DRAFT 071624 - GEOMETRY OF MINIMUM VARIANCE PORTFOLIO ESTIMATION ERROR

Introduction

Consider a positive definite symmetric nxn covariance matrix $\mathbb{M}$. For concreteness, we may take as an example $\mathbb{M} \equiv$ the return covariance matrix for portfolios in n securities, with r_i the dollar return on the investment x_i in security s_i. A point $x \in \mathbb{R}^n$ then represents a portfolio with total size (or market value) of Σx_i. For similarity with the traditional Markowitz framework of percentage returns and portfolio weights, we take for analysis a portfolio of size $1, so the $ investments translate easily into portfolio weights.

The traditional framework's main benefit is that by dispensing with the absolute size of investments, both return and risk for all investments, individual securities and portfolios alike, can be graphed in two dimensions. However, the price of this simplicity is projection and the associated loss of some interesting geometry, as we show below. The price of that geometry is representing portfolios as points in $\mathbb{R}^n$.

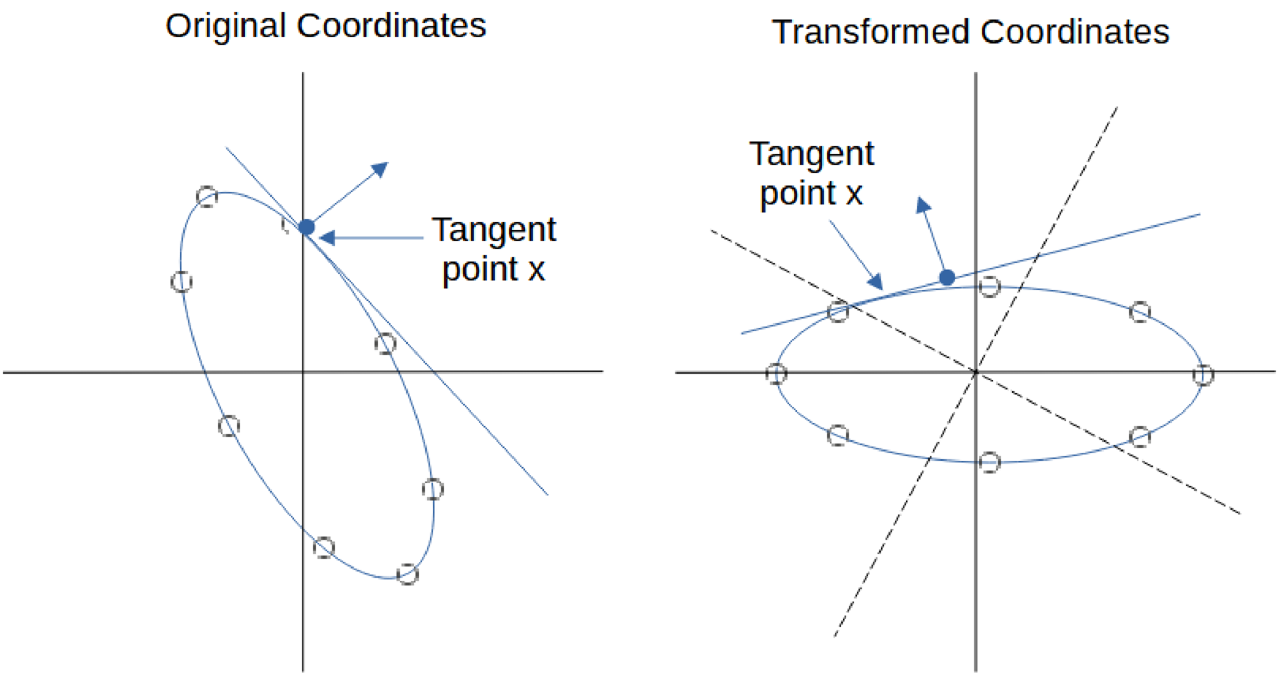
Instead of representing risk on a one-dimensional x-axis as in the traditional framework, we represent risk by variance, shown by ellipsoidal contours, equivalently, *isovariance ellipsoids*, given by $\{x|x'\mathbb{M}x = c\}$, c a constant. Noting that scaling a portfolio by k increases its variance by k^2, we see that the variances of portfolios with size $1 constitute the intesections of all such ellipsoidal contours above some threshold. That threshold is identified with the smallest contour that intersects the size hyperplane $\mathbb{H}_1 \equiv \{x|\Sigma x_i = 1\}$, which is the unique ellipsoid tangent to $\mathbb{H}_1$. Note that $\mathbb{H}_1 \equiv \{x|e'x = 1\}$, where e is the vector with all its entries $= 1$. (*NB* Although we work exclusively with this "full investment" constraint, other constraints could be handled similarly.)

Analysis

Orthogonal Transformation

In our framework, the hyperplane $\mathbb{H}_1$ is simple (indeed, like all hyperplanes, it may be identified with its normal vector) but the ellipsoids are complex, representing as they do level contours for a positive definite symmetric (covariance) matrix. We therefore simplify $\mathbb{M}$ by diagonalizing it via an orthogonal transformation. In the transformed coordinate system the matrix $\mathbb{M}$ becomes a diagonal matrix of eigenvalues $\equiv \Lambda$ while $\mathbb{H}_1$ transforms to a hyperplane $\mathbb{H}_m$ which we identify with its normal vector, an arbitrary vector $m | m'm = n$.

In two dimensions, the transformation may be visualized as follows:



Except where otherwise stated, we will work in the transformed coordinate system.

Tangent Point

In the transformed coordinate system, denote by $\mathbb{L}_m$ the tangent ellipsoid to $\mathbb{H}_m$. $\mathbb{L}_m$ is defined by $\{x | x'\Lambda = c\}$ for some c. Let x_0 be the point at which $\mathbb{L}_m, \mathbb{H}_m$ are tangent. x_0 satisfies the following two equations:

$$\begin{gathered} \Lambda x_0 = km \\ m'x_0 = 1 \end{gathered} \tag{1A,1B}$$

The first equation (1A) follows from standard matrix differentiation and says that at x_0 the normal to $\mathbb{L}_m$ is a multiple of m, the normal to the transformed constraint line, while the second equation (1B) says that x_0 is on the constraint line, which follows because an orthogonal transformation is an isometry.

From the equation (1A):

$$x_0 = k\Lambda^{-1}m \tag{2}$$

Plugging this into equation (1B):

$$
\begin{aligned}
km'\Lambda^{-1}m = 1 \\
\implies k = \frac{1}{m'\Lambda^{-1}m}
\end{aligned}
\tag{3}
$$

Thus,

$$
x_0 = \frac{\Lambda^{-1}m}{m'\Lambda^{-1}m} \tag{4}
$$

Note also pre-multiplying Equation (2) by $x'\Lambda$ we also have another expression for k showing k equals the variance of $\mathbb{L}_0$:

$$
\begin{aligned}
x_0'\Lambda x_0 = kx_0'\Lambda\Lambda^{-1}m \\
= kx_0'm = k \times 1 = k \\
\implies k = x_0'\Lambda x_0
\end{aligned}
$$

Comparing with Equation (3), the variance at x_0, and thus at slope m, is equal to:

$$
\frac{1}{m'\Lambda^{-1}m} \tag{2}
$$

Example

For a simple computational example in $\mathbb{R}^2$, apply Equation (4) with

$m' = (\alpha, \bar{\alpha}), \bar{\alpha} = \sqrt{1-\alpha^2}, \alpha = cos(\theta)$, where $\theta \equiv$ angle of rotation for the orthogonal transformation, and let

$\Lambda_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix}, Z > 1 \implies \Lambda_0^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{Z} \end{bmatrix}$

We have from equation (4):

$$
\begin{aligned}
x_0 = \frac{(\alpha, \bar{\alpha}/Z)}{m'\Lambda_0^{-1}m} \\
= \frac{(\alpha, \bar{\alpha}/Z)}{\alpha^2 + \bar{\alpha}^2/Z} \\
= \frac{(Z\alpha, \bar{\alpha})}{(Z-1)\alpha^2 + 1}
\end{aligned}
$$

With $\Lambda = \Lambda_0$ and m as in the example above we have

$$Var = \frac{1}{\alpha^2 + \bar{\alpha}^2/Z} = \frac{Z}{(Z-1)\alpha^2 + 1}$$

Considering $\alpha_1, \alpha_2, Z = 0, \frac{1}{\sqrt{2}}, 10$, respectively, we see that where the eigenvalues differ by an order of magnitude, a rotation of $\frac{\pi}{4}$ can change variance by a factor of $\frac{10}{0.275} = 5.5$. Indeed, because the curvature of the ellipse at α_1 is zero, small changes in m result in (potentially very) large changes in variance and weights.

By contrast, a further rotation of $\frac{\pi}{4}$ changes variance by a fraction of only $\frac{1.8}{1} = 1.8$.

We therefore see that the effect of eigenvector estimation error on estimated variance error depends on the ground truth, more specifically, on the curvature of the isovariance surface at the ground truth tangency point ("base point"). Since the constraint introduces tangency, this conclusion is intuitive - the curvature is the rate of change in slope, and thus the location of the tangent point. Moreover, because for ellipsoids the curvature itself changes, the portfolio weight and variance estimation and will both exhibit bias.