Algebraic Topology I P.S. 1

October 18, 2023

1. 1

2. (a) From the two short exact sequences induced from the inclusion of complexes, we get two long exact sequences. Define f to be $f = q_* \circ \partial$ in the diagram below, where $H_n(C_1)$ are identified with each other. Also define g to be equal to ∂ as below.
   Since the rows are exact, $\ker g = \text{im } q_*$ and since $\text{im } f \subset \text{im } q_* = \ker g$, this implies that $g \circ f = 0$.

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & H_{n+1}(C_1) \arrow[r, "i_*"] & H_{n+1}(C) \arrow[r, "q_*"] & H_{n+1}(C/C_1) \arrow[r, "\partial"] \arrow[d, "f"] & H_n(C_1) \arrow[r, "i_*"] \arrow[dll, dashed] & \dots \\
\dots \arrow[r, "g"] & H_n(C_0) \arrow[r, "i_*"] & H_n(C_1) \arrow[r, "q_*"] & H_n(C_1/C_0) \arrow[r, "g"] & H_{n-1}(C_0) \arrow[r, "i_*"] & \dots
\end{tikzcd}$$

   (b) We define $d : \ker f \to \text{coker } g$ as follows: take $x \in \ker f$, then by definition $f(x) = q_* \circ \partial(x) = 0 \in H_n(C_1/C_0)$. By $\ker q_* = \text{im } i_*$, there exists a $y \in H_n(C_0)$ such that $i_*(y) = \partial(x)$ then $y \mapsto y \text{ im } g \in \text{coker } g$. This gives us d.

$$\begin{tikzcd}
 & & & \ker f \arrow[d, hook] \arrow[dddll, "d"] & & \\
\dots \arrow[r, "\partial"] & H_{n+1}(C_1) \arrow[r, "i_*"] & H_{n+1}(C) \arrow[r, "q_*"] & H_{n+1}(C/C_1) \arrow[r, "\partial"] \arrow[d, "f"] & H_n(C_1) \arrow[r, "i_*"] \arrow[dll, dashed] & \dots \\
\dots \arrow[r, "g"] & H_n(C_0) \arrow[r, "i_*"] \arrow[d, two heads] & H_n(C_1) \arrow[r, "q_*"] & H_n(C_1/C_0) \arrow[r, "g"] & H_{n-1}(C_0) \arrow[r, "i_*"] & \dots \\
 & \text{coker } g & & & &
\end{tikzcd}$$