Implicit Bias of Gradient Descent based Adversarial Training on Separable Data

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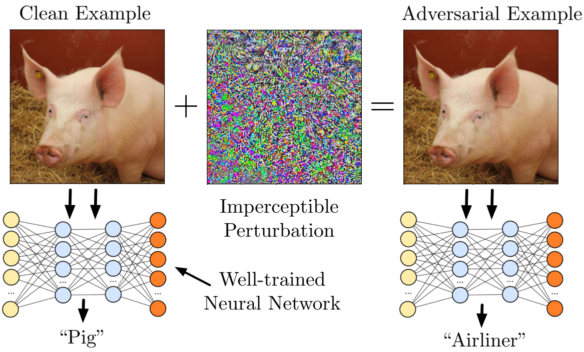
Adversarial Examples



All current deep neural network (DNN) models are subject to adversarial examples.

Training Robust Models

Adversarial training directly minimize the worst-case loss for a given perturbation set Δ:

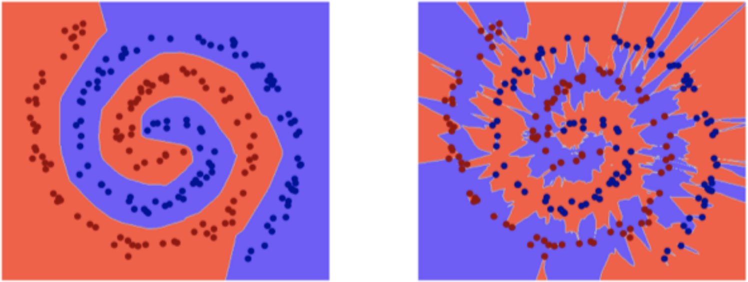
$$\theta_{\text{robust}} = \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max_{\delta_i \in \Delta} \ell(x_i + \delta_i, y_i, \theta).$$

Question: How does adversarial training promote robustness?

We propose to study from a computational perspective – implicit bias of the optimization algorithm.

Implicit Bias

Neural network can easily overfit training data. Training algorithm biases toward a certain kind of solutions.



(a). (b).

Implicit Bias of Algorithms: network (a) is learnt by SGD (Smooth Boundary). Both networks overfits training data. Only network (a) generalizes well.

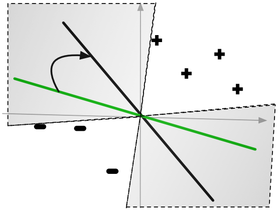
Training a Linear Classifier

Directly analyzing DNNs is beyond current technical limit.
▷ A simplified yet non-trivial example, training a linear classifier on linearly separable data $\{(x_i, y_i)\}_{i=1}^n$. We aim to solve

$$\min_{\theta \in \mathbb{R}^d} \mathcal{L}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i x_i^\top \theta), \ell \text{ exponential/logistic loss.}$$

• Only the **direction** of the linear classifier is important.
• There is **no finite minimizer** of $\mathcal{L}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i x_i^\top \theta)$. But there exists infinite amount of solutions at infinity.

Implicit Bias of Gradient Descent



$\theta^t / \|\theta^t\|_2$: Standard Training
θ_2: Maximum ℓ_2 Margin Classifier

Classifiers within the shaded cone overfit the clean examples.

Gradient descent converges in direction to the ℓ_2 norm max margin classifier (Soudry et al 2017; Ji and Telgarsky 2018):

$$1 - \left\langle \theta^t / \left\|\theta^t\right\|_2, \theta_2 \right\rangle = \mathcal{O}(\log n / \log t),$$

where θ_q (here $q = 2$) and the optimal value γ_q is defined by:

$$\theta_q = \underset{\|\theta\|_p = 1}{\arg\max} \min_{i=1,\ldots,n} y_i x_i^\top \theta, \quad \text{with } 1/p + 1/q = 1.$$

GDAT – Gradient Descent based Adversarial Training

GDAT on Separable Data with ℓ_q Perturbation

Input: Data points $\{(x_i, y_i)\}_{i=1}^n$, perturbation level $c < \gamma_q$ and step sizes $\{\eta^t\}_{t=0}^{T-1}$.

Init: Set $\theta^0 = 0$.

For $t = 0 \dots T-1$:

For $i = 1 \dots n$, $\widehat{\delta}_i = \arg\max_{\|\delta_i\|_q \le c} \ell(y_i(x_i + \delta_i)^\top \theta^t)$.

Set $\widetilde{x}_i = x_i + \widehat{\delta}_i$, for $i = 1 \dots n$.

Update $\theta^{t+1} = \theta^t - (\eta^t/n) \cdot \sum_{i=1}^n \nabla \ell(y_i \widetilde{x}_i \theta^t)$.

Questions: Can we characterize the implicit bias of GDAT on separable data? How is it related to adversarial robustness?

GDAT Adapts to Adversary Examples

Consider the following large margin classifier:

$$\theta_{q,c} = \arg\max_{\|\theta\|_2=1} \min_{i=1,\ldots,n} \min_{\|\delta_i\|_q \leq c} y_i(x_i+\delta_i)^\top \theta.$$

Robustness: $\theta_{q,c}$ is in the same direction to the solution of

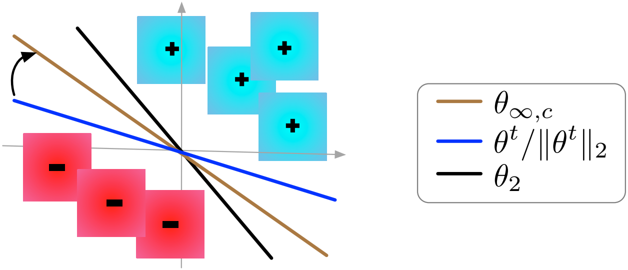
$$\min_{\theta\in\mathbb{R}^d} \|\theta\|_2 \quad \text{s.t. } y_i \widetilde{x}_i^\top \theta \geq 1 \text{ for all } \|\widetilde{x}_i - x_i\|_q \leq c, \forall i = 1 \ldots n.$$

GDAT Adapts to Adversary Examples

Theorem (Informal)

Let perturbation level $c < \gamma_q$, Then

$$1 - \left\langle \theta^t / \left\|\theta^t\right\|_2, \theta_{q,c} \right\rangle = \mathcal{O}\left(\log n / \log t\right).$$



GDAT Accelerates Convergence ($q = 2$)

Theorem (Informal)

Let c and number of iterations T satisfy $\gamma_2 - c = \mathcal{O}\left(\frac{\log^2 T}{T}\right)^{1/2}$,
We have $\theta_{2,c} = \theta_2$, and

$$1 - \left\langle \theta^T / \left\|\theta^T\right\|_2, \theta_2 \right\rangle = \mathcal{O}\left(\frac{\log T}{\sqrt{T}}\right).$$

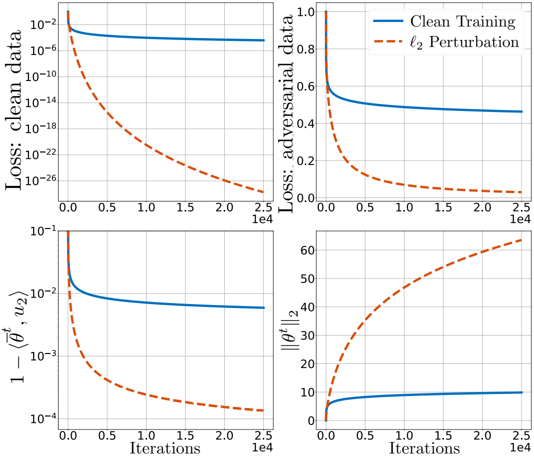
Exponential Acceleration by GDAT

Corollary: Convergence on clean loss by GDAT is almost exponentially faster than GD.

- GDAT: $\mathcal{L}(\theta_T) = \mathcal{O}\left(\exp(-\sqrt{T}/\log T)\right)$
- GD: $\mathcal{L}(\theta_T) = \mathcal{O}(1/T)$

Empirical Study

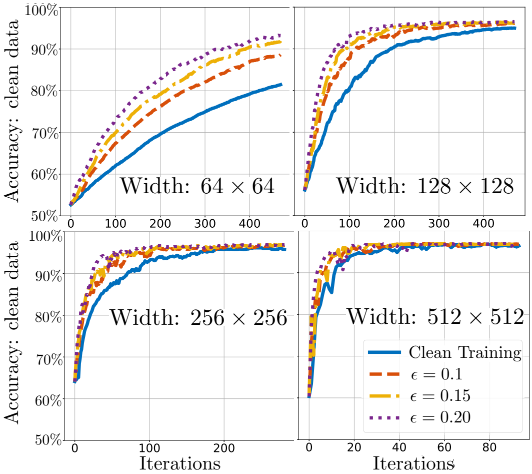
Linear Classifiers: We generate data with $\gamma_2 = 1$. We set $c = 0.95$. $\eta = 0.1$ for GDAT and $\eta = 1$ for standard training.



Clean Training v.s. GDAT (ℓ_2 perturbation)

Empirical Study

Neural Networks: We use MNIST dataset. The width of hidden layer varies in $\{64 \times 64, 128 \times 128, 256 \times 256, 512 \times 512\}$. We use ℓ_∞ perturbation with perturbation level $\epsilon \in \{0.1, 0.15, 0.20\}$.



Thank you!