



M2177.003100

Deep Learning

[2: Neuron Modeling]

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Outline

Introduction

Logistic Regression

Backprop Demystified

Minibatch Processing

Summary

References

- *Deep Learning* by Goodfellow, Bengio and Courville [▶ Link](#)
 - ▶ Chapter 6
- *Pattern Recognition and Machine Learning* by Bishop
 - ▶ Chapter 5: Neural Networks
- online resources:
 - ▶ *Deep Learning Specialization (coursera)* [▶ Link](#)
 - ▶ *Stanford CS231n: CNN for Visual Recognition* [▶ Link](#)
 - ▶ *Machine Learning Yearning* [▶ Link](#)

Outline

Introduction

Logistic Regression

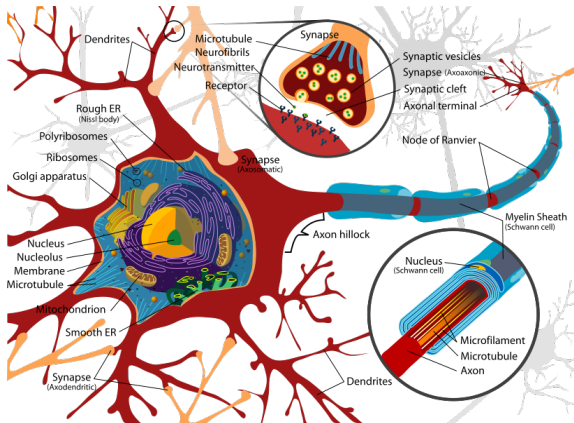
Backprop Demystified

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Neuron

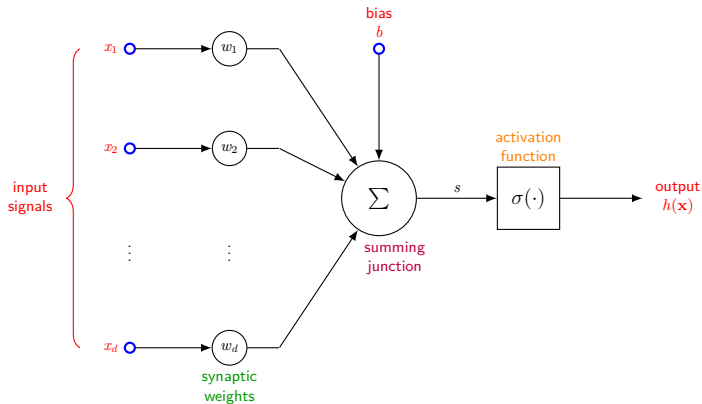
- electrically excitable cell in animal brains
 - ▶ processes and transmits information through electrical/chemical signals



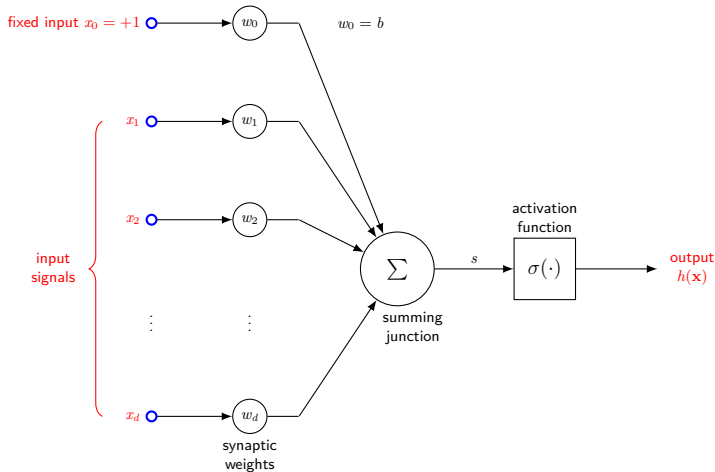
(source: <http://en.wikipedia.org/wiki/Neuron>)

Modeling a neuron

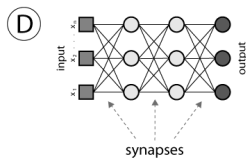
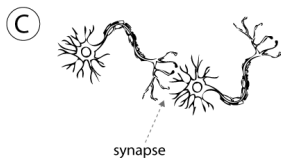
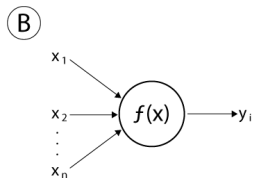
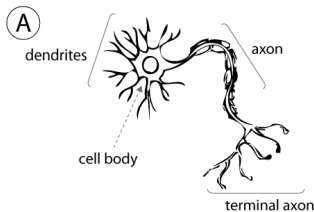
- three basic elements
 1. **synapses** (with weights)
 2. **adder** (input vector \rightarrow scalar)
 3. **activation function** (possibly nonlinear)



- alternative representation (w_0 for bias b):



Human neuron vs artificial neuron



(a) human neuron

(b) artificial neuron

(c) biological neural net

(d) artificial neural net

(source: Matarollo, 2013)

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Linear models

- core of linear models
 - ▶ signal (weighted sum) $z = \text{_____}$: combines input variables linearly
 - ▶ we have seen two models based on this
- 1. linear regression
 - ▶ signal itself = output
 - ▶ for predicting real (unbounded) response
- 2. linear classification
 - ▶ signal is thresholded at zero to produce ± 1 output
 - ▶ for binary decisions

Logistic regression as a neuron model

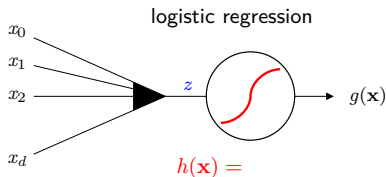
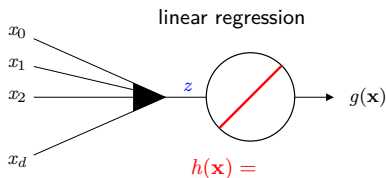
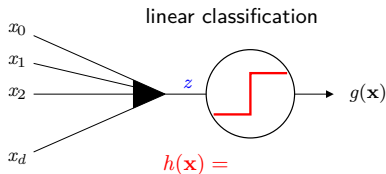
- we review the third linear model:
 - ▶ outputs *probability* of a binary response
e.g. heart attack or not, dead or alive
 - ▶ returns 'soft labels' (probability)
- this model: called *logistic regression*
 - ▶ output: *real* (like regression) but *bounded* (like classification)
- comparison: linear classification vs logistic regression
 - ▶ both deal with a binary event
 - ▶ logistic regression: allowed to be uncertain
 - ⇒ intermediate values between 0 and 1 reflect this uncertainty
- in early neural nets:

logistic regression unit =

Recall: linear models

- based on
“signal” z :

$$z = \sum_{i=0}^d w_i x_i$$



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Formulation

- problem

- ▶ given: $\mathbf{x} \in \mathbb{R}^{n_x}$
- ▶ want: $\hat{y} = P(y = 1 \mid \mathbf{x})$

- model

- ▶ $\hat{y} =$
- ▶ parameters:

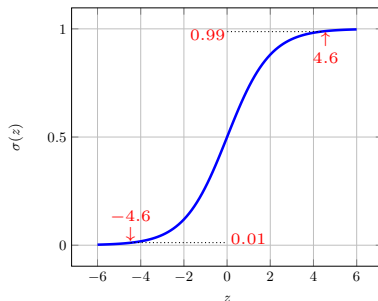
$$\mathbf{w} \in \mathbb{R}^{n_x}$$

$$b \in \mathbb{R}$$

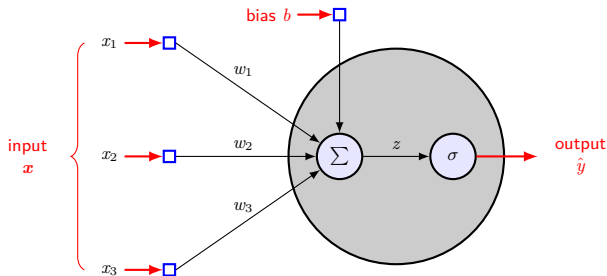
- ▶ (logistic) sigmoid:

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\sigma'(z) = \sigma(z)(1 - \sigma(z))$$



- computational graph (simplified)



$$\hat{y} = \sigma(\underbrace{\mathbf{w}^\top \mathbf{x} + b}_{\triangleq z})$$

- ▶ weighted sum z : “signal”

$$\begin{aligned} z &= \sum_{i=1}^{n_x} w_i x_i + b \\ &= \mathbf{w}^\top \mathbf{x} + b \end{aligned}$$

Probabilistic interpretation

- given training set drawn independently from p_{data}

$$\mathbb{X} = \{(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots, (\mathbf{x}^{(m)}, y^{(m)})\}$$

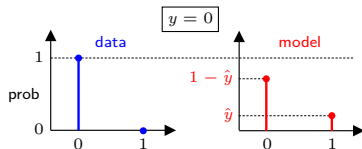
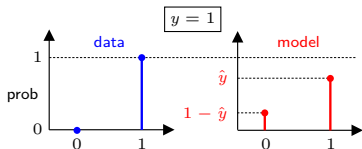
- consider label $y^{(i)} \in \{0, 1\}$ as (hard) probability

$$y \equiv P(y = 1 | \mathbf{x}) \Rightarrow P(y = 0 | \mathbf{x}) = 1 - P(y = 1 | \mathbf{x}) = 1 - y$$

- then

- observed pmf $\hat{p}_{\text{data}} \in \left\{ \underbrace{y}_{P(y=1|\mathbf{x})}, \underbrace{1-y}_{P(y=0|\mathbf{x})} \right\}$ ← training ____

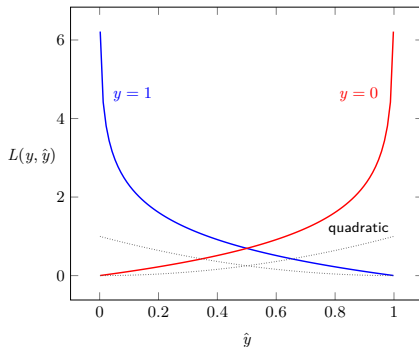
- fitted pmf $p_{\text{model}} \in \left\{ \underbrace{\hat{y}}_{P(y=1|\mathbf{x})}, \underbrace{1-\hat{y}}_{P(y=0|\mathbf{x})} \right\}$ ← predicted by ____



Loss (= pointwise error)

- cross entropy for two pmfs p and q

$$\begin{aligned} H(p, q) &= -\mathbb{E}_p[\log q] \\ &= -\sum_k p(k) \log q(k) \end{aligned}$$



- we use _____ (= logistic loss, cross-entropy loss):

$$L(y, \hat{y}) = -y \log \hat{y} - (1 - y) \log(1 - \hat{y})$$

- ▶ if $y = 1 \Rightarrow L(y, \hat{y}) = -\log \hat{y} \Rightarrow$ want \hat{y} large ($\hat{y} \rightarrow 1$)
- ▶ if $y = 0 \Rightarrow L(y, \hat{y}) = -\log(1 - \hat{y}) \Rightarrow$ want \hat{y} small ($\hat{y} \rightarrow 0$)

Cost function

- simply _____ pointwise loss:

$$\begin{aligned} J(\mathbf{w}, b) &= -\mathbb{E}_{\mathbf{y} \sim \hat{p}_{\text{data}}(\mathbf{y} | \mathbf{x})} \log p_{\text{model}}(\mathbf{y} | \mathbf{x}) \\ &= \frac{1}{m} \sum_{i=1}^m L(\mathbf{y}^{(i)}, \hat{\mathbf{y}}^{(i)}) \\ &= -\frac{1}{m} \sum_{i=1}^m \left[\mathbf{y}^{(i)} \log \hat{\mathbf{y}}^{(i)} + (1 - \mathbf{y}^{(i)}) \log(1 - \hat{\mathbf{y}}^{(i)}) \right] \end{aligned} \quad (1)$$

- bad news: no known closed-form equation to optimize
- good news: this cost function is convex \Rightarrow global minimum exists
- we will show:

$$\begin{aligned} \frac{\partial J}{\partial w_j} &= \frac{1}{m} \sum_{i=1}^m \left(\sigma(\mathbf{w}^\top \mathbf{x}^{(i)} + b) - y^{(i)} \right) x_j^{(i)} \\ \frac{\partial J}{\partial b} &= \frac{1}{m} \sum_{i=1}^m \left(\sigma(\mathbf{w}^\top \mathbf{x}^{(i)} + b) - y^{(i)} \right) \end{aligned}$$

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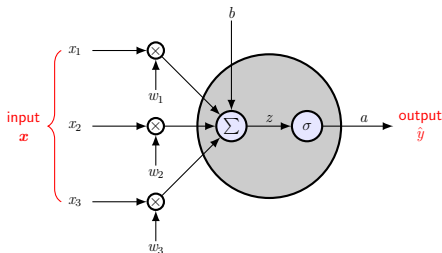
Summary

Training a neuron

- find w and b that minimize cost function $J(w, b)$ in (1)

- ▶ use iterative optimization

i.e. gradient descent



- repeat the following:

1. forward propagation

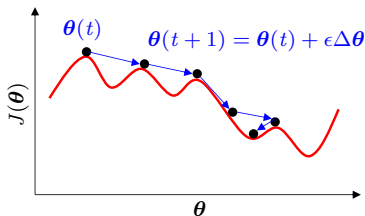
- ▶ pick an example (x, y) and feed x to the neuron
 - ▶ the net returns $\hat{y} \Rightarrow$ generates loss $L(y, \hat{y})$

2. backward propagation ("backprop"): **gradient pump**

- ▶ propagate error $L(y, \hat{y})$ at the output to w and b
 - ▶ update w and b using the propagated ____

Gradient descent

- a general technique for minimizing twice-differentiable function
e.g. cost function $J(\theta)$
- idea: $J(\theta)$ is a 'surface' in parameter space
 - ▶ start from somewhere on J (initial location is critical)
 - ▶ roll down the surface, decreasing J step by step
- two things to decide at each step
 - ▶ which direction? $\Rightarrow \Delta\theta = -\nabla J(\theta)$
 - ▶ how much? $\Rightarrow \epsilon$ ()



- parameter update:

$$\begin{aligned}\theta(t+1) &= \theta(t) + \epsilon \Delta\theta \\ &= \theta(t) - \epsilon \nabla J(\theta)\end{aligned}$$

or

$$\boxed{\theta \leftarrow \theta - \epsilon \nabla J(\theta)}$$

Gradient descent algorithm

algorithm 1 gradient descent

- 1: initialize θ
 - 2: **while** stopping criterion not met **do**
 - 3: sample m examples: $\mathbb{X}_m = \{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(m)}, y^{(m)})\}$
 - 4: compute gradient estimate: $\hat{\mathbf{g}} \leftarrow \frac{1}{m} \nabla_{\theta} \sum_{i=1}^m L(y^{(i)}, \hat{y}^{(i)})$ $\triangleright m$ forward props
 - 5: apply update: $\theta \leftarrow \theta - \epsilon \hat{\mathbf{g}}$ $\triangleright \epsilon$: learning rate
 - 6: **end while**
-

- three variants (N : total number of examples)
 - ▶ $m = 1$: stochastic gradient descent (sgd)
 - ▶ $1 < m < N$: _____ sgd (typical m : 64, 128, 256, 512)
 - ▶ $m = N$: batch gradient descent

Training logistic regression by backprop

- minimize the cost function by gradient descent

$$J(\mathbf{w}, b) = -\frac{1}{m} \sum_{i=1}^m \left[y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right]$$

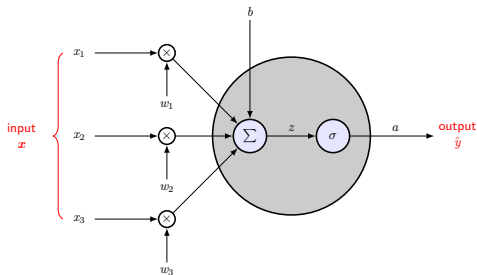
- repeat over training examples

► ϵ : learning rate

$$\mathbf{w} \leftarrow \mathbf{w} - \epsilon \nabla_{\mathbf{w}} J$$

$$b \leftarrow b - \epsilon \nabla_b J$$

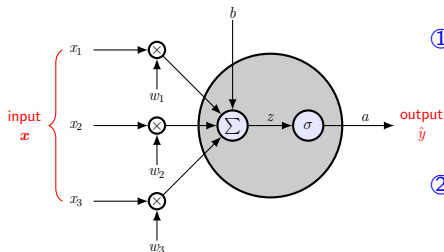
- forward prop:



$$z = \mathbf{w}^\top \mathbf{x} + b$$

$$\hat{y} = \sigma(z) \triangleq a$$

- some math (in the order of information flow):



① activation a

$$\hat{y} = \sigma(\mathbf{w}^\top \mathbf{x} + b) \triangleq a$$

② cost function J

$$\begin{aligned} J &= -y \log \hat{y} - (1 - y) \log(1 - \hat{y}) \\ &= -y \log a - (1 - y) \log(1 - a) \end{aligned}$$

③ gradient at activation a (output)

$$\frac{\partial J}{\partial a} = \frac{\partial J}{\partial \hat{y}} = -\frac{y}{a} + \frac{(1 - y)}{1 - a}$$

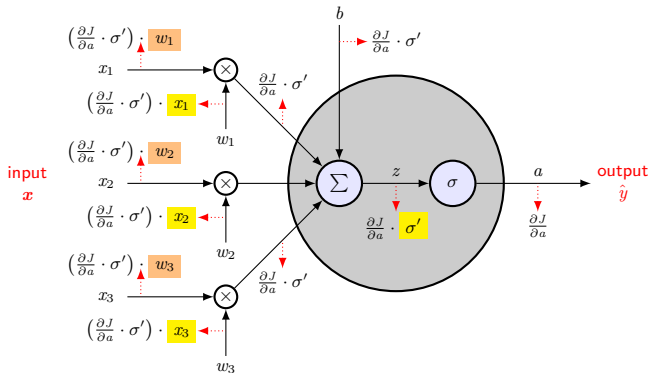
④ local gradient at sigmoid σ

$$\sigma' = \sigma(1 - \sigma) = a(1 - a)$$

⑤ gradient at signal z

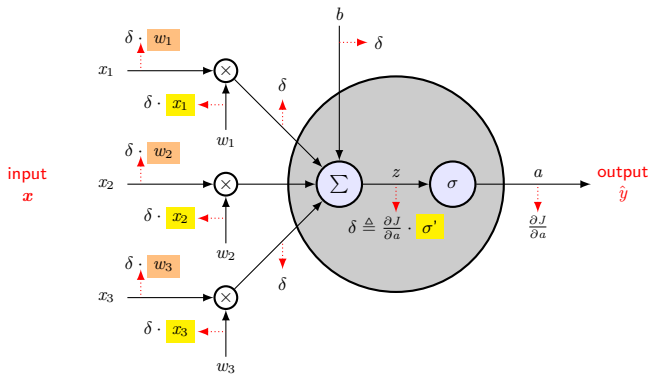
$$\begin{aligned} \frac{\partial J}{\partial z} &= \frac{\partial J}{\partial a} \frac{\partial a}{\partial z} = \frac{\partial J}{\partial a} \cdot \sigma' \\ &= \left(-\frac{y}{a} + \frac{(1 - y)}{1 - a} \right) \cdot a(1 - a) \\ &= a - y \triangleq \delta \quad \text{[“ error”]} \end{aligned}$$

- backprop:



$$\frac{\partial J}{\partial z} = \frac{\partial J}{\partial a} \cdot \sigma' = a - y \triangleq \delta$$

- backprop (simplified):



$$\frac{\partial J}{\partial z} = \frac{\partial J}{\partial a} \cdot \sigma' = a - y \triangleq \delta$$

SGD equations (1 example)

- cost function

$$J(\mathbf{w}, b) = L(\mathbf{y}, \mathbf{a}) = -y \log a - (1 - y) \log(1 - a)$$

- compute gradient:

$$\frac{\partial J}{\partial w_1} = x_1 \cdot \frac{\partial J}{\partial z} = x_1 (\quad) = x_1 \delta$$

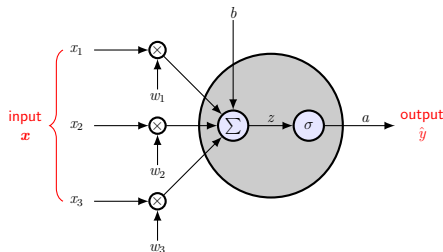
$$\frac{\partial J}{\partial w_2} = x_2 \cdot \frac{\partial J}{\partial z} = x_2 (a - y) = x_2 \delta$$

$$\frac{\partial J}{\partial b} = \frac{\partial J}{\partial z} = (a - y) = \delta$$

- apply update:

$$w_i \leftarrow w_i - \epsilon \frac{\partial J}{\partial w_i} = w_i - \epsilon \cdot x_i \cdot \delta = w_i - \epsilon x_i (a - y)$$

$$b \leftarrow b - \epsilon \frac{\partial J}{\partial b} = b - \epsilon \cdot \delta = b - \epsilon (a - y)$$



Minibatch SGD equations

- cost function

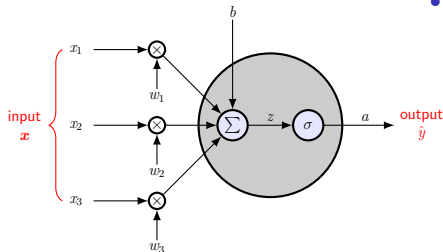
$$J(\mathbf{w}, b) = \frac{1}{m} \sum_{i=1}^m L(y^{(i)}, \mathbf{a}^{(i)})$$

- signal, activation, delta error

$$z^{(i)} = \mathbf{w}^\top \mathbf{x}^{(i)} + b$$

$$a^{(i)} = \sigma(z^{(i)})$$

$$\delta^{(i)} = a^{(i)} - y^{(i)}$$



- compute gradient:

$$\frac{\partial J}{\partial w_1} = \frac{1}{m} \sum_{i=1}^m x_1^{(i)} \delta^{(i)}$$

$$\frac{\partial J}{\partial w_2} = \frac{1}{m} \sum_{i=1}^m x_2^{(i)} \delta^{(i)}$$

$$\frac{\partial J}{\partial b} = \frac{1}{m} \sum_{i=1}^m \delta^{(i)}$$

- apply update:

$$w_1 \leftarrow w_1 - \epsilon \frac{\partial J}{\partial w_1}$$

$$w_2 \leftarrow w_2 - \epsilon \frac{\partial J}{\partial w_2}$$

$$b \leftarrow b - \epsilon \frac{\partial J}{\partial b}$$

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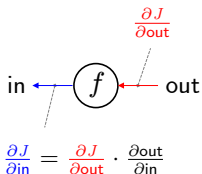
Single-gate backprop

- $\text{out} = f(\text{in})$

forward



backprop

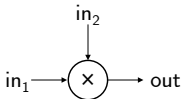


$$\begin{aligned} \frac{\partial J}{\partial \text{in}} &= \underbrace{\frac{\partial J}{\partial \text{out}}}_{\text{output gradient}} \cdot \underbrace{\frac{\partial \text{out}}{\partial \text{in}}}_{\text{local gradient}} \\ &= \frac{\partial J}{\partial \text{out}} \cdot f'(\text{in}) \end{aligned}$$

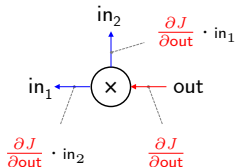
Multiplication

- $\text{out} = \text{in}_1 \cdot \text{in}_2$

forward



backprop



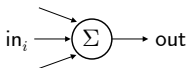
$$\begin{aligned}\frac{\partial J}{\partial \text{in}_1} &= \frac{\partial J}{\partial \text{out}} \cdot \frac{\partial \text{out}}{\partial \text{in}_1} \\ &= \underbrace{\frac{\partial J}{\partial \text{out}}}_{\text{output gradient}} \cdot \underbrace{\text{in}_2}_{\text{local gradient}}\end{aligned}$$

$$\begin{aligned}\frac{\partial J}{\partial \text{in}_2} &= \frac{\partial J}{\partial \text{out}} \cdot \frac{\partial \text{out}}{\partial \text{in}_2} \\ &= \frac{\partial J}{\partial \text{out}} \cdot \text{in}_1\end{aligned}$$

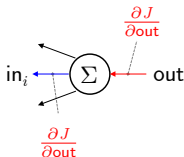
Summation

- $\text{out} = \sum_i \text{in}_i$

forward



backprop



- $\text{sum (forward)} \Leftrightarrow \text{_____ (backprop)}$

$$\begin{aligned}\frac{\partial J}{\partial \text{in}_i} &= \frac{\partial J}{\partial \text{out}} \cdot \frac{\partial \text{out}}{\partial \text{in}_i} \\ &= \underbrace{\frac{\partial J}{\partial \text{out}}}_{\text{output gradient}} \cdot \underbrace{1}_{\text{local gradient}} \\ &= \frac{\partial J}{\partial \text{out}}\end{aligned}$$

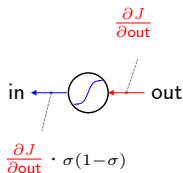
Sigmoid

- $\text{out} = \sigma(\text{in})$

forward



backprop

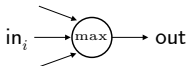


$$\begin{aligned}\frac{\partial J}{\partial \text{in}} &= \frac{\partial J}{\partial \text{out}} \cdot \frac{\partial \text{out}}{\partial \text{in}} \\ &= \underbrace{\frac{\partial J}{\partial \text{out}}}_{\text{output gradient}} \cdot \underbrace{\sigma'(\text{in})}_{\text{local gradient}} \\ &= \frac{\partial J}{\partial \text{out}} \cdot [\sigma(\text{in}) (1 - \sigma(\text{in}))]\end{aligned}$$

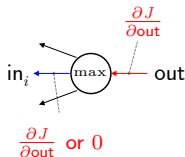
Max

- $\text{out} = \max_i \{\text{in}_i\}$

forward



backprop



$$\frac{\partial J}{\partial \text{in}_i} = \frac{\partial J}{\partial \text{out}} \cdot \underbrace{\frac{\partial \text{out}}{\partial \text{in}}}_{1 \text{ or } 0}$$

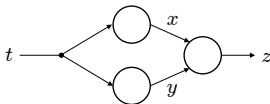
$$= \begin{cases} \frac{\partial J}{\partial \text{out}} & \text{if } \text{in}_i \text{ is max} \\ 0 & \text{otherwise} \end{cases}$$

- $\text{max (forward)} \Leftrightarrow \text{mux (backprop)}$

Backprop through fanout

- multivariable chain rule

forward

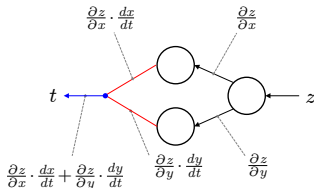


let

$$x = x(t), \quad y = y(t)$$

$$z = f(x, y)$$

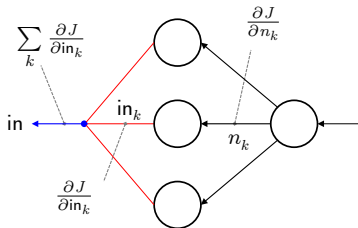
backprop



then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

- fanout



assuming

$$\mathcal{E} = f(n_1, \dots, n_k, \dots)$$

and

$$n_k = n_k(\text{in})$$

gives

$$\begin{aligned} \frac{\partial J}{\partial \text{in}} &= \sum_k \frac{\partial J}{\partial n_k} \cdot \frac{\partial n_k}{\partial \text{in}} \\ &= \sum_k \frac{\partial J}{\partial \text{in}_k} \end{aligned}$$

where

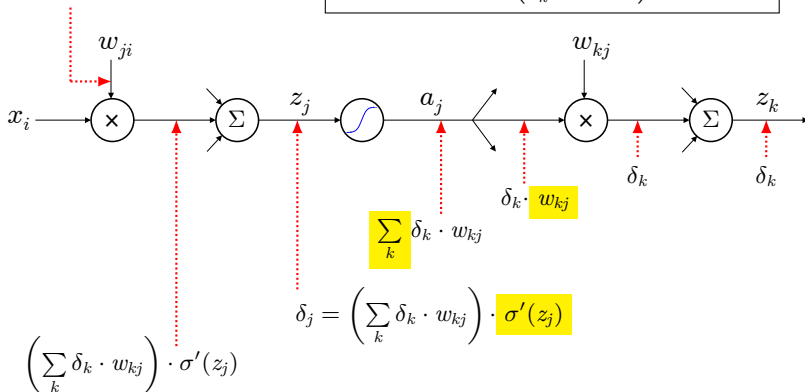
$$\text{in}_k \triangleq \text{input to } n_k$$

- fanout (forward) \Leftrightarrow ____ (backprop)

Example

- computing $\frac{\partial J}{\partial w_{ji}}$

$$\left(\sum_k \delta_k \cdot w_{kj} \right) \cdot \sigma'(z_j) \cdot x_i \Rightarrow \boxed{\frac{\partial J}{\partial w_{ji}} = \delta_j \cdot x_i = \left(\sum_k \delta_k \cdot w_{kj} \right) \cdot \sigma'(z_j) \cdot x_i}$$



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Arranging minibatch

- two options to arrange m examples

- ▶ in **columns**

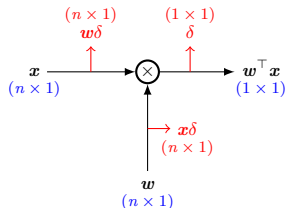
$$\mathbf{X}_{||} = \begin{bmatrix} \left. \begin{array}{c} | \\ \mathbf{x}^{(1)} \\ | \end{array} \right| & \left. \begin{array}{c} | \\ \mathbf{x}^{(2)} \\ | \end{array} \right| & \dots & \left. \begin{array}{c} | \\ \mathbf{x}^{(m)} \\ | \end{array} \right| \end{bmatrix}$$

- ▶ in **rows** (“ ”)

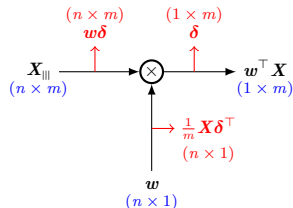
$$\mathbf{X}_{\equiv} = \begin{bmatrix} \text{---} & \mathbf{x}^{(1)\top} & \text{---} \\ \text{---} & \mathbf{x}^{(2)\top} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{x}^{(m)\top} & \text{---} \end{bmatrix}$$

Weighted sum

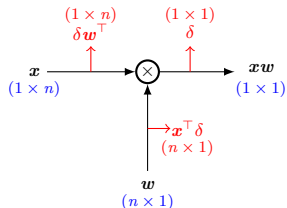
- 1 example (column):



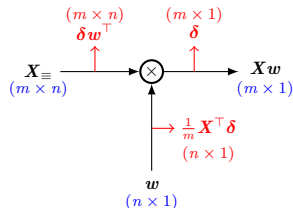
- size- m minibatch (column):



- 1 example (row):

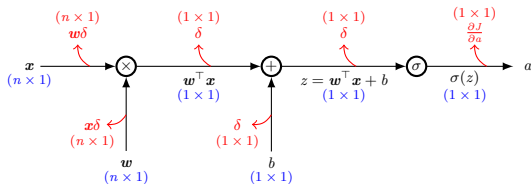


- size- m minibatch (row):

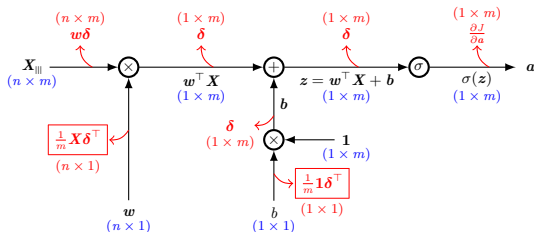


Forward/backward prop (column-wise)

- 1 example:



- size- m minibatch:



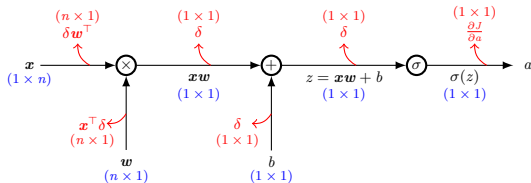
algorithm 2

logistic regression (col)

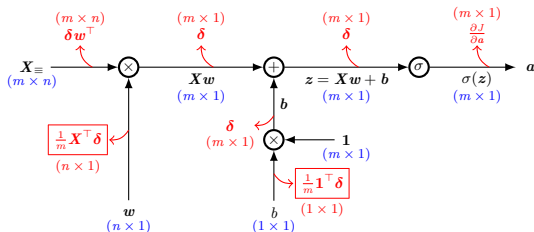
- 1: initialize w, b
 - 2: **while** necessary **do**
 - 3: $z = w^\top X + b$
 - 4: $a = \sigma(z)$
 - 5: $\frac{\partial J}{\partial z} \triangleq \delta = a - y$
 - 6: $\frac{\partial J}{\partial w} = \frac{1}{m} X \delta^\top$
 - 7: $\frac{\partial J}{\partial b} = \frac{1}{m} \mathbf{1} \delta^\top$
 - 8: $w \leftarrow w - \epsilon \frac{\partial J}{\partial w}$
 - 9: $b \leftarrow b - \epsilon \frac{\partial J}{\partial b}$
 - 10: **end while**
 - 11: **return** w, b
-

Forward/backward prop (row-wise)

- 1 example:



- size- m minibatch:



algorithm 3

logistic regression (row)

- 1: initialize w, b
 - 2: **while** necessary **do**
 - 3: $z = Xw + b$
 - 4: $a = \sigma(z)$
 - 5: $\frac{\partial J}{\partial z} \triangleq \delta = a - y$
 - 6: $\frac{\partial J}{\partial w} =$
 - 7: $\frac{\partial J}{\partial b} =$
 - 8: $w \leftarrow w - \epsilon \frac{\partial J}{\partial w}$
 - 9: $b \leftarrow b - \epsilon \frac{\partial J}{\partial b}$
 - 10: **end while**
 - 11: **return** w, b
-

Outline

Introduction

Logistic Regression

Backprop Demystified

Minibatch Processing

Summary

Summary

- neuron: brain cell for information processing
 - ▶ model: synaptic weights, adder, nonlinear activation function
- logistic regression: a linear model to probability estimation
 - ▶ parameterized by weights and bias: $\theta = (w, b)$
 - ▶ used as a neuron model in early neural nets
 - ▶ log loss: $L(y, \hat{y}) = -y \log \hat{y} - (1 - y) \log(1 - \hat{y})$
 - ▶ cost function $J(\theta)$: average loss from training examples
 - ▶ training: iterative optimization (such as gradient descent)
- gradient descent: a general, iterative optimization technique
 - ▶ update equation: $\theta \leftarrow \theta - \epsilon \nabla_{\theta} J(\theta)$
 - ▶ unit of gradient estimation: batch (all), minibatch (m), stochastic (1)
 - ▶ neural nets: gradients are provided by back propagation (backprop)