Multi-task Learning with a Shared Annotator

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Contents

List of Figures

List of Tables

Abstract

We introduce a new multi-task framework, in which K online learners are sharing a single annotator with limited bandwidth. On each round, each of the K learners receives an input, and makes a prediction about the label of that input. Then, a shared (stochastic) mechanism decides which of the K inputs will be annotated. The learner that receives the feedback (label) may update its prediction rule, and we proceed to the next round. We develop online algorithms for multi-task binary classification that learns in this setting, and bound its performances in the worst-case setting. The algorithms apply an exploration-exploitation approach in order to allocate the limited feedback in the way that reduces the total number of errors. Additionally, we show that our algorithm can be used to solve two bandits problems: contextual bandits, and dueling bandits with context, both allowed to decouple exploration and exploitation. Empirical study with OCR data , vowel prediction (VJ project) and NLP -sentiment analysis data shows that our algorithms outperforms algorithms that use uniform allocation, and essentially makes more (accuracy) for the same labour of the annotator.

Abbreviations and Notations

$SHAMPO$	—	SHared Annotator for Multiple PrOblems
$\|\mathbf{u}\|$	—	ℓ_2-norm of the vector $\mathbf{u}$
$ECOC$	—	Error COrrecting Code
OCR	—	Optical Character Recognition
NLP	—	Natural language processing
$MFCC$	—	Mel-Frequency Cepstral Coefficients

Chapter 1

Introduction

Machine learning is a field of computer science that concerned with data processing and the ability of the computer to learn from this data. One main objective of this field is the development of algorithms capable of inference based on observable data, such as text documents, pictures, audio, video etc...

In the *Supervised Learning* setting, the input of the learning algorithm are input-label pairs. The goal of the algorithm is to learn the underlying connection between the inputs and their labels, thus being able to *predict* a label for a previously unseen input. When the possible label values are from a discrete finite set, this learning problem is called *classification*. The basic classification problem is the *binary classification*, i.e. classifying each data instance into one of only two possible classes. In contradiction to the multiclass classification task, in the binary case is more simple because, eliminating one class, gives you the correct one straightforward. However, this is more difficult in the *multiclass classification*, when even when we eliminate one possible class, we yet have some more classes the we need to decide which of these classes is the correct one.

1.1   Online learning

One of the main features of the classification problem is the way how the data is been collected. In some application, the labeled data is been collected first, such that we have an access to entire training dataset at once. Then we use this whole examples collection as an input to a *Batch learning* algorithm and learn a classification model about this problem. However, in a lot of real life application

3

this is not the case. In some problems like spam filtering, there is a flow of data that is transmitted in sequence and it takes time to collect the a large amount of data to learn from it, so we don't want to wait too long before we can have a decent prediction about the continuing incoming examples. For those application we use *Online Learning* based algorithms. In this setting, at any time we keep the learned model in memory, and update it when a new labeled example is coming in. Unlike the *Batch learning*, in *Online Learning* at any time, we the learner perception about the classification task become stronger when the time is pass by.

The *Online Learning* is performed in rounds, where in each round t, the algorithm gets an input instance $\mathbf{x}_t$ in some domain $\mathcal{X}$ and predicts a correspond measure, $\hat{p}_t$ based on the algorithm decision rule. This measure can be in the label domain, $\mathcal{Y}$, or can be mapped into $\hat{y}_t$ which is the predicted label in $\mathcal{Y}$. After predicting the label, the true label (y_t in the labels domain, $\mathcal{Y}$) is revealed and the learner suffers a non negative loss of $l\left(\hat{p}_t, y_t\right)$ that measures how much the prediction is compatible with the true label. The desired property of such function is to generate low values when the prediction is close to the actual label in some sense, and high values when the opposite is true. Then, the algorithm update its decision rule based on the past known data and the revealed label.

1.2 Selective Sampling

Usually, in an online binary learning task setting, we improve the prediction over the time, which means that the algorithm have less and lees prediction mistakes when it updates its model. Sometimes, annotating the data consume expensive resources, like time, money or manpower, and we would like to avoid using this resources when we can. In other words, we would like to avoid querying labels for the input examples when it is possible. For example, if we update the model only when there is a prediction mistake (as in Perceptron), we actually don't really use the information about the correct label when there is no need to update. In such cases, it will be helpful to assess every time how much we sure about our prediction, and no update should be done, so no query should be issued, or if we not sure about the prediction, hence we should issue a query and update the model using the update rule an the correct label. This approach, that queries labels only for selected examples is called *Selective sampling.*

1.3 Multi Task with Shared Annotator

In supervised learning setting, the main bottleneck is the need to annotate data. A common protocol is problem centric: first collect data or inputs automatically (with low cost), and then pass it on to a user or an expert to be annotated. Annotation can be outsourced to the crowed by a service like Mechanical Turk (like google's recaptcha project), or performed by experts as in the Linguistic data Consortium. Then, this data may be used to build models, either for a single task or many tasks. This approach is not making optimal use of the main resource - the annotator - as some tasks are harder than others, yet we need to give the annotator the (amount of) data to be annotated for each task a-priori.

Another aspect of this problem is the need to adapt systems to individual users, to this end, such systems may query the user for the label of some input, yet, if few systems will do so independently, the user will be flooded with queries, and will avoid interaction with those systems. For example, sometimes there is a need to annotate news items from few agencies. One person cannot handle all of them, and only some items can be annotated, which ones? Our setting is designed to handle exactly this problem, and specifically, how to make best usage of annotation time. This settings can also handle with the case when we want to limit the updates number, for example if we have a lot of of clients that generate data, but only one server with a limited computation power is allocated to process the received data and we want to limit the amount of updates for all tasks.

We propose a new framework of online multi-task learning with a shared annotator. Here, algorithms are learning few tasks simultaneously, yet they receive feedback using a central mechanism that trades off the amount of feedback (or labels) each task receives. We derive a specific algorithm based on the good-old Perceptron algorithm, called SHAMPO (SHared Annotator for Multiple PrOblems) for binary classification and analyze it in the mistake bound model, showing that our algorithm may perform well compared with methods that observe all annotated data. We then show how to reduce few contextual bandit problems into our framework, and provide specific bounds for such settings. We evaluate our algorithm with four different datasets for OCR, vowel prediction (VJ) and document classification, and show that it can improve performance either on average over all tasks, or even if their output is combined towards a single shared task, such as multi-class prediction. We conclude with discussion of related work, and few of the many routes to extend this work.

Chapter 2

The Shared Annotator Setting

In our setting, there are K binary classification tasks to be learned simultaneously. In opposed to the common multi-task classification settings, here, no dependency between the tasks is assumed during the analysis, but the tasks can be dependent as well. The model learning is performed in rounds as an online learning algorithm, as following: On each round t, there are K input-label pairs $(\mathbf{x}_{i,t}, y_{i,t})$, one for each classification task, where $i = 1, 2, \ldots, K$ is the task index and t is the step index. The inputs $\mathbf{x}_{i,t} \in \mathbb{R}^{d_i}$ are vectors, and the labels $y_{i,t} \in \{-1, +1\}$ are binary. In the general case, the input-spaces for each problem may be different, and inputs may have different number of elements. Yet, in order to simplify the notation and without loss of generality, from now on, in our analysis we assume that for all of the tasks, $\mathbf{x}_{i,t} \in \mathbb{R}^d$. i.e. all the tasks are in the same dimension and $d_i = d$ holds for all tasks. In practice, since the proposed algorithms use the margin that is affected by the vector norm, there is a need to scale all the vectors into a ball.

On round t, the learning algorithm receives K input vectors $\mathbf{x}_{i,t}$ for $i = 1, \ldots, K$ tasks and produce K binary-labels output $\hat{y}_{i,t}$, where $\hat{y}_{i,t} \in \{-1, +1\}$ is the label predicted for the input $\mathbf{x}_{i,t}$ of task i. The algorithm then chooses a task $J_t \in \{1, \ldots, K\}$ and asks from an annotator its true-label $y_{J_t,t}$ for that task J_t. Unlike the usual online multi-task setting, and due to the limitation in our case, it does not observe any other label. Then, the algorithm updates its models, using the received feedback, and proceeds to the next round (and inputs). For the ease of the calculations below, we denote by K indicators $Z_t = (Z_{1,t}, \ldots, Z_{K,t})$, the identity of the task which was queried by the algorithm on round t, and set $Z_{J_t,t} = 1$ and $Z_{i,t} = 0$ for $i \neq J_t$. Clearly from the definition, the condition, $\sum_i Z_{i,t} = 1, \forall i, t$, always holds. In order to use it in the analysis below, we define

the notation $E_{t-1}[x]$ as well, to be the conditional expectation $E[x|Z_1, \ldots Z_{t-1}]$ given all previous choices of the tasks to be queried.

Step	1	2	3	4	5
Task 1	Q	NQ	Q	NQ	NQ
Task 2	NQ	NQ	Q	Q	NQ
Task 3	Q	NQ	Q	NQ	NQ
Task 4	Q	Q	NQ	NQ	Q

Table 2.1: In selective sampling we focus on when to issue a query for a single task (a row).

Step	1	2	3	4	5
Task 1	Q	NQ	NQ	NQ	NQ
Task 2	NQ	NQ	NQ	Q	NQ
Task 3	NQ	NQ	NQ	NQ	Q
Task 4	NQ	Q	NQ	NQ	NQ

Table 2.2: In the SHAMPO setting we focus on when to issue a query for a single step (a column).

Schematic illustration of a single iteration of multi-task algorithms is shown in Fig. 2.1. The top panel shows the standard setting of online multi-task algorithms with a shared annotator, that labels all inputs, which are fed to the corresponding algorithms to update corresponding models. The bottom panel shows the SHAMPO algorithm, which couples labeling annotation and learning process, and synchronizes a single annotation per round. At most one task performs an update per round, the one with the annotated input.

As we have mentioned before, our setting is similar to the Selective Sampling case, in a sense that at every round we ask a single binary question. However, while the Selective Sampling algorithm asks "when" to issue a query, at each time step, our SAHMPO algorithm asks on "which one" of the tasks to issue a query. As a result, when we run multi-task problem as an independent Selective sampling tasks, we allow at each round, to query more than one query, or even less than one query, i.e. not asking for a feedback at all. The SHAMPO setting on the other
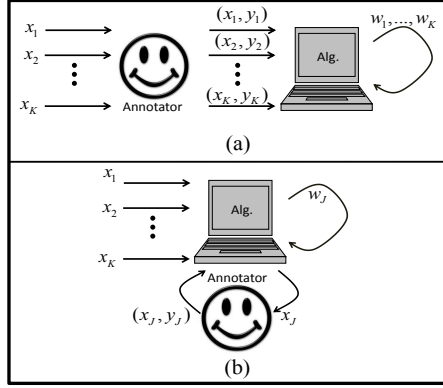
Figure 2.1: Illustration of a single iteration of multi-task algorithms (a) standard setting, shared annotator labels all inputs, and algorithms update models. (b) SHAMPO algorithm couples labeling annotation and learning process, and synchronizes a single annotation per round.

hand, is more limited, since for every round, the algorithm issue one, and only one query. An example for the difference between the Selective Sampling multi-task problem and the SHAMPO setting is Table 2.1 and Table 2. Where the cells are assigned with 'Q' (Queried) for tasks and rounds that the algorithm issue a query, and with 'NQ' (Not Queried) on tasks and rounds that the algorithm decided not to issue a query.

Chapter 3

SHMPO (SHared Annotator for Multiple Problems) Algorithms

We now describe algorithms for multi-task learning with a shared annotator setting, that uses linear models. Two steps are yet to be specified: how to pick a task to be labeled and how to perform an update, once the true label for that task is given. Our first algorithm is based on the perceptron algorithm

3.1 SHAMPO perceptron

We focus now on linear-functions of the form $f(\mathbf{x}) = \mathrm{sign}(p)$ for the quantity $p = \mathbf{w}^\top \mathbf{x}$, $\mathbf{w} \in \mathbb{R}^d$, which often called the *margin*. Specifically, the algorithm maintains a set of K weight vectors, $\mathbf{w}_{i,t}$, $i \in \{1, \dots, K\}$. On round t, the algorithm predicts a label for each one of the tasks, $\hat{y}_{i,t} = \mathrm{sign}(\hat{p}_{i,t})$ where $\hat{p}_{i,t} = \mathbf{w}_{i,t-1}^\top \mathbf{x}_t$. On rounds for which the label of some task J_t is queried, the algorithm update the model of the queried task only, such that for the rest of the tasks, $i \neq J_t$, we have $\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1}$ and no update is made for those unqueried tasks.

We say that the algorithm has a prediction mistake on task i in round t if $y_{i,t} \neq \hat{y}_{i,t}$, and denote this event by the indicator $M_{i,t} = 1$, otherwise, if there is no mistake, we set $M_{i,t} = 0$. The goal of the algorithm is to minimize the cumulative mistakes, $\sum_t \sum_i M_{i,t}$. Models are also evaluated using the *Hinge-loss*. Specifically, let $\mathbf{u}_i \in \mathbb{R}^d$ be some vector associated with problem i. We denote the Hinge-loss of it, with respect to some input-label pair $(\mathbf{x}_{i,t}, y_{i,t})$, by

$\ell_{\gamma,i,t}(\mathbf{u}_i) = \left(\gamma - y_{i,t}\mathbf{u}_i^\top \mathbf{x}_{i,t}\right)_+$, where, $(x)_+ = \max\{x, 0\}$, and $\gamma > 0$ is some parameter. The cumulative loss over all tasks and a sequence of n input steps, is $L_{\gamma,n} = L_\gamma(\{\mathbf{u}_i\}) = \sum_{t=1}^n \sum_{i=1}^K \ell_{\gamma,i,t}(\mathbf{u}_i)$. We also use the following expected hinge-loss over the random queried choices of the algorithm,

$$\bar{L}_{\gamma,n} = \bar{L}_{\{\mathbf{u}_i\}} = \mathrm{E}\left[\sum_t^n \sum_{i=1}^K M_{i,t} Z_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i)\right]. \tag{3.1}$$

Now, we proceed by describing our algorithm and specify how to choose a task to query its label, and how to perform an update.

In order to select a task to query on, the algorithm uses the absolute margin $\hat{p}_{i,t}$. We can see $|\hat{p}_{i,t}|$ as the certainty measure of the label prediction. intuitively if $|\hat{p}_{i,t}|$ is small, then there is uncertainty about the labeling of $\mathbf{x}_{i,t}$, and vice-versa for large values of $|\hat{p}_{i,t}|$. Similar argument was used by Tong and Koller [2000] for picking an example to be labeled in batch active learning. Prima facie, under this claim, at each point we should query the true label for the task that corresponds to the smallest margin. Yet, if the model $\mathbf{w}_{i,t-1}$ is not accurate enough, due to small number of observed examples, this estimation may be rough, and may lead to a wrong conclusion. We thus perform an exploration-exploitation strategy, and query tasks randomly, with a bias towards tasks with low margin $|\hat{p}_{i,t}|$.
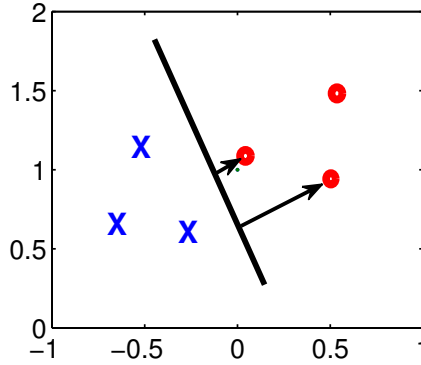

Figure 3.1: Example for the using of the margin as a certainty measure. The circles and crosses represent two different classes and the separate hyperplane is shown. One of the circles examples have lower margin than the other, so its label can easily been replaced with the other one.

To the best of our knowledge, exploration-exploitation usage in this context of choosing an examples to be labeled (e.g. in settings such as semi-supervised learning or selective sampling) is novel and new. In order to get this property, we induced a distribution over the tasks in the time step t, such that the probability to issue a query on the task j ($j = 1, \ldots, K$) is:

$$\Pr\left[J_t = j\right] = \frac{\left(b + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|\right)^{-1}}{D_t}$$

$$\text{for } D_t = \sum_{i=1}^{K} \left(b + |\hat{p}_{i,t}| - \min_m |\hat{p}_{m,t}|\right)^{-1}. \tag{3.2}$$

where $b \geq 0, b \in \mathbb{R}$ is a tradeoff parameter, between exploration and exploitation. Clearly, it is a distribution, since $\Pr\left[J_t = j\right] \geq 0$ and $\sum_j \Pr\left[J_t = j\right] = 1$. When we examine the extreme cases of b we see that for $b \to 0$ we have $\Pr\left[J_t = j\right] \to 1$ and for the task with minimal margin, $J_t = \arg\min_{i=1}^{K} |\hat{p}_{i,t}|$, and $\Pr\left[J_t = j\right] \to 0$ for all the rest. In this case , pure exploitation is being made. The pure exploration happens when $b \to \infty$, then the distribution is uniform, $\Pr\left[J_t = j\right] = 1/K, \quad \forall j$. Fig. **??** shows an example of this distribution for the case of two tasks ($K = 2$) at an arbitrary step t^*. For visualization purpose, we fix $|\hat{p}_{2,t^*}| = 0.5$ and see how the probability of task 1 to be chosen, is affected by varying b and $|\hat{p}_{1,t^*}|$ values. Three different vertical areas can be easily seen in the graph. The upper (green) area, where $b >> \max\{\hat{p}_{1,t^*}, \hat{p}_{2,t^*}\}$, shows uniform distribution ($\Pr\left[J_{t^*} = 2\right] = \Pr\left[J_{t^*} = 1\right] = 1/K = 0.5$) which represents the exploration over the tasks. In the lower area, the probability is compatible with the exploitation method and is changing between probability 1 in the left, and probability 0 to the right with a sharp threshold at $|\hat{p}_{1,t^*}| = 0.5$, which is very close to the delta distribution $\Pr\left[J_{t^*} = 1\right] = \mathbb{I}\left[\hat{p}_{1,t^*} < \hat{p}_{2,t^*}\right]$. Whereas, the intermediate area, is the exploration-exploitation area that is given by a distribution that is biased toward the lowest margin task.

As noted above we denote by $Z_{i,t} = 1$ iff $i = J_t$. The update of the algorithm is performed with the perceptron rule, that is $\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + M_{J_t,t} \, y_{J_t,t} \, \mathbf{x}_{J_t,t}$ and $\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1}$ for $i \neq J_t$. For simplicity of presentation we write the update for all of the tasks in one term as, $\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1} + Z_{i,t} \, M_{i,t} \, y_{i,t} \, \mathbf{x}_{i,t}$. One can notice that although this notation uses labels for all tasks, only the label of the task J_t is used in practice, as for other tasks $Z_{i,t} = 0$.

We call this algorithm *SHAMPO* for SHared Annotator for Multiple PrOblems.
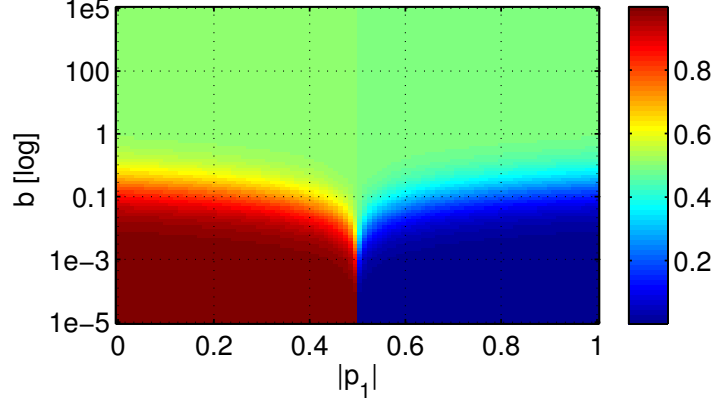
11

Figure 3.2: Example of the distribution over 2 tasks.

The pseudo-code of this algorithm appears in Alg. 3.2.

The following theorem states that the expected cumulative number of mistakes that the algorithm makes, may not be higher than for an algorithm that observes the labels of all inputs.

Theorem 1 *If SHAMPO algorithm runs on K tasks with K parallel example pair sequences $(\mathbf{x}_{i,1}, y_{i,1}), ...(\mathbf{x}_{i,n}, y_{i,n}) \in \mathbb{R}^d \times \{-1, 1\}$, $i = 1, ..., K$ with input parameter $b > 0$, then for all $\gamma > 0$, all $\mathbf{u}_i \in \mathbb{R}^d$ and all $n \geq 1$, there exists $0 < \delta \leq K$, such that,*

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{\delta}{\gamma}\left[\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{(2b + X^2)^2 \tilde{U}^2}{8\gamma b}\right],$$

where $X = \max_{i,t}\|\mathbf{x}_{i,t}\|$, $\tilde{U}^2 = \sum_{i=1}^{K}\|\mathbf{u}_i\|^2$ and the expectation is over the random choices of the algorithm.

Proof: Fix n examples sequences, $(\mathbf{x}_{i,1}, y_{i,1}), ..., (\mathbf{x}_{i,n}, y_{i,n})$ for each one of the K tasks. Let t be certain trial and i to be an update task on this trial, such that $M_{i,t} = 1$. We can write,

$$\gamma - \ell_{\gamma,i,t}(\mathbf{u}_i) = \gamma - \left(\gamma - y_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}\right)_+$$
$$\leq y_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}$$

12

Algorithm 3.1 SHAMPO: SHared Annotator for Multiple PrOblems.

Parameters: $b \in \mathbb{R} > 0$.
Initialize: $\mathbf{w}_{i,0} = \mathbf{0}$ for $i = 1, \ldots, K$

for $t = 1, 2, ..., n$ **do**

    1. Observe K instance vectors, $\mathbf{x}_{i,t}$, $(i = 1, \ldots, K)$.
    2. Compute margins $\hat{p}_{i,t} = \mathbf{w}_{i,t-1}^{\top} \mathbf{x}_{i,t}$.
    3. Predict K labels, $\hat{y}_{i,t} = \text{sign}(\hat{p}_{i,t})$.
    4. Draw task J_t with the distribution:

$$\Pr\left[J_t = j\right] = \frac{\left(b + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|\right)^{-1}}{D_t}, Eq. \text{ (??)}$$

$$D_t = \sum_i \left(b + |\hat{p}_{i,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|\right)^{-1}.$$

    5. Query the true label ,$y_{J_t,t} \in \{-1, 1\}$.
    6. Set the indicator $M_{J_t,t} = 1$ iff $y_{J_t,t} \neq \hat{y}_{J_t,t}$.
    7. Update with the perceptron rule:

$$\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + M_{J_t,t}\, y_{J_t,t}\, \mathbf{x}_{J_t,t} \tag{3.3}$$
$$\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1} \text{ for } i \neq J_t$$

end for
Output: $\mathbf{w}_{i,n}$ for $i = 1, \ldots, K$.

$$= y_{i,t}\left(\mathbf{u}_i + \mathbf{w}_{i,t-1} - \mathbf{w}_{i,t-1}\right)^T \mathbf{x}_{i,t}$$
$$= y_{i,t}\mathbf{w}_{i,t-1}^T \mathbf{x}_{i,t} + \left(\mathbf{u}_i - \mathbf{w}_{i,t-1}\right)^T y_{i,t}\mathbf{x}_{i,t}$$
$$= y_{i,t}\mathbf{w}_{i,t-1}^T \mathbf{x}_{i,t} + \left(\mathbf{u}_i - \mathbf{w}_{i,t-1}\right)^T \left(\mathbf{w}_{i,t} - \mathbf{w}_{i,t-1}\right)$$
$$= y_{i,t}\mathbf{w}_{i,t-1}^T \mathbf{x}_{i,t} + \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{1}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2$$
$$= y_{i,t}\hat{p}_{i,t} + \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{1}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2.$$

The last inequality holds for all $\gamma > 0$ and for all $\mathbf{u}_i \in \mathbb{R}^d$, so we can replace γ and $\mathbf{u}_i$ by their scaling $\alpha\gamma$ and $\alpha\mathbf{u}_i$ respectively, where $\alpha > 0$ is a scaling parameter that will be determined shortly. Since the inequality is computed on an update task, $y_{i,t}\hat{p}_{i,t} \leq 0$ holds , i.e. $y_{i,t}\hat{p}_{i,t} = -|\hat{p}_{i,t}|$, and we get

$\alpha\gamma + |\hat{p}_{i,t}| \leq$

$$\alpha\ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{1}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{1}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{1}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2.$$

Now, in order to generalize the inequality, we multiply it by the indicator $M_{i,t}Z_{i,t}$, which means that the inequality holds for trials and tasks where there is an update, as stated above.

$$M_{i,t}Z_{i,t}(\alpha\gamma - |\hat{p}_{i,t}|) \leq M_{i,t}Z_{i,t}\alpha\ell_{\gamma,i,t}(\mathbf{u}_i) +$$
$$\frac{M_{i,t}Z_{i,t}}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{M_{i,t}Z_{i,t}}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{M_{i,t}Z_{i,t}}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2.$$

Yet, in the other cases, when no update is performed and $M_{i,t}Z_{i,t} = 0$, the equality $\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1}$ holds as well, so we can rid off the indicator for any one of the last three terms and we have

$$M_{i,t}Z_{i,t}(\alpha\gamma + |\hat{p}_{i,t}|) \leq M_{i,t}Z_{i,t}\alpha\ell_{\gamma,i,t}(\mathbf{u}_i) +$$
$$\frac{1}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{1}{2}\|\alpha\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{M_{i,t}Z_{i,t}}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2.$$
$$(3.4)$$

Next, we sum the inequality above, over t, recall the facts that $\mathbf{w}_{i,0} = 0$ and $\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2 \leq X^2$ for $X = \max_{i,t}\|\mathbf{x}_{i,t}\|$ to get,

$$\sum_{t=1}^{n} M_{i,t}Z_{i,t}\left(\alpha\gamma + |\hat{p}_{i,t}| - \frac{X^2}{2}\right) \leq \alpha\sum_{t=1}^{n} M_{i,t}Z_{i,t}\ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{\alpha^2}{2}\|\mathbf{u}_i\|^2. \quad (3.5)$$

Substituting $\alpha = (2b + X^2)/2\gamma$ (where $b \in \mathbb{R}, \ b > 0$) in Eq. (3.6), we get

$$\sum_{t=1}^{n} M_{i,t}Z_{i,t}(b + |\hat{p}_{i,t}|) \leq \frac{2b + X^2}{2\gamma}\sum_{t=1}^{n} M_{i,t}Z_{i,t}\ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{(2b + X^2)^2}{8\gamma^2}\|\mathbf{u}_i\|^2.$$

Now, we subtract a non negative quantity $\sum_{t=1}^{n} M_{i,t}Z_{i,t}\min_j |\hat{p}_{j,t}|$ from the l.h.s.

and get,

$$\sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(b + |\hat{p}_{i,t}| - \min_{j} |\hat{p}_{j,t}| \right) \leq$$

$$\frac{2b + X^2}{2\gamma} \sum_{t=1}^{n} M_{i,t} Z_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{(2b + X^2)^2}{8\gamma^2} \|\mathbf{u}_i\|^2 . \tag{3.6}$$

At this point we take the expectation of all the terms. Recall that the conditional expectation of $Z_{i,t}$ is $(b + |\hat{p}_{i,t}| - \min_{j} |\hat{p}_{j,t}|)^{-1}/D_t$ and that $M_{i,t}$ and $\hat{p}_{i,t}$ are measurable with respect to the σ-algebra that generated by $Z_1, ... Z_{t-1}$. We start with the left term,

$$\mathbb{E} \left[\sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(b + |\hat{p}_{i,t}| - \min_{j} |\hat{p}_{j,t}| \right) \right] =$$

$$\mathbb{E} \left[\mathbb{E}_{t-1} \left[\sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(b + |\hat{p}_{i,t}| - \min_{j} |\hat{p}_{j,t}| \right) \right] \right] =$$

$$\mathbb{E} \left[\sum_{t=1}^{n} M_{i,t} \left(b + |\hat{p}_{i,t}| - \min_{j} |\hat{p}_{j,t}| \right) \mathbb{E}_{t-1} \left[Z_{i,t} \right] \right] =$$

$$\mathbb{E} \left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t} \right] .$$

Substituting the last term in the expectation of Eq. (3.6),

$$\mathbb{E} \left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t} \right] \leq \frac{2b + X^2}{2\gamma} \bar{L}_{\gamma,i,n}(u_i) + \frac{(2b + X^2)^2}{8\gamma^2} \|\mathbf{u}_i\|^2 . \tag{3.7}$$

Since that the normalization factor is a sum of positive components, hence D_t is maximal when each of the components gets its maximal value. This happens when $|\hat{p}_{m,t}| = |\hat{p}_{j,t}| \quad \forall m, j \in \{1, .., K\}$. This allows us to bound the normalization factor as follows,

$$D_t = \sum_{m=1}^{K} \left(b + |\hat{p}_{m,t}| - \min_{j} |\hat{p}_{j,t}| \right)^{-1} \leq \sum_{m=1}^{K} b^{-1} = \frac{K}{b} . \tag{3.8}$$

Since $M_{i,t} > 0 \ \forall i, t$, thus there exists $\delta_i \in \mathbb{R} > 0$ such that

$$\mathbb{E}\left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t}\right] = \frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n} M_{i,t}\right]. \tag{3.9}$$

and

$$\frac{b}{\delta_i} \geq \min \frac{1}{D_t} = \frac{1}{\max D_t} \geq \frac{b}{K}.$$

Which implies that $0 < \delta_i \leq K$.

Plugging Eq. (3.9) in Eq. (3.7) we get,

$$\frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{2b + X^2}{2\gamma}\bar{L}_{\gamma,i,n}(\mathbf{u}_i) + \frac{\left(2b + X^2\right)^2}{8\gamma^2}\|\mathbf{u}_i\|^2. \tag{3.10}$$

Summing up the last inequality over all K tasks and setting $\delta = \max_i \delta_i$ yields,

$$\frac{1}{\delta}\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{1}{\gamma}\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{\left(2b + X^2\right)^2}{8\gamma^2 b}\sum_{i=1}^{K}\|\mathbf{u}_i\|^2, \tag{3.11}$$

which concludes the proof. ∎

One can use the bound to tune the algorithm for an optimal value of b. However, this may not be possible as unfortunately $\bar{L}_{\gamma,n}$ depends implicitly on the value of b[1]. Alternatively, we can take a loose estimate of $\bar{L}_{\gamma,n}$, and replace it with $L_{\gamma,n}$ (which is $\sim K$ times larger). The optimal value of b can be now calculated,

$$b = \frac{X^2}{2}\sqrt{1 + \frac{4\gamma L_{\gamma,n}}{U^2 X^2}}.$$

Substituting this value in the bound of Eq. (??) with $L_{\gamma,n}$ leads to the following bound,

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{\delta}{\gamma}\left[L_{\gamma,n} + \frac{U^2 X^2}{2\gamma} + \frac{U^2}{2\gamma}\sqrt{1 + \frac{4\gamma L_{\gamma,n}}{U^2 X^2}}\right]$$

which has the same dependency in the number of inputs n as algorithm that ob-

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[1] Similar issue appears also after the discussion of Theorem 1 in a different context Cesa-Bianchi et al. [2006c].

serves all of them.

We continue now with an analysis of an extension of our algorithm that allows more than a single query per round. In this setting, the ability of the annotator to annotate data instances is less limited. Here, we allow the algorithm to query κ labels at a time ($\kappa < K$), instead of one. On each iteration t, the modified algorithm samples without repetitions κ labels to be annotated, and perform the same update as of Eq. (**??**). Formally, on each round we have $\sum_i Z_{i,t} = \kappa$ for $Z_{i,t} \in \{0,1\}$ where the first task-index to be queried is drawn according to Eq. (3.2). The second task is drawn from the same distribution, eliminating the first choice and adjusting the normalization factor, and so on. Once κ problems are drawn, the algorithm receives κ labels for the κ corresponding inputs, and updates the κ models according to Eq. (**??**).

Corollary 2 *If SHAMPO algorithm gets feedback for κ tasks on each round, instead of only a single problem, the expected cumulative weighted mistakes is bounded as follows*

$$
\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{1}{C(\kappa,K)}\left(\frac{1}{\gamma}\left(1+\frac{X^2}{2b}\right)\bar{L}_{\gamma,n}^{\kappa} + \kappa\frac{\left(2b+X^2\right)^2}{8\gamma^2 b}\tilde{U}^2\right) ,
$$

where $C(\kappa,K) = \sum_{j=K-\kappa+1}^{K}\frac{1}{j}$, and $\bar{L}_{\gamma,n}^{\kappa}$ is the expected loss of K models $\{\mathbf{u}_i\}$ over the κ instances that are annotated per round t.

Proof: We follow the proof of Theorem **??** until Eq. (3.11). We repeat the process κ times, and get the equivalent inequality for sampling κ problems without repetitions,

$$
\left(\sum_{j=K-\kappa+1}^{K}\frac{1}{j}\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{1}{\gamma}\left(1+\frac{X^2}{2b}\right)\bar{L}_{\gamma,n}^{\kappa} + \kappa\frac{\left(2b+X^2\right)^2}{8\gamma^2 b}U^2 ,
$$

(3.12)

where all expectations are now with respect to the sampling repetitions, and specifically $\bar{L}_{\gamma,n}^{\kappa}$ is the expected loss of a set of linear models $\{\mathbf{u}_i\}$ where κ problems are sampled rather than a single one. ∎

It can be shown that this bound is a generalized form of Theorem **??**. For $\kappa = 1$ we get the bound of Theorem **??**, while for $\kappa = K$ we have $C(\kappa,K) \approx \log(K)$, thus recovering the bound of K Perceptron trained in parallel, with a sightly ($logK$)

worse dependency in the number of tasks.

3.2 Aggressive version

So far we used the margin to build the distribution over the tasks, but in the update stage, we used the same regular perceptron update rule, that makes an update only when there is a prediction mistake. However, we already said that the margin can tells us about the uncertainty in the prediction, so it may be helpful to proceed with this approach in the update stage as well and make the sharp update threshold a little bit softer.

In this version of SHAMPO algorithm, the update is been preformed not only when there is a mistake, but also in the case of correct prediction with low margin, i.e. low certainty. We define this margin to be $\lambda \in \mathbb{R} > 0$. In addition to the previous mistake indicator, $M_{i,t}$, we introduce one more indicator, $G_{i,t}$. We keep the notation of $M_{i,t} = 1$ when there is wrong prediction on the task i in round t and $M_{i,t} = 0$ otherwise, and set $G_{i,t} = 1$ when there is no prediction mistake on the task i in round t, but the margin is lower than our threshold, i.e. $0 < \hat{p}_{i,t} < \lambda$, therefore an aggressive update takes place, and $G_{i,t} = 0$ otherwise. An update is performed if either there is a mistake ($M_{J_i,t} = 1$) or the margin is low ($G_{J_i,t} = 1$). Note that these events are mutually exclusive thus, for simplicity, we define the update indicator which is $U_{i,t} = M_{i,t} + G_{i,t}$. This indicator is $U_{i,t} = 1$ on update trials and $U_{i,t} = 1$ when no update has been made.

Theorem 3 *If Aggressive SHAMPO algorithm runs on K problems with K parallel example pair sequences $(\mathbf{x}_{i,1}, y_{i,1}), ...(\mathbf{x}_{i,n}, y_{i,n}) \in \mathbb{R}^d \times \{-1, 1\}$, $i = 1, ..., K$ with input parameters $b > 0$, and $0 \leq \lambda \leq b/2$ then for all $\gamma > 0$, all $\mathbf{u}_i \in \mathbb{R}^d$ and all $n \geq 1$, there exists $0 < \delta \leq K$, such that,*

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{\delta}{\gamma}\left[\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{\left(2b + X^2\right)^2 \tilde{U}^2}{8\gamma b}\right.$$

$$\left. + \left(2\frac{\lambda}{b} - 1\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} G_{i,t}\right]\right],$$

where $X = \max_{i,t} \|\mathbf{x}_{i,t}\|$, $\tilde{U}^2 = \sum_{i=1}^{K} \|\mathbf{u}_i\|^2$ and the expectation is over the random choices of the algorithm.

18

Algorithm 3.2 SHAMPO aggressive perceptron.

Parameters: $b, \lambda \in \mathbb{R} > 0$.
Initialize: $\mathbf{w}_{i,0} = \mathbf{0}$ for $i = 1, \ldots, K$

for $t = 1, 2, \ldots, n$ **do**

1. Observe K instance vectors, $\mathbf{x}_{i,t}$, $(i = 1, \ldots, K)$.
2. Compute margins $\hat{p}_{i,t} = \mathbf{w}_{i,t-1}^{\top} \mathbf{x}_{i,t}$.
3. Predict K labels, $\hat{y}_{i,t} = \text{sign}(\hat{p}_{i,t})$.
4. Draw problem J_t with the distribution:

$$\Pr\left[J_t = j\right] = \frac{\left(b + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|\right)^{-1}}{D_t}, Eq. \textbf{(??)}$$

$$D_t = \sum_i \left(b + |\hat{p}_{i,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|\right)^{-1}.$$

5. Query the true label ,$y_{J_t,t} \in \{-1, 1\}$.
6. Set the indicator $M_{J_t,t} = 1$ iff $y_{J_t,t} \neq \hat{y}_{J_t,t}$.
7. Iff $M_{J_t,t} = 0$ and $|p_{J_t,t}| < \lambda$, set the indicator $G_{J_t,t} = 1$
8. For $U_{i,t} = M_{i,t} + G_{i,t}$ update with the perceptron rule:

$$\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + U_{J_t,t} \, y_{J_t,t} \, \mathbf{x}_{J_t,t} \tag{3.13}$$
$$\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1} \text{ for } i \neq J_t$$

end for
Output: $\mathbf{w}_{i,n}$ for $i = 1, \ldots, K$.

The theorem above, states that when the update is aggressive, the bound on the expected number of errors can be tighter than the non aggressive algorithm because we have added a new term term to the bound which can be negative.

In order to achieve tighter bound on the expected cumulative mistakes than we had in the regular SHAMPO perceptron algorithm (Thm. 1), we need to restrict the aggressive margin λ to be $\lambda \leq b/2$. Indeed, one can say that this bound shows that the best choice of λ is $\lambda = 0$, i.e. it seems that aggressive update is not a good idea. However, in this case, $G_{i,t} = 0$, $\forall i, t$ which means that this would not be the best choice. This discrimination actually shows that there is a sweet point where $0 < \lambda < b/2$ such that the bound will be the tightest as this bound allows .

Proof: We follow the proof of $Thm.$ 3.1 on an update trial, reminding that now the update indicator is $U_{i,t} = 1$, until we get,

$$\gamma - \ell_{\gamma,i,t}(\mathbf{u}_i) \leq$$
$$y_{i,t}\hat{p}_{i,t} + \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t-1}\|^2 - \frac{1}{2}\|\mathbf{u}_i - \mathbf{w}_{i,t}\|^2 + \frac{1}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2 .$$

Whereas in the non aggressive version proof we claimed that $y_{i,t}\hat{p}_{i,t} = -|\hat{p}_{i,t}|$ in an update trial, here, it is not true when an aggressive update takes place since we update when the prediction is correct as well. For this reason, we remain with the term $y_{i,t}\hat{p}_{i,t}$ as is for now and we will deal with it later. In addition, in this proof, we change the update indicator from $M_{i,t}$ to $U_{i,t}$. Considering these two changes, we proceed the proof until $Eq.$ (3.6). Now we have,

$$\sum_{t=1}^{n} U_{i,t}Z_{i,t}\left(b - y_{i,t}\hat{p}_{i,t} - \min_j |\hat{p}_{j,t}|\right) \leq$$
$$\frac{2b + X^2}{2\gamma}\sum_{t=1}^{n} U_{i,t}Z_{i,t}\ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{(2b + X^2)^2}{8\gamma^2}\|\mathbf{u}_i\|^2 . \tag{3.14}$$

Since the inequality is computed on an update task, there is a need to discriminate between the two possible options: either $y_{i,t}\hat{p}_{i,t} \leq 0$ holds when there is prediction mistake ($M_{i,t} = 1$), i.e. $y_{i,t}\hat{p}_{i,t} = -|\hat{p}_{i,t}|$, or $0 \leq y_{i,t}\hat{p}_{i,t} \leq \lambda$ holds when the prediction is correct, but an aggressive update occurs ($G_{i,t} = 1$) and in that case $y_{i,t}\hat{p}_{i,t} = |\hat{p}_{i,t}|$.

At this point we take the expectation of all the terms as before taking into consideration the two cases. Recall that the conditional expectation of $Z_{i,t}$ is $(b + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|)^{-1}/D_t$ and that $M_{i,t}$ and $G_{i,t}$ (therefor also $U_{i,t}$) and $\hat{p}_{i,t}$ are measurable with respect to the σ-algebra that generated by $Z_1, ...Z_{t-1}$. We start

with the left term,

$$\mathbb{E}\left[\sum_{t=1}^{n} U_{i,t} Z_{i,t}\left(b - y_{i,t}\hat{p}_{i,t} - \min_j |\hat{p}_{j,t}|\right)\right]$$

$$= \mathbb{E}\left[\mathbb{E}_{t-1}\left[\sum_{t=1}^{n} U_{i,t} Z_{i,t}\left(b - y_{i,t}\hat{p}_{i,t} - \min_j |\hat{p}_{j,t}|\right)\right]\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{n} U_{i,t}\left(b - y_{i,t}\hat{p}_{i,t} - \min_j |\hat{p}_{j,t}|\right)\mathbb{E}_{t-1}\left[Z_{i,t}\right]\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{n} \frac{(M_{i,t} + G_{i,t})}{D_t}\frac{(b - y_{i,t}\hat{p}_{i,t} - \min_j |\hat{p}_{j,t}|)}{b + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{n} \frac{1}{D_t}\left(M_{i,t} + \frac{b - |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}{b + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}G_{i,t}\right)\right].$$

Substituting the last term in the expectation of Eq. (3.14) we get,

$$\mathbb{E}\left[\sum_{t=1}^{n} \frac{1}{D_t}\left(M_{i,t} + \frac{b - |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}{b + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}G_{i,t}\right)\right] \leq$$

$$\frac{2b + X^2}{2\gamma}\bar{L}_{\gamma,i,n}(u_i) + \frac{\left(2b + X^2\right)^2}{8\gamma^2}\|\mathbf{u}_i\|^2 \, . \tag{3.15}$$

Now we bound the factor that is multiplied by $G_{i,t}$ using the fact that $G_{i,t} = 1$ only when $|\hat{p}_{i,t}| < \lambda$,

$$\left(1 - 2\frac{\lambda}{b}\right) = \frac{b - 2\lambda}{b} \leq \frac{b - |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}{b + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|} \leq 1. \tag{3.16}$$

As stated before, there exists $\delta_i \in \mathbb{R}$, $0 < \delta_i \leq K$ such that Eq. (3.9) holds. Plugging this in Eq. (3.15) we get,

$$\frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{2b + X^2}{2\gamma}\bar{L}_{\gamma,i,n}(\mathbf{u}_i)$$

$$+ \frac{\left(2b + X^2\right)^2}{8\gamma^2}\|\mathbf{u}_i\|^2 + \frac{b}{\delta_i}\left(2\frac{\lambda}{b} - 1\right)\mathbb{E}\left[\sum_{t=1}^{n} G_{i,t}\right]. \tag{3.17}$$

Summing up the last inequality over all K tasks and setting $\delta = \max_i \delta_i$ yields,

$$\frac{1}{\delta} \mathbb{E} \left[\sum_{i=1}^{K} \sum_{t=1}^{n} M_{i,t} \right] \leq \frac{1}{\gamma} \left(1 + \frac{X^2}{2b} \right) \bar{L}_{\gamma,n}$$
$$+ \frac{\left(2b + X^2 \right)^2}{8\gamma^2 b} \sum_{i=1}^{K} \| \mathbf{u}_i \|^2 + \frac{1}{\delta} \left(2\frac{\lambda}{b} - 1 \right) \mathbb{E} \left[\sum_{i=1}^{K} \sum_{t=1}^{n} G_{i,t} \right]$$

Which concludes the proof ∎

3.3   prior distribution over tasks

We saw before that b is a tradeoff parameter between exploitation and uniform distribution over the tasks (exploration). It is possible that we have prior knowledge about the tasks, which tasks are harder than the others. In this case, we would like to add this prior knowledge to the distribution such that in the exploration will be done with a prior distribution. This is especially important in the first steps of the algorithms when instead starting in pure uniform exploration (since we initialize our model vectors, $\mathbf{w}_{i,0} = \mathbf{0}$) we can start with a reasonable prior distribution. In order to do that we change the probability to be

$$\Pr \left[J_t = j \right] = \frac{1}{D_t} \frac{a_j}{\left(b + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}| \right)}$$

where $a_j > 1$ $\forall j \in \{1, \cdots, K\}$ and D_t is the normalization factor. We can use this distribution with the SHAMPO perceptron algorithm, but we would like to analyse the bound of the SHAMPO aggressive perceptron algorithm because it's more global.

Theorem 4 *If Aggressive SHAMPO algorithm with prior runs on K problems with K parallel example pair sequences $(\mathbf{x}_{i,1}, y_{i,1}), ...(\mathbf{x}_{i,n}, y_{i,n}) \in \mathbb{R}^d \times \{-1, 1\}$, $i = 1, ..., K$ with input parameters $b > 0$, and $\lambda > 0$ and $a_i > 1$ $\forall i \in \{1, \cdots, K\}$ then for all $\gamma > 0$, all $\mathbf{u}_i \in \mathbb{R}^d$ and all $n \geq 1$,*

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}M_{i,t}\right] \leq \frac{\sum_{j=1}^{K}a_j}{\gamma}\left[\left(1+\frac{X^2}{2b}\right)\bar{L}_{\gamma,n}+\frac{\left(2b+X^2\right)^2\tilde{U}^2}{8\gamma b}\right]$$
$$+\left(2\frac{\lambda}{b}-1\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}a_iG_{i,t}\right],$$

where $X = \max_{i,t}\|\mathbf{x}_{i,t}\|$, $\tilde{U}^2 = \sum_{i=1}^{K}\|\mathbf{u}_i\|^2$ *and the expectation is over the random choices of the algorithm.*

This bound brings an advantage and disadvantage. We can reduce the last term in the bound choosing the right weights for a_i, but then, we suffer the factor of $\sum_{j=1}^{K}a_j$ in rest part of the bound.

Proof: We follow the proof of Thm. 3 until Eq. (3.14), than we take the expectation of this term. First, we take the expectation of the left hand side where now the conditional expectation of $Z_{i,t}$ changed to

$$\mathbb{E}\left[Z_{i,t}\right] = \frac{1}{D_t}\frac{a_i}{\left(b+|\hat{p}_{i,t}|-\min_{m=1}^{K}|\hat{p}_{m,t}|\right)}.$$

Doing that, the expectation becomes

$$\mathbb{E}\left[\sum_{t=1}^{n}U_{i,t}Z_{i,t}\left(b-y_{i,t}\hat{p}_{i,t}-\min_j|\hat{p}_{j,t}|\right)\right]$$
$$=\mathbb{E}\left[\mathbb{E}_{t-1}\left[\sum_{t=1}^{n}U_{i,t}Z_{i,t}\left(b-y_{i,t}\hat{p}_{i,t}-\min_j|\hat{p}_{j,t}|\right)\right]\right]$$
$$=\mathbb{E}\left[\sum_{t=1}^{n}\frac{a_i}{D_t}\left(M_{i,t}+\frac{b-|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}{b+|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}G_{i,t}\right)\right].$$

Reminding that $a_i \geq 1 \forall i$, we can bound $M_{i,t} \leq M_{i,t}a_i$ and get

$$\mathbb{E}\left[\sum_{t=1}^{n}\frac{1}{D_t}\left(M_{i,t}+\frac{b-|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}{b+|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}G_{i,t}a_i\right)\right] \leq$$
$$\frac{2b+X^2}{2\gamma}\bar{L}_{\gamma,i,n}(u_i)+\frac{\left(2b+X^2\right)^2}{8\gamma^2}\|\mathbf{u}_i\|^2 .$$

(3.18)

We use now the same bound as in Eq. (3.16) and plug it into the left side of the inequality,

$$\mathbb{E}\left[\sum_{t=1}^{n}\frac{1}{D_t}\left(M_{i,t}+\left(1-2\frac{\lambda}{b}\right)G_{i,t}a_i\right)\right]\leq$$

$$\mathbb{E}\left[\sum_{t=1}^{n}\frac{1}{D_t}\left(M_{i,t}+\frac{b-|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}{b+|\hat{p}_{j,t}|-\min_j|\hat{p}_{j,t}|}G_{i,t}a_i\right)\right].$$

Since $b/2 \geq \lambda$ we have that $M_{i,t}+\left(1-2\frac{\lambda}{b}\right)G_{i,t}a_i \geq 0$, thus there exists δ_i such that,

$$\mathbb{E}\left[\sum_{t=1}^{n}\frac{1}{D_t}\left(M_{i,t}+\left(1-2\frac{\lambda}{b}\right)G_{i,t}a_i\right)\right]=$$

$$\frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n}\left(M_{i,t}+\left(1-2\frac{\lambda}{b}\right)G_{i,t}a_i\right)\right],\quad (3.19)$$

and

$$\frac{b}{\delta_i}\geq\min\frac{1}{D_t}=\frac{1}{\max D_t}\geq\frac{b}{\sum_{j=1}^{K}a_j},$$

where the last inequality follows from,

$$D_t=\sum_{j=1}^{K}\frac{a_j}{\left(b+|\hat{p}_{j,t}|-\min_{m=1}^{K}|\hat{p}_{m,t}|\right)}\leq\frac{\sum_{j=1}^{K}a_j}{b}.\quad (3.20)$$

The previous bound implies that $0<\delta_i\leq\sum_{i=1}^{K}a_i$.

Combining Eq. (3.18) and Eq. (3.19),

$$\frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n}M_{i,t}\right]\leq\frac{2b+X^2}{2\gamma}\bar{L}_{\gamma,i,n}(\mathbf{u}_i)$$

$$+\frac{\left(2b+X^2\right)^2}{8\gamma^2}\|\mathbf{u}_i\|^2+\frac{b}{\delta_i}\left(2\frac{\lambda}{b}-1\right)a_i\mathbb{E}\left[\sum_{t=1}^{n}G_{i,t}\right].\quad (3.21)$$

Summing up the last inequality over all K tasks and setting $\delta = \max_i \delta_i$ yields,

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{\delta}{\gamma}\left[\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{\left(2b + X^2\right)^2}{8\gamma b}\sum_{i=1}^{K}\|\mathbf{u}_i\|^2\right]$$
$$+ \left(2\frac{\lambda}{b} - 1\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} a_i G_{i,t}\right],$$

(3.22)

which concludes the proof. $\blacksquare$

We see that when this bound is a global bound for the last two algorithms. If we set $a_i = 1, \ \forall i \in \{1, \cdots, K\}$ we get the same bound as SHAMPO aggressive algorithm and if we update only when a prediction mistake occurs, it means that $G_{i,t} = 0, \ \forall i \in \{a, \cdots, K\}, t > 0$ and we get the bound of the ordinary SHAMPO perceptron algorithm.

3.4 Kernel-based SHAMPO

SHAMPO algorithm can also be used with Kernels in order to get a non linear classifiers in case of non separated data set by transforming the feature vector $\mathbf{x}$ to $\Phi(\mathbf{x})$. Define the binary parameter $\alpha_{i,t} = Z_{i,t}M_{i,t}$, we can rewrite the weight vector $\mathbf{w}_{i,t}$ in its dual form as a linar combination of the input vectors

$$\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1} + Z_{i,t}M_{i,t}y_{i,t}\mathbf{x}_{i,t} = \sum_{s=1}^{t} Z_{i,s}M_{i,s}y_{i,s}\mathbf{x}_{i,s} = \sum_{s=1}^{t} \alpha_{i,s}y_{i,s}\mathbf{x}_{i,s}.$$

Now wee can rid of the weight algorithm and only store $\alpha_{i,t}$ values. Doing that, we get a new representation of the margin

$$\hat{p}_{i,t} = \mathbf{w}_{i,t-1}^{\top}\mathbf{x}_{i,t} = \left(\sum_{s=1}^{t} \alpha_{i,s}y_{i,s}\mathbf{x}_{i,s}\right)^{\top}\mathbf{x}_{i,t} = \sum_{s=1}^{t} \alpha_{i,s}y_{i,s}\left(\mathbf{x}_{i,s}^{\top}\mathbf{x}_{i,t}\right) = \sum_{s=1}^{t} \alpha_{i,s}y_{i,s}K(\mathbf{x}_{i,s}, \mathbf{x}_{i,t})$$

where $K(\mathbf{x}, \boldsymbol{y}) = \Phi(\mathbf{x})^{\top}\Phi(\boldsymbol{y})$ is the kernel function that represents the inner product of a high dimension space.

3.5 SHAMPO perceptron with Adaptive b parameter

As it has been proved in the analysis and will be shown in experiments, the parameter b can tune the distribution over the tasks between exploration and exploitation, yet, there is still a need to supply the algorithm with this input parameter. So far, all of the tasks sheared the same parameter b which was determined at the beginning of the algorithm and remained constant during the whole process. For this reason, once the initial parameters were supplied, the margin was the only factor that had a varying influence on the distribution. Now, we would like to suggest another algorithm that allow us to dominate the distribution using the number of updates for each task. We assume as in Cesa-Bianchi et al Cesa-Bianchi et al. [2006c] that the algorithm makes more mistakes on harder task than easier one, so a natural choice at any time, can be one that set higher probability to the tasks that made more prediction mistakes up to the same time.

The adaptive algorithm is similar to the one in Fig. **??** with three variation in order to comply with this intuition. Here we introduce two more variables per task. First, the scalar $N_{i,n} = \sum_{t=1}^{n} Z_{i,t} M_{i,t}$ that holds the number of updates that has been done for the ith task up to time n. For simplicity of analysis we will write it in it's recursive way $N_{i,t} = N_{i,t-1} + 1$. The second variable is $X_{i,t}$ which is holds at the end of each cycle, the maximal norm of the input vectors that were involved in the updates. The algorithm still have a parameter β to be determined, however, it has less influence on the distribution than the parameter b in Fig. **??**. Using those three new parameter we introduce the adaptive version of b

$$b_{i,t-1} = \beta X_{i,t} \sqrt{1 + N_{i,t-1}}$$

The adaptive version of the SHAMPO perceptron algorithm is described in Fig. 3.3.

Theorem 5 *If adaptive SHAMPO algorithm runs on K tasks with K parallel example pair sequences $(\mathbf{x}_{i,1}, y_{i,1}), ...(\mathbf{x}_{i,n}, y_{i,n}) \in \mathbb{R}^d \times \{-1, 1\}$, $i = 1, ..., K$ with input parameter $b > 0$, then for all $\gamma > 0$, all $\mathbf{u}_i \in \mathbb{R}^d$ and all $n \geq 1$, there exists $0 < \delta \leq K$, such that,*

$$\mathbb{E}\left[\sum_{i=1}^{K} \sum_{t=1}^{n} M_{i,t}\right] \leq \delta \left[\frac{\delta B^2}{2} + \frac{1}{\gamma}\bar{L}_{\gamma,n} + \frac{KR}{2\beta} + B\sqrt{\frac{\delta B^2}{4} + \frac{1}{\gamma}\bar{L}_{\gamma,n} + \frac{KR}{2\beta}}\right]$$

Algorithm 3.3 Adaptive SHAMPO algorithm

Parameters: $\beta \in \mathbb{R} > 0$.
Initialize: $\mathbf{w}_{i,0} = \mathbf{0}$ for $i = 1, \ldots, K$

for $t = 1, 2, \ldots, n$ **do**

1. Observe K instance vectors, $\mathbf{x}_{i,t}$, $(i = 1, \ldots, K)$.
2. Compute margins $\hat{p}_{i,t} = \mathbf{w}_{i,t-1}^{\top} \mathbf{x}_{i,t}$.
3. Predict K labels, $\hat{y}_{i,t} = \text{sign}(\hat{p}_{i,t})$.
4. Set $X_{i,t} = \max\left(X_{i,t-1}, \|\mathbf{x}_{i,t}\|\right)$
5. Draw task J_t with the distribution:

$$\Pr\left[J_t = j\right] = \frac{1}{D_t} \frac{b_{i,t}}{b_{i,t} + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|},$$

$$D_t = \sum_i \frac{b_{i,t}}{b_{i,t} + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}|}.$$

Where $b_{i,t} = \beta X_{i,t}^2 \sqrt{1 + N_{i,t}}$.
6. Query the true label ,$y_{J_t,t} \in \{-1, 1\}$.
7. Set the indicator $M_{J_t,t} = 1$ iff $y_{J_t,t} \neq \hat{y}_{J_t,t}$.
8. Update with the perceptron rule:

$$\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + M_{J_t,t}\, y_{J_t,t}\, \mathbf{x}_{J_t,t}$$
$$\mathbf{w}_{i,t} = \mathbf{w}_{i,t-1} \text{ for } i \neq J_t$$
$$N_{J_t,t} = N_{J_t,t} + M_{J_t,t}$$

end for
Output: $\mathbf{w}_{i,n}$ for $i = 1, \ldots, K$.

---

where $R = \max_i \left(\|u_i\| X_i\right)/\gamma$, $B = \left(R + \frac{2+3R}{2\beta}\right)$ and the expectation is over the random choices of the algorithm.

The proof here is more extended version of the proof in Theorem **??** . We will start here from Eq. (3.6). here, we will replace the constant scaling parameter, formerly α, with a time varying factor $c_{i,t-1}/\gamma$, where $c_{i,t} \geq 0$ is defined as following

$$c_{i,t-1} = \frac{1}{2}\left(\max\left\{X_{i,t-1}, \|\mathbf{x}_{i,t}\|\right\}\right) + b_{i,t-1}.$$

Plugging the new scaling factor into the given inequality we get

$$M_{i,t}Z_{i,t}(c_{i,t-1} + |\hat{p}_{i,t}|) \le M_{i,t}Z_{i,t}\frac{c_{i,t-1}}{\gamma}\ell_{\gamma,i,t}(\mathbf{u}_i)+$$

$$\frac{1}{2}\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t-1}\right\|^2 - \frac{1}{2}\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2 + \frac{M_{i,t}Z_{i,t}}{2}\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2.$$

From the update rule of the algorithm we can bound $\|\mathbf{w}_{i,t-1} - \mathbf{w}_{i,t}\|^2 \le \|\mathbf{x}_{i,t}\|^2$. This inequality holds for any time step, and particular for an update iteration. Using this bound and dividing the inequality by b_{t-1} yields

$$M_{i,t}Z_{i,t}\left(\frac{c_{i,t-1} + |\hat{p}_{i,t}| - \|\mathbf{x}_{i,t}\|^2/2}{b_{i,t-1}}\right) \le M_{i,t}Z_{i,t}\frac{c_{i,t-1}}{b_{i,t-1}}\frac{\ell_{\gamma,i,t}(\mathbf{u}_i)}{\gamma}$$

$$+ \frac{1}{2b_{i,t-1}}\left(\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t-1}\right\|^2 - \left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2\right). \tag{3.23}$$

Now, we decompose the last term of the inequality to two differences,

$$\frac{1}{2b_{i,t-1}}\left(\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t-1}\right\|^2 - \left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2\right) =$$

$$\frac{1}{2b_{i,t-1}}\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t-1}\right\|^2 - \frac{1}{2b_{i,t}}\left\|\frac{c_{i,t}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2 +$$

$$\frac{1}{2b_{i,t}}\left\|\frac{c_{i,t}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2 - \frac{1}{2b_{i,t-1}}\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2.$$

At this point expand the last two terms and set
and we get

$$\frac{1}{2b_{i,t}}\left\|\frac{c_{i,t}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2 - \frac{1}{2b_{i,t-1}}\left\|\frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t}\right\|^2$$

$$= \frac{\|\mathbf{u}_i\|^2}{2\gamma^2}\left(\frac{c_{i,t}^2}{b_{i,t}} - \frac{c_{i,t-1}^2}{b_{i,t-1}}\right) + \frac{\mathbf{u}_i^T\mathbf{w}_{i,t}}{\gamma}\left(\frac{c_{i,t-1}}{b_{i,t-1}} - \frac{c_{i,t}}{b_{i,t}}\right)$$

$$+ \frac{\|\mathbf{w}_{i,t}\|^2}{2}\left(\frac{1}{b_{i,t}} - \frac{1}{b_{i,t-1}}\right)$$

We know that by definition, $b_{i,t}$ is non decreasing over time such that the last term

is always negative and can be waved. In addition, the definition of $c_{i,t}$ leads to

$$\frac{c_{i,t}}{b_{i,t}} = \frac{1}{2\beta\sqrt{1 + N_{i,t}}} + 1$$

which, combining with the fact that $N_{i,t}$ is nondecreasing over time as well, we get the inequality

$$\frac{c_{i,t}}{b_{i,t}} \le \frac{c_{i,t-1}}{b_{i,t-1}}$$

Combining these facts and using $Cauchy - Schwarz$ the inequality becomes,

$$\frac{1}{2b_{i,t}} \left\| \frac{c_{i,t}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2 - \frac{1}{2b_{i,t-1}} \left\| \frac{c_{i,t-1}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2$$
$$\le \frac{\|\mathbf{u}_i\|^2}{2\gamma^2} \left(\frac{c_{i,t}^2}{b_{i,t}} - \frac{c_{i,t-1}^2}{b_{i,t-1}} \right) + \frac{\|\mathbf{u}_i\| \, \|\mathbf{w}_{i,t}\|}{\gamma} \left(\frac{c_{i,t-1}}{b_{i,t-1}} - \frac{c_{i,t}}{b_{i,t}} \right).$$

(3.24)

Here, recall that on an update trial $y_{i,t}\mathbf{w}_{i,t-1}^T\mathbf{x}_{i,t} \le 0$ we define $X_i = \max_t \|\mathbf{x}_{i,t}\|$ and use again the update rule and bound the weight vector norm with the number of updates ,

$$\|\mathbf{w}_{i,t}\|^2 = \|\mathbf{w}_{i,t-1} + M_{i,t}Z_{i,t}y_{i,t}\mathbf{x}_{i,t}\|^2$$
$$= \|\mathbf{w}_{i,t-1}\|^2 + 2M_{i,t}Z_{i,t}y_{i,t}\mathbf{w}_{i,t-1}^T\mathbf{x}_{i,t} + M_{i,t}Z_{i,t} \|\mathbf{x}_{i,t}\|^2$$
$$\le \|\mathbf{w}_{i,t-1}\|^2 + M_{i,t}Z_{i,t} \|\mathbf{x}_{i,t}\|^2$$
$$\le \|\mathbf{w}_{i,t-1}\|^2 + M_{i,t}Z_{i,t}X_i^2$$

Applying this inequality recursively, combined with the initial value $\mathbf{w}_{i,0} = 0$ leads us to

$$\|\mathbf{w}_{i,t}\| \le X_i\sqrt{N_{i,t}} \tag{3.25}$$

which holds for any i and t. Combining the last inequality into Eq. (3.24) we get

$$\frac{1}{2b_{i,t}} \left\| \frac{c_{i,t}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2 - \frac{1}{2b_{i,t-1}} \left\| \frac{c_{i,t-1}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2 \le$$
$$\frac{\|\mathbf{u}_i\|^2}{2\gamma^2} \left(\frac{c_{i,t}^2}{b_{i,t}} - \frac{c_{i,t-1}^2}{b_{i,t-1}} \right) + \frac{\|\mathbf{u}_i\| \, X_i\sqrt{N_{i,t}}}{\gamma} \left(\frac{c_{i,t-1}}{b_{i,t-1}} - \frac{c_{i,t}}{b_{i,t}} \right).$$

(3.26)

We will write now, a simpler bound on the last term. First we note that on t that $M_{i,t}Z_{i,t} = 1$, i.e. in the case of update, $N_{i,t} = N_{i,t-1} + 1$. Using the bound $\sqrt{1+x} - \sqrt{x} \le \frac{1}{2\sqrt{x}}$, yields

$$
\begin{aligned}
\sqrt{N_{i,t}} \left(\frac{c_{t-1}}{b_{t-1}} - \frac{c_t}{b_t} \right) &= \frac{\sqrt{N_{i,t}}}{2\beta} \left(\frac{1}{\sqrt{1 + N_{i,t-1}}} - \frac{1}{\sqrt{1 + N_{i,t}}} \right) \\
&= \frac{\sqrt{N_{i,t}}}{2\beta} \left(\frac{1}{\sqrt{N_{i,t}}} - \frac{1}{\sqrt{1 + N_{i,t}}} \right) \\
&= \frac{1}{2\beta} \frac{\sqrt{1 + N_{i,t}} - \sqrt{N_{i,t}}}{\sqrt{1 + N_{i,t}}} \\
&\le \frac{1}{4\beta} \frac{1}{\sqrt{N_{i,t}}\sqrt{1 + N_{i,t}}} \\
&\le \frac{1}{4\beta} \frac{1}{N_{i,t}}.
\end{aligned}
$$

If $M_{i,t}Z_{i,t} = 0$, the left hand side becomes 0 because in such case, $c_{i,t} = c_{i,t-1}$ and $b_{i,t} = b_{i,t-1}$, which means that this bound holds for all i and t and we can write it as

$$
\sqrt{N_{i,t}} \left(\frac{c_{t-1}}{b_{t-1}} - \frac{c_t}{b_t} \right) \le \frac{M_{i,t}Z_{i,t}}{4\beta} \frac{1}{N_{i,t}}.
$$

Plugging the last bound and Eq. (3.26) into Eq. (3.23) gives us the next inequality that holds for any $i, t, \gamma > 0$, and $u_i \in \mathbb{R}^d$,

$$
\begin{aligned}
M_{i,t}Z_{i,t} \left(\frac{c_{i,t-1} + |\hat{p}_{i,t}| - \|\mathbf{x}_{i,t}\|^2/2}{b_{i,t-1}} \right) &\le M_{i,t}Z_{i,t} \frac{c_{i,t-1}}{b_{i,t-1}} \frac{\ell_{\gamma,i,t}(\mathbf{u}_i)}{\gamma} \\
&+ \frac{1}{2b_{i,t-1}} \left\| \frac{c_{i,t-1}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t-1} \right\|^2 - \frac{1}{2b_{i,t}} \left\| \frac{c_{i,t}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2 \\
&+ \frac{\|\mathbf{u}_i\|^2}{2\gamma^2} \left(\frac{c_{i,t}^2}{b_{i,t}} - \frac{c_{i,t-1}^2}{b_{i,t-1}} \right) + \frac{\|\mathbf{u}_i\| \, X_i}{\gamma} \frac{M_{i,t}Z_{i,t}}{4\beta} \frac{1}{N_{i,t}}.
\end{aligned}
$$

Recall the definition of $c_{i,t}$, we can write $c_{i,t-1} - \|\mathbf{x}_{i,t}\|^2/2 \ge b_{i,t-1}$ and the

30

ineqality gets the form

$$
M_{i,t} Z_{i,t} \left(\frac{b_{i,t-1} + |\hat{p}_{i,t}|}{b_{i,t-1}} \right) \leq M_{i,t} Z_{i,t} \frac{c_{i,t-1}}{b_{i,t-1}} \frac{\ell_{\gamma,i,t}(\mathbf{u}_i)}{\gamma}
$$

$$
+ \frac{1}{2b_{i,t-1}} \left\| \frac{c_{i,t-1}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t-1} \right\|^2 - \frac{1}{2b_{i,t}} \left\| \frac{c_{i,t}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,t} \right\|^2
$$

$$
+ \frac{\|\mathbf{u}_i\|^2}{2\gamma^2} \left(\frac{c_{i,t}^2}{b_{i,t}} - \frac{c_{i,t-1}^2}{b_{i,t-1}} \right) + \frac{\|\mathbf{u}_i\| X_i}{\gamma} \frac{M_{i,t} Z_{i,t}}{4\beta} \frac{1}{N_{i,t}}.
$$

Summing both sides over $t = 1, \cdots, n$, and taking into consideration the initialization, $w_0 = 0$ we obtain,

$$
\sum_{t=1}^n M_{i,t} Z_{i,t} \left(\frac{b_{i,t-1} + |\hat{p}_{i,t}|}{b_{i,t-1}} \right) \leq \frac{1}{\gamma} \sum_{t=1}^n M_{i,t} Z_{i,t} \frac{c_{i,t-1}}{b_{i,t-1}} \ell_{\gamma,i,t}(\mathbf{u}_i)
$$

$$
+ \frac{\|\mathbf{u}_i\|^2}{2\gamma^2} \frac{c_{i,n}^2}{b_{i,n}} - \frac{1}{2b_{i,n}} \left\| \frac{c_{i,n}}{\gamma} \mathbf{u}_i - \mathbf{w}_{i,n} \right\|^2 + \frac{\|\mathbf{u}_i\| X_i}{4\beta\gamma} \sum_{t=1}^n \frac{M_{i,t} Z_{i,t}}{N_{i,t}}.
$$

$$
(3.27)
$$

Now, we will bound the terms in the right hand side. The first term can be bounded as following,

$$
\frac{1}{\gamma} \frac{c_{i,t-1}}{b_{i,t-1}} \ell_{\gamma,i,t}(\mathbf{u}_i) = \frac{1}{\gamma} \left(\frac{1}{2\beta\sqrt{1 + N_{i,t-1}}} + 1 \right) \ell_{\gamma,i,t}(\mathbf{u}_i)
$$

$$
\leq \frac{1}{2\gamma\beta\sqrt{1 + N_{i,t-1}}} (\gamma + \|u_i\| X_i) + \frac{\ell_{\gamma,i,t}(\mathbf{u}_i)}{\gamma}
$$

$$
= \frac{1}{2\beta\sqrt{1 + N_{i,t-1}}} \left(1 + \frac{\|u_i\| X_i}{\gamma} \right) + \frac{\ell_{\gamma,i,t}(\mathbf{u}_i)}{\gamma}
$$

Here, we used the fact that $\ell_{\gamma,i,t}(\mathbf{u}_i) \leq \gamma + \|u_i\| X_i$. Now we proceed with the two

31

middle terms of Eq. (3.27) and get the bound

$$
\frac{\|\mathbf{u}_i\|^2}{2\gamma^2}\frac{c_{i,n}^2}{b_{i,n}} - \frac{1}{2b_{i,n}}\left\|\frac{c_{i,n}}{\gamma}\mathbf{u}_i - \mathbf{w}_{i,n}\right\|^2 = \frac{c_{i,n}u_i^T\mathbf{w}_{i,n}}{b_{i,n}\gamma} - \frac{\|\mathbf{w}_{i,n}\|^2}{2b_{i,n}}
$$

$$
\leq \frac{c_{i,n}u_i^T\mathbf{w}_{i,n}}{b_{i,n}\gamma}
$$

$$
\leq \frac{c_{i,n}}{b_{i,n}}\frac{\|u_i\|\,\|\mathbf{w}_{i,n}\|}{\gamma}
$$

$$
\leq \left(\frac{1}{2\beta\sqrt{1+N_{i,n}}} + 1\right)\frac{\|u_i\|X_i\sqrt{N_{i,n}}}{\gamma},
$$

where we used Eq. (3.25) in the last step. Putting all together in Eq. (3.27) gives us

$$
\sum_{t=1}^{n} M_{i,t}Z_{i,t}\left(\frac{b_{i,t-1}+|\hat{p}_{i,t}|}{b_{i,t-1}}\right)
$$

$$
\leq \frac{1}{2\beta}\left(1 + \frac{\|u_i\|X_i}{\gamma}\right)\sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{\sqrt{1+N_{i,t-1}}} + \frac{1}{\gamma}\sum_{t=1}^{n}M_{i,t}Z_{i,t}\ell_{\gamma,i,t}(\mathbf{u}_i)
$$

$$
+ \left(\frac{1}{2\beta\sqrt{1+N_{i,n}}} + 1\right)\frac{\|u_i\|X_i\sqrt{N_{i,n}}}{\gamma} + \frac{\|\mathbf{u}_i\|\,X_i}{4\beta\gamma}\sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{N_{i,t}}.
$$

The first and the last sums in the right hand side can get more elegant form. Recall $M_{i,t}Z_{i,t} = 1$ implies $N_{i,t} = N_{i,t-1} + 1$, we can write

$$
\sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{\sqrt{1+N_{i,t-1}}} = \sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{\sqrt{N_{i,t}}} = \sum_{r=1}^{N_{i,n}}\frac{1}{\sqrt{r}} \leq 2\sqrt{N_{i,n}}.
$$

Now, we apply the same bound on the second sum, but with more loose bound,

$$
\sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{N_{i,t}} \leq \sum_{t=1}^{n}\frac{M_{i,t}Z_{i,t}}{\sqrt{N_{i,t}}} \leq 2\sqrt{N_{i,n}}.
$$

Defining $R_i = \left(\|u_i\| X_i\right)/\gamma$, and combining the last three bounds together we get

$$\sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(\frac{b_{i,t-1} + |\hat{p}_{i,t}|}{b_{i,t-1}} \right)$$

$$\leq \frac{1}{\gamma} \sum_{t=1}^{n} M_{i,t} Z_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{1}{\beta} \left(1 + \frac{3R_i}{2} \right) \sqrt{N_{i,n}}$$

$$+ \frac{R_i}{2\beta} + R_i \sqrt{N_{i,n}}.$$

Now, we subtract non negative term from the left side, sum the inequality over all tasks and take the expectation.

$$\mathbb{E} \left[\sum_{i=1}^{K} \sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(\frac{b_{i,t-1} + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}{b_{i,t-1}} \right) \right]$$

$$\leq \frac{1}{\gamma} \mathbb{E} \left[\sum_{i=1}^{K} \sum_{t=1}^{n} M_{i,t} Z_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i) \right] + \mathbb{E} \left[\sum_{i=1}^{K} \left(R_i + \frac{2 + 3R_i}{2\beta} \right) \sqrt{N_{i,n}} \right]$$

$$+ \mathbb{E} \left[\sum_{i=1}^{K} \frac{R_i}{2\beta} \right].$$

First we look at the left hand side of the inequality.

$$\sum_{i=1}^{K} \mathbb{E} \left[\sum_{t=1}^{n} M_{i,t} Z_{i,t} \left(\frac{b_{i,t-1} + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|}{b_{i,t-1}} \right) \right] = \sum_{i=1}^{K} \mathbb{E} \left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t} \right]$$

The bound of D_t now is

$$D_t = \sum_{i=1}^{K} \frac{b_{i,t-1}}{b_{i,t-1} + |\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}|} \leq K$$

Since $M_{i,t} > 0 \forall i, t$, thus there exists $\delta_i \in \mathbb{R} > 0$ such that

$$\sum_{i=1}^{K} \mathbb{E} \left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t} \right] = \sum_{i=1}^{K} \frac{1}{\delta_i} \mathbb{E} \left[\sum_{t=1}^{n} M_{i,t} \right] \leq \frac{1}{\delta} \mathbb{E} \left[\sum_{i=1}^{K} \sum_{t=1}^{n} M_{i,t} \right]$$

When $\delta = max_i\delta_i$. Next we handle the middle term of the rhs.

$$\mathbb{E}\left[\sum_{i=1}^{K}\left(R_i + \frac{2+3R_i}{2\beta}\right)\sqrt{N_{i,n}}\right] \leq \mathbb{E}\left[\sum_{i=1}^{K}\left(R + \frac{2+3R}{2\beta}\right)\sqrt{N_{i,n}}\right]$$

$$= \left(R + \frac{2+3R}{2\beta}\right)\mathbb{E}\left[\sum_{i=1}^{K}\sqrt{N_{i,n}}\right],$$

Where $R = \max_i\left(\|u_i\|X_i\right)/\gamma$. Now, considering that $N_{i,n}$ is non-negative and and exploiting the concavity of the square root function we can continue as following

$$\mathbb{E}\left[\sum_{i=1}^{K}\sqrt{N_{i,n}}\right] \leq \mathbb{E}\left[\sqrt{K\sum_{i=1}^{K}N_{i,n}}\right] = \sqrt{K}\mathbb{E}\left[\sqrt{\sum_{i=1}^{K}N_{i,n}}\right]$$

$$\leq \sqrt{K}\sqrt{\mathbb{E}\left[\sum_{i=1}^{K}N_{i,n}\right]}.$$

Here we used the bound for sum of squares, $\sum_{i=1}^{K}\sqrt{a_i} \leq \sqrt{K\sum_{i=1}^{K}a_i}$ for $a_i \in \mathbb{R}, a_i \geq 0$ combining with Jensen inequality. Moreover, we can continue, recall that by definition, $N_{i,n} = \sum_{t=1}^{n}Z_{i,t}M_{i,t}$ and $Z_{i,t}M_{i,t} \leq M_{i,t}$ such that

$$\sqrt{K}\sqrt{\mathbb{E}\left[\sum_{i=1}^{K}N_{i,n}\right]} \leq \sqrt{K}\sqrt{\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}M_{i,t}\right]}.$$

Plugging the result in the inequality, we get

$$\frac{1}{\delta}\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}M_{i,t}\right]$$

$$\leq \frac{1}{\gamma}\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}M_{i,t}Z_{i,t}\ell_{\gamma,i,t}(\mathbf{u}_i)\right] + \mathbb{E}\left[\sum_{i=1}^{K}\left(R_i + \frac{2+3R_i}{2\beta}\right)\sqrt{N_{i,n}}\right] + \mathbb{E}\left[\sum_{i=1}^{K}\frac{R_i}{2\beta}\right]$$

$$\leq \frac{1}{\gamma}\bar{L}_{\gamma,n} + \left(R + \frac{2+3R}{2\beta}\right)\sqrt{K}\sqrt{\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}M_{i,t}\right]} + \frac{KR}{2\beta}$$

34

which is quadratic inequality in $\sqrt{\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right]}$. Solving this equation gives us the bound for the cumulative mistakes

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right] \leq \delta\left[\frac{\delta B^2}{2} + \frac{1}{\gamma}\bar{L}_{\gamma,n} + \frac{KR}{2\beta} + B\sqrt{\frac{\delta B^2}{4} + \frac{1}{\gamma}\bar{L}_{\gamma,n} + \frac{KR}{2\beta}}\right]$$

■

3.6 Second Order SHAMPO

Te second order version of the perceptron algorithm, was proposed and analyzed by Cesa-Bianchi et al. [2005]. This algorithm was adopted to the online binary classification settings from the ridge-regression (squared loss and Euclidean regularization) framework as described by Hoerl and Kennard [1970] and Vovk [1997], and further analyzed in Azoury and Warmuth [2001] and Forster and Warmuth [2002]. It was also shown that this variation has the effect of reducing the number of mistakes compare to the first order version. Cesa-Bianchi et al. [2006a] also adapted the second order perceptron algorithm to the selective sampling case, which was extended later by Crammer [2014].

In this algorithm, we store and update two quantities per task instead of one, the matrix $A_{i,t} = I + \sum_t M_{i,t} Z_{i,t} \mathbf{x}_{i,t} \mathbf{x}_{i,t}^T$ and the vector $\mathbf{w}_{i,t} = \sum_t M_{i,t} Z_{i,t} \mathbf{x}_{i,t} y_{i,t}$. Whereas, the probability to issue a query on task i in time t is similar to the first order SHAMPO algorithm, the update that is done here in the case of queried mistake, is the second order perceptron update. The pseudo code of the second order SHAMPO is shown in Alg. 3.4

Theorem 6 *If second order SHAMPO algorithm runs on K tasks with K parallel example pair sequences $(\mathbf{x}_{i,1}, y_{i,1}), ... (\mathbf{x}_{i,n}, y_{i,n}) \in \mathbb{R}^d \times \{-1, 1\}$, $i = 1, ..., K$ with input parameter $b > 0$, then for all $\gamma > 0$, all $\mathbf{u}_i \in \mathbb{R}^d$ and all $n \geq 1$, there exists $0 < \delta \leq K$, such that,*

$$\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right]$$

$$\leq \frac{\delta}{\gamma}\bar{L}_{\gamma,n}(\mathbf{u}_i) + \frac{\delta b}{2\gamma^2}\sum_{i=1}^{K}\mathbf{u}_i^T \mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + \frac{\delta}{2b}\sum_{i=1}^{K}\sum_{k=1}^{d}\mathbb{E}\left[\ln\left(1 + \lambda_{i,k}\right)\right],$$

35

where $A_{i,n} = I + \sum_{t=1}^{n} M_{i,t} Z_{i,t} \mathbf{x}_{i,t} \mathbf{x}_{i,t}^T$, $\lambda_{i,k}$ is the ith eigenvalue of the matrix $A_{i,n}$ and the expectation is over the random choices of the algorithm.

Algorithm 3.4 Second order SHAMPO

Parameters: $b \in \mathbb{R} > 0$.
Initialize: $\mathbf{w}_{i,0} = \mathbf{0}$, $A_{i,0} = I$

for $t = 1, 2, ..., n$ **do**

    1. Observe K instance vectors, $\mathbf{x}_{i,t}$, $(i = 1, \ldots, K)$.
    2. Compute $\hat{p}_{i,t} = \mathbf{x}_{i,t}^T \left(A_{i,t-1} + \mathbf{x}_{i,t} \mathbf{x}_{i,t}^T \right)^{-1} \mathbf{w}_{i,t-1}$.
    3. Predict K labels, $\hat{y}_{i,t} = \text{sign}(\hat{p}_{i,t})$.
    4. Draw problem J_t with the distribution:

$$\Pr\left[J_t = j \right] = \frac{\left(b + |\hat{p}_{j,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}| \right)^{-1}}{D_t},$$

$$D_t = \sum_{i} \left(b + |\hat{p}_{i,t}| - \min_{m=1}^{K} |\hat{p}_{m,t}| \right)^{-1}.$$

    Where $m, i \in \{1, ..., K\}$ and D_t is the normalization factor.
    5. Query the true label ,$y_{J_t,t} \in \{-1, 1\}$.
    6. Set the indicator $M_{J_t,t} = 1$ iff $y_{J_t,t} \neq \hat{y}_{J_t,t}$.
    7. Update with the perceptron rule:
    8. Update:

$$\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + M_{J_t,t} \, \mathbf{x}_{J_t,t} y_{J_t,t} \qquad (3.28)$$
$$A_{J_t,t} = A_{J_t,t-1} + M_{J_t,t} \mathbf{x}_{J_t,t} \mathbf{x}_{J_t,t}^T$$

end for
Output: $\mathbf{w}_{i,n}$ for $i = 1, \ldots, K$.

Proof: Define the regularized cumulative square loss of the updated rounds on the task i up to the round t by

$$\Phi_{i,t}(\mathbf{u}_i) = \frac{1}{2} \|\mathbf{u}_i\|^2 + \frac{1}{2} \sum_{s=1}^{t} Z_{i,t} M_{i,t} \left(y_{i,t} - \mathbf{u}_i^T \mathbf{x}_{i,t} \right)^2.$$

It was proved by Forster [1999] in the linear regression case and adopted by Cesa-

Bianchi et al. [2006c] and Crammer Crammer [2014] for the second order classification that of the single task,

$$\frac{1}{2}Z_{i,t}M_{i,t}\left(y_{i,t}-\hat{p}_{i,t}\right)^2 = \inf_{\mathbf{u}_i}\Phi_{i,t+1}(\mathbf{u}_i) - \inf_{\mathbf{u}_i}\Phi_{i,t}(\mathbf{u}_i) + \frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t}^{-1}\mathbf{x}_{i,t}$$
$$- \frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t-1}^{-1}\mathbf{x}_{i,t}\hat{p}_{i,t}^2.$$

We can drop now the last term which is nonnegative because $A_{i,t-1}$ is positive definite matrix and so that $A_{i,t-1}^{-1}$. Now, we sum up the equation over t and get

$$\sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\left(y_{i,t}-\hat{p}_{i,t}\right)^2 \le \inf_{\mathbf{u}_i}\Phi_{i,n+1}(\mathbf{u}_i) - \inf_{\mathbf{u}_i}\Phi_{i,1}(\mathbf{u}_i) + \sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t}^{-1}\mathbf{x}_{i,t}$$

$$\le \Phi_{i,n+1}(\mathbf{u}_i) + \sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t}^{-1}\mathbf{x}_{i,t}$$

$$\le \frac{1}{2}\|\mathbf{u}_i\|^2 + \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}M_{i,t}\left(y_{i,t}-\mathbf{u}_i^T\mathbf{x}_{i,t}\right)^2 + \sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t}^{-1}\mathbf{x}_{i,t}$$

since $\inf_{\mathbf{u}_i}\Phi_{i,1}(\mathbf{u}_i) = 0$. We now expand the squares and the inequality becomes

$$\sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\left(\hat{p}_{i,t}^2 - 2\hat{p}_{i,t}y_{i,t}\right) \le \frac{1}{2}\|\mathbf{u}_i\|^2$$
$$+ \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}M_{i,t}\left(\mathbf{u}_i^T\mathbf{x}_{i,t}\right)^2 + \sum_{t=1}^{n}Z_{i,t}M_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t} + \sum_{t=1}^{n}\frac{Z_{i,t}M_{i,t}}{2}\mathbf{x}_{i,t}^T A_{i,t}^{-1}\mathbf{x}_{i,t}.$$

(3.29)

Now we handle the right hand side of the inequality one by one. First we start by writing the two first terms in the form

$$\frac{1}{2}\|\mathbf{u}_i\|^2 + \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}M_{i,t}\left(\mathbf{u}_i^T\mathbf{x}_{i,t}\right)^2 = \frac{1}{2}\mathbf{u}_i^T\left(I + \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}M_{i,t}\mathbf{x}_{i,t}\mathbf{x}_{i,t}^T\right)\mathbf{u}_i$$
$$= \frac{1}{2}\mathbf{u}_i^T A_{i,n}\mathbf{u}_i$$

(3.30)

Denote the k_i^{th} eigenvalue of the matrix $A_{i,n}$ by $1 + \lambda_{i,k}$, as proved by Forster [1999], we bound the last term by

$$\frac{1}{2} \sum_{t=1}^{n} Z_{i,t} M_{i,t} \mathbf{x}_{i,t}^T A_{i,t}^{-1} \mathbf{x}_{i,t} \le \frac{1}{2} \sum_{t=1}^{n} \ln \left(\frac{\det A_{i,t}}{\det A_{i,t-1}} \right)$$
$$= \frac{1}{2} \ln \left(\frac{\det A_{i,n}}{\det A_{i,0}} \right) = \frac{1}{2} \ln \left(\det A_{i,n} \right) = \frac{1}{2} \sum_{k=1}^{d} \ln \left(1 + \lambda_{i,k} \right). \tag{3.31}$$

Plugging Eq. (3.30) and Eq. (3.31) back into Eq. (3.29) and dropping the positive term from the left hand side, recall that on round when there is a mistake ($M_{i,t} = 1$), $\hat{p}_{i,t} y_{i,t} \le 0$, we obtain

$$\sum_{t=1}^{n} Z_{i,t} M_{i,t} \left(|\hat{p}_{i,t}| + \mathbf{u}_i^T \mathbf{x}_{i,t} y_{i,t} \right) \le \frac{1}{2} \mathbf{u}_i^T A_{i,n} \mathbf{u}_i + \frac{1}{2} \sum_{k=1}^{d} \ln \left(1 + \lambda_{i,k} \right).$$

By the definition of hinge loss, $\gamma - \ell_{\gamma,i,t}(\mathbf{u}_i) \le y_{i,t} \mathbf{u}_i^T \mathbf{x}_{i,t}$. Replacing the vu_i with it's scaling $\frac{b}{\gamma} \mathbf{u}_i$ yields

$$\sum_{t=1}^{n} Z_{i,t} M_{i,t} \left(|\hat{p}_{i,t}| + b \right)$$
$$\le \frac{b}{\gamma} \sum_{t=1}^{n} Z_{i,t} M_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{b^2}{2\gamma^2} \mathbf{u}_i^T A_{i,n} \mathbf{u}_i + \frac{1}{2} \sum_{k=1}^{d} \ln \left(1 + \lambda_{i,k} \right).$$

Now, we subtract a non negative quantity $\sum_{t=1}^{n} M_{i,t} Z_{i,t} \min_j |\hat{p}_{j,t}|$ from the left hand side and get,

$$\sum_{t=1}^{n} Z_{i,t} M_{i,t} \left(|\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}| + b \right)$$
$$\le \frac{b}{\gamma} \sum_{t=1}^{n} Z_{i,t} M_{i,t} \ell_{\gamma,i,t}(\mathbf{u}_i) + \frac{b^2}{2\gamma^2} \mathbf{u}_i^T A_{i,n} \mathbf{u}_i + \frac{1}{2} \sum_{k=1}^{d} \ln \left(1 + \lambda_{i,k} \right).$$

Now we take the expectation on both inequality sides. First we start from the left

side. As in the first order proof,

$$\mathbb{E}\left[\sum_{t=1}^{n} Z_{i,t} M_{i,t}\left(|\hat{p}_{i,t}| - \min_j |\hat{p}_{j,t}| + b\right)\right] = \mathbb{E}\left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t}\right].$$

Taking the expectation from the right hand side the equation becomes

$$\mathbb{E}\left[\sum_{t=1}^{n} \frac{M_{i,t}}{D_t}\right] \leq \frac{b}{\gamma}\bar{L}_{\gamma,i,n}(\mathbf{u}_i) + \frac{b^2}{2\gamma^2}\mathbf{u}_i^T \mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + \frac{1}{2}\sum_{k=1}^{d} \mathbb{E}\left[\ln\left(1 + \lambda_{i,k}\right)\right].$$

We use now Eq. (3.9) and get

$$\frac{b}{\delta_i}\mathbb{E}\left[\sum_{t=1}^{n} M_{i,t}\right] \leq \frac{b}{\gamma}\bar{L}_{\gamma,i,n}(\mathbf{u}_i) + \frac{b^2}{2\gamma^2}\mathbf{u}_i^T \mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + \frac{1}{2}\sum_{k=1}^{d} \mathbb{E}\left[\ln\left(1 + \lambda_{i,k}\right)\right]$$

We conclude the proof by summing up the last inequality over all K tasks and setting $\delta = \max \delta_i$, such that

$$\frac{1}{\delta}\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} M_{i,t}\right]$$
$$\leq \frac{1}{\gamma}\bar{L}_{\gamma,n}(\mathbf{u}_i) + \frac{b}{2\gamma^2}\sum_{i=1}^{K}\mathbf{u}_i^T \mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + \frac{1}{2b}\sum_{i=1}^{K}\sum_{k=1}^{d} \mathbb{E}\left[\ln\left(1 + \lambda_{i,k}\right)\right].$$

3.7   Second Order Adaptive SHAMPO

All the algorithms that were shown so far, exploited the uncertainty of the algorithm prediction using the low margin rule. Another approach of estimating the certainty of prediction could be comparing the input example with all the previous tagged examples. If the algorithm already received similar examples before, and requested their labels, probably the prediction that is based on those similar examples, can be considered with high confidence. Crammer [2014] used the same approach in the selective sampling setting, by introducing a distribution that depends on both margin and this uncertainty measure. An aggressive update was applied as well, based on the same measure.

39

Algorithm 3.5 Second order aggressive SHAMPO.

Parameters: $c \in \mathbb{R} > 0$.
Initialize: $\mathbf{w}_{i,0} = \mathbf{0}$, $A_0 = I$

for $t = 1, 2, ..., n$ **do**

    1. Observe K instance vectors, $\mathbf{x}_{i,t}$, $(i = 1, \ldots, K)$.

    2. Compute $\hat{p}_{i,t} = \mathbf{x}_{i,t}^T \left(A_{i,t-1} + \mathbf{x}_{i,t}\mathbf{x}_{i,t}^T \right)^{-1} \mathbf{w}_{i,t-1}$.

    3. Predict K labels, $\hat{y}_{i,t} = \text{sign}(\hat{p}_{i,t})$.

    4. Compute $r_{i,t} = \mathbf{x}_{i,t}^T A_{i,t-1}^{-1} \mathbf{x}_{i,t}$.

    5. Draw problem J_t with the distribution:

$$\Pr\left[J_t = j\right] = \frac{1}{D_t} \frac{2c}{2c + \left(\Theta\left(|\hat{p}_{j,t}|, r_{j,t}\right)\right)_+},$$

$$D_t = 2c \sum_i \left(2c + \left(\Theta\left(|\hat{p}_{i,t}|, r_{i,t}\right)\right)_+\right)^{-1}.$$

    Where $m, i \in \{1, ..., K\}$ and D_t is the normalization factor.

    6. Set $Z_{J,t} = 1$, $Z_{i,t} = 0$, $\forall i \neq J$.

    7.   **if** $\Theta\left(|\hat{p}_{J_t,t}|, r_{i,t}\right) \geq 0$ **then**

    8.   **if** $y_{J_t,t} \neq \hat{y}_{J_t,t}$ **then**

    9. $U_{J_t,t} = 1$

   10.   **else**

   11. $U_{J_t,t} = 0$

   12.   **end if**

   13.   **else**

   14.   set $U_{J_t,t} = 1$

   15.   **end if**

   16. Update:

$$\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + U_{J_t,t}\, \mathbf{x}_{J_t,t} y_{J_t,t} \tag{3.32}$$

$$A_{J_t,t} = A_{J_t,t-1} + U_{J_t,t}\mathbf{x}_{J_t,t}\mathbf{x}_{J_t,t}^T$$

end for
Output: $\mathbf{w}_{i,n}$ for $i = 1, \ldots, K$.

We adapt this approach to our setting of multitask learning sharing a single annotator. First, we introduce another quantity that is the projection of the vector $\mathbf{x}_{i,t}$ on the inverse of the covariance-like matrix $A_{i,t-1}$

$$r_{i,t} = \mathbf{x}_{i,t}^T A_{i,t-1}^{-1} \mathbf{x}_{i,t}$$

We can think about this quantity as an uncertainty measure. Where large values of $r_{i,t}$ represent high uncertainty, whereas small value of the same quantity are related to low uncertainty, since the algorithm already received labeled that are close to $\mathbf{x}_{i,t}$. In the edge case, when the example $\mathbf{x}_{i,t}$ is orthogonal to all previous queried examples that came from the same task, $r_{i,t} = 1$.

Define the function

$$\Theta\left(|\hat{p}_{i,t}|, r_{i,t}\right) = (1 + r_{i,t})\,\hat{p}_{i,t}^2 + 2\,|\hat{p}_{i,t}| - \frac{r_{i,t}}{1 + r_{i,t}} \tag{3.33}$$

Proof:

Before proofing the Thm. we will use the next inequality. define $x \in [0,1]$

$$\sqrt{1 - x} + \sqrt{1 + x} \le 2. \tag{3.34}$$

From the concavity of $\sqrt{1 + x}$ we see that $\sqrt{1 + x} \le 1 + \frac{1}{2}x$. Using this inequality twice, we get $\sqrt{1 - x} + \sqrt{1 + x} \le 1 - \frac{1}{2}x + 1 + \frac{1}{2}x = 2$. Define the regularized cumulative square loss of the updated rounds on the task i up to the round t

$$\Phi_{i,t}(\mathbf{u}_i) = \frac{1}{2}\,\|\mathbf{u}_i\|^2 + \frac{1}{2}\sum_{s=1}^{t} Z_{i,t} U_{i,t}\left(y_{i,t} - \mathbf{u}_i^T \mathbf{x}_{i,t}\right)^2.$$

Simlar to the proof of Thm .3 of Cesa-Bianchi et al Cesa-Bianchi et al. [2006c] and Forster ????,

$$
\begin{aligned}
\frac{1}{2} Z_{i,t} U_{i,t}\left(y_{i,t} - \hat{p}_{i,t}\right)^2 =& \inf_{\mathbf{u}_i} \Phi_{i,t+1}(\mathbf{u}_i) - \inf_{\mathbf{u}_i} \Phi_{i,t}(\mathbf{u}_i) + \frac{Z_{i,t} U_{i,t}}{2} \mathbf{x}_{i,t}^T A_{i,t}^{-1} \mathbf{x}_{i,t} \\
&- \frac{Z_{i,t} U_{i,t}}{2} \mathbf{x}_{i,t}^T A_{i,t-1}^{-1} \mathbf{x}_{i,t} \hat{p}_{i,t}^2 \\
=& \inf_{\mathbf{u}_i} \Phi_{i,t+1}(\mathbf{u}_i) - \inf_{\mathbf{u}_i} \Phi_{i,t}(\mathbf{u}_i) + \frac{Z_{i,t} U_{i,t}}{2} \frac{r_{i,t}}{1 + r_{i,t}} - \frac{Z_{i,t} U_{i,t}}{2} r_{i,t} \hat{p}_{i,t}^2
\end{aligned}
$$

Now, we sum up the equation over t,

$$\frac{1}{2}\sum_{t=1}^{n}Z_{i,t}U_{i,t}\left(y_{i,t}-\hat{p}_{i,t}\right)^2 = \inf_{\mathbf{u}_i}\Phi_{i,n+1}(\mathbf{u}_i) + \sum_{t=1}^{n}\frac{Z_{i,t}U_{i,t}}{2}\frac{r_{i,t}}{1+r_{i,t}} - \sum_{t=1}^{n}\frac{Z_{i,t}U_{i,t}}{2}r_{i,t}\hat{p}_{i,t}^2$$

$$\leq \frac{1}{2}\left\|\mathbf{u}_i\right\|^2 + \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}U_{i,t}\left(y_{i,t}-\mathbf{u}_i^T\mathbf{x}_{i,t}\right)^2 + \sum_{t=1}^{n}\frac{Z_{i,t}U_{i,t}}{2}\frac{r_{i,t}}{1+r_{i,t}}$$

$$- \sum_{t=1}^{n}\frac{Z_{i,t}U_{i,t}}{2}r_{i,t}\hat{p}_{i,t}^2.$$

For simplification we will define

$$A_{i,n} = I + \sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{x}_{i,t}\mathbf{x}_{i,t}^T.$$

Expanding the squares we get,

$$\frac{1}{2}\sum_{t=1}^{n}Z_{i,t}U_{i,t}\left(\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} - \frac{r_{i,t}}{1+r_{i,t}} + r_{i,t}\hat{p}_{i,t}^2\right)$$

$$\leq \frac{1}{2}\left\|\mathbf{u}_i\right\|^2 + \frac{1}{2}\sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}\mathbf{x}_{i,t}^T\mathbf{u}_i - \sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}$$

$$= \frac{1}{2}\mathbf{u}_i^T\left(I + \sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{x}_{i,t}\mathbf{x}_{i,t}^T\right)\mathbf{u}_i - \sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}$$

$$= \frac{1}{2}\mathbf{u}_i^T A_{i,n}\mathbf{u}_i - \sum_{t=1}^{n}Z_{i,t}U_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}.$$

$$(3.35)$$

The vectors $\mathbf{u}_i$ can be replaced with their scaled version, $c\mathbf{u}_i$. Introducing the trivial inequality, $1 - x \leq \max\{1 - x, 0\}$ we get

$$cZ_{i,t}U_{i,t}\left(1 - \mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right) \leq cZ_{i,t}U_{i,t}\max\left\{1 - \mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}, 0\right\}$$

$$-cZ_{i,t}U_{i,t}\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t} \leq -cZ_{i,t}U_{i,t} + cZ_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right).$$

$$(3.36)$$

Rearranging an plugging Eq. (3.37) and Eq. (3.35)

$$
\frac{1}{2} \sum_{t=1}^{n} Z_{i,t} U_{i,t} \left(\hat{p}_{i,t}^2 - 2 y_{i,t} \hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t} \hat{p}_{i,t}^2 + 2c \right)
$$

$$
\leq \frac{c^2}{2} \mathbf{u}_i^T A_{i,n} \mathbf{u}_i + c \sum_{t=1}^{n} Z_{i,t} U_{i,t} \ell \left(\mathbf{u}_i^T \mathbf{x}_{i,t} y_{i,t} \right)
$$

(3.37)

Recall that $U_{i,t} = M_{i,t} + G_{i,t}$ we will split the inequality into two different cases. First we will take into consideration the cases when an error update was performed, i.e. $M_{i,t} = 1$, in which we have $-y_{i,t} \hat{p}_{i,t} = |\hat{p}_{i,t}|$. In this case we need to consider also two subcases, when $\Theta(|\hat{p}_{i,t}|, r_{i,t}) \geq 0$ and when $\Theta(|\hat{p}_{i,t}|, r_{i,t}) < 0$. Beginning with the former subcase, recall that for this case $\mathbb{E}_{t-1}[Z_{i,t}] = \frac{1}{D_t} \frac{2c}{2c + (\Theta(|\hat{p}_{j,t}|, r_{j,t}))}$, we get

$$
\mathbb{E} \left[Z_{i,t} U_{i,t} \left(\hat{p}_{i,t}^2 - 2 y_{i,t} \hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t} \hat{p}_{i,t}^2 + 2c \right) \right]
$$

$$
= \mathbb{E} \left[\mathbb{E}_{t-1}[Z_{i,t}] U_{i,t} \left(\hat{p}_{i,t}^2 - 2 y_{i,t} \hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t} \hat{p}_{i,t}^2 + 2c \right) \right]
$$

$$
= 2c \mathbb{E} \left[\frac{1}{D_t} U_{i,t} \right].
$$

When $\Theta(|\hat{p}_{i,t}|, r_{i,t}) < 0$ the conditional expectation becomes $\mathbb{E}_{t-1}[Z_{i,t}] = \frac{1}{D_t}$ and the thus,

$$
\mathbb{E} \left[Z_{i,t} U_{i,t} \left(\hat{p}_{i,t}^2 - 2 y_{i,t} \hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t} \hat{p}_{i,t}^2 + 2c \right) \right]
$$

$$
= \mathbb{E} \left[\mathbb{E}_{t-1}[Z_{i,t}] U_{i,t} \left(\hat{p}_{i,t}^2 + 2 |\hat{p}_{i,t}| - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t} \hat{p}_{i,t}^2 + 2c \right) \right]
$$

$$
\geq \mathbb{E} \left[\mathbb{E}_{t-1}[Z_{i,t}] U_{i,t} \left(-\frac{r_{i,t}}{1 + r_{i,t}} + 2c \right) \right]
$$

$$
\geq 2c \mathbb{E} \left[\frac{U_{i,t}}{D_t} \right] - \frac{r_{i,t}}{1 + r_{i,t}} \mathbb{E} \left[\frac{U_{i,t}}{D_t} \right]
$$

$$
\geq 2c \mathbb{E} \left[\frac{1}{D_t} U_{i,t} \right] - \frac{r_{i,t}}{1 + r_{i,t}} \mathbb{E} \left[\frac{1}{D_t} \right].
$$

Now we examine the case where an update was performed, but there was no mis-

take. In this case, $0 \le y_{i,t}\hat{p}_{i,t}$ and the aggressive update was performed. Recall the bound on the margin for such case and using Eq. (3.37) , we bound the margin as follows

$$0 \le y_{i,t}\hat{p}_{i,t} \le \theta(r_{i,t}) = \frac{-1 + \sqrt{1 + r_{i,t}}}{1 + r_{i,t}}.$$

We can bound now,

$$\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t}\hat{p}_{i,t}^2 + 2c$$

$$= (1 + r_{i,t})\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} + \frac{r_{i,t}}{1 + r_{i,t}} - 2\frac{r_{i,t}}{1 + r_{i,t}} + 2c$$

$$= f(y_{i,t}\hat{p}_{i,t}) - 2\frac{r_{i,t}}{1 + r_{i,t}} + 2c$$

where $f(y_{i,t}\hat{p}_{i,t}) = (1 + r_{i,t})\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} + \frac{r_{i,t}}{1+r_{i,t}}$ is a quadratic convex function with two non-negative roots $\frac{1 \pm \sqrt{1-r_{i,t}}}{1+r_{i,t}}$ and we know that the margin is lower than the smaller root , $y_{i,t}\hat{p}_{i,t} \le \frac{1 - \sqrt{1-r_{i,t}}}{1+r_{i,t}}$ which leads to the inequality $f(y_{i,t}\hat{p}_{i,t}) \ge 0$ so we bound

$$\mathbb{E}\left[Z_{i,t}U_{i,t}\left(\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} - \frac{r_{i,t}}{1 + r_{i,t}} + r_{i,t}\hat{p}_{i,t}^2 + 2c\right)\right]$$

$$= \mathbb{E}\left[\mathbb{E}_{t-1}\left[Z_{i,t}\right]U_{i,t}\left(f(y_{i,t}\hat{p}_{i,t}) - 2\frac{r_{i,t}}{1 + r_{i,t}} + 2c\right)\right]$$

$$\ge \mathbb{E}\left[\mathbb{E}_{t-1}\left[Z_{i,t}\right]U_{i,t}\left(-2\frac{r_{i,t}}{1 + r_{i,t}} + 2c\right)\right]$$

$$\ge 2c\mathbb{E}\left[\frac{U_{i,t}}{D_t}\right] - \frac{2r_{i,t}}{1 + r_{i,t}}\mathbb{E}\left[\frac{U_{i,t}}{D_t}\right]$$

$$\ge 2c\mathbb{E}\left[\frac{1}{D_t}U_{i,t}\right] - \frac{2r_{i,t}}{1 + r_{i,t}}\mathbb{E}\left[\frac{1}{D_t}\right].$$

Summarize the results we get,

$$\frac{1}{2}\sum_{t=1}^{n}\mathbb{E}\left[Z_{i,t}U_{i,t}\left(\hat{p}_{i,t}^2 - 2y_{i,t}\hat{p}_{i,t} - \frac{r_{i,t}}{1+r_{i,t}} + r_{i,t}\hat{p}_{i,t}^2 + 2c\right)\right]$$

$$\geq c\sum_{t\in\mathcal{M}}\mathbb{E}\left[\frac{1}{D_t}U_{i,t}\right] + c\sum_{t\in\mathcal{G}}\mathbb{E}\left[\frac{1}{D_t}U_{i,t}\right] \qquad (3.38)$$

$$-\frac{1}{2}\sum_{t\in\mathcal{A}\cap\mathcal{M}}\frac{r_{i,t}}{1+r_{i,t}}\mathbb{E}\left[\frac{1}{D_t}\right] - \sum_{t\in\mathcal{A}\cap\mathcal{G}}\frac{r_{i,t}}{1+r_{i,t}}\mathbb{E}\left[\frac{1}{D_t}\right]$$

Combining the result of the last inequality with the expectation of the Eq. (3.37), recall that $\sum_{t\in\mathcal{M}}U_{i,t} = M_i$, and $\sum_{t\in\mathcal{G}}U_{i,t} = G_i$ we get,

$$c\sum_{t\in\mathcal{M}}\mathbb{E}\left[\frac{1}{D_t}U_{i,t}\right] + c\sum_{t\in\mathcal{G}}\mathbb{E}\left[\frac{1}{D_t}U_{i,t}\right] - \frac{1}{2}\sum_{t\in\mathcal{A}\cap\mathcal{M}}\frac{r_{i,t}}{1+r_{i,t}}\mathbb{E}\left[\frac{1}{D_t}\right]$$

$$-\sum_{t\in\mathcal{A}\cap\mathcal{U}}\frac{r_{i,t}}{1+r_{i,t}}\mathbb{E}\left[\frac{1}{D_t}\right] \leq \frac{c^2}{2}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + c\sum_{t=1}^{n}\mathbb{E}\left[Z_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right)\right]$$

The normalization can also been bound by

$$D_t = 2c\sum_{i=1}^{K}\left(2c + \left(\Theta\left(|\hat{p}_{i,t}|, r_{i,t}\right)\right)_{+}\right)^{-1} \leq 2c\sum_{m=1}^{K}\frac{1}{2c} = K$$

which leads to

$$\mathbb{E}\left[M_i\right] + \mathbb{E}\left[G_i\right] - \frac{1}{2c}\mathbb{E}\sum_{t\in\mathcal{A}\cap\mathcal{M}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right] - \frac{1}{c}\mathbb{E}\sum_{t\in\mathcal{A}\cap\mathcal{U}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right]$$

$$\leq \frac{Kc}{2}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + K\sum_{t=1}^{n}\mathbb{E}\left[Z_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right)\right]$$

$$\mathbb{E}\left[M_i\right] \leq \frac{Kc}{2}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + K\sum_{t=1}^{n}\mathbb{E}\left[Z_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right)\right] - \mathbb{E}\left[G_i\right]$$

$$+ \frac{1}{2c}\mathbb{E}\sum_{t\in\mathcal{A}\cap\mathcal{M}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right] + \frac{1}{c}\mathbb{E}\sum_{t\in\mathcal{A}\cap\mathcal{U}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right]$$

$$\leq \frac{Kc}{2}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + K\mathbb{E}\left[\sum_{t=1}^{n}Z_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right)\right] - \mathbb{E}\left[G_i\right]$$

$$+ \frac{1}{c}\sum_{t\in\mathcal{A}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right]$$

Now, we can summarize the inequality over all of the tasks,

$$\mathbb{E}\left[M\right] \leq \frac{cK}{2}\sum_{i=1}^{K}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + K\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n}Z_{i,t}U_{i,t}\ell\left(\mathbf{u}_i^T\mathbf{x}_{i,t}y_{i,t}\right)\right] - \mathbb{E}\left[G\right]$$

$$+ \frac{1}{c}\mathbb{E}\sum_{i=1}^{K}\sum_{t\in\mathcal{A}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right]$$

$$\mathbb{E}\left[M\right] \leq \frac{cK}{2}\sum_{i=1}^{K}\mathbf{u}_i^T\mathbb{E}\left[A_{i,n}\right]\mathbf{u}_i + K\bar{L}_{1,n} - \mathbb{E}\left[G\right]$$

$$+ \frac{1}{c}\mathbb{E}\sum_{i=1}^{K}\sum_{t\in\mathcal{A}}\left[\frac{r_{i,t}}{1+r_{i,t}}\right]$$

Chapter 4

From Multi-task to Contextual Bandits

Although our algorithm is designed for many binary-classification tasks that can be independent, it can also be applied in two settings of contextual bandits, when decoupling exploration and exploitation is allowed as in Yu and Mannor [2009], Avner et al. [2012]. The problem of this setting is predicting a label $\hat{Y}_t \in \{1, \ldots, C\}$ given an input $\mathbf{x}_t$. As before, the algorithm works in rounds. On round t the algorithm receives an input $\mathbf{x}_t$ and outputs multicalss label $\hat{Y}_t \in \{1, \ldots, C\}$. Then, it queries for some information about the label via a single binary "yes-no" question, and uses the feedback to update its model. We consider here two forms of binary questions.

4.1 One-vs-Rest

The first setting is termed *one-vs-rest*. The algorithm asks if the true label is some label $\bar{Y}_t \in \{1, \ldots, C\}$, possibly not the predicted label, i.e. it may be the case that $\bar{Y}_t \neq \hat{Y}_t$. Given the response whether $\bar{Y}_t$ is the true label Y_t, the algorithm updates its models. The reduction we perform is by introducing K tasks, one per class. The problem of the learning algorithm for task i is to decide whether the true label is class i or not. Given the output of all the binary classifiers, the algorithm generates a single multi-class prediction to be the single label for which the output of the corresponding binary classifier is positive. If such class does not exist, or there are more than one classes as such, a random prediction is used, i.e., given an input $\mathbf{x}_t$

we define $\hat{Y}_t = \arg\max_i \hat{y}_{i,t}$, where ties are broken arbitrarily. The label to be queried is $\bar{Y}_t = J_t$, i.e. the problem index that SHAMPO is querying. We analyze the performance of this reduction as a multiclass prediction algorithm.

Corollary 7 *Assume the SHAMPO algorithm is executed as above with $K = C$ one-vs-rest problems, on a sequence $(\mathbf{x}_1, Y_1), ...(\mathbf{x}_n, Y_n) \in \mathbb{R}^d \times \{1, ..., C\}$, and input parameter $b > 0$ and prior $1 \le a_i \forall i$. Then for all $\gamma > 0$ and all $\mathbf{u}_i \in \mathbb{R}^d$, there exists $0 < \delta \le \sum_{i=1}^{C} a_i$ such that the expected number of multi-class errors is bounded as follows*

$$\mathbb{E}\left[\sum_t \llbracket Y_t \ne \hat{Y}_t \rrbracket \right]$$

$$\le \frac{\delta}{\gamma}\left[\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{(2b + X^2)^2 U^2}{8\gamma b}\right] + \left(2\frac{\lambda}{b} - 1\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} a_i G_{i,t}\right],$$

where $\llbracket I \rrbracket = 1$ if the predicate I is true, and zero otherwise.

The corollary follows directly from Theorem **??** by noting that, $\llbracket Y_t \ne \hat{Y}_t \rrbracket \le \sum_i M_{i,t}$. That is, there is a multiclass mistake if there is at least one prediction mistake of one of the one-vs-rest problems. The closest setting is contextual bandits, yet we allow decoupling of exploration and exploitation. Ignoring this decoupling, the Banditron algorithm Kakade et al. [2008] is the closest to ours, with a regret of $O(T^{2/3})$. Hazan et al Hazan and Kale [2011] proposed an algorithm with $O(\sqrt{T})$ regret but designed for the log loss, with coefficient that may be very large, and another Crammer and Gentile [2013] algorithm has $O(\sqrt{T})$ regret with respect to prediction mistakes, yet they assumed stochastic labeling, rather than adversarial.

4.2 One-vs-One

In the second setting, termed by *one-vs-one*, the algorithm picks two labels $\bar{Y}_t^+, \bar{Y}_t^- \in \{1\ldots C\}$, possibly both not the predicted label. The feedback for the learner is three-fold: it is $y_{J_t,t} = +1$ if the first alternative is the correct label, $\bar{Y}_t^+ = Y_t$, $y_{J_t,t} = -1$ if the second alternative is the correct label, $\bar{Y}_t^- = Y_t$, and it is $y_{J_t,t} = 0$ otherwise (in this case there is no error and we set $M_{J_t,t} = 0$). The reduction we perform is by introducing $K = \binom{C}{2}$ tasks, one per pair of classes. The goal of the learning algorithm for a task indexed with two labels (y_1, y_2) is to decide which

one is the correct label, given it is one of the two. Given the output of all (binary) classifiers the algorithm generates a single multi-class prediction using a tournament in a round-robin approach Fürnkranz [2002]. If there is no clear winner, a random prediction is used. We now analyze the performance of this reduction as a multiclass prediction algorithm.

Corollary 8 *Assume the SHAMPO algorithm is executed as above, with $K = \binom{C}{2}$ one-vs-one problems, on a sequence $(\mathbf{x}_1, Y_1), ... (\mathbf{x}_n, Y_n) \in \mathbb{R}^d \times \{1, ..., C\}$, and input parameter $b > 0$ and prior $1 \leq a_i \forall i$. Then for all $\gamma > 0$ and all $\mathbf{u}_i \in \mathbb{R}^d$, there exists $0 < \delta \leq \sum_{i=1}^{\binom{C}{2}} a_i$ such that the expected number of multi-class errors can be bounded as follows*

$$
\mathbb{E}\left[\sum_t [\![Y_t \neq \hat{Y}_t]\!]\right] \leq \frac{2}{(\binom{C}{2} - 1)/2 + 1} \times
$$
$$
\left\{\frac{\delta}{\gamma}\left[\left(1 + \frac{X^2}{2b}\right)\bar{L}_{\gamma,n} + \frac{(2b + X^2)^2 U^2}{8\gamma b}\right] + \left(2\frac{\lambda}{b} - 1\right)\mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=1}^{n} a_i G_{i,t}\right]\right\}
$$

The corollary follows directly from Theorem **??** by noting that the multiclass error is bounded by the multitask errors with a constant factor as following,

$$
[\![Y_t \neq \hat{Y}_t]\!] \leq \frac{2}{(\binom{C}{2} - 1)/2 + 1} \sum_{i=1}^{\binom{C}{2}} M_{i,t}.
$$

The last fact is derived from Allwein et al. [2001] analysis of the training error with the corresponding hamming distance of the ECOC matrix for the One-vs-One setting. Note, that the bound is essentially independent of C as the coefficient in the bound is upper bounded by 4.

We conclude this section with two algorithmic modifications, we employed in this setting. Currently, when the feedback is zero, there is no update of the weights, because there are no errors. This causes the algorithm to effectively ignore such examples, as in these cases the algorithm is not modifying any model, furthermore, if such example is repeated, a problem with possibly "0" feedback may be queried again.

$$
\mathbf{w}_{J_t,t} = \mathbf{w}_{J_t,t-1} + [\![y_{J_t,t} \neq 0]\!] y_{J_t,t} \mathbf{x}_{J_t,t} + [\![y_{J_t,t} = 0]\!] \eta \hat{y}_{J_t,t} \mathbf{x}_{J_t,t} \,,
$$

for some $\eta > 0$. In other words, if there is a possible error (i.e. $y_{J_t,t} \neq 0$) the update follows the Perceptron's rule. Otherwise, the weights are updated such that the absolute margin will increase, as

$$
\begin{aligned}
|\mathbf{w}_{J_t,t}^\top \mathbf{x}_{J_t,t}| &= \left(\mathbf{w}_{J_t,t-1} + \eta \hat{y}_{J_t,t} \mathbf{x}_{J_t,t} \right)^\top \mathbf{x}_{J_t,t} \\
&= |\mathbf{w}_{J_t,t-1}^\top \mathbf{x}_{J_t,t} + \eta \mathrm{sign}(\mathbf{w}_{J_t,t-1}^\top \mathbf{x}_{J_t,t}) \|\mathbf{x}_{J_t,t}\|^2| \\
&= |\mathbf{w}_{J_t,t-1}^\top \mathbf{x}_{J_t,t}| + \eta \|\mathbf{x}_{J_t,t}\|^2 > |\mathbf{w}_{J_t,t-1}^\top \mathbf{x}_{J_t,t}|.
\end{aligned}
$$

We call this method *one-vs-one-weak*, as it performs weak updates for zero feedback. The second alternative is not to allow 0 value feedback, and if this is the case, to set the label to be either $+1$ or -1, randomly. We call this method *one-vs-one-random*.

Chapter 5

Experiments

We evaluated the SHAMPO algorithm using four different datasets: USPS, MNIST (both OCR), Vocal Joystick (VJ, vowel recognition) and document classification (NLP). The USPS dataset, contains $7,291$ training examples and $2,007$ test examples, each one of them is a 16×16 pixels gray-scale images converted to a 256 dimensional vector. The MNIST dataset with 28×28 gray-scale images, contains $60,000$ examples in the training set and $10,000$ in the testing set. In both cases there are 10 possible labels, the 0 to 9 digits. The VJ project was built to enable individuals with motor impairments to use vocal parameters to control objects on a computer screen (buttons, sliders, etc.) and ultimately electro-mechanical instruments (e.g., robotic arms, wireless home automation devices). The VJ tasks is to predict a vowel from eight possible vowels. Each example is a frame of spoken value described with 13 MFCC coefficients transformed into 26 features. There are $572,911$ training examples and $236,680$ test examples. We created binary tasks from these multi-class datasets using two reductions: One-vs-Rest setting and One-vs-One setting. For example, in both USPS and MNIST there are 10 binary one-vs-rest tasks and 45 binary one-vs-one tasks. The NLP document classification is one-vs-one binary classification include of spam filtering, news items and news-group classification, sentiment classification, and product domain categorization. A total of 36 binary prediction tasks over all, with a total of $252,609$ examples, and input dimension varying between $8,768$ and $1,447,866$. Details of the individual binary tasks can be found elsewhere Crammer et al. [2012]. This yielded seven collections (USPS, MNIST and VJ; each as one-vs-rest or one-vs-one) and document classification.

5.1 Hard and Easy Tasks

First, we wanted to show that indeed, the SHMAPO algorithms are optimal for different distributions of hard and easy tasks and also tasks from different domains and dimensions. For that reason, we created an eighth collection, named MIXED, which consists of 40 tasks: 10 random tasks from each one of the four basic datasets (one-vs-one versions). Now, from each one of the eight dataset collections we generated between 6 to 10 combinations (or problems), each problem was created by sampling between 2 and 8 tasks which yielded a total of 64 multitask problems. We tried to diversify problems difficulty by including both hard and easy binary classification problems. For example, from the VJ dataset we generated 10 multitask problems with up to 5 tasks each, and different number of hard and easy tasks : 1 (easy task) + 1 (hard task), 1+2, 1+3, 1+4, 2+1, 2+2, 2+3, 3+1, 3+2, 4+1. The hardness of a binary problem is evaluated by the number of mistakes the Perceptron algorithm performs on the problem where for each multitask problem, we sample uniformly from each group of tasks (hard and easy) .

We evaluated two baselines as well as our algorithm. Algorithm *uniform* picks a random task to be queried and updated (corresponding to $b \to \infty$), *exploit* which picks the tasks with the lowest absolute margin (i.e. the "hardest instance"), this combination corresponds to $b \approx 0$ of SHAMPO. We tried for SHAMPO 13 values for b, equally spaced on a logarithmic scale between 10^{-7} and 10^5. All algorithms made a single pass over the training data. We ran two versions of the our First Order algorithm: plain version, without aggressiveness (updates on mistakes only, $\lambda = 0$) and an Aggressive version $\lambda = b/2$ (we tried lower values of λ as in the bound, but we found that $\lambda = b/2$ gives better results), both with uniform prior $(a_i = 1)$. We used separate training set and a test set, to build a model and evaluate it.

The results are evaluated using 2 quantities. First, the average test error (over all the dataset combinations) and the average score. For each combination we assigned a score of 1 to the algorithm with the lowest test error, and a score of 2, to the second best, and all the way up to a score of 6 to the algorithm with the highest test error.

Results are summarized in Table 5.1. In general *exploit* is better than *uniform* (for majority of the datasets) and aggressive algorithm is better than non-aggressive one. Aggressive SHAMPO yields the best results both evaluated as average (over tasks per combination and over combinations). Remarkably, even in the mixed

Table 5.1: Test errors percentage . Scores are shown in parenthesis.

	Aggressive $\lambda = b/2$			Plain		
Dataset	*exploit*	*SHAMPO*	*uniform*	*exploit*	*SHAMPO*	*uniform*
VJ 1 vs 1	5.22 (2.9)	**4.57 (1.1)**	5.67 (3.9)	5.21 (2.7)	6.93 (4.6)	6.26 (5.8)
VJ 1 vs Rest	13.26 (3.5)	**11.73 (1.2)**	12.43 (2.5)	13.11 (3.0)	14.17 (5.0)	14.71 (5.8)
USPS 1 vs 1	3.31 (2.5)	**2.73 (1.0)**	19.29 (6.0)	3.37 (2.5)	4.83 (4.0)	5.33 (5,0)
USPS 1 vs Rest	5.45 (2.8)	**4.93 (1.2)**	10.12 (6.0)	5.31 (2.0)	6.51 (4.0)	7.06 (5.0)
MNIST 1 vs 1	1.08 (2.3)	**0.75 (1.0)**	5.9 (6.0)	1.2 (2.7)	1.69 (4.1)	1.94 (4.9)
MNIST 1 vs Rest	4.74 (2.8)	**3.88 (1.0)**	10.01 (6.0)	4.44 (2.8)	5.4 (3.8)	6.1 (5.0)
NLP documents	19.43 (2.3)	**16.5 (1.0)**	23.21 (5.0)	19.46 (2.7)	21.54 (4.7)	21.74 (5.3)
MIXED	2.75 (2.4)	**2.06 (1.0)**	13.59 (6.0)	2.78 (2.6)	4.2 (4.3)	4.45 (4.7)
Mean score	(2.7)	**(1.1)**	(5.2)	(2.6)	(4.3)	(5.2)

dataset (where tasks are of different nature: images, audio and documents), the aggressive SHAPO improves over uniform (4.45% error) and the aggressive-exploit baseline (2.75%), and achieves a mean test error of 2.06%.

Indeed, Table 5.1 shows that aggressive SHAMPO outperforms better alternatives. Yet, we claim that a good prior may improve results. We compute prior over the 45 USPS one-vs-one tasks and 10 USPS one-vs-rest, by running the perceptron algorithm on 1000 examples and computing the number of mistakes. We set the prior to be proportional to this number. We then reran aggressive SHAMPO with prior, comparing it to aggressive SHAMPO with no prior (i.e. $a_i = 1$). The results are summarized in Table 5.2. The prior improves performance in the USPS 1 vs 1 collection and USPS 1 vs Rest, when evaluated using score-rank, on averaged it is slightly worse than aggressive SHAMPO with not prior. Aggressive SHAMO with prior achieves average error of 1.47 (vs. 2.73 with no prior) on 1-vs-1 USPS and 4.97 (vs 4.93) on one-vs-rest USPS, with score rank of 1.0 (vs 2.9) and 1.7 (vs 2.0) respectively. Fig. 5.4(c) and Fig. 5.4(d) shows the test error for all values of b we evaluated. A good prior is shown to outperform the case $a_i = 1$ for most values of b. Again, we chose a specific prior generator system, however, another method or another prior knowledge may lead to better results.

Table 5.2: Test errors percentage . Scores are shown in parenthesis.

	Aggressive $\lambda = b/2$			Aggressive $\lambda = b/2$ **with prior**		
Dataset	*exploit*	*SHAMPO*	*uniform*	*exploit*	*SHAMPO*	*uniform*
USPS 1 vs 1	3.31 (3.9)	2.73 (2.9)	19.29 (5.6)	1.92 (2.2)	**1.47 (1.0)**	17.66 (5.4)
USPS 1 vs Rest	5.45 (3.5)	**4.93** (2.0)	10.12 (5.7)	5.23 (2.8)	4.97 **(1.7)**	9.64 (5.3)

Fig. 5.5(e) and Fig. 5.5(g) show the test error of the three algorithms on two of document classification combinations, with four and eight tasks. Clearly, not only

SHAMPO performs better, but it does so on each task individually. (Our analysis above bounds the total number of mistakes over all tasks.)

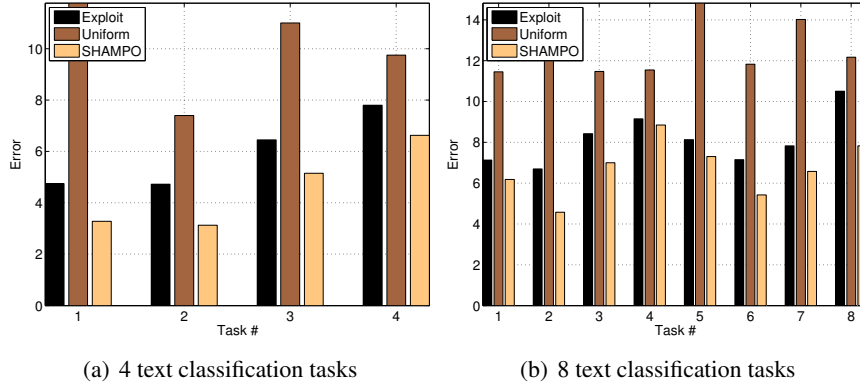(a) 4 text classification tasks   (b) 8 text classification tasks

Figure 5.1: Test error of aggressive SHAMPO on (a) four and (b) eight binary text classification tasks. Three algorithms are evaluated: uniform, exploit, and aggressive SHAMPO with $\lambda = b/2$.

5.2 Multi-task Binary Classification

First, we evaluated all the mentioned variations of SHAMPO algorithms. We ran a single pass over all of the training examples for all datasets and 11 equally spaced values of b between 10^{-5} and 10^{5}. The algorithms performance was evaluated using mean of accumulated training error of all tasks and the mean of test errors over all of the tasks for each datasets. Since the NLP dataset doesn't have a test set, we evaluated those quantities using 4 cross validation folds. We repeated the experiments 100 times and took the mean of all algorithm runs. For simplicity, we denote each one of those algorithms the abbreviations: First Order (FO), First Order AGgressive (AG) with two different values of λ, $\lambda = b/2$ and $\lambda = b/4$, First Order with PRior (PR), First Order ADaptive (AD), Second Order (SO) and Second Order AGgressive (SOAG). Recall, in the adaptive algorithm the constant b parameter is replaced with the constant β.

Fig. 5.5 shows the mean cumulative training errors over all tasks. The behavior is similar for all of the datasets. We can see that as expected, in the exploration interval ($b \to \inf$) all the algorithms shows high error rate, where in the vast major-

ity, this is also the highest error. In the exploitation interval ($b \rightarrow 0$), we see lower cumulative error than the exploration case for the most of the datasets. This makes sense, since we query (and update) more on the harder tasks. Between those two extreme cases, there is a point where cumulative error is lower. This point is where b gets its optimal value. This value can change between datasets and algorithms and next we see a convenient experimental way to find it. There are algorithms and datasets (e.g. VJ one-vs-one in Fig. 5.4(e)), that this point converges into the exploitation area, it may depends on the examples noise and the number of examples in the dataset. As we claimed before, a pure exploitation approach may not be the best choice, at least on the first online steps, especially for problem with small number of examples. can see the same phenomena in Fig. 5.4 which shows the mean test error for all tasks and all round for different b values and different tasks. Here, we added one more algorithm, the Watch All algorithm (WA), which runs K parallel online - perceptron algorithms, one per task, while each one of them query and watches all of the tasks labels, without any limitation on the feedback. We show this algorithm here as a benchmark, to measure how much SHAMPO algorithms looses or gain performance with respect to the algorithm that can watch all the labels.

Wen we compare all of the algorithms, there is no such absolute winner, however, it is obvious from results that the aggressive algorithms shows better performance than the rest of them. For most of the algorithms, the aggressive version of the second order (SOAG) seems to preform better then the rest. It is probably due to the fact that it draws tasks to query on, based on two measurements ($r_{i,t}$ and $\hat{p}_{i,t}$) and also make an aggressive update. The second best algorithm is the first order aggressive algorithm (AG) which performs the best on a limited b interval because the aggressiveness depends strongly on b. The first order algorithm with prior (PR) doesn't show an improvement over the ordinary first order (FO) algorithm. In fact, we already mentioned before than better prior may lead to an improvements in the results. The adaptive b algorithm (AD), does not show an improvement over FO as well.

Now, we focus on the tasks that the algorithm chooses to annotate on each iteration for various values of b. Fig. 5.3 shows the total number of mistakes that first order SHAMPO algorithm made during training time on all datasets. We show here two quantities: fraction of mistakes over all training examples (denoted by "All tasks" - blue) and fraction of mistakes over only queried examples (denoted by "Queried tasks" - dashed red). In pure exploration (large values of b) both

quantities are the same, as the choice of task to be labeled is independent of the task and example, and essentially the fraction of mistakes in queried examples is a good estimate of the fraction of mistakes over all examples. The other extreme is when performing pure exploitation (low values of b), here, the fraction of mistakes made on queried examples went up, while the overall fraction of mistakes went down. This indicates that the algorithm indeed focuses its queries on the harder inputs, which in turn, improves overall training mistake. There is a sweet point of b (that is changed between datasets), for which SHAMPO is still focusing on the harder examples, yet reduces the total fraction of training mistakes even more. The existence of such tradeoff is predicted by Theorem 3.1. So far we saw algorithms and bounds that depends on the value of b and proof that an optimal value that get a minimal accumulative train error exists. The inevitable question is: how to find such optimal value? We suggested an adaptive algorithm, yet, this algorithm have an initial value to set. Is there a practical way to find this value? Recall, the only prediction mistakes that the algorithm knows about, are the mistakes of the queried tasks. When we look at the accumulative mistakes that the algorithm made in Fig. 5.3, it is possible to find an answer for this question. The value of b which minimizes the mean test error, is about the point for which there is a change in the error of queried examples (the only quantity SHAMPO observes), which provides a rough rule-of-thumb to pick b automatically on-the-fly.

Another perspective of the phenomena is that for values of $b \ll \infty$ SHAMPO focuses on the harder examples, is illustrated in Fig. 5.6 where test error vs number of queries is plotted for each task in all the datasets. We show three cases: uniform, exploit and a mid-value of $b \approx 0.01$ which tradeoffs exploration and exploitation. We take the MNIST one-vs-one data set as an example and refer a few points. First performing uniform querying, all tasks have about the same number of queries (266), close to the number of examples per problem (12, 000), divided by the number of problems (45). Second, when having a tradeoff between exploration and exploitation, harder problems (as indicated by test error) get more queries than easier problems. For example, the four problems with test error greater than 6% get at least 400 queries, which is about twice the number of queries received by each of the 12 problems with test error less than 1%. Third, as a consequence, SHAMPO performs equalization, giving the harder problems more labeled data, and as a consequence, reduces the error of these problems, however, is not increasing the error of the easier problems which gets less queries (in fact it reduces the test error of almost all 45 problems!). The tradeoff mechanism of SHAMPO, reduces the test
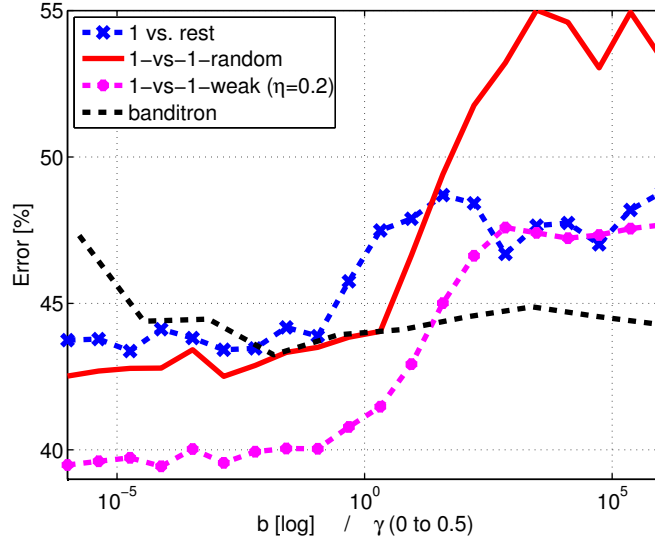
Figure 5.2: Multi-class error on VJ data for three bandit reductions and the banditron.

error of each problem by more than 40% compared to full exploration. Fourth, exploits performs similar equalization, yet in some hard tasks it performs worse than SHAMPO. This could be because it overfits the training data, by focusing on hard-examples too much, as SHAMPO has a randomness mechanism.

5.3 Reduction of Multi-task to Contextual Bandits

We also evaluated SHAMPO as a contextual bandit algorithm, by breaking a multi-class problem into few binary tasks, and integrating their output into a single multi-class problem. We focus on the VJ data, as there are many examples, and linear models perform relatively well Lin et al. [2009]. We implemented all three reductions mentioned in Chapter 4, namely, *one-vs-rest*, *one-vs-one-random* which picks a random label if the feedback is zero, *one-vs-one-weak* (which performs updates to increase confidence when the feedback is zero), where we set $\eta = 0.2$, and the Banditron algorithm Kakade et al. [2008]. All algorithms show the existence of a tradeoff between exploration and exploitation, where *one-vs-one-random* is most sensitive to the choice of parameters, yet it also achieves the best results, for

a large range of values for b. The *one-vs-rest* reduction and the Banditron have a test error of about 43.5%, and the *one-vs-one-random* of about 42.5%. Finally, *one-vs-one-weak* achieves an error of 39.4%. This is slightly worse than PLM Lin et al. [2009] with test error of 38.4% (and higher than MLP with 32.8%), yet all of these algorithms observe only one bit of feedback per example, while both MLP and PLM observe 3 bits (as class identity can be coded with 3 bits for 8 classes). We claim that our setting can be easily used to adapt a system to individual user, as we only need to assume the ability to recognize three words, such as three letters. Given an utterance of the user, the system may ask: "Did you say (a) 'a' like 'bad' (b) 'o' like in 'book') (c) none". The user can communicate the correct answer with no need for a another person to key in the answer.
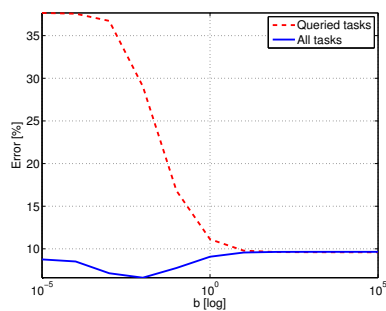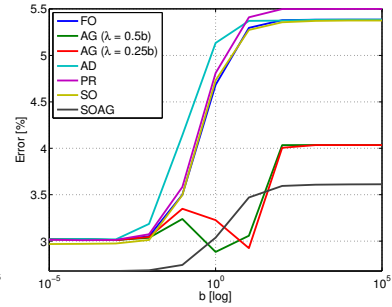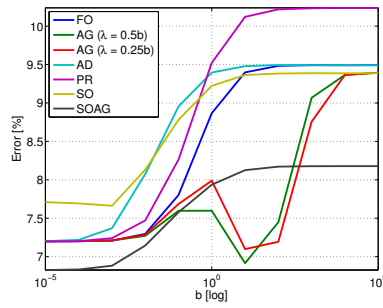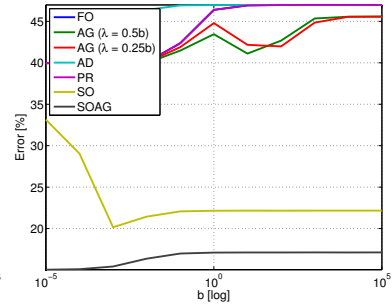
(a) MNIST one-vs-one
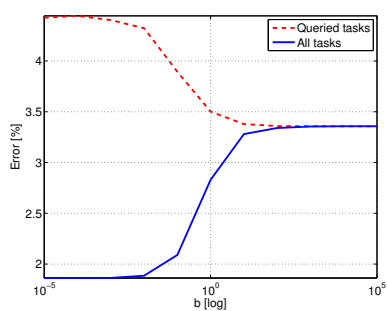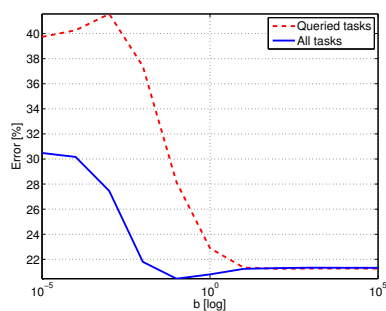
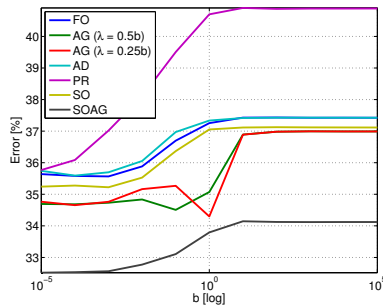(b) MNIST one-vs-rest
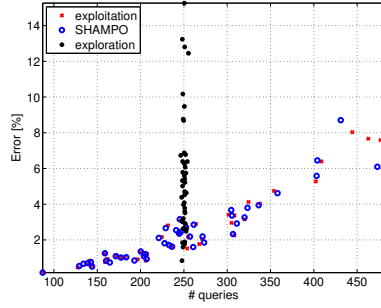
(c) USPS one-vs-one
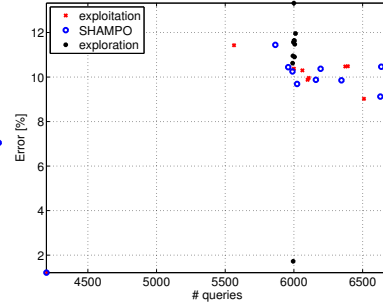
(d) USPS one-vs-rest

(e) VJ one-vs-one

(f) VJ one-vs-rest

(g) NLP one-vs-one

59
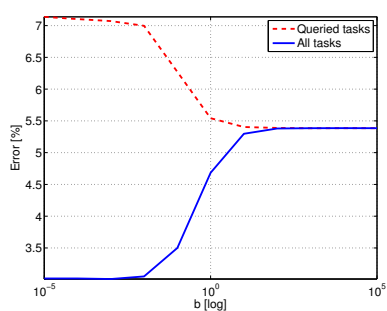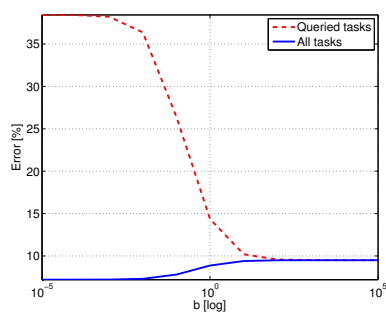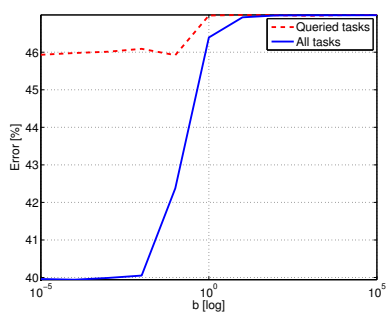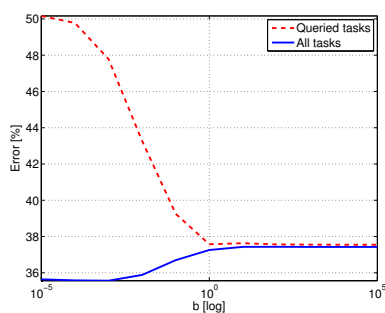
Figure 5.3: train error queried vs all -FO algorithm
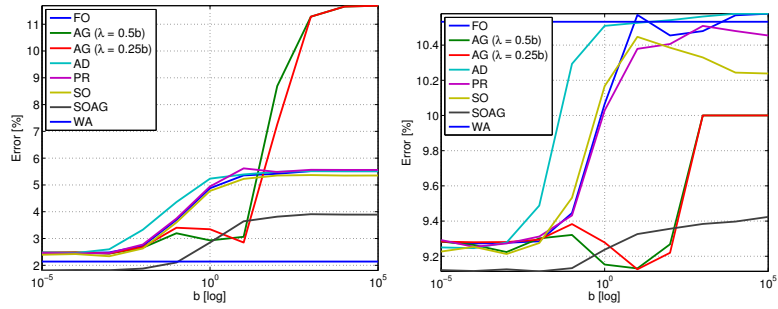
(a) MNIST one-vs-one

(b) MNIST one-vs-rest
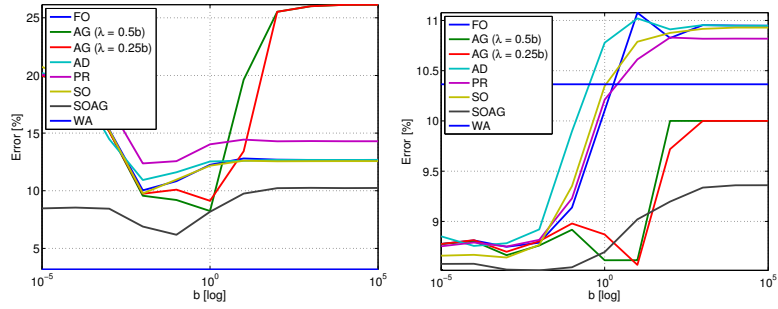
(c) USPS one-vs-one
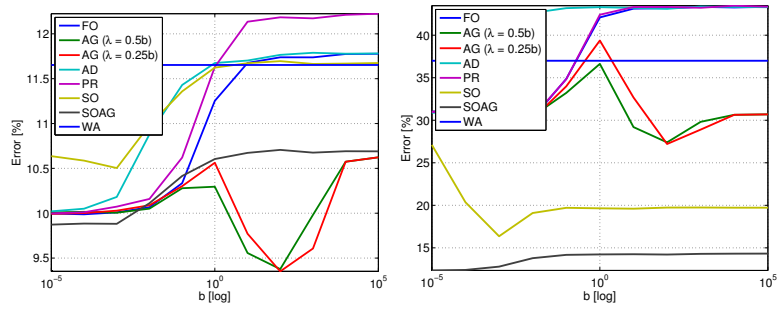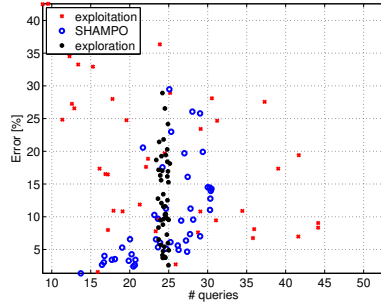
(d) USPS one-vs-rest

(e) VJ one-vs-one

(f) VJ one-vs-rest

(g) NLP ove-vs-one
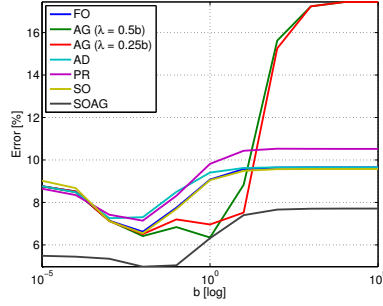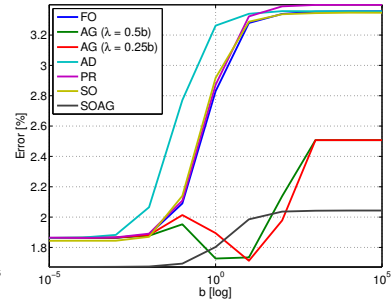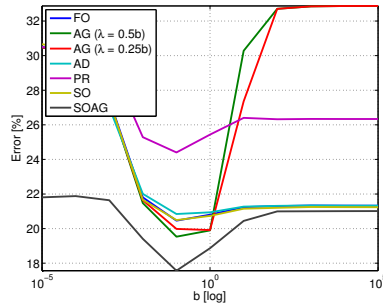
60

Figure 5.4: test error

(a) MNIST one-vs-one

(b) MNIST one-vs-rest

(c) USPS one-vs-one

(d) USPS one-vs-rest

(e) VJ one-vs-one

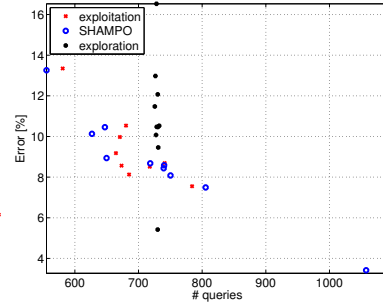(f) VJ one-vs-rest

(g) NLP one-vs-one

61

Figure 5.5: train
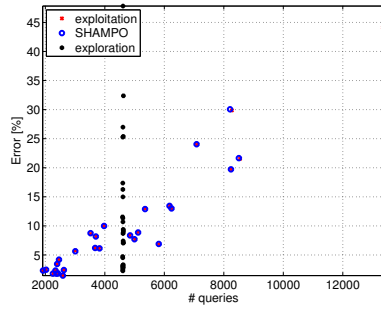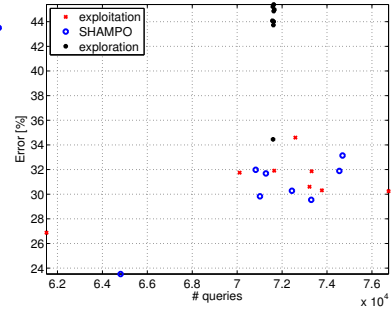
(a) MNIST one-vs-one

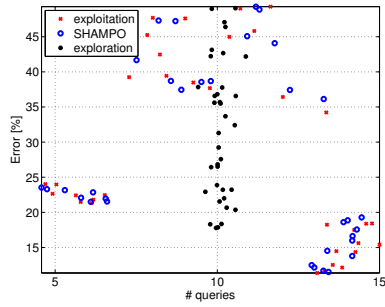(b) MNIST one-vs-rest

(c) USPS one-vs-one

(d) USPS one-vs-rest

(e) VJ one-vs-one

(f) VJ one-vs-rest

(g) NLP one-vs-one

62

Figure 5.6: scatter kjkulk jkhj

Chapter 6

Related Work

In the past few years there is a large volume of work on multi-task learning, which clearly we can not cover here. The reader is referred to a recent survey on the topic by Pan and Yang [2010]. Most of this work is focused on exploring relations between tasks, that is, find similarities and dissimilarities between tasks, and use it to share data directly ,e.g. Crammer and Mansour [2012], or model parameters as in Evgeniou and Pontil [2004], Daumé et al. [2010], Argyriou et al. [2008]. In the online settings, there are only a handful of work on multi-task learning. Dekel et al. [2006] consider the setting where all algorithms are evaluated using a global loss function, and all work towards the shared goal of minimizing it. Lugosi et al. [2009], assume that there are constraints on the predictions of all learners, and focus in the expert setting. Agarwal et al. [2008], formalize the problem in the framework of stochastic convex programming with few matrix regularization, each captures some assumption about the relation between the models. Cavallanti et al. [2010] and Cesa-Bianchi et al. [2006b], assume a known relation between tasks which is exploited during learning. Unlike these approaches, we assume the ability to share an annotator rather than data or parameters, thus our methods can be applied to problems with no common input space. One can expand our algorithm in the case of a related task, and update all the tasks of the model on each round, based on the queried task and not only the queried task.

In the other hand, there are also works that have been done on binary classification with partial feedback . Our analysis is derived from Cesa-Bianchi et al. [2006c], yet they focus in selective sampling (see also Cesa-Bianchi et al. [2009], Dekel et al. [2010], Crammer [2014]), that is, making individual binary decisions of whether to query, while our algorithm always query, and needs to decide for

which task to query on each round. Selective sampling algorithms such as first and second order selective sampling perceptron and BBQ of Cesa-Bianci et al Cesa-Bianchi et al. [2006c, 2009] deals with binary selective sampling algorithms. Dekel et al Dekel et al. [2010] have also a multiple teachers selective sampling algorithm, whit setting that is similar to SHAMPO with κ updates per round , but is force all of the inputs to be in the same space.

Dietterich and Bakiri [1995] introduced the use of ECOC matrix to reduce a multi-class classification problem into a multi-task problem we use this method, combines with SHAMPO algorithms to get multi-class classification. There have been recent work in contextual bandits each with slightly different assumptions. Algorithms like contextual bandits such the Banditron of Kakade et al. [2008] and Newtron of Hazan and Kale [2011] used the exploration exploitation method in order to reduce the regret, but in their setting, for each round the choice is also the prediction, so it suitable to our setting only for the case of One-vs-Rest. To the best of our knowledge, we are the first to consider decoupled exploration and exploitation in this context.

Finally, there is recent work in learning with relative or preference feedback in various settings as in Yue et al. [2009, 2012], Yue and Joachims [2011], Shivaswamy and Joachims [2011]. Unlike this work, our work allows again decoupled exploitation and exploration, and also non-relevant feedback.

Chapter 7

Summary and Conclusions

We proposed a new framework for online multi-task learning, where learners share a single annotator. We presented an algorithm (SHAMPO) that works in this settings and analyzed it in the mistake-bound model. We also suggested a few more variation of the SHAMPO algoritm: Aggressive version, Adaptive version, Prior version and Second Order version. Mistake bound analysis was done for all of the proposed algorithms. Then, we showed how learning in such model can be used to learn in contextual-bandits setting with few types of feedback. Empirical results show that our algorithms does better than only random uniform choice of feedback for the same price. It focuses the annotator on the harder instances, and is improving performance in various tasks and settings.

Further work can be done to improve those algorithms. We have shown that there exist b optimal value that may vary for different multitask problems. Now, after we know that such parameter exist, we can try to find a way to select the optimal value for every dataset. We can also investigate ways to generate good priors for the SHAMPO with prior algorithm or derive another distribution that select the task to be queried at every step, not only by the margin and the number of queried up to the same step, but also consider the count of the mistaken queried for every task. The algorithm prediction results may be improved when we assume similarity between tasks, such that an update can be done not only to the queried task but also to all of the tasks, based on the tasks relations.

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