Problem Sheet 3

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1. Let F o Y o X be a fibre sequence with $F = \mathbb{R}P^{\infty} = K(\mathbb{Z}_2, 1)$ and $X = \mathbb{C}P^{\infty} = K(\mathbb{Z}, 2)$. First we can note that $[K(\mathbb{Z}, 2), K(\mathbb{Z}_2, 2)] \cong H^2(K(\mathbb{Z}, 2), \mathbb{Z}_2) \cong Z_2$ by the Universal coefficient theorem and $H^2(\mathbb{C}P^{\infty}) = \mathbb{Z}$. Hence there are two maps $X \to K(\mathbb{Z}_2, 2)$ up to homotopy determine by multiplication on H^2 or H_2 by an element of \mathbb{Z}_2 which we call f_0, f_1 . Consider then the homotopy fibres of f_0, f_1 denoted F_0, F_1 .

By application of Hurewicz we find the first few homology groups for $K(\mathbb{Z}_2,1)$ and $K(\mathbb{Z},2)$.

$$H_0(K(\mathbb{Z}_2,1),\mathbb{Z}) = \mathbb{Z}$$

$$\pi_1(K(\mathbb{Z}_2,1)) \cong H_1(K(\mathbb{Z}_2,1),\mathbb{Z}) \cong \mathbb{Z}_2$$

$$0 = \pi_2(K(\mathbb{Z}_2,1)) \twoheadrightarrow H_2(K(\mathbb{Z}_2,1),\mathbb{Z}) = 0$$

$$H_0(K(\mathbb{Z},2),\mathbb{Z}) = \mathbb{Z}$$

$$\pi_2(K(\mathbb{Z},2)) \cong H_2(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z}$$

$$0 = \pi_3(K(\mathbb{Z},2)) \twoheadrightarrow H_3(K(\mathbb{Z},2),\mathbb{Z}) = 0$$

Then by Universal coefficient theorem, the first few cohomology groups are $H^*(K(\mathbb{Z}_2,1)) = \mathbb{Z}, \mathbb{Z}_2, 0$ and $H^*(K(\mathbb{Z}_2,1)) = \mathbb{Z}, 0, \mathbb{Z}, 0$. This gives us E_2 -page.

	0	0	0	0
	\mathbb{Z}_2	0	\mathbb{Z}_2	0
	\mathbb{Z}	0	\mathbb{Z}	0

2. Consider map $F: S^k \to S^k$ of degree $n, b \in S^k$. Call the homotopy fibre $\mathsf{hofib}_b(f)$ of f at b, F. The fibre sequence $F \to S^k \xrightarrow{f} S^k$ induces a long exact sequence,

$$\dots \to \pi_{n+1}(S^k) \to \pi_n(F) \to \pi_n(S^k) \to \pi_n(S^k) \to \dots$$

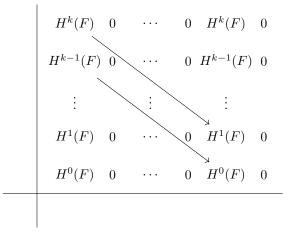
1

 $\pi_n(S^k)$ vanishes for $n \leq k-1$ so we have

$$\dots \to \pi_{k+1}(S^k) \to \pi_k(F) \to \pi_k(S^k) \xrightarrow{f_*} \pi_k(S^k) \to \pi_{k-1}(F) \to \pi_{k-1}(S^k) \to \dots$$
$$\dots \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \pi_{k-1}(F) \to 0 \to \dots$$

(The induced map on homotopy groups is by the Hurewicz map on S^k). As this is an exact sequence, $\pi_{k-1}(F) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ and $\pi_n(F) = 0$ for n < k-1. Thus F is k-2-connected so by Hurewicz $H_{k-1}(F) = \mathbb{Z}_n$, $H_j(F) = 0$ for $j \le k-2$. By UCT $H^j(F) = 0$ for $1 \le j \le k-1$

The E_2 -page of the cohomological Serre spectral sequence is



The only non-trivial differential are d_k and looking at the E_{∞} page the only non-trival terms will be $E_{\infty}^{0,0}=\mathbb{Z}$ and $E_{\infty}^{k,0}=E_{\infty}^{0,k}=\mathbb{Z}$. Obviously then $H^0(F)=\mathbb{Z}$, and as $H^j(F)=0$ for $1\leq j,\leq k-1$, $H^k(F)=\mathbb{Z}$. All other terms on the E_{∞} page vanish and for degree reasons $H^k(F)\cong H^{k+k-1}(F)\cong H^{k+2(k-1)}(F)\cong \ldots$ and by induction we then see $H^{k+j\cdot(k-1)}(F)\cong H^k(F)$ for all $j=0,1,\ldots$ and all other $H^m(F)$ vanish. Of course from the multiplicative structure, we have a multiplication $H^q(F)\otimes H^{q'}(F)\to H^{q+q'}(F)$. The only non-trivial cohomology groups $H^0(F),H^{k+j(k-1)}(F)$ have multiplication $H^{k+j(k-1)}(F)\otimes H^{k+j'(k-1)}(F)\to H^{k+(j+j')(k-1)}(F)$ but the output always vanishes.