Problem Sheet 3

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1. Let $F \to Y \to X$ be a fibre sequence with $F = \mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$ and $X = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. First we can note that $[K(\mathbb{Z}, 2), K(\mathbb{Z}_2, 2)] \cong H^2(K(\mathbb{Z}, 2), \mathbb{Z}_2) \cong Z_2$ by the Universal coefficient theorem and $H^2(\mathbb{C}P^\infty) = \mathbb{Z}$. Hence there are two maps $X \to K(\mathbb{Z}_2, 2)$ up to homotopy determine by multiplication on H^2 or H_2 by an element of $\mathbb{Z}_2$ which we call f_0, f_1. Consider then the homotopy fibres of f_0, f_1 denoted F_0, F_1.

   By application of Hurewicz we find the first few homology groups for $K(\mathbb{Z}_2, 1)$ and $K(\mathbb{Z}, 2)$.

$$\begin{aligned} H_0(K(\mathbb{Z}_2,1),\mathbb{Z}) &= \mathbb{Z} \\ \pi_1(K(\mathbb{Z}_2,1)) \cong & H_1(K(\mathbb{Z}_2,1),\mathbb{Z}) \cong \mathbb{Z}_2 \\ 0 = \pi_2(K(\mathbb{Z}_2,1)) \twoheadrightarrow & H_2(K(\mathbb{Z}_2,1),\mathbb{Z}) = 0 \\ H_0(K(\mathbb{Z},2),\mathbb{Z}) &= \mathbb{Z} \\ \pi_2(K(\mathbb{Z},2)) \cong & H_2(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z} \\ 0 = \pi_3(K(\mathbb{Z},2)) \twoheadrightarrow & H_3(K(\mathbb{Z},2),\mathbb{Z}) = 0 \end{aligned}$$

   Then by Universal coefficient theorem, the first few cohomology groups are $H^*(K(\mathbb{Z}_2, 1)) = \mathbb{Z}, \mathbb{Z}_2, 0$ and $H^*(K(\mathbb{Z}_2, 1)) = \mathbb{Z}, 0, \mathbb{Z}, 0$. This gives us E_2-page.

$$\begin{array}{c|cccc} & 0 & 0 & 0 & 0 \\ & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ \hline & & & & \end{array}$$

2. Consider map $F : S^k \to S^k$ of degree n, $b \in S^k$. Call the homotopy fibre $\mathrm{hofib}_b(f)$ of f at b, F. The fibre sequence $F \to S^k \xrightarrow{f} S^k$ induces a long exact sequence,

$$\ldots \to \pi_{n+1}(S^k) \to \pi_n(F) \to \pi_n(S^k) \to \pi_n(S^k) \to \ldots.$$

$\pi_n(S^k)$ vanishes for $n \leq k-1$ so we have

$$\ldots \to \pi_{k+1}(S^k) \to \pi_k(F) \to \pi_k(S^k) \xrightarrow{f_*} \pi_k(S^k) \to \pi_{k-1}(F) \to \pi_{k-1}(S^k) \to \ldots$$
$$\ldots \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \pi_{k-1}(F) \to 0 \to \ldots$$

(The induced map on homotopy groups is by the Hurewicz map on S^k). As this is an exact sequence, $\pi_{k-1}(F) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ and $\pi_n(F) = 0$ for $n < k-1$. Thus F is $k-2$-connected so by Hurewicz $H_{k-1}(F) = \mathbb{Z}_n$, $H_j(F) = 0$ for $j \leq k-2$. By UCT $H^j(F) = 0$ for $1 \leq j \leq k-1$

The E_2-page of the cohomological Serre spectral sequence is

$$\begin{tikzcd}
H^k(F) \arrow[rrrrddd] & 0 & \cdots & 0 & H^k(F) & 0 \\
H^{k-1}(F) \arrow[rrrrddd] & 0 & \cdots & 0 & H^{k-1}(F) & 0 \\
\vdots & & \vdots & & \vdots & \\
H^1(F) & 0 & \cdots & 0 & H^1(F) & 0 \\
H^0(F) & 0 & \cdots & 0 & H^0(F) & 0
\end{tikzcd}$$

The only non-trivial differential are d_k and looking at the E_∞ page the only non-trival terms will be $E_\infty^{0,0} = \mathbb{Z}$ and $E_\infty^{k,0} = E_\infty^{0,k} = \mathbb{Z}$. Obviously then $H^0(F) = \mathbb{Z}$, and as $H^j(F) = 0$ for $1 \leq j, \leq k-1$, $H^k(F) = \mathbb{Z}$. All other terms on the E_∞ page vanish and for degree reasons $H^k(F) \cong H^{k+k-1}(F) \cong H^{k+2(k-1)}(F) \cong \ldots$ and by induction we then see $H^{k+j\cdot(k-1)}(F) \cong H^k(F)$ for all $j = 0, 1, \ldots$ and all other $H^m(F)$ vanish.
Of course from the multiplicative structure, we have a multiplication $H^q(F) \otimes H^{q'}(F) \to H^{q+q'}(F)$. The only non-trivial cohomology groups $H^0(F)$, $H^{k+j(k-1)}(F)$ have multiplication $H^{k+j(k-1)}(F) \otimes H^{k+j'(k-1)}(F) \to H^{k+(j+j')(k-1)}(F)$ but the output always vanishes.