Algebraic Topology I PS5

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1. (a) Consider the fibration on the two-sheeted cover $S^0 \to S^1 \xrightarrow{p} S^1$. Note $H_0(S^0;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$, so $H_*(S^1; H_0(S^0;\mathbb{Z})) = \mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}, 0\ldots$. We pick a cell structure on S^1 with 1 cell e_1, attached to 0-cell e_0. Looking at the local coefficient system $H_0(F_{(-)};\mathbb{Z})$ on S^1 induced by the fibration. The differentials

$$d_1(\sigma, g) = (\sigma\circ d_0, (\sigma_{0,1})_* g) - (\sigma\circ d_1, g)$$

$$d_2(\sigma, g) = (\sigma\circ d_0, (\sigma_{0,1})_* g) - (\sigma\circ d_1, g) + (\sigma\circ d_2, g)$$

Let σ have image e_1, so $d_1(\sigma, m) = (e_0, (\sigma_{0,1})_* m - m)$. σ lifts to $\tilde{\sigma}$ a 1-simplex with image a half-circle on S^1. Since $H_0(F_x;\mathbb{Z}) \cong H_0(S^0;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$, this map induces a map $H_0(F_{e_0};\mathbb{Z}) \to H_0(F_{e_0'};\mathbb{Z})$, $(m,n)\to(n,m)$ where $e_0' \in S^1$ is antipodal to $e_0 \in S^1$. So $d_1(\sigma,(m,n)) = (e_0, (n-m, m-n))$. This show

$$H_0(S^1; H_0(F_{(-)};\mathbb{Z}) = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\text{im } d_1} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\text{im } d_1} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\times(1,-1)} \ncong \mathbb{Z}\oplus\mathbb{Z} = H_0(S^1; H_0(S^0;\mathbb{Z})).$$

Similarly it can also be show that d_2 is onto and so

$$H_1(S^1; H_0(F_{(-)};\mathbb{Z}) = \ker d_2 \cong \mathbb{Z} \ncong H_1(S^1; H_0(S^0;\mathbb{Z}))$$

   (b) Let (X,x) be based connected CW, and universal cover $q : \tilde{X} \to X$. Suppose $\pi = \pi_1(X,x)$ acts on Y CW, with induced action on homology α. Consider fibre bundle $Y \to \tilde{X}\times_\pi Y \xrightarrow{q\times x} X$. Take the action $\beta : \pi \times H_*(Y;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ where $[\gamma]$ acts on $H_*(Y;\mathbb{Z})$ by

2. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with B path connected. Let $\{c_j\} \in H^*(E;\mathbb{Z})$ with only finitely many in any degree, such that $\{i^* c_j\}$ form a $\mathbb{Z}$ basis for the cohomology of $H^*(F;\mathbb{Z})$. Note first that this condition implies that the induced map is $i^* : H^q(E;\mathbb{Z}) \to H^q(F;\mathbb{Z})$ is a surjection. Since $H^*(F;\mathbb{Z})$ is freely generated, there is a right inverse $j : H^*(F;\mathbb{Z}) \to H^*(E;\mathbb{Z})$.
We have the natural Serre spectral sequence $E_2^{p,q} = H^p(B; H^q(F,\mathbb{Z})) \implies H^{p+q}(E;\mathbb{Z})$, and also a Serre sequence arising from the fibration $F \to F \to *$, $\tilde{E}_2^{p,q} = H^p(*; H^q(F;\mathbb{Z})) \implies H^{p+q}(F;\mathbb{Z})$. The following diagram commutes:

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, equal] & F \arrow[r] \arrow[d, "i"] & * \arrow[d] \\ F \arrow[r, "i"] & E \arrow[r, "p"] & B \end{tikzcd}$$

Then by naturality argument i induces a map on spectral sequences $E_r^{p,q} \to \tilde{E}_r^{p,q}$ which converges on the E_∞ page to i^*. $\tilde{E}_2$-page only has non-trivial entries $E_2^{0,q}$ so collapses on the E_2-page. So the composition of edge maps

$$\begin{tikzcd}
H^q(E;\mathbb{Z}) \arrow[r, two heads] \arrow[d, "i^*"] & E_\infty^{0,q} = E_{q+1}^{0,q} \arrow[r, hook] \arrow[d] & E_q^{0,q} \arrow[r, hook] \arrow[d] & \dots \arrow[r, hook] & E_2^{0,q} = H^q(F;\mathbb{Z}) \arrow[d, equal] \\
H^q(F;\mathbb{Z}) \arrow[r, equal] & H^q(F;\mathbb{Z}) \arrow[r, equal] & H^q(F;\mathbb{Z}) \arrow[r, equal] & \dots \arrow[r, equal] & H^q(F;\mathbb{Z})
\end{tikzcd}$$

is just i^*. Similarly, we can show that the composition of maps $H^p(B;\mathbb{Z}) \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \dots \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(E;\mathbb{Z})$ is equal to $p^* : H^p(B;\mathbb{Z}) \to H^p(E;\mathbb{Z})$ using the fibre sequence $* \to B \to B$.
Now as i^* is surjective, each inclusion map must also be a bijection, so the $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ vanish for all r. Also on the E_2-page, $H^q(F;\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module. So by UCT $H^p(B;H^q(F;\mathbb{Z})) = H^p(B;\mathbb{Z}) \otimes H^q(F;\mathbb{Z})$. Since d_r is zero on both the p and q axes for all r, by multiplicative structure d_r is zero everywhere to the sequence collapses on the E_2 page. This is seen by induction, . Then $H^p(B;\mathbb{Z}) \otimes H^q(F;\mathbb{Z}) \to H^{p+q}(E;\mathbb{Z})$, $x \otimes y \mapsto p^*(x) \smile j(y)$ is an isomorphism. It can be seen from this then that $\{c_j\}$ is a basis for the $H^*(B;\mathbb{Z})$-module $H^*(E;\mathbb{Z})$.