Algebraic Topology I Problem Sheet 2

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1. (a) Suppose $S^n \to E \to B$ is a fibre sequence, $n \neq 0$ and B simply connected. By Serre's theorem there exists a Serre spectral sequence

$$E^2_{p,q} = H_p(B; H_q(S^n; \mathbb{Z})) \implies H_{p+q}(E; \mathbb{Z}).$$

Since $H_q(S^n) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & \text{else} \end{cases}$,, then $E^2_{p,q} = H_p(B; H_q(S^n; \mathbb{Z})) = H_p(B; \mathbb{Z})$ for $p = 0, n$ and vanishes for other p. The E^2-page is given by

$$\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
n+1 & 0 & 0 & 0 & 0 \\
n & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\
n-1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 \\
0 & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

The only differentials are d^{n+1} given by $d^{n+1} : H_p(B; \mathbb{Z}) \mapsto H_{p-n-1}(B, \mathbb{Z})$ and $E^\infty = E^{n+2}$.

Since $E^{n+2} \cong H_*(E^{n+1})$, we get an exact sequence

$$0 \to E^\infty_{p,0} \to H_p(B) \xrightarrow{d^{n+1}} H_{p-n-1}(B) \to E^\infty_{p-n-1,n} \to 0$$

We have that $E^\infty_{p-k,k} = F_p^{p-k}/F_p^{p-k-1} = 0$ for $k = 1, \ldots, n-1$ so $E^\infty_{p,0} = \left(F_p^p/F_p^0\right) \ldots \left(F_p^{p-n+1}/F_p^{p-n}\right) = F_p^p/F_p^{p-n}$. Moreover, $E^\infty_{p-n-k,n+k} = F_p^{p-n-k}/F_p^{p-n-k-1} = 0$ for all $k = 1, 2, \ldots$ so take it up to $k = p-n$, then combining as before $E^\infty_{p-n,n} = F^{p-n}_p$. Note $F^p_p = H_p(E;\mathbb{Z})$. Thus there is an exact sequence

$$0 \to E^\infty_{p-n,n} \to H_p(E;\mathbb{Z}) \to E^\infty_{p,0} \to 0$$

Combining this with the previous sequence give us a long exact sequence

$$\ldots \to H_{p-n}(B) \to H_p(E) \to H_p(B) \to H_{p-n-1}(B) \to H_{p-1}(E) \to \ldots$$

Exactness is obvious from the exactness of the two short exact sequences and identification of $E^\infty_{p,q}$.

(b) Similarly, suppose $F \xrightarrow{i} E \to S^n$ is a fibre sequence, $n \neq 0,1$. By Serre's theorem there exists a Serre spectral sequence

$$E^2_{p,q} = H_p(S^n; H_q(F;\mathbb{Z})) \implies H_{p+q}(E;\mathbb{Z}).$$

Since $H_p(S^n) = \begin{cases} \mathbb{Z} & p = 0,n \\ 0 & \text{else} \end{cases}$, and $E^2_{p,q} = H_p(S^n; H_q(F;\mathbb{Z})) = H_q(F;\mathbb{Z})$ by the universal coefficient theorem,
then the E^2-page is

2	$H_2(F)$	0	$\cdots$	0	$H_2(F)$	0	$\cdots$
1	$H_1(F)$	0	$\cdots$	0	$H_1(F)$	0	$\cdots$
0	$H_0(F)$	0	$\cdots$	0	$H_0(F)$	0	$\cdots$
	0	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$

Hence the only differentials are d^n given by $d^n : H_q(F) \mapsto H_{q+n-1}$ and $E^\infty = E^{n+1}$.
Since $E^{n+1} \cong H_*(E^n)$, we get an exact sequence

$$0 \to E^\infty_{n,q-n} \to H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \to E^\infty_{0,q-1} \to 0$$

We have that $E^\infty_{k,q-k} \cong F^k_q / F^{k-1}_q = 0$ for all $k = 1, 2, \ldots, n-1$, so $E^\infty_{n,q-n} \cong F^n_q / F^0_q$ and since $F^n_q = H_q(E;\mathbb{Z})$, $E^\infty_{0,q} = F^0_q$ we also have an exact sequence

$$0 \to E^\infty_{0,q} \to H_q(E;\mathbb{Z}) \to E^\infty_{n,q-n} \to 0$$

Combining the two sequences we get a long exact sequence

$$\ldots \to H_q(F;\mathbb{Z}) \xrightarrow{i_*} H_q(E;\mathbb{Z}) \to H_{q-n}(F;\mathbb{Z}) \to H_{q-1}(F;\mathbb{Z}) \to H_{q-1}(E;\mathbb{Z}) \xrightarrow{i_*} \ldots$$