PARTIAL PERMUTATION AND ALTERNATING SIGN MATRIX POLYTOPES

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Abstract. We define and study a new family of polytopes which are formed as convex hulls of partial alternating sign matrices. We use machinery developed in the study of sign matrix polytopes to determine the inequality descriptions, facet enumerations, and face lattices of these polytopes. We also study partial permutohedra that we show arise naturally as projections of these polytopes. We directly prove vertex and facet enumerations and also characterize the face lattices of partial permutohedra in terms of chains in the Boolean lattice.

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1. Introduction

Many examples of polytopes are either *simple* (every vertex is contained in the minimal number of facets), such as the n-cube, or *simplicial* (every proper face is a simplex), such as the tetrahedron. A quintessential example of a non-simple and non-simplicial polytope is the nth Birkhoff polytope (for $n \geq 3$), defined as the convex hull of $n \times n$ permutation matrices [4, 22]. Another such example is the nth alternating sign matrix polytope (for $n \geq 3$), defined as the convex hull of $n \times n$ alternating sign matrices [3, 21]. In this paper, we define and study a more general class of polytopes composed as convex hulls of $m \times n$ *partial alternating sign matrices*, denoted $PASM(m, n)$. We use machinery developed in this study of sign matrix polytopes [19] to determine the inequality descriptions of these polytopes, as well as facet enumerations and a description of their face lattices. Finally, we

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investigate the *partial permutohedron*, $\mathcal{P}(m,n)$, and show it is a projection of the polytopes from the first part of the paper.

Below we state our main results. Our first set of main results involves the partial alternating sign matrix polytope $PASM(m,n)$, while our second concerns the partial permutohedron $\mathcal{P}(m,n)$. For each set, we find an inequality description, enumerate the facets, and characterize the face lattice.

Theorem 4.6. *$PASM(m,n)$ consists of all $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$0 \leq \sum_{i'=1}^{i} X_{i'j} \leq 1, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n,$$

$$0 \leq \sum_{j'=1}^{j} X_{ij'} \leq 1, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Theorem 4.8. *The number of facets of $PASM(m,n)$ equals $4mn - 3m - 3n + 5$.*

Theorem 4.17. *Let F be a face of $PASM(m,n)$ and $\mathcal{M}(F)$ be equal to the set of partial alternating sign matrices that are vertices of F. The map $\psi : F \mapsto g(\mathcal{M}(F))$ induces an isomorphism between the face lattice of $PASM(m,n)$ and the set of sum-labelings of $\Gamma_{(m,n)}$ ordered by containment. Moreover, the dimension of F equals the number of regions of $\psi(F)$.*

The following three theorems from Section 5 comprise our second set of main results.

Theorem 5.9. *$\mathcal{P}(m,n)$ consists of all vectors $u \in \mathbb{R}^m$ such that:*

$$\sum_{i \in S} u_i \leq \binom{n+1}{2} - \binom{n-k+1}{2}, \qquad \text{where } S \subseteq [m], |S| = k \neq 0, \text{ and}$$

$$u_i \geq 0, \qquad \text{for all } 1 \leq i \leq m.$$

Theorem 5.10. *The number of facets of $\mathcal{P}(m,n)$ equals $m + 2^m - 1 - \sum_{r=1}^{m-n} \binom{m}{m-r}$.*

We use the result [13, Prop. 56] that $\mathcal{P}(m,m)$ is a graph associahedron called the *stellohedron* to prove an alternate characterization of its face lattice in terms of chains in the Boolean lattice.

Theorem 5.22. *The face lattice of $\mathcal{P}(m,m)$ is isomorphic to the lattice of chains in $\mathcal{B}_m$, where $C < C'$ if C' can be obtained from C by iterations of (1) and/or (2) from Lemma 5.20. A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

We conjecture a similar face lattice characterization for $\mathcal{P}(m,n)$ in the case $m \neq n$.

Conjecture 5.23. *Faces of $\mathcal{P}(m,n)$ are in bijection with chains in $\mathcal{B}_m$ whose difference between largest and smallest nonempty subsets is at most $n-1$. A face of $\mathcal{P}(m,n)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

We furthermore connect these polytopes by showing in that $PASM(m,n)$ projects to $\mathcal{P}(n,m)$, by a similar technique used to show that alternating sign matrix polytopes project to permutohedra [21]. Our last main result is as follows; here ϕ_z is the map that multiplies a matrix by z on the left and $\mathcal{P}_z$ is a generalized partial permutohedron determined by z.

Theorem 5.26. *Let z be a strictly decreasing vector in $\mathbb{R}^m$. Then $\phi_z(PASM(m,n)) = \mathcal{P}_z(n,m)$.*

Finally, we have computed the volume and Ehrhart polynomials of the polytopes studied in this paper. We note that the Ehrhart polynomials we were able to compute have positive coefficients, and have found the following result and conjecture regarding the volume of the partial permutohedron.

Theorem 5.27. *$\mathcal{P}(2, n)$ has normalized volume equal to $2n^2 - 1$.*

Conjecture 5.28. *$\mathcal{P}(m, 2)$ has normalized volume equal to $3^m - m$.*

Our outline is as follows. In Section 2, we introduce definitions and notation for the families of matrices used to create our polytopes. In Section 3, we summarize known results on partial permutation polytopes. In Section 4, we define partial alternating sign matrix polytopes and determine their inequality descriptions, facet enumerations, and face lattice description. In Section 5, we define partial permutohedra. We then determine inequality descriptions and facet enumerations and characterize its face lattice using chains in the Boolean lattice. We show that the polytopes from Sections 3 and 4 project to these partial permutohedra. Finally, at the end of each of Sections 3, 4, and 5, we discuss volumes.

2. Matrices

In this section, we discuss matrices which generalize permutation matrices and alternating sign matrices, then in the next section, we study their corresponding polytopes.

2.1. Partial permutation matrices.

We begin with the following definition.

Definition 2.1. An $m \times n$ *partial permutation matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{0, 1\}$ such that:

$$\sum_{i'=1}^{m} M_{i'j} \in \{0,1\}, \qquad \text{for all } 1 \le j \le n. \tag{2.1}$$

$$\sum_{j'=1}^{n} M_{ij'} \in \{0,1\}, \qquad \text{for all } 1 \le i \le m. \tag{2.2}$$

We denote the set of all $m \times n$ partial permutation matrices $P_{m,n}$.

Remark 2.2. Partial permutations matrices are sometimes called *subpermutation matrices*, see, for example, [5]. We choose the terminology partial permutation since we do consider our matrices as objects in their own right, rather than as submatrices of larger (square) permutation matrices. The use of the term partial permutation is consistent with literature on square partial permutation matrices, such as [6]. Rectangular partial permutation matrices are mentioned in [16].

We now enumerate partial permutation matrices, using standard counting arguments.

Proposition 2.3. *Let $m \le n$. Then $P_{m,n}$ and $P_{n,m}$ are each enumerated by*

$$\sum_{k=0}^{m} \binom{m}{k} (n)_k, \tag{2.3}$$

where $(n)_k$ denotes the falling factorial $(n)_k := n(n-1)(n-2)\cdots(n-k+1)$.

Proof. For any partial permutation matrix in $P_{m,n}$, there can be at most one 1 in each row and column, and all other entries must be 0. We begin by choosing which k rows will have a 1. There are $\binom{m}{k}$ ways to do this. Label these rows $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$. Now consider which columns will contain these ones. There are n possible columns for the 1 in row r_{i_1}. This leaves $n-1$ possible columns for the 1 in row r_{n_2}. Continuing in this manner, there will be $n-t+1$ columns available for the 1 in row r_{i_t} for $1 \le t \le k$. So there are $(n)_k$ possibilities of columns for each of the $\binom{m}{k}$ choices of rows. Since $m \le n$, there can be at most m ones in the matrix, so summing k over these values gives us the total number of matrices in $P_{m,n}$. $P_{n,m}$ has the same cardinality as $P_{m,n}$, since there is a bijection given by transposing the matrix, so the result follows. $\square$

Example 2.4. The 13 elements of $P_{2,3}$ are:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad M_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad M_{10} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2.2. **Partial alternating sign matrices.** In this subsection, we consider a superset of partial permutation matrices.

Definition 2.5. An $m \times n$ *partial alternating sign matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{-1, 0, 1\}$ such that:

$$\sum_{i'=1}^{i} M_{i'j} \in \{0, 1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.4}$$

$$\sum_{j'=1}^{j} M_{ij'} \in \{0, 1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.5}$$

We denote the set of all $m \times n$ partial alternating sign matrices as $PASM_{m,n}$.

Remark 2.6. The set of matrices in $PASM_{m,n}$ with no -1 entries is the set $P_{m,n}$ of partial permutation matrices.

Example 2.7. $PASM_{2,3}$ consists of the 13 matrices from Example 2.4 plus the following four additional matrices:

$$M_{14} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \qquad M_{15} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad M_{16} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad M_{17} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Remark 2.8. Partial alternating sign matrices are a subset of *sign matrices*, which differ from Definition 2.5 in that each row partial sum is not restricted to $\{0, 1\}$ as in (2.5), but may equal any non-negative integer. See [19] for information about polytopes whose vertices are sign matrices and Lemma 4.9 for the relationship between these polytopes.

The cardinality of $PASM_{n,n}$ is given by OEIS sequence A202751 [1]. It is unlikely that there exists a product formula for $|PASM_{m,n}|$, since, for example, $|PASM_{6,6}| = 1442764 = 2^2 \cdot 373 \cdot 967$.

Remark 2.9. $n \times n$ partial alternating sign matrices were studied in a different context by Fortin [10]. He showed that, with a certain poset structure, the lattice of partial alternating sign matrices is the MacNeille completion of the poset of partial permutations (which he called partial injective functions). This is analogous to the result of Lascoux [12] that the lattice of $n \times n$ alternating sign matrices is the MacNeille completion of the strong Bruhat order on S_n.

3. Partial permutation polytopes

In this section, we give the definition of partial permutation polytopes, review their inequality descriptions, and give the enumeration of their vertices and facets. We also remark on its volume and conjecture a formula for $m = 2$.

Definition 3.1. Let $PPerm(m, n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^{mn}$, of all the matrices in $P_{m,n}$. Call this the *(m,n)-partial permutation polytope.*

Remark 3.2. The dimension of $PPerm(m, n)$ is mn. To see this, let $U_{i,j}$ be the $m \times n$ matrix with (i, j) entry equal to 1 and zeros elsewhere. Note that $U_{i,j} \in PPerm(m, n)$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. Since $PPerm(m, n)$ contains each of these mn unit vectors, its dimension equals the ambient dimension mn.

The set of $n \times n$ doubly substochastic matrices is the convex hull of all $n \times n$ subpermutation matrices [15]. This result is easily extendable to the $m \times n$ case, which we state below.

Theorem 3.3. *$PPerm(m, n)$ consists of all $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$X_{ij} \geq 0, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n, \tag{3.1}$$

$$\sum_{j'=1}^{n} X_{ij'} \leq 1, \qquad \text{for all } 1 \leq i \leq m, \tag{3.2}$$

$$\sum_{i'=1}^{n} X_{i'j} \leq 1, \qquad \text{for all } 1 \leq j \leq n. \tag{3.3}$$

We can also easily count the vertices and facets.

Proposition 3.4. *The vertices of $PPerm(m, n)$ are exactly the matrices in $P_{m,n}$, so $PPerm(m, n)$ has $\sum_{k=0}^{m} \binom{m}{k} (n)_k$ vertices.*

Proof. It is easily seen that each matrix in $P_{m,n}$ is an extreme point of $PPerm(m, n)$. Then Proposition 2.3 gives the enumeration. □

Proposition 3.5. *The number of facets of $PPerm(m, n)$ equals $mn + m + n$.*

Proof. The number of facets of a polytope is given by the number of linear independent inequalities of its inequality description. (3.1) gives mn inequalities, while (3.2) gives m and (3.3) gives n more inequalities. These inequalities are linearly independent, so the total number of linearly independent inequalities (and thus facets) is $mn + m + n$. □

Remark 3.6. Note that any partial permutation matrix can be reinterpreted as an incidence vector of some matching. Thus $PPerm(m, n)$ is a *matching polytope*, a family of polytopes which have previously been studied. In [2, 8], adjacency conditions of vertices were studied. For a nice summary and proof of these results, see [18, Chapter 25].

Remark 3.7. The normalized volume of $PPerm(m, n)$ for small values of m and n is given in Figure 1, computed using SageMath [20]. Due to the large size of the polytopes, further computations are not easily obtained. Note that there does not appear to be a nice formula for these volumes.

Conjecture 3.8. *The normalized volume of $PPerm(n, 2)$ (or equivalently $PPerm(2, n)$) is equal to $\binom{2n}{n} - n$.*

We have confirmed this conjecture for $m \leq 14$ using SageMath.

Remark 3.9. We have used SageMath to compute the Ehrhart polynomials for $PPerm(m, n)$ for $m, n \leq 5$ and note that in all of these cases their coefficients are positive.

n \ m	1	2	3	4	5
1	1	1	1	1	1
2	1	4	17	66	247
3	1	17	642	22148	622791
4	1	66	22148	12065248	5089403019
5	1	247	622791	5089403019	53480547965190

FIGURE 1. The normalized volume of $PPerm(m,n)$ for small values of m and n

4. Partial alternating sign matrix polytopes

In this section, we define partial alternating sign matrix polytopes. We give an inequality description and facet enumeration in Subsection 4.1. In Subsection 4.2, we determine the face lattice. We also compute the volume for small values of m and n in Subsection 4.3.

4.1. **Vertices, facets, inequality description.**

In this subsection, we give the definition of partial alternating sign matrix polytopes. In Proposition 4.3, we determine the vertices. We prove an inequality description in Theorem 4.6. Then in Theorem 4.8, we enumerate the facets.

Definition 4.1. Let $PASM(m,n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^{mn}$, of all the matrices in $PASM_{m,n}$. Call this the (m,n)-*partial alternating sign matrix polytope.*

Remark 4.2. $PASM(m,n)$ contains $PPerm(m,n)$, since, as noted in Remark 2.6, the set of partial alternating sign matrices $PASM_{m,n}$ contains all the partial permutation matrices $P_{n,m}$. So the dimension of $PASM(m,n)$ is mn, since by Remark 3.2, this is the dimension of $PPerm(m,n)$.

Proposition 4.3. *The vertices of $PASM(m,n)$ are exactly the matrices in $PASM_{m,n}$.*

Proof. In [19, Theorem 4.3], a hyperplane is constructed that separates a given $m \times n$ sign matrix from all other $m \times n$ sign matrices. Since $m \times n$ partial alternating sign matrices are a subset of $m \times n$ sign matrices, this hyperplane must separate a given $m \times n$ partial alternating sign matrix from all others. The hyperplane construction is as follows. Let M be an $m \times n$ partial alternating sign matrix and $C_M = \{(i,j) \ : \ \sum_{i'=1}^{i} M_{i'j} = 1\}$. Then the hyperplane in $\mathbb{R}^{mn}$ that separates M from the other elements of $PASM_{m,n}$ is $\sum_{(i,j)\in C_M} \sum_{i'=1}^{i} X_{i'j} - \sum_{(i,j)\notin C_M} \sum_{i'=1}^{i} X_{i'j} = |C_M| - \frac{1}{2}$. Thus the vertices of $PASM(m,n)$ are the $m \times n$ partial alternating sign matrices. □

We now give the following definitions from [19], which we will use in the proof of Theorem 4.6.

Definition 4.4 ([19, Definition 3.3])**.** We define the $m \times n$ *grid graph* $\Gamma_{(m,n)}$ as follows. The vertex set is $V(m,n) := \{(i,j) : 1 \le i \le m+1, 1 \le j \le n+1\}$. We separate the vertices into two categories. We say the *internal vertices* are $\{(i,j) : 1 \le i \le m, 1 \le j \le n\}$ and the *boundary vertices* are $\{(m+1,j)$ and $(i,n+1) : 1 \le i \le m, 1 \le j \le n\}$. The edge set is:

$$E(m,n) := \begin{cases} (i,j) \text{ to } (i+1,j) & 1 \le i \le m, 1 \le j \le n \\ (i,j) \text{ to } (i,j+1) & 1 \le i \le m, 1 \le j \le n. \end{cases}$$

Edges between internal vertices are called *internal edges* and any edge between an internal and boundary vertex is called a *boundary edge*. We draw the graph with i increasing to the right and j increasing down, to correspond with matrix indexing.

Definition 4.5 ([19, Definition 3.4])**.** Given an $m \times n$ matrix X, we define a labeled graph, $\hat{X}$, which is a labeling of the vertices and edges of $\Gamma_{(m,n)}$ from Definition 4.4. The internal vertices (i,j), $1 \le i \le m$, $1 \le j \le n$ are each labeled with the corresponding entry of X: $\hat{X}_{ij} = X_{ij}$.

The horizontal edges from (i, j) to $(i, j+1)$ are each labeled by the corresponding row partial sum $r_{ij} = \sum_{j'=1}^{j} X_{ij'}$ $(1 \le i \le m, 1 \le j \le n)$. Likewise, the vertical edges from (i, j) to $(i+1, j)$ are each labeled by the corresponding column partial sum $c_{ij} = \sum_{i'=1}^{i} X_{i'j}$ $(1 \le i \le m, 1 \le j \le n)$.

The following theorem gives an inequality description of $PASM(m, n)$. The proof uses a combination of ideas from [19, 21].

Theorem 4.6. *$PASM(m, n)$ consists of all $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$0 \le \sum_{i'=1}^{i} X_{i'j} \le 1, \qquad \textit{for all } 1 \le i \le m, 1 \le j \le n, \tag{4.1}$$

$$0 \le \sum_{j'=1}^{j} X_{ij'} \le 1, \qquad \textit{for all } 1 \le i \le m, 1 \le j \le n. \tag{4.2}$$

Proof. First we need to show that any $X \in PASM(m, n)$ satisfies (4.1) - (4.2). Then $X = \sum_\gamma \mu_\gamma M_\gamma$ where $\sum_\gamma \mu_\gamma = 1$ and the $M_\gamma \in PASM(m, n)$. Since we have a convex combination of partial alternating sign matrices, by Definition 2.5, we obtain (4.1) - (4.2) immediately. Thus $PASM(m, n)$ fits the inequality description.

Let X be a real-valued $m \times n$ matrix satisfying (4.1) - (4.2). We wish to show that X can be written as a convex combination of partial alternating sign matrices in $PASM_{m,n}$, so that X is in $PASM(m, n)$.

Consider the corresponding labeled graph $\hat{X}$ of Definition 4.5. We will construct a trail in $\hat{X}$ all of whose edges are labeled by inner numbers and show it is a simple path or cycle. (A number α is *inner* if $0 < \alpha < 1$.) Let $r_{i0} = 0 = c_{0j}$ for all i, j. Then for all $1 \le i \le \lambda_1, 1 \le j \le n$, we have $\hat{X}_{ij} = r_{ij} - r_{i,j-1} = c_{ij} - c_{i-1,j}$. Thus,

$$r_{ij} + c_{i-1,j} = c_{ij} + r_{i,j-1}. \tag{4.3}$$

Note that if there are no inner edge labels, then X is already a partial alternating sign matrix. If there exists i or j such that $\hat{X}_{i,n+1}$ or $\hat{X}_{m+1,j}$ is inner, begin constructing the trail at the adjacent boundary edge. If no such i or j exist, start the trail on any vertex, say $\hat{X}_{ij}$ adjacent to an edge with inner label. By (4.3), at least one of $c_{(i\pm1,j)}, r_{(i,j\pm1)}$ is also inner, so we may begin forming a trail by moving through edges with inner labels. From the starting point, construct the trail as follows. Go along a row or column from the starting point along edges with inner labels. Continue in this manner until either (1) you reach a vertex adjacent to an edge that was previously in the trail, or (2) you reach a new boundary edge. If (1), then the part of the trail constructed between the first and second time you reached that vertex will be a simple cycle. That is, we cut off any part that was constructed before the first time that vertex was reached. If (2), then the starting point for the trail must have been a boundary edge, since there is at least one boundary vertex with inner label. Thus our trail is actually a path.

Label the corners of the path or cycle (not the boundary vertices) alternately $(+)$ and $(-)$. Set ℓ^+ equal to the largest number that we could subtract from the $(-)$ entries and add to the $(+)$ entries while still satisfying (4.1) - (4.2). Construct a matrix X^+ by subtracting and adding in this way. X^+ is a matrix which stills satisfies (4.1) - (4.2) and which has least one more non-inner edge label than X.

Now give opposite labels to the corners and set ℓ^- equal to the largest number we could subtract from $(-)$ entries and add to $(+)$ entries while still satisfying (4.1) - (4.2). Add and subtract in a similar way to create X^-, another matrix satisfying (4.1) - (4.2) and which has at least one more non-inner edge label than X.

Both X^+ and X^- satisfy (4.1) - (4.2) by construction. Also by construction,

$$X = \frac{\ell^-}{\ell^+ + \ell^-} X^+ + \frac{\ell^+}{\ell^+ + \ell^-} X^-$$

and $\frac{\ell^-}{\ell^+ + \ell^-} + \frac{\ell^+}{\ell^+ + \ell^-} = 1$. So X is a convex combination of the two matrices X^+ and X^- that still satisfy the inequalities and are each at least one step closer to being partial alternating sign matrices, since that have at least one more partial sum attaining its maximum or minimum bound. By repeatedly applying this procedure, X can be written as a convex combination of partial alternating sign matrices.

See Figure 2 and Example 4.7 for an example of this construction. $\square$

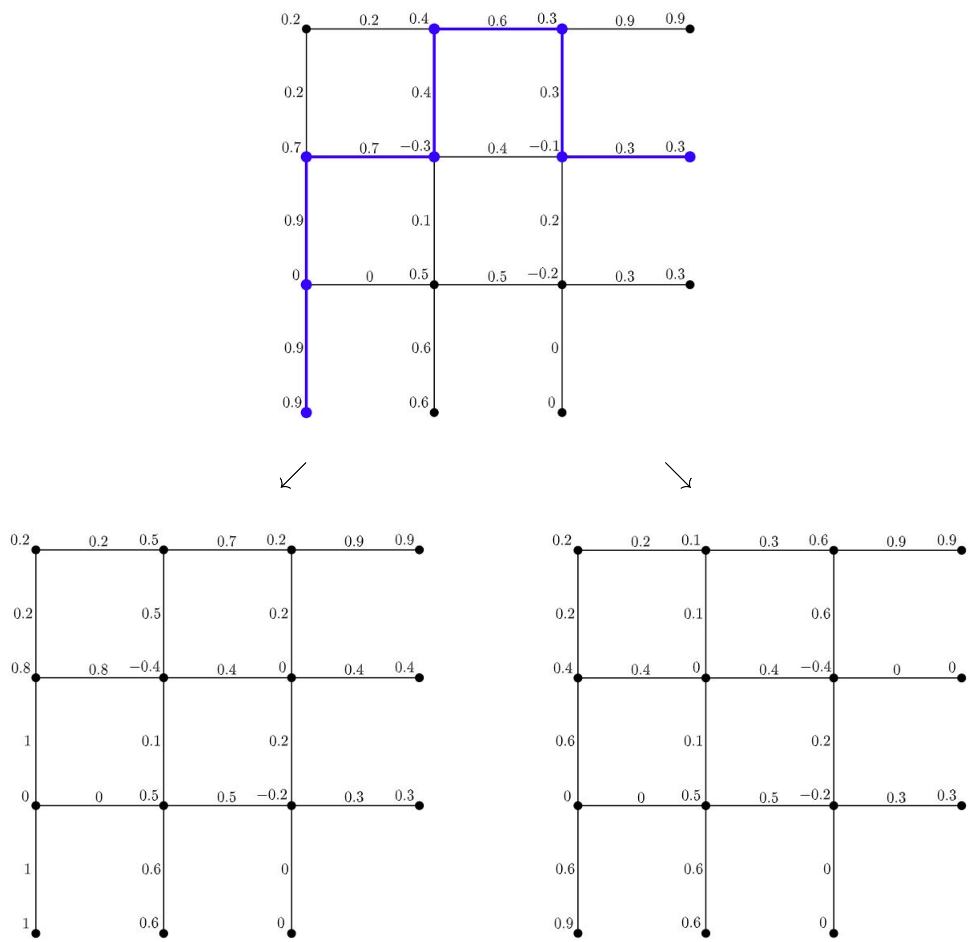

Figure 2. An example of the path construction described in the proof of Theorem 4.6. The bold blue edges and vertices are those included in the path.

Example 4.7. Let $X = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.7 & -0.3 & -0.1 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. Then by the construction described in the proof of Theorem 4.6 and shown in Figure 2, X can be decomposed as $X = \frac{0.3}{0.1+0.3} X^+ + \frac{0.1}{0.1+0.3} X^-$, where

$X^+ = \begin{pmatrix} 0.2 & 0.5 & 0.2 \\ 0.8 & -0.4 & 0 \\ 9 & 0.5 & -0.2 \end{pmatrix}$ and $X^- = \begin{pmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0 & -0.4 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. In this step of decomposing, $\ell^+ = 0.1$ and $\ell^- = 0.3$. Continuing the process of decomposition, one could write X as a convex combination of partial alternating sign matrices.

Theorem 4.6 gives a simple inequality description, but it is not a *minimal* inequality description. That is, some of the inequalities in (4.1) and (4.2) are redundant. In the following theorem, we determine these redundancies to count the inequalities that determine facets.

Theorem 4.8. *The number of facets of PASM(m,n) equals $4mn - 3m - 3n + 5$.*

Proof. There are $4mn$ total inequalities given in (4.1) and (4.2). We show how $3m+3n-5$ of these are redundant.

First, we have that $0 \leq X_{1j}$ from the column partial sums, so $0 \leq \sum_{j'=1}^{j} X_{1j'}$ for $1 \leq j \leq n$ are all unnecessary. This is n total redundant inequalities.

From the column partial sums we already have that $0 \leq X_{11}$ and from row partial sums we have $0 \leq X_{21}$. Together these imply $0 \leq X_{11} + X_{21}$. Similarly, all of the partial column sums $0 \leq \sum_{i'=1}^{i} X_{i'1}$ for $2 \leq i \leq m$ are implied by the partial row sums $0 \leq X_{i'1}$. This gives another $m-1$ redundancies.

Now note that $\sum_{i'=1}^{m} X_{i'1} \leq 1$, and that $0 \leq X_{m1}$. This implies that $\sum_{i'=1}^{m-1} X_{i'1} \leq 1 - X_{m1} \leq 1$. Similarly, all of the $m-1$ inequalities of the form $\sum_{i'=1}^{i} X_{i'1} \leq 1$ for $1 \leq i < m$ are all implied by the partial row sums $0 \leq X_{i'1}$. This gives us another $m-1$ redundant inequalities. By a similar argument, we will also have that the $n-1$ inequalities of the form $\sum_{j'=1}^{j} X_{1j'} \leq 1$ for $1 \leq j < n$ are implied by the partial column sums $0 \leq X_{1j'}$. This give us another $n-1$ redundant inequalities.

Finally, note that $\sum_{i'=1}^{i-1} X_{i'1} \geq 0$ and $\sum_{i'=1}^{i} X_{i'1} \leq 1$. This implies that $X_{i1} \leq 1 - \sum_{i'=1}^{i-1} X_{i'1} \leq 1$ for $2 \leq i \leq m$. This gives us another $m-1$ redundancies. By a similar argument we also have that $\sum_{j'=1}^{j-1} X_{1j'} \geq 0$ and $\sum_{j'=1}^{j} X_{1j'} \leq 1$ combine to imply that $X_{1j} \leq 1$ for $2 \leq j \leq n$. This gives $n-1$ additional redundancies.

Overall, this means that at most $4mn - 3m - 3n + 5$ of the original $4mn$ inequalities are necessary. The linearly independent inequalities that remain are as follows:

$$\sum_{j'=1}^{j} X_{ij} \geq 0, \qquad \text{for all } 2 \leq i \leq m \text{ and } 1 \leq j \leq n \tag{4.4}$$

$$\sum_{j'=1}^{j} X_{ij} \leq 1, \qquad \text{for all } 2 \leq i \leq m \text{ and } 2 \leq j \leq n \tag{4.5}$$

$$\sum_{i'=1}^{i} X_{ij} \geq 0, \qquad \text{for all } 1 \leq i \leq m \text{ and } 2 \leq j \leq n \tag{4.6}$$

$$\sum_{i'=1}^{i} X_{ij} \leq 1, \qquad \text{for all } 2 \leq i \leq m \text{ and } 2 \leq j \leq n \tag{4.7}$$

$$\sum_{i=1}^{m} X_{i1} \leq 1 \tag{4.8}$$

$$\sum_{j=1}^{n} X_{1j} \leq 1 \tag{4.9}$$

$$X_{11} \geq 0 \tag{4.10}$$

These $4mn-3m-3n+5$ necessary inequalities determine the number of facets of $PASM(m,n)$. $\square$

4.2. **Face lattice.** In this subsection, we characterize the face lattice of $PASM(m,n)$ in Theorem 4.17, using *sum-labelings* of the graph $\Gamma(m,n)$ (see Definition 4.4).

Recall from Remark 2.8 that partial alternating sign matrices are a subset of sign matrices [19]. It was shown in [19, Theorem 5.3] that the convex hull of $m \times n$ sign matrices, denoted $P(m,n)$ has inequality description as in Theorem 4.6, except in (4.2) the ≤ 1 is not present. More specifically, we have the following relation.

Lemma 4.9. *$PASM(m,n)$ is the intersection of $P(m,n)$ with the subspace of $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$\sum_{i'=1}^{i} X_{i'j} \leq 1, \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{4.11}$$

We now state some definitions and a lemma that will help prove Theorem 4.17 describing the face lattice of $PASM(m,n)$. This theorem is analogous to [19, Theorems 7.15 and 7.16] which describe the face lattice of $P(m,n)$. The proof is also similar.

Recall $\hat{M}$ from Definition 4.5.

Definition 4.10. A *basic sum-labeling* of $\Gamma_{(m,n)}$ is a labeling of the edges of $\Gamma_{(m,n)}$ with 0 or 1 such that the edge labels equal the corresponding edge labels of $\hat{M}$ for some $M \in PASM_{m,n}$.

Remark 4.11. Recall we can recover any matrix from its column partial sums, thus basic sum-labelings of $\Gamma_{(m,n)}$ are in bijection with partial alternating sign matrices $PASM_{m,n}$.

Definition 4.12. Let δ and δ' be labelings of the edges of $\Gamma_{(m,n)}$ with 0, 1, or $\{0,1\}$. Define the *union* $\delta \cup \delta'$ as the labeling of $\Gamma_{(m,n)}$ such that each edge is labeled by the union of the corresponding labels on δ and δ'. Define *intersection* $\delta \cap \delta'$ and *containment* $\delta \subseteq \delta'$ similarly.

Definition 4.13. A *sum-labeling* δ of $\Gamma_{(m,n)}$ is either the empty labeling of $\Gamma_{(m,n)}$ (denoted $\emptyset$) or a labeling of the edges of $\Gamma_{(m,n)}$ with 0, 1, or $\{0,1\}$ such that there exists a set S of basic sum-labelings of $\Gamma_{(m,n)}$ such that $\delta = \bigcup_{\delta' \in S} \delta'$.

Definition 4.14. Given $M \in PASM_{m,n}$, let $g(M)$ denote the sum-labeling of $\Gamma_{(m,n)}$ associated to M. Given a collection of partial alternating sign matrices $\mathcal{M} = \{M_1, M_2, \ldots, M_r\} \subseteq PASM_{m,n}$, define the map $g(\mathcal{M}) = \displaystyle\bigcup_{i=1}^{r} g(M_i)$.

Definition 4.15. Given a sum-labeling δ, consider the planar graph G composed of the edges of δ labeled by the two-element set $\{0,1\}$ (and all incident vertices), where we regard any external edges on the right and bottom as meeting at a point in the exterior. We say a *region* of δ is defined as a planar region of G, excluding the exterior region. Let $\mathcal{R}(\delta)$ denote the number of regions of δ. (For consistency we set $\mathcal{R}(\emptyset) = -1$.)

See Figure 3 for an example of a sum-labeling of $\Gamma_{(2,3)}$ with 4 regions.

Lemma 4.16. *Suppose a sum-labeling δ has $\mathcal{R}(\delta) = \omega$. If $\delta \subset \delta'$ then $\mathcal{R}(\delta') > \omega$.*

Proof. By convention, the empty labeling has $\mathcal{R}(\emptyset) = -1$. If δ is a basic sum-labeling, $\mathcal{R}(\delta) = 0$, as there are no edges labeled $\{0,1\}$ in a basic sum-labeling. Suppose a sum-labeling δ has $\mathcal{R}(\delta) = \omega > 0$. We wish to show if $\delta \subset \delta'$ then $\mathcal{R}(\delta') > \omega$. $\delta \subset \delta'$ implies that the labels of each edge of δ are subsets of the labels of each edge of δ', where at least one of these containments is strict. So there is an edge in δ' labeled $\{0,1\}$ that was labeled 0 or 1 in δ. So δ' contains a basic sum

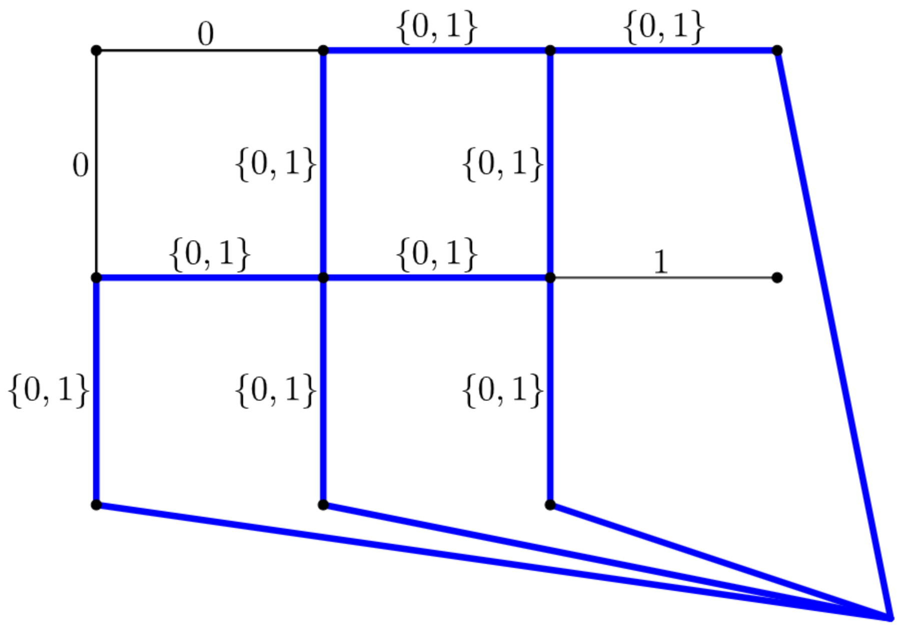

FIGURE 3. The sum-labeling of $\Gamma_{(2,3)}$ which is $g(M_3) \cup g(M_{13}) \cup g(M_{15})$, where M_3, M_{13}, and M_{15} are as in Examples 2.4 and 2.7. Edges labeled $\{0,1\}$ are bolded and blue to accentuate the regions.

labeling β' that differs from all the basic sum labelings in δ at edge e. Let β denote a basic sum labeling such that $\beta \subseteq \delta$. By Equation (4.3), at least one edge label of β' adjacent to e must also differ from the corresponding edge label of β. By iterating this (as in the proof of Theorem 4.6), β' differs from β by at least one simple path (connecting boundary vertices) or cycle of differing partial sums. This path or cycle appears as edges labeled by $\{0,1\}$ in δ', and at least one of these edges was not labeled by $\{0,1\}$ in δ. So δ' has at least one new region. Therefore, $\mathcal{R}(\delta') > \omega$. ☐

We are now ready to state and prove the main theorem of this subsection.

Theorem 4.17. *Let F be a face of $PASM(m,n)$ and $\mathcal{M}(F)$ be equal to the set of partial alternating sign matrices that are vertices of F. The map $\psi : F \mapsto g(\mathcal{M}(F))$ induces an isomorphism between the face lattice of $PASM(m,n)$ and the set of sum-labelings of $\Gamma_{(m,n)}$ ordered by containment. Moreover, dim $F = \mathcal{R}(\psi(F))$.*

Proof. Let F be a face of $PASM(m,n)$. Then $g(\mathcal{M}(F))$ is a sum-labeling of $\Gamma_{(m,n)}$ since $g(\mathcal{M}(F)) = \bigcup_{i=1}^{r} g(M_i)$ is a union of basic sum-labelings. We now construct the inverse of ψ, call it φ. Given a sum-labeling ν of $\Gamma_{(m,n)}$, let $\varphi(\nu)$ be the face that results as the intersection of the facets corresponding to the edges of ν with label 0 or 1.

We wish to show $\psi(\varphi(\nu)) = \nu$. First, we show $\nu \subseteq \psi(\varphi(\nu))$. Let $M \in PASM(m,n)$ such that $g(M) \subset \nu$ is a basic sum-labeling. M is in the intersection of the facets that yields $\varphi(\nu)$, since otherwise $g(M)$ would not be a basic sum-labeling such that $g(M) \subset \nu$. Thus $g(M) \subseteq \psi(\varphi(\nu))$ as well. So $\nu \subseteq \psi(\varphi(\nu))$.

Next, we show $\psi(\varphi(\nu)) \subseteq \nu$. Suppose not. Then there exists some edge e of $\Gamma_{(m,n)}$ whose label in $\psi(\varphi(\nu))$ strictly contains the label of e in ν. The label of e in ν is 0 or 1 and the label of e in $\psi(\varphi(\nu))$ is $\{0,1\}$. Let γ denote the label of e in ν. As in the previous case, the facet corresponding to the label γ on e would have been one of the facets intersected to get $\varphi(\nu)$. Therefore the matrix partial column sum corresponding to edge e would be fixed as γ in each partial alternating sign matrix in $\varphi(\nu)$. So in the union $\psi(\varphi(\nu))$, that edge label would be the union of the edge labels of all the partial alternating sign matrices in $\varphi(\nu)$, and this union would be γ. This is a contradiction. Thus $\nu = \psi(\varphi(\nu))$.

Let F_1 and F_2 be faces of $PASM(m,n)$ such that $F_1 \subset F_2$. Then F_1 is an intersection of F_2 and some facet hyperplanes. In other words, F_1 is obtained from F_2 by setting at least one of the inequalities in Theorem 4.6 to an equality. We have that $\psi(F_1)$ is obtained from $\psi(F_2)$ by changing at least edge label of $\{0,1\}$ to a label of 0 or 1. Therefore we have $\psi(F_1) \subset \psi(F_2)$.

Conversely, suppose that $\psi(F_1) \subset \psi(F_2)$. Recall the inverse of ψ is φ, where for any sum-labeling ν of $\Gamma_{(m,n)}$, $\varphi(\nu)$ is the face of $PASM(m,n)$ that results as the intersection of the facets corresponding to the edges of ν with labels 0 or 1. Now if $\psi(F_1) \subset \psi(F_2)$, the edges of $\psi(F_1)$ with label $\{0,1\}$ are a subset of such edges of $\psi(F_2)$, so the edges of $\psi(F_2)$ with labels of either 0 or 1 are a subset of such edges of $\psi(F_1)$. So $\varphi(\psi(F_1))$ is an intersection of the facets intersected in $\varphi(\psi(F_2))$ and one or more additional facets. Thus $F_1 = \varphi(\psi(F_1)) \subset \varphi(\psi(F_2)) = F_2$.

Now, we prove the dimension claim. Recall from Remark 4.2 that $\dim(PASM(m,n)) = mn$. Since ψ is a poset isomorphism, ψ maps a maximal chain of faces $F_0 \subset F_1 \subset \cdots \subset F_{mn}$ to the maximal chain $\psi(F_0) \subset \psi(F_1) \subset \cdots \subset \psi(F_{mn})$ in the sum-labelings of $\Gamma_{(m,n)}$. The sum-labeling whose labels are all equal to $\{0,1\}$ contains all other sum-labelings, and this sum-labeling has mn regions. Thus the result follows by Lemma 4.16. $\square$

4.3. **Volume.** The normalized volume of $PASM(m,n)$ for small values of m and n is given in Figure 4 (computed in SageMath). Due to the large size of the polytopes, further computations are not easily obtained. Note that there does not appear to be a nice formula for the volume.

m \ n	1	2	3	4
1	1	1	1	1
2	1	6	43	308
3	1	43	5036	696658
4	1	308	696658	3106156252

FIGURE 4. The normalized volume of $PASM(m,n)$ for small values of m and n.

Remark 4.18. We have used SageMath to compute the Ehrhart polynomials for $PASM(m,n)$ for $m,n \leq 4$ and note that in all of these cases their coefficients are positive.

5. Partial permutohedron

In this section, we study partial permutohedra that arise naturally as projections of $PPerm(m,n)$ and $PASM(m,n)$. After giving the definition, we count vertices and facets and find an inequality description in Subsection 5.1. Then in Subsection 5.2, we note the relation between the partial permutohedron and the stellohedron and give a new combinatorial description of its face lattice. We show in Subsection 5.3 that partial permutation and partial alternating sign matrix polytopes project to partial permutohedra. Finally, in Subsection 5.4, we give a result and conjecture on volume.

5.1. Vertices, facets, inequality description.

In this subsection, we first give the definition of partial permutohedra. We enumerate the vertices in Proposition 5.6 and the facets in Theorem 5.10 and prove an inequality description in Theorem 5.9.

Definition 5.1. Given a partial permutation matrix $M \in P_{m,n}$, its *one-line notation* $w(M)$ is a word $w_1 w_2 \dots w_m$ where $w_i = j$ if there exists j such that $M_{ij} = 1$ and 0 otherwise.

Example 5.2. Let $M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$. Then $w(M) = 3502$.

Proposition 5.3. *$w(P_{m,n})$ can be characterized as the set of all words of length m whose entries are in $\{0, 1, \dots, n\}$ and whose non-zero entries are distinct.*

Proof. By definition, any matrix in $P_{m,n}$ has m rows and n columns with at most one 1 in any given row or column. Thus its image under w will be a word of length m with entries in $\{0, 1, \dots, n\}$ such that the non-zero entries are all distinct. It follows from the definition of w that this map is bijective. $\square$

Definition 5.4. Let $\mathcal{P}(m, n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$, of the words in $w(P_{m,n})$. Call this the *(m, n)-partial permutohedron.*

Definition 5.5. Let $z \in \mathbb{R}^n$ be a vector with distinct nonzero entries. Define $\phi_z : \mathbb{R}^{m \times n} \to \mathbb{R}^m$ as $\phi_z(X) = zX$. Also define $w_z(P_{m,n})$ as the set of all words of length m whose entries are in $\{0, z_1, z_2, \dots, z_n\}$ and whose nonzero entries are distinct. Then $\mathcal{P}_z(m, n)$ is the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$, of the words in $w_z(P_{m,n})$.

Note that we will not use Definition 5.5 until later in this section, but the upcoming results about the structure of partial permutohedra can also be extended to $\mathcal{P}_z$ polytopes.

Proposition 5.6. *The number of vertices of $\mathcal{P}(m, n)$ equals*

$$\sum_{k=\max(m-n,0)}^{m} \frac{m!}{k!}. \tag{5.1}$$

Proof. The extreme points of $\mathcal{P}(m, n)$ are those whose nonzero entries are maximized. That is, if k is the number of zeros, the $(m-k)$ non-zero entries must be precisely $\{n, n-1, \dots, n-(m-k)+1\}$. Now, since there are m total entries and k zeros, there are $\frac{m!}{k!}$ distinct vectors whose $m-k$ nonzero elements are maximized. $\square$

For the proof of the next theorem, and for that of Theorem 5.26, we need the concept of (weak) majorization [14].

Definition 5.7 ([14, Definition A.2])**.** Let u and v be vectors of length N. Then $u \prec_w v$ (that is, u is *weakly majorized* by v) if

$$\sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]}, \text{ for all } 1 \leq k \leq N \tag{5.2}$$

where the vector $\left(u_{[1]}, u_{[2]}, \dots, u_{[N]}\right)$ is obtained from u by rearranging its components so that they are in decreasing order (and similarly for v).

Proposition 5.8 ([14, Proposition 4.C.2])**.** *For vectors u and v of length n, $u \prec_w v$ if and only if u lies in the convex hull of the set of all vectors z which have the form $z = \left(\varepsilon_1 v_{\pi(1)}, \dots, \varepsilon_n v_{\pi(n)}\right)$, where π is a permutation and each ε_i is either 0 or 1.*

Theorem 5.9. *$\mathcal{P}(m,n)$ consists of all vectors $u \in \mathbb{R}^m$ such that:*

$$\sum_{i\in S} u_i \leq \binom{n+1}{2} - \binom{n-k+1}{2}, \qquad \text{where } S \subseteq [m], |S| = k \neq 0, \text{ and} \tag{5.3}$$

$$u_i \geq 0, \qquad \text{for all } 1 \leq i \leq m. \tag{5.4}$$

Proof. First, note that if $P \in P_{m,n}$, then $w(P)$ satisfies (5.3) and (5.4). This is because the largest values that may appear are the m largest non-negative integers less than or equal to n, and the non-zero integers must be distinct. Since $w(P)$ satisfies the inequalities for any P, so must any convex combination.

Now, suppose $x \in \mathbb{R}^m$ satisfies (5.3) and (5.4). We will proceed by using Proposition 5.8. Fix n and let $v = (n, n-1, n-2, \dots, 1, 0, \dots, 0)$ be the decreasing vector in $\mathbb{R}^m$ whose largest entry is n, and whose subsequent non-zero entries decrease by 1 and for which all other entries are 0. Note that if $n \geq m$, then v will have no 0 entries: it will be $(n, n-1, \dots, n-m+1)$. Since x satisfies (5.3) and (5.4), it is by definition weakly majorized by v; note in particular that (5.3) requires that the sum of the k largest entries is never more than the k largest integers less than or equal to n. But now the convex hull described in Proposition 5.8 is actually $\mathcal{P}(m,n)$, thus $x \in \mathcal{P}(m,n)$. ☐

Theorem 5.10. *The number of facets of $\mathcal{P}(m,n)$ equals $m + 2^m - 1 - \sum_{r=1}^{m-n} \binom{m}{m-r}$.*

Proof. There are $2^m - 1$ total inequalities given in (5.3), and m inequalities given in (5.4). Note that $\binom{n-k+1}{2} = 0$ whenever $k \geq n$. When $m > n$, there are $m-n$ values of k such that $\binom{n-k+1}{2} = 0$, creating redundancies. For each r between 1 and $m-n$, we have redundant inequalities for the subsets of $[m]$ of size $m-r$. These are counted by $\binom{m}{m-r}$.

When $m \leq n$, none of the inequalities in (5.3) are redundant, since $\binom{n-k+1}{2} = 0$ may only be satisfied by $k = n$. ☐

Remark 5.11. When $m \geq n$, the number of facets of $\mathcal{P}(m,n)$ can be written as: $m + \sum_{r=m-n+1}^{m} \binom{m}{m-r}$

5.2. **Face lattice.** In this subsection, we give a combinatorial description of the face lattice of $\mathcal{P}(m,m)$ in Theorem 5.22 involving chains in the Boolean lattice. We furthermore state Conjecture 5.23, which extends this characterization to $m \neq n$.

We begin by relating $\mathcal{P}(m,m)$ to a specific graph associahedron, the stellohedron. But first, we need the following definitions.

Definition 5.12 ([7, Definition 2.2])**.** Let G be a graph. A *tube* is a proper nonempty set of vertices of G whose induced graph is a proper, connected subgraph of G. There are three ways that two tubes t_1 and t_2 may interact on the graph:

(1) Tubes are *nested* if $t_1 \subset t_2$.
(2) Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.
(3) Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of G is a set of tubes of G such that every pair of tubes is compatible. A *k–tubing* is a tubing with k tubes.

Definition 5.13 ([9, Definition 2])**.** For a graph G, the *graph associahedron* $\mathrm{Assoc}(G)$ is a simple, convex polytope whose face poset is isomorphic to the set of tubings of G, ordered such that $T < T'$ if T obtained from T' by adding tubes.

Of particular interest to us is the graph associahedron of the star graph, $\mathrm{Assoc}(K_{1,m})$, also called the *stellohedron*.

Definition 5.14. The *star graph* (with $m+1$ vertices) is the complete bipartite graph $K_{1,m}$. We label the lone vertex $*$, and call it the *inner vertex*. We label the other m vertices $x_1, x_2, \ldots, x_m$, and call them *outer vertices*.

Remark 5.15. Note that if G has n nodes, vertices of $\mathrm{Assoc}(G)$ correspond to maximal tubings of G (*i.e.* $(n-1)$–tubings), and in general, faces of dimension k correspond to $(n-k-1)$-tubings of G. Thus for the star graph $K_{1,m}$, which has $m+1$ nodes, vertices of $\mathrm{Assoc}(K_{1,m})$ correspond to m-tubings, and in general, faces of dimension k correspond to $(m-k)$–tubings.

We examine the polytope $\mathrm{Assoc}(K_{1,m})$ through the lens of partial permutations, which allows us to understand it in a different way. Lemmas 5.19 and 5.20 and Corollary 5.21, which culminate in Theorem 5.22, shed light on a way to view these tubings, and thus the faces of the stellohedron, as certain chains in the Boolean lattice. Furthermore, in Conjecture 5.23 we describe what we think happens for $\mathcal{P}(m,n)$, where $m \neq n$. But first, we review the following result that relates $\mathcal{P}(m,m)$ to the stellohedron; this can be found, in other language, in [13]; we describe the explicit map in the remark below.

Theorem 5.16 ([13, Proposition 56])**.** *$\mathcal{P}(m,m)$ is a realization of $\mathrm{Assoc}(K_{1,m})$.*

Remark 5.17. The map which sends maximal tubings of $K_{1,m}$ to the vertices of $\mathcal{P}(m,m)$ is as follows. Let T be a maximal tubing of $K_{1,m}$, and for each outer vertex x_i, let t_i be the smallest tube containing x_i. Then the coordinate in $\mathbb{R}^m$ corresponding to T is $(|t_1|-1, |t_2|-1, \ldots, |t_m|-1)$. Note that two tubes of the star graph are compatible only if they each contain a single outer vertex and do not contain $*$, or one is contained in the other. So a maximal tubing will have r tubes which are singleton outer vertices and tubes of each size from $r+1$ to $m+1$. Moreover, the tube of size $r+1$ must contain each of the r singleton outer vertices along with the inner vertex. Thus such a tubing gets mapped to a coordinate in $\mathbb{R}^m$ with r zeros and whose non-zero entries are $\{m, m-1, \ldots, r+1\}$, which is a vertex of $\mathcal{P}(m,m)$.

One can view a tubing instead as its corresponding spine, defined below. This will help in our goal of describing a bijection between tubings of the star graph and chains in the Boolean lattice.

Definition 5.18. Let T be a tubing of the star graph. The *spine* of T is the poset of tubes of T ordered by inclusion, whose elements are labeled not by the tubes themselves but by the set of new vertices in each tube. For simplicity, we will use the label i in place of x_i.

Spines are defined (in more generality) in [13, Remark 10] and are called B-trees in [17, Definition 7.7]. See Figure 5 for examples of tubings with their corresponding spines, as well as their corresponding chains from the bijection in the following lemma. The *Boolean lattice* B_m is the lattice of all subsets of $[m]$.

Lemma 5.19. *Tubings of $K_{1,m}$ are in bijection with chains in the Boolean lattice $\mathcal{B}_m$.*

Proof. Given a spine S of a tubing T of $K_{1,m}$, we can recover the corresponding chain in the Boolean lattice as follows. The bottom element of the chain is the subset including anything that is grouped with $*$ in S. Each subsequent subset is made by adding in the elements in the next level of S, until we reach the top level. Any elements not used in the subsets of the chain will be those that appear below the $*$ in S.

Starting with a chain $C \in \mathcal{B}_m$, we can recover the corresponding spine S (and thus the tubing) by reversing this process. Any elements not in the maximal subset of C will be in the bottom level of S as singletons. Any elements in the minimal chain of C will appear with $*$ in S. The new elements that appear in each subsequent subset in C appear together as a new level in S. Once we have S, we can, of course, recover T. $\square$

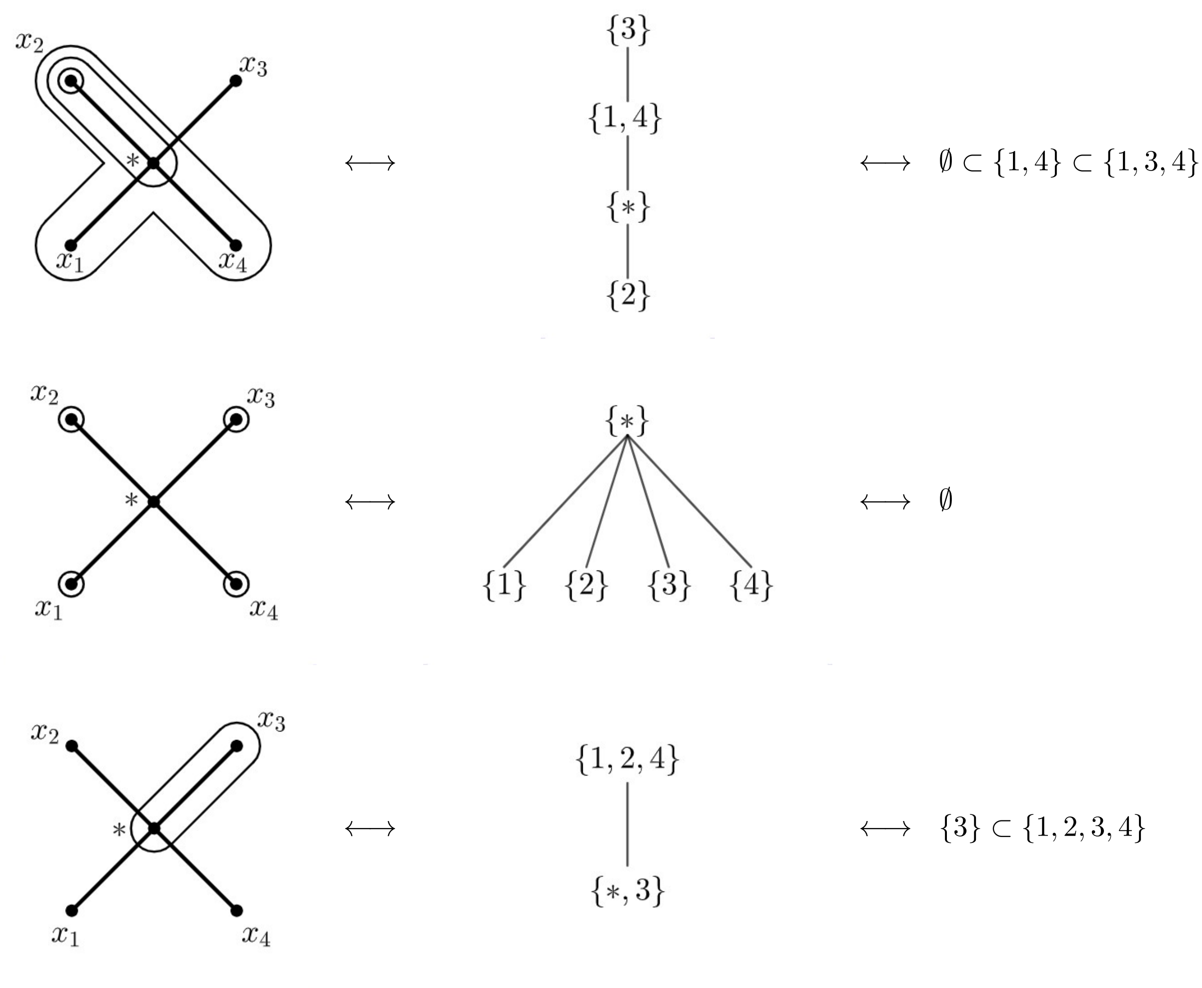

FIGURE 5. Examples of tubings of $K_{1,4}$ along with their corresponding spines (see Definition 5.18) and chains in $\mathcal{B}_4$ (via the bijection in Lemma 5.19)

Lemma 5.20. *Let T be a k-tubing and T' be a $(k+j)$-tubing of $K_{1,m}$, and let C and C' be their corresponding chains in $\mathcal{B}_m$ via the bijection in Lemma 5.19. $T \subset T'$ if and only if C' can be obtained from C by j iterations of the following:*

(1) *adding a non-maximal subset, or*
(2) *removing the same element from every subset.*

Proof. Consider $T \subset T'$, *i.e.* T' is obtained from T by adding tubes. Suppose T and T' differ by adding a single tube, that is, $T = \{t_1, t_2, \ldots, t_k\}$ and $T' = \{t_1, t_2, \ldots, t_k, t'\}$. Let S and S' be their corresponding spines, and let C and C' be their corresponding chains. First note that by the nature of the star graph, a tube either is a singleton outer vertex, x_i, or contains the inner vertex, $*$. Note that a singleton x_i and the singleton $*$ cannot coexist as tubes in a tubing since they are not compatible (they are adjacent).

First consider the case that t' was a singleton outer vertex, x_i. This means that in S, i was grouped with $*$, while in S', $\{i\}$ now appears below $*$. On the level of chains, this means that i is removed from all of the subsets in C to obtain C'.

Now consider the case that t' was not a singleton outer vertex. Then it necessarily contains $*$. In this case, S' has a new level which was not present in S. In particular, this level contains $*$ (and possibly other labels). A new level containing $*$ corresponds to a non-maximal subset being added on the level of chains. In other words, C' is obtained from C by adding a non-maximal subset.

Now suppose T and T' differ by more than one tube, say T is a k-tubing and T' is a $(k+j)$-tubing for some j. Then T' is obtained from T by adding one tube at a time, j times, and thus C' is obtained from C by j iterations of (1) and/or (2) above. □

We now give a description of the dimension of a face in terms of its corresponding chain. Given a chain $C \in \mathcal{B}_m$, we say a rank j is *missing* from C if there is no subset of size j in C and there is a subset of size greater than j in C.

Corollary 5.21. *A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

Proof. We know that adding a tube reduces the dimension of the corresponding face by one. Also, by Lemma 5.20, we know that adding a tube corresponds to either adding a non-maximal subset or removing an element from every subset in the corresponding chain. In either case, this reduces the number of missing ranks in the chain by one. So, having k missing ranks in the chain corresponds to having $m-k$ tubes, which by definition of the graph associahedron corresponds to a face being of dimension k. □

The theorem below follows directly from the above lemmas and corollary.

Theorem 5.22. *The face lattice of $\mathcal{P}(m,m)$ is isomorphic to the lattice of chains in $\mathcal{B}_m$, where $C < C'$ if C' can be obtained from C by iterations of (1) and/or (2) from Lemma 5.20. A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

As chains in the Boolean lattice are generally more familiar objects than tubings of graphs, presenting results in terms of these chains is conceptually helpful. In fact, because of the description of the faces of $\mathcal{P}(m,m)$ in terms of chains, we are able to form the following conjecture for $\mathcal{P}(m,n)$.

Conjecture 5.23. *Faces of $\mathcal{P}(m,n)$ are in bijection with chains in $\mathcal{B}_m$ whose difference between largest and smallest nonempty subsets is at most $n-1$. A face of $\mathcal{P}(m,n)$ is of dimension k if and only if the corresponding chain has k missing ranks*

Remark 5.24. This conjecture has been tested and verified for $m,n \leq 4$.

5.3. **Projection from partial alternating sign matrix polytopes.** In this subsection, we show that the partial permutohedron is a projection of both $PPerm(m,n)$ (in Theorem 5.25) and $PASM(m,n)$ (in Theorem 5.26). Recall ϕ_z from Definition 5.5.

Theorem 5.25. *$\phi_z(PPerm(m,n)) = \mathcal{P}_z(n,m)$.*

Proof. First we need to show $\mathcal{P}_z(n,m) \subseteq \phi_z(PPerm(m,n))$. Suppose $v \in \mathcal{P}_z(n,m)$. We wish to show $v \in \phi_z(PPerm(m,n))$. By definition, $v = \sum \lambda_i w_i$ where the sum is over all length n words w_i whose entries are in $\{0, z_1, z_2, \ldots, z_m\}$ and whose nonzero entries are distinct. But $w_i = zX_i$ where $X_i \in P_{m,n}$. So $v = \sum \lambda_i z X_i = z \sum \lambda_i X_i$, which proves our claim.

Then we need to show that $\phi_z(PPerm(m,n)) \subseteq \mathcal{P}_z(n,m)$. Define $\hat{z}$ as z with $n-m$ zeros appended if $n \geq m$ and as the largest $m-n$ components of z if $n < m$. Let $X = \{x_{ij}\}$ be an $m \times n$ partial permutation matrix. Then, by Proposition 5.8, the proof will be completed by showing $zX \prec_w \hat{z}$ since the convex hull described will then be $\mathcal{P}_z(n,m)$. So, by Definition 5.7, we need to show:

$$\sum_{j=1}^{k} (zX)_{[j]} \leq \sum_{j=1}^{k} \hat{z}_{[j]}, \text{ for } 1 \leq k \leq n.$$

This is true, since each component of the vector zX is either 0 or z_i for some $1 \le i \le m$, because each column of X has at most one nonzero entry. $\square$

Theorem 5.26. *Let z be a strictly decreasing vector in $\mathbb{R}^m$. Then $\phi_z(PASM(m,n)) = \mathcal{P}_z(n,m)$.*

Proof. Let z be a strictly decreasing vector in $\mathbb{R}^m$ and let $n \ge m$. It follows from Theorem 5.25 and $PPerm(m,n) \subseteq PASM(m,n)$ that $\mathcal{P}_z(n,m) \subseteq \phi_z(PASM(m,n))$. Thus it only remains to be shown that $\phi_z(PASM(m,n)) \subseteq \mathcal{P}_z(n,m)$.

As in the previous theorem, define $\hat{z}$ as z with $n-m$ zeros appended if $n \ge m$ and as the largest $m-n$ components of z if $n < m$. Let $X = \{x_{ij}\}$ be an $m \times n$ partial alternating sign matrix. Then, by Proposition 5.8, the proof will be completed by showing $zX \prec_w \hat{z}$ since the convex hull described will then be $\mathcal{P}_z(n,m)$. So, by Definition 5.7, we need to show:

$$\sum_{j=1}^{k} (zX)_{[j]} \le \sum_{j=1}^{k} \hat{z}_j, \text{ for } 1 \le k \le n. \tag{5.5}$$

To prove this, we will show that $\sum_{j\in J}(zX)_j \le \sum_{j=1}^{|J|} \hat{z}_j$ given any $J \subseteq \{1,\dots,n\}$, so that, in particular, $\sum_{j=1}^{|J|}(zX)_{[j]} \le \sum_{j=1}^{|J|} \hat{z}_j$.

We will need to verify the following:

$$\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} \le \min(\ell, |J|), \text{ for } 1 \le \ell \le m \tag{5.6}$$

To prove this, note that

$$\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} = \sum_{j\in J}\sum_{i=1}^{\ell} x_{ij} \le |J|$$

since $\displaystyle\sum_{i=1}^{\ell} x_{ij} \le 1$. But since $\displaystyle\sum_{i=1}^{\ell} x_{ij} \ge 0$ and $\displaystyle\sum_{j=1}^{n} x_{ij} \in \{0,1\}$, we also have that:

$$\sum_{j\in J}\sum_{i=1}^{\ell} x_{ij} \le \sum_{j=1}^{n}\sum_{i=1}^{\ell} x_{ij} = \sum_{i=1}^{\ell}\sum_{j=1}^{n} x_{ij} \le \ell.$$

Now using (5.6) we see that

$$\begin{aligned}
\sum_{j\in J}(zX)_j &= \sum_{j\in J}\sum_{i=1}^{m} z_i x_{ij} = \sum_{i=1} z_i \sum_{j\in J} x_{ij} = \sum_{\ell=1}^{m-1}(z_\ell - z_{\ell+1})\sum_{i=1}\sum_{j\in J} x_{ij} + z_m \sum_{i=1}^{m}\sum_{j\in J} x_{ij} \\
&\le \sum_{\ell=1}^{m-1}(z_\ell - z_{\ell+1})\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} + z_m \min(m,|J|) \qquad \text{by (5.6)} \\
&= \sum_{\ell=1}^{\min(m,|J|)-1}(z_\ell - z_{\ell+1})\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} + \sum_{\ell=\min(m,|J|)}^{m-1}(z_\ell - z_{\ell+1})\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} + z_m \min(m,|J|) \\
&\le \sum_{\ell=1}^{\min(m,|J|)-1}(z_\ell - z_{\ell+1})\ell + \sum_{\ell=\min(m,|J|)}^{m-1}(z_\ell - z_{\ell+1})|J| + z_m \min(m,|J|)
\end{aligned}$$

by (5.6) and since $z_\ell \geq z_{\ell+1}$. Furthermore, this equals

$$\sum_{\ell=1}^{\min(m,|J|)} z_\ell \qquad \text{by telescoping sums,}$$

$$= \sum_{\ell=1}^{|J|} \hat{z}_\ell, \qquad \text{since the last } n-m \text{ entries of } \hat{z} \text{ are zero in the case } m < n.$$

Thus $zX \prec_w \hat{z}$ and so zX is contained in the convex hull of the partial permutations of z. Therefore $\phi_z(PASM(m,n)) = \mathcal{P}_z(n,m)$. ☐

5.4. **Volume.** Regarding the volume of $\mathcal{P}(m,n)$, we have the following theorem for $m = 2$ and conjecture for $n = 2$. We also give normalized volume computations for $m, n \leq 7$ in Figure 6.

Theorem 5.27. *$\mathcal{P}(2,n)$ has normalized volume equal to $2n^2 - 1$.*

Proof. $\mathcal{P}(2,n)$ is a 2-dimensional polytope whose extreme points consist of exactly $(0,0)$, $(n,0)$, $(0,n)$, $(n,n-1)$, and $(n-1,n)$. This forms an $n \times n$ square with one corner "cut off" by the line segment connecting $(n,n-1)$ to $(n-1,n)$. We can explicitly calculate the area of this region to be $n^2 - \frac{1}{2}$. To obtain the normalized volume we multiply by $\dim(\mathcal{P}(2,n))! = 2!$ giving us $2n^2 - 1$. Refer to Figure 7 for the case $m = n = 2$. ☐

m \ n	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	7	17	31	49	71	97
3	1	24	129	342	699	1236	1989
4	1	77	954	4554	12666	27882	53370
5	1	238	6521	59040	262410	751380	1741950
6	1	723	42207	707669	5295150	22406130	65379150
7	1	2180	264501	7975502	99170254	651354480	2657217150

FIGURE 6. Some normalized volume computations for $\mathcal{P}(m,n)$

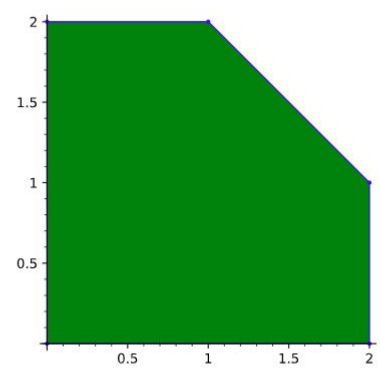

FIGURE 7. A plot of $\mathcal{P}(2,2)$

Conjecture 5.28. *$\mathcal{P}(m,2)$ has normalized volume equal to $3^m - m$.*

Using SageMath, we have confirmed this conjecture for $m \leq 50$.

Remark 5.29. We have used SageMath to compute the Ehrhart polynomials for $\mathcal{P}(m,n)$ for $m,n \leq 7$ and note that in all of these cases their coefficients are positive.

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