

3. (a) Consider the category of sheaves on abelian groups  $C = \text{Sh}_{\text{ab}}(X)$ .  $C$  is obviously a preadditive category, since the composition of homomorphism  $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ ,  $\mathcal{G}(U) \rightarrow \mathcal{H}(U)$  for each  $U \subset X$  has the structure of an abelian group, then so does  $\text{Hom}(A, B)$ .  $\underline{0}$ , the constant sheaf with value at 0 is both a final and initial object in  $C$ . Given sheaves  $\mathcal{F}, \mathcal{G}$  we can define the direct sum as  $\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$  with obvious projection and embedding morphisms, hence  $C$  is additive as well. Given some morphism of sheaves in  $C$ ,  $f : \mathcal{F} \rightarrow \mathcal{G}$  define subsheaf  $\text{Ker}(f)$  as

$$\text{Ker}(f)(U) = \{s \in \mathcal{F} \mid f(s) = 0 \in \mathcal{G}(U)\}$$

which is a kernel in  $C$ . Similarly, consider the pre-sheaf on abelian groups  $\mathcal{H}(f)$  by  $\mathcal{H}(U) = \mathcal{G}(U)/f(\mathcal{F}(U))$ , sheafify to  $\text{Coker}(f) := \overline{\mathcal{H}}$ . Since for  $p \in X$ ,  $\text{Coker}(f)_p = \text{Coker}(f_p)$  then the map  $\mathcal{G} \rightarrow \text{Coker}(f)$  is an epimorphism and if  $g : \mathcal{G} \rightarrow \mathcal{K}$  is a  $C$ -morphism such that  $g \circ f$  vanishes then  $g$  induces a morphism between  $\mathcal{G}(U)/f(\mathcal{F}(U))$  to  $\mathcal{K}(U)$  for all  $U \subset X$  open. By universal property we get  $\text{Coker}(f) \rightarrow \mathcal{K}$  which is then unique by surjectivity above. Hence it is a coker.

Since  $C$  is an abelian category, all limits and colimits exist in general.

- (b) Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves on abelian groups. Assume first that  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is surjective for all  $p \in X$ , then if  $g_1, g_2 : \mathcal{G} \rightrightarrows \mathcal{H}$  (where  $\mathcal{H}$  is another sheaf) such that  $g_1 \circ f = g_2 \circ f$ , this also becomes equal on stalks. Since  $f_p$  is surjective, it is an epimorphism in  $\mathbf{Ab}$  thus  $(g_1)_p = (g_2)_p$  and since the maps are equal on stalks they are equal on the sheaf so  $g_1 = g_2$ . Conversely, assume  $f$  is an epimorphism and pick  $p \in X$ . Define a sheaf  $\mathcal{H}$  as follows: if  $p \in U \subset X$ , then  $\mathcal{H}(U) = \text{coker}(f_p)$ , and  $\mathcal{H}(U)$  vanishes when  $p \notin U$ . Restriction maps are either the identity if both sets contain  $p$  and the zero map otherwise. Then define  $g : \mathcal{G} \rightarrow \mathcal{H}$ , sending  $\mathcal{G}(U)$  to  $\mathcal{H}(U)$  by  $s \mapsto [s]$  if  $p \in U$  and  $s \mapsto 0$  otherwise. Further define  $g_1, g_2 : \mathcal{G} \rightrightarrows \mathcal{H} \oplus \mathcal{H}$  as  $g_1 = (g, 0)$ ,  $g_2 = (0, g)$ . Now if  $f_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$  is not surjective then  $\mathcal{H}$  is non-trivial so  $g_1 \neq g_2$ . However  $g_1 \circ f = g_2 \circ f$  so this can not be the case. Hence  $f_p$  must be surjective.
- (c) Define  $\mathcal{F} := i_{-,*}\underline{\mathbb{Z}} \oplus i_{+,*}\underline{\mathbb{Z}}$ ,  $\mathcal{G} := i_*\underline{\mathbb{Z}}$