Recursive Methods in Long-Only Minimum Variance Portfolio Weight Computations

The Search for Tractable Analytic Solutions

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Blessings of Dimensionality Workshop

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Minimum Variance Solution

Background

A Factor Model for Stock Returns

$$Y = \beta X + Z \; (stochastic) \tag{1}$$

$$Y = \beta X^T + Z \; (n \; observations) \tag{2}$$

$$\Omega = \beta K \beta^T + D(for \; diagonal), K = X^T X \tag{3}$$

$$Z, Y \sim p \times n, X \sim n \times k, \beta \sim p \times k,$$

$$K \sim k \times k, D \sim p \times p, \Omega \sim p \times p$$

$$(by \; Woodbury) \implies$$

$$\Omega^{-1} = D^{-1} - D^{-1}\beta\left(K^{-1} + \beta^T D^{-1}\beta\right)^{-1}\beta^T D^{-1} \tag{4}$$

One Factor, Full Investment

(Tractable, Quasi-) Analytic Solution for MinVar Weights

Set $k = 1 \implies$ "factors" $=$ the (one) market factor. [1] 34:

A key analytic result derived in the appendix is that, under the assumption of the single-[market factor] model for security returns, the weight for individual securities in the unconstrained minimum-variance portfolio is:

$$w_i = \frac{\sigma^2_{MV}}{\sigma_{\epsilon,i^2}} \left(1 - \frac{\beta_i}{\beta_{LS}}\right) \tag{5}$$

where, the w_i are the weights in the minimum variance portfolio in the given p securities and, familiarly [2],:

$\beta_i =$ ex ante market beta for security i,

$\sigma^2_{\epsilon,i} =$ ex ante idiosyncratic return variance for security i,

$\sigma^2_{MV} =$ ex ante return variance of the minimum variance portfolio, and

$\beta_{LS} =$ **long–short** threshold beta

Quasi-Analytic Solution MV Weights (cont.)

$$\beta_{LS} \equiv \frac{\frac{1}{\sigma_M^2} + \sum \frac{\beta_i^2}{\sigma_{\epsilon,i}^2}}{\sum \frac{\beta_i}{\sigma_{\epsilon,i}^2}} \tag{6}$$

A key insight from Equation (6) *is that systematic rather than idiosyncratic risk dictates whether an individual security has a negative weight in an unconstrained optimization.*[1]

A companion insight from Equation (5) is that the sign of a security's weight in the minimum variance depends only on a comparison between the security's β and a constant $= \beta_{LS}, sign(w_i) = sign(\beta_{LS} - \beta_i)$.

One Factor, Full Investment & Long-Only

Quasi-analytic Solution for Long-Only MinVar Weights

An even more novel mathematical result from the appendix is that the basic form of Equation (6) *is preserved in long-only constrained optimizations. [1]*

Claim 1: There exists a β_L such that:

$$w_i = \frac{\sigma^2_{LMV}}{\sigma^2_{\epsilon,i}} \left(1 - \frac{\beta_i}{\beta_L}\right) \textit{ if } \beta_i < \beta_L \textit{ else } 0 \qquad (7)$$

where

- w_i are the weights in the optimal long-only minimum variance portfolio in the p securities,
- σ^2_{LMV} = ex ante return variance of the long-only minimum-variance portfolio, and
- β_L = long-only threshold beta, defined below.

Quasi-Analytic Solution LO MV Weights (cont.)

$$\beta_L \equiv \frac{\frac{1}{\sigma_M^2} + \sum_{\beta_i < \beta_L} \frac{\beta_i^2}{\sigma_{\epsilon,i}^2}}{\sum_{\beta_i < \beta_L} \frac{\beta_i}{\sigma_{\epsilon,i}^2}} \tag{8}$$

- "Quasi" because β_L appears on the both sides "inseparably".
- Suggests an iterative construction, starting with $\beta_L = max(\beta_i)$ and lowering it until equality.
- This in turn suggests a "stationary point", as follows:

Claim 2: $\beta_L = \beta_{LS}$ for the securities with non-negative weights in the MV portfolio.

[O]ptimal weights can be calculated without numerical search routines simply by sorting securities from low to high ex ante beta and examining the running sums. [1]

Preliminary Derivations

General Solution with Full Investment

$w \equiv portfolio\ weights \implies Variance(w) = w^T \Omega w, where\ \Omega \equiv Cov(X)$

- Objective: $Argmin_w \left(w^T \Omega w\right)$
- Minimum Variance with no constraints: $\implies w \equiv 0$
- Minimum Variance with full investment: $Argmin_w \left(\bullet \mid \mathbf{1}^T w = 1\right)$

$$\begin{aligned}
&\implies \frac{\partial}{\partial w, \lambda}\left(w^T \Omega w + 2\lambda(1 - \mathbf{1}^T w)\right) = 0 \\
&\frac{\partial}{\partial w} : \implies w^T \Omega = \lambda \mathbf{1}^T \implies w = \lambda \Omega^{-1} \mathbf{1} \\
&+\frac{\partial}{\partial \lambda} : \implies 1 = \mathbf{1}^T w = \lambda \mathbf{1}^T \Omega^{-1} \mathbf{1} \\
&\implies \frac{1}{\lambda} = \mathbf{1}^T \Omega^{-1} \mathbf{1} \implies w = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}
\end{aligned} \tag{9}$$

One-Factor Model Without Matrix Inversion

- In the context of the one-factor model from [1], recall the factor model on slide 5 and particularly Equation (4).
- K = the scalar market factor variance which may be inverted as $\frac{1}{\sigma_M^2}$ to machine accuracy.

One-Factor Model Without Matrix Inversion (cont.)

Setting $D^{-1} = Diag\left(\frac{1}{\sigma_\epsilon^2}\right)$ and writing $\frac{1}{\sigma_\epsilon^2} \equiv Diag\left(\frac{1}{\sigma_\epsilon^2}\right)$, we may rewrite Equation (4) as:

$$\Omega^{-1} = \frac{1}{\sigma_\epsilon^2}\left(\mathbf{I} - \beta\left(\frac{1}{\sigma_M^2} + \beta^T \frac{1}{\sigma_\epsilon^2}\beta\right)^{-1} \beta^T \frac{1}{\sigma_\epsilon^2}\right) \tag{10}$$

- The inner term to invert is a scalar, and may be written as $\frac{1}{\sigma_M^2} + \frac{\beta}{\sigma_\epsilon}^T \frac{\beta}{\sigma_\epsilon} = \frac{1}{\sigma_M^2} + \sum \frac{\beta_i^2}{\sigma_{\epsilon,i}^2}$
- We therefore have $\Omega^{-1} = \frac{1}{\sigma_\epsilon^2}\left(\mathbf{I} - \beta \frac{\frac{\beta^T}{\sigma_\epsilon^2}}{\frac{1}{\sigma_M^2} + \sum \frac{\beta_i^2}{\sigma_{\epsilon,i}^2}}\right)$

One-Factor Model Without Matrix Inversion (cont.)

- Finally we have from Equation (9):

$$(\mathbf{1}^T\Omega^{-1}\mathbf{1})w = \Omega^{-1}\mathbf{1} = \frac{1}{\sigma_\epsilon^2}\left(\mathbf{1} - \beta\frac{\sum\frac{\beta_i}{\sigma_{\epsilon,i}^2}}{\frac{1}{\sigma_M^2} + \sum\frac{\beta_i^2}{\sigma_{\epsilon,i}^2}}\right)$$

- We find the minimum variance $\equiv \sigma_{MV}^2$, also from Equation (9)

$$\sigma_{MV}^2 = w_{MV}^T\Omega w_{MV} = \left(\frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\right)^2\mathbf{1}^T\Omega^{-1}\Omega\Omega^{-1}\mathbf{1} = \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}$$

  completing the derivation of Equations (5), (6) and (8).//

- *Note*: As a *normalization factor*, σ_{MV}^2 may be computed indirectly.

MVLO - One Factor, Full Investment, Long-Only: Review and Preliminary Conclusions

- *Benefit:* An analytic (closed-form) solution with no matrix inversion in a limited, though prominent, case.
- *Contra:* However, estimating a factor model frequently requires matrix inversion.
- *Contra:* The Woodbury formula can be unstable.
- *Benefit:* Equations (5) and (6) provide insight into relationships between minimum variance portfolio construction and βs, as described in Slide 8.
- *Benefit:* Extends naturally to long-only case in Equation (8).
- *Contra:* No rigorous proofs yet for $k > 1$ (multi-factor).

2-Factor Model

Introduction

Introduction

Choosing two factors illustrates in the simplest context the main differences between one-factor and multi-factor models. Returning to Equation (10):

- Let β be a $p \times 2$ matrix of (orthogonal) factors. (Simpler analysis, is automatic from PCA-based models.)
- Replace $\frac{1}{\sigma_M^2}$ with $K^{-1}, K \equiv$ factor covariance $= 2 \times 2$ *diagonal* with entries the variances of the two factors.
- Using Equation (9), we post-multiply by $\mathbf{1}$ and have the following for the minvar weights, up to scaling constant σ_{MV}^2:

$$w \propto \Omega^{-1}\mathbf{1} = \frac{1}{\sigma_\epsilon^2}\left(\mathbf{I} - \beta\left(K^{-1} + \beta^T \frac{1}{\sigma_\epsilon^2}\beta\right)^{-1} \frac{\beta^T}{\sigma_\epsilon^2}\right)\mathbf{1}$$

Minimum Variance Weights

Multiplying terms in the outer parentheses by the final **1**.

$$w \propto \frac{1}{\sigma_\epsilon^2}\left(\mathbf{1} - \beta\left(K^{-1} + \frac{\beta}{\sigma_\epsilon^2}^T \beta\right)^{-1} \Sigma \frac{\beta_i}{\sigma_{\epsilon,i}^2}\right) \qquad (11)$$

where $\Sigma \frac{\beta_i}{\sigma_{\epsilon,i}^2}$ is a 2-vector = sum of the β_i for each security, normalized by the corresponding specific variance.
To emphasize the similarity with Equation (9), we may write this (very) loosely as

$$w \propto \frac{1}{\sigma_\epsilon^2}\left(\mathbf{1} - \frac{\beta}{\boldsymbol{\beta}_{LS}}\right), \boldsymbol{\beta}_{\boldsymbol{LS}} = \frac{K^{-1} + \frac{\beta}{\sigma_\epsilon^2}^T \beta}{\Sigma \frac{\beta_i}{\sigma_{\epsilon,i}^2}} = 2 \times 1 \qquad (12)$$

Long-Only Minimum Variance Weights

- A more accurate statement is

$$w \propto \frac{1}{\sigma_\epsilon^2}(\mathbf{1} - \beta \times q), q = \left(K^{-1} + \frac{\beta}{\sigma_\epsilon^2}^T \beta \right)^{-1} \Sigma \frac{\beta_i}{\sigma_{\epsilon,i}^2} = 2 \ vector \tag{13}$$

- Again the sign of a security weight depends only on a comparison of the corresponding beta with a constant = q (for quotient).
- In this 2-factor case, the comparison is a dot product, and the sign is determined by in which of the two half-planes determined by $\beta_i \cdot q = 1$ β_i lies.
- Generalization to k-factors is immediate.

Example

2023 Daily Return Covariance Matrix (x 10,000) for Eleven Vanguard ETFs

Materials
Consumer Discretionary
Consumer Staples
Energy
Financials
Information Technology
Health Care
Industrials
REIT
Telecom Services
Utilities

	VAW	VCR	VDC	VDE	VFH	VGT	VHT	VIS	VNQ	VOX	VPU
VAW	1.19	0.93	0.41	0.89	0.97	0.77	0.48	0.94	0.99	0.77	0.58
VCR	0.93	1.54	0.41	0.42	0.93	1.15	0.49	0.89	1.02	1.19	0.46
VDC	0.41	0.41	0.49	0.27	0.40	0.30	0.35	0.39	0.48	0.33	0.49
VDE	0.89	0.42	0.27	2.16	0.90	0.32	0.39	0.74	0.60	0.36	0.39
VFH	0.97	0.93	0.40	0.90	1.24	0.75	0.49	0.92	0.95	0.80	0.51
VGT	0.77	1.15	0.30	0.32	0.75	1.40	0.40	0.75	0.79	1.10	0.32
VHT	0.48	0.49	0.35	0.39	0.49	0.40	0.56	0.46	0.54	0.45	0.46
VIS	0.94	0.89	0.39	0.74	0.92	0.75	0.46	0.98	0.93	0.75	0.52
VNQ	0.99	1.02	0.48	0.60	0.95	0.79	0.54	0.93	1.61	0.85	0.94
VOX	0.77	1.19	0.33	0.36	0.80	1.10	0.45	0.75	0.85	1.45	0.41
VPU	0.58	0.46	0.49	0.39	0.51	0.32	0.46	0.52	0.94	0.41	1.23

Convergence in 3 Iterations

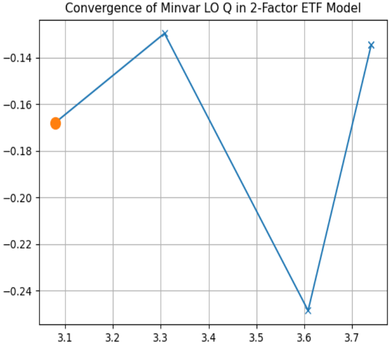

Figure 1: Monotonic on market factor (x-axis) but not second factor

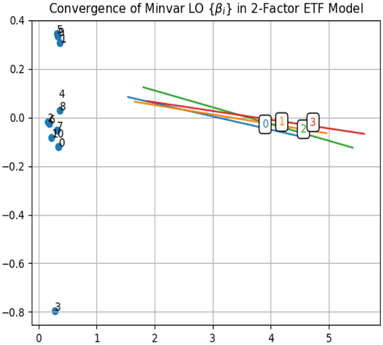

Figure 2: Converges to Consumer Staples, Health Care, Utilities

Comparison with Convex Optimization and Perturbation Analysis

	cvxopt	fixed_point
VAW	-0.0000	0.0000
VCR	-0.0000	0.0000
VDC	0.5233	0.5233
VDE	0.0000	0.0000
VFH	-0.0000	0.0000
VGT	-0.0000	0.0000
VHT	0.4108	0.4108
VIS	-0.0000	0.0000
VNQ	-0.0000	0.0000
VOX	-0.0000	0.0000
VPU	0.0659	0.0659

Figure 3: Stationary Point Method has Full Precision

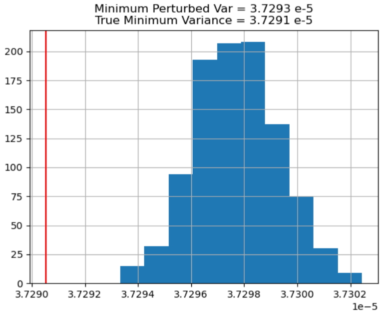

Figure 4: Mind the Gap!

Understanding the CST Stationary Point as Optimization Geometry

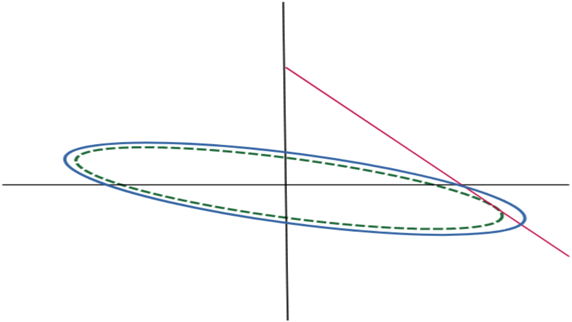

Figure 5: Move Towards Constraint, Along Constraint (Simplex) or Other?

Bibliography

[1] Harindra de Silva Roger Clarke and Steven Thorley. "Minimum-Variance Portfolio Composition". In: *Journal of Portfolio Management* 37.2 (Winter 2011), pp. 32–46.

[2] William Sharpe. "A Simplified Model for Portfolio Analysis". In: *Management Science* 9.2 (1963), pp. 277–293.

Appendix: Theorems, Proofs & Open Questions

Intuition for Proof

- It's all about weighted averages pulling one way, and a constant term pulling the other!

$$\text{Equation (8): } \beta_L \equiv \frac{\frac{1}{\sigma_M^2} + \sum_{\beta_i<\beta_L} \frac{\beta_i^2}{\sigma_{\epsilon,i}^2}}{\sum_{\beta_i<\beta_L} \frac{\beta_i}{\sigma_{\epsilon,i}^2}} = \frac{\frac{1}{\sigma_M^2} + \sum_{\beta_i<\beta_L} \frac{\beta_i}{\sigma_{\epsilon,i}^2}\beta_i}{\sum_{\beta_i<\beta_L} \frac{\beta_i}{\sigma_{\epsilon,i}^2}}$$

$$= \frac{\frac{1}{\sigma_M^2} + \sum_{\beta_i<\beta_L} u_i\beta_i}{\sum_{\beta_i<\beta_L} u_i}, u_i \geq 0 \equiv \mathit{weight}_i(\leftarrow \mathit{quasi}) \tag{14}$$

$$= \frac{\frac{1}{\sigma_M^2}}{\sum_{\beta_i<\beta_L} u_i} + \mathit{weighted\text{-}average}_u(\beta_i) \tag{15}$$

- $\beta_L \downarrow \implies$ First term $\uparrow$, second term $\downarrow$, $\implies \beta_L \downarrow, \uparrow, \rightarrow ?$

One Factor

Definitions of q and Q

- *Assumption for this Subsection*: Let c, a_i, b_i be real numbers for $i \in (1, 2 ... n)$ with $c, a_i > 0$ and b_i monotonically increasing (b_i can be negative).
- *Definitions*: $\forall k \leq n$, a function $q(k) \equiv \frac{c + \Sigma_1^k a_i b_i}{\Sigma_1^k a_i}$ and a companion discrete function $Q(k) = max\{i | q(k) \geq b_i\}$
- *Equivalent Second Definition for Q*: $b_{Q(k)} \leq q(k) < b_{Q(k)+1}$. Proof: By the definition of Q(k), $q(k) \geq b_{Q(k)}$. By the maximality of Q(k), $q(k) < b_{Q(k)+1}$
- *Comment*: With $b_0, b_{n+1} = -\infty, \infty$, the $\{b_i\}$ *partition* $\mathbb{R}$ via the half-open intervals $[b_i, b_{i+1})$ indexed by i. Q(k) is then the partition index into which q(k) falls.
- *Definition*: k is a *stationary (fixed) point* of Q & q means $Q(k) = k \iff b_k \leq q(k) < b_{k+1}$

Lemma 1 to Theorem 1

- *Assumptions*: Let c, a_i, b_i be real numbers for $i \in (1, 2...n)$ with $c, a_i > 0$ and b_i monotonically increasing (b_i can be negative).
- *Lemma 1*: Let c, a_i, b_i be as in the Assumption above. Then $sign(q(k) - b_k) = sign(q(k-1) - b_k)$.
- *Proof of Lemma 1*:

$$
\begin{aligned}
&sign\left(q(k) - b_k\right) = sign\left(\frac{c + \Sigma_1^k a_i b_i}{\Sigma_1^k a_i} - b_k\right) \\
=&sign\left(c + \Sigma_1^k a_i b_i - b_k \times \Sigma_1^k a_i\right) \text{ (because } a_i > 0)) \\
=&sign\left(c + \Sigma_1^{k-1} a_i b_i + a_k b_k - b_k \times \Sigma_1^{k-1} a_i - b_k a_k)\right) \\
=&sign\left(c + \Sigma_1^{k-1} a_i b_i - b_k \times \Sigma_1^{k-1} a_i\right) \\
=&sign\left(q(k-1) - b_k\right) \text{ (again because } a_i > 0)
\end{aligned}
$$

Theorem 1

- *Theorem 1*: With $b_{n+1} \equiv \infty, k^* \equiv max\{k|q(k) \geq b_k\}$ is a stationary point.
- Proof: The Theorem is true if $k^* = n$, so we may assume $k^* < n$. By definition of k^*,$q(k^*) \geq b_k^*$ and from k^*'s maximality $q(k^*+1) < b_{k^*+1}$. By Lemma 1 with $k = k^* + 1$, $q(k^*) < b_{k^*+1}$. We therefore have $b_{k^*} \leq q(k^*) < b_{k^*+1}$.//
- For the balance of this subsection we will assume $\exists k^* < n$, except where otherwise stated.
- With $c = \frac{1}{\sigma_M^2}, a_i = \frac{\beta_i}{\sigma_{\epsilon,i}^2}, b_i = \beta_i$, Theorem 1 proves Claims 1 and 2.

Work in Process

q-Monotonicity

Lemma 2: $sign\,(q(k+1) - q(k)) = sign\,(\Sigma_{i=1..k} a_i(b_{k+1} - b_i) - c)$
Proof: Let (1) N(k) be the numerator, and D(k) the denominator, of q(k), (2) n(k) $\equiv a_k b_k$ be the final term in the sum in N(k) and (3) d(k) $\equiv a_k$ be the final term in D(k). Then:

$$sign\,(q(k+1) - q(k)) = sign\left(\frac{N(k+1)}{D(k+1)} - \frac{N(k)}{D(k)}\right)$$

$= sign(N(k+1)D(k) - N(k)D(k+1))$ (since D is always positive)

$= sign(N(k)D(k) + n(k+1)D(k) - N(k)D(k) - N(k)d(k+1))$

$= sign(n(k+1)D(k) - N(k)d(k+1))$

$= sign(a_{k+1}b_{k+1}D(k) - a_{k+1}N(k)) = sign(a_{k+1})sign(b_{k+1}D(k) - N(k))$

$= sign(b_{k+1}D(k) - N(k)) = sign\,(\Sigma_{i=1..k} a_i(b_{k+1} - b_i) - c)$ //

Corollary 1: $sign\,(q(k+1) - q(k))$ changes at most once, from negative to positive, possibly with an intermediate 0 value. Proof:$\{b_i\}$ strictly increasing $\implies \Sigma_{i=1..k} a_i(b_{k+1} - b_i) - c$ strictly increasing.

q-Monotonicity (cont.)

- *q-Montonicity Lemma*: q(k) is either (a) monotonically increasing, (b) monotonically decreasing, or (c) first montonically decreasing and then monotonically increasing. In each case, the monotonicity is strict except possibly for one k with q(k+1) = q(k).

  Proof: See Corollary 1.
- *Corollary 2*: $sign\left(q(k+1) - q(k)\right) = sign\left(b_{k+1} - q(k)\right)$
- Proof: From the last line of the proof of Lemma 2:

$$sign\left(q(k+1) - q(k)\right) = sign\left(b_{k+1}\Sigma_{i=1..k}a_i - \Sigma_{i=1..k}a_i b_i - c\right)$$

$$= sign\left(b_{k+1} - \frac{\Sigma_{i=1..k}a_i b_i + c}{\Sigma_{i=1..k}a_i}\right) \text{(because } \Sigma_{i=1..k}a_i > 0\text{)}$$

$$= sign(b_{k+1} - q(k))//$$

- *Note to Corollary 2*: More careful computation shows $sign\left(q(k+1) - q(k)\right) = \alpha \times sign(b_{k+1} - q(k)), \alpha = \frac{a_{k+1}}{D(k+1)} < 1.$

Lemma 3: q() is Increasing After a Stationary Point

- *Lemma 3*: k stationary $\implies q(k+1) > q(k)$
  Proof: k stationary $\implies q(k) < b_{k+1} \implies q(k) < q(k+1)$ by Corollary 2.
- *Lemma 4*: k' stationary $\implies$ q is montonically increasing over the interval [k',n]. Proof: By Lemma 3 and q-Montonicity.
- *Q-Monotonicity Lemma*: k' stationary $\implies$ Q is weakly monotonically increasing over the interval [k',n].
  Proof: By Lemma 4 and the definition of Q,
  $\forall i \geq k', q(i+1) > q(i) \geq b_{Q(i)} \implies Q(i+1) \geq Q(i)$ by the maximality of Q(i+1).

CST Process

- Define the *CST process (CST)*[1] as follows:
  1. Set k = n
  2. If $Q(k) \geq k$, stop.
  3. Else (re)set $k = Q(k)$ and repeat starting at Step 2.
- *Notes*:
  1. k can decrease by more than 1 in a CST iteration.
  2. Let k^* be the stationary point of Theorem 1. By Q-Monotonicity, Q(k) is bounded from below by $k^* = Q(k^*)$ over the interval $[k^*, n]$.
- *Lemma 5* The CST iterations are weakly monotonically decreasing.

  Proof: We need only consider the CST prior to termination $\implies Q(k_i) = k_{i+1} < k_i$ from Step 2. Then Q-Monotonicity $\implies Q(k_i) \geq Q(k_{i+1})$.//

[1] *See* [1] Eq. (A-6) and following text.

Lemma for Theorem 2

Lemma 6: Set $b_{n+1} = \infty$. CST terminates at a stationary point.
Proof:

- Let k^* be as in Theorem 1. We have seen that the integers in the set $\{Q(k_i) = k_{i+1}\}$ generated by CST decrease monotonically (Lemma 5) and are bounded from below by $k^* \implies$ the CST stops at some k' with $Q(k') \geq k^*$.
- $k' = n \implies Q(n) \geq n \implies n$ is a stationary point $\implies k' = n = k^*$ by k^*'s maximality.
- Alternatively $k' < n \implies \exists k$ previous to k' in the CST with $Q(k) = k'$.
- CST stopped at k', not k $\implies k > k'$ which $\implies$ by Q-Montonocity $Q(k) = k' \geq Q(k')$, but since CST stopped at k' we also have $Q(k') \geq k'$ which then $\implies Q(k') = k'.//$

Theorem 2

Theorem 2. The CST terminates with $k' = k^*$.

- *Proof*: If $k^* = n$ the CST terminates immediately and the Theorem is true.
- Assume alternatively $k^* < n$. The CST creates a chain $\{k_{1,2..t}\}$ with $k_1 = n, Q(k_i) = k_{i+1} < k_i, Q(k') = k'$.
- By Lemma 6, this chain terminates with a stationary point $k' = k_t$. By k^*'s maximality $k^* \geq k'$ and k^* cannot appear as an element in this chain before k_t because k^* stationary $\implies$ the CST would have stopped at k^*.
- Assume k^* falls between two elements of the chain $\implies \exists i, 1 \leq i < t | k_i > k^* > k_{i+1}$. But then $k^* > k_{i+1} = Q(k_i) \wedge k^* = Q(k^*)$ (since k^* is stationary) $\implies Q(k^*) > Q(k_i)$, contradicting Q-Monotonicity since $k_i > k^*$.
- The only remaining possibility is $k^* = k_t = k'$.

Theorem 3: Argmin(q) is Stationary

Proof:

- $i = argmin(q)$ & q-Monotonicity $\implies q(i-1) \geq q(i) \leq q(i+1)$ with at most one equality.
- Assume the second inequality is strict so $q(i) < q(i+1)$.
  1. Corollary 2 $\implies q(i-1) \geq b_i, \implies q(i) \geq b(i)$ by Lemma 1.
  2. Applying Corollary 2 again we have $q(i) < b_{i+1}$ completing the proof in this case.
- Alternatively, if $q(i) = q(i+1)$ then $i+1$ also $= argmin(q)$; we have by q-Monotonicity $q(i) \leq q(i+1) < q(i+2)$ and we find from the previous argument $i+1$ is stationary.//

Multi-Factor

Current Status of Long-Only Minimum Variance Weights via Stationary Points

- The form of Equation (13) and the "symbolic" Equation (12) suggest an iterative procedure similar to the CST used in the one factor case.
- Task: Generalize the monotonicity properties of One Factor Models to Multi-Factor Models.
- Task: Identify appropriate stationary point convergence mechanism.
- Open Research Problems!

Some Open Questions.

Open questions include:

- The multi-factor CST converges if we allow *pruning* $\equiv$ at each iteration remove securities with negative weights from the current and *all future* iterations. *But* convergence stationary point might represent an under-inclusive portfolio:
    - What is the region for convergence without pruning?
    - What is the region for full optimization with pruning?
    - Are these regions connected?
    - When are sub-portfolios of minimum variance portfolios (suboptimally) minimum variance? Is there an associated calculus?
- What are the connections with convex optimization?
- What is the geometry?