem Since M, N are complete, exp is defined everywhere. Define f (locally) though the relation $f(\exp_m(v)) = \exp_n(\phi(v))$ - this is indeed well defined on a normal neighbourhood $\mathcal{U}$ of m.
Set $\bar{f} = f|_{\mathcal{U}}$, and define f on $\exp_m(v)$ via continuation of $\bar{f}$ along $\sigma(t) := \exp_m(tv)$. If there exists $v, w \in T_m M$ such that $\exp_m(v) = \exp_n(v)$ then the paths $\sigma(t) = \exp_m(tv)$ and $\tau(t) = \exp_m(tw)$ are homotopic rel ∂I due to M being simply-connected. Then be the previous lemmma, it is well defined as $f(\exp_m(v)) = f(\exp_m(w))$. As f is a continuation along paths it is a local isometry. M being complete implies that f is a smooth covering (+ localisation) by a previous lemma. Hence this is Riemann covering. □

Theorem 1.9. *Let M be a complete, simply-connected Riemannian manifold then the following are equal.*

1. *M is a symmetric space.*
2. *M is a locally symmetric space.*
3. *Any curvature preserving linear isometry $\phi : T_x M \to T_y M$ is induced by a (unique) linear isometry $f : M \to M$ such that $f(x) = y$, $df_x = \phi$.*

Proof. $1 \implies 2$ is trivial, $2 \implies 3$ hold by the previous theorem. For $3 \implies 1$, we apply 3 to $-\mathrm{Id} : T_p M \to T_p M$. □

Remark. Not all symmetric spaces are simply-connected.

Example.
- $\mathbb{R}P^n$ is a non-simply-connected symmetric space.
- $T^n = (S^1)^n$ is a symmetric space (obviously not 1-connected).

Proposition 1.10. *Let M be a symmetric space, the M is complete.*

Proof. □

Definition 1.3. Let M be a smooth manifold. We say that a group action $G \curvearrowright M$ is *transitive* if $\forall p, q$ there exists $g \in G$ such that $g(p) = q$.

Proposition 1.11. *Let M be a symmetric space. Then Iso(M) acts transitively.*

Proof. □

Definition 1.4. An isometry $f : M \to M$ is called a *transvection* if there exists $p \in M$ and geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = f(p)$ such that f realises parallel transport along γ.

Remark. If M is not flat, then parallel transport really depends on the curve (or geodesic)

Proposition 1.12. *Let M be a symmetric space. For any $p, q \in M$ and any geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, there exists an isometry realising parallel transport along γ.*

Proof. □