Comments on Homework #2
AMath 574, Winter Quarter, 2015

Problem #6.3

Suppose Q_i^n approaches Q_ℓ and Q_r as $i \longrightarrow \pm\infty$. Then $TV(Q^n) = |Q_r - Q_\ell|$. We are assuming Q^{n+1} has the same limits.

Note than in general, $|a| + |b| + |c| = |a + b + c|$ if and only if a, b, and c all have the same sign. Otherwise the left hand side is strictly larger than the right.

Note also that for any choice of i and any grid function Q approaching the same limiting values (whether monotone or not),

$$TV(Q) =\geq |Q_r - Q_{i+1}| + |Q_{i+1} - Q_i| + |Q_i - Q_\ell|$$

If the three terms on the right hand side all have the same sign, then they collapse to $|Q_r - Q_\ell|$. But if $Q_{i+1} - Q_i$ has a different sign than $Q_r - Q_\ell$ for any choice of i, then we would conclude that $TV(Q) > |Q_r - Q_\ell|$.

So if Q^{n+1} is not monotone, then $TV(Q^{n+1}) > TV(Q^n)$, which violates the assumption that the method is TVD. Hence any TVD method preserves monotonicity.

Problem #6.4

There are several ways to do this, but the easiest might be to note that it is enough to consider the cells where Q_i is a local maximum or minimum. Let i_k be the indices of such cells. Then

$$TV(Q) = \sum_k |Q_{i_k} - Q_{i_{k-1}}|.$$

Then choose points ξ_k in these cells so that $q(\xi_k) \geq Q_{i_k}$ in the local maximum cells, or so that $q(\xi_k) \leq Q_{i_k}$ in the local minimum cells (always possible since the cell average lies between the max and min in the cell).

Then

$$TV(q) \geq \sum_k |q(\xi_k) - q(\xi_{k-1})| \geq \sum_k |Q_{i_k} - Q_{i_{k-1}}| = TV(Q).$$

Problem #6.5

Note that if $\tilde{q}(x)$ is the reconstruction with minmod slopes σ_i in the ith cell, then $\tilde{q}(x)$ is always continuous at the cell centers x_i (where $\tilde{q}(x_i) = Q_i$), and so

$$TV(\tilde{q}) = \sum_i TV_i(\tilde{q}),$$

where $TV_i(\tilde{q})$ is the total variation over the interval from x_{i-1} to x_i.

The choice of minmod slope guarantees that $\tilde{q}(x)$ is monotone on this interval and hence $TV_i(\tilde{q}) = |Q_i - Q_{i-1}|$. It follows that

$$TV(\tilde{q}) = \sum_i TV_i(\tilde{q}) = \sum_i |Q_i - Q_{i-1}| = TV(Q).$$

To show the monotonicity on each interval, note for example that if $Q_{i-1} \leq Q_i$ then both σ_{i-1} and σ_i satisfy $0 \leq \sigma \leq (Q_i - Q_{i-1})/\Delta x$ and so

$$Q_{i-1} \leq Q_{i-1} + \frac{\Delta x}{2}\sigma_{i-1} \leq Q_i - \frac{\Delta x}{2}\sigma_i \leq Q_i.$$