Alg Geo I PS 4

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1. Firstly we compute the Eilenberg-MacLane space $K(\mathbb{Z}_2,3)$. To do that first we look at the path-loop fibration $K(\mathbb{Z}_2,1) \to * \to K(\mathbb{Z}_2,2)$. For simplicity we denote $K_n = K(\mathbb{Z}_2,n)$. As $K_1 = \mathbb{R}P^{\infty}$, the first few homology and cohomology groups are as follows,

$$H_*(K_1; \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2$$

$$H^*(K_1; \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0$$

$$H_*(K_1; \mathbb{Z}_2) = \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$$

$$H^*(K_1; \mathbb{Z}_2) = \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$$

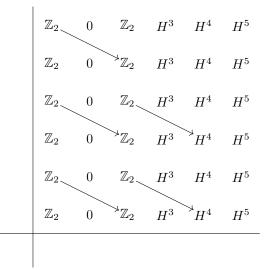
$$H_*(K_2; \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0$$

$$H^*(K_2; \mathbb{Z}) = \mathbb{Z}, 0, 0, \mathbb{Z}_2$$

$$H_*(K_2; \mathbb{Z}_2) = \mathbb{Z}_2, 0$$

$$H^*(K_2; \mathbb{Z}_2) = \mathbb{Z}_2, 0$$

We know that $H^*(K_1; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ the polynomial ring on one generator. Thus by UCT we can see that $H^p(K_2; H^q(K_1; \mathbb{Z}_2)) = H^p(K_2; \mathbb{Z}_2) \otimes H^q(K_1; \mathbb{Z}_2)$. The cohomological Serre spectral sequence on \mathbb{Z}_2 of the fibre sequence,



Take the generator $x\in E_2^{0,1}$ of $H^*(K_1;\mathbb{Z}_2),\ x$ is a transgressive element and by vanishing at E_∞ page, $H^2(K_2;\mathbb{Z}_2)=\mathbb{Z}_2$ with generator $y=d_2(x)$. Then $x^2\in E_2^{0,2},\ x^4\in E_2^{0,4}$ are both transgressive hence $d_2(x^2)=d_2(x^4)=0$ and $d_2(x^3)=yx^2,\ d_2(x^5)=yx^4$. Also the multiplicative structure $E_2^{2,q}=\mathbb{Z}_2$ is generated by yx^q . Hence $E_2^{2,2},\ E_2^{2,4}$ vanish at

the E_{∞} page. By the transgression, $E_2^{2,1}$, $E_2^{2,3}$, $E_2^{2,5}$ do not vanish at the E^3 page, but $d_2(yx) = y^2$, $d_2(yx^3) = y^2x^2$, $d_2(yx^5) = y^2x^4$. In particular $E_2^{4,0}$ has a subgroup generated by y^2 . Note that y has degree 2, so we take the 2nd order Steenrod square such that $\operatorname{Sq}^2(y) = y^2$. Similarly, x has degree one and so $\operatorname{Sq}^1(x) = x^2$ so $z = d_3(x^2) = \operatorname{Sq}^1(d_3(x)) = \operatorname{Sq}^1(y)$ and hence again $E_2^{3,0}$ contains \mathbb{Z}_2 generated by $\operatorname{Sq}^1(y)$. By a similar process, $E_2^{5,0}$ is \mathbb{Z}_2 generated by $\operatorname{Sq}^2(y)$.

For the integral co-homological sequence;

\mathbb{Z}_2	0	0		0		
0	0	0	0	0	0	0
\mathbb{Z}_2	0	0		0		
0	0	0	0	0	0	0
\mathbb{Z}_2	0	0	\mathbb{Z}_2	0		
0	0 d_3	0	0	0	0	0
\mathbb{Z}	0	0	\mathbb{Z}_2	0	H^5	H^6

Which converges to the E^{∞} page which vanishes everywhere except $E^{\infty}_{0,0} = \mathbb{Z}$. Note that $E^2_{4,0}$ vanishes as there is no-nontrival differential to it. Looking at the mulitplicative structure, the generator $x \in E^2_{0,2} = \mathbb{Z}_2$ maps to the generator $y = d_3(x) \in \mathbb{Z}$ and $d_3(x^2) = 2xy$ so $d_3 : E^2_{0,4} \to E^2_{3,2}$ is not an isomorphism and so for vanishing at the E^{∞} page H^6 must be non-empty. Note further that $d_3(xy) = d_3(x)y = y^2$ so it is clear that $H^6 = \mathbb{Z}_2$.

From UCT we have an exact sequence $0 \to \mathbb{Z}_2 \to H^3(K_2; H^2(K_1)) = H^3(K_2; \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0$. Also there is a d_5 differential $d_5 : \mathbb{Z}_2 \to H^5(K_2)$ so $H^5(K_2) = \mathbb{Z}_4$. From this we can find the homology groups $H_*(K_2) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}_2$.

Similarly we do this for $K_2 \to * \to K_3$. We already have that $H_*(K_3) = \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0, H^*(K_3) = \mathbb{Z}, 0, 0, 0, \mathbb{Z}_2, 0$. Again as before $H^4(K_3; H^3(K_2)) = \mathbb{Z}_2 \otimes \mathbb{Z}_2$

\mathbb{Z}_2	0	0	0		0
\mathbb{Z}_2	0 0 0	0	0		0
0	0	0	0	0	0
\mathbb{Z}_2	0	0	0	$\mathbb{Z}_2\otimes\mathbb{Z}_2$	0
0	0	0	0	0	0
0	0	0	0	0	0
\mathbb{Z}	0	0	0	\mathbb{Z}_2	0

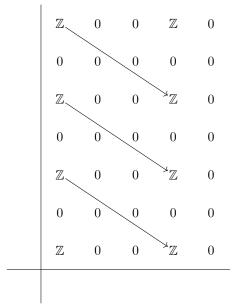
Similarly to before we get that the only differential hitting $E_2^{0,5}$ is d_6 : $E_2^{0,5} = \mathbb{Z}_2 \to E_2^{6,0}$ and since both these vanish at the E_∞ page, it is an isomrphism, thus $H^6(K_3,\mathbb{Z}_2) = \mathbb{Z}_2$. Finally, we can apply UCT again to get homology groups $H_*(K_3) = \mathbb{Z}, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2$.

Since S^3 is CW and 2-connected, there exists a Whitehead tower up to degree 4

$$K(\mathbb{Z}_2,3)$$
 $K(\mathbb{Z},2)$
$$\downarrow \qquad \qquad \downarrow$$

$$X_4 \longrightarrow X_3 \longrightarrow X_2 = S^3$$

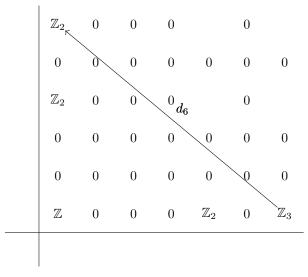
By long exact sequences on the fibrations $K(\mathbb{Z}_2,3) \to X_4 \to X_3$ and $K(\mathbb{Z},2) \to X_3 \to S^3, \pi_5(S^3) = \pi_5(X_4)$ and by Hurewicz $\pi_5(X_4) = H_5(X_4)$ so it suffices to compute the fifth homology group of X_4 . First we look at X_3 , i.e the fibration $K(\mathbb{Z},2) \to X_3 \to S^3$. Note $\mathbb{C}P^\infty = K(\mathbb{Z},2)$ so $H^q(K(\mathbb{Z},2)) = \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \ldots$ Then cohomological Serre spectral sequence has E_2 -page $E_2^{p,q} = H^p(S^3, H^q(K(\mathbb{Z},2))) \Longrightarrow H^{p+q}(X_3)$



Non-trivial differentials are all d_3 then if $x \in E_2^{0,2} = H^2(K(\mathbb{Z},2)) = \mathbb{Z}$ is a generator then $y = d_3(x)$ is a generator for $E_2^{3,0} = H^3(S^3)$. By the multiplicative structure xy generates $E_2^{3,2}$ and as $d_3(x^2) = d_3(x)x + xd_3(x) = 2xy$ so x^2 is twice a generator of $E_2^{0,4} = H^4(K(\mathbb{Z},2))$. Similarly x^3 is thrice a generator of $H^6(K(\mathbb{Z},2))$ since $d_3(x^3) = 3x^2y$.

As there are no more larger degree differentials after this $E_4 = E_{\infty}$, and so the first four cohomology group are easily seen to be $H^n(X_3) = \mathbb{Z}, 0, 0, 0$ and due to the maps $\mathbb{Z} \xrightarrow{\times 2, \times 3} \mathbb{Z}$, $H^4(X_3) = 0$, $H^5(X_3) = \mathbb{Z}_2$, $H^6(X_3) = 0$ and $H^7(X_3) = \mathbb{Z}_3$. Then by UCT we see that the first few homology groups are $H_n(X_3) = \mathbb{Z}, 0, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}_3$.

Now looking at the homological Serre spectral sequence for $K(\mathbb{Z}_2,3) \to X_4 \to X_3$.



Since the only morphism $\mathbb{Z}_3 \to \mathbb{Z}_2$ is the trivial map, it follows that at the

$$E^{\infty}$$
-page $H_5(X_4) = E_{0.5}^{\infty} = \mathbb{Z}_2$. Hence $\pi_5(S^3) = \mathbb{Z}_2$.

2. Let $X = \operatorname{map}(S^1, S^3)$, take base-point $* \in S^1$ and consider the evaluation map $\operatorname{ev}_* : X \to S^3$, $f \mapsto f(*)$. Take base-point $b \in S^3$ and fibre sequence $F := \operatorname{ev}_*^{-1}(b) \to X \xrightarrow{\operatorname{ev}_*} S^3$. Note that F is just the loop space on S^3 . Every map $S^1 \to S^3$ is null-homotopic, X is path connected. From the fibre sequence, there is a long exact sequence of homotopy groups. Since $\pi_0(S^3) = \pi_1(S^3) = \pi_2(S^3) = 0$ and the loop-space F is simply connected (in fact contractible), $\pi_1(X) = 0$. Looking at the long exact sequence

$$\dots \to \pi_3(F) \to \pi_3(X) \to \pi_3(S^3) = \mathbb{Z} \to \pi_2(F) \to \pi_2(X) \to 0 \to \dots$$

ev₀ has a right inverse $\sigma: S^3 \to X$ taking $x \in S^3$ to the constant map to x. Hence the map $\pi_3(X) \to \pi_3(S^3)$ is surjective, so $\pi_3(S^3) \to \pi_2(F)$ is the zero map. Therefore $\pi_2(X) = \pi_2(F)$ (also follows from splitting) and as $F \cong \Omega(S^3)$, $\pi_2(\Omega(S^3)) \cong \pi_3(S^3) = \mathbb{Z}$. By Hurewicz then $H_2(X) = H_2(F) = \mathbb{Z}$ and $H^2(X) = H^2(F) = \mathbb{Z}$ from UCT. We know that $H^*(F)$ is the divided power algebra on one generator $\Gamma(x)$ so looking at the cohomological Serre spectral sequence,

0	0	0	0	0
\mathbb{Z}_{\diagdown}	0	0	\mathbb{Z}	0
0	0	0	0	0
\mathbb{Z}_{\diagdown}	0	0	$\searrow_{\mathbb{Z}}$	0
0	0	0	0	0
\mathbb{Z}	0	0	$\searrow_{\mathbb{Z}}$	0

We see the only non-trival differentials are $d_3: H^0(S^3; H^{2k}(F)) \to H^3(S^3; H^{2k-2}(F))$. The case k=1 must be the zero map as $H^3(X) \cong E^{3,0}_\infty \cong H^3(S^3)/\mathrm{im}(d_3)$ so the map $H^3(S^3) \to H^3(X)$ induced by the right inverse from before is injective (since it has left inverse ev_{*}).

Then as $H^*(F)$ is the divided power algebra on one generator $\Gamma(x)$, the multiplicative structure shows inductively that all d_3 are zero maps. Hence we get

$$H^n(X) = \begin{cases} 0 & \text{for } n = 1\\ \mathbb{Z} & \text{else} \end{cases}$$