

# Learning Drifting Data Using Selective Sampling

## Machine Learning Seminar

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March 26, 2014

# On-line Learning

On-line learning setting:

- Data revealed round after round
- Learner has to make prediction on-line
- At each round  $t$ :
  - 1 Instance  $x_t$  observed
  - 2 Prediction  $\hat{y}_t$  issued
  - 3 True value  $y_t$  revealed
- Used for many tasks from information filtering to market analysis

# Selective Sampling

- In some tasks acquiring true value (or label)  $y_t$  can be costly or complicated
- Algorithm can achieve similar results without knowing true labels  $y_t$
- Only some of the labels are queried. others remain unknown
- Queries are issued according to confidence of the learner regarding prediction at round  $t$

# Drifting Data

- In some problems data changes over time
- Optimal function from family of function learned, is not fixed
- Amongst different solutions known in literature:
  - Time windows for classifying or drift detection
  - Detecting drift using error distribution
  - Forgetting strategies

# Drifts vs Shifts

Data can change gradually (drift) or suddenly (shift or switch).

Example - market analysis:

- Gradual drift - Sub prime crisis effect on mid 2000's trading algorithms for USA market
- Sudden Switch (shift) - Russian invasion to Crimea effect on 2014 trading algorithms for Eastern Europe market

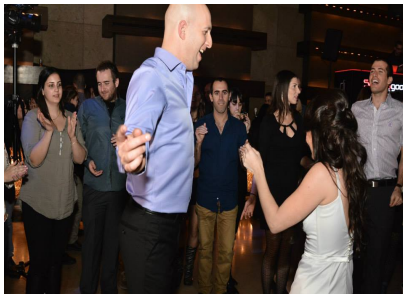


# Drifts vs Shifts

Data can change gradually (drift) or suddenly (shift or switch).

Example - targeted advertisement:

- Gradual drift - person's new skiing hobby effect on advertisement algorithms
- Sudden Switch (shift) - person's engagement effect on advertisement algorithms



# Suggested Method

- Use selective sampling to detect switch
- Exploit notion of confidence provided by selective sampling
- Avoid unnecessary loss of information while overcoming switch effect
- Time windows used for change detection but not for classification

# Expected Objectives

Approaching the problem of shifting concept in an on-line learning classification or regression setting, using selective sampling principles to overcome switch, the following objectives are set:

- 1 Detect the switch
- 2 If switch is undetected - assure that the harm caused by the switch is minor
- 3 Small probability for false detections



# Problem Setting - On-line Classification

Problem setup:

- $\mathbf{x}_t \in R^d$
- $y_t \in \{\pm 1\}$

Assumptions:

- $\|\mathbf{x}_t\| = \|\mathbf{u}\| = \|\mathbf{v}\| = 1, \mathbf{u}, \mathbf{v} \in R^d$
- For  $t \leq \tau$  holds  $E[y_t] = \mathbf{u}^\top \mathbf{x}_t$
- For  $t > \tau$  holds  $E[y_t] = \mathbf{v}^\top \mathbf{x}_t$

# Problem Setting

- At each round  $t$  instance  $x_t$  is observed
- Prediction  $\hat{y}_t$  is issued
- Regret  $R_t$  is suffered
- True label  $y_t$  can be queried

Linear classification is used to issue prediction:

$$\hat{y}_t = \text{sign} \left\{ \mathbf{w}_t^\top \mathbf{x}_t \right\} \quad (1)$$

# RLS Estimator

$w_t$  would be the RLS estimator (Chesa-Bianchi et al. 2004, 2006, 2009) solving the following problem:

$$w_t = \min_{w \in R^d} \left\{ \sum_{i=1}^n \left( y_i - w^\top x_i \right)^2 + \|w\|^2 \right\} \quad (2)$$

with  $n = N_t$  being the number of queries issued until round  $t - 1$

# RLS Estimator

The solution to equation 2 is:

$$\mathbf{w}_t = \left( I + S_{t-1} S_{t-1}^T + \mathbf{x}_t \mathbf{x}_t^T \right)^{-1} S_{t-1} Y_{t-1} \quad (3)$$

Where:

- $S_{t-1} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in R^{d \times n}$
- $Y_{t-1} = (y_1, \dots, y_n) \in R^n$ .

# RLS Estimator

Equivalent formulation:

$$\mathbf{w}_t = \mathbf{A}_t^{-1} \mathbf{b}_t \quad (4)$$

Where:

- $\mathbf{A}_t = \left( \mathbf{I} + \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^\top + \mathbf{x}_t \mathbf{x}_t^\top \right) \in \mathbb{R}^{d \times d}$
- $\mathbf{b}_t = \sum_{i=1}^n y_i \mathbf{x}_i \in \mathbb{R}^d$

$\mathbf{A}_t$  can be viewed as covariance or "confidence" matrix

# BBQ Algorithm - Querying Labels

Cesa-Bianchi, Gentile, Orabona 2009:

Selective sampling algorithm -

- Set  $\kappa \in (0, 1)$
- Calculate  $r_t = \mathbf{x}_t^\top A_t^{-1} \mathbf{x}_t$
- If  $r_t > t^{-\kappa}$  label  $y_t$  is queried
- If  $r_t \leq t^{-\kappa}$  label  $y_t$  remains unknown

# RLS Estimator and BBQ Algorithm Properties

Assuming standard, no switch setting, ( $u = v$ ):

- Logarithmic cumulative regret  $R_T$ :

$$R_T \leq O(d \ln T) \quad (5)$$

- Reduced number of queried labels  $N_T$ :

$$N_T \sim O(dT^\kappa \ln T) \quad (6)$$

- Controlled estimator bias  $B_t$ :

$$B_t = \mathbf{w}_t^\top \mathbf{x}_t - \mathbb{E} \left[ \mathbf{w}_t^\top \mathbf{x}_t \right] \leq r_t + \sqrt{r_t} \quad (7)$$

# Effect of Switch

- Switch from  $\mathbf{u}$  to  $\mathbf{v}$  at round  $\tau$  increases bias bound:

$$B_t \leq r_t + \sqrt{r_t} + N_\tau \|\mathbf{v} - \mathbf{u}\| \sqrt{r_t} \quad (8)$$

- The bias  $B_t$  controls the regret  $R_T$ . So switch at round  $\tau$  increases regret bound:

$$R_T \leq O \left( \left\{ \|\mathbf{v} - \mathbf{u}\|^2 \tau^{2\kappa} (d \ln \tau)^2 + 1 \right\} d \ln T \right) \quad (9)$$



# Using Selective Sampling to Detect Switch

- When switch occurs low cumulative regret can no longer be expected
- Selective sampling approach measures estimator's confidence regarding prediction on given instance  $x_t$
- When confidence on instance  $x_t$  is high prediction should be close to optimal
- Evaluating prediction on "high confidence" instances can be used to detect change

# Using Selective Sampling to Detect Switch

Confidence factor  $r_t = \mathbf{x}_t^\top A_t^{-1} \mathbf{x}_t$  :

- Small  $r_t$  - high confidence regarding instance  $\mathbf{x}_t$
- Large  $r_t$  - high uncertainty (low confidence) regarding instance  $\mathbf{x}_t$

$r_t$  controls both the bias  $B_t$  and the instantaneous regret  $R_t$ :

- If  $r_t$  is large, low regret  $R_t$  can not be assured, switch or no switch.
- If  $r_t$  is small, low regret  $R_T$  should be expected. Unless a switch had occurred...

# Using Selective Sampling to Detect Switch

Main idea - evaluate performance on instances with small  $r_t$  to detect switch.

Bad performance will indicate that switch had occurred.

- Performance cannot be evaluated comparing prediction  $\hat{y}_t$  to label  $y_t$  due to noise
- Even if optimal classifier  $v$  is known - error probability will be  $\frac{1 - |v^\top x_t|}{2}$
- Prediction will be evaluated comparing to optimal classifier  $|w_t^\top x_t - v^\top x_t|$

# Windowed Demo Classifiers

- Problem - optimal classifier  $v$  is unknown.
- Solution - estimate optimal classifier  $v$  with demo classifier  $h_t$ .
- Demo classifier  $h_t$  constructed from a window of last  $L$  rounds and should estimate  $v$  well enough
- Performance of  $w_t$  comparing to  $h_t$  will evaluate  $|w_t^\top x_t - v^\top x_t|$  and indicate possible switch

# Construction of Windowed Demo Classifier

- Parameter - initial window length  $L_0 > 0$
- Calculate window length  $L_t = L_0 + \sqrt{t}$
- At round  $t$  select a window of last  $L_t$  instances

- Calculate  $A_{L_t} = \left( I + \sum_{l=t-L_t}^{t-1} \mathbf{x}_l \mathbf{x}_l^\top \right), b_{L_t} = \sum_{l=t-L_t}^{t-1} y_l \mathbf{x}_l$

- Construct demo classifier  $h_t = (A_{L_t} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} b_{L_t}$

# Resolution of Windowed Demo Classifier

- Demo classifier  $h_t$  constructed at round  $t$  from a window of last  $L_t$  instances
- Demo classifier  $h_t$  will be used to evaluate next  $KL_t$  instances
- Next demo classifier  $h_{t_{next}}$  will be constructed at round  $KL_t + 1$
- Only  $\frac{T}{K}$  labels will be queried
- Switch detection resolution reduced from  $L_t$  to  $KL_t$

# Algorithm for Detecting Switch

- Calculate estimator  $\mathbf{w}_t = A_t^{-1} \mathbf{b}_t$ 
  - Where  $A_t = \left( I + \sum_{i=1}^{N_t} \mathbf{x}_i \mathbf{x}_i^\top + \mathbf{x}_t \mathbf{x}_t^\top \right)$ ,  $\mathbf{b}_t = \sum_{i=1}^{N_t} y_i \mathbf{x}_i$
- Calculate demo classifier  $\mathbf{h}_t = (A_{L_t} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} \mathbf{b}_{L_t}$ 
  - $A_{L_t} = \left( I + \sum_{l=m_t-L_t}^{m_t} \mathbf{x}_l \mathbf{x}_l^\top \right)$ ,  $\mathbf{b}_{L_t} = \sum_{l=m_t-L_t}^{m_t} y_l \mathbf{x}_l$
- Calculate  $r_t = \mathbf{x}_t^\top A_t^{-1} \mathbf{x}_t$
- Calculate  $r_{L_t} = \mathbf{x}_t^\top (A_{L_t} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} \mathbf{x}_t$

# Algorithm for Detecting Switch

- Parameter -  $\delta \in (0, 1)$
- Calculate  $\delta_t = \frac{\delta}{t(t+1)}$
- Calculate  $C_t = |\mathbf{w}_t^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t|$
- Calculate:

$$K_t = \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_t} + r_t + \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_{L_t}} + r_{L_t}$$

- If  $C_t > K_t$  declare switch and restart classifier  $w_t$  from zero. Else continue to next round



# Algorithm's Main Result

- If  $C_t > K_t$ 
  - If switch occurred it is detected
  - If no switch occurred  $\Pr [C_t > K_t] \leq (1 - 2\delta)$  - to be proved
- If  $C_t \leq K_t$ 
  - If no switch occurred, no change applied to standard setting
  - If a switch occurred and undetected as  $C_t \leq K_t$ , additional regret caused would be small - to be proved

# Algorithm's Main Result

Main result:

- 1 If a switch occurs - algorithm detects it, or assures it causes small harm
- 2 No switch occurs - no false detection

# Proving Main Result

Proof structure as follows:

- ① Proving undetected switch will cause low regret:
  - Bounding instantaneous regret
  - Summing to cumulative regret
- ② Proving probability for false positives is small

# Proving Main Result - Instantaneous Regret Bound

Instantaneous regret  $R_t$  controlled by the term  $|\mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t|$ :

$$\begin{aligned} R_t &= \Pr \left[ y_t \mathbf{w}_t^\top \mathbf{x}_t < 0 \right] - \Pr \left[ y_t \mathbf{v}^\top \mathbf{x}_t < 0 \right] \leq \\ &\varepsilon I_{\{|\mathbf{v}^\top \mathbf{x}_t| < \varepsilon\}} + \Pr \left[ \left| \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right| \geq \varepsilon \right] \end{aligned} \quad (10)$$

$|\mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t|$  can be bounded by triangle inequality:

$$\begin{aligned} \left| \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right| &\leq \left| \mathbf{w}_t^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| + \left| \mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| \\ &= C_t + \left| \mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| \end{aligned} \quad (11)$$

# Proving Main Result - Instantaneous Regret Bound

- $C_t$  is bounded by  $K_t$  as a switch was not detected:

$$C_t \leq K_t = \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_t} + r_t + \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_{L_t}} + r_{L_t} \quad (12)$$

- From properties of RLS estimator (Cesa-Bianchi at all) applied to demo classifier  $h_t$ :

$$\left| \mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| \leq \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_{L_t}} + r_{L_t} \quad (13)$$

With probability  $1 - \delta_t$ .

# Proving Main Result - Instantaneous Regret Bound

Combining bounds on  $C_t$  and on  $|\mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t|$ :

$$\begin{aligned} |\mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t| &\leq \sqrt{r_t} \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) + r_t \\ &+ 2\sqrt{r_{L_t}} \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) + 2r_{L_t} \end{aligned} \quad (14)$$

# Proving Main Result - Instantaneous Regret Bound

Using identities:

- $\Pr[A] = \mathbb{E}[I_A]$
- $I_{\{x < 1\}} \leq e^{1-x}$

final bound instantaneous regret  $R_t$  achieved:

$$\begin{aligned} R_t &\leq \varepsilon I_{\{|\mathbf{v}^\top \mathbf{x}_t| < \varepsilon\}} + \Pr \left[ \left| \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right| \geq \varepsilon \right] \quad (15) \\ &\leq \varepsilon I_{\{|\mathbf{v}^\top \mathbf{x}_t| < \varepsilon\}} + 2 \exp \left\{ 1 - \frac{\alpha_{\varepsilon,t}}{r_{L_t}} \right\} + 2 \exp \left\{ 1 - \frac{\beta_\varepsilon}{r_{L_t}} \right\} \\ &\quad + \exp \left\{ 1 - \frac{\alpha_{\varepsilon,t}}{r_t} \right\} + \exp \left\{ 1 - \frac{\beta_\varepsilon}{r_t} \right\} + \delta_t \end{aligned}$$

# Proving Main Result - Cumulative Regret Bound

Cumulative regret  $R_T$  is given by:

$$R_T = \sum_{t=1}^T R_t \quad (16)$$

Cumulative regret  $R_T$  will be bounded by summing over bound of instantaneous regret  $R_t$ .

Calculation outline:

- 1 Summation over the  $r_t$  terms - separate calculation for:
  - Rounds  $t$  for which with  $r_t \leq t^{-\kappa}$
  - Rounds  $t$  for which with  $r_t > t^{-\kappa}$
- 2 Summation over the  $r_{L_t}$  terms.
- 3 Deriving final bound



# Proving Main Result - Cumulative Regret Bound

Summation over  $r_t$  terms - for rounds with  $r_t > t^{-\kappa}$  -

- Identity  $\exp\{-x\} \leq \frac{1}{ex}$  gives:

$$\sum_{t=T_1, r_t > t^{-\kappa}}^T \exp\left\{1 - \frac{\alpha_{\varepsilon, t}}{r_t}\right\} \leq \frac{1}{\alpha_{\varepsilon, T}} \sum_{t=T_1, r_t > t^{-\kappa}}^T r_t$$

- The result  $r_t \leq \left(1 - \frac{\det A_{t-1}}{\det A_t}\right)$  (Cesa-Bianchi et. al 2004) yields:

$$\frac{1}{\alpha_{\varepsilon, t}} \sum_{t=T_1, r_t > t^{-\kappa}}^T r_t \leq \frac{1}{\alpha_{\varepsilon, t}} \sum_{t=T_1, r_t > t^{-\kappa}}^T \left(1 - \frac{\det A_{t-1}}{\det A_t}\right)$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_t$  terms - for rounds with  $r_t > t^{-\kappa}$  -

- Identity  $1 - x \leq -\ln x$  (for  $x \leq 1$ ) gives:

$$\frac{1}{\alpha_{\varepsilon,t}} \sum_{t=T_1, r_t > t^{-\kappa}}^T \left(1 - \frac{\det A_{t-1}}{\det A_t}\right) \leq -\frac{1}{\alpha_{\varepsilon,t}} \sum_{t=T_1, r_t > t^{-\kappa}}^T \ln \left(\frac{\det A_{t-1}}{\det A_t}\right)$$

- Computing the sum will give final expression:

$$-\frac{1}{\alpha_{\varepsilon,t}} \sum_{t=T_1, r_t > t^{-\kappa}}^T \ln \left(\frac{\det A_{t-1}}{\det A_t}\right) \leq \frac{1}{\alpha_{\varepsilon,t}} \{d \ln T - \ln (\det A_{T_1})\}$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_t$  terms - for rounds with  $r_t \leq t^{-\kappa}$  -

- Substituting  $r_t \leq t^{-\kappa}$  and replacing sum with integral yields:

$$\begin{aligned}
 \sum_{t=T_1, r_t > t^{-\kappa}}^T \exp \left\{ 1 - \frac{\alpha_{\varepsilon, t}}{r_t} \right\} &\leq e \sum_{t=T_1, r_t > t^{-\kappa}}^T \exp \left\{ -\frac{\alpha_{\varepsilon, t}}{t^{-\kappa}} \right\} \\
 &\leq e \int_{T_1}^T \exp \{ -\alpha_{\varepsilon, T} t^{\kappa} \} dt = \\
 &= \frac{e}{\kappa (\alpha_{\varepsilon, T})^{\frac{1}{\kappa}}} \left( \Gamma \left\{ \frac{1}{\kappa}, \alpha_{\varepsilon, T} T_1^{\kappa} \right\} - \Gamma \left\{ \frac{1}{\kappa}, \alpha_{\varepsilon, T} T^{\kappa} \right\} \right)
 \end{aligned}$$

- Last equality follows from the identity:

$$\int \exp \{ a z^s \} dz = -\frac{z(-a z^s)^{-\frac{1}{s}}}{s} \Gamma \left\{ \frac{1}{s}, -a z^s \right\} \quad (17)$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_{L_t}$  terms -

Matrix Chernoff bound - for a series of random, i.i.d PSD matrices  $Z_k \in R^{d \times d}$  holds:

$$\Pr \left[ \lambda_{\min} \left\{ \sum_k Z_k \right\} \leq (1 - \gamma) \mu_{\min} \right] \leq d \left( \frac{e^{-\gamma}}{(1 - \gamma)^{(1-\gamma)}} \right)^{\frac{\mu_{\min}}{\rho}} \quad (18)$$

where:

- $\gamma \in (0, 1)$
- $\mu_{\min} = \lambda_{\min} \{ \sum_k \mathbb{E} [Z_k] \}$
- $\lambda_{\max} \{ \mathbb{E} [Z_k] \} \leq \rho$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_{L_t}$  terms -

- Assumption: smallest eigenvalue of covariance matrix grows linearly -  $\lambda_{\min} \left\{ \sum_{k=1}^L \mathbb{E} [\mathbf{x}_k \mathbf{x}_k^\top] \right\} \sim O\left(\frac{L}{d}\right)$
- Using Chernoff matrix bound on  $Z_k = \mathbf{x}_k \mathbf{x}_k^\top$ , under the above assumption, yields:

$$\lambda_{\min} \{A_{L_t}\} = \lambda_{\min} \left\{ I + \sum_{k=1}^{L_t} \mathbf{x}_k \mathbf{x}_k^\top \right\} > (1 - \gamma) \frac{L_t}{d} + 1 \quad (19)$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_{L_t}$  terms -

Using the bound and identities below:

- For unit normed  $\mathbf{x}$ :  $\mathbf{x}^\top M \mathbf{x} \leq \lambda_{\max}\{M\}$
- $\lambda_{\max}\{M\} = \frac{1}{\lambda_{\min}\{M^{-1}\}}$
- $\lambda_{\min}\{A_{L_t}\} > (1 - \gamma) \frac{L_t}{d}$

we get:

$$r_{L_t} = \mathbf{x}_t^\top \left( A_{L_t} + \mathbf{x}_t \mathbf{x}_t^\top \right)^{-1} \mathbf{x}_t \leq \frac{d}{(L_t + 2)(1 - \gamma)} \quad (20)$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_{L_t}$  terms -

- Replacing  $L_t = L_0 + \sqrt{t}$  into the bound would yield:

$$r_{L_t} \leq \frac{d}{(L_0 + \sqrt{t} + 2)(1 - \gamma)} \quad (21)$$

- Substituting bound  $r_{L_t}$  into the sum over regret  $R_T$  bound:

$$\sum_{t=T_1}^T \exp \left\{ 1 - \frac{\alpha_{\varepsilon,t}}{r_{L_t}} \right\} \leq e \sum_{t=T_1, r_t > t^{-\kappa}}^T \exp \left\{ -\hat{\alpha}_{\varepsilon,t} (L_0 + \sqrt{t}) \right\}$$

# Proving Main Result - Cumulative Regret Bound

Summation over  $r_{L_t}$  terms -

Now replacing sum with integral and solving as before yields:

$$e \sum_{t=T_1}^T \exp \left\{ -\hat{\alpha}_{\varepsilon,t} \left( L_0 + \sqrt{t} \right) \right\} \leq \frac{2e}{(\tilde{\alpha}_{\varepsilon,T})^2} \left( \Gamma \left\{ 2, \tilde{\alpha}_{\varepsilon,T} (T_1 - L_0)^{\frac{1}{2}} \right\} - \Gamma \left\{ 2, \tilde{\alpha}_{\varepsilon,T} (T - L_0)^{\frac{1}{2}} \right\} \right)$$



# Proving Main Result - Cumulative Regret Bound

- Summing all developed bounds yields:

$$R_T \leq O\left(d \{\ln T\}^2\right) \quad (22)$$

- Cumulative regret controlled and small
- Bound overcomes switch effect - square logarithmic bound in  $T$  comparing to more than linear bound in  $\tau$

# Proving Main Result - No False Positives

- Reminder -switch detection if  $C_t > K_t$
- Assuring no false detection - if no switch occurs than  $C_t \leq K_t$
- From triangle inequality:

$$C_t = \left| \mathbf{w}_t^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| \leq \left| \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right| + \left| \mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t \right| \quad (23)$$

# Proving Main Result - No False Positives

- Reminder - from RLS estimator properties, holds with probability  $1 - 2\delta_t$ :

- $|\mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t| \leq \sqrt{r_t} \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) + r_t$

- $|\mathbf{v}^\top \mathbf{x}_t - h_t^\top \mathbf{x}_t| \leq \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_{L_t}} + r_{L_t}$

- Substituting this into bound:

$$\begin{aligned} C_t &\leq \sqrt{r_t} \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) + r_t \\ &+ \left( \sqrt{2 \ln \frac{2}{\delta_t}} + 1 \right) \sqrt{r_{L_t}} + r_{L_t} \leq K_t \end{aligned}$$

# Proving Main Result - No False Positives

- Last result assures that if no switch occurred  $C_t \leq K_t$  with probability  $1 - 2\delta_t$
- Thus the probability for a false detection at round  $t$  is  $2\delta_t$
- Using union bound - probability for a false detection throughout the algorithm is  $2\delta$

# Simulation Results

Synthetic data Matlab simulation:

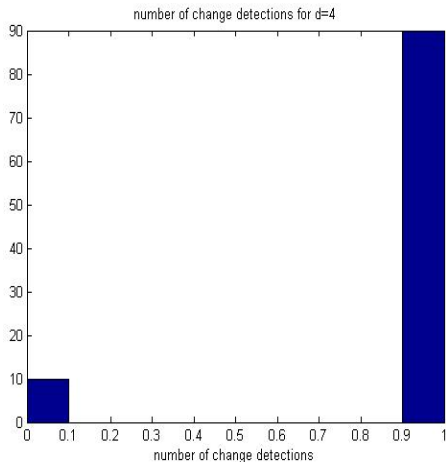
- $T = 10^5$ ,  $\mathbf{x}_t \in R^4$ ,  $\kappa = 0.7$ ,  $L_0 = 500$ ,  $K = 6$
- Instances  $\mathbf{x}_t$  drawn randomly from Gaussian distribution and then normalized
- Labels  $y_t$  drawn from Bernoulli distribution with  $p_t = \frac{1 - \mathbf{u}^\top \mathbf{x}_t}{2}$

Results, averaged after 100 runs of the algorithm:

- Optimal classifier - 28.81% error
- BBQ RLS estimator classifier - 35.76% error
- Suggested switch detection algorithm - 29.58% error

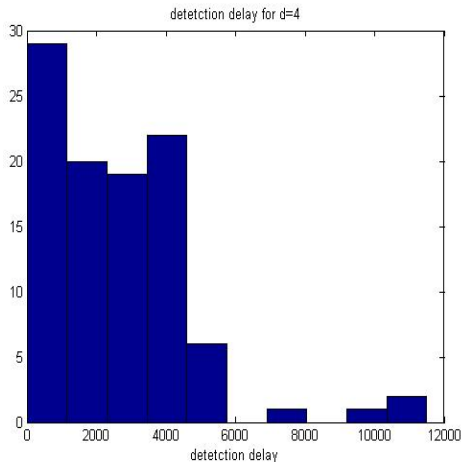
# Simulation Results - No false Detection

Number of switch detections, in 100 runs of the algorithm:



# Simulation Results - Switch Detected Relatively Fast

Distribution of switch delay, in 100 runs of the algorithm:



# Problem Setting - On-line Regression

- Analysis for regression is similar to one presented for classification.
- Differences between the two problems will be discussed.

Problem setup:

- $\mathbf{x}_t \in R^d$
- $y_t \in R$



# Problem Setting - On-line Regression

Assumptions:

- $\|x_t\| = \|u\| = \|v\| = 1, u, v \in R^d$

- for  $t \leq \tau$  holds:

$$y_t = u^\top x_t + \eta_t \text{ and } E[y_t] = u^\top x_t$$

- for  $t > \tau$  holds:

$$y_t = v^\top x_t + \eta_t \text{ and } E[y_t] = v^\top x_t$$

- $\eta_t$  i.i.d noise with  $E[\eta_t] = 0, \text{var}\{\eta_t\} = \sigma^2$

- $|\eta_t| \leq Z_\eta, Z_\eta$  is known

# Regret Definition

- Linear regression is to issue prediction:

$$\hat{y}_t = \mathbf{w}_t^\top \mathbf{x}_t \quad (24)$$

- As in classification  $\mathbf{w}_t$  is the RLS estimator:  $\mathbf{w}_t = A_t^{-1} \mathbf{b}_t$
- Major difference - instantaneous regret definition:

$$\begin{aligned} R_t &= (y_t - \hat{y}_t)^2 - \left( y_t - \mathbf{v}^\top \mathbf{x}_t \right)^2 \\ &= \left( y_t - \mathbf{w}_t^\top \mathbf{x}_t \right)^2 - \left( y_t - \mathbf{v}^\top \mathbf{x}_t \right)^2 \\ &= \left( \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right)^2 - 2 \left( y_t - \mathbf{v}^\top \mathbf{x}_t \right) \left( \mathbf{w}_t^\top \mathbf{x}_t - \mathbf{v}^\top \mathbf{x}_t \right) \end{aligned}$$

# Regret Bounds

- RLS properties used to bound  $|w_t^\top x_t - v^\top x_t|$
- To bound the cumulative regret  $R_T$ :
  - 1  $\sum_{t=T_1}^T (w_t^\top x_t - v^\top x_t)^2$  will be bounded using RLS properties
  - 2 Azuma's inequality will be used to bound  $\sum_{t=T_1}^T (y_t - v^\top x_t) (w_t^\top x_t - v^\top x_t)$

# Main Result for Regression

- If switch occurs it is either detected or low regret assured
- In case of non detection -  $R_T \leq O\left(\sqrt{d}T^{(1-\frac{2\kappa+1}{4})} \ln T\right)$
- Improving expected bound due to effect of switch:  
$$R_T \leq O\left(d^2 \tau^{2\kappa} \{\ln T\}^2 T^{1-\kappa}\right)$$
- Result close to bound in no switch case -  $O\left(T^{1-\kappa} \ln T\right)$
- Probability for false detection -  $1 - 2\delta$
- Note - in regression problem selective sampling increases regret

# Simulation Results

# Problem

# Proposed Method

# Main Results



# Model Limitations

# Future Work