Koszul duality*

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1 Introduction

These notes follow the course Advanced Topics in Topology - Koszul Duality, having taken place in the winter semester 2025 at the University of Bonn. It was taught by Dr. Yuqing Shi. These are my personal notes and as such may contain errors, both typographical and factual. I am thankful to David Bowman, Carl Foth and Yordan Toshev for their help in preparing and correcting these notes.

This course aims to construct an analogue to Kozsul duality found in the setting of homological algebra over higher-dimensional algebra.

2 Algebraic Koszul duality

2.1 The Bar construction

In homological algebra, the Bar construction is a way of constructing a resolution of some k-algebra object A. While the construction can be done in any nice monoidal abelian category (and generalisations) we will first exhibit it for the nicest one, namely Mod_k, the category of k-modules for some field k.

Let k be a field. Consider an augmented associative graded k-algebra $A = \bigoplus_{i\geq 0} A_i$, with augmentation $\epsilon : A \to k$ and augmentation ideal $I(A) \coloneqq \ker(\epsilon)$[1]. We define a chain complex of graded k-modules $B_*(A, A)$ by $B_h(A, A) \coloneqq A \otimes I(A)^{\otimes h} \otimes A$.
For $a, a' \in A$ and $a_i \in I(A)$, $1 \leq i \leq h$ denote the element $a \otimes a_1 \otimes \ldots \otimes a_h \otimes a'$ by $a[a_1|\ldots|a_h]a'$.

Remark. Note that there are several different gradings of which to keep track. Taking the element $a[a_1|\ldots|a_h]a' \in B_h(A, A)$ we define the

- *(internal degree)*: $\deg^i(a[a_1|\ldots|a_h]a') = \deg(a) + \sum_{j=1}^h \deg(a_j) + \deg(a')$,
- *(height degree)*: $\deg^h(a[a_1|\ldots|a_h]a') = h$,
- *(total grading)*: $\deg \coloneqq \deg^i + \deg^h$.

Let $B_h(A, A)_i$ denote the submodule of $B_h(A, A)$ generated by elements of internal degree $\deg^i = i$. The differential is then

$$\begin{aligned} \partial : B_h(A, A)_i &\to B_{h-1}(A, A)_i \\ a[a_1|\ldots|a_h]a' &\mapsto (-1)^{e_0} a a_1[a_2|\ldots|a_h]a' + \sum_{j=1}^{h-1} (-1)^{e_j} a[a_1|\ldots|a_j a_{j+1}|\ldots|a_h]a' \\ &\qquad - (-1)^{e_{h-1}} a[a_1|\ldots|a_{h-1}]a_h a' \end{aligned}$$

where $e_0 = \deg(a)$ and $e_j = \deg(a[a_1|\ldots|a_j])$ for $1 \leq j \leq h-1$.

[1] Note in particular A splits as $k \otimes I(A)$.

Definition 2.1.1 (Two-sided bar construction)**.** Let L be a left A-module and R a right A-module. Define $\mathrm{Bar}_*(R, A, L) := R \otimes_A B_*(A, A) \otimes_A L$. If $L, R = k$ (with A-module structure coming from the augmentation) then we write $\mathrm{Bar}_*(A) := \mathrm{Bar}_*(k, A, k)$.

Remark. For a left A-module L, $\mathrm{Bar}(A, A, L)$ is a resolution of L by free left A-modules. Similarly for a right A-module R, $\mathrm{Bar}(R, A, A)$ is a resolution of R by free right A-modules. By taking this resolutions, we easily compute

$$\mathrm{Ext}^*_A(R, L^\vee) \cong H^*(\mathrm{Hom}_A(\mathrm{Bar}_*(R, A, A), L^\vee))$$

$$\mathrm{Ext}^*_A(L, R^\vee) \cong H^*(\mathrm{Hom}_A(\mathrm{Bar}_*(A, A, L), R^\vee))$$

A proof of this can be found in [Zha19] Theorem 11.

Exercise. Show that

$$\mathrm{Ext}^*_A(k, k) \cong H_*(\mathrm{Bar}_*(k, A, k))^\vee \cong \left(\mathrm{Tor}^A_*(k, k)\right)^\vee .$$

Exercise. Show that $\mathrm{Bar}_*(k, A, k)$ is a differentially graded coalgebra by explicitly writing down the comultiplication map. Conclude that $\mathrm{Ext}_A(k, k)$ is a k-algebra.

2.2 Koszul Algebras

Consider as in the previous section an augmented associative graded k-algebra A with augmentation map $\epsilon : A \to k$.

Definition 2.2.1. Given a k-vector space V, the *tensor algebra* of V, denoted $T(V)$, is a graded algebra with underlying vector space

$$T(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

with the natural multiplication given by the tensor product. Note that we define $T_n(V) = V^{\otimes n}$ for $n \geq 1$ and $T_0(V) = V^0 = k$.

Remark. $T(V)$ is naturally an augmented k-algebra via the projection map $\epsilon : T(V) \to V^0 = k$.

Definition 2.2.2. A *presentation* of A is a pair (V, α) with V a graded k-vector space and an epimorphism of augmented k-algebras $\alpha : T(V) \twoheadrightarrow A$.

Definition 2.2.3. A is said to be

(i) *linear-quadratic* if A admits a presentation (V, α) such that $\ker(\alpha)$ is generated by elements in $V \oplus V \otimes V \subset T(V)$. Equivalently, $A \cong T(V)/I$ where I is a two-sided ideal generated by homogeneous elements in degrees 1 and 2.

(ii) *quadratic* if A admits a presentation (V, α) such that $\ker(\alpha)$ is generated by elements in $V \otimes V \subset T(V)$. Equivalently, $A \cong T(V)/I$ where I is a two-sided ideal generated by homogeneous elements in degree 2.

Remark. Note that $T(V)$ inherits a grading from V and two others similarly to before. Explicitly for $x_1 \otimes \ldots \otimes x_n \in T(V)$,

- *(internal grading)*: $\deg^i(x_1 \otimes \ldots \otimes x_n) := \sum_{i=1}^n \deg(x_i)$,
- *(length grading)*: $\deg^\ell(x_1 \otimes \ldots \otimes x_n) = n$,
- *(total grading)*: $\deg := \deg^i + \deg^\ell$.

If A is quadratic with presentation (V, α), then it inherits a length grading from $T(V)$ as follows: An element $a \in A$ is of length degree $\deg^\ell(a) = n$ if there exists an $v \in T_n(V)$ such that $a = \alpha(v)$. (Is this well defined??) Essentially this is saying that $A = \bigoplus_{n \geq 0} \alpha(T_n(V))$.

Definition 2.2.4. Let A be a linear quadratic algebra with presentation (V, α). Define an augmented graded k-algebra $E(A)$ as follows:
Let $T_{-1}(V) = 0$, $T_{\leq \ell}(V) := \bigoplus_{n=0}^{\ell} T_n(V)$ for $\ell \geq 1$ and define

$$E(A) := \bigoplus_{\ell \geq 0} \alpha(T_{\leq \ell}(V))/\alpha(T_{\leq \ell-1}(V)).$$

- Note that $E(A)$ receives an induced augmented graded k-algebra structure from A.
- If $(x_i)_{i \in I}$ forms a basis of V, then $\ker(\alpha)$ is of the form $\langle \sum_n c_n x_n + \sum_{\ell,m} c_{\ell,m} x_\ell \otimes x_m \rangle$ for some constants $c_n, c_{\ell,m} \in k$. Thus $E(A)$ admits a presentation $\alpha' : T(V) \twoheadrightarrow E(A)$ such that $\ker \alpha' = \langle \sum c_{\ell,m} x_\ell \otimes x_m \rangle$.

In this case $E(A)$ is a bigraded quadratic algebra called the *associated quadratic algebra* of A.

Proposition 2.2.5. *The canonical map $A \to E(A)$ is an isomorphism if A is quadratic.*

Proof. Trivial and an exercise. □

Notation. For a k-module A we call $H^*(A) := \mathrm{Ext}^*_A(k, k)$, the *$A$-cohomology.*
Similarly we call $H_*(A) := \mathrm{Tor}^A_*(k, k)$, the *$A$-homology.*
Note that if A is bigraded (for example when it is graded and comes with a presentation) then $\mathrm{Ext}^*_A(k, k)$ comes with a natural trigrading $H^{h,\ell,i}(A)$, where the variables h, ℓ, i correspond to the homological, length and internal grading respectively.

Definition 2.2.6. (i) A quadratic algebra A is called *Koszul* if $H^{h,\ell,i}(A) = 0$ whenever $h \neq \ell$.

(ii) A linear-quadratic algebra A is called *Koszul* if the associated quadratic algebra $E(A)$ is Koszul.

Assume from now on that every graded k-vector space V will be degree-wise finite dimensional. Recall the dual k-vector space $V^\vee$ defined as $\mathrm{Hom}_k(V, k)$ and its natural grading.

Proposition 2.2.7. *Let A be a quadratic algebra with presentation (V, α) and let $\{x_i\}_{i\in I}$ be a basis of V. Then,*

(i) A is generated as an algebra by $\{a_i := \alpha(x_i)\}_{i\in I}$,

(ii) $H^{1,1,*}(A)$ has a basis $\{\alpha_i\}_{i\in I}$ where $\alpha_i \in H^{1,1,q}(A)$ corresponds to $a_i \in A_{1,q}$, and

(iii) A is Koszul if and only if $H^*(A)$ is generated as an algebra by $H^{1,1,*}(A)$.

Proof. (i) This is clear as $(x_i)_{i\in I}$ generates $T(V)$ as an algebra and α is a surjection.

(ii) Recall that $\mathrm{Ext}_A(k,k) \cong H^*(\mathrm{Bar}_*(A)^\vee)$ and the bar construction

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & I(A)^{\otimes 2} & \longrightarrow & I(A) & \xrightarrow{\ 0\ } & k & \longrightarrow & 0 \\ & & [a_i|a_j] & \longmapsto & (-1)^{\deg(a_i)}a_ia_j & & & & \\ & & & & a_k & \longmapsto & 0 & & \end{array}$$

is by a previous remark a free resolution of k. The dual complex looks like

$$\begin{array}{ccccccccc} \cdots & \longleftarrow & \left(I(A)^{\otimes 2}\right)^\vee & \longleftarrow & I(A)^\vee & \xleftarrow[\ 0\ ]{} & k & \longleftarrow & 0 \\ & & 0 & \longleftarrow\!\!\shortmid & a_i^\vee & & & & \end{array}$$

and so the result follows as the generators a_i of A lie in $I(A)$.

(iii) Assume that A is a Koszul algebra then $H^h(\mathrm{Bar}_\ell(A))$ is concentrated in $h = \ell$. We see that $\mathrm{Bar}_h(A)_\ell^\vee$ for $h = \ell$ is generated by the ℓ-fold product of $\mathrm{Bar}_1(A)_1^\vee$. Then by taking cohomology, we see that $H^{h,*,*}(A) = H^{h,h,*}(A)$ is generated by $H^{1,1,*}$. Conversely, $H^*(A)$ being generated by $H^{1,1,*}(A)$ is the same as $H^{h,\ell,*}(A)$ being concentrated on $h = \ell$, hence A is Koszul.

□

Example. Let V be a degree-wise finite dimensional graded k-vector space with basis $(x_i)_{i\in I}$. Then $T(V)$ is quadratic.

Theorem 2.2.8 (Dold-Kan Correspondence)**.** *Let $\mathcal{A}$ be an abelian category. Then there is an equivalence $N : Fun(\Delta^{op}, \mathcal{A}) \to Ch_{\geq 0}(\mathcal{A})$ between the category of simplical objects in $\mathcal{A}$ and the category of non-negatively graded chain complexes in $\mathcal{A}$.*

Remark. The term *bar construction* comes not, as is commonly hypothesised, because of its discovery at a bar, but because of its use of vertical bars to denote tensor products. In the pre-LaTeXera, it was difficult to typeset elements of $A \otimes I(A)^{\otimes h} \otimes A$, choosing instead to write such elements as $a_0[a_1|\dots|a_h]a_{h+1}$.

Proposition 2.2.9. *Let A, A' be quadratic Koszul algebras over k, then $A \otimes_k A'$ is also quadratic and Koszul.*

Proof. Let $\{a_i\}_{i\in I}$ and $\{a'_j\}_{j\in J}$ generate A and A' respectively as in Proposition 2.2.7. Then $\{a_i \otimes 1\}_{i\in I} \cup \{1 \otimes a_j\}_{j\in J}$ generate $A \otimes A'$ subject to

$$(1 \otimes a'_j)(a_i \otimes 1) = (-1)^{\deg(a_i)\deg(a'_j)}(a_i \otimes 1)(1 \otimes a'_j)$$

as well as the relations stemming from A and A'. $A \otimes A'$ is then a quadratic algebra, and furthermore $H^*(A \otimes A') = H^*(A) \otimes H^*(A')$ so $A \otimes A'$ is also a Koszul algebra. $\square$

Example. Let V be a degree-wise finite dimensional graded k-vector space with basis $\{x_i\}_{i\in I}$. We claim that the quadratic algebra $T(V)$ a Koszul algebra:
Define sV to be a bigraded k-vector space with underlying vector space structure V and grading defined for $x \in sV$ as $\deg_{sV}(x) = (1, \deg_V(x))$. Then $\mathrm{Ext}_{T(V)}(k,k) \cong \mathrm{Triv}(sV) \cong T(sV)/(x_i \otimes x_j \forall i,j \in I)$ where $\mathrm{Triv}(sV)$ is the trivial or null algebra on sV generated by $\{x_i\}_{i\in I}$. The claim then follows by computing $\mathrm{Ext}_{T(V)}(k,k)$ using the bar complex $\mathrm{Bar}(k, T(V), k)$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T(V)^{\otimes 3}_{\geq 1} & \longrightarrow & T(V)^{\otimes 2}_{\geq 1} & \longrightarrow & T(V)_{\geq 1} & \xrightarrow{0} k \longrightarrow 0 \\
 & & [x_i|x_j|x_k] & \longmapsto & \pm[x_i x_j|x_k] \pm [x_i|x_j x_k] & & & \\
 & & & & [x_i|x_j] & \longmapsto & \pm x_i \otimes x_j & \\
 & & & & & & x_i & \longmapsto 0
\end{array}$$

where the dual $\mathrm{Bar}(k, T(V), k)^\vee$ is

$$\begin{array}{ccccccc}
\cdots \longleftarrow & T(V^\vee)^{\otimes 2}_{\geq 1} & \longleftarrow & T(V)^\vee_{\geq 1} & \xleftarrow[0]{} & k & \longleftarrow 0 \\
 & 0 & \longleftarrow\!\shortmid & x_i^\vee & & & \\
0 \longleftarrow\!\shortmid & [x_i^\vee|x_j^\vee] & \longleftarrow\!\shortmid & (x_i x_j)^\vee & & &
\end{array}$$

Then in degrees $n \geq 2$, $H^n(\mathrm{Bar}(k, T(V), k)^\vee) = 0$ (where n is the homological degree) so

$$H^*(\mathrm{Bar}(k, T(V), k)^\vee) \cong T(x_i^\vee)/(x_i^\vee \otimes x_j^\vee | \forall i,j).$$

Exercise. Prove the following isomorphisms:

- $\mathrm{Ext}_{\mathrm{Triv}(V)}(k,k) \cong T(sV)$
- $\mathrm{Ext}_{k(V)}(k,k) \cong \bigwedge(sV)^2$
- $\mathrm{Ext}_{\bigwedge(V)}(k,k) \cong k(sV)$
- $\mathrm{Ext}_{k(\langle x\rangle)}(k,k) \cong \bigwedge(s\langle x\rangle) \cong T(s\langle x\rangle)/((s\langle x\rangle)^2)$

[2] Here $k(V) \cong T(V)/(x_i \otimes x_j - x_j \otimes x_i)$ and $\bigwedge(V) \cong T(V)/(x_i \otimes x_j + x_j \otimes x_i)$ i.e the usual symmetric and exterior algebras.

Exercise. Show that for $\mathrm{char}(k) \neq 2$, $\mathrm{Ext}_{\bigwedge(\langle x \rangle)}(k,k) \cong k(s\langle x \rangle)$.

Let A be a quadratic algebra over k with presentation (V, α) where V is a degree-wise finitely-generated graded k-vector space. Pick a basis $\{x_i\}_{i \in I}$ of V and set $a_i \coloneqq \alpha(x_i)$ for $i \in I$.

Definition 2.2.10. Let B be a basis of A (as a graded k-vector space) consisting of elements 1, a_i for all $i \in I$ and mononomials of the form $a_{i_1} a_{i_2} \dots a_{i_n}$ where $a_{i_j} \in \{a_i\}_{i \in I}$ for all $1 \leq j \leq n$. A set $S \subset U \coloneqq \bigcup_{n=1}^{\infty} I^{\times n}$ is called a *labelling set* for B if for all $a \in B$ such that $a \neq 1$, there exists a unique $(i_1, i_2, \dots, i_n) \in S$ such that $a = a_{i_1} a_{i_2} \dots a_{i_n}$.
The pair (B, S) is called a *labelled basis* for A.

Notation. Given a tuple $J \in U$, we denote the length of J (i.e number of elements in J) by $|J|$.

Remark. A labelled basis (B, S) exists for A.

Set a labelled basis (B, S) for A. Each mononomial of the form $a_k a_\ell$ for $k, \ell \in I$ can be expressed uniquely in the form

$$a_k a_\ell = \sum_{(i,j) \in S} f\begin{pmatrix} k, \ell \\ i, j \end{pmatrix} a_i a_j \tag{2.2.1}$$

where f is a k-valued function on a domain of definition being some obvious subset of $I^{\times 4}$[3].
The relations given by (2.2.1) are called the *admissible relations* for A with respect to (B, S). Let $B^\vee \coloneqq \{1, \alpha(i) \coloneqq a_i^\vee, \alpha(i_1, i_2, \dots, i_n) \coloneqq (a_{i_1} a_{i_2} \dots a_{i_n})^\vee\}_{i, i_j \in I}$ be the dual basis for $A^\vee$ i.e if $(i_1, i_2, \dots, i_n) \in S \setminus \{1\}$ for some $n \geq 1$ then

$$\alpha(i_1, i_2, \dots, i_n)(a) = \begin{cases} 1 & \text{if } a = a_{i_1} a_{i_2} \dots a_{i_n} \\ 0 & \text{else} \end{cases}$$

Notation. By (semi-)abuse of notation we denote by α_i the cohomology class in $H^{1.1.*}(A) = \mathrm{Ext}_A^{1,1,*}(k,k)$ represented by $\alpha(i)$.

Theorem 2.2.11. *Let A be a quadratic Koszul algebra with presentation (V, α) and take some labelled basis (B, S) of A. Then $H^*(A) = \mathrm{Ext}_A^*(k,k)$ is generated by $(\alpha_i)_{i \in I}$ subject to the following relation:*

$$(-1)^{v_{i,j}} \alpha_i \alpha_j + \sum_{(k,\ell) \in U \setminus S} (-1)^{v_{k,\ell}} f\begin{pmatrix} k, \ell \\ i, j \end{pmatrix} \alpha_k \alpha_\ell = 0$$

for all $(i,j) \in S$, where $v_{k,\ell} = \deg(\alpha_k) + (\deg(\alpha_k) - 1)(\deg(\alpha_\ell) - 1)$.

[3]The domain could be all of I^4 but this was not made clear in the lecture.

We defer the proof to after collecting some more machinery.
First recall that if V is a graded k-vector space $V = \bigoplus_{i=0}^{\infty} V_i$, V is said to be *degree-wise finite dimensional* if $\dim V_i < \infty$ for all $i \geq 0$. If V is bigraded (or even multigraded) by the degree of an element in $V = \bigoplus_{i,j\geq 0} V_{i,j}$ we will mean the *total degree.*
The linear dual of V, denoted $V^\vee$ is defined degree-wise by

$$(V^\vee)_n = \mathrm{Hom}_k(V_n, k) \cong \mathrm{Hom}_k\left(\bigoplus_{i+j=n} V_{i,j}, k\right) \cong \bigoplus_{i+j=n} \mathrm{Hom}_k(V_{i,j}, k)$$

For a degree-wise finite dimensional vector space V, the tensor algebra $T(V)$ is also degree-wise finite dimensional. Hence a k-algebra admitting a presentation $\alpha : T(V) \twoheadrightarrow A$ is also degree-wise finite dimensional.

In general if (C_*, ∂) is a chain complex of graded degree-wise finite dimensional k-vector spaces, then we can define the dual chain complex $(C^* := \mathrm{Hom}_k(C_*, k), k), \delta)$ given by $\delta(f)(x) = (-1)^{\deg(f)+1}(f(\partial(x))$.

We now look at the dual of $\mathrm{Bar}(k, A, k)$. Recall that we assume that A is an augmented associative k-algebra with multiplication $\mu : A \otimes A \to A$.

$$\begin{array}{ccccccc}
 & I(A)^{\otimes h} & & I(A)^{\otimes(h-1)} & & \\
 & \shortparallel & & \shortparallel & & \\
\cdots \longrightarrow & \mathrm{Bar}(A)_h & \xrightarrow{\quad\partial\quad} & \mathrm{Bar}(A)_{h-1} & \longrightarrow & \cdots \\
 & [x_1|\dots|x_h] & \longmapsto & \sum_{i=1}^{h-1}(-1)^{e_i}[x_1|\dots|x_i x_{i+1}|\dots|x_h] & &
\end{array}$$

where $e_i = \deg([a_1|\dots|a_i])$.

We define $C(A) := \mathrm{Bar}(k, A, k)^\vee$ with $\delta'(f)(x) = (-1)^{\deg(f)+1} f(\partial(x))$ and δ naturally chosen such that the following diagram commutes.

$$\begin{array}{ccccccc}
\cdots \longleftarrow & \mathrm{Hom}(I(A)^{\otimes h}, k) & \xleftarrow{\delta'} & \mathrm{Hom}(I(A)^{\otimes(h-1)}, k) & \longleftarrow \cdots \\
 & \| & & \| & \\
\cdots \longleftarrow & \mathrm{Hom}(I(A), k)^{\otimes h} & \xleftarrow{\delta} & \mathrm{Hom}(I(A), k)^{\otimes(h-1)} & \longleftarrow \cdots \\
 & \| & & \| & \\
\cdots \longleftarrow & (I(A)^\vee)^{\otimes h} & \xleftarrow{\quad\delta\quad} & (I(A)^\vee)^{\otimes(h-1)} & \longleftarrow \cdots
\end{array}$$

The linear dual of the multiplication map is $\mu^\vee : A^\vee \to (A \otimes A)^\vee \cong A^\vee \otimes A^\vee$. It sends $\alpha \in A^\vee$ to an element of the form $\sum_r \alpha'_r \otimes \alpha''_r$. We can go a bit further that this, by looking at role of the (co)-augmentation: since

$$A^\vee \to A^\vee \otimes A^\vee = (k \oplus I(A)^\vee) \otimes (k \oplus I(A)^\vee) = k \oplus (k \otimes I(A)^\vee) \oplus (I(A)^\vee \otimes k) \oplus (I(A)^\vee \otimes I(A)^\vee)$$

we can exhibit $\mu^\vee$ (now denoted as Δ which sends $\alpha \in A^\vee$ to an element of the form $c + \sum \alpha'_r \otimes \alpha''_r + \sum \beta_r + \sum \beta'_r$. We can then introduce the *modified comultiplication*

$$\widetilde{\Delta} : I(A)^\vee \to I(A)^\vee \otimes I(A)^\vee$$

given by $\alpha \mapsto \sum \alpha'_r \otimes \alpha''_r$. Then the differential $\delta : (I(A)^\vee)^{\otimes(h-1)} \to (I(A)^\vee)^{\otimes h}$ from above is given by

$$\alpha = [\alpha_1|\dots|\alpha_{h-1}] \mapsto \sum_{i=1}^{h-1} \sum_r (-1)^{\xi_{i,r}} [\alpha_1|\dots|\alpha_{i-1}|\alpha'_{i,r}|\alpha''_{i,r}|\alpha_{i+1}|\dots\alpha_{h-1}]$$

where $\xi_{i,r} = \deg([\alpha_1|\dots|\alpha_{i-1}|\alpha'_{i,r}])$.

Proof. Theorem 2.2.11. Since A is a Koszul algebra, $H^*(A) = \mathrm{Ext}^*_A(k,k)$ is generated (as an algebra) by the given $(\alpha_i)_{i\in I}$ and $\mathrm{Ext}^{h,\ell,*}_A(k,k) = 0$ for $h \neq \ell$. To check the relations, we first note that $C^{p,p-1}$ is spanned by elements in the form $[\alpha(i_1)|\dots|\alpha(i_j,i_{j+1})|\dots|\alpha(i_p)]$ where $i_j \in I$ for all $1 \le j \le p$.

$$\begin{array}{ccccccc} \cdots \longleftarrow & C(A)^{p,p} & \xleftarrow[\delta]{\qquad\qquad} & C(A)^{p-1,p} & \longleftarrow & \cdots \\ & [\alpha(i_1)|\dots|\alpha_{i_p}] & \longleftarrow\!\shortmid & [\alpha(i_1)|\dots|\alpha(i_n,i_m)|\dots|\alpha(i_{p-1})] & & \end{array}$$

Evaluation of δ on elements of this form depends on the evaluation of $\widetilde{\Delta}$ on $\alpha(i)$ and on $\alpha(k,\ell)$ for $i,k,\ell \in I$. The relations are thus generated by $\delta(\alpha(m,n))$ for $(m,n) \in S$ (by the admissible relations (2.2.1)).

A basis for $A^\vee \otimes A^\vee$ is given by $\alpha(m)\otimes\alpha(n)$ for $(m,n) \in S$, and for all $\alpha \in A^\vee$, $\mu^\vee(\alpha) = \sum_{m,n} c(m,n)(\alpha(m)\otimes\alpha(n))$ where $c(m,n) \in k$. For $\alpha \in I(A)^\vee$,

$$\widetilde{\Delta}(\alpha) = \sum_{\substack{(m,n)\in S\times S \\ m,n\neq}} c(m,n)(\alpha(m)\otimes\alpha(n))$$

Now let $\alpha = \alpha(m,n)$ and recall that we have $\widetilde{\Delta}(\alpha) \in I(A)^\vee \otimes I(A)^\vee \cong \mathrm{Hom}(I(A)\otimes I(A),k)$. Then $\widetilde{\Delta}(\alpha)(x\otimes y) = \alpha(xy)$, so $c(m,n) = 0$ if $|m|$ or $|n|$ is not equal to 1, thus $\widetilde{\Delta}(\alpha) = \sum_{k,\ell\in I} c(k,\ell)(\alpha(k)\otimes\alpha(\ell))$ must satisfy

$$\alpha(a_k a_\ell) = \widetilde{\Delta}(\alpha)(a_k\otimes a_\ell) = \begin{cases} 1 & \text{if } k=m, \ell=n \\ f\binom{k,\ell}{m,n} & \text{if } (k,\ell)\notin S \\ 0 & \text{else} \end{cases}$$

Then $c(m,n) = (-1)^{\deg(\alpha(n))\deg(\alpha(m))}$, $c(h,\ell) = (-1)^{\deg(\alpha(h))\deg(\alpha(\ell))} f\binom{k,\ell}{m,n}$, and

$$\begin{aligned} \delta(\alpha(m,n)) =& -(-1)^{\deg(\alpha(m))}c(m,n)[\alpha(m)|\alpha(n)] + \sum_{(h,\ell)\in(I\times I)\setminus S} (-1)^{\deg(\alpha(h))}c(h,\ell)[\alpha(h)|\alpha(\ell)] \\ =& (-1)^{1+\deg(\alpha(m))+\deg(\alpha(m))\deg(\alpha(n))}[\alpha(m)|\alpha(n)] \\ &+ \sum_{(h,\ell)} (-1)^{1+\deg(\alpha(h))+\deg(\alpha(h))\deg(\alpha(\ell))} f\begin{pmatrix} h,\ell \\ m,n \end{pmatrix}[\alpha(h)|\alpha(\ell)] \end{aligned}$$

$\square$

2.3 PBW-algebras

We would like a nice way to determine when a quadratic algebra is in fact a Koszul algebra. It turns out that there is a completely algebraic sufficient condition for this.

Definition 2.3.1. If $a \in B$ and $(i_1, \ldots, i_n) \in S$ such that $a = a_{i_1} \ldots a_{i_n}$, we say that $(i_1, \ldots, i_n)$ is a (S-)label of a.

Remark. If I is a well-ordered (and countable) set then one can define a well ordering on $U := \bigcup_{m=0}^{\infty} I^{\times m}$ by length followed by lexicographical ordering as follows:
take $I = \{i_1 < i_2 < \ldots\}$ and let $\sigma = (i_1, \ldots, i_n), \tau = (i'_1, \ldots, i'_{n'}) \in U$ then if $n < n'$, $\sigma \leq \tau$, and otherwise if there exists a minimal $1 \leq j \leq n = n'$ such $i_j \leq i'_j$ and $\forall k < j$ $i_k = i'_k$ then $\sigma \leq \tau$.

Henceforth we shall assume that I is a well-ordered set.

Definition 2.3.2. We say that labelled basis for A, (B, S) is a *Poincaré-Birkhoff-Witt basis* (*PBW-basis*) if the following holds:

(i) For all $(i_1, \ldots, i_k), (j_1, \ldots, j_\ell) \in S$ we either have

  (a) $(i_1, \ldots, i_k, j_1, \ldots, j_\ell) \in S$, or

  (b) $a := a_{i_1} \ldots a_{i_k} a_{j_1} \ldots a_{j_\ell}$ is (by applications of (2.2.1)) of the form $\sum_{L \in S, |L| = k+\ell} c_L a_L$ such that for every $L \in S$ appearing in this sum, $(i_1, \ldots, i_k, j_1, \ldots, j_\ell) < L$.

(ii) For every $k > 2$, $(i_1, \ldots, i_k) \in S$ if and only if for each j with $1 \leq j < k$, the sequence $(i_1, \ldots, i_j)$ and $(i_{j+1}, \ldots, i_k)$ are both in S.

Definition 2.3.3. We say that A is a *PBW-algebra* if there exists a PBW-basis for A.

Exercise. Find a set of PBW-bases for the tensor algebra $T(V)$, the polynomial algebra $P(V)$ and the exterior algebra $\bigwedge(V)$.

Theorem 2.3.4. *If A is a PBW-algebra, then A is a (quadratic) Koszul algebra.*

The idea of the proof is going to be to filter the cobar construction $C^{s,p}$ using the labels for a PBW basis and then show that outside of the diagonals $s = p$ the quotients of this filtration will have vanishing cohomology. It then will follow by a standard homolological algebra argument that $H^{s,p} = 0$ unless $s = p$.

Proof. Fix a PBW basis (B, S) of A. We want to give a filtration on the cobar complex $C(A) = \mathrm{Bar}(k, A, k)^\vee$ using the labels from S. To that end denote $C^{s,p} := C(A)^{s,p}$ and recall that $C^{s,p}$ is generated by elements of the form $[y_1 | \ldots | y_h]$ such that $y_i \in I(A)^\vee$ and $\sum_{i=1}^{h} \deg^\ell(y_i) = p$. For any $L \in S$ such that $|L| = p$ define $F_L C^{s,p}$ as the submodule of $C^{s,p}$ generated by elements of the form

$$\alpha = [\alpha(i_1, \ldots, i_{k_1}) | \alpha(i_{k_1+1}, \ldots, i_{k_2}) | \ldots | \alpha(i_{k_{s-1}+1}, \ldots, i_{k_s})] \tag{2.3.1}$$

where $\alpha(i_{k_j+1}, \ldots, i_{k_{j+1}}) \in B^\vee$ (the dual basis) for all $0 \leq j \leq s-1$ and $(i_1, \ldots, i_{k_1}, i_{k_1+1}, \ldots, i_p) \leq L$ (we define $k_0 = 1$ and $k_s = p$). For each such L, we claim that $F_L C^{*,*}$ is a subcomplex of $C^{*,*}$. To see this, we need to show that for all elements $\alpha \in F_L C^{s,p}$ of the form (2.3.1), $\delta(\alpha)$ lies in $F_L C^{s+1,p}$. For all $0 \leq j \leq s-1$, we have that

$$\widetilde{\Delta}(\alpha(i_{k_j+1}, \dots, i_{k_{j+1}})) = \sum_{\substack{K,J \in S \\ |K|+|J|=k_{j+1}-k_j}} c\binom{K, J}{(i_{k_j+1}, \dots, i_{k_{j+1}})} \alpha(K) \otimes \alpha(J)$$

We need to show that $L \geq (i_1, \dots, i_{k_1}, i_{k_1+1}, \dots, i_{k_2}, K, J, i_{k_{j+1}+1}, \dots, i_p)$. To that end for all $0 \leq j \leq s-1$ and all $K, J \in S$ such that $|K| + |J| = k_{j+1} - k_j$,

(i) If $(K, J) \in S$, then $c\binom{K,J}{L} = 0$ if $(K, J) \neq (i_{k_j+1}, \dots, i_{k_{j+1}})$.

(ii) If $(K, J) \notin S$, then

$$a_{(K,J)} = \sum_{\substack{L \in S \\ |L|=|K \setminus J|}} C\binom{K, J}{L} a_L$$

(iii) $(K, J) = (i_{k_j+1}, \dots, i_{k_{j+1}})$ implies that $C(K, J) = \pm 1$.

Then by Definition 2.3.2 of the PBW-basis, $(i_{k_j+1}, \dots, i_{k_{j+1}}) \geq (K, J)$. Hence $F_L C^{*,*}$ defines a bounded below increasing filtration on $C^{*,*}$.

Define $F_{L-1}C \coloneqq \bigcup_{J \in S,\, J<L} F_J C$. δ induces a map, also called δ, of the quotient

$$\delta : \frac{F_L C}{F_{L-1}C} \to \frac{F_L C}{F_{L-1}C}$$

We claim that under this induced differential $H^{s,p}(F_L C / F_{L-1}C) = 0$ for $s \neq p$.

(i) The case where $s > p$ is clear

(ii) For $1 \leq s < p$ we can define a chain homotopy

$$\Phi : F_L C^{s,p} / F_{L-1} C^{s,p} \to F_L C^{s-1,p} / F_{L-1} C^{s-1,p}$$

between $\mathrm{id}_{F_L C / F_{L-1} C}$ and the null map (see [Pri70] 5.3) such that $\Phi \circ \delta(x) + \delta \circ \Phi(x) = \mathrm{id}(x)$ for all $x \in F_L^{s,p}$ with $s < p$. Then the cohomology vanishes and we are done.

$\square$

Proposition 2.3.5. *Let A be a linear quadratic algebra with presentation (V, α) and basis $(x_i)_{i \in I}$, and recall the associated quadratic algebra $E(A)$. Then*

(i) $E(A)$ is a graded k-algebra, and

(ii) $E(A)$ is a quadratic algebra with a presentation (V, α') given by

$$\ker(\alpha') = \left\{ \sum_{j,k \in I} f_{jk}(x_j \otimes x_k) \right\}$$

where the f_{jk} come from $\ker(\alpha) = \left\{ \sum_{i \in I} f_i x_i + \sum_{j,k \in I} f_{jk}(x_j \otimes x_k) \right\}$.

Proof. Diagram chase.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\alpha_{\leq n-1}) \arrow[r, hook] \arrow[d, hook] & \ker(\alpha_{\leq n}) \arrow[r, two heads] \arrow[d, hook] & \ker(\alpha_{\leq n})/\ker(\alpha_{\leq n-1}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T_{\leq n-1}(V) \arrow[r, hook] \arrow[d, two heads] & T_{\leq n}(V) \arrow[r, two heads] \arrow[d, two heads] & T_{\leq n}(V)/T_{\leq n-1}(V) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \alpha(T_{\leq n-1}(V)) \arrow[r, hook] \arrow[d] & \alpha(T_{\leq n}(V)) \arrow[r, two heads] \arrow[d] & \alpha(T_{\leq n}(V))/\alpha(T_{\leq n-1}(V)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

2.4 The Steenrod algebra

One of the more interesting (to a topologist that is) examples of a Koszul algebra is going to turn out to be the Steenrod algebra of stable cohomology operations. We will quickly construct and show this to be the case.
First we recall the following concepts from topology:

(i) For a group G and positive integer n, an *Eilenberg-MacLane space* is a connected topological space denoted by $K(G, n)$ such that $\pi_n(K(G, n)) \cong G$ and $\pi_i(K(G, n)) = 0$ for all other $i \neq n$.

(ii) The *Brown representability theorem* (for reduced singular cohomology) states that for a topological space X and abelia group A, there is an isomorphism $\widetilde{H}^n(X; A) \cong [X, K(A, n)]_*$ between the n-th reduced cohomology group of X in A and the set of pointed homotopy classes of maps $X \to K(A, n)$. Under this identification we get an the image of $[1_{K(A,n)}]$ is an element $\iota_A \in \widetilde{H}^n(K(A, n); A)$ called the *fundamental class*. From this we can get a map

$$(X \xrightarrow{f} K(A, n)) \mapsto (\widetilde{H}^n(K(A, n); A) \xrightarrow{f^*} \widetilde{H}^n(X; A))$$

where f^* sends $\iota_A \to f^*(\iota_A)$.

(iii) The *relative Künneth formula*: fix a field $\mathbb{F}$ and take CW-pairs (X, A), (Y, B) then

$$H^*(X \times Y, (A \times Y) \cup (X \times B); \mathbb{F}) \cong H^*(X, A; \mathbb{F}) \otimes_F H^*(Y, B; \mathbb{F})$$

if either $H^*(X, A; \mathbb{F})$ or $H^*(Y, B; \mathbb{F})$ are finite dimensional as $\mathbb{F}$-modules.

Definition 2.4.1. A *cohomology operation* of type (n, m, A, B) for $n, m \in \mathbb{Z}$ and A, B abelian groups is a natural transformation

$$\theta : \widetilde{H}^n(-; A) \implies \widetilde{H}^m(-; B).$$

We call θ *stable* if θ commutes with suspensions.

Definition 2.4.2. For all $i \in \mathbb{N}$, the (mod 2) *ith Steenrod square* Sq^i is a collection of stable cohomology operations $\mathrm{Sq}^i : \widetilde{H}^n(-; \mathbb{F}_2) \to \widetilde{H}^{n+i}(-; \mathbb{F}_2)$ for all $n \in N$ satisfying the following conditions:

(i) Sq^0 is the identity transformation.

(ii) For a space X and $u \in \widetilde{H}^i(X; \mathbb{F}_2)$, $\mathrm{Sq}^i(u) = u^2$.

(iii) For a space X and $u \in \widetilde{H}^n(X; \mathbb{F}_2)$, $\mathrm{Sq}^i(u) = 0$ for all $i > n$.

(iv) The following identity (called the *Cartan formula*) holds for all $i \in \mathbb{N}$ and $x, y \in \widetilde{H}^*(X; \mathbb{F}_2)$ for any space X:

$$\mathrm{Sq}^i(xy) = \sum_{j+k=i} \mathrm{Sq}^j(x) \smile \mathrm{Sq}^k(y).$$

Remark. Denote $\mathrm{Sq} \coloneqq \sum_{i \geq 0} \mathrm{Sq}^i : \widetilde{H}^*(-; \mathbb{F}_2) \to \widetilde{H}^*(-; \mathbb{F}_2)$. Note that the Cartan formula makes this map a ring homomorphism.

Theorem 2.4.3. *The Steenrod squares are uniquely characterised by the above axioms.*

We now give an explicit construction of the Steenrod squares, following [Mil88]:
Let us take a subgroup G of the symmetric group on n elements Σ_n (we shall mainly be concerned with the case $\mathbb{F}_2 = \Sigma_2 \subset \Sigma_2$). We get a universal pricipal G-bundle $G \to EG \to BG$ where the total space EG is a weakly contractible space with a free right G-action and $BG \coloneqq EG/G$ called the *classifying space* for G. Pick a point $e \in EG$ and let $b \in BG$ denote its image.

Example. Let $G = \mathbb{F}_2$, then $EF_2 \cong S^\infty \cong \mathrm{colim}_{n\to\infty} S^n$ and $BF_2 \cong \mathbb{R}P^\infty$

Let (X, x_0) be a pointed CW complex. Fix some $n \in \mathbb{N}$, then we can find a filtration of the n-fold product $X^n \coloneqq X^{\times n}$ given by $F_k(X^n) = \{(x_1, \ldots, x_n) \in X^n |$ at most k components differ from x_0

$$\begin{array}{ccccccc}
F_0(X^n) & \hookrightarrow & F_1(X^n) & \hookrightarrow \ldots \hookrightarrow & F_{n-1}(X^n) & \hookrightarrow & F_n(X^n) \\
\| & & \| & & \| & & \| \\
\mathrm{pt} & & \bigvee_{i=1}^n X & & \text{‘fat wedge’} & & X^n
\end{array}$$

Remark. (i) G acts on X^n by permuting the components. Moreover, this action preserves the filtration.

(ii) $X^n/F_{n-1}(X^n) \cong X^{(n)} \coloneqq \bigwedge_{i=1}^n X$.

Recall the associated bundle construction:
Given a principal G-bundle $G \to P \to B$ and a CW complex Y with a G-action on it, one can construct a fibre bundle $Y \to P \times_G Y \to B$, where $P \times_G Y \coloneqq P \times Y/N_G$ ($(p, y) \sim (pg, g^{-1}y)$ for $g \in G$). Applying this to $EG \to BG$ and X^n we get

$$\begin{tikzcd}
F_{n-1}(X^n) \arrow[r, hook] \arrow[d] & X^n \arrow[d] \\
EG \times_G F_{n-1}(X^n) \arrow[r, hook] \arrow[d] & EG \times_G X^n \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

Exercise. Show that $EG \times_G X^n / EG \times_G F_{n-1}(X^n) \cong (EG)_+ \wedge X^{(n)}$ which we will denote by $D_G(X)$.
(Hint) $X \times Y/X \times Z \cong X \times (Y/Z)/X \times \mathrm{pt} \cong X_+ \wedge (Y/Z)$.

Proposition 2.4.4. *(i) The map $D_G(-) : Top_* \to Top_*$ is functorial.*

(ii) There exists a natural transformation $i_{(-)} : (-)^{(n)} \implies D_G(-)$ induced by $\bar{i}_X : X^n \to EG \times_G X^n$, $x \mapsto (e, x)$ which then descends to the quotient $i_X : X^{(n)} \to B_G(X)$. In other words for any $f : X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
X^{(n)} \arrow[r, "f^{\wedge n}"] \arrow[d, "i_X"'] & Y^{(n)} \arrow[d, "i_Y"] \\
D_G(X) \arrow[r, "D_G(f)"] & D_G(Y)
\end{tikzcd}$$

Lemma 2.4.5. *Let Z be a pointed CW complex such that $\widetilde{H}^i(Z; \mathbb{F}) = 0$ for all $i < q$ for $\mathbb{F}$ some field, and that $\widetilde{H}^q(Z; \mathbb{F})$ is a finite dimensional $\mathbb{F}$-vector space. Then*

$$\widetilde{H}^i(D_G(Z); \mathbb{F}) = \begin{cases} 0 & \text{if } i < nq \\ \left(\widetilde{H}^q(Z; \mathbb{F})^{\otimes n}\right)^G & \text{if } i = nq \end{cases}$$

Moreover the induced map $i_X^* : \tilde{H^{nq}}(D_G(Z); \mathbb{F}) \to \tilde{H^{nq}}(Z^{(n)}; \mathbb{F})$ is the inclusion G-fixed points $(\widetilde{H}^q(Z; \mathbb{F})^{\otimes n})^G \hookrightarrow \widetilde{H}^q(Z; \mathbb{F})^{\otimes n}$ where the G acts by permutation.

Notation. Fix a prime p. Denote $K_q \coloneqq K(\mathbb{F}_p, q)$ and $\widetilde{H}^q(-) \coloneqq \widetilde{H}^q(-; \mathbb{F}_p)$. By Hurewicz and the Universal Coefficient theorems: $\widetilde{H}^i(K_q) = 0$ for $i < q$ and $\widetilde{H}^q(K_q) = \mathbb{F}_p$. By the previous lemma,

$$\widetilde{H}^{nq}(D_G(K_q)) \cong (\widetilde{H}^q(K_q)^{\otimes n})^G \cong (F_p^{\otimes n})^G$$

Remark. The canonical map $\mathbb{F}_p^o times \xrightarrow{\simeq} F_p$, where G acts by permutation on the domain and trivially on the codomain, is G-equivariant. Hence sG acts trivially on $\mathbb{F}_p^{\otimes n}$, and so every point is fixed. Then map $i^* : \tilde{H^{nq}}(D_G(K_q)) \to \tilde{H^{nq}}(K_q^{(n)})$ is an isomorphism.
$\tilde{H^{nq}}(K_q^{(n)})$ contains $\iota_q^{\wedge n}$, the n-fold smash product of the fundamental class $\iota_q \in \widetilde{H}^q(K_q)$.

Corollary 2.4.6. *There exists a unique class $P_G(\iota_q) \in \tilde{H^{nq}}(D_G(K_q))$ such that $i^* P_G(\iota_q) = \iota_q^{\wedge n}$. Equivalently, there is (up to homotopy) a unique map $P_G(\iota_q)$ such that the following diagram commutes (up to homotopy)*

$$\begin{tikzcd}
K_q^{(n)} \arrow[r, "\iota_q^{\wedge n}"] \arrow[d, "i"'] & K_{nq} \\
D_G(K_q) \arrow[ur, dashed, "P_G(\iota_q)"'] &
\end{tikzcd}$$

Let us return to (X, x_0) a pointed CW complex with no further cohomological assumptions and pick $u \in \widetilde{H}^q(X)$ which is represented by a map $X \to K_q$ also called u. We have the following diagram

$$\begin{tikzcd}
X^{(n)} \arrow[r, "u^{\wedge n}"'] \arrow[d, "i"'] & K_q^{(n)} \arrow[r, "\iota_q^{\wedge n}"] \arrow[d, "i"'] & K_{nq} \\
D_G(X) \arrow[r, "D_G(u)"] & D_G(K_q) \arrow[ur, dashed, "P_G(\iota_q)"'] &
\end{tikzcd}$$

which further induces

$$\begin{tikzcd}
\tilde{H^{nq}}(X^{(n)}) & \tilde{H^{nq}}(K_q^{(n)}) \arrow[l, "(u^{\wedge n})^*"'] & \tilde{H^{nq}}(K_{nq}) \ni \iota_{nq} \arrow[l, "(\iota_q^{\wedge n})^*"'] \arrow[dl, dashed, "(P_G(\iota_q))^*"] \\
\tilde{H^{nq}}(D_G(X)) \arrow[u, "i^*"] & \tilde{H^{nq}}(D_G(K_q)) \arrow[l, "(D_G(u))^*"'] \arrow[u, "i^*", "\cong"'] &
\end{tikzcd}$$

This defines a (unique) natural transformation $P_G : \widetilde{H}^q(-;\mathbb{F}_p) \to \tilde{H^{nq}}(D_G(-);\mathbb{F}_p)$ given by $u \mapsto (D_G(u))(P_G(\iota_q))$ such that $i^*(P_G(u)) = u^{\wedge n} \in \widetilde{H}^{nq}(X^{(n)})$.

Finally consider the G-equivariant diagonal map $\Delta : X \to X^{(n)}$ where G acts trivially on the domain and by permutation on the codomain. This induces

$$\begin{tikzcd}
(EG)_+ \wedge_G X \arrow[r, "\Delta"] \arrow[d, equal] & (EG)_+ \wedge_G X^{(n)} \arrow[d, equal] \\
(BG)_+ \wedge X \arrow[r, "j"] & D_G(X)
\end{tikzcd}$$

Thus for any class $u \in \widetilde{H}^q(X)$ we map to a class $j^*(P_G(u)) \in \widetilde{H}^{nq}((BG)_+ \wedge X)$.

We now specialise to the case of $G = \mathbb{F}_2 = \Sigma_2$, $n = 2$ and $p = 2$. In this case $BG = \mathbb{R}P^\infty$ and $\widetilde{H}^*(BG_+) = H^*((BG)_+) = \mathbb{F}_2[x]$ where $\deg x = 1$. By Künneth, $\widetilde{H}^*((BG)_+ \wedge X) \cong H^*(BG) \otimes \widetilde{H}^*(X)$ thus given $u \in \widetilde{H}^q(X)$ we can write

$$j^* P_G(u) = \sum_{i=-q}^{q} x^{q-i} \otimes F^i(u), \qquad \text{where } F^i(u) \in \widetilde{H}^{q+i}(X)$$

s

Definition 2.4.7. We define the *Steenrod squares* $\mathrm{Sq}^i(u) \coloneqq F^i(u)$ for $u \in \widetilde{H}^{q+i}(X)$.

Proposition 2.4.8. *The Steenrod squares define as above have the following properties:*

(i) $Sq^i : \widetilde{H}^q(-) \to \widetilde{H}^{q+i}(-)$ *is a natural transformation.*

(ii) $Sq^i = 0$ *for* $-q \leq i \leq 0$.

(iii) For $u \in \widetilde{H}^q(X)$, $Sq^i(u) = 0$ *for* $i > q$.

(iv) For $u \in \widetilde{H}^q(X)$, $Sq^i(u) = u^2$.

Proof (Sketch).

$$\begin{tikzcd} (BG)_+ \wedge X \arrow[r, "j"] & D_G(X) \\ S^0 \wedge X \arrow[u, "k\wedge \mathrm{id}"] \arrow[r, "\Delta"] & X^{(2)} \arrow[u, "i"'] \end{tikzcd}$$

where $k : S^0 \to (BG)_+$ is a pointed map that sends basepoint to basepoint and the other point in S^0 to b. □

We want to show that the Sq^i's satisfy the Cartan formula. To that end define the map

$$\delta : D_G(X\wedge Y) = (EG)_+\wedge_G(X\wedge Y)^{(2)} \to \Big((EG)_+ \wedge_G X^{(2)}\Big)\wedge(EG)_+\wedge_G Y^{(2)} = D_G(X)\wedge D_G(Y)$$

given by $(z,(x_1,y_1),(x_2,y_2)) \mapsto (z,(x_1,x_2),z,(y_1,y_2))$. We then fit this into the commutative diagram

$$\begin{tikzcd} (X\wedge Y)^{(2)} \arrow[r, "i"] \arrow[d, "T"] & D_G(X\wedge Y) \arrow[d, "\delta"] & (BG)_+ \wedge (X\wedge Y) \arrow[l, "j"'] \arrow[d, "\Delta_{(BG)_+}"] \\ X^{(2)} \wedge Y^{(2)} \arrow[r, "i\wedge i"] & D_G(X)\wedge D_G(Y) & (BG)_+ \wedge X \wedge (BG)_+ \wedge Y \arrow[l, "j\wedge j"'] \end{tikzcd}$$

Lemma 2.4.9. *For* $u \in \widetilde{H}^q(X)$, $v \in \widetilde{H}^q(Y)$ *we have* $\delta^*(P_G(u) \wedge P_G(v)) = P_G(u \wedge v)$.

Proof. Assume that $X \cong K(G,p)$ and $Y = K(G,q)$ and $u = \iota_p$, $v = \iota_q$. Then consider the induced cohomology of the left square in the above diagram

$$\begin{tikzcd} H^{2(p+q)}\left((X\wedge Y)^{(2)}\right) & H^{2(p+q)}(D_G(X\wedge Y)) \arrow[l, "i^*"'] \\ H^{2(p+q)}\left(X^{(2)}\wedge Y^{(2)}\right) \arrow[u, "T^*"] & H^{2(p+q)}(D_G(X)\wedge D_G(Y)) \arrow[l, "(i\wedge i)^*"'] \arrow[u, "\delta^*"] \end{tikzcd}$$

then by Lemma 2.4.5 i^* is a monomorphism and so the result follows. □

Proposition 2.4.10. $Sq^i(u \wedge v) = \sum_{j+k=i} Sq^j(u) \otimes Sq^k(v)$.

Proof.

$$\begin{aligned}
\sum_{i=-(p+q)}^{p+q} t^{p+q-i} \otimes \mathrm{Sq}^i(u \wedge v) &= j^*(P_G(u \wedge v)) \\
&= j^*(\delta^*(P_G(u) \wedge P_G(v))) \\
&= \Delta^*_{(BG)_+}(j \wedge j)^*(P_G(u) \wedge P_G(v)) \\
&= \Delta^*_{(BG)_+}\left((\sum_j x^{p-j} \otimes \mathrm{Sq}^j(u)) \wedge (\sum_k x^{q-k} \otimes \mathrm{Sq}^k(v)) \right)
\end{aligned}$$

□

Exercise. (i) Show $\mathrm{Sq}^0(e) = e$ where e is a generator of $\widetilde{H}^1(S^1; \mathbb{F}_2)$.

(ii) Show Sq^i commutes with suspension.

(iii) Show $\mathrm{Sq}^0 = \mathrm{id}$.

Exercise. Show that Sq^i is a group homomorphism.
Hint: use the adjunction $\Sigma \dashv \Omega$ and the previous exercise.

Theorem 2.4.11 (Serre)**.** *Let $\mathcal{A}$ be the graded $\mathbb{F}_2$-algebra generated by all stable cohomology operations on $\widetilde{H}^*(-; \mathbb{F}_2)$. Then $\mathcal{A}$ is generated by $\{Sq^i\}_{i \geq 1}$ subject to the Adem relations:*

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

for all $0 < a < 2b$.

Definition 2.4.12. $\mathcal{A}$ as above is called the (mod 2) *Steenrod algebra.*

Corollary 2.4.13. *$\mathcal{A}$ is a linear-quadratic graded $\mathbb{F}_2$ algebra.*

Proof (Sketch). The Adem relations are only linear or quadratic in its generators so as

$$\mathcal{A} \cong T(\{\mathrm{Sq}^i\}_{i \geq 1})/\{\text{Adem relations}\},$$

$\mathcal{A}$ is a linear quadratic algebra. □

We want to show that $\mathcal{A}$ is in fact a Koszul algebra. In view of this we will find a PBW-basis of the associated algebra $E(\mathcal{A})$.

Definition 2.4.14. A finite sequence $(\mathrm{Sq}^{i_1}, \ldots, \mathrm{Sq}^{i_n})$ is *admissible* if $i_j \geq 2i_{j+1}$ for all $1 \leq j \leq n-1$.

Theorem 2.4.15 (Serre-Cartan)**.** *The set $B_{\mathcal{A}} := \{Sq^I = Sq^{i_1} \cdot \ldots \cdot Sq^{i_n} \mid I \text{ admissible}\}$ forms an additive basis of $\mathcal{A}$.*

Proposition 2.4.16. *Order the Steenrod squares by $Sq^1 < Sq^2 < \ldots$, then the pair*

$$(B_{\mathcal{A}}, S \coloneqq \{I = (i_1, \ldots, i_n) \mid i_j \geq 2i_{j+1}\ \forall 1 \leq j \leq n-1\})$$

is PBW basis for the associated quadratic algebra $E(\mathcal{A})$ of $\mathcal{A}$.

Proof. We want to show that the pair is both:

(i) A labelled basis. This is clear.

(ii) A PBW basis:

(a) For $J \coloneqq (j_1, \ldots, j_n)$, $J' \coloneqq (j'_1, \ldots, j'_n)$ both in S , either

i. $(J, J') = (j_1, \ldots, j_n, j'_1, \ldots, j'_n) \in S$: this is clearly the case if $j_n \geq 2j'_1$.

ii. If this is not the case then $j_n < 2j'_1$. We can thus write $\mathrm{Sq}^{(J,J')} = \mathrm{Sq}^J \mathrm{Sq}^{J'}$ which is admissible via the Adem relations. We want to show that the labelling of the mononomials in the admissible expression for $\mathrm{Sq}^{(J,J')}$ is bigger than (J, J'). It suffices to check that the Adem relations $(a, b) < (a + b - j, j)$ for all $j \geq 1$.

(b) If $(i_1, \ldots, i_n) \in S$ such that $n > 2$ then any partition $(i_1, \ldots, i_k)$ and $(i_{k+1}, \ldots, i_n)$ both are in S.

□

Corollary 2.4.17. *$\mathcal{A}$ is a Koszul algebra.*

We would like to compute $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ given we already know the related $\mathrm{Ext}_{E(\mathcal{A})}(\mathbb{F}_2, \mathbb{F}_2)$. For this we will need the Koszul complex.

2.5 The Koszul Complex

Here again we want to let A be a linear quadratic algebra over a field k with presentation $\alpha : T(V) \twoheadrightarrow A$ and the associated quadratic algebra $E(A)$ with presentation $\alpha' : T(V) \twoheadrightarrow E(A)$.

Note that by the definition of $E(A)$, there is a natural injection $E(A)_{1,*} \hookrightarrow I(A)$. This induces an injection of k-modules

$$i : \mathrm{Bar}(E(A))_{p,p,*} \hookrightarrow \mathrm{Bar}(A)_{p,p,*}$$

Recall that we defined

$$H_*(A) \coloneqq \mathrm{Tor}^A_{*,*}(k, k) \cong H_*(\mathrm{Bar}(A))$$
$$H^*(A) \coloneqq \mathrm{Ext}^{*,*}_A(k, k) \cong H^*(\mathrm{Bar}(A)^\vee)$$

and we have another map

$$H_{p,p}(E(A)) \overset{j}{\hookrightarrow} \mathrm{Bar}(E(A))_{p,p} \overset{i}{\hookrightarrow} \mathrm{Bar}(A)_{p,p}$$

coming from the fact that that every class of $H_{p,p}(E(A))$ can be represented by a cycle of $\mathrm{Bar}(E(A))_{p,p}$ uniquely: in $E(A)$, there are no $p+1$-chains of length p.

Notation. We denote the $E(A)$-homology and cohomology at the diagonals by

$$\mathcal{K}_{p,*}(A) := H_{p,p}(E(A))$$
$$\mathcal{K}^{p,*}(A) := H^{p,p}(E(A))$$

Theorem 2.5.1. *Let A be as above.*

(i) One can define a morphism $d_p : \mathcal{K}_p(A) \to \mathcal{K}_{p-1}(A)$ for all $p \geq 0$ such that the map $i \circ j : \mathcal{K}_p(A) \to \mathrm{Bar}(A)_{p,p}$ induces a morphism

$$(\mathcal{K}_*(A), d_*) \to (\mathrm{Bar}(A), \partial)$$

such that this is an injective morphism of differentially graded coalgebras.

(ii) The degree-wise dual morphism

$$(\mathrm{Bar}(A)^\vee, \delta) \to (\mathcal{K}^*(A), d^*)$$

is a morphism of differentially graded algebras.

(iii) If A is a Koszul algebra, then

$$H_*(\mathcal{K}_*(A), d_*) \cong H_*(A)$$
$$H^*(\mathcal{K}^*(A), d^*) \cong H^*(A)$$

Definition 2.5.2. We call $(\mathcal{K}_*(A), d)$ (respectively $(\mathcal{K}^*(A), d^*)$) the *Koszul complex* (respectively the *co-Koszul complex*) for A.

Now let (B_A, S) be labelled basis for A. The admissible relations for the linear quadratic algebra A are

$$a_k a_\ell = \sum_{m \in S} f\binom{k,l}{m} a_m + \sum_{(i,j) \in S} f\binom{k,l}{i,j} a_i a_j \tag{2.5.1}$$

Consider the dual basis $(B_A^\vee, S)$. This induces a labelled basis $(B_{E(A)}, S)$ for $E(A)$ as well as a dual basis $(B_{E(A)}^\vee, S)$ defined in the same way, i.e

$$B_{E(A)}^\vee = \{1, \beta(i), \beta(i_1, \ldots, i_n)\}_{i, i_1, \ldots, i_n \in I}$$

such that for all $b \in B_{E(A)}$,

$$\beta(i_1, \ldots, i_n)(b) = \begin{cases} 1 & \text{if } b = b_{i_1} \ldots b_{i_n} \\ 0 & \text{else} \end{cases}$$

Theorem 2.5.3. *Let A be a Koszul algebra in the form given above. Then $(\mathcal{K}^*(A), d^*)$ is a differentially graded algebra generated by $\{\beta_i\}_{i\in I}$ [4] subject to the relations*

$$(-1)^{v_{i,j}}\beta_i\beta_j + \sum_{(k,\ell)\in U\setminus S} (-1)^{v_{k,\ell}} f\binom{k,\ell}{i,j}\beta_k\beta_l = 0$$

for every pair $(i,j)\in S$ and such that the differentials satisfy

$$d^*(\beta_m) = \sum_{(k,\ell)\in U\setminus S} (-1)^{v_{k,\ell}} f\binom{k,\ell}{m}\beta_k\beta_\ell$$

Here for $(i,j)\in U$, $v_{i,j} = \deg(\beta_i) + (\deg(\beta_i)-1)(\deg(\beta_j)-1)$.[5]

Proof. [Pri70] Theorem 4.6. □

Remark. If A is quadratic, then $d^* = 0$.

Applying this all to the Steenrod algebra $\mathcal{A}$, we are given that $(\mathcal{K}^*(\mathcal{A}), d^*)$ is generated by σ_i for $i \geq 1$ subject to the relations

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j\geq 2b}^{\lfloor \frac{a+b}{2}\rfloor} \binom{a-j-1}{j-2b}\mathrm{Sq}^j\mathrm{Sq}^{a+b-j}$$

with differential

$$d(\mathrm{Sq}^a) = \sum_{j=1}^{\lfloor \frac{2a}{3}\rfloor} \binom{a-j-1}{j}\mathrm{Sq}^j\mathrm{Sq}^{a-j}$$

Remark. $(\mathcal{K}(\mathcal{A}), d^*)$ is isomorphic to (mod 2) Λ-algebra which is isomorphic to the E_1 page of the *restricted lower central series spectral sequence* converging to $\pi_*(\mathbb{S})^\wedge_2$.

Going back to our construction,

$$\begin{tikzcd}
H_{p,p}(A) \arrow[r, hook, "j"] \arrow[d, dashed, "\exists! d"'] & \mathrm{Bar}(E(A))_{p,p} \arrow[r, hook, "i"] \arrow[d, "d"] & \mathrm{Bar}(A)_{p,p} \arrow[d, "\partial"] \\
H_{p-1,p-1}(A) \arrow[r, hook, "j"] & \mathrm{Bar}(E(A))_{p-1,p-1} \arrow[r, hook, "j"] & \mathrm{Bar}(A)_{p-1,p-1}
\end{tikzcd}$$

Note that a priori, the bottom map i has codomain $\mathrm{Bar}(A)_{p-1,p-1}$ but the image satisfies the linear quadratic relations (2.5.1) hence is contained within $\mathrm{Bar}(A)_{p-1,p}$. We want to construct the $d_p : \mathcal{K}_p(A) \to \mathcal{K}_{p-1}(A)$ above. All $x \in H_{p,p}(A)$ can be represented by an element, also denoted by x, in $\mathrm{Bar}(E(A))_{p,p} = \langle [b_{i_1}|\dots|b_{i_p}]\rangle_{i_j\in I}$, which we can then write as

$$x = \sum_{i=(i_1,\dots,i_p)\in I^p} f_i[b_{i_1}|\dots|b_{i_p}]$$

[4] Recall that β_i is the element in $H^{1,1,*}(E(A))$ represented by $\beta(i)$.

[5] Recall that $U = \bigcup_{n=0}^\infty I^{\times n}$

and the differential of x is

$$\begin{aligned}\partial_{E(A)}(x) &= \sum_{j=1}^{p-1}\sum_{i\in I^p}(-1)^{e_j} f_i[b_{i_1}|\dots|b_{i_j}b_{i_{j+1}}|\dots|b_{i_p}] \\ &= \sum_{j=1}^{p-1}\sum_{\substack{J:=(i_1,\dots,i_{j-1})\in I^{j-1}\\ J':=(i_{j+2},\dots,i_p)\in I^{p-j-1}}} f(J,i_j,i_{j+1},J')[b_{i_1}|\dots|b_{i_j}b_{i_{j+1}}|\dots|b_{i_p}]\end{aligned}$$

Hence $\partial_{E(A)}(x)=0$ is equivalent to saying that for all $(J,J')\in I^{j-1}\times I^{p-j-1}$,

$$\begin{aligned}&\sum_{(i_j,i_{j+1})\in I^2} f_{(J,i_j,i_{j+1},J')}[b_{i_1}|\dots|b_{i_j}b_{i_{j+1}}|\dots|b_{i_p}]=0\\ \iff &\sum_{(i_j,i_{j+1})\in I^2}(-1)^{e_j} f_{(J,i_j,i_{j+1},J')}b_{i_j}b_{i_{j+1}}=0\end{aligned}$$

where we have denoted $J=(i_1,\dots,i_{j-1})$ and $J'=(i_{j+2},\dots,i_p)$. This is a quadratic relation on $E(A)$. So in A there exists $f\binom{J,J'}{j}\in k$ such that

$$\sum_{\ell\in I} f\binom{J,J'}{\ell}a_\ell+\sum_j(-1)^{e_j} f_{(J,i_j,i_{j+1},J')}a_{i_j}a_{i_{j+1}}=0$$

holds in A.

Definition 2.5.4. For $x\in \mathrm{Bar}(E(A))_{p,p}$ as above, the differential is defined as

$$d(x)=\sum_{j=1}^{p-1}\sum_{(J,J')\in I^{j-1}\times I^{p-j-1}}\sum_{(i_j,i_{j+1})\in I^2}(-1)^{e_j} f\binom{J,i_j,i_{j+1},J'}{k}[b_{i_1}|\dots|b_{j-1}|b_k|b_{j+2}|\dots|b_{i_p}]$$

which is contained in $\mathrm{Bar}(E(A))_{p-1,p-1}$, and where $f\binom{J,i_j,i_{j+1},J'}{k}=f\binom{J,J'}{k}\cdot f_{(J,b_j,b_{j+1},J')}$.

Proposition 2.5.5. *For $x\in \mathrm{Bar}(E(A))_{p,p}$ as above,*

(i) $\partial_{E(A)}(d(x))=0$ i.e $d(x)$ represents a homology class in $H_{p-1,p-1}(E(A))$.

(ii) $\partial\circ i=i\circ d$ so $d^2=0$ and ι becomes a morphism of chain complexes

$$\iota:(\mathcal{K}_*(A),d)\to(\mathrm{Bar}_*(A),\partial_A).$$

Proof.

$$\begin{tikzcd}
H_{p,p}(E(A)) = Z(\mathrm{Bar}(A)_{p,p}) \arrow[r, hook, "j"] \arrow[d, "\Delta"] & \mathrm{Bar}(E(A))_{p,p} \arrow[r, hook, "i"] \arrow[d, "\Delta"] & \mathrm{Bar}(A)_{p,p} \arrow[d, "\Delta"] \\
Z\left(\bigoplus_{r\neq s} \mathrm{Bar}(E(A))_{r,r} \otimes \mathrm{Bar}(E(A))_{s,s}\right) \arrow[r, hook, "j"] & \bigoplus_{r+s=p} \mathrm{Bar}(E(A))_{r,r} \otimes \mathrm{Bar}(E(A))_{s,s} \arrow[r, hook, "i\otimes i"] & \bigoplus_{r+s=p} \mathrm{Bar}(A)_{r,r} \otimes \mathrm{Bar}(A)_{s,s} \\
\bigoplus_{r+s=p} H_{r,r}(A) \otimes H_{s,s}(A) \arrow[u, "h", "\cong"'] \arrow[ur, hook, "j\otimes j"'] & &
\end{tikzcd} \tag{2.5.2}$$

□

Definition 2.5.6. $\Delta_{\mathcal{K}_p(A)} = h^{-1} \circ \Delta$.

We now look at the co-Koszul complex $(\mathcal{K}^*(A), d^*) = (\mathcal{K}_*(A)^\vee, (d_*)^\vee)$.

Corollary 2.5.7. *The linear dual of ι, $(C(A) := \mathrm{Bar}(A)^\vee, \partial) \to (\mathcal{K}^*(A), d^*)$ is an morphism of differentially graded algebras.*

To determine $d : \mathcal{K}^p(A) \to \mathcal{K}^{p+1}(A)$, it suffices to know the evaluations of d on $\beta_i \in H^{1,1}(E(A)) \cong (E(A)_{1,*})^\vee$ where $i \in I$. This gives us that $d^1 : K^1(A) \to K^2(A)$ is induced by the composition

$$E(A)_1^\vee \xrightarrow{\pi^\vee} I(A)^\vee \xrightarrow{\widetilde{\Delta}} I(A)^\vee \otimes I(A)^\vee \xrightarrow{i^\vee \otimes i^\vee} E(A)_1^\vee \otimes E(A)_1^\vee$$

where $\pi : I(A) \to E(A)_1$ is a splitting of $i : E(A)_1 \hookrightarrow I(A)$ obtained by choosing a basis for A.

2.6 Koszul Duality

Definition 2.6.1. Let A be any augmented bigraded degree-wise finite k-algebra were k is a field. Define a bigraded k-algebra $D(A)$ as $D^{s,i}(A) = H^{s,s,i}(A)$.

Exercise. Show that $D(A)$ is a quadratic algebra.
Hint: $T((A_{1,*}^\vee)/\{\text{quadratic relations}\}$.

Theorem 2.6.2. *There exists a natural morphism of bigraded algebras $\Phi : D(D(A)) \to A$ induced by natural isomorphisms $\psi_q : ((A_{1,q})^\vee)^\vee \to A_{1,q}$ for all $q \geq 0$. If A is quadratic then Φ is an isomorphism.*

Proof. Since $D(A)$ is a quadratic algebra, it suffices to define maps $\Phi_1 : D^{1,*}(D(A)) \to A_{1,*}$ and $\Phi_2 : D^{2,*}(D(A)) \to A_{2,*}$ such that they satisfy

$$\begin{tikzcd}
D^{1,*}(D(A)) \otimes D^{1,*}(D(A)) \arrow[r, "\Phi_1\otimes\Phi_1"] \arrow[d, "\mu_{D(D(A))}"'] & A_{1,*} \otimes A_{1,*} \arrow[d, "\mu_A"] \\
D^{2,*}(D(A)) \arrow[r, "\Phi_2"'] & A_{2,*}
\end{tikzcd}$$

To that end, define $\Phi_1 : D^{1,*}(D(A)) \cong A_{1,*}^{\vee\vee} \xrightarrow{\psi_*} A_{1,*}$ and construct Φ_2 from $\Phi_1 \otimes \Phi_1 \cong \Psi \otimes \Psi$ as follows

$$\begin{tikzcd}
& D^{2,*}(A) \arrow[r, dashed, equal] & D^{2,*}(D(A)) \arrow[r, hook, dashed, "\Phi_2", "④"'] & A_{2,*} \\
D^{1,*}(D(A)) \otimes D^{1,*}(D(A)) \arrow[r, equal] & D^{1,*}(A)^\vee \otimes D^{1,*}(A)^\vee \arrow[u, "\mu_{D(D(A))}"] \arrow[r, equal] & A_{1,*}^{\vee\vee} \otimes A_{1,*}^{\vee\vee} \arrow[u, two heads, dashed, "③"'] \arrow[r, "\Psi\otimes\Psi"] & A_{1,*} \otimes A_{1,*} \arrow[u, "\mu_A"'] \\
& D^{2,*}(A)^\vee \arrow[u, "\mu^\vee_{D(A)}"] \arrow[r, dashed, "\simeq ②", "①"'] & R^{\vee\vee} \arrow[u, hook, "i^{\vee\vee}"'] \arrow[r, "\simeq"] & R := \ker(\mu_A) \arrow[u, hook, "i"'] \\
& & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

- For ①,② note that for $i^\vee : A_{1,*}^\vee \otimes A_{1,*}^\vee \twoheadrightarrow R^\vee$ we have $R^\vee \cong D^{2,*}(A)$ and we can identify $i^\vee$ with the multiplication $D^{1,*}(A) \otimes D^{1,*}(A) \to D^{2,*}(A)$.
- ③ follows from the commutativity given by ②.
- ④ is obtained by diagram chase.

Now if we let A be quadratic, then Φ is surjective as A is generated by $A_{1,*}$ and Φ is injective since the relations of A are generated by elements in $A_{2,*}$ and we know that Φ_2 is injective. $\square$

Corollary 2.6.3. *If A is a quadratic Koszul algebra and $H^*(A)$ is Koszul, then $H^*(H^*(A)) \xrightarrow{\simeq} A$.*

3 The theory of operads

We shall now recall the theory of 1-categorical operads. To that purpose we shall assume that we are working with ordinary categories unless stated otherwise for the remainder of this section.

Definition 3.0.1. Let $\mathbb{V}$ be a symmetric monoidal category. An *operad* $\mathcal{O}$ with values in $\mathbb{V}$ consists of

(i) a set $\mathcal{C}\mathrm{ol}(\mathcal{O})$ of *colours*, and

(ii) for every pair $((c_i)_{i=1}^r, c)$ of a colour $c \in \mathcal{C}\mathrm{ol}(\mathcal{O})$ and r-tuple of colours $(c_i)_{i=1}^r \in \mathcal{C}\mathrm{ol}(\mathcal{O})^r$ an object $\mathcal{O}\left((c_i)_{i=1}^r; c\right) \in \mathbb{V}$,
together with the following maps for every pair of $((c_i)_{i=1}^r, c)$ of colour $c \in \mathcal{C}\mathrm{ol}(\mathcal{O})$ and r-tuple of colours $(c_i)_{i=1}^r \in \mathrm{Col}(\mathcal{O})^r$, $r \in \mathbb{N}$:

(iii) A unit map $1_c : 1_{\mathbb{V}} \to \mathcal{O}(c, c)$ for all $c \in \mathrm{Col}(\mathcal{O})$.

(iv) A morphism

$$\mathcal{O}\left((c_i)_{i=1}^r; c\right) \otimes \left(\bigotimes_{i=1}^{r} \mathcal{O}\left((c_{i,j})_{j=1}^{m_i}; c_i\right)\right) \to \mathcal{O}\left((c_{1,j_1})_{j_1=1}^{m_1}, \dots, (c_{r,j_r})_{j_r=1}^{m_r}; c\right)$$

in $\mathbb{V}$ called a *composition map*, for every r-tuple $\left((c_{i,j})_{j=1}^{m_i}\right)_{i=1}^{r}$ of finite sequences $(c_{i,j})_{j=1}^{m_i}$ of colours.

(v) A morphism $\sigma^* : \mathcal{O}(c_1, \dots, c_r; c) \to \mathcal{O}(c_{\sigma(1)}, \dots, c_{\sigma(r)}; c)$ in $\mathbb{V}$ for each element σ in the symmetric group $\mathfrak{S}_r$.
satisfying the following axioms [6]:

(vi) TODO

Definition 3.0.2. A *one-coloured operad* is an operated $\mathcal{O}$ with values in $\mathbb{V}$ whose set of colours contains only one element.

Remark. One should think of the object $\mathcal{O}(c_1, \dots, c_r; c)$ as describing an operation having r-number of inputs of 'types' $c_1, \dots, c_r$ and one output of 'type' c. We call this an *operation of $\mathcal{O}$ of arity r.*
Assume that $\mathcal{O}$ is a one-coloured operad with $\mathrm{Col}(\mathcal{O}) = \{c\}$. Then for every $r \in \mathbb{N}$ and r-tuple of colours $(c)_{i=1}^r$ we denote

$$\mathcal{O}(r) \coloneqq \mathcal{O}\left((c)_{i=1}^r; c\right).$$

Note that $\mathcal{O}(r)$ admits a right $\mathfrak{S}_r$-action for every $r \in \mathbb{N}$.

Definition 3.0.3. Let $\mathcal{O}, \mathcal{P}$ be operads with values in $\mathbb{V}$. An operad map $f : \mathcal{O} \to \mathcal{P}$ consists of

(i) a morphism $f : \mathcal{C}\mathrm{ol}(\mathcal{O}) \to \mathcal{C}\mathrm{ol}(\mathcal{P})$ of sets, and

(ii) a morphism
$$f\left((c_i)_{i=1}^r; c\right) : \mathcal{O}\left((c_i)_{i=1}^r; c\right) \to \mathcal{P}\left((f(c_i))_{i=1}^r; f(c)\right)$$
in $\mathbb{V}$, for every operation of $\mathcal{O}$ of arity r, and for all $r \in \mathbb{N}$

such that they are compatible with the structure maps of $\mathcal{O}$ and $\mathcal{P}$.

Definition 3.0.4. An operad map $f : \mathcal{O} \to \mathcal{P}$ is an operad *inclusion* if

(i) $f : \mathcal{C}\mathrm{ol}(\mathcal{O}) \to \mathcal{C}\mathrm{ol}(\mathcal{P})$ is injective, and

[6] Morally speaking we want these morphism to satisfy the conditions that

(a) the composition maps are associative and unital,

(b) for every $r \in N$ and colour c, the set $\{\sigma^* \mid \sigma \in \mathfrak{S}_r\}$ of morphism induces a right $\mathfrak{S}_r$-action on the set $\{Oo\left((c_i)_{i=1}^r; c\right) \mid (c_i)_{i=1}^r \in \mathrm{Col}(\mathcal{O})^r\}$ of objects, and

(c) the right symmetric group actions are compatible with the composition maps

(ii) all the morphisms $f\left((c_i)_{i=1}^r; c\right)$ are isomorphisms in $\mathbb{V}$

We say that $\mathcal{O}$ is a *suboperad* of $\mathcal{P}$.

Example. Let $\mathcal{C}$ be a symmetric monoidal category enriched over $\mathbb{V}$. Let S be a set of objects in C. We can define that *mapping operad* $\mathrm{Map}(\mathcal{C}, S)$ with values in $\mathbb{V}$ as follows:

(i) The set of colours $\mathcal{C}\mathrm{ol}(\mathrm{Map}(\mathcal{C}, S))$ is the set S.

(ii) For each $X \in S$, define the operation of arity 0 as

$$\mathrm{Map}(\mathcal{C}, S)(0, X) \coloneqq \mathrm{Map}_{\mathcal{C}}(1_C, X).$$

(iii) For each $r \in \mathbb{N}$, $r \geq 1$, an operation of arity r is defined as

$$\mathrm{Map}(\mathcal{C}, S)\left((X_i)_{i=1}^r; X\right) \coloneqq \mathrm{Map}_{\mathcal{C}}(\mathcal{C}, S)\left(\otimes_{i=1}^r X_i, X\right)$$

for each $X_1, \ldots, X_r, X \in S$.

Exercise. Show that $\mathrm{Map}(\mathcal{C}, S)$ is an operad.

If $S = \{X\}$ then denote $\mathrm{Map}(\mathcal{C}, \{X\})$ by $\mathrm{End}(X)$ called the *endomorphism operad* of X. Note that its operation of arity r is $\mathrm{End}(X)(r) = \mathrm{Map}_{\mathcal{C}}(X^{\otimes r}, X)$.

Definition 3.0.5. Let $\mathbb{V}$ be a closed symmetric monoidal category and $\mathcal{C}$ a symmetric monoidal category enriched over $\mathbb{V}$. We say that $\mathcal{C}$ is *copowered* over $\mathbb{V}$ if the following conditions hold:

(i) $\mathcal{C}$ is tensored over $\mathbb{V}$: For all $V \in \mathbb{V}$ there exists a functor $V \otimes - : \mathcal{C} \to \mathcal{C}$ such that $V' \otimes (V \otimes C) \cong (V' \otimes_{\mathbb{V}} V) \otimes C$ and $(V \otimes C) \otimes_{\mathcal{C}} C' \cong V \otimes (C \otimes_{\mathcal{C}} C')$ for all $V, V' \in \mathbb{V}$ and $C, C' \in \mathcal{C}$.

(ii) The functor $V \otimes -$ defined above satisfies a natural isomorphism $\mathrm{Map}_{\mathcal{C}}(V \otimes C, C') \cong \mathrm{Map}_{\mathbb{V}}(V, \mathrm{Map}_{\mathcal{C}}(C, C'))$ in $\mathbb{V}$ for each $V \in \mathbb{V}$ and $C, C' \in \mathcal{C}$.

Definition 3.0.6. Let $\mathbb{V}$ be a closed symmetric monoidal category and $\mathcal{C}$ a symmetric monoidal category enriched over $\mathbb{V}$. Let $\mathcal{O}$ be an operad with values in $\mathbb{V}$. An $\mathcal{O}$*-algebra* in $\mathcal{C}$ is a set $S_{\mathcal{O}} = \{X_i\}_{i \in \mathcal{C}\mathrm{ol}(\mathcal{O})}$ of objects in $\mathcal{C}$ together with a map $\mathcal{O} \to \mathrm{Map}(\mathcal{C}, S_{\mathcal{O}})$ of operads with values in $\mathbb{V}$.

Example. Let $\mathcal{O}$ be a one-coloured operad with values in $\mathbb{V}$. An $\mathcal{O}$-algebra is then an object $X \in \mathcal{C}$ together with a stucture map

$$\mathcal{O}(r) \to \mathrm{Map}_{\mathcal{C}}(X^{\otimes r}, X)$$

for each $r \geq 0$, with compatibility with the structure maps of $\mathcal{O}$ and $\mathrm{End}(X)$. If $\mathcal{C}$ is copowered over $\mathbb{V}$, then the above map is equivalently an object $X \in \mathcal{C}$ along with a map

$$\mathcal{O}(r) \otimes X^{\otimes r} \to X$$

for each $r \geq 0$ with (omitted) suitable compatibilities.

Example. The *trivial operad* Triv is a one-coloured operad with values in **Set** (as a symmetric monoidal category) where

$$\mathrm{Triv}(r) := \begin{cases} \{\mathrm{pt}\}, & r = 1 \\ \emptyset, & \text{else} \end{cases}.$$

The structure maps are obvious.

Exercise. Show that every object X is a symmetric monoidal category admits a unique trivial algebra structure.

Example. The *unital operad* E_0 is a one-coloured operad with values in **Set** where

$$E_0(r) := \begin{cases} \{\mathrm{pt}\}, & r = 0, 1 \\ \emptyset, & \text{else} \end{cases}.$$

An E_0-algebra in a symmetric monoidal category $\mathcal{C}$ is an object $X \in \mathcal{C}$ together with a morphism $1_{\mathcal{C}} \to X$ (unit map).

Example. The *associative operad* Ass is a one-coloured operad with values in **Set** defined as follows:
Let $M\langle x_i \rangle_{i=1}^r$ be the free monoid in **Set** generated by r letters x_i. Denote by $M(x_i)_{i=1}^r$ the subset of mononomials of length r where each x_i appears exactly once. Then $\mathbf{Ass}(r) := M(x_i)_{i=1}^r \cong \mathfrak{S}_r$ (where $S_0 = \{\mathrm{pt}\}$). The composition maps

$$\mathfrak{S}_r \times \mathfrak{S}_{b_1} \times \ldots \times \mathfrak{S}_{b_r} \to \mathfrak{S}_{b_1 + \ldots + b_r}$$

are given by the so-called *block permutations* as follows: Take a set S of $b_1 + \ldots + b_r$ elements and consider a decomposition $S = \sqcup_{i=1}^r S_i$ where $S_i = \{a_{i,1}, \ldots, a_{i,b_i}\}$ is a set of b_i elements. Given a tuple $(\sigma_0, \ldots, \sigma_r)$ with $\sigma_0 \in \mathfrak{S}_r$ and $\sigma_i \in \mathfrak{S}_{b_i}$ for $1 \leq i \leq r$ we define a element $\sigma \in \mathfrak{S}_{b_1 + \ldots + b_r}$ permuting S by setting $\sigma : a_{i,j} \mapsto a_{\sigma_0(i), \sigma_i(j)}\}$.[7]

Exercise. $(\mathrm{Ass}(r))_{r \geq 0}$ is an operad for $\mathcal{C}$ a symmetric monoidal category.

Exercise. An Ass-algebra is an object $X \in \mathcal{C}$ together with a unit map $\epsilon : \to X$ and a multiplication $\mu : X \otimes_{\mathcal{C}} X \to X$ such that μ is associative and unital.
In other words X us an *associative algebra object* in $\mathcal{C}$.

Example. The *left module operad* LM is an operad with values in **Set** defined as follows:

(i) $\mathcal{C}\mathrm{ol}(\mathrm{LM}) := \{a, m\}$

(ii) For any $1 \leq j \leq r$ define

$$\mathrm{LinOrd}(r) := \{i_1 < \ldots < i_r \mid \{i_1, \ldots, i_r\} = \{1, \ldots, r\}\} \tag{3.0.1}$$

$$\mathrm{LinOrd}(r, j) := \{i_1 < \ldots < i_r \mid \{i_1, \ldots, i_r\} = \{1, \ldots, r\}, i_r = j\} \tag{3.0.2}$$

[7] Basically we are decomposing S into r 'blocks' of size b_i, and having σ_0 permute the blocks while σ_i permutes only S_i.

$$\mathrm{LM}\left((c_i)_{i=1}^r\right) := \begin{cases} \mathrm{LinOrd}(r), & c = c_i = a\ \forall i,\ 1 \le i \le r \\ \mathrm{LinOrd}(r,j), & \text{if } c = m = c_j = m \text{ for exactly one } j \text{ and } c_i = a\ \forall i, 1 \le i \le r, i \ne j \\ \emptyset, & \text{else} \end{cases}$$

(iii) The structure maps are given by restricting the structure maps of Ass to $\mathrm{LinOrd}(r, j)$ as follows:

$$\begin{gathered} \mathrm{LinOrd}(r,j) \times (\mathrm{LinOrd}(b_1) \times \ldots \mathrm{LinOrd}(b_{j-1}) \times \mathrm{LinOrd}(b_j, k) \times \ldots \times \mathrm{LinOrd}(b_r)) \\ \downarrow \\ \mathrm{LinOrd}(\textstyle\sum_{i=1}^r b_j, k + \sum_{i=1}^{j-1} b_i) \end{gathered}$$

$$\big(i_1 < \ldots < i_r, (i_{1,1} < \ldots < i_{1,b_1}, \ldots, i_{j,1} < \ldots < i_{j,b_j}, \ldots)\big) \mapsto (???)$$

Finish writing down the maps above.

(iv) Given a permutation $\sigma \in \mathfrak{S}_r$:

(a)

$$\begin{aligned} \sigma^* : &\mathrm{LM}\left((c_i)_{i=1}^r; c\right) \to \mathrm{LM}\left((c_{\sigma(i)})_{i=1}^r; c\right) \\ &(i_1 < \ldots < i_r) \mapsto (i_{\sigma(1)} < \ldots i_{\sigma(r)} \end{aligned}$$

(b) Finish this???

Exercise. Show that Ass is a suboperad of LM.

Exercise. Show that an algebra over LM in a symmetric monoidal category $\mathcal{C}$ is a pair (A, M) such that A is an associative algebra object in $\mathcal{C}$ and M is a left module over A.

3.1 Operads via SymSeq

Recall that if $\mathbb{V}$ is a bicomplete symmetric monoidal category, then the functor category $\mathrm{Fun}(\mathbb{V}, \mathbb{V})$ admits a monoidal structure via composition of functors and the identity natural transformation.

Definition 3.1.1. A monad T on a symmetric monoidal category $\mathbb{V}$ is an associative algebra object in the monoidal category $\mathrm{Fun}(\mathbb{V}, \mathbb{V})$. In other words it is an object $T \in \mathbb{V}$ with maps $\mu : T \circ T \to T$ and $\iota : 1_{\mathbb{V}} \implies T$ such that μ is associative and unital.

Definition 3.1.2. (i) The *category of finite sets*, denoted Fin, has as objects finite sets in the form $\underline{n} := \{1, \ldots, n\}$ for all $n \in \mathbb{N}$ and morphisms maps of finite sets.

(ii) Denote by $\mathrm{Fin}^{\simeq}$ the maximal subgrouoid of Fin with $\mathrm{Ob}(\mathrm{Fin}^{\simeq}) = \mathrm{Ob}(\mathrm{Fin}^{\simeq})$ and $\mathrm{Mor}(\mathrm{Fin}^{\simeq}) = \{\varphi \in \mathrm{Fin} \mid \varphi \text{ is an isomorphism}\}$.

(iii) The *category of finite pointed sets*, denoted Fin_*, has objects finite pointed sets in the form $\langle n\rangle := \{\mathrm{pt}, 1, \dots, n\}$ for all $n \in \mathbb{N}$ and morphisms pointed maps of pointed finite sets.

Definition 3.1.3. Let $\mathbb{V}$ be a symmetric monoidal category. The *category of symmetric sequences* $\mathrm{SymSeq}(\mathbb{V})$ is the functor category $\mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathbb{V})$. An object $M \in \mathrm{SymSeq}(\mathbb{V})$ is called a *symmetric sequence* in $\mathbb{V}$. Moreover, we denote $M(\underline{r})$ by $M(r)$ for $\underline{r} \in \mathrm{Fin}^{\simeq}$.

Example. (i) Given $X \in \mathbb{V}$ define a symmetric sequence $X^{\mathfrak{S}}$ where $X^{\mathfrak{S}}(1) := X$ and $X^{\mathfrak{S}}(r) = \emptyset_{\mathbb{V}}$ (the initial object in V).

(ii) Let $\mathcal{O}$ be a one-coloured operad with values in $\mathbb{V}$. The sequence $M_{\mathcal{O}} := (\mathcal{O}(r))_{r\in\mathbb{N}}$ is a symmetric sequence in $\mathbb{V}$.

Construction: Let $\mathcal{C}$ be a cocomplete symmetric monoidal category copowered over a closed symmetric monoidal category $\mathbb{V}$. Then every symmetric sequence M in $\mathbb{V}$ induces a functor

$$F_M : \mathcal{C} \to \mathcal{C},\ X \mapsto \coprod_{r\in\mathbb{N}} \left(M(r) \otimes X^{\otimes r}\right)_{\mathfrak{S}_r}$$

.

Theorem 3.1.4 (Kelly)**.** *Consider the construction above, the following statements hold:*

(i) There exists a functor $\circledcirc : \mathrm{SymSeq}(\mathbb{V}) \times \mathrm{SymSeq}(\mathbb{V}) \to \mathrm{SymSeq}(\mathbb{V})$ *called the composition product such that* $(\mathrm{SymSeq}(\mathbb{V}), \circledcirc, (1_{\mathbb{V}})^{\mathfrak{S}})$ *is a monoidal category.*

(ii) Considering $\mathrm{SymSeq}(\mathbb{V})$ *as a monoidal category as above, the functor* $F_{(-)} : \mathrm{SymSeq}(\mathbb{V}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ *is a monoidal functor.*

(iii) The is a bijective correspondence between one-coloured operads with values in $\mathbb{V}$ *and associative algebras on* $(\mathrm{SymSeq}(\mathbb{V}), \circledcirc, (1_{\mathbb{V}})^{\mathfrak{S}})$*, given by* $\mathcal{O} \mapsto M_{\mathcal{O}}$*.*

(iv) Under this correspondence, an algebra in $\mathcal{C}$ *over an operad* $\mathcal{O}$ *with values in* $\mathbb{V}$ *is a left module over the associated monad* $F_{M_{\mathcal{O}}}$*.*

Proof. We shall postpone this proof until the ∞-categorical case - see ???. $\square$

3.2 The category of operators

Definition 3.2.1. Let $\mathcal{O}$ be an operad with values in $(\mathrm{Set}, \times, \{\mathrm{pt}\})$. The *category of operators* $\mathcal{O}^{\otimes}$ associated to $\mathcal{O}$ consists of the following data:

(i) An object $\mathcal{O}^{\otimes}$ is a finite sequence of colours of $\mathcal{O}$.

(ii) A morphism $f \in \mathrm{Hom}_{\mathcal{O}^{\otimes}}\left((c_i)_{i=1}^m, (d_j)_{j=1}^{\ell}\right)$ consists of a pair $(\alpha, (\phi_1, \dots, \phi_\ell))$ where

   (a) $\alpha : \langle m\rangle \to \langle \ell\rangle$ is a morphism of finite pointed sets, and

   (b) $\phi_k \in \mathcal{O}\left((c_i)_{i\in\alpha^{-1}(k)}; d_k\right)$ for $k = 1, \dots, \ell$ where $(c_i)_{i\in\alpha^{-1}(k)}$ denotes the subsequence of $(c_i)_{i=1}^m$ such that indices map to j under α.

(iii) The composition of morphisms in $\mathcal{O}^{\otimes}$ is given pairwise by the composition of morphisms of pointed sets and the composition map of operations of $\mathcal{O}$.

Definition 3.2.2. We call a morphism of pointed finite sets $i : \langle m \rangle \to \langle n \rangle$ *inert* if for all $j \in \langle n \rangle$ such that $j \neq \mathrm{pt}$ then $|i^{-1}(k)| = 1$.

Definition 3.2.3. For $1 \leq i \leq n$, let $\rho_i : \langle m \rangle \to \langle 1 \rangle$ be the inert morphism sending i to 1 and everything else to the basepoint pt.

Definition 3.2.4. Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor and $n \in \mathbb{N}$. Define the subcategory $\mathcal{C}_{\langle n \rangle}$ of $\mathcal{C}$ as having objects $x \in \mathrm{ob}(\mathcal{C})$ such that $p(x) = \langle n \rangle$ and morphisms $f : X \to Y$ if $p(f) = \mathrm{id}_{\langle n \rangle}$ in $\mathcal{C}$.

Definition 3.2.5. Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor. A morphism $f : X \to Y$ in $\mathcal{C}$ is *p-cocartesian* if for all tuples (Z, g, α) where $Z \in \mathrm{ob}(\mathcal{C})$, $g : X \to Z$ a morphism in $\mathcal{C}$ and $\alpha : p(Y) \to p(Z)$ a morphism in Fin_* such that $p(g) = \alpha \circ p(f)$, there exists a unique morphism $h : Y \to Z$ in $\mathcal{C}$ such that $g = hf$. In other words if you have

$$\begin{tikzcd} & p(X) \arrow[dl, "p(f)"'] \arrow[dr, "p(g)"] & \\ p(Y) \arrow[rr, "\alpha"'] & & p(Z) \end{tikzcd}$$

then there is a unique lift

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, dashed, "\exists! h"'] & & Z \end{tikzcd}$$

Proposition 3.2.6. *The category $\mathcal{O}^{\otimes}$ is equipped with a functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ satisfying the following:*

(i) For all objects $(c_i)_{i=1}^m$ of $\mathcal{O}^{\otimes}$ and all inert morphisms $i : \langle n \rangle \to \langle \ell \rangle$ in Fin_* there exists a unique (up to equivalence) p-cocartesian lift $\bar{i} : (c_j)_{j=1}^m \to (c_k)_{k=1}^{\ell}$ of i - i.e $\bar{i}$ is p-cocartesian and $p(\bar{i}) = i$.

(ii) For all $m \in \mathbb{N}$ and $1 \leq n \leq m$ the inert morphism ρ_n induces a functor $R_{m,n} : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, $(c_j)_{j=1}^m \mapsto c_n$ by taking the p-cocartesian lifts of ρ_i.

(iii) For all $m \in \mathbb{N}$ such that $m \geq 1$, the sequence $(R_{m,i})_{i=1}^m$ of functors induces an equivalence of categories $\mathcal{O}^{\otimes}_{\langle m \rangle} \xrightarrow{\sim} \left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{\times m}$.

Proof (Sketch). TODO $\square$

Remark. Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor. For a tuple (α, X, Y) where $\alpha : \langle m \rangle \to \langle \ell \rangle$ is a morphism in Fin_*, X an object of $\mathcal{C}_{\langle m \rangle}$ and Y an object of $\mathcal{C}_{\langle \ell \rangle}$, let $\mathrm{Hom}^{\alpha}_{\mathcal{C}}(X, Y)$ denote the subset of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms f which lift to α i.e $p(f) = \alpha$.

Definition 3.2.7. Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor. We say that $\mathcal{C}$ is a *category of operations* if $(\mathcal{C}, p)$ satisfies conditions i), ii), iii) of the previous proposition.

Proposition 3.2.8. *Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor such that $\mathcal{C}$ is a category of operations. Then there exists an operad $\mathcal{O}_\mathcal{C}$ with values in Set whose associated category $\mathcal{O}_\mathcal{C}^\otimes$ of operators is equivalent to $\mathcal{C}$.*

Proof (Sketch). TODO ☐

Corollary 3.2.9. *There is a bijection*

$$\{\textit{Operads with vaules in Set}\} \xrightarrow{\sim} \{\textit{categories of operations } \mathcal{C} \to \mathrm{Fin}_*\}$$

Proposition 3.2.10. *Let $\mathcal{O}, \mathcal{P}$ be operads with values in Set. The data of an operadic map $\mathcal{O} \to \mathcal{P}$ is the same as a functor $F : \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ of categories of operations such that*

$$\begin{tikzcd} \mathcal{O}^\otimes \arrow[rr, "F"] \arrow[dr] & & \mathcal{P}^\otimes \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

commutes and F preserves p-cocartesian morphisms and are lifts of inert maps.

Todo3

- Decide on mathcal or mathbb for 1-cats
- Do Comm?

4 ∞-categorical operads

4.1 ∞-categories

We begin this section with a (reasonably small) exposition on the language and techniques of the theory of ∞-categories.

Definition 4.1.1. Let Δ be the category of non-empty totally ordered finite sets, whose morphisms are maps that preserves the order. Up to isomorphism each object of Δ is of the form

$$[n] := \{0 < 1 < \ldots < n\}$$

Definition 4.1.2. A *simplicial set* is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. We call the functor category $\mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set})$ the *category of simplical sets*, denoted sSet. Morphisms in this category are simply natural transformations. Given some $X \in \mathrm{sSet}$ we shall denote $X_n := X([n])$, called the *n-simplices of X*.

Example. The simplical set Δ^n given by the functor $\Delta^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$, $\Delta([m]) = \mathrm{Hom}_\Delta([m], [n])$ is called the *standard n-simplex*.

(i) Fix a non-negative integer n. Take some $\mathcal{J} \subset \mathcal{P}(\{0, \dots, n\})$ and define

$$\Delta^{\mathcal{J}} : \Delta\mathrm{op} \to \mathrm{Set}, \ [m] \mapsto \{f : [m] \to [n] \mid \exists J \in \mathcal{J} \text{ such that } \mathrm{im}(f) \subset J\}$$

This is a generalisation of the previous case, as for certain choices of $\mathcal{J}$:

- Taking $\mathcal{J} = \mathcal{P}(\{0, \dots, n\})$, we have $\Delta^{\mathcal{J}} = \Delta^n$.
- Taking $\mathcal{J} = \mathcal{P}(\{0, \dots, n\}) \setminus \{\{0, \dots, n\}\}$, we get $\partial\Delta \coloneqq \Delta^{\mathcal{J}}$, called the *boundary of the n-simplex.*
- Fix some $0 \le i \le n$, and take $\mathcal{J}_i = \mathcal{P}(\{0, \dots, n\}) \setminus \Big\{\{0, \dots, n\}, \{0, \dots, \hat{i}, \dots, n\}\Big\}$. We call $\Lambda^n_i \coloneqq \Delta^{\mathcal{J}}$ the *i-horn of* Δ^n.

Example. We define a functor $\chi : \Delta \to \mathrm{Cat}$ which sends the finite set $[n]$ to the poset category of $[n]$[8] and a morphism of sets $f : [m] \to [n]$ to the obvious functor between the post categories.
Pick some (small) category $\mathcal{C}$. Then consider

$$N(\mathcal{C}) : \Delta^{\mathrm{op}} \xrightarrow{\chi^{\mathrm{op}}} \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}}(-,\mathcal{C})} \mathrm{Set}$$

which sends $[n] \in \Delta$ to $\mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$. This is called the (ordinary) *nerve* of $\mathcal{C}$. For $n \ge 1$ the n-simplices of $N(\mathcal{C})$ will look like

$$N(\mathcal{C})_n = \{(f_1, \dots, f_n) \mid f_i \in \mathrm{Mor}(\mathcal{C}) \text{ and } \mathrm{dom}(f_{i+1}) = \mathrm{cod}(f_i) \text{ for all } 1 \le i \le n-1\}$$

Example. Define the *topological n-simplex*

$$|\Delta^n| \coloneqq \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } 0 \le t_i \,\forall i\}$$

This gives us a functor

$$\rho : \Delta \to \mathrm{Top}, \ [n] \mapsto |\Delta^n|$$

where a morphism $[m] \xrightarrow{d} [n]$ gets sent to the map $|\Delta^m| \to |\Delta^n|$, $(t_0, \dots, t_m) \mapsto (s_0, \dots, s_n)$ where $s_i = \sum_{j \in d^{-1}(i)} t_j$.

Example. We can get simplical sets that come from toplogical spaces. Given a space $Y \in \mathrm{Top}$ consider the simplical set

$$\mathrm{Sing}(Y) : \Delta^{\mathrm{op}} \xrightarrow{\rho^{\mathrm{op}}} \mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Top}}(-,Y)} \mathrm{Set}$$

which sends $[n] \to \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, Y)$. This is called the *singular simplical set* of Y.

[8] I.e the category with object elements of $[n]$ and giving a unique morphisms between $i, j \in [n]$ if and only if $i \le j$. It will also be denoted as $[n]$.

Definition 4.1.3. The functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ defined by sending $X \in \mathrm{sSet}$ to

$$|X| = \mathrm{colim}_{\Delta^n \to X} |\Delta^n|$$

where the indexing is take over all morphims $\Delta^n \to X$ for all n, is called the *geometric realisation* functor.

Remark. There exists an adjunction $|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$.

Definition 4.1.4. Let X be a simplical set. We say that X is a *Kan complex* if for all non-negative n, all $0 \leq i \leq n$ and a map of simplical sets $f : \Lambda^n_i \to X$, the following diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd} \tag{4.1.1}$$

admits a (possibly not unique) lift $\tilde{f} : \Delta^n \to X$ making this diagram commutative. The diagram above is called the *horn filler* of $\Lambda^n_i \xrightarrow{f} X$.

This at first inscrutable definition is made somewhat clearer by the following proposition:

Proposition 4.1.5. *Take a space* $Y \in \mathrm{Top}$*. Then* $\mathrm{Sing}(Y)$ *is a Kan complex.*

Proof. By the $|-| \dashv \mathrm{Sing}$ adjunction it suffices to show that

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r, "g"] \arrow[d, hook] & Y \\ |\Delta^n| \arrow[ur, dashed, "\tilde{g}"'] & \end{tikzcd}$$

admits a lift $\tilde{g}$. This is equivalently asking the map $Y \to *$ to be a Serre fibration which is the case, so we are done. □

Proposition 4.1.6. *Take a category* $\mathcal{C} \in \mathrm{Cat}$*.*

(i) For all integers $n \geq 2$ *and all* $0 < j < n$ *there is a* ***unique*** *horn filler of all* $\Lambda^n_i \to N(\mathcal{C})$*.*

(ii) $N(\mathcal{C})$ *is a Kan complex if and only if* $\mathcal{C}$ *is a groupoid.*

Proof. TODO □

Definition 4.1.7. A (model for an) ∞-*groupoid* is a Kan complex.

We want an $(\infty, 1)$-category to be a 'weak category enriched in Kan complexes' - for example a simplicially enriched category where mapping spaces are Kan complexes.

Notation. We denote the category of all small categories enriched in sSet by Cat_Δ.

Definition 4.1.8. For some $n \geq 0$ we define a simplically enriched category $\mathbb{C}[\Delta^n]$ as follows:
The objects of $\mathbb{C}[\Delta^n]$ is the set $[n] = \{0, \dots, n\}$. For the morphisms, first consider for all $0 \leq i, j \leq n$ the poset

$$P_{i,j} := \{I \subset [n] \mid i, j \in I, \text{ and } k \in I \text{ if and only if } i \leq k \leq j\}.$$

Then for $i, j \in [n]$ we define the mapping space

$$\mathrm{Map}_{\mathbb{C}[\Delta^n]}(i, j) := N(P_{i,j}).$$

The composition of morphisms is induced by the map $P_{j,k} \times P_{i,j} \to P_{i,k}$ sending $(I, J) \mapsto I \cup J$.

Definition 4.1.9. We define the *simplicial nerve* $\mathcal{N} : \mathrm{Cat}_\Delta \to \mathrm{sSet}, \mathcal{N}(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathbb{C}[\Delta^*], \mathcal{C}) : \Delta^{\mathrm{op}} \to \mathrm{Set}$ where $[n] \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathbb{C}[\Delta^n], \mathcal{C})$.

Proposition 4.1.10. *Let $C \in \mathrm{Cat}_\Delta$ be a simplicially enriched category such that for all $x, y \in \mathrm{Ob}(\mathcal{C})$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex. Then $N(\mathcal{C})$ satsifies the inner horn filling property - i.e its satisfies 4.1.1 for all $n \geq 2$ and $0 < i < n$.*

Proof. [Lur09] Proposition 1.1.5.10. □

Definition 4.1.11. If a simplical set X satisfies the inner horn filling property as above then we call it a *quasi-category*. A quasi-category is a (model for an) ∞*-category*.

The category of simplical sets sSet is cartesian closed. This can be seen as

$$\mathrm{sSet}(\Delta^n, X) \cong X([n]) = X_n$$

by the Yoneda lemma, so for $X, Y \in \mathrm{sSet}$ we can define

$$\left(Y^X\right)_n = \mathrm{sSet}(\Delta^n, Y^X) := \mathrm{sSet}(X \times \Delta^n, Y)$$

which gives us an internal hom $Y^X \in \mathrm{sSet}$. In fact has the structure of a closed monoidal category induced from the cartesian product on Set.

Notation. Define the category Kan as the full subcategory of sSet spanned by Kan complexes. It can be shown that Kan is in fact simplically enriched and so we can define $\mathcal{H}\mathrm{o} := \mathbb{N}(\mathrm{Kan})$ the ∞*-category of homotopy types.*

Now take $\mathcal{C}$ to be some ∞-category.

Notation. We call elements of $\mathcal{C}_0$ *objects* of $\mathcal{C}$. Similarly, elements of $\mathcal{C}_1$ are called *morphisms.* More generally, elements of $\mathcal{C}_2$ are called *n-morphisms* or *n-simplices.* Since $\mathcal{C}$ is a simplicial set, given a morphism $f \in \mathcal{C}_1$ (i.e a morphism of the form $\tilde{f} : \Delta^1 \to \mathcal{C}$) we can 'recover' the *domain* $\tilde{f}(0) = d_1(f)$ and the *codomain* $\tilde{f}(1) = d_0(f)$ which we shall call x and y respectively. We can then express f in a more familiar way as $f : x \to y$ (here hiding all of its 'higher' information).

Now take two morphisms of $\mathcal{C}$, $f : x \to y$, $g : y \to z$. We can define a map $\varphi : \Lambda^2_1 \to \mathcal{C}$ which sends $0 \mapsto x$, $1 \mapsto y$, $2 \mapsto z$ and $(0 \to 1) \mapsto f$, $(1 \to 2) \mapsto g$. Graphically this looks like

$$\begin{tikzcd} & z & \\ x \arrow[rr, "f"'] & & y \arrow[ul, "g"'] \end{tikzcd}$$

Then the inner horn lifting condition says that we can find a map $\tilde{\varphi} : \Delta^2 \to \mathcal{C}$ that factors through φ. This say that we can find a (not necessarily unique) morphism $h : x \to z$ in $\mathcal{C}$ and $\sigma \in \mathcal{C}_2$ such that $\tilde{\varphi}$ looks like

$$\begin{tikzcd} & z & \\ x \arrow[ur, "h"] \arrow[rr, "f"'] & {} \arrow[u, phantom, "\sigma"] & y \arrow[ul, "g"'] \end{tikzcd}$$

The choice of σ here is important because it 'witnesses' the 'composition' of f and g to h. We can (with caution) denote $h \coloneqq g \circ f$.

Remark. The choice of h above is 'unique up to homotopy'[9].

Definition 4.1.12. Take $f, g : x \to y$ be morphisms in $\mathcal{C}$. We say that f is *homotopic* to g if there exists a 2-simplex σ and $\epsilon : \Delta^2 \to \mathcal{C}$ of the form

$$\begin{tikzcd} & z & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & {} \arrow[u, phantom, "\sigma"] & y \arrow[ul, "id"'] \end{tikzcd}$$

We denote this as $f \simeq g$.

Proposition 4.1.13. *The relation f is homotopic to g if and only if $f \simeq g$ is an equivalence relation on the set of morphisms in $\mathcal{C}$.*

Remark. Given morphisms $f : x \to y$, $g : y \to z$ and $h : z \to w$, by doing a inner horn filling argument for $n = 3$ we can see that $(h \circ g) \circ f \simeq h \circ (g \circ f)$. In other words, composition of morphisms is associative up to homotopy.

Definition 4.1.14. Given an ∞-category, we can define the associated *homotopy category* of $\mathcal{C}$, denoted $h(\mathcal{C})$, as having the same objects of $\mathcal{C}$ and morphism

$$\mathrm{Hom}_{h(\mathcal{C})}(x, y) = \mathcal{C}_1 / \simeq$$

the set of equivalence classes of morphisms of $\mathcal{C}$ under homotopy equivalence.

Exercise. (i) Given a space $X \in \mathrm{Top}$, show that $h(\mathrm{Sing}(X)) = \Pi_1(X)$ (the fundamental groupoid for X).

[9] In slightly more detail, the space of such choices is 'contractible' i.e homotopic to a point.

(ii) If $\mathcal{D}$ is a 1-category, then $h(N(\mathcal{D}))$ is equivalent to $\mathcal{D}$.

Definition 4.1.15. A morphism f in $\mathcal{C}$ is an *equivalence* is it becomes an isomorphism in $h(\mathcal{C})$.

Proposition 4.1.16. *Take a ∞-category $\mathcal{C}$ and simplical set K, then the mapping simplicial set $\mathrm{Map}_{\mathrm{sSet}}(K, \mathcal{C})$ is an ∞-category.*

Proof. [Lur09] Proposition 1.2.7.3. □

Notation. When $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, will denote the mapping simplicial set $\mathrm{Map}_{\mathrm{sSet}}(\mathcal{C}, \mathcal{D})$ above as $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$.

Definition 4.1.17. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is

(i) *essentially surjective* if the associated functor of 1-categories $hF : h(\mathcal{C}) \to h(\mathcal{D})$ is essentially surjective,

(ii) *fully faithful* if hF is fully faithful, and

(iii) an *equivalence* if it is fully faithful and essentially surjective.

Definition 4.1.18. Let S,S' be simplicial sets. We define the *join* of S and S', denoted $S \star S'$ as follows:
Pick a non-empty finite linearly ordered set I, we set

$$S \star S'(I) := S(I) \sqcup S'(I) \sqcup \left(\coprod_{J \sqcup J' = I} S(J) \times S'(J') \right)$$

where J, J' are taken over non-empty finite linearly ordered sets. In other words, $S \star S'$ is a simplical set such that:

- The objects of $S \star S'$ are the disjoint union of objects in S and S'.
- The morphisms are the disjoint union of morphisms in S and S' as well as a 1-simplex $x \to y$ for each $x \in S$ and $y \in S'$.
- in general
$$(S \star S')_n = S_n \sqcup S'_n \sqcup \coprod_{i+j=n-1} X_i \times Y_j$$

Remark. Note that this construction is in general not symmetric.

Proposition 4.1.19. *For 1-categories $\mathcal{D}, \mathcal{D}'$ we have $N(D \star D') \cong N(D) \star N(D')$ where $D \star D'$ is the usual join in 1-categories.*

Proposition 4.1.20. *Let $\mathcal{C}$, $\mathcal{C}'$ be ∞-categories, then $\mathcal{C} \star \mathcal{C}'$ is an ∞-category.*

Definition 4.1.21. Let $S \in \mathrm{sSet}$. We define the *left cone* over S as $S^{\triangleleft} := \Delta^0 \star S$. Similarly we define the *right cone* over S as $S^{\triangleright} := S \star \Delta^0$.

Definition 4.1.22. Let $p : S' \to S$ be a morphism of simplicial sets. We define the simplicial set

$$S_{p/} : \Delta^{\mathrm{op}} \to \mathrm{Set},\ [n] \mapsto \{f : S' \star \Delta^n \to S \mid f|_{S'} = p\}$$

where the restriction of f to S' is on the n-simplices of $S' \star \Delta^n$ contained solely in S'. $S_{p/}$ is called the *undercategory* of p. Similarly we define

$$S_{/p} : \Delta^{\mathrm{op}} \to \mathrm{Set},\ [n] \mapsto \{g : \Delta^n \star S' \to S \mid g|_{S'} = p\}$$

called the *overcategory* of p.

Proposition 4.1.23. *For all $Y \in \mathrm{sSet}$ we have*

$$\mathrm{sSet}(Y, S_{/p}) \cong \mathrm{sSet}_p(Y \star S', S)$$

where the right hand is the space of all maps of simplicial sets such that their restriction to S' is exactly p. A similar thing holds for $S_{p/}$.[10]

Proof. First note that the equation holds whenever the have $Y = \Delta^n$. Moreover we have $Y = \mathrm{colim}_{\Delta^n \to Y} \Delta^n$, hence

$$\mathrm{sSet}(\mathrm{colim}_{\Delta^n \to Y} \Delta^n, S_{/p}) \cong \lim_{\Delta \to Y} \mathrm{sSet}(\Delta^n, S_{/p}) \cong \lim_{\Delta \to Y} \mathrm{sSet}(\Delta^n \star S', S) \cong \mathrm{sSet}(Y \star S', S)$$

□

Proposition 4.1.24. *Let $p : S \to \mathcal{C}$ be a morphism between $S \in \mathrm{sSet}$ and $\mathcal{C}$ and ∞-category. Then $\mathcal{C}_{p/}$ and $\mathcal{C}_{p/}$ are ∞-categories.*

4.2 The theory of ∞-operads

Remark. Let Fin, Fin_* and $\mathrm{Fin}^{\simeq}$ denote the ∞-categories $N(\mathrm{Fin})$, $N(\mathrm{Fin}_*)$ and $N(\mathrm{Fin}^{\simeq})$ respectively. Note that $\mathrm{Fin}^{\simeq}$ is the maximal ∞-groupoid of Fin.

Definition 4.2.1. Let $p : \mathcal{C} \to \mathrm{Fin}_*$ be a functor of ∞-categories.

(i) Define $\mathcal{C}_{\langle n \rangle} \subset \mathcal{C}$ (is this a full subcat?) as the pullback

$$\begin{tikzcd} \mathcal{C}_{\langle n \rangle} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ \Delta^0 \arrow[r] & Fin_* \end{tikzcd}$$

(ii) For a morphism $\alpha : \langle m \rangle \to \langle \ell \rangle$ in Fin_*, $X \in \mathcal{C}_{\langle m \rangle}$ and $Y \in \mathcal{C}_{\langle \ell \rangle}$, define the ∞-subgroupoid $\mathrm{Map}^{\alpha}_{\mathcal{C}}(X, Y)$ as the union of connected components of $\mathrm{Map}_{\mathcal{C}}(X, Y)$ where $f \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ if and only if $p(f) \simeq \alpha$.

[10] In fact by the universal property of simplicial sets, these equations can be taken to be the definition of the over/under category of p.

(iii) p-cocartesian morphism??

Definition 4.2.2. An ∞-*operad* is an ∞-category $\mathcal{O}^{\otimes}$ together with a functor $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ of ∞-categories satisfying the following conditions:

(i) For every inert morphism i in Fin_*, there exists a p-cocartesian morphism $\bar{i}$ in $\mathcal{O}^{\otimes}$ such that $p(\bar{i}) \simeq i$.

(ii) For a tuple (α, C, D) where $\alpha : \langle m \rangle \to \langle \ell \rangle$ is a morphism in Fin_*, $\mathcal{C} \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and $D \in \mathcal{O}^{\otimes}_{\langle \ell \rangle}$, and let $(\bar{\rho}_i : D \to D)_{i=1}^{\ell}$ be a sequence of p-cocartesian lifs of rho_i. Then there exists an equivalence

$$\mathrm{Map}^{\alpha}_{\mathcal{O}^{\otimes}}(C, D) \xrightarrow{\simeq} \prod_{i=1}^{\ell} \mathrm{Map}^{\rho_i \circ \alpha}_{\mathcal{O}^{\otimes}}(C, D_i)$$

of ∞-groupoids induced by composition.

(iii) For every $m \in \mathbb{N}$, $m \geq 1$ and m-tuple of objects $(C_1, \ldots, C_m) \in \left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{\times m}$ then there exists an objects $X \in \mathcal{O}^{\otimes}_{\langle m \rangle}$ and a p-cocartesian lift $\overline{\rho_i} : C \to C_i$ of ρ_i for every $1 \leq i \leq n$.

We call $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ the ∞-groupoid of *colours* of $\mathcal{O}^{\otimes}$.

Remark. This gives us a natural generalisation of the ordinary categories of operators. In particular, for every $m \geq 1$ we can get an equivalence $\mathcal{O}^{\otimes}_{\langle m \rangle} \simeq \left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{m}$ from 4.2.2.iii which is essentially surjective and 4.2.2.ii show that this functor is fully faithfull.

Definition 4.2.3. A *one-colour* ∞-*operads* is an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ along with an essentially surjective functor $\Delta^0 \to \mathcal{O}^{\otimes}_{\langle 1 \rangle}$.

Example. Let $\mathcal{O}$ be an operad with values in Set and its associated category of operads $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$. By taking the nerve, $N(p) : N(\mathcal{O}^{\otimes}) \to N(\mathrm{Fin}_*) = \mathrm{Fin}_*$ exhibits $N(\mathcal{O}^{\otimes})$ as an ∞-operad. As a consequence we have the following ∞-operads

(i) The *trivial* ∞-*operad* $\mathrm{Triv}^{\otimes}$ with structure map $p : \mathrm{Triv}^{\otimes} \to \mathrm{Fin}*$ coming from the canonical inclusion $\mathrm{Triv}^{\otimes} \hookrightarrow \mathrm{Fin}_*$.

(ii) The *unital* ∞-*operad* $\mathcal{E}_0^{\otimes}$ with $p : \mathcal{E}_0^{\otimes} \mathrm{Fin}_*$ induced by the canonical inclusion $\mathrm{E}_0^{\otimes} \hookrightarrow \mathrm{Fin}_*$.

(iii) The *associative* ∞-*operad* Ass.

(iv) The *commutative* ∞-*operad* Com.

(v) The *left module* ∞-*operad* $\mathcal{LM}$.

Definition 4.2.4. A *simplical operad* is an operad with values in $(\mathrm{sSet}, \times, \mathrm{pt})$ i.e sSet with the cartesian symmetric monoidal structure.

Remark. Let $\mathcal{O}$ be a simplical operad. Then the category $\mathcal{O}^{\otimes}$ is simplicially enriched (write how).

Proposition 4.2.5. *Assume $\mathcal{O}$ is a simplical operad such that $\mathcal{O}\left((c_i)_{i=1}^r; c\right)$ is a Kan complex for each $((c_i)_{i=1}^r; c)$ then the simplical nerve $N(\mathcal{O}^{\otimes})$ of the category $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ of operators together with the induce functor $N(p)$ is an ∞-operad.*

Definition 4.2.6. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. We say a morphism f in $\mathcal{O}^{\otimes}$ is *inert* if $p(f)$ is inert and f is p-cocartesian

Definition 4.2.7. Let $q_{\mathcal{O}} : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ and $p_{\mathcal{P}} : \mathcal{P}^{\otimes} \to \mathrm{Fin}_*$ be ∞-operads. A *morphism of ∞-operads* from $\mathcal{O}^{\otimes}$ to $\mathcal{P}^{\otimes}$ is a morphism of simplicial sets $f : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ such that

(i) the diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "q_{\mathcal{O}}"'] & & \mathcal{P}^{\otimes} \arrow[dl, "q_{\mathcal{P}}"] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

commutes, and

(ii) f preserves inert morphisms.

Definition 4.2.8. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of ∞-categories. We say that q exhibits $\mathcal{C}^{\otimes}$ as an $\mathcal{O}$-monoidal ∞-category if the composition $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

Remark. For any $X \in \mathcal{O}^{\otimes}$, denote $\mathcal{C}_X^{\otimes}$ as the pullback

$$\begin{tikzcd} \mathcal{C}_X^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\otimes} \arrow[d] \\ \Delta^0 \arrow[r] & \mathcal{O}^{\otimes} \end{tikzcd}$$

of fibres over X.

Proposition 4.2.9. *Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of ∞-categories. Then q is an $\mathcal{O}$-monoidal category if and only if every sequence $(\overline{\rho_i} : C \to C_i)$ of p-cocartesian lifts $\overline{\rho_i}$ of ρ_i induces an equivalence*

$$\mathcal{C}_C^{\otimes} \xrightarrow{\simeq} \prod_{i=1}^m \mathcal{C}_{C_i}^{\otimes}$$

of ∞-categories for each $m \geq 1$.

Definition 4.2.10. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category. The *underlying $\mathcal{O}$-monoidal ∞-category* $\mathcal{C}$ of $\mathcal{C}^{\otimes}$ is defined as the pullback

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\otimes} \arrow[d] \\ \mathcal{O}_{\langle 1 \rangle}^{\otimes} \arrow[r] & \mathcal{O}^{\otimes} \end{tikzcd}$$

Definition 4.2.11. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. For $\mathcal{O}$-monoidal ∞-categories $q_{\mathcal{C}} : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q_{\mathcal{D}} : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$, an $\mathcal{O}$-monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is an ∞-operad map from $f : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ which carries $q_{\mathcal{C}}$-cocartesian morphisms to $q_{\mathcal{D}}$-cocartesian morphisms.

Definition 4.2.12. Let $\mathcal{D}$ be an ∞-category that admits finite products and $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ an ∞-operad. An $\mathcal{O}$-monoid in $\mathcal{D}$ is a functor $F : \mathcal{O}^{\otimes} \to \mathcal{D}$ such that every sequence $(\overline{\rho_i} : C \to C_i)_{i=1}^m$ of p-cocartesian lifts $\overline{\rho_i}$ of ρ_i induces an equivalence

$$F(C) \xrightarrow{\simeq} \prod_{i=1}^{m} F(C_i)$$

in $\mathcal{D}$, for every $m \geq 1$.

Remark. This is the generalisation of the 'Segal condition' for a commutative topological monoid to an arbitrary ∞-operad.
Straightening unstraightening?

(Recall?) for $n \in \mathbb{N}$ the morphism $f_r : \langle r \rangle \to \langle 1 \rangle$ where $f_r^{-1}(\mathrm{pt}) = \{\mathrm{pt}\}$.

Definition 4.2.13. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad. For $r \in \mathbb{N}$, an *r-ary operation* $f_r(\mathcal{O})$ consists of the following data:

(i) A colour C and sequence $(C_i)_{1 \leq i \leq r}$ of colours of $\mathcal{O}^{\otimes}$.

(ii) An object $C_{\underline{r}}$ of $\mathcal{O}^{\otimes}_{\langle r \rangle}$ corresponding to $(C_i)_{1 \leq i \leq r}$ under the equivalence $\mathcal{O}^{\otimes}_{\langle r \rangle} \xrightarrow{\simeq} \left(\mathcal{O}^{\otimes}_{\langle 1 \rangle}\right)^{\times r}$.

(iii) A morphism $f_r(\mathcal{O}) : C_{\underline{r}} \to C$ such that $p(f_r(\mathcal{O})) \simeq f_r$.

Remark. Interpretation: an $\mathcal{O}$-monoid object in $\mathcal{E}$ consists of the following data:

(i) For each colour $C \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ an object $X_C \coloneqq F(C)$ in $\mathcal{E}$.

(ii) For each r-ary operation $f_r(\mathcal{O})$ an '$\mathcal{O}$-multiplication'

$$(f_r)_* : X_{C_1} \times \ldots X_{C_r} \simeq F(C_{\underline{r}}) \to F(C) = X_C$$

induced by the morphism f_r.

(iii) Suitable compatibilities among the $\mathcal{O}$-multiplication maps (up to homotopy) described by the evaluations of morphisms of $\mathcal{O}^{\otimes}$ under F.

Example. Consider the ∞-operad Com. A Com-monoid is an object $X = F(\langle 1 \rangle)$ of $\mathcal{E}$ together with a multiplication $X \times X \xrightarrow{\simeq} F(\langle 2 \rangle) \xrightarrow{f_*} F(\langle 1 \rangle) = X$ induced by the morphism $f : \langle 2 \rangle \to \langle 1 \rangle$, and a unit map $F(\{\mathrm{pt}\}) \to X$ where the multiplication is commutative and unital (up to homotopy). In other words this is an ∞-categorical version of a commutative monoid in $\mathcal{E}$.

Definition 4.2.14. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category and $f : \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ an ∞-operad map. A *$\mathcal{P}$-algebra in $\mathcal{C}$* is a map $\alpha : \mathcal{P}^{\otimes}\mathcal{C}^{\otimes}$ of ∞-operads such that $q \circ \alpha \simeq f$. The ∞-category $\mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$ is the full ∞-subcategory of $\mathcal{F}\mathrm{un}_{/\mathcal{O}^{\otimes}}(\mathcal{P}^{\otimes}, \mathcal{C}^{\otimes})$ (the ∞-category of functors over $\mathcal{O}^{\otimes}$) spanned by $\mathcal{P}$-algebras in $\mathcal{C}$, called the *∞-category of $\mathcal{P}$-algebras.*

Remark.

- If $f = \mathrm{id}_{\mathcal{O}^{\otimes}}$ then we write $\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}) \coloneqq \mathcal{A}\mathrm{lg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$.
- If $\mathcal{O}^{\otimes}$ is Com and f is the structure map of $\mathcal{P}^{\otimes}$, then we write $\mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C}) \coloneqq \mathcal{A}\mathrm{lg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$.

Remark. Let $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal ∞-category. This is equivalent to q being an $\mathcal{O}$-monoid in Cat_{∞}. Hence the $\mathcal{O}$-multiplications on q are functors $\otimes_{\mathcal{C}} : \mathcal{C}_{C_1} \times \dots \mathcal{C}_{C_r} \to \mathcal{C}_C$ for every r-tuple $(C_i)_{1 \le i \le r}$ of colours in $\mathcal{O}^{\otimes}$ and colour C of $\mathcal{O}$.
An $\mathcal{O}$-algebra in $\mathcal{C}$ is an ∞-operad map $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ along with a commutative diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "A"] \arrow[dr, "\mathrm{id}"'] & & \mathcal{C}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

which we can consider as the following data:

- For every colour $C \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$, an object $X_C \coloneqq A(C) \in \mathcal{C}_C$.
- For ever r-ary operation $f_r(\mathcal{O}) : C_{\underline{r}} \to C$, a morphism

  $$m_r : X_{C_1} \otimes_{\mathcal{C}} \dots \otimes_{\mathcal{C}} X_{C_r} \to X_C$$

  obtained by setting $X_{\underline{r}} = A(C_{\underline{r}})$ and seeing there are two morphisms lifting $f_r(\mathcal{O})$,

  $$\overline{f_r(\mathcal{O})} : X_{\underline{r}} \to X_{C_1} \otimes \dots X_{C_r} \text{ and } A(f_r(\mathcal{O})) : X_{\underline{r}} \to X_C$$

  where $\overline{f_r(\mathcal{O})}$ the q-cocartesian lift. Then by the universal property of q-cocartesian morphisms this induces m_r.
- Compatibility of the 'multiplications' m_r (up to homotopy), obtained from the operations of $\mathcal{O}^{\otimes}$ and the universal property if the cocartesian fibrations q.

  Definition 4.2.15. A *monoidal* (respectively *symmetric monoidal*) *∞-category* is an Ass-monoidal (respectively Com-monoidal) ∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$.

(i) By the previous remark the underlying ∞-category of a monoidal (respectively symmetric monoidal) ∞-category is equipped with a *monoidal* (respectively *symmetric monoidal*) *product* $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a *monoidal* (respectively *symmetric monoidal*) *unit* $1_{\mathcal{C}} \in \mathcal{C}$ where $\otimes_{\mathcal{C}}$ is (up to homotopy) associative, commutative and unital.

(ii) A *monoidal* (respectively *symmetric monoidal*) *functor* is a Ass-monoidal (respectively Com-monoidal) functor between monoidal (respectively symmetric monoidal) ∞-categories.

(iii) A *lax monoidal* (respectively *lax symmetric monoidal*) *functor* between monoidal (respectively symmetric monoidal) ∞-categories is an ∞-operad map between the underlying ∞-operads of two monoidal (respectively symmetric monoidal) ∞-categories.

Definition 4.2.16. An *associative algebra* in a monoidal ∞-category $\mathcal{C}$ is an element of the ∞-category $\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C})$.

Example. Take an ∞-category $\mathcal{C}$, then there exists a monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})^{\otimes} \to \mathrm{Ass}^{\otimes}$ where the underlying ∞-category is the functor ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$. Sinces as a quasicategory $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C}) := \mathrm{Map}_{\mathrm{sSet}}(\mathcal{C},\mathcal{C})$ we have

$$\circ : \mathrm{Map}_{\mathrm{sSet}}(\mathcal{C},\mathcal{C}) \times \mathrm{Map}_{\mathrm{sSet}}(\mathcal{C},\mathcal{C}) \to \mathrm{Map}_{\mathrm{sSet}}(\mathcal{C},\mathcal{C})$$

given by composition of morphisms of simplical sets which gives $\mathrm{Map}_{\mathrm{sSet}}(\mathcal{C},\mathcal{C})$ a strict monoidal structure if and only if we have $\mathrm{Ass}^{\otimes} \xrightarrow{F} \mathrm{sSet}$ such that F satsifies the Segal condition. Then ....

Definition 4.2.17. A *monad* of an ∞-category $\mathcal{C}$ is an associative algebra object in $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{C})$.

Recall the left module ∞-operad $\mathrm{LM}^{\otimes}$ and the inclusion $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$ of ∞-operads. Let $\mathcal{C}^{\otimes} \to \mathrm{LM}^{\otimes}$ be a LM-monoidal ∞-category.

Definition 4.2.18. An object in $\mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C})$ is called a *left module object* in $\mathcal{C}$.

Such a left module object is a pair $(A, M) \in \mathcal{C}_a \times \mathcal{C}_m$ together with morphisms $A \otimes_a A \to A$ and $A \otimes M \to M$ satisfying certain compatibility conditions. Moreover A is an associative algebra object of $\mathcal{C}_a$ and M is exhibited as a left module over A.

Proposition 4.2.19. *The map* $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$ *induces a functor*

$$\mathrm{forg}_m : \mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{\mathrm{Ass}/\mathrm{LM}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{/Ass}(\mathcal{C}_a)$$

Remark. forg_m 'sends' (A, M) to A.

Definition 4.2.20. Take an associative algebra $A \in \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C})$.

(i) The ∞-category $\mathcal{LM}\mathrm{od}_A(\mathcal{C}_m)$ is defined as the pullback

$$\begin{tikzcd}
\mathcal{LM}\mathrm{od}_A(\mathcal{C}_m) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C}) \arrow[d, "\mathrm{forg}_m"] \\
\Delta^0 \arrow[r, "A"'] & \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)
\end{tikzcd}$$

(ii) There exists a forgetful functor $\mathrm{forg}_A : \mathcal{LM}\mathrm{od}_A(\mathcal{C}_m) \to \mathcal{C}_m$ defined as the composition

$$\begin{array}{ccccccc}
\mathcal{LM}\mathrm{od}_A(\mathcal{C}_m) & \longrightarrow & \mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C}) & \longrightarrow & \mathcal{F}\mathrm{un}(\mathrm{LM}^{\otimes}_{\langle 1 \rangle}, \mathcal{C}^{\otimes}) & \xrightarrow{\mathrm{ev}_m} & \mathcal{C}_m \\
 & & (\mathrm{LM}^{\otimes} \to \mathcal{C}) & \xmapsto{\text{restrict}} & F & \longmapsto & F(m)
\end{array}$$

Proposition 4.2.21. *The functor* forg_m *is a cartesian fibration.*

Proof. [Lur17] Corollary 4.2.3.2. $\square$

Proposition 4.2.22. *Given a morphism* $f : A \to B$ *in* $\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)$*, there is an induced commutative diagram*

$$\begin{tikzcd} \mathcal{LM}\mathrm{od}_B(\mathcal{C}_m) \arrow[rr, "f^*"] \arrow[dr, "\mathrm{forg}_B"'] & & \mathcal{LM}\mathrm{od}_A(\mathcal{C}_m) \arrow[dl, "\mathrm{forg}_A"] \\ & \mathcal{C}_m & \end{tikzcd}$$

Proof. We have that forg_m is a cartesian fibration. By straightening, this corresponds to a functor $\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)^{\mathrm{op}} \to \mathrm{Cat}_\infty$, that sends $A \mapsto \mathcal{LM}\mathrm{od}_A(\mathcal{C})$ and $f \mapsto f^*$. Let $g : (B, M_1) \to (A, M_2)$ be a morphism in $\mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C})$. Then g is a forg_m-cartesian morphism if and only if the induced $M_1 \to M_2$ is an equivalence. $\square$

Example. Let $\mathcal{C}$ be an ∞-category. There exists an LM-monoidal ∞-category $\mathcal{M}(\mathcal{C})^\otimes \to \mathrm{LM}^\otimes$ such that $\mathcal{M}(\mathcal{C})_a \simeq \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$ and $\mathcal{M}(\mathcal{C})_m \simeq \mathcal{C}$. This is saying that an object in $\mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{M}(\mathcal{C}))$ is a pair (T, M) where T is a monad on $\mathcal{C}$ and M is a left module over T in $\mathcal{C}$.

Proposition 4.2.23. *Let* $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ *be an adjunction of* ∞*-categories.*

(i) The composition $G \circ F$ *admits the structure of a monad on* $\mathcal{C}$ *as follows: the unit map is just the adjuction unit* $\mathrm{id}_\mathcal{C} \implies G \circ F$ *and the monoidal structure* $(G \circ F) \circ (G \circ F) \to G \circ F$ *is induced by the adjunction co-unit* $F \circ G \implies \mathrm{id}_\mathcal{D}$*.*

(ii) The functor G *admits a factorisation*

$$\begin{tikzcd} \mathcal{C} & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, dashed, "G'"] \\ & \mathcal{LM}\mathrm{od}_{G \circ F}(\mathcal{C}) \arrow[ul, "\mathrm{forg}_{G \circ F}"] & \end{tikzcd}$$

Proof. [Lur17] Proposition 4.7.3.3. $\square$

Definition 4.2.24. An adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is monadic if $G' : \mathcal{D} \to \mathcal{LM}\mathrm{od}_{G \circ F}(\mathcal{C})$ is an equivalence of ∞-categories.

Proposition 4.2.25. *Let* $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ *be an adjunction of* ∞*-categories and let* $T : \mathcal{C} \to \mathcal{C}$ *be a monad on* $\mathcal{C}$ *such that* G *admits a factorisation*

$$G \simeq \mathrm{forg}_T \circ G'_T : D \xrightarrow{G'_T} \mathcal{LM}\mathrm{od}_T(\mathcal{C}) \xrightarrow{\mathrm{forg}_T} \mathcal{C}$$

Then there exists a morphism $h : T \to G \circ F$ of monads, unique up to contractible choice such that

$$\begin{tikzcd}
\mathcal{C} & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "G'"] \arrow[ddl, bend left, "G'_T"] \\
& \mathcal{LM}\mathrm{od}_{G\circ F}(\mathcal{C}) \arrow[ul, "\mathrm{forg}_{G\circ F}"'] \arrow[d, dashed, "h*"] & \\
& \mathcal{LM}\mathrm{od}_T(\mathcal{C}) \arrow[uul, bend left, "\mathrm{forg}_T"] &
\end{tikzcd}$$

commutes.

Proof. Use the universal property of $G \circ F$ being the endomorphism object of G - see [Lur17] §4.7.1 and Lemma 4.7.3.1. ☐

(SMALLNESS?)

Definition 4.2.26. An ∞-category $\mathcal{C}$ is called:

(i) *essentially small* if $\mathcal{C}$ is equivalent to a small ∞-category,

(ii) *locally small* if for all $X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is essentially small,

(iii) *κ-accessible* for a given regular cardinal κ if $\mathcal{C}$

  (a) is locally small,

  (b) admits κ-filtered colimits[11],

  (c) contains an essentially small ∞-subcategory $\mathcal{C}'$ such that every object $X' \in \mathcal{C}'$ is κ-compact[12], and

  (d) every object of $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{C}'$,

(iv) *acessible* if it is κ-accessible for some regular cardinal κ,

(v) *presentable* if it is accessible and cocomplete.

Proposition 4.2.27. *A presentable ∞-category $\mathcal{C}$ is also complete.*

Proof. [Lur09] Corollary 5.5.2.4. ☐

Theorem 4.2.28. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories, then*

(i) the functor F has a right adjoint if and only if it preserves small colimits, and

(ii) the functor F has a left adjoint if and only if it is accessible and preserves small limits.

[11] a colimit over a κ-filtered ∞-category $\mathcal{D}$ - i.e a ∞-category $\mathcal{D}$ such that for all κ-small $K \in \mathrm{sSet}$ and every morphism $p : K \to \mathcal{D}$ there is a morphism $\hat{p} : K^{\triangleright} \to \mathcal{D}$ extending p.

[12] the functor $\mathrm{Map}_{\mathcal{C}}(X', -)$ preserves κ-filtered colimits

Aside on size things - 'Set theory for Category theory'

Definition 4.2.29. Let $\widehat{\mathrm{CAT}}_\infty$ be the ∞-category of (all) ∞-categories. Define the ∞-subcategories $\mathrm{Pr}^L, \mathrm{Pr}^R \subset \widehat{\mathrm{CAT}}_\infty$ where

- the objects in both are presentable ∞-categories,
- a morphism $F : \mathcal{C} \to \mathcal{D}$ is in Pr^L if F preserves all small colimits, and
- $G : \mathcal{C} \to \mathcal{D}$ is in Pr^R is F preserves all small limits and κ-filtered colimits for some regular cardinal κ.

Remark. There is an equivalence of ∞ categories $\left(\mathrm{Pr}^L\right)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Pr}^R$ defined by $\mathcal{C} \mapsto \mathcal{C}$ on objects and $F \mapsto G$ where G is a right adjoint to F. (Make precise?)

Definition 4.2.30. Let $p : \mathcal{O}^\otimes \to \mathrm{Fin}_*$ be an ∞-operad, $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ an $\mathcal{O}$-monoidal ∞-category and $\mathcal{K}$ a set of simplical sets. We say that $\mathcal{C}$ is *compatible with $\mathcal{K}$-indexed colimits* if for every $K \in \mathcal{K}$

- K-indexed colimits in $\mathcal{C}^\otimes_{\langle m \rangle}$ exists for all $m \geq 1$, and
- the $\mathcal{O}$-monoidal tensor product $\otimes_\mathcal{C}$ preserves K-indexed colimits in each variable.

Definition 4.2.31. An ∞-category $\mathcal{C}^\otimes$ together with a functor $q : \mathcal{C}^\otimes \to \mathrm{Com}^\otimes$ is a *presentable symmetric monoidal ∞-category*[13] if

(i) q exhibits $\mathcal{C}^\otimes$ as a symmetric monoidal category,

(ii) $\mathcal{C}^\otimes$ is compatible with small colimits, and

(iii) the underlying ∞-category is a presentable ∞-category.

Proposition 4.2.32. *The ∞-category of presentable ∞-categories Pr^L can be endowed with a symmetric monoidal structure $\mathrm{Pr}^\otimes \to \mathrm{Com}^\otimes$ where $\mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Pr}^L)$ is the ∞-category of presentable symmetric monoidal ∞-categories.*

Proof. TODO □

Remark. Take $\mathcal{C}, \mathcal{D} \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Pr}^L)$, then denote $\mathcal{F}\mathrm{un}^\otimes_{\mathrm{Pr}^L}(\mathcal{C}, \mathcal{D}) := \mathcal{M}\mathrm{or}_{\mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Pr}^L)}(\mathcal{C}, \mathcal{D})$ the ∞-category of small colimit preserving symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

For the remainder of this section we shall assume that $\mathcal{C}$ is a presentable symmetric monoidal ∞-category.

Definition 4.2.33. Define the ∞-category of symmetric sequences in $\mathcal{C}$, $\mathrm{SymSeq}(\mathcal{C})$, as the functor ∞-category $\mathcal{F}\mathrm{un}(\mathrm{Fin}^\simeq, \mathcal{C})$. For $F \in \mathrm{SymSeq}(\mathcal{C})$ and $\underline{r} = \{1, \dots, r\} \in \mathrm{Fin}^\simeq$ denote $F(\underline{r})$ by $F(r)$.

Example. Let $X \in \mathcal{C}$.

[13] Also called a *psmc*

- The symmetric sequence $X^{\mathfrak{S}}$ in $\mathcal{C}$ has $X^{\mathfrak{S}}(1) := X$ and $X^{\mathfrak{S}}(r)$ is the inital object of $\mathcal{C}$ for all $r \neq 1$.
- The symmetric sequence $\underline{X}$ in $\mathcal{C}$ has $\underline{X}(0) := X$ and $\underline{X}(r)$ is the initial object of $\mathcal{C}$ for all $r \neq 0$.

Remark (Construction). We now construct a monoidal structure on $\mathrm{SymSeq}(\mathcal{C})$, with the *composition product* $\circledcirc : \mathrm{SymSeq}(\mathcal{C}) \times \mathrm{SymSeq}(\mathcal{C}) \to \mathrm{SymSeq}(\mathcal{C})$. We will need that:

(i) The ∞-category $\mathrm{Fin}^{\simeq}$ has a symmetric monoidal structure coming from the coproduct in sets[14]

(ii) The homotopy category $\mathcal{H}\mathrm{o}$ is the free presentable ∞-category generated by a point under small colimits.

(iii) $\mathrm{SymSeq}(\mathcal{H}\mathrm{o})$ admits a symmetric monoidal structure by Day convolution (see §6.3). With this structure $\mathrm{SymSeq}(\mathcal{H}\mathrm{o})$ is the free presentable symmetric monoidal ∞-category generated by the unit symmetric sequence $1_{\mathcal{H}}\mathrm{o}^{\mathfrak{S}}$, with monoidal unit the symmetric sequence of the point, $\underline{pt}$.

Finally the observation

Proposition 4.2.34. *Let* $\mathcal{D}$ *be an* ∞*-category,* $\mathrm{PSh}(()\mathcal{D}) := \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \mathcal{H}\mathrm{o})$ *the* ∞*-category of presheaves on* $\mathcal{D}$*. Then*

$$\mathcal{F}\mathrm{un}^{L}(PSh(\mathcal{D}), \mathcal{E}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}(\mathcal{D}, \mathcal{E})$$

is an equivalence of ∞*-categories.*

Proof. [Lur09] Theorem 5.1.5.6. □

Precisely, the ∞-category $\mathrm{PSh}(\mathcal{C})$ of presheaves on a symmetric monoidal category $\mathcal{C}$ can be equipped with the Day convolution and satisfies the following universal property:

$$\mathcal{F}\mathrm{un}^{L,\otimes}(\mathrm{PSh}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{\otimes}(\mathcal{C}, \mathcal{D}) \quad \text{for all } \mathcal{D} \text{ symmetric monoidal categories admiting small colimits}$$

Given a presentable symmetric monoidal ∞-category, we can endow the ∞-category $\mathrm{SymSeq}(\mathcal{C})$ with the structure of a presentable symmetric monoidal category via the Day convolution (now denoted $\circledcirc$), where evaluation on the generator induces

$$\begin{tikzcd}
\mathrm{Fun}_{\mathrm{Pr}^L}(\mathcal{H}\mathrm{o}, \mathcal{C}) \arrow[r, "\mathrm{ev}_{\mathrm{pt}}"] \arrow[d] & \mathcal{C} \arrow[d, "\underline{(-)}"] \\
\mathrm{Fun}_{\mathrm{Pr}^L}(\mathrm{SymSeq}(\mathcal{H}\mathrm{o}), \mathrm{SymSeq}(\mathcal{C})) \arrow[r, "\mathrm{ev}_{\underline{\mathrm{pt}}}"'] & \mathrm{SymSeq}(\mathcal{C})
\end{tikzcd}$$

In the same vein as $\mathrm{SymSeq}(\mathcal{H}\mathrm{o})$, we claim that $\mathrm{SymSeq}(\mathcal{C})$ with the Day convolution $\circledast$ is the free $\mathcal{C}$-linear presentable symmetric monoidal ∞-category generated by $1_{\mathcal{C}}^{\mathfrak{S}}$ i.e

[14] Given this structure, $\mathrm{Fin}^{\simeq}$ is the free symmetric monoidal ∞-category generated by the one-point set.

- $\mathrm{SymSeq}(\mathcal{C}) \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}))$
- For all $\mathcal{D} \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}}))$, we have

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathrm{Pr}^{\mathrm{L}},\mathcal{C}-\mathrm{lin}}(\mathrm{SymSeq}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathcal{D}$$

given by $ev : F \mapsto F(1^{\mathfrak{S}}_{\mathcal{C}})$.

Remark. We give/recall some facts about module categories: Let $\mathcal{D}$ be a symmetric monoidal ∞-category and $A \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathcal{D})$.

(i) One can define a symmetric monoidal ∞-category $\mathrm{Mod}_A(\mathcal{D})^{\otimes} \to \mathrm{Com}^{\otimes}$ whose underlying ∞-category $\mathrm{Mod}_A(\mathcal{D})$ of modules over A in $\mathcal{D}$ is [Lur17] §3.4.1, the unit of which $1_{\mathrm{Mod}_A}(\mathcal{D}) \simeq A$.

(ii) Let $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞-operad, then [Lur17] Corollary 3.4.1.7. gives

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathrm{Mod}_A(\mathcal{D})) \xrightarrow{\sim} \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{D})_{A/}$$

(iii) Assume $\otimes_{\mathcal{D}}$ preserves geometric realisations in each variable separately, then for $f : A \to B$ in $\mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathcal{D})$ the forgetful functor

$$U : \mathrm{Mod}_B(D) \to \mathrm{Mod}_A(D)$$

admits a symmetric monoidal left adjoint $- \otimes_A B$.

Corollary 4.2.35. *For all $A \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathcal{D})$, the unit map $1_{\mathcal{D}} \to A$ in $\mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathcal{D})$ induces an adjunction*

$$- \otimes_{1_{\mathcal{D}}} B : \mathrm{Mod}_{1_{\mathcal{D}}}(\mathcal{D}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Mod}_A(\mathcal{D}) : U$$

Proof. [Lur17] §2.4.9 □

A sketch of the claim is seen by taking $\mathrm{SymSeq}(\mathcal{H}\mathrm{o}) \in \mathcal{A}\mathrm{lg}_{\mathrm{Com}}(\mathrm{Pr}^{\mathrm{L}})$ and looking at

$$\mathcal{F}\mathrm{un}^{\otimes}_{\mathrm{Pr}^{\mathrm{L}}}(\mathrm{SymSeq}(\mathcal{H}\mathrm{o}), \mathcal{E}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}^{\mathfrak{S}}}} \mathcal{E}$$

TODO ———

Proposition 4.2.36. *(i) $\circledcirc$ induces the monoidal functor*

$$\mathrm{SymSeq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathrm{SymSeq}(\mathcal{C}), \mathrm{SymSeq}(\mathcal{C}))$$

given by $F \mapsto (G \mapsto F \circledcirc G)$.

(ii) For $X \in \mathcal{C}$, we have

$$(F \circledcirc \underline{X})(r) \simeq \begin{cases} \coprod_{n\geq 0} (F(n) \otimes X^{\otimes n})_{\mathfrak{S}_n}, & \text{for } r = 0 \\ \text{initial object of } \mathcal{C}, & \text{else} \end{cases}$$

(iii) Consider $\mathcal{C}$ as a full ∞-subcategory of $\mathrm{SymSeq}(\mathcal{C})$ *via the functor* $\underline{(-)} : \mathcal{C} \to \mathrm{SymSeq}(\mathcal{C})$*. We then have a functor*

$$\mathrm{SymSeq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$$

given by $F \mapsto (X \mapsto F \circledcirc \underline{X})$*. This functor is monoidal.*

Proof. TODO □

Definition 4.2.37. An ∞-*operad with values in* $\mathcal{C}$ is an object $\mathcal{O} \in \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathrm{SymSeq}(\mathcal{C}))$. We denote the ∞-category of ∞-operads with values in $\mathcal{C}$ as $\mathcal{O}\mathrm{pd}(\mathcal{C}) \coloneqq \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathrm{SymSeq}(\mathcal{C}))$. An $\mathcal{O}$-*algebra in* $\mathcal{C}$ is a left module over the associated monad $\mathcal{T}_{\mathcal{O}}$. We denote the ∞-category of $\mathcal{O}$-algebras for some ∞-operad $\mathcal{O}$ as $\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \coloneqq \mathcal{LM}\mathrm{od}_{\mathcal{T}_{\mathcal{O}}}(\mathcal{C})$.

Proposition 4.2.38. *For all morphisms of ∞-operads $f : \mathcal{P} \to \mathcal{O}$, we obtain and adjunction $f_! \dashv f^*$ such that the following diagram commutes*

$$\begin{tikzcd}
f_! : \mathcal{A}\mathrm{lg}_{\mathcal{P}}(\mathcal{C}) \arrow[rr, shift left] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\mathrm{forg}_{\mathcal{P}}"] & & \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) : f^* \arrow[ll, shift left] \arrow[dl, shift left, "\mathrm{forg}_{\mathcal{O}}"] \\
& \mathcal{C} \arrow[ul, shift left, "\mathrm{free}_{\mathcal{P}}"] \arrow[ur, shift left, "\mathrm{free}_{\mathcal{O}}"] \arrow[ul, phantom, "\perp"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

Where the adjunction $\mathrm{free}_{\mathcal{O}} \dashv \mathrm{forg}_{\mathcal{O}}$ *is induced by* $1^{\mathfrak{S}}_{\mathcal{C}} \to \mathcal{O}$*.*

Definition 4.2.39. An *augmentation* of an ∞-operad $\mathcal{O} \in \mathcal{O}\mathrm{pd}(\mathcal{C})$ is a morphism $\mathcal{O} \to 1^{\mathfrak{S}}_{\mathcal{C}} =: \mathrm{Triv}_{\mathcal{C}}$. An *augmented ∞-operad with values in $\mathcal{C}$* is an ∞-operad with an augmentation. We denote the ∞-category of augmented ∞-operads as $\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C})$.

Example. Given an augmented ∞-operad $\mathcal{O}$ with augmentation $\mathcal{E} : \mathcal{O} \to 1^{\mathfrak{S}}_{\mathcal{C}}$, then $\mathcal{E}$ induces an adjunction

$$\mathcal{E}_! \coloneqq \mathrm{indu}_{\mathcal{O}} : \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C} : \mathrm{triv}_{\mathcal{O}}$$

Informally, the right adjoint tells us that every element $X \in \mathcal{C}$ gets an left module structure via the morphism $\mathcal{T}_{\mathcal{O}}(X) \to T_{1^{\mathfrak{S}}_{\mathcal{C}}}(X) \to X$ - 'X has trivial $\mathcal{O}$-multiplication'.
The left adjoint has the property that

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{E}_!(Y), X) \simeq \mathrm{Map}_{\mathcal{A}\mathrm{lg}_{\mathcal{O}}}(Y, \mathrm{triv}_{\mathcal{O}}(X).$$

A morphism $Y \to \mathrm{triv}_{\mathcal{O}}$ must send decomposable elements in Y (i.e 'things in Y obtained by $\mathcal{O}$-multiplication of elements') to zero.

4.3 Operadic Koszul duality

We want to relate the comonad $\mathrm{indu}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}}$ in $\mathcal{C}$ with the ∞-operad $\mathcal{O}$.

Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞ operad, $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category. We can construct a 'canonical'[15] $\mathcal{O}$-monoidal structure on $\mathcal{C}^{\mathrm{op}}$ as follows:
The cocartesian fibration q corresponds to a functor $F : \mathcal{O}^{\otimes} \to \mathrm{CAT}_{\infty}$ satisfying the Segal condition. Composing $(-)^{\mathrm{op}}$ with F gives us $F' \coloneqq (-)^{\mathrm{op}} \circ F : \mathcal{O}^{\otimes} \to \mathrm{CAT}_{\infty}$ which satsifies the Segal condition. Thus F' corresponds to a cocartesian fibration $(q^{\vee})^{\mathrm{op}} : (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathcal{O}^{\otimes}$ exhibiting $\mathcal{C}^{\mathrm{op}}$ as an $\mathcal{O}$-monoidal ∞-category.

Definition 4.3.1. Let $p : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ be an ∞ operad and $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ an $\mathcal{O}$-monoidal ∞-category. An *$\mathcal{O}$-coalgebra object X in $\mathcal{C}$* is an $\mathcal{O}$-algebra object in the opposite $\mathcal{O}$-monoidal category $\mathcal{C}^{\mathrm{op}}$. Denote the ∞-category of $\mathcal{O}$-coalgebras on $\mathcal{C}$ as

$$\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}) \coloneqq (\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$$

Remark. The interpretation of this definition is that $X \in \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C})$ is an object $X \in \mathcal{C}$ together with comultiplication maps $\mathcal{O}(r) \to \mathrm{Map}_{\mathcal{C}}(X, X^{\otimes r})$ which are compatible with each other up to coherent homotopy. This generalises the corresponding 1-categorical notation.

Example. Let $\mathcal{C}^{\otimes} \to \mathrm{LM}^{\otimes}$ be a LM-monoidal ∞-category i.e exhibiting $\mathcal{C}_m$ as left tensored over the monoidal ∞-category $\mathcal{C}_a$. Define the ∞-category of *left comodules* $\mathrm{co}\mathcal{LM}\mathrm{od}(\mathcal{C})$ as the ∞-category $(\mathcal{A}\mathrm{lg}_{/\mathrm{LM}}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$. There then exists a forgetful functor

$$\mathrm{forg}_m : \mathrm{co}\mathcal{LM}\mathrm{od}(\mathcal{C}) \to \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)$$

induced by the inclusion $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$.
Take $B \in \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)$, then we define the ∞-category of *left B-comodules*

$$\mathrm{co}\mathcal{LM}\mathrm{od}_B(\mathcal{C}) \coloneqq \mathrm{co}\mathcal{LM}\mathrm{od}(\mathcal{C}) \times_{\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}_a)} \{B\}$$

Proposition 4.3.2. *In the previous example, assume that*

(i) the ∞-category $\mathcal{C}_a$ is presentable, and

(ii) the functor $B \otimes - : \mathcal{C}_m \to \mathcal{C}_m$ preserves κ-filtered colimits for each uncountable regular cardinal κ such that $\mathcal{C}_m$ is κ-accessible.

Then the ∞-category $\mathrm{co}\mathcal{LM}\mathrm{od}_B(\mathcal{C})$ is presentable.

Proof (Sketch). The ∞-categories $\mathcal{LM}\mathrm{od}(\mathcal{C})$ and $\mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C})$ are presentable by [Pé22] Proposition 2.8 (also see [Lur17] (????)). Then the result follows as presentable ∞-categories are closed under small limits in Pr^L and said limits can be computed in the ∞-category CAT_{∞} (see [Lur09] Proposition 5.5.3.13). □

Now assume that $\mathcal{C}$ is a presentable symmetric monoidal ∞-category.

[15] Meaning we want the monoidal structure to be the same on the objects of $\mathcal{C}^{\mathrm{op}}$ as on the objects of $\mathcal{C}$

Definition 4.3.3. An ∞-*cooperad* with values in $\mathcal{C}$ is a coassociative coalgebra object $\mathrm{SymSeq}(\mathcal{C})$. We denote the ∞-category of ∞-cooperads with values in $\mathcal{C}$ as

$$\mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{C}) \coloneqq \mathrm{co}\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathrm{SymSeq}(\mathcal{C}))$$

A *comonad* in $\mathcal{C}$ is a coassociative colalgebra in the functor ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$. Recall the monoidal functor $\mathrm{SymSeq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C})$, $F \mapsto (\underline{X} \mapsto F \circledcirc \underline{X}$. For a ∞-cooperad $\mathcal{L}$ we have the associated comonad $T_{\mathcal{L}} \coloneqq \mathcal{L} \circledcirc \underline{(-)}$

Remark. Working in the opposite setting, we also obtain the notion of a *comonadic adjunction.* For example, given an adjunction $F \dashv G$, $F \circ G$ becomes a comonad in $\mathcal{D}$ and satisfies a corresponding universal property.

Definition 4.3.4. Let $\mathcal{L}$ e a ∞-cooperad with values in $\mathcal{C}$. A *conilpotent dived power coalgebra over* $\mathcal{L}$ is a left comodule object in $\mathcal{C}$ over the comonad $T_{\mathcal{L}}$. Denote the ∞-category of conilpotent divided powers as

$$\mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \coloneqq \mathrm{co}\mathcal{L}\mathcal{M}\mathrm{od}_{T_{\mathcal{L}}}(\mathcal{C})$$

Remark. A conilpotent divided power $X \in \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$ is the data of an object $X \in \mathcal{C}$ together with a comultiplication map

$$X \to \coprod_{r \geq 0} (\mathcal{L}(r) \otimes X^{\otimes r})_{\mathfrak{S}}$$

that is coassociative and counital up to coherent homotopy.
Recall that for $\mathcal{O} \in \mathcal{O}\mathrm{pd}(\mathcal{C})$ an $\mathcal{O}$-algebra in $\mathcal{C}$ together with structure maps

$$\coprod_{r \geq 0} (\mathcal{O}(r) \otimes X^{\otimes r})_{\mathfrak{S}} \to X$$

satisfying compatibility conditions. Informally, we want a $\mathcal{L}$-coalgebra in $\mathcal{C}$ to looks something like and $\mathcal{L}$-algebra in the opposite category $\mathcal{C}^{\mathrm{op}}$ so an object $Y \in \mathcal{C}$ with structure maps

$$Y \to \prod_{r \geq 0} (\mathcal{L}(r) \otimes Y^{\otimes r})^{\mathfrak{S}}$$

since colimts become limits and arrows are reversed in the opposite category. This is fine in the 1-categorical case, however ∞-categorically, this becomes more involved since the functor

$$\mathrm{SymSeq}(\mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}),\ M \to \prod_{r \geq 0} (M(r) \otimes (-)^{\otimes r})^{\mathfrak{S}}$$

is not oplax monoidal (in particular it does not send ∞-operads to comonads) but it is lax monoidal.

Proposition 4.3.5. *Let* $\mathcal{L} \in \mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{C})$.

(i) There exists a forgetful functor

$$\mathrm{forg}_{\mathcal{L}} : \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \to \mathcal{C}$$

sending a $\mathcal{L}$-coalgebra to its underlying object in $\mathcal{C}$.

(ii) For any morphism $u : \mathcal{L} \to \mathcal{K}$ of ∞-cooperads, there is an induced functor u_* and right adjoint $u^!$

$$u_* : \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{K}}(\mathcal{C}) : u^!$$

such that $\mathrm{forg}_{\mathcal{K}} \circ u_* \simeq \mathrm{forg}_{\mathcal{L}}$ *and u_* preserves all small colimits.*

Definition 4.3.6. A *coaugmented ∞-cooperad* is an ∞-operad $\mathcal{L} \in \mathrm{co}\mathcal{O}\mathrm{pd}(C)$ together with a *coaugmentation* i.e a morphism $1^{\mathfrak{S}}_{\mathcal{C}} \to \mathcal{L}$ of ∞-cooperads. We denote the ∞-category of coaugmented ∞-cooperads as $\mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{coaug}}(\mathcal{C})$.

Remark. Take $\mathcal{L}, \mathcal{K} \in \mathrm{co}\mathcal{O}\mathrm{pd}(\mathcal{C})$, where $\mathcal{K}$ is coagumented.

(i) The counit $\mathcal{L} \to 1^{\mathfrak{S}}_{\mathcal{C}}$ induces the adjunction

$$\mathrm{forg}_{\mathcal{L}} : \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C} : \mathrm{cofree}_{\mathcal{L}}$$

(ii) The coaugmentation $1^{\mathfrak{S}}_{\mathcal{C}} \to \mathcal{K}$ induces the adjunction

$$\mathrm{triv}_{\mathcal{K}} : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{K}}(\mathcal{C}) : \mathrm{prim}_{\mathcal{K}}$$

Proposition 4.3.7. *Let $\mathcal{O} \to 1^{\mathfrak{S}}_{\mathcal{C}} \in \mathcal{O}\mathrm{pd}$.....*

Recall the relative tensor product from 6.4 The ∞-categorical generalisation of this [Lur17] §4.4. From this we get that the ∞-categorical tensor product with the expected property exists. The construction for this is generally more complicated that the ordinary case, where it is given by the coequaliser of the obvious diagram $M \otimes B \otimes N \rightrightarrows M \otimes B$. Instead it is done by taking the geometric realisation of the simplicial bimodule given by

$$\mathrm{Bar}(M,B,N) \coloneqq \; M \otimes B^{\otimes 2} \otimes N \; \begin{smallmatrix} \xrightarrow{\alpha_{B,M}} \\ \xleftarrow{\mu_B} \\ \xrightarrow{\epsilon} \\ \xrightarrow[\alpha_{B,N}]{} \end{smallmatrix} \; M \otimes B \otimes N \; \begin{smallmatrix} \xrightarrow{\alpha_{B,M}} \\ \longleftarrow \\ \xrightarrow[\alpha_{B,N}]{} \end{smallmatrix} \; M \otimes B$$

called the *two sided Bar construction* (see [Lur17] Theorem 4.4.2.8.).

Theorem 4.3.8. *Let $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ be a monoidal ∞-category, and further assume that $\mathcal{C}^{\otimes}$ is compatible with geometric realisation of simplical objects. The the relative tensor product is*

(i) associative, in particular there exist canonical equivalences $(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P)$, and

(ii) unital, in particular there exists canonical equivalences $A \otimes_A M \simeq M \simeq M \otimes_B B$.

Proof. See [Lur17] Proposition 4.4.3.14. and Proposition 4.4.3.16. $\square$

We now give the Bar construction of an augmented associative algebra.
Let us take $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ a monoidal ∞-category and $A \xrightarrow{\mathcal{E}} 1_{\mathcal{C}}^{\mathfrak{S}} \in \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C})$. The augmentation $\mathcal{E}$ induces a forgetful functor

$$\rho : \mathcal{C} \simeq {}_{1_{\mathcal{C}}}\mathrm{BMod}_{1_{\mathcal{C}}} \to {}_A\mathrm{BMod}_A$$

Definition 4.3.9. We say that a morphism $f : A \to \rho(C)$ in ${}_A\mathrm{BMod}_A$ exhibits $\mathcal{C}$ as the *Bar construction on A* if f induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, D) \xrightarrow{\sim} \mathrm{Map}_{{}_A\mathrm{BMod}_A}(A, \rho(D))$$

for all $D \in \mathcal{C}$.

Remark. If the Bar construction on A exists, then it is also unique up to contractable choice.

Example. If ρ admits a left adjoint $F \dashv \rho$ then the Bar construction on A exists and is given by $F(A)$. Assume that the monoidal ∞-category $\mathcal{C}$ is compatible with the geometric realisation of simplicial objects. Then ρ admits a left adjoint F that is given by $F(M) = 1 \otimes_A M \otimes_A 1$: thus $F(A) = 1 \otimes_A A \otimes_A 1$ is equivalent to $1 \otimes_A 1$.

Proposition 4.3.10. *Assume that $\mathcal{C}$ admits geometric realisations of simplical objects. Then $\mathrm{Bar}(A)$ exists and is given by the geometric realisation of the two-sided Bar construction $B(1, A, 1)$. In particular $\mathrm{Bar}(A) \simeq 1 \otimes_A 1$.*

Lemma 4.3.11. *Let $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ be a monoidal ∞-category as before and A an associative algebra object of $\mathcal{C}$. Then there exists a simplical object $X \in {}_A\mathrm{BMod}_A$ such that*

(i) $|X_*|$ exists and $|X_*| \simeq A$ in ${}_A\mathrm{BMod}_A$ and,

(ii) for all $n \geq 0$, $X_n \simeq A \otimes A^{\otimes n} \otimes A \in {}_A\mathrm{BMod}_A$ i.e X_n is the free A-A-bimodule given by $A^{\otimes n}$.

Proof. First assume that $\mathcal{C}$ is compatible with geometric realisations of simplicial objects. Then we have the equivalence $A \simeq A \otimes_A A$ by unitality and we know that $A \otimes_A A \simeq |\mathrm{Bar}(A, A, A)|$.
In general it suffices to have a fully faithful monoidal functor $\mathcal{C} \xrightarrow{\iota} \mathcal{C}'$ such that the monoidal ∞-category is compatible with geometric realisations of simplical objects. Assuming such an ι exists, then the induced $\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}')$ gives us $\iota(A) \in \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C})$ and by the first case

$$\iota(A) \simeq \iota(A) \otimes_{\iota(A)} \iota(A) \simeq |\mathrm{Bar}(\iota(A), \iota(A), \iota(A))|$$

Note that $\mathrm{Bar}(\iota(A), \iota(A), \iota(A)) : \Delta^{\mathrm{op}} \to {}_{\iota(A)}\mathrm{BMod}_{\iota(A)}$ factors through ${}_A\mathrm{BMod}_A \hookrightarrow {}_{\iota(A)}\mathrm{BMod}_{\iota(A)}$ hence

$$\mathrm{Bar}(\iota(A), \iota(A), \iota(A))_n \simeq \iota(A) \otimes \iota(A)^{\otimes n} \otimes \iota(A)$$

To see that such an ι exists we can take the Yoneda embedding $\mathcal{C} \xrightarrow{j} \mathrm{PSh}(\mathcal{C})$ - we can do this by [Lur17] Corollary 4.8.1.12 which states that $\mathrm{PSh}(\mathcal{C})$ admits that Day convolution monoidal structure which is compatible with small limits and such that j is monoidal. $\square$

Proof of Proposition 4.3.10. We say that an element $M \in {}_A\mathrm{BMod}_A$ is *good* if the functor $F_M : \mathcal{C} \to \mathcal{H}\mathrm{o}$, $\mathcal{C} \mapsto \mathrm{Map}_{{}_A\mathrm{BMod}_A}(M, \rho(\mathcal{C}))$ is corepresentable - i.e there exists an object $X_M \in \mathcal{C}$ such that $F_M \simeq \mathrm{Map}_{\mathcal{C}}(X_M, -)$. We see immediately see that every free bimodule $M = A \otimes M_0 \otimes A$ for some $M_0 \in \mathcal{C}$ is good: it is corepresentable by M_0. Let $D : \Delta^{\mathrm{op}} \to {}_A\mathrm{BMod}_A(\mathcal{C})$ be a simplicial diagram of good objects in ${}_A\mathrm{BMod}_A$. Then $\mathrm{colim}_{\Delta^{\mathrm{op}}} D$ is also good:

$$\mathrm{Map}_{{}_A\mathrm{BMod}_A}(\mathrm{colim}_{\Delta^{\mathrm{op}}} D, \rho(\mathcal{C})) \simeq \lim_{\Delta^{\mathrm{op}}} \mathrm{Map}_{{}_A\mathrm{BMod}_A}(D_n, \rho(\mathcal{C})) \simeq \lim_{\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(X_{D_n}, \mathcal{C}) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_{\Delta^{\mathrm{op}}} X_{D_n}, \mathcal{C})$$

Thus the map $\mathrm{Map}_{{}_A\mathrm{BMod}_A}(\mathrm{colim}_{\Delta^{\mathrm{op}}} D, \rho(-))$ is corepresented by $\mathrm{colim}_{\Delta^{\mathrm{op}}} X_{D_n}$.
By the lemma $A \simeq |X_*|$ and X_* is a free A-A-bimodule, so A is good. Then $\mathrm{Bar}(A)$ exists, given by the object in $\mathcal{C}$ corepresenting the functor F_A. $\square$

Theorem 4.3.12. *Assume that $\mathcal{C}$ admits geometric realisations. Then the assignment $A \mapsto \mathrm{Bar}(A)$ satisfies the following properties:*

(i) $\mathrm{Bar}(A)$ *admits the structure of a coaugmented coassociative coalgebra object of $\mathcal{C}$.*

(ii) $\mathrm{Bar}(-)$ *is functorial and*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C}) \arrow[rr, "\mathrm{Bar}"] \arrow[dr, "\mathrm{Bar}"'] & & \mathcal{C} \\ & \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C}) \arrow[ur] & \end{tikzcd}$$

(iii) Assume that $\mathcal{C}$ admits totalisations of cosimplical objects. Then $\mathrm{Bar} : \mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C}) \to \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C})$ *admits a right adjoint* coBar *given by*

$$\mathrm{coBar}(Y) = \lim \left(1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; Y \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; Y \otimes Y \dots \right)$$

For $\mathcal{C}$ a presentable symmetric monoidal ∞-category, applying Theorem 4.3.12 to $(\mathrm{SymSeq}(\mathcal{C}), \circledcirc)$ and $(\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}), \circ)$ gives us the adjunctions

$$\mathrm{Bar} : \mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) : \mathrm{coBar}$$

$$\mathrm{Bar} : \mathrm{Monad}^{\mathrm{aug}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{coMonad}^{\mathrm{aug}}(\mathcal{C}) : \mathrm{coBar}$$

Proposition 4.3.13. *The following diagram commutes*

$$\begin{tikzcd} \mathcal{O} \arrow[d, mapsto] \\ T : X \mapsto \coprod_{r\geq 0} (\mathcal{O}(r) \otimes X^{\otimes r})_{\mathfrak{S}_r} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) \arrow[r, "\mathrm{Bar}"] \arrow[d, "T_{(-)}"'] & \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C}) \arrow[d, "T_{(-)}"] \\ \mathrm{Monad}^{\mathrm{aug}}(\mathcal{C}) \arrow[r, "\mathrm{Bar}"'] & \mathrm{coMonad}^{\mathrm{aug}}(\mathcal{C}) \end{tikzcd}$$

Proof. $T_{(-)}$ preserves begin augmented, $T_{(-)}$ is given by a small colimit construction and Bar is a left adjoin functor preserving small colimits. Moreover $\otimes_{\mathcal{C}}$ preserves small colimits in each variable. □

Definition 4.3.14. A *pairing* of ∞-categories is a triple $(\mathcal{C}, \mathcal{D}, \lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D})$ where $\mathcal{C}, \mathcal{D}, \mathcal{M} \in \mathrm{Cat}_\infty$ and λ is a right fibration.

Notation. For a pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$, denote $\lambda(M) = (C_M, D_M)$ for $M \in \mathcal{M}$ and $C_M \in \mathcal{C}$, $D_M \in \mathcal{D}$.

Definition 4.3.15. Let $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ be a pairing. We say that an element $M \in \mathcal{M}$ is *left universal* if it is a final object of $\mathcal{M} \times_{\mathcal{C}} \{C_M\}$ in the obvious way. Similarly, M is said to be *right universal* if it is a final object of $\mathcal{M} \times_{\mathcal{D}} \{D_M\}$.
λ is said to be *left representable* if for all $C \in \mathcal{C}$ there exists a left universal object $M_C \in \mathcal{M}$ lifting C - i.e $(\mathrm{proj}_{\mathcal{C}} \circ \lambda)(M_C) \simeq \mathcal{C}$. Similarly, λ is said to be *right representable* if for all $D \in \mathcal{D}$ there exists a right universal object $M_D \in \mathcal{D}$ lifting D.

Remark. As such a pairing λ is a right fibration, by straightening-unstraightening it is classified by a functor $\chi : \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{H}\mathrm{o}$.

Remark. A paring λ is left representable if and only if for all $C \in \mathcal{C}$ the functor $\chi|_C : \{C\} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{H}\mathrm{o}$, $D \mapsto \chi(C, D)$ is representable. Let $M_C \in \mathcal{M}$ be the left universal object lifting C, then $\chi|_C \simeq \mathrm{Map}_{\mathcal{D}}(-, (\mathrm{proj}_D \circ \lambda)(M_C))$. We want to show that $\chi(C, D') \simeq \mathrm{Map}_{\mathcal{D}}(D', D_C)$. We can view χ as a map $\mathcal{C}^{\mathrm{op}} \to \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \mathcal{H}\mathrm{o})$ given by the composition

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathfrak{D}^l_\lambda"'] & & \mathcal{F}\mathrm{un}(\mathcal{D}^{\mathrm{op}}, \mathcal{H}\mathrm{o}) \\ & \mathcal{D} \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where $\mathfrak{D}^l_\lambda$ is characterised by the property that $\mathrm{Map}_{\mathcal{D}}(D, \mathfrak{D}^r_\lambda(C)) \simeq \chi(C, D)$. $\mathfrak{D}^r$ is called the *(right) duality functor associated to* λ: it sends all objects $C \in \mathcal{C}$ to an object $\mathfrak{D}^r_\lambda(C)$ that represents the functor $\chi|_{\mathcal{C}}$. A similar thing is done for right representable pairings, giving us $\mathcal{D}^{\mathrm{op}} \to \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{H}\mathrm{o})$ defined by the *(left) duality functor associated to* λ $\mathfrak{D}^l_\lambda$. Note that if λ is both left and right representable then we have an adjunction $(\mathfrak{D}^l_\lambda)^{\mathrm{op}} : \mathcal{D} \leftrightarrow \mathcal{C}^{\mathrm{op}} : \mathfrak{D}^r_\lambda$.

Example. Take a simplicial set $K : \Delta^{\mathrm{op}} \to \mathrm{Set}$. Recall that Δ is the category of non-empty finite ordered sets with weakly order preserving maps. Given $I \in \Delta$ let I^{op} have the same underlying set as I but with reversed ordering and so consider the join $I \star I^{\mathrm{op}}$ with induced order within I and I^{op} and $i \leq j$ whenever $i \in I$ and $j \in I^{\mathrm{op}}$. Define the functor

$$Q : \Delta \to \Delta \; I \mapsto I \star I^{\mathrm{op}}$$

and then the simplical set $\mathcal{T}\mathrm{wArr}(K) : \Delta^{\mathrm{op}} \xrightarrow{Q} \Delta^{\mathrm{op}} \xrightarrow{K} \mathrm{Set}$ which sends $[n] \in \Delta$ to some $\sigma \in K_{2n+1}$.
Taking $\mathcal{C}$ to be an ∞-category then $\mathcal{T}\mathrm{wArr}(\mathcal{C})$ has

- 0-simplices: morphisms in $\mathcal{C}$,
- 1-simplices: diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "g"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

  where f, g are objects of $\mathcal{T}\mathrm{wArr}(\mathcal{C})$,

- n-simplices:

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] & X_1 \arrow[l] \arrow[d, "f_1"'] & \cdots \arrow[l] & X_n \arrow[l] \arrow[d, "f_n"] \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \end{tikzcd}$$

Example. Let $\mathcal{C}$ be an ordinary category, then $\mathcal{T}\mathrm{wArr}(N(\mathcal{C})) \simeq N(\mathrm{TwArr}(\mathcal{C}))$.

Remark. Given a simplicial set K, the natural inclusions $I \hookrightarrow I \star I^{\mathrm{op}} \hookleftarrow I^{\mathrm{op}}$ induce morphisms $K \leftarrow \mathcal{T}\mathrm{wArr}(K) \to K^{\mathrm{op}}$, and moreover a morphism $\lambda_K : \mathcal{T}\mathrm{wArr}(K) \to K \times K^{\mathrm{op}}$

Proposition 4.3.16. *Let $\mathcal{C}$ be an ∞-category, then $\lambda_{\mathcal{C}} : \mathcal{T}\mathrm{wArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a right fibration. Hence $\mathcal{T}\mathrm{wArr}(\mathcal{C})$ is an ∞-category*

Proposition 4.3.17. *[Lur17] Proposition 5.2.1.3.*

Proposition 4.3.18. *Let $\mathcal{C}$ be an ∞-category, $\lambda_{\mathcal{C}}$ as before. Then the following are equivalent.*

(i) An object $M \in \mathcal{T}\mathrm{wArr}(\mathcal{C})$ is left universal.

(ii) An object $M \in \mathcal{T}\mathrm{wArr}(\mathcal{C})$ is right universal.

(iii) An object $M \in \mathcal{T}\mathrm{wArr}(\mathcal{C})$ considered as a morphism in $\mathcal{C}$ is an equivalence.

Proposition 4.3.19. *[Lur17] Proposition 5.2.1.10.*

Corollary 4.3.20. *The pairing $\lambda_{\mathcal{C}}$ is left and right representable.*

Proof. For all $C \in \mathcal{C}$ pick $M_C = \mathrm{id}_C$. $\square$

Remark. $\lambda_{\mathcal{C}}$ being left representable gives us

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathfrak{D}^r_{\lambda_{\mathcal{C}}}"'] & & \mathcal{F}\mathrm{un}((\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathcal{H}\mathrm{o}) \\ & \mathcal{C}^{\mathrm{op}} \arrow[ur, hook, "j"'] & \end{tikzcd}$$

Similarly λ_C being right representable gives us

$$\begin{tikzcd} \mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathfrak{D}^l_{\lambda_{\mathcal{C}}}"'] & & \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{H}\mathrm{o}) \\ & \mathcal{C} \arrow[ur, hook, "j"'] & \end{tikzcd}$$

Proposition 4.3.21. *Let $\mathcal{C}$ be an ∞-category and $\chi_{\lambda_{\mathcal{C}}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{H}\mathrm{o}$ classify the pairing $\lambda : \mathcal{T}\mathrm{wArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. Then the induced functor $\mathcal{C} \to \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{H}\mathrm{o})$ is equivalent to the Yoneda embedding.*

Proof. [Lur17] Proposition 5.2.1.11 and [Lur09] §5.1.3. □

Definition 4.3.22. Let $\mathcal{O}^{\otimes}$ be an ∞-operad. A *pairing of $\mathcal{O}$-monoidal ∞-categories* is a triple

$$(p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}, q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}, \lambda : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \otimes_{\mathrm{Ass}^{\otimes}} \mathcal{D}^{\otimes})$$

where $\mathcal{C}, \mathcal{D}$ are $\mathcal{O}$-monoidal ∞-categories and λ is an $\mathcal{O}$-monoidal morphism such that

(i) λ is a *categorical fibration*: λ has the right lifting property in sSet over all monomorphic categorical equivalences.

(ii) $\lambda_X : M^{\otimes}_X \to \mathcal{C}^{\otimes}_X \times \mathcal{D}^{\otimes}_X$ is a right fibration for all colours $X \in \mathcal{O}^{\otimes}$.

Definition 4.3.23. A morphism from a pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ to a pairing $\lambda' : \mathcal{M}' \to \mathcal{C}' \times \mathcal{D}'$ is a triple

$$(\alpha : \mathcal{C} \to \mathcal{C}', \beta : \mathcal{D} \to \mathcal{D}', \gamma : \mathcal{M} \to \mathcal{M}')$$

such that

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\gamma"] \arrow[d, "\lambda"'] & \mathcal{M}' \arrow[d, "\lambda"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "\alpha \times \beta"] & \mathcal{C}' \times \mathcal{D}' \end{tikzcd}$$

commutes. We can then denote CPair_{Δ} as the category whose objects are pairings and morphisms are defined above.

Lemma 4.3.24. *The category CPair_{Δ} is canonically simplicially enriched.*

Proof. [Lur17] Construction 5.2.1.14. □

We can then define the *∞-category of pairing of ∞-categories* as $\mathcal{C}\mathrm{Pair} := \mathcal{N}(\mathrm{CPair}_{\Delta})$.

Definition 4.3.25. Take $\lambda, \lambda' \in \mathcal{C}\mathrm{Pair}$ such that λ and λ' are left representable pairings. We say that a morphism $\gamma : \lambda \to \lambda'$ is *left representable* if γ preserves left universal objects. We define *right presentable* morphisms analogously. We denote the ∞-subcategory of left presentable pairings with left presentable morphisms as $\mathcal{C}\mathrm{Pair}^L$. We define $\mathcal{C}\mathrm{Pair}^R$ analogously.

Remark. If $\gamma : \lambda \to \lambda'$ is a left representable functor, then the following diagram commutes

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\alpha^{\mathrm{op}}"] \arrow[d, "\mathfrak{D}^r_\lambda"'] & \mathcal{C}'^{\mathrm{op}} \arrow[d, "\mathfrak{D}^r_{\lambda'}"] \\ \mathcal{D} \arrow[r, "\beta"] & \mathcal{D}' \end{tikzcd}$$

Definition 4.3.26. A pairing $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ is *perfect* if it is equivalent to a pairing $\mathcal{T}\mathrm{wArr}(\mathcal{C}_0) \xrightarrow{\lambda_{\mathcal{C}_0}} \mathcal{C}_0 \times \mathcal{C}_0^{\mathrm{op}}$ for some ∞-category $\mathcal{C}_0$. We denote the subcategory $\mathcal{C}\mathrm{Pair}^{\mathrm{perf}} \subset \mathcal{C}\mathrm{Pair}$ of perfect pairings with morphism being right (equivalently left) representable morphisms between perfect pairings.

Proposition 4.3.27. *Let $\mathcal{C}$ be a ∞-category, $\lambda_{\mathcal{C}} : \mathcal{T}\mathrm{wArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ as before and $\mu : \mathcal{M} \to \mathcal{D} \times \mathcal{E}$ an element of $\mathcal{C}\mathrm{Pair}^R$. Then we have natural maps*

$$\mathrm{Map}_{\mathcal{C}\mathrm{Pair}^R}(\lambda_{\mathcal{C}}, \mu) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \mathrm{Map}_{\mathcal{C}\mathrm{Pair}^R}(\mu, \lambda_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathcal{C})$$

are equivalences.

Proof. [Lur17] Proposition 5.2.1.18. □

Corollary 4.3.28. *Consider the forgetful functors $\mathcal{C}\mathrm{Pair}^{\mathrm{perf}} \xrightarrow{p_l} \mathrm{Cat}_\infty$, $(\mu : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) \mapsto \mathcal{C}$ and $\mathcal{C}\mathrm{Pair}^{\mathrm{perf}} \xrightarrow{p_r} \mathrm{Cat}_\infty$, $(\mu : \mathcal{M} \to \mathcal{C} \times \mathcal{D}) \mapsto \mathcal{D}^{\mathrm{op}}$. Then*

(i) p_l admits a right adjoint $q_l : C \mapsto \lambda_{\mathcal{C}}$, and

(ii) p_r admits a left adjoint $q_r : \mathcal{D}^{\mathrm{op}} \mapsto \lambda_{\mathcal{D}^{\mathrm{op}}}$. Moreover the adjunctions $p_l \dashv q_l$ and $q_r \dashv p_r$ are equivalences of ∞-categories.

Remark. An $\mathcal{O}$-monoidal pairing corresponds to an $\mathcal{O}$-monoid object $\mathcal{O}^{\otimes} \to \mathrm{Cat}_\infty \simeq \mathcal{C}\mathrm{Pair}^{\mathrm{perf}}$. Then the pairing $\lambda_{\mathcal{C}} : \mathcal{T}\mathrm{wArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ admits the structure of an $\mathcal{O}$-monoid object of $\mathcal{C}\mathrm{Pair}^{\mathrm{perf}}$ - i.e an $\emptyset$-monoidal pairing $\mathcal{T}\mathrm{wArr}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\mathrm{op}})^{\otimes}$.

Remark. Let $\lambda^{\otimes} : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}$ be an $\mathcal{O}$-monoid object in $\mathcal{C}\mathrm{Pair}$, then $\mathcal{A}\mathrm{lg}_{/\mathcal{O}}(M) \to \mathcal{A}\mathrm{lg}_{/\mathcal{O}}(\mathcal{C}) \times \mathcal{A}\mathrm{lg}/\mathcal{O}(\mathcal{D})$ is a pairing of ∞-categories. If $\mathcal{C}$ is a monoidal ∞-category, we specialise to $\mathcal{O}^{\otimes} = \mathrm{Ass}^{\otimes}$ for $\lambda^{\otimes}_{\mathcal{C}} : \mathcal{T}\mathrm{wArr}(\mathcal{C}) \to \mathcal{C}^{\otimes} \times_{\mathrm{Ass}^{\otimes}} (\mathcal{C}^{\mathrm{op}})^{\otimes}$ which then gives us $\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}}) : \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{T}\mathrm{wArr}(\mathcal{C})) \to \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}) \times \mathcal{A}\mathrm{lg}/\mathrm{Ass}(\mathcal{C}^{\mathrm{op}})$.

Theorem 4.3.29. *In view of the above remark:*

(i) $\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}}) \in \mathcal{C}\mathrm{Pair}$.

(ii) Assume $1_{\mathcal{C}} \in \mathcal{C}$ is final and $\mathcal{C}$ admits geometric realisations. Then $\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})$ is left representable so we get $\mathfrak{D}^r_{\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})} : \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C})^{\mathrm{op}} \to \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}^{\mathrm{op}})$, and the composition $\mathrm{forg}_{\mathrm{Ass}} \circ \mathfrak{D}^r_{\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})}$ is equivalent to the Bar construction.

(iii) Assume $1_{\mathcal{C}} \in \mathcal{C}$ is initial, $\mathcal{C}$ admits totalisation. Then $\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})$ is right representable so we get $\mathfrak{D}^l_{\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})} : \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathcal{C})$. We denote the composition $\mathrm{coBar} := (\mathrm{forg}_{\mathrm{Ass}} \circ \mathfrak{D}^l_{\mathcal{A}\mathrm{lg}(\lambda_{\mathcal{C}})})$.

Corollary 4.3.30. *If $\mathcal{C}$ is a monoidal ∞-category that admits both geometric realisation and totalisation of cosimplicial objects, then we get an adjunction*

$$\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C}) \simeq \mathcal{A}\mathrm{lg}_{\mathrm{Ass}}(\mathcal{C}_{1//1}) \underset{\mathfrak{D}^l_{\mathcal{A}\mathrm{lg}(\lambda)}}{\overset{\mathfrak{D}^r_{\mathcal{A}\mathrm{lg}(\lambda)}}{\rightleftarrows}} \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathrm{Ass}}(\mathcal{C}_{1//1}) \simeq \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{aug}}_{/\mathrm{Ass}}(\mathcal{C})$$

Proof (Idea). Trivial pairing has left universal object + construct monoidal category $A\mathrm{BMod}_A$. ☐

Proposition 4.3.31 (Francis-Gaitsgory [FG11]). *Let $\mathcal{O} \in \mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C})$, and $f : \mathrm{Bar}(\mathcal{O}) \to \mathcal{L}$ where $\mathcal{L} \in \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{aug}}(\mathcal{C})$. From this we get*

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "g"] \arrow[dr] & & \mathrm{coBar}(\mathcal{L}) \\ & \mathrm{coBar}(\mathrm{Bar}(\mathcal{O})) \arrow[ur, "\mathrm{coBar}(f)"'] & \end{tikzcd}$$

Then the following compostion of functors

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\mathrm{indu}'_{\mathcal{O}}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C}) \xrightarrow{f_*} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$$

and

$$\mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \xleftarrow{g^*} \mathcal{A}\mathrm{lg}_{\mathrm{coBar}(\mathcal{O})}(\mathcal{C}) \xleftarrow{\mathrm{prim}'_{\mathcal{L}}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathcal{L}}(\mathcal{C})$$

are mutually adjoint.

Proof (Idea). Denote the adjoints $L \dashv \mathrm{prim}'_{\mathcal{L}}$ and $g_! \dashv g^*$.

- Construct a morphism $L \circ g_! \to f_* \circ \mathrm{indu}'_{\mathcal{O}}$.
- Show that $(L \circ g_!)(\mathrm{free}_{\mathcal{O}}(X)) \simeq (f_* \circ \mathrm{indu}'_{\mathcal{O}})(\mathrm{free}_{\mathcal{O}}(X))$.
- Recall that every $\mathcal{O}$-algebra X is a small colimit of $\mathrm{free}_{\mathcal{O}}$-algebras (geometric realisation of simplicial diagram of $\mathrm{free}_{\mathcal{O}}$-algebras).

For details, refer to [FG11] §3.3. and Corollary 3.3.13. ☐

Example. In the previous proposition take $f \coloneqq \mathrm{id}_{\mathrm{Bar}(\mathcal{O})} : \mathrm{Bar}(\mathcal{O}) \to \mathrm{Bar}(\mathcal{O})$, $g : \mathcal{O} \to \mathrm{coBarBar}(\mathcal{O})$, then we have an adjunction

$$\mathrm{indu}'_{\mathcal{O}} : \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{coBar}(\mathcal{O})}(\mathcal{C}) : g^* \circ \mathrm{prim}'_{\mathrm{Bar}(\mathcal{O})}$$

We now take a small detour to look at the stable ∞-category of spectra.

Definition 4.3.32. A ∞-category $\mathcal{E}$ is called *stable* if

(i) it is *pointed*: there exists an initial and final object that are equivalent (called the *zero object*),

(ii) every morphism f in $\mathcal{E}$ admits a *fibre*: the pullback

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and a *cofibre*: the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{cofib}(f) \end{tikzcd}$$

(iii) every *fibre sequence*: a pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

is a *cofibre sequence*: a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Z \end{tikzcd}$$

and vice-versa.

Example. Define the ∞-category of *spectra* $\mathrm{Sp} \coloneqq \lim(\cdots \xrightarrow{\Omega} \mathcal{H}\mathrm{o}_* \xrightarrow{\Omega} \mathcal{H}\mathrm{o}_*)$. An object E in Sp is the data of (X_i) a sequence of homotopy types such that $\Omega X_{i+1} \simeq X_i$. In other words, a spectrum E is an infinite loop object in $\mathcal{H}\mathrm{o}_*$.

Proposition 4.3.33. *(i)* Sp *is stable, then adjunction* $\Sigma : \mathrm{Sp} \leftrightarrows \mathrm{Sp} : \Omega$ *is an equivalence.*

(ii) Sp *is presentable. Moreover it is the free presentable stable ∞-category generated by* $\mathbb{S} := \Sigma^\infty(S^0)$. *The for a stable presentable ∞-category $\mathcal{C}$, the map*

$$\mathcal{F}\mathrm{un}^L(\mathrm{Sp}, \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathbb{S}}} \mathcal{C}, \ F \mapsto F(\mathbb{S})$$

is an equivalence.

We shall see that there exists a natural symmetric monoidal structure on Sp.

Proposition 4.3.34. *(i) There exists a symmetric monoidal ∞-category* $\mathrm{Sp}^{\otimes} \to \mathrm{Com}^{\otimes}$ *whose underlying ∞-category is* Sp *such that*

(a) $- \otimes_{\mathrm{Sp}} -$ *preserves small colimits in each variable, and*

(b) the unit if the sphere spectrum §.

(ii) For all presentable symmetric monoidal ∞-categories $\mathcal{C}^{\otimes} \to \mathrm{Com}^{\otimes}$ such that $\mathcal{C}$ is stable, then there exists a symmetric monoidal functor $\mathrm{Sp}^{\otimes} \to \mathcal{C}^{\otimes}$ such that the underlying functor $F : \mathrm{Sp} \to \mathcal{C}$ is in Pr^L and $F(\mathbb{S}) \simeq 1_{\mathcal{C}}$. Hence F is unique up to contractible choice.

Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category such that $\mathcal{C}$ is stable.

Definition 4.3.35. An ∞-operad $\mathcal{O}$ with values in $\mathcal{C}$ is

(i) *unital* if $\mathcal{O}(0) \simeq 1_{\mathcal{C}}$

(ii) *non-unital* if $\mathcal{O}(0) \simeq 0_{\mathcal{C}}$ (the zero object of $\mathcal{C}$),

(iii) *reduced* if $\mathcal{O}$ is non-unital and $\mathcal{O}(1) \simeq 1_{\mathcal{C}}$.

Remark. (i) Any reduced ∞-operad $\mathcal{O}$ is augmented - $\mathcal{O} \to 1_{\mathcal{C}}^{\mathfrak{S}}$ is given by

$$\mathcal{O}(r) \to 1_{\mathcal{C}}^{\mathfrak{S}}(r) \simeq 0_{\mathcal{C}}\mathcal{O}(1) \xrightarrow{\mathrm{id}} 1_{\mathcal{C}}^{\mathfrak{S}}$$

(ii) The associated monad $\mathcal{T}_{\mathcal{O}}$ for a non-unital $\mathcal{O}$ is equivalent to

$$\mathcal{T}_{\mathcal{O}} \simeq \coprod_{r \geq 1} (\mathcal{O}(r) \otimes X^{\otimes r})_{\mathfrak{S}_r}$$

(iii) For unital $\mathcal{O} \in \mathcal{O}\mathrm{pd}(\mathcal{C})$ and $X \in \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C})$, we have a map $1_{\mathcal{C}} \simeq \mathcal{O}(0) \to X$ i.e a 'unit map'.

Theorem 4.3.36 (Heuts [Heu24])**.** *Let $\mathcal{C}$ be a stable presentable symmetric monoidal ∞-category, then*

$$\mathrm{Bar} : \mathcal{O}\mathrm{pd}^{\mathrm{red}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{O}\mathrm{pd}^{\mathrm{red}}(\mathcal{C}) : \mathrm{coBar}$$

becomes an equivalence of ∞-categories. We also have an adjunction

$$\mathrm{indu}'_{\mathcal{O}} : \mathcal{A}\mathrm{lg}_{\mathcal{O}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{co}\mathcal{A}\mathrm{lg}^{\mathrm{ndp}}_{\mathrm{Bar}(\mathcal{O})}(\mathcal{C}) : \mathrm{prim}'_{\mathcal{O}}$$

Example. Consider the monoidal structure $(\mathrm{SymSeq}(\mathrm{Sp})), \circledcirc)$. We can obtain examples from classical operads in Top, Sp. Let $\mathbb{T}\mathrm{op}$ denote the category Top with the Quillen model structure. Then $(\mathrm{SymSeq}(\mathbb{T}\mathrm{op}), \circledcirc)$ has an induced model structure, hence there exists a monoidal equivalence

$$N(\mathrm{SymSeq}(\mathbb{T}\mathrm{op}^0))[W^{-1}] \to \mathrm{SymSeq}(\mathcal{H}\mathrm{o})$$

where $\mathbb{T}\mathrm{op}^0$ is the subcategory of cofibre-fibrant objects and W some class of morphisms. Indeed if $\mathcal{O}$ is an operad in $\mathbb{T}\mathrm{op}$ then we get an induced $\mathcal{O} \in \mathcal{O}\mathrm{pd}(\mathcal{H}\mathrm{o})$ such that $\mathcal{O}(r) \simeq \mathrm{Sing}(\mathcal{O}(r))$.
We can get $\mathrm{Com}_{\mathrm{Sp}}$ (or $\mathrm{Com}_{\mathcal{C}}$ for some presentable symmetric monoidal category $\mathcal{C}$) from the operad Com in Top (defined by $\mathrm{Com}(r) \simeq \mathrm{pt}$): it is $\mathrm{Com} \in \mathcal{O}\mathrm{pd}(\mathcal{H}\mathrm{o})$ such that $\mathrm{Com}(r) \simeq \mathrm{pt}$.
$\mathcal{H}\mathrm{o}$ is the free presentable ∞-category generated by pt so for all presentable symmetric monoidal categories $\mathcal{C}$, $\mathcal{F}\mathrm{un}^L(\mathcal{H}\mathrm{o}, \mathcal{C}) \xrightarrow{\mathrm{ev}_{\mathrm{pt}}} \mathcal{C}$ is an equivalence, so there exists $F : \mathcal{H}\mathrm{o} \to \mathcal{C}$ given by sendind pt to $1_{\mathcal{C}}$ in Pr^L.

Proposition 4.3.37. *F as above induces a monoidal functor* $\mathrm{SymSeq}(\mathcal{H}\mathrm{o}) \to \mathrm{SymSeq}(\mathcal{C})$ *given by $M \mapsto F(M)$ such that $F(M)(r) \simeq F(M(r))$. This further induces a map which we shall also denote as F,*

$$F : \mathcal{O}\mathrm{pd}(\mathcal{H}\mathrm{o}) \to \mathcal{O}\mathrm{pd}(\mathcal{C}),\ \mathcal{O} \mapsto F(\mathcal{O}) \text{ such that } F(\mathcal{O}(r)) \simeq F(\mathcal{O}(r))$$

F then sends Com *to* $\mathrm{Com}_{\mathcal{C}} \coloneqq F(\mathrm{Com}_{\mathcal{H}\mathrm{o}})$ *i.e* $\mathrm{Com}_{\mathcal{C}}(r) \simeq 1_{\mathcal{C}}$. *If $\mathcal{C} = \mathrm{Sp}$ then $F = \Sigma^{\infty}_{+}$.*

Proposition 4.3.38. *Let $\mathcal{O} \in \mathrm{co}\mathcal{O}\mathrm{pd}(\mathrm{Sp})$. The symmetric sequence $\mathcal{O}^{\vee}$ defined as $\mathcal{O}^{\vee}(r) \coloneqq \mathrm{Map}_{\mathrm{Sp}}(\mathcal{O}(r), \mathbb{S}) \simeq \emptyset(r)^{\vee}$ is an ∞-operad with values in* Sp.

Proof (Sketch). $\mathrm{SymSeq}(\mathrm{Sp}) = \mathcal{F}\mathrm{un}(\mathrm{Fin}^{\simeq}, \mathrm{Sp})$ admits a level-wise symmetric monoidal structure induced by $- \otimes_{\mathrm{Sp}} -$, namely

$$(M \otimes_{\mathrm{lev}} N)(r) \simeq M(r) \otimes_{\mathrm{Sp}} N(r)$$

We claim that

$$- \otimes_{\mathrm{lev}} - : (\mathrm{SymSeq}(\mathrm{Sp}) \times \mathrm{SymSeq}(\mathrm{Sp}), \circledcirc \otimes \circledcirc) \to (\mathrm{SymSeq}(\mathrm{Sp}), \circledcirc)$$

is lax (in fact also oplax) monoidal. This is shown in Proposition 3.9 of [BCN24]. Consider the functors

$$X \longmapsto \mathrm{Map}_{\mathrm{SymSeq}(\mathrm{Sp})}(X \otimes_{\mathrm{lev}} -, \mathrm{Com}_{\mathrm{Sp}})$$

$$\begin{tikzcd}
X \arrow[dr, mapsto] & \mathrm{SymSeq}(\mathrm{Sp})^{\mathrm{op}} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathcal{F}\mathrm{un}(\mathrm{SymSeq}(\mathrm{Sp}^{\mathrm{op}}), \mathcal{H}\mathrm{o}) \\
& X^{\vee} & \mathrm{SymSeq}(\mathrm{Sp}) \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

$\mathrm{Map}_{\mathrm{SymSeq(Sp)}}(X \otimes_{\mathrm{lev}} -, \mathrm{Com}_{\mathrm{Sp}})$ is representable by $X^\vee$ because

$$\mathrm{Map}_{\mathrm{Sp}}(X \otimes_{\mathrm{Sp}} Y, \mathbb{S}) \simeq \mathrm{Map}_{\mathrm{Sp}}(Y, \mathrm{Map}(X, \mathbb{S}))$$

for all $Y \in \mathrm{Sp}$.
$(\mathrm{SymSeq(Sp)}, \circledcirc)$ induces a monoidal structure on $(\mathcal{F}\mathrm{un}(\mathrm{SymSeq(Sp)}^{\mathrm{op}}, \mathcal{H}\mathrm{o}), - \circledcirc_{\mathrm{Day}} -)$. + the Yoneda embedding j is already monoidal with respect tot $\circledcirc$ and $\circledcirc_{\mathrm{Day}}$. It suffices to show that F is lax monoidal which, by the universal property of the Day convolution is requiring that

$$\mathrm{SymSeq}(\mathcal{C})^{\mathrm{op}} \times \mathrm{SymSeq}(\mathcal{C})^{\mathrm{op}} \xrightarrow{- \otimes_{\mathrm{lev}} -} \mathrm{SymSeq(Sp)}^{\mathrm{op}} \xrightarrow{\mathrm{Map}(-, \mathrm{Com}_{\mathrm{Sp}})} \mathcal{H}\mathrm{o}$$

is lax monoidal. Note that we already showed that $- \otimes_{\mathrm{lev}} -$ is lax monoidal. Hence we claim that $\mathrm{Map}(-, \mathrm{Com}_{\mathrm{Sp}})$ is lax monoidal. This can be seen as under the Yoneda embedding j, $\mathrm{Com}_{\mathrm{Sp}} \mapsto \mathrm{Map}(-, \mathrm{Com}_{\mathrm{Sp}})$ is an associative algebra monoid structure[16], and hence lax monoidal. In fact F' induces

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathrm{SymSeq(Sp)}^{\mathrm{op}}) \arrow[r, "\simeq"] \arrow[d, "F'^*"'] & \mathrm{co}\mathcal{O}\mathrm{pd(Sp)}^{\mathrm{op}} \arrow[d, "(-)^\vee"] \\
\mathcal{A}\mathrm{lg}_{/\mathrm{Ass}}(\mathrm{SymSeq(Sp)}) \arrow[r, "\simeq"] & \mathcal{O}\mathrm{pd(Sp)}
\end{tikzcd}$$

□

Definition 4.3.39. We define the *spectrial Lie ∞-operad* as $\mathcal{L}\mathrm{ie} := \mathrm{Bar}(\mathrm{Com}_{\mathrm{Sp}})^\vee \in \mathcal{O}\mathrm{pd(Sp)}$.

TODO

- Pullback diagram?
- cats of operators/operations?
- ∞-p-cocartesian
- Give list of examples for the cats of operators 5.2.1.7
- bar construction arrow placement
- Fill in the details for HA bar/cobar construction.
- free 'universal condition' for $\mathrm{SymSeq}(\mathcal{H}\mathrm{o})$.

[16]Something wrong here.

5 Applications

We now come to the applications of all this.

$$\begin{tikzcd}
 & & & \mathcal{H}\mathrm{o}_{\mathbb{Q}}^{\geq 2} \arrow[rr, "L_{\mathbb{Q}}"'] \arrow[dl, "W"'] \arrow[dr, "\tilde{C}_*(-;\mathbb{Q})"] & & \mathcal{H}\mathrm{o}_{*}^{\geq 2} \arrow[dr, "\Sigma^\infty(-)\otimes H\mathbb{Q}"] & \\
\mathcal{DGLA}(\mathcal{C}\mathrm{h}_{\mathbb{Q}})^{\geq 1} \arrow[dd, shift right, "\mathrm{forg}_{\mathcal{L}\mathrm{ie}}"'] & & \mathcal{A}\mathrm{lg}_{\mathcal{L}\mathrm{ie}}(\mathrm{Sp}_{Q})^{\geq 2} \arrow[ll, "\Omega DK_{\mathcal{L}\mathrm{ie}}"'] \arrow[rr, shift left, "\mathrm{indu}'_{\mathcal{L}\mathrm{ie}}"] \arrow[ddr, shift right, "\mathrm{forg}_{\mathcal{L}\mathrm{ie}}"'] & & \mathrm{co}\mathcal{A}\mathrm{lg}_{\mathrm{Com}}^{\mathrm{ndp}}(\mathrm{Sp}_{\mathbb{Q}}) \arrow[ll, shift left, "\mathrm{prim}_{\mathrm{Com}}"] \arrow[rr, "\mathrm{forg}"] \arrow[ddl, shift left, "\mathrm{prim}_{\mathrm{Com}}"] & & \mathrm{Sp}_{\mathbb{Q}} \\
 & & & & & & \\
\mathcal{C}\mathrm{h}_{\mathbb{Q}} \arrow[uu, shift right, "\mathrm{free}_{\mathcal{L}\mathrm{ie}}"'] & & \mathrm{Sp}_{\mathbb{Q}} \arrow[ll, "\simeq", "DK"'] & \mathrm{Sq}_{\mathbb{Q}} \arrow[l, "\simeq", "\Omega"'] \arrow[uul, shift right, "\mathrm{free}_{\mathcal{L}\mathrm{ie}}"'] \arrow[uur, shift left, "\mathrm{triv}_{\mathrm{Com}}"] & & &
\end{tikzcd}$$

6 Appendix

6.1 Recap on homological algebra

Here we give a brief overview of some of the definitions and tools from homological algebra that we use in the course of these notes. We are mainly interested in working over the category of R-modules for some ring R, but all of these constructions work equally well over an arbitrary abelian category $\mathcal{C}$.

Definition 6.1.1. Given an abelian category $\mathcal{C}$ a *chain complex* (C_*, d_*) is a collection $\{C_n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$ and morphisms $\{d_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that $d_n \circ d_{n+1} = 0$. This is often displayed as

$$\dots \xleftarrow[d_{-1}]{} C_{-1} \xleftarrow[d_0]{} C_0 \xleftarrow[d_1]{} C_1 \xleftarrow[d_2]{} C_2 \xleftarrow[d_3]{} \dots$$

Dually we define a *cochain complex* (C^*, d^*) to be a collection $\{C^n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{C}$ and morphisms $\{d^n : C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ such that $d^{n+1} \circ d_n = 0$. We also display this as

$$\dots \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \dots$$

Definition 6.1.2. The *homology* of a chain complex (C_*, d_*) is given by a collection of $\{H_n(C_*)\}_{n\in\mathbb{Z}}$ where $H_n(C_*) \coloneqq \ker(d_n)/\mathrm{im}(d_{n+1})$. Similarly, the *cohomology* of a cochain complex (C^*, d^*) is a collection of $\{H^n(C^*)\}_{n\in\mathbb{Z}}$ where $H^n(C^*) \coloneqq \ker(d^n)/\mathrm{im}(d^{n-1})$.

Definition 6.1.3. A sequence of composable morphisms

$$\dots \xrightarrow{f_{-2}} D_{-1} \xrightarrow{f_{-1}} D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \dots$$

is said to be *exact*, if $\ker(f_n) = \mathrm{im}(f_{n-1})$ for every n.

Definition 6.1.4. Given an object $M \in \mathcal{C}$, a *(left) resolution* of M is an sequence $\{f_n : D_n \to D_{n-1}\}_{n \in \mathbb{N}}$ and a map $\epsilon : D_0 \to M$ called an *augmentation map*, such that

$$\dots \xrightarrow{f_3} D_2 \xrightarrow{f_2} D_1 \xrightarrow{f_1} D_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is an exact sequence. An analogous construction works for a *(right) resolution* of M.

Definition 6.1.5. An object $I \in \mathcal{C}$ is said to be *injective* if for every monomorphism $f : M \to N$ and morphism $g : M \to I$, f factors through g - i.e there exists a morphisms $h : N \to I$ such that $g = h \circ f$.

$$\begin{tikzcd} M \arrow[r, hook, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ I & \end{tikzcd}$$

Dually, an object $P \in \mathcal{C}$ is said to be *projective* if for every epimorphism $f : M \to N$ and morphism $g : P \to N$, f factors through g.

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"'] & N \end{tikzcd}$$

Remark. A resolution of M is called injective (respectively projective, flat, free etc. whenever this makes sense) if all of the objects D_n for $n \in \mathbb{N}$ are injective (respectively projective, flat, free etc.).

We now want to specialise to the case that $\mathcal{C} = \mathrm{Mod}_R$.

Proposition 6.1.6. *The category* Mod_R *has enough injectives i.e for every object* $M \in \mathrm{Mod}_R$*, there exists a morphism* $M \to I$ *such that* I *injective. Similarly,* Mod_R *also has enough projectives.*

Corollary 6.1.7. *Every* R*-module has an injective resolution and a projective resolution.*

Definition 6.1.8. Let M and N be an R-modules and $0 \to N \to I^0 \to I^1 \to \dots$ an injective resolution of N. Consider the induced cochain complex

$$0 \longrightarrow \mathrm{Hom}_R(M, I^0) \longrightarrow \mathrm{Hom}_R(M, I^1) \longrightarrow \dots$$

given by remove the first term and taking $\mathrm{Hom}_R(M, -)$. For each $n \in \mathbb{N}$ define $\mathrm{Ext}^n_R(M, N)$ as the nth cohomology group of this complex i.e $H^n(\mathrm{Hom}_R(M, I^*))$. Similarly, take a projective resolution $\dots P_1 \to P_0 \to M \to 0$ and consider the induced chain complex

$$\dots \longrightarrow P_1 \otimes_R N \longrightarrow P_0 \otimes_R N \longrightarrow 0$$

given be removing its co-first term and taking $- \otimes_R N$. For each $n \in \mathbb{N}$ define $\mathrm{Tor}^R_n(M, N)$ as the nth homology group of this complex i.e $H_n(P_* \otimes_R N)$.

Proposition 6.1.9. *The above constructions of the* Tor *and* Ext *functors are well-defined: in particular they do not depend on the choice of projective and injective resolutions.*

6.2 Straightening/Unstraightening

Example. Given a morphism $X \to S$ of sets, we clearly have $X =?$???

Example. Let $F : \mathcal{C} \to \mathrm{Cat}$ be a functor. We define a object $\int F$ in $\mathrm{Cat}_{/\mathcal{C}}$ as having objects of the form (x, c) where $x \in \mathrm{ob}(\mathcal{C})$ and $x \in \mathrm{ob}(\mathbb{F}(x))$, and a morphism $(c, x) \to (d, y)$ being a tuple $(f : c \to d,\ \alpha : F(f)(x) \to y)$ of morphisms in $\mathcal{C}$ and $F(d)$ respectively.

Proposition 6.2.1. *The object $p : \int F \to \mathcal{C}$ in $\mathrm{Cat}_{/\mathcal{C}}$ satisfies the following: For every morphism $f : c \to d$ in $\mathcal{C}$ and $x \in \int F$ such that $p(x) = c$, there exists a morphism $\tilde{f} : x \to y$ such that*

(i) $p(\tilde{f}) = f$ and,

(ii) For all morphisms $g : d \to c$ in $\mathcal{C}$ and $w : x \to z$ lifting $g \circ f$, there exists a unique $\tilde{g} : y \to z$ such that $p(\tilde{g}) = g$ and $\tilde{g} \circ \tilde{f} = w$. In other words

$$\begin{tikzcd} \int F \arrow[d, "p"] \\ \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} & & z \\ x \arrow[r, "\tilde{f}"'] \arrow[urr, "w"] & y \arrow[ur, dashed, "\exists!\tilde{g}"'] & c \\ c \arrow[r, "f"'] \arrow[urr, "g\circ f"] & d \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Pick $\tilde{f} := (f, \mathrm{id}) : (c, x) \to (d, F(f)(x))$.??? $\square$

Definition 6.2.2. Such an $\tilde{f}$ is called a *cocartesian morphism.*

Exercise. (i) Cocartesian morphisms are closed under composition.

(ii) Cocartesian morphisms are unique.

Remark. There is an bijection

$$\mathrm{Fun}^{\mathrm{pseudo}}(\mathcal{C}, \mathrm{Cat}) \simeq \mathrm{coCart}(\mathcal{C})$$

between the set of pseudo-functors[17] $\mathcal{C} \to \mathrm{Cat}$ and cocartesian morphisms on $\mathcal{C}$.

6.3 The Day convolution

This section largely follows [Lur17] §2.2.6, so refer there for more information.
Consider symmetric monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_D)$ and assume that $\mathcal{C}$ is small and $\mathcal{D}$ admit all small colimits. For two functors $F, G \in \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ we define

[17] https://ncatlab.org/nlab/show/pseudofunctor

the *Day convolution* of F and G, denoted by $F \circledast G$ to be the left Kan extension of the diagram below

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \arrow[r, "F \times G"] \arrow[d, "\otimes_{\mathcal{C}}"'] & \mathcal{D} \otimes \mathcal{D} \arrow[r, "\otimes_{\mathcal{D}}"] & \mathcal{D} \\
\mathcal{C} \arrow[urr, "\mathrm{Lan}_{\otimes_{\mathcal{C}}}(\otimes_{\mathcal{D}} \circ F \times G)"'] & &
\end{tikzcd}$$

More explicitly, the Day convolution is given by

$$F \circledast G : \mathcal{C} \to \mathcal{D} \; z \mapsto \operatorname*{colim}_{x \otimes_{\mathcal{C}} y \to z} F(x) \otimes_{\mathcal{D}} G(y)$$

This gives us a functor $\circledast : \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D}) \times \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D}) \to \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ given by $(F, G) \mapsto F \circledast G$. Assuming that $\otimes_{\mathcal{D}}$ preserves small colimits in each variable, then we have the following properties of the Day convolution:

(i) $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ can be given the structure of a symmetric monoidal category with the underlying product being the Day convolution $\circledast$.

(ii) The category $\mathrm{CAlg}(\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D}))$ of commutative algebra objects of $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ is equivalent to the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

   Exercise. Describe the symmetric monoidal structure of the Day convolution structure.

6.4 The relative tensor product

Consider A, B, C associative $\mathbb{Z}$-algebras and M a $A - B$ bimodule, N a $B - C$-bimodule, and X an $A - C$-bimodule.
A *bilinear pairing* from (M, N) to X is a morphism $F : M \otimes_{\mathbb{Z}} N \to X$ of $A - C$-bimodules satisfying the property that

$$\begin{tikzcd}
A \otimes M \otimes N \arrow[r, "\mathrm{id}_A \otimes F"] \arrow[d, "\alpha_{A,M} \otimes \mathrm{id}_N"] & A \otimes X \arrow[d, "\alpha_{A,X}"'] & M \otimes N \otimes C \arrow[r, "\mathrm{id}_M \otimes \alpha_{C,N}"] \arrow[d, "F \otimes \mathrm{id}_N"] & M \otimes N \arrow[d, "F"'] & M \otimes B \otimes N \arrow[r, "\mathrm{id}_M \otimes \alpha_{B,N}"] \arrow[d, "\alpha_{B,M} \otimes \mathrm{id}_N"] & M \otimes N \arrow[d, "F"'] \\
M \otimes N \arrow[r, "F"] & X & X \otimes C \arrow[r, "\alpha_{C,X}"] & X & M \otimes N \arrow[r, "F"] & X
\end{tikzcd}$$

commute.

Theorem 6.4.1. *For M and N as above there exists an A-C-bimodule $M \otimes_B N$ called the relative tensor product of M with N over B such that* $\mathrm{Hom}_{{}_A\mathrm{BMod}_C}(M \otimes_B N, X) \cong$ *Bilinear$(M \otimes N, X)$ for all $X \in {}_A\mathrm{BMod}_C$.*

TODO

- prescript indentation
- bar construction arrow placement

7 Solutions

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