Definition 0.1. *Let $\mathfrak{g}$ be a Lie algebra on a field K and V be an n-dimensional vector space over K. And, Let $\phi : \mathfrak{g} \times V \longrightarrow V$ be a K-bilinear map. Hereinafter, we write $\phi(x, v) = x \cdot v(\forall x \in \mathfrak{g}, \forall v \in V)$. Then, (V, ϕ) is $\mathfrak{g}$-module such that*

$$\phi([x, y], v) = [x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \forall x, y \in \mathfrak{g}, \forall v \in V.$$

Definition 0.2. *Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. Then, a p-dimensional* ***cochain*** *of $\mathfrak{g}$ with values in V is a p-linear alternating mapping from $\underbrace{\mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g}}_{p}$ to V.*
However, if p is 0, 0-cochain of $\mathfrak{g}$ is a constant mapping from $\mathfrak{g}$ to V.

Hereinafter, let $\mathfrak{g}, V$ be finite dimension and $C^p(\mathfrak{g}, V)$ be a p-cochain space of $\mathfrak{g}$. That is,

$$\begin{aligned}
&C^p(\mathfrak{g}, V) \cong \mathrm{Hom}(\Lambda^p \mathfrak{g}, V), \quad \phi \mapsto e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto \phi(e_{i_1}, \ldots, e_{i_p}) && (p \geqq 1)\\
&C^0(\mathfrak{g}, V) = V && (p = 0)\\
&C^p(\mathfrak{g}, V) = 0 && (p < 0)
\end{aligned}$$

($e_1, \ldots, e_n$ are basis of $\mathfrak{g}$, $1 \leqq i_1 < \cdots < i_p \leqq n$). Also, let $C^*(\mathfrak{g}, V)$ be a direct sum of all $C^p(\mathfrak{g}, V)$. Then, we will define $\theta(x)$, $i(y)$ as follows. Firstly, we will confirm that for $x, x_1, \ldots, x_p$ in $\mathfrak{g}$ and Φ in $C^p(\mathfrak{g}, V)$,

$$(x \cdot \Phi)(x_1, \ldots, x_p) = x \cdot \Phi(x_1, \ldots, x_p) - \sum_{1 \leqq i \leqq p} \Phi(x_1, \ldots, x_{i-1}, [x, x_i], \ldots, x_p) \quad (*)$$

defines a $\mathfrak{g}$-module structure in $C^p(\mathfrak{g}, V)$. In fact,

$$
\begin{aligned}
&(x \cdot (y \cdot \Phi))(x_1, \ldots, x_p) - (y \cdot (x \cdot \Phi))(x_1, \ldots, x_p) \\
&= x \cdot ((y \cdot \Phi)(x_1, \ldots, x_p)) - \sum_i (y \cdot \Phi)(x_1, \ldots, [x, x_i], \ldots, x_p) \\
&- y \cdot ((x \cdot \Phi)(x_1, \ldots, x_p)) + \sum_i (x \cdot \Phi)(x_1, \ldots, [y, x_i], \ldots, x_p) \\
&= x \cdot \{y \cdot (\Phi(x_1, \ldots, x_p)) - \sum_j \Phi(x_1, \ldots, [y, x_j], \ldots, x_p)\} \\
&- \sum_i \{y \cdot (\Phi(x_1, \cdots, [x, x_i], \ldots, x_p) - \sum_{j \neq i} \Phi(x_1, \ldots, [y, x_j], \ldots, [x, x_i], \ldots, x_p) \\
&- \Phi(x_1, \ldots, [y, [x, x_i]], \ldots, x_p)\} \\
&- y \cdot \{x \cdot (\Phi(x_1, \ldots, x_p)) - \sum_j \Phi(x_1, \ldots, [x, y_j], \ldots, x_p\} \\
&- \sum_i \{x \cdot \{\Phi(x, \cdots, [y, x_i], \ldots, x_p) - \sum_{i \neq j} \Phi(x_1, \ldots, [x, x_j], \ldots, [y, x_j], \ldots, x_p) \\
&- \Phi(x_1, \ldots, [x, [y, x_j]], \ldots, x_p)\} \\
&= [x, y] \cdot (\Phi(x_1, \ldots, x_p)) - \sum_i \Phi(x_1, \ldots, [x, y], x_i], \ldots, x_p) \\
&= ([x, y] \cdot \Phi)(x_1, \ldots, x_p).
\end{aligned}
$$

We write θ as a representation corresponding to this $(*)$. Also, for y in $\mathfrak{g}$, we define $i(y)$ as a homomorphism from C^p to C^{p-1} such as the following.

$$(i(y)\Phi)(x_1, \ldots, x_{p-1}) = \Phi(y, x_1, \ldots, x_{p-1}).$$

Lemma 0.1. *For x, y in $\mathfrak{g}$,*

$$\theta(x) \circ i(y) - i(y) \circ \theta(x) = i([x, y]).$$

Proof. Substituting Φ,

$$\begin{aligned}
(Leftside) &= ((\theta(x) \circ i(y))\Phi - (i(y) \circ \theta(x))\Phi)(x_1, \ldots, x_{p-1}) \\
&= (\theta(x)(i(y)\Phi))(x_1, \ldots, x_{p-1}) - (i(y)(\theta(x)\Phi))(x_1, \ldots, x_{p-1}) \\
&= x(i(y)\Phi)(x_1, \ldots, x_{p-1}) \\
&- \sum_{1 \leqq i \leqq p-1} (i(y)\Phi)(x_1, \ldots, x_{i-1}, [x, x_i], \ldots, x_{p-1}) - (\theta(x)\Phi)(y, x_1, \ldots, x_{p-1}) \\
&= x\Phi(y, x_1, \ldots, x_{p-1}) - \sum_{1 \leqq i \leqq p-1} \Phi(y, x_1, \ldots, x_{i-1}, [x, x_i], \ldots, x_{p-1}) \\
&- \big\{ x\Phi(y, x_1, \ldots, x_{p-1}) - \Phi([x, y], x_1, \ldots, x_{p-1}) \\
&- \sum_{1 \leqq i \leqq p-1} \Phi(y, x_1, \ldots, x_{i-1}, [x, x_i], \ldots, x_p) \big\} \\
&= \Phi([x, y], x_1, \ldots, x_{p-1}) \\
&= (i([x, y])\Phi)(x_1, \ldots, x_{p-1}) = (Rightside).
\end{aligned}$$

$\square$

Definition 0.3. *Let $\mathfrak{g}$ be a Lie algebra and V be $\mathfrak{g}$-module. Then, we define endomorphism d as*

$$\begin{aligned}
d : C^*(\mathfrak{g}, V) &\longrightarrow C^*(\mathfrak{g}, V) \\
d\Phi(x) &= x \cdot \Phi \quad \text{for} \quad \forall \Phi \in C^0(\mathfrak{g}, V),\ \forall x \in \mathfrak{g} \\
d\Phi(x_1, \ldots, x_{p+1}) &= \sum_{1 \leqq s \leqq p+1} (-1)^{s+1} x_s \cdot (\Phi(x_1, \ldots, \hat{x}_s, \ldots, x_{p+1})) \\
&+ \sum_{1 \leqq s < t \leqq p+1} (-1)^{s+t} \Phi([x_s, x_t], x_1, \ldots, \hat{x}_s, \ldots, \hat{x}_t, \ldots, x_{p+1}) \\
&\text{for} \quad \forall \Phi \in C^p(\mathfrak{g}, V) (p \geqq 1),\ \forall x_1, \ldots, x_{p+1} \in \mathfrak{g}.
\end{aligned}$$

*d maps an element of $C^p(\mathfrak{g}, V)$ to $C^{p+1}(\mathfrak{g}, V)$. This d is called **coboundary operator**.*

Theorem 0.1. *d satisfies $d^2 = 0$.*

Fistly, we will prove this lemma.

Lemma 0.2.

$$\begin{aligned}
&\text{(i)} \quad \theta(x) = i(x) \circ d + d \circ i(x) \\
&\text{(ii)} \quad d \circ \theta(x) = \theta(x) \circ d \quad \text{for} \quad \forall x \in \mathfrak{g}.
\end{aligned}$$

Proof. About (i),
For $x, x_1, \dots, x_p$ in $\mathfrak{g}$,

$$\begin{aligned}
((i(x)\circ d)(\Phi))(x_1,\dots,x_p) &= (i(x)(d\Phi))(x_1,\dots,x_p) \\
&= d\Phi(x, x_1,\dots,x_p) \\
&= x\cdot(\Phi(x_1,\dots,x_p)) - \sum_{1\leqq s\leqq p}(-1)^{s+1}\theta(x_s)\Phi((x, x_1,\dots,\hat{x}_s,\dots,x_p)) \\
&+ \sum_{1\leqq s\leqq p}(-1)^{s+1}\Phi([x, x_s], x_1,\dots,\hat{x}_s,\dots,x_p) \\
&+ \sum_{1\leqq s<t\leqq p}(-1)^{s+t}\Phi([x_s, x_t], x, x_1,\dots,\hat{x}_s,\dots,\hat{x}_t,\dots,x_p) \\
&= x\cdot(\Phi(x_1,\dots,x_p)) \\
&- \Big\{\sum_{1\leqq s\leqq p}(-1)^{s+1}\Big(i(x)(\theta(x_s)\Phi) - i([x, x_s])\Phi\Big)(x_1,\dots,\hat{x}_s,\dots,x_p) \\
&+ \sum_{1\leqq s<t\leqq p}(-1)^{s+t}(i(x)\Phi)([x_s, x_t], x_1,\dots,\hat{x}_s,\dots,x_p)\Big\} \\
&= (\theta(x) - d\circ i(x))(\Phi(x_1,\dots,x_p)).
\end{aligned}$$

About (ii), let Φ be a 0-cohchain. Then,

$$\begin{aligned}
(\theta(x)\circ d\Phi)(y) &= x(y\Phi) - [x, y]\Phi \\
&= x(y\Phi) - x(y\Phi) - y(x\Phi) = y(x\Phi) = (d\theta(x)\Phi)(y) \\
&\therefore\ \theta(x)d\Phi = d\theta(x)\Phi
\end{aligned}$$

Now, we assume that this relationship holds up to $p-1$. That is, for Φ in $C^{p-1}(\mathfrak{g}, V)$, $d(\theta(x)\Phi) = \theta(x)d\Phi$. Assuming that Φ is an element of $C^p(\mathfrak{g}, V)$,

$$\begin{aligned}
(i(y)(d\theta(x) - \theta(x)d))\Phi &= (i(y)(d\theta(x))\Phi - i(y)\theta(x)d\Phi \\
&= (\theta(y) - di(y))\theta(x)\Phi + (i([x, y]) - \theta(x)i(y))d\Phi \\
&= \theta(y)\theta(x)\Phi - di(y)\theta(x)\Phi + i([x, y])d\Phi - \theta(x)i(y)d\Phi \\
&= \theta(y)\theta(x)\Phi + d(i([x, y]) - \theta(x)i(y))\Phi \\
&+ \big\{\theta([x, y]) - di([x, y])\big\}\Phi - \big\{\theta(x)(\theta(y) - di(y))\big\}\Phi \\
&= (\theta(x)d - d\theta(x))(i(y)\Phi) = 0 \\
&\therefore\ (d\theta(x) - \theta(x)d)\Phi = 0.
\end{aligned}$$

□

then, we will prove theorem 0.1.

Proof. We prove by induction on p. Firstly, for $\Phi \in C^0(\mathfrak{g}, V), \forall x, y \in \mathfrak{g}$,

$$\begin{aligned}((d \circ d)(\Phi))(x, y) &= x \cdot (d\Phi(y)) - y \cdot (d\Phi(x)) - d\Phi([x, y]) \\ &= 0.\end{aligned}$$

Secondary, we suppose that the above equation holds up to $p-1$. Then, for $\Phi \in C^p(\mathfrak{g}, V), x_1, \ldots, x_{p+2} \in \mathfrak{g}$,

$$\begin{aligned}((d \circ d)(\Phi))(x_1, \ldots, x_{p+2}) &= (i(x_1) \circ d \circ d)(\Phi)(x_2, \ldots, x_{p+2}) \\ &= ((\theta(x_1) - d \circ i(x_1)) \circ d)(\Phi)(x_2, \ldots, x_{p+2}) \\ &= (\theta(x_1) \circ d - d \circ (\theta(x_1) - d \circ i(x_1)))(\Phi)(x_2, \ldots, x_{p+2}) \\ &= ((d \circ d \circ i(x_1)(\Phi)(x_2, \ldots, x_{p+2}) \\ &= (d \circ d)(i(x_1)(\Phi))(x_2, \ldots, x_{p+2})\end{aligned}$$

Therefore, $d^2 = 0$. □

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