Homework 4

Hai Zhu

due date: February 25, 2015

1. Problem 1.

Consider the scalar conservation law $q_t + f(q)_x = 0$ with $f(q) = \sqrt{q}$, which is convex as long as we only consider states $q > 0$.

(a) Consider this problem with data

$$q(x,0) = \begin{cases} 4 & \text{if } 0 < x < 1, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Determine the time t_s when the shock and rarefaction wave first begin to interact and the solution $q(x, t_s)$.

(b) Use Clawpack to verify your solution. Set up a problem similar to what you did on the Programming problem of Homework #3. Choose the domain, mesh size, and limiter to illustrate the solution in a convincing manner. Save the code required in a directory hw4/sqrtflux and add a file README.txt with any instructions or comments on the solution.

(c) Repeat parts (a) and (b) for the initial data

$$q(x,0) = \begin{cases} 4 & \text{if } 0 < x < 1, \\ 0.01 & \text{otherwise.} \end{cases} \tag{2}$$

For the programming part, also do the following:

- Set clawdata.verbosity = 1 in setrun.py so that it prints out information every time step about the size of the time step. Comment on what you observe relative to a similar experiment with the initial data (1).
- Try using the Lax-Wendroff method (no limiter) for this problem and comment on what happens, relative to a similar experiment with the initial data (1).

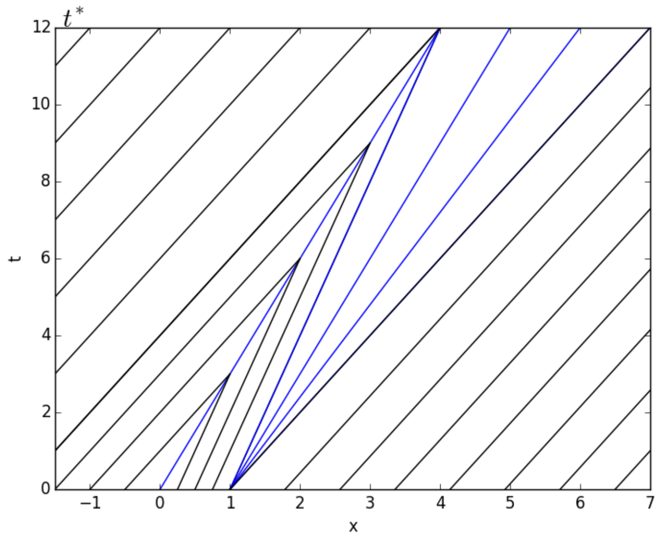
Solution

(a) We want to solve $q_t + \frac{1}{2}q^{-1/2}q_x = 0$ by using characteristics. The slope of characteristics should be:

$$x'(t) = \frac{1}{2}q^{-1/2}$$

Since along characteristics q is constant, which should be the same as initial value $q(x,0)$ when we trace back along characteristics.

$$x'(t) = \begin{cases} 1/4 & \text{if } 0 < x < 1 \\ 1/2 & \text{otherwise} \end{cases}$$



As we can tell from the figure of characteristics, shock should form at $t = 0$, $x = 0$, and rarefaction wave should form as $t = 0$, $x = 1$.

Shock speed can de derived from Rankine-Hugoniot condition

$$s(q_l - q_r) = f(q_l) - f(q_r)$$
$$\Rightarrow s = \frac{1}{3}$$

which gives us shock position formula

$$x(t) = \frac{1}{3}t$$

The left edge of rarefaction fan should start from $x = 1$ with slope $1/4$, which means we can determine the time when shock and rarefaction wave intersect by solving the following

$$x(t) = \frac{1}{3}t$$
$$x(t) = 1 + \frac{1}{4}t$$

and get $t_s = 12$.

Now we can determine the similarity solution for rarefaction wave. Assume solution has the form $\tilde{q}((x-1)/t)$, then substitute into scalar conservation law. We can get the following solution

$$q(x,t) = \tilde{q}((x-1)/t) = (\frac{2(x-1)}{t})^{-2}$$

For $t_s = 12$, the profile of solution $q(x,t_s)$ has the following expression

$$q(x,t_s) = \begin{cases} (\frac{x-1}{6})^{-2} & \text{if } 4 < x < 7 \\ 1 & \text{otherwise} \end{cases}$$

(b) For the programming part, I use f-wave version of algorithms. I think it should be the same for 1-equation scalar case.

As for implementation of Lax-Wendroff method, numerical solution will have oscillation.

(c) For initial data (2), shock and rarefaction wave form at the same place with different speed and wedge

$$\begin{aligned} s(q_l - q_r) &= f(q_l) - f(q_r) \\ \Rightarrow s &= \frac{190}{399} \end{aligned}$$

Left edge of rarefaction wave is $x(t) = \frac{1}{4}t + 1$, and right edge of rarefaction wave is $x(t) = 5t + 1$.
Thus t_s can be derived by solving the following equations

$$\begin{aligned} x(t) &= \frac{190}{399}t \\ x(t) &= 1 + \frac{1}{4}t \end{aligned}$$

and get $t_s = \frac{84}{19}$.

The similarity solution is still the same formula on a fan section with a different right edge

$$q(x,t) = \tilde{q}((x-1)/t) = (\frac{2(x-1)}{t})^{-2}$$

For $t_s = \frac{84}{19}$, the profile of solution $q(x,t_s)$ has the following expression

$$q(x,t_s) = \begin{cases} (\frac{19(x-1)}{42})^{-2} & \text{if } 40/19 < x < 439/19 \\ 0.01 & \text{otherwise} \end{cases}$$

As for the programming part, it turns out if I use Lax-Wendroff method, numerical solution will break down. Without using limiter, the oscillation will lead to negative value since the initial value is very close to 0. And that would cause trouble when we compute shock speed.

- As for the choice of domain, I make sure that we can see all the states as wave propagate. The choice of limiter doesn't seem to affect the time when shock intersects with rarefaction wave. While the refinement of x domain seems to make the time closer to t_s.
- The (2) initial data can also be implemented in the same directory by uncommenting the initial data in qinit.f.

$\blacksquare$

2. Problem 2.

Consider the linear system $q_t + Aq_x = 0$ with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

(a) Determine the eigenvalues and eigenvectors of A and sketch the integral curves of the eigenvectors in the phase plane. (And recall that for a linear system these are also the Hugoniot loci.)

(b) Consider the Cauchy problem for this system (no boundaries) with initial data

$$q(x,0) = \begin{cases} [4,4]^T & \text{if } x < 1, \\ [1,1]^T & \text{if } 1 \le x \le 3, \\ [2,1]^T & \text{if } x > 3. \end{cases}$$

(c) Set up a Clawpack Riemann solver for this problem and use the initial condition above as a test of your code. (modify acoustics_1d_example1). Solve it over a large enough domain to see all the states you expect to see in the exact solution, and use extrapolation boundary conditions.
Include some plots from your solution in your writeup.
Commit the files needed to produce these plots, in a directory hw4/linsys.
Please make sure comments in the code are relevant to the problem being solved and clean up things not needed for this code.

Solution

(a) Eigenvalues:

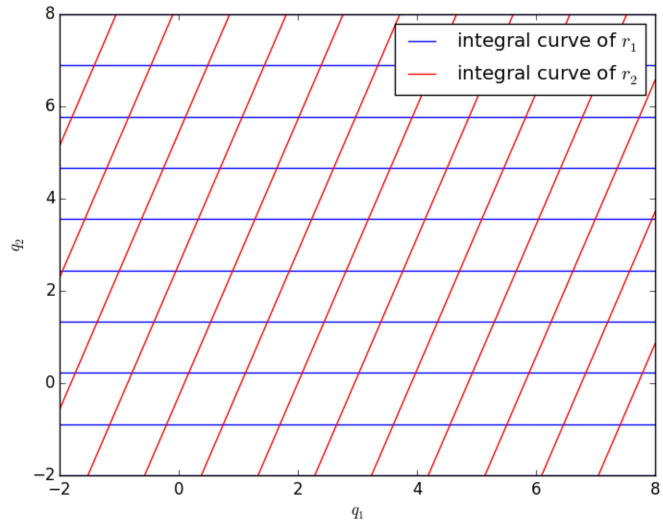
$$|\lambda I - A| = \begin{vmatrix} \lambda + 1 & -1 \\ & \lambda - 2 \end{vmatrix} = (\lambda + 1)(\lambda - 2) = 0$$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = 2.$

The following is the matrix for eigenvectors, and its inverse (in order of their corresponding eigenvalues)

$$R = \begin{bmatrix} 1 & 1/3 \\ & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & -1/3 \\ & 1 \end{bmatrix}$$

The following is the sketch of the integral curves of the eigenvectors in the phase plane.



(b) First, find the coefficients for this initial value under basis generated by eigenvectors in (a).

$$R^{-1}\begin{bmatrix}4 & 1 & 2\\ 4 & 1 & 1\end{bmatrix} = \begin{bmatrix}8/3 & 2/3 & 5/3\\ 4 & 1 & 1\end{bmatrix}$$

Consider the following initial value problem for the decoupled waves

$$\tilde{q}(x,0) = \begin{cases} [8/3, 4]^T & 1 < x \\ [2/3, 1]^T & 1 \le x \le 3 \\ [5/3, 1]^T & 3 < x \end{cases}$$

The first component travels with speed -1, and the second component travels with speed 2. Thus the general solution has the following form

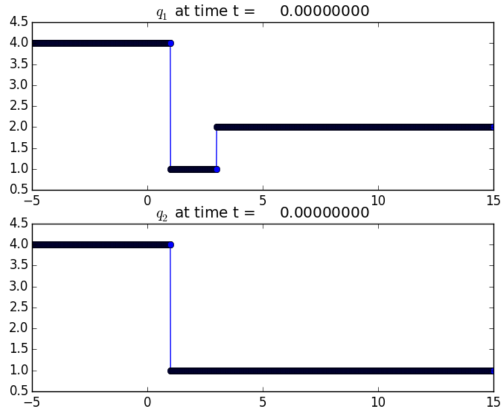
$$\tilde{q}(x,t) = \begin{bmatrix}\tilde{q}_1(x+t,0)\\ \tilde{q}_2(x-2t,0)\end{bmatrix}$$

$$\Rightarrow q(x,t) = R\cdot\tilde{q}(x,t) = \begin{bmatrix}\tilde{q}_1(x+t,0) + 1/3\tilde{q}_2(x-2t,0)\\ \tilde{q}_2(x-2t,0)\end{bmatrix}$$

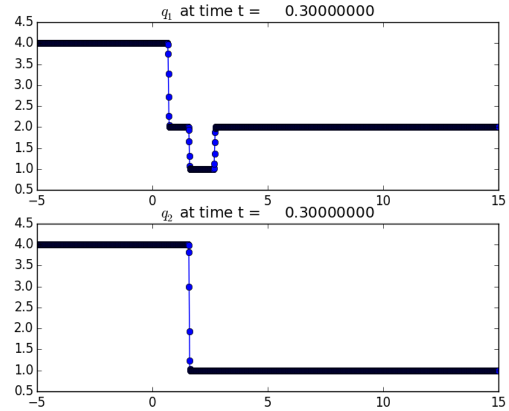
where

$$\tilde{q}(x,0) = \begin{bmatrix}\tilde{q}_1(x,0)\\ \tilde{q}_2(x,0)\end{bmatrix} = \begin{cases} [8/3, 4]^T & 1 < x \\ [2/3, 1]^T & 1 \le x \le 3 \\ [5/3, 1]^T & 3 < x \end{cases}$$

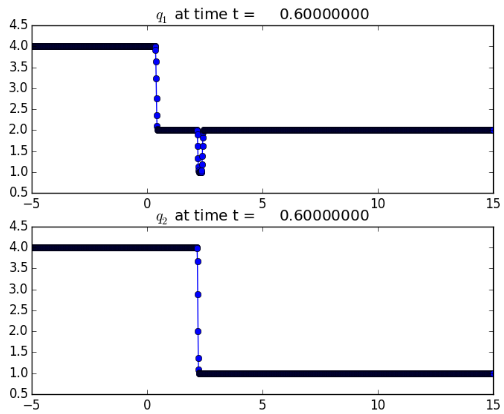
(c) The followings are some plots from my solution.



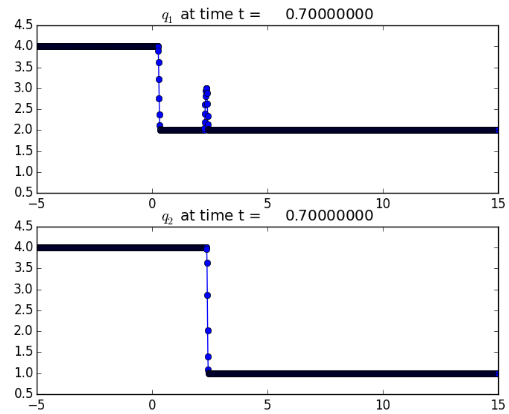
(a) $t = 0$



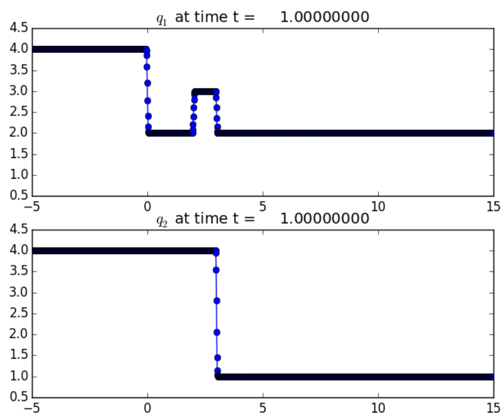
(b) $t = 0.3$



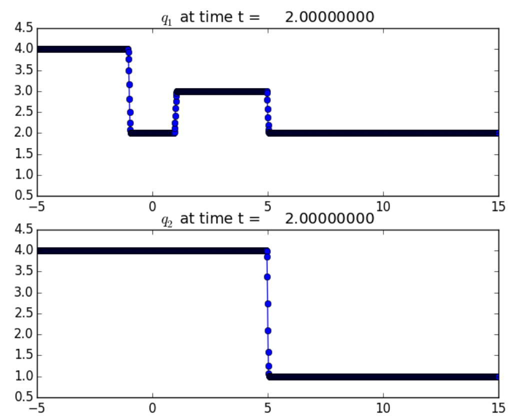
(c) $t = 0.6$



(d) $t = 0.7$



(e) $t = 1$



(f) $t = 2$

Figure 1

3. Problem 13.11

The variable-coefficient scalar advection equation $q_t + (u(x)q)_x = 0$ can be viewed as a hyperbolic system of two equations,

$$\begin{aligned} q_t + (uq)_x &= 0, \\ u_t &= 0, \end{aligned}$$

where we now view $u(x,t) \equiv u(x)$ as a second component of the system.

(a) Determine the eigenvalues and eigenvectors of the Jacobian matrix for this system.

(b) Show that both fields are linearly degenerate, and that in each field the integral curves and Hugoniot loci coincide. Plot the integral curves of each field in the $q-u$ plane.

(c) Indicate the structure of a general Riemann solution in the $q-u$ plane for the case $u_l, u_r > 0$. Relate this to Figure 9.1.

Solution

(a) Matrix form of this system

$$\begin{bmatrix} q \\ u \end{bmatrix}_t + \begin{bmatrix} u & q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ u \end{bmatrix}_x = 0$$

Thus eigenvalues are $\lambda^1 = 0$, $\lambda^2 = u$. And the corresponding eigenvectors are

$$r^1 = \begin{bmatrix} q \\ -u \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) For $\lambda^1 = 0$, $\nabla\lambda^1 = (0,0)$, thus $\nabla\lambda^1 \cdot r^1 = 0$.
For $\lambda^2 = u$, $\nabla\lambda^2 = (0,1)$, thus $\nabla\lambda^2 \cdot r^2 = 0$.
Thus both fields are linearly degenerate.

Integral curve for r^1,

$$r^1 = \begin{bmatrix} q \\ -u \end{bmatrix} = \begin{bmatrix} \frac{dq}{dt} \\ \frac{du}{dt} \end{bmatrix}$$

Thus $q(t) = A\exp(t)$, $u(t) = B\exp(-t)$.

Integral curve for r^2,

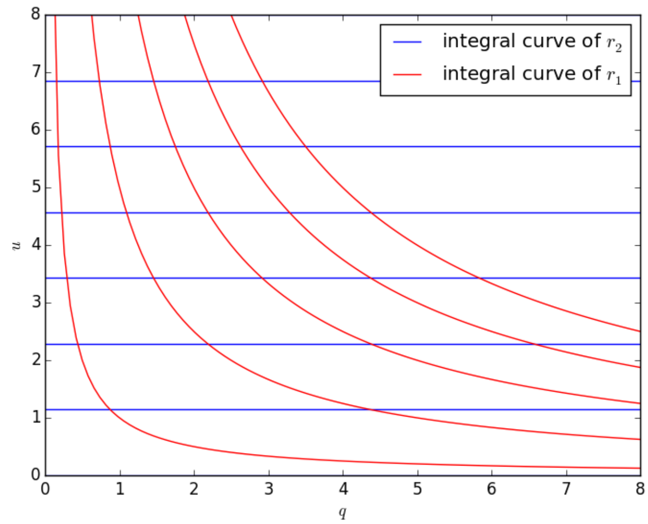
$$r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dq}{dt} \\ \frac{du}{dt} \end{bmatrix}$$

Thus $q(t) = q_0 + t$, $u(t) = u_0$.

As for Hugoniot loci, we need to use Rankine-Hugoniot condition

$$\begin{aligned} &s(q_* - q) = u_* q_* - uq \\ &s(u_* - u) = 0 \end{aligned}$$

If $s = 0$, then we have $u_* q_* = uq$. It is the same as integral curve for r^1
If $u_* = u$, then it is the same as integral curve for r^2.
The following is the sketch of the integral curves of the eigenvectors in the phase plane.



(c) Each left state and right state in $q - u$ plane is connected by a middle state which can be derived from integral curves of r^1 and r^2 in (b).

The middle state can be found by starting from left state along integral curve of r_1 (red curve), arriving at the point which has the same u as right state. And then go along integral curve of r_2 to the right state.

The way I understand this, any two states can be connected by an intermediate state which has the same moving speed as right state, since the integral curves of r_2 are constants of u. So we should expect a shock in q between right and intermediate states. As for the left state, if u_l is smaller than u_m, then we should expect a rarefaction wave in q. If u_l is greater than u_m, then we should expect a shock in q.

∎

4. Problem 15.2.

Suppose an HLL approximate Riemann solver of the form discussed in Approximate Riemann solver section is used, but with $s^1_{i-1/2} = -\Delta x/\Delta t$ and $s^2_{i-1/2} = \Delta x/\Delta t$. These are the largest speeds that can be used with this grid spacing and still respect the CFL condition, so these should be upper bounds on the physical speeds. Show that if this approximate Riemann solver is used in the first-order Godunov method, then the result is the Lax-Friedrichs method.

Solution

Suppose $s^1_{i-1/2} = -\Delta x/\Delta t$ and $s^2_{i-1/2} = \Delta x/\Delta t$. Then the middle state to make sure conservation property is given by

$$\begin{aligned}\hat{Q}_{i-1/2} &= \frac{f(Q_i) - f(Q_{i-1}) - \frac{\Delta x}{\Delta t}Q_i - \frac{\Delta x}{\Delta t}Q_{i-1}}{-\frac{\Delta x}{\Delta t} - \frac{\Delta x}{\Delta t}} \\ &= -\frac{\Delta t}{2\Delta x}[f(Q_i) - f(Q_{i-1})] + \frac{1}{2}(Q_i + Q_{i-1})\end{aligned}$$

Thus we can define wave flux

$$\begin{aligned}\mathcal{A}^+\Delta Q_{i-1/2} &= s^2_{i-1/2}\mathcal{W}^2_{i-1/2} \\ &= \frac{\Delta x}{\Delta t}[Q_i + \frac{\Delta t}{2\Delta x}(f(Q_i) - f(Q_{i-1})) - \frac{1}{2}(Q_i + Q_{i-1})] \\ \mathcal{A}^-\Delta Q_{i-1/2} &= s^1_{i-1/2}\mathcal{W}^1_{i-1/2} \\ &= -\frac{\Delta x}{\Delta t}[-\frac{\Delta t}{2\Delta x}(f(Q_i) - f(Q_{i-1})) + \frac{1}{2}(Q_i + Q_{i-1}) - Q_{i-1}]\end{aligned}$$

Then plug in first order Godunov method

$$\begin{aligned}\Rightarrow Q^{n+1}_i &= Q^n_i - \frac{\Delta t}{\Delta x}\{\frac{\Delta x}{\Delta t}[Q^n_i + \frac{\Delta t}{2\Delta x}(f(Q^n_i) - f(Q^n_{i-1})) - \frac{1}{2}(Q^n_i + Q^n_{i-1})] \\ &\qquad - \frac{\Delta x}{\Delta t}[-\frac{\Delta t}{2\Delta x}(f(Q^n_i) - f(Q^n_{i-1})) + \frac{1}{2}(Q^n_i + Q^n_{i-1}) - Q^n_{i-1}]\} \\ &= \frac{1}{2}(Q^n_{i-1} + Q^n_{i+1}) - \frac{\Delta t}{2\Delta x}[f(Q^n_{i+1}) - f(Q^n_{i-1})]\end{aligned}$$

The result is the Lax-Friedrichs method.

∎