Learning Drifting Data Using Selective Sampling

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Outline

On-line Learning

- Used for many different tasks:
  - Information filtering
  - Market analysis
  - To be added
- Data revealed round after round
- Learner has to make prediction on-line

On-line Learning

- At each round t:
    1. Instance x_t observed
    2. Prediction $\hat{y}_t$ issued
    3. Regret R_t suffered
    4. True value y_t revealed
- Regret definition

$$R_t = \mathcal{L}\left\{\hat{y}_t, y_t\right\} - \mathcal{L}\left\{\bar{y}_t, y_t\right\}$$

$\bar{y}_t$ - optimal prediction, $\mathcal{L}\left\{,\right\}$ - loss function

Selective Sampling

- Acquiring true value (or label) y_t can be costly or complicated
- Algorithms can achieve similar results without knowing all true labels y_t
- Only some of the labels are queried. Others remain unknown
- Queries are issued according to confidence of the algorithm

Drifting Data

- In some problems data varies over time
- The optimal function from family of functions learned, is not fixed
- Approaches to handle drifting data :
  - Detect the drift
  - Adjust algorithm to drift setting

Drifting Data - Related Work

- Time windows:
  - For drift detection
  - For prediction
- Detecting drift using error distribution
- Forgetting strategies

Drifts vs Shifts

Data can change gradually (drift) or suddenly (shift or switch).

Example - market analysis:

- Gradual drift - effect of Israeli real estate bubble
- Abrupt Switch (shift) - effect of Russian invasion to Crimea

In this work

- Effect of switch on on-line learning problems is approached
- Linear classification and linear regression selective sampling settings are examined
- Selective sampling principles suggested to overcome switch effect

Suggested Method

- Implement selective sampling approach to overcome switch in an on-line learning setting
- Strategy of dealing with switch:
  1. Detect the switch
  2. If switch is undetected - assure that the harm caused by the switch is minor
  3. Small probability for false detections

Suggested Method

- Exploit notion of confidence provided by selective sampling to handle switch
- Avoid unnecessary loss of information while overcoming switch effect
- Time windows used for change detection but not for classification
- Selective sampling is also used in original context - only part of the labels are queried

Problem Setting - On-line Classification

Problem setup:

- $\boldsymbol{x}_t \in R^d$
- $y_t \in \{\pm 1\}$

Assumptions on instances:

- $\|\boldsymbol{x}_t\| = 1$

Problem Setting - On-line Classification

Assumptions on label distribution:

- $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1,\ \boldsymbol{u}, \boldsymbol{v} \in R^d$
- For $t \leq \tau$ holds:
  - $\mathrm{E}\left[y_t\right] = \boldsymbol{u}^\top \boldsymbol{x}_t$
  - $\Pr\left[y_t = 1\right] = \frac{1+\boldsymbol{u}^\top \boldsymbol{x}_t}{2}$
- For $t > \tau$ holds:
  - $\mathrm{E}\left[y_t\right] = \boldsymbol{v}^\top \boldsymbol{x}_t$
  - $\Pr\left[y_t = 1\right] = \frac{1+\boldsymbol{v}^\top \boldsymbol{x}_t}{2}$

Problem Setting

- At each round t instance $\boldsymbol{x}_t$ observed
- Prediction $\hat{y}_t$ issued
- Regret R_t suffered
- True label y_t can be queried

Linear classification is used to issue prediction:

$$\hat{y}_t = \operatorname{sign}\left\{\boldsymbol{w}_t^\top \boldsymbol{x}_t\right\}$$

RLS Estimator

RLS estimator (Cesa-Bianchi et al. 2004, 2006, 2009) used:

$$\boldsymbol{w}_t = A_t^{-1} b_t$$

Where:

- $A_t = \left(I + \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^\top + \boldsymbol{x}_t \boldsymbol{x}_t^\top \right) \in R^{d \times d}$
- $b_t = \sum_{i=1}^{n} y_i \boldsymbol{x}_i \in R^d$

$n = N_t$ - number of queries issued until round $t-1$

A_t can be viewed as covariance or "confidence " matrix

BBQ Algorithm - Querying Labels

Cesa-Bianchi, Gentile, Orabona 2009:

Selective sampling algorithm -

- Set $\kappa \in (0,1)$
- Calculate $r_t = \boldsymbol{x}_t^\top A_t^{-1} \boldsymbol{x}_t$
- If $r_t > t^{-\kappa}$ label y_t is queried and A_t, b_t are updated
- If $r_t \leq t^{-\kappa}$ label y_t remains unknown and no update performed

RLS Estimator and BBQ Algorithm Properties

Assuming standard, no switch setting, $(\boldsymbol{u} = \boldsymbol{v})$:

- Logarithmic cumulative regret R_T:

$$R_T \leq O\left(d \ln T\right) + f\left\{\kappa\right\}$$

- Reduced number of queried labels N_T:

$$N_T \sim O\left(dT^{\kappa} \ln T\right)$$

- Controlled estimator bias B_t:

$$B_t = \boldsymbol{w}_t^\top \boldsymbol{x}_t - \mathrm{E}\left[\boldsymbol{w}_t^\top \boldsymbol{x}_t\right] \leq r_t + \sqrt{r_t}$$

Effect of Switch

- Switch from $\boldsymbol{u}$ to $\boldsymbol{v}$ at round τ increases bias bound:

$$B_t \leq r_t + \sqrt{r_t} + N_\tau \|\boldsymbol{v} - \boldsymbol{u}\| \sqrt{r_t}$$

- The bias B_t controls the regret R_t. So switch at round τ increases regret bound:

$$R_T \leq O\left(\left\{\|\boldsymbol{v} - \boldsymbol{u}\|^2 \tau^{2\kappa} \left(d \ln \tau\right)^2 + 1\right\} d \ln T\right) + \Gamma\left\{\frac{1}{\kappa}\right\}$$

Using Selective Sampling to Detect Switch

- When switch occurs low cumulative regret can no longer be expected
- Selective sampling approach measures estimator's confidence regarding prediction on give instance x_t
- When confidence on instance x_t is high, prediction should be close to optimal
- Evaluating prediction on "high confidence" instances can be used to detect change

Using Selective Sampling to Detect Switch

Confidence factor $r_t = \boldsymbol{x}_t^\top A_t^{-1} \boldsymbol{x}_t$:

- Small r_t - high confidence regarding instance $\boldsymbol{x}_t$
- Large r_t - high uncertainty (low confidence) regarding instance $\boldsymbol{x}_t$

r_t controls both the bias B_t and the instantaneous regret R_t:

- If r_t is large, low regret R_t can not be assured, switch or no switch.
- If r_t is small, low regret R_T should be expected. Unless a switch had occurred...

Using Selective Sampling to Detect Switch

Main idea - evaluate performance on instances with small r_t to detect switch.

Bad performance will indicate that switch had occurred.

- Performance cannot be evaluated comparing prediction $\hat{y}_t$ to label y_t due to noise
- Even if optimal classifier $\boldsymbol{v}$ is known - error probability will be $\frac{1-\left|\boldsymbol{v}^\top \boldsymbol{x}_t\right|}{2}$
- Prediction will be evaluated comparing to optimal classifier $\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right|$

Windowed Demo Classifiers

- Problem - optimal classifier $\boldsymbol{v}$ is unknown.
- Solution - estimate optimal classifier $\boldsymbol{v}$ with demo classifier h_t.
- Demo classifier h_t constructed from a window of last L rounds and should estimate $\boldsymbol{v}$ well enough
- Performance of $\boldsymbol{w}_t$ comparing to h_t will evaluate $\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right|$ and indicate possible switch

Construction of Windowed Demo Classifier

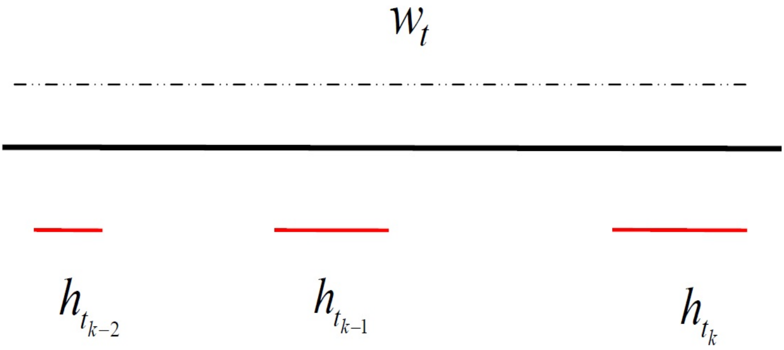
- Parameter - initial window length $L_0 > 0$
- Calculate window length $L_t = L_0 + \sqrt{t}$
- At round t select a window of last L_t instances
- Calculate $A_{L_t} = \left(I + \sum_{l=t-L_t}^{t-1} \boldsymbol{x}_l \boldsymbol{x}_l^\top \right), b_{L_t} = \sum_{l=t-L_t}^{t-1} y_l \boldsymbol{x}_l$
- Construct demo classifier $h_t = \left(A_{L_t} + \boldsymbol{x}_t \boldsymbol{x}_t^\top \right)^{-1} b_{L_t}$

Resolution of Windowed Demo Classifier

- Demo classifier h_t constructed at round t from a window of last L_t instances
- Demo classifier h_t will be used to evaluate next KL_t instances
- Next demo classifier $h_{t_{next}}$ will be constructed at round $KL_t + 1$
- Only $\frac{T}{K}$ labels will be queried
- Switch detection resolution reduced from L_t to KL_t

Construction and Resolution of Demo Classifier

Demo classifier setup:



Algorithm for Detecting Switch

- Calculate estimator $\boldsymbol{w}_t = A_t^{-1} b_t$
  - Where $A_t = \left(I + \sum_{i=1}^{N_t} \boldsymbol{x}_i \boldsymbol{x}_i^\top + \boldsymbol{x}_t \boldsymbol{x}_t^\top \right), b_t = \sum_{i=1}^{N_t} y_i \boldsymbol{x}_i$
- Calculate demo classifier $h_t = \left(A_{L_t} + \boldsymbol{x}_t \boldsymbol{x}_t^\top \right)^{-1} b_{L_t}$
  - $A_{L_t} = \left(I + \sum_{l=m_t-L_t}^{m_t} \boldsymbol{x}_l \boldsymbol{x}_l^\top \right), b_{L_t} = \sum_{l=m_t-L_t}^{m_t} y_l \boldsymbol{x}_l$
- Calculate $r_t = \boldsymbol{x}_t^\top A_t^{-1} \boldsymbol{x}_t$
- Calculate $r_{L_t} = \boldsymbol{x}_t^\top \left(A_{L_t} + \boldsymbol{x}_t \boldsymbol{x}_t^\top \right)^{-1} \boldsymbol{x}_t$

Algorithm for Detecting Switch

- Parameter - $\delta \in (0,1)$
- Calculate $\delta_t = \frac{\delta}{t(t+1)}$
- Calculate $C_t = \left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$
- Calculate:

$$K_t = \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_t} + r_t + \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_{L_t}} + r_{L_t}$$

- If $C_t > K_t$ declare switch and restart classifier w_t from zero. Else continue to next round

Algorithm for Detecting Switch

- If $C_t > K_t$
  - If switch occurred it is detected
  - If no switch occurred $\Pr[C_t > K_t] \leq 2\delta$ - to be proved
- If $C_t \leq K_t$
  - If no switch occurred, no change applied to standard setting
  - If a switch occurred and undetected as $C_t \leq K_t$, additional regret caused would be small - to be proved

Algorithm's Main Result

Main result:

1. If a switch occurs - algorithm detects it, or assures it causes small harm
2. No switch occurs - no false detection

Proving Main Result

Proof structure as follows:

1. Proving undetected switch will cause low regret:
   - Bounding instantaneous regret
   - Summing to bound cumulative regret
2. Proving probability for false positives is small

Proving Main Result - Instantaneous Regret Bound

Instantaneous regret R_t controlled by the term $|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t|$:

$$R_t = \Pr\left[y_t \boldsymbol{w}_t^\top \boldsymbol{x}_t < 0\right] - \Pr\left[y_t \boldsymbol{v}^\top \boldsymbol{x}_t < 0\right] \leq$$
$$\varepsilon I_{\{|\boldsymbol{v}^\top \boldsymbol{x}_t| < \varepsilon\}} + \Pr\left[\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| \geq \varepsilon\right]$$

$|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t|$ can be bounded by triangle inequality:

$$\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| \leq \left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right| + \left|\boldsymbol{v}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$$
$$= C_t + \left|\boldsymbol{v}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$$

Proving Main Result - Instantaneous Regret Bound

- C_t is bounded by K_t as a switch was not detected:

$$C_t \le K_t = \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_t} + r_t + \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_{L_t}} + r_{L_t} \quad (1)$$

- From properties of RLS estimator (Cesa-Bianchi et al.) applied to demo classifier h_t:

$$\left|\boldsymbol{v}^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right| \le \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_{L_t}} + r_{L_t} \quad (2)$$

With probability $1 - \delta_t$.

Proving Main Result - Instantaneous Regret Bound

Combining bounds on C_t and on $\left|\boldsymbol{v}^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$:

$$\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| \le \sqrt{r_t}\left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right) + r_t$$
$$+2\sqrt{r_{L_t}}\left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right) + 2r_{L_t}$$

Proving Main Result - Instantaneous Regret Bound

Using identities:

- $\Pr[A] = \mathrm{E}[I_A]$
- $I_{\{x<1\}} \leq e^{1-x}$

final bound instantaneous regret R_t achieved:

$$
\begin{aligned}
R_t &\leq \varepsilon I_{\{|\boldsymbol{v}^\top \boldsymbol{x}_t|<\varepsilon\}} + \Pr\left[\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| \geq \varepsilon\right] \\
&\leq \varepsilon I_{\{|\boldsymbol{v}^\top \boldsymbol{x}_t|<\varepsilon\}} + 2\exp\left\{1 - \frac{\alpha_{\varepsilon,t}}{r_{L_t}}\right\} + 2\exp\left\{1 - \frac{\beta_\varepsilon}{r_{L_t}}\right\} \\
&+ \exp\left\{1 - \frac{\alpha_{\varepsilon,t}}{r_t}\right\} + \exp\left\{1 - \frac{\beta_\varepsilon}{r_t}\right\} + \delta_t
\end{aligned}
$$

Proving Main Result - Cumulative Regret Bound

Cumulative regret R_T is given by:

$$R_T = \sum_{t=1}^{T} R_t$$

Cumulative regret R_T will be bounded by summing over bound of instantaneous regret R_t.

Calculation outline:

1. Summation over the r_t terms - separate calculation for:
   - Rounds t for which with $r_t \leq t^{-\kappa}$
   - Rounds t for which with $r_t > t^{-\kappa}$
2. Summation over the r_{L_t} terms.
3. Deriving final bound

Proving Main Result - Cumulative Regret Bound

Summation over r_t terms - for rounds with $r_t > t^{-\kappa}$ -

- Identity $\exp\{-x\} \leq \frac{1}{ex}$ gives:

$$\sum_{t=T_1, r_t > t^{-\kappa}}^{T} \exp\left\{1 - \frac{\alpha_{\varepsilon,t}}{r_t}\right\} \leq \frac{1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} r_t$$

- The result $r_t \leq \left(1 - \frac{\det A_{t-1}}{\det A_t}\right)$ (Cesa-Bianchi et. al 2004) yields:

$$\frac{1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} r_t \leq \frac{1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \left(1 - \frac{\det A_{t-1}}{\det A_t}\right)$$

Proving Main Result - Cumulative Regret Bound

Summation over r_t terms - for rounds with $r_t > t^{-\kappa}$ -

- Identity $1 - x \leq -\ln x$ (for $x \leq 1$) gives:

$$\frac{1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \left(1 - \frac{\det A_{t-1}}{\det A_t}\right) \leq \frac{-1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \ln\left(\frac{\det A_{t-1}}{\det A_t}\right)$$

- Computing the sum will give final expression:

$$-\frac{1}{\alpha_{\varepsilon,T}} \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \ln\left(\frac{\det A_{t-1}}{\det A_t}\right) \leq \frac{1}{\alpha_{\varepsilon,T}} \left\{d \ln T - \ln\left(\det A_{T_1}\right)\right\}$$

Proving Main Result - Cumulative Regret Bound

Summation over r_t terms - for rounds with $r_t \leq t^{-\kappa}$ -

- Substituting $r_t \leq t^{-\kappa}$ and replacing sum with integral yields:

$$\sum_{t=T_1, r_t > t^{-\kappa}}^{T} \exp\left\{1 - \frac{\alpha_{\varepsilon,t}}{r_t}\right\} \leq e \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \exp\left\{-\frac{\alpha_{\varepsilon,t}}{t^{-\kappa}}\right\}$$

$$\leq e \int_{T_1}^{T} \exp\left\{-\alpha_{\varepsilon,T} t^{\kappa}\right\} dt =$$

$$= \frac{e}{\kappa \left(\alpha_{\varepsilon,T}\right)^{\frac{1}{\kappa}}} \left(\Gamma\left\{\frac{1}{\kappa}, \alpha_{\varepsilon,T} T_1^{\kappa}\right\} - \Gamma\left\{\frac{1}{\kappa}, \alpha_{\varepsilon,T} T^{\kappa}\right\}\right)$$

- Last equality follows from the identity:

$$\int \exp\{a z^s\}\, dz = -\frac{z(-a z^s)^{-\frac{1}{s}}}{s} \Gamma\left\{\frac{1}{s}, -a z^s\right\}$$

Proving Main Result - Cumulative Regret Bound

Summation over r_{L_t} terms -

Matrix Chernoff bound - for a series of random, i.i.d PSD matrices $Z_k \in R^{d \times d}$ holds:

$$\Pr\left[\lambda_{\min}\left\{\sum_k Z_k\right\} \leq (1-\gamma)\,\mu_{\min}\right] \leq d\left(\frac{e^{-\gamma}}{(1-\gamma)^{(1-\gamma)}}\right)^{\frac{\mu_{\min}}{\rho}}$$

where:

- $\gamma \in (0,1)$
- $\mu_{\min} = \lambda_{\min}\left\{\sum_k \mathrm{E}\left[Z_k\right]\right\}$
- $\lambda_{\max}\left\{\mathrm{E}\left[Z_k\right]\right\} \leq \rho$

Proving Main Result - Cumulative Regret Bound

Summation over r_{L_t} terms -

- Assumption: smallest eigenvalue of covariance matrix grows linearly - $\lambda_{\min}\left\{\sum_{k=1}^{L} \mathrm{E}\left[\boldsymbol{x}_k\boldsymbol{x}_k^\top\right]\right\} \sim O\left(\frac{L}{d}\right)$
- Using Chernoff matrix bound on $Z_k = \boldsymbol{x}_k\boldsymbol{x}_k^\top$, under the above assumption, yields:

$$\lambda_{\min}\left\{A_{L_t}\right\} = \lambda_{\min}\left\{I + \sum_{k=1}^{L_t} \boldsymbol{x}_k\boldsymbol{x}_k^\top\right\} > (1-\gamma)\frac{L_t}{d} + 1$$

Proving Main Result - Cumulative Regret Bound

Summation over r_{L_t} terms -

Using the bound and identities below:

- For unit normed $\boldsymbol{x}$: $\boldsymbol{x}^\top M \boldsymbol{x} \le \lambda_{max}\{M\}$
- $\lambda_{max}\{M\} = \frac{1}{\lambda_{min}\{M^{-1}\}}$
- $\lambda_{\min}\{A_{L_t}\} > (1-\gamma)\frac{L_t}{d}$

we get:

$$r_{L_t} = \boldsymbol{x}_t^\top \left(A_{L_t} + \boldsymbol{x}_t \boldsymbol{x}_t^\top\right)^{-1} \boldsymbol{x}_t \le \frac{d}{(L_t+2)(1-\gamma)}$$

Proving Main Result - Cumulative Regret Bound

Summation over r_{L_t} terms -

- Replacing $L_t = L_0 + \sqrt{t}$ into the bound would yield:

$$r_{L_t} \leq \frac{d}{\left(L_0 + \sqrt{t} + 2\right)(1-\gamma)}$$

- Substituting bound r_{L_t} into the sum over regret R_T bound:

$$\sum_{t=T_1}^{T} \exp\left\{1 - \frac{\alpha_{\varepsilon,t}}{r_{L_t}}\right\} \leq e \sum_{t=T_1, r_t > t^{-\kappa}}^{T} \exp\left\{-\hat{\alpha}_{\varepsilon,t}\left(L_0 + \sqrt{t}\right)\right\}$$

Proving Main Result - Cumulative Regret Bound

Summation over r_{L_t} terms -

Now replacing sum with integral and solving as before yields:

$$e\sum_{t=T_1}^{T}\exp\left\{-\hat{\alpha}_{\varepsilon,t}\left(L_0+\sqrt{t}\right)\right\}\leq$$
$$\frac{2e}{\left(\tilde{\alpha}_{\varepsilon,T}\right)^2}\left(\Gamma\left\{2,\tilde{\alpha}_{\varepsilon,T}\left(T_1-L_0\right)^{\frac{1}{2}}\right\}-\Gamma\left\{2,\tilde{\alpha}_{\varepsilon,T}\left(T-L_0\right)^{\frac{1}{2}}\right\}\right)$$

Proving Main Result - Cumulative Regret Bound

- Summing all developed bounds yields:

$$R_T \leq O\left(d \left\{\ln T\right\}^2\right)$$

- Cumulative regret controlled and small
- Bound overcomes switch effect - square logarithmic bound in T comparing to more than linear bound in τ

Proving Main Result - No False Positives

- Reminder -switch detection if $C_t > K_t$
- Assuring no false detection - if no switch occurs than $C_t \leq K_t$
- From triangle inequality:

$$C_t = \left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right| \leq \left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| + \left|\boldsymbol{v}^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$$

Proving Main Result - No False Positives

- Reminder - from RLS estimator properties, holds with probability $1 - 2\delta_t$:
  - $\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right| \leq \sqrt{r_t}\left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right) + r_t$
  - $\left|\boldsymbol{v}^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right| \leq \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_{L_t}} + r_{L_t}$
- Substituting this into bound:

$$C_t \leq \sqrt{r_t}\left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right) + r_t + \left(\sqrt{2\ln\frac{2}{\delta_t}} + 1\right)\sqrt{r_{L_t}} + r_{L_t} \leq K_t$$

Proving Main Result - No False Positives

- Last result assures that if no switch occurred $C_t \leq K_t$ with probability $1 - 2\delta_t$
- Thus the probability for a false detection at round t is $2\delta_t$
- Using union bound - probability for a false detection throughout the algorithm is 2δ

Simulation Results

Synthetic data Matlab simulation:

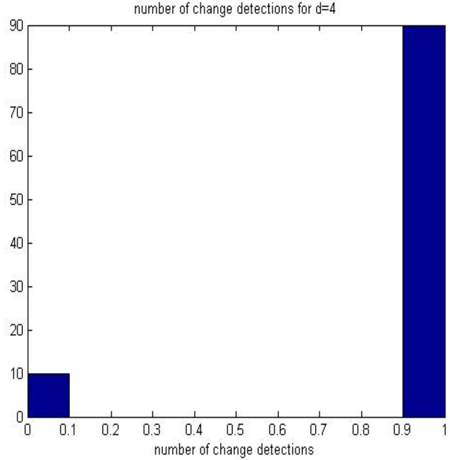
- $T = 10^5$, $\boldsymbol{x}_t \in R^4$, $\kappa = 0.7$, $L_0 = 500$, $K = 6$
- Instances x_t drawn randomly from Gaussian distribution and then normalized
- Labels y_t drawn from Bernoulli distribution with $p_t = \frac{1+\boldsymbol{u}^\top \boldsymbol{x}_t}{2}$

Results, averaged after 100 runs of the algorithm:

- Optimal classifier - 28.81% error
- BBQ RLS estimator classifier - 35.76% error
- Suggested switch detection algorithm - 29.58% error

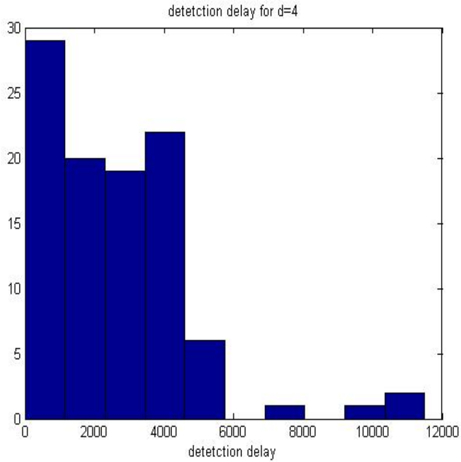
Simulation Results - No false Detection

Number of switch detections, in 100 runs of the algorithm:



Simulation Results - Switch Detected Relatively Fast

Distribution of switch delay, in 100 runs of the algorithm:



Problem Setting - On-line Regression

- Analysis for regression is similar to one presented for classification.
- Differences between the two problems will be discussed.

Problem setup:

- $\boldsymbol{x}_t \in R^d$
- $y_t \in R$

Problem Setting - On-line Regression

Assumptions:

- $\|\boldsymbol{x}_t\| = \|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1, \boldsymbol{u}, \boldsymbol{v} \in R^d$
- for $t \leq \tau$ holds:

  $y_t = \boldsymbol{u}^\top \boldsymbol{x}_t + \eta_t$ and $\mathrm{E}\left[y_t\right] = \boldsymbol{u}^\top \boldsymbol{x}_t$
- for $t > \tau$ holds:

  $y_t = \boldsymbol{v}^\top \boldsymbol{x}_t + \eta_t$ and $\mathrm{E}\left[y_t\right] = \boldsymbol{v}^\top \boldsymbol{x}_t$
- η_t i.i.d noise with $\mathrm{E}\left[\eta_t\right] = 0, var\left\{\eta_t\right\} = \sigma^2$
- $|\eta_t| \leq Z_\eta$, Z_η is known

On-line Regression - Regret Definition

- Linear regression is to issue prediction:

$$\hat{y}_t = \boldsymbol{w}_t^\top \boldsymbol{x}_t$$

- As in classification $\boldsymbol{w}_t$ is the RLS estimator: $\boldsymbol{w}_t = A_t^{-1} b_t$

- Major difference - instantaneous regret definition:

$$\begin{aligned} R_t &= (y_t - \hat{y}_t)^2 - \left(y_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)^2 \\ &= \left(y_t - \boldsymbol{w}_t^\top \boldsymbol{x}_t\right)^2 - \left(y_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)^2 \\ &= \left(\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)^2 - 2\left(y_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)\left(\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right) \end{aligned}$$

Regret Bounds

- RLS properties used to bound $\left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right|$
- To bound the cumulative regret R_T:
  1. $\sum_{t=T_1}^{T} \left(\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)^2$ will be bounded using RLS properties
  2. Azuma's inequality will be used to bound $\sum_{t=T_1}^{T} \left(y_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)\left(\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{v}^\top \boldsymbol{x}_t\right)$

Main Result for Regression

- If switch occurs it is either detected or low regret assured
- In case of non detection - $R_T \leq O\left(\sqrt{d}T^{\left(1-\frac{2\kappa+1}{4}\right)}\ln T\right)$
- Improving expected bound due to effect of switch:
  $R_T \leq O\left(d^2\tau^{2\kappa}\left\{\ln T\right\}^2 T^{1-\kappa}\right)$
- Result close to bound in no switch case - $O\left(T^{1-\kappa}\ln T\right)$
- Probability for false detection - 2δ
- Note - in regression problem selective sampling increases regret

Simulation Results

Proposed Method

- Problem approached - switch in data at on-line linear classification and regression settings
- Proposed solution -
  1. Using confidence notion of selective sampling for switch detection
  2. Constructing demo classifier h_t from recent time window
  3. Difference between estimator and demo classifier $C_t = \left|\boldsymbol{w}_t^\top \boldsymbol{x}_t - h_t^\top \boldsymbol{x}_t\right|$ indicates switch
  4. Difference considered with respect to confidence r_t about instance $\boldsymbol{x}_t$

Main Results

- Algorithm either detects switch or assurers low regret in case of non-detection
- Low probability for false detection
- Simulations on synthetic data results:
  - Most switches were detected in relativity short time
  - Error reduced close to optimal result
  - No false detections

Model Limitations

- Strong assumptions on instances x_t distribution
- Strong assumptions on label y_t distribution
- Union bound used on non independent events
- In regression setting noise bound Z_η assumed to be known
- Demo classifier construction increases number of queries

Future Work

- Introducing proposed methods and concepts with adjustment to drift detection
- Constructing demo classifier from recent queried labels without further sampling
- Weakening assumptions on data

Thank You for Your Time

Questions?