# 习题七

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2024.3.27

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**Exercise 8**

令$t = x - 2y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -2\frac{\partial u}{\partial t}$$

故

$$2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = 0$$

□

**Exercise 9**

(1)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = f'(r)\frac{y}{r}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} = f'(r)\frac{z}{r}$$

(2)

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} \\ &= g'_x(x,y,z)\cos\varphi\cos\theta + g'_y(x,y,z)\sin\varphi\cos\theta + g'_z(x,y,z)\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial \varphi} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi} \\ &= -g_x'(x,y,z)r\sin\varphi\cos\theta + g_y'(x,y,z)r\cos\varphi\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} \\ &= -g_x'(x,y,z)\cos\varphi\sin\theta - g_y'(x,y,z)r\sin\varphi\sin\theta + g_z'(x,y,z)r\cos\theta\end{aligned}$$

(3) 为防止混淆,记$h(x,\theta)$中的$x$为$p$满足$p = x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = h_x'(x,\theta) + h_\theta'(x,\theta)\frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = -h_\theta'(x,\theta)\frac{4x^2y}{(x^2+y^2)^2}$$

□

**Exercise 10**

正确的答案如下:

我们假设在新的自变量下

$$S = R(T,V)$$

于是

$$\begin{aligned}dU &= \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV \\ &= \frac{\partial U}{\partial S}\left(\frac{\partial S}{\partial T}dT + \frac{\partial S}{\partial V}dV\right) + \frac{\partial U}{\partial V}dV\end{aligned}$$

从而我们有

$$\frac{\partial A}{\partial T} = \frac{\partial (U-TS)}{\partial T} = \frac{\partial U}{\partial S}\frac{\partial S}{\partial T} - S - T\frac{\partial S}{\partial T} = -S$$

$$\frac{\partial A}{\partial V} = \frac{\partial U}{\partial S}\frac{\partial S}{\partial V} + \frac{\partial U}{\partial V} - T\frac{\partial S}{\partial V} = -p$$

□

**Exercise 11**

定义从$\mathbb{R}^2 \to \mathbb{R}^2$的映射

$$\varphi : \begin{cases} p = x - t \\ q = 0 \end{cases}$$

我们记$g(x,t)=u(x,t)-u(\varphi(x,t))$,那么有

$$\begin{aligned}\frac{\partial g}{\partial x}=\frac{\partial u}{\partial x}-\frac{\partial u(\varphi(x,t))}{\partial x}&=\frac{\partial u}{\partial x}-\frac{\partial u(\varphi(x,t))}{\partial p}\frac{\partial p}{\partial x}-\frac{\partial u(\varphi(x,t))}{\partial q}\frac{\partial q}{\partial x}\\&=\frac{\partial u}{\partial x}-\frac{\partial u(\varphi(x,t))}{\partial p}-\frac{\partial u(\varphi(x,t))}{\partial q}\cdot 0\\&=\frac{\partial u}{\partial x}-\frac{\partial u}{\partial p}\\&=0\end{aligned}$$

同理有

$$\begin{aligned}\frac{\partial g}{\partial t}=\frac{\partial u}{\partial t}-\frac{\partial u(\varphi(x,t))}{\partial t}&=\frac{\partial u}{\partial t}-\frac{\partial u(\varphi(x,t))}{\partial p}\frac{\partial p}{\partial t}-\frac{\partial u(\varphi(x,t))}{\partial q}\frac{\partial q}{\partial t}\\&=\frac{\partial u}{\partial t}+\frac{\partial u(\varphi(x,t))}{\partial p}-\frac{\partial u(\varphi(x,t))}{\partial q}\cdot 0\\&=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial p}\\&=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\\&=0\end{aligned}$$

利用有限增量公式的推论,可知$g(x,t)\equiv C$,又因为$g(x,0)=u(x,0)-u(x-0,0)=0$,故

$$g(x,t)\equiv 0$$

即

$$u(x,t)=u(\varphi(x,t))=u(x-t,0)=f(x-t)$$

□

**Exercise 12**

将等式两侧都看出关于$t$的函数,对$t$求导分别得到

$$\frac{df(tx,ty)}{dt}=\frac{\partial f(tx,ty)}{\partial(tx)}\frac{\partial tx}{\partial t}+\frac{\partial f(tx,ty)}{\partial(ty)}\frac{\partial ty}{\partial t}=\frac{\partial f(tx,ty)}{\partial(tx)}x+\frac{\partial f(tx,ty)}{\partial(ty)}y$$

$$\frac{dt^nf(x,y)}{dt}=nt^{n-1}f(x,y)$$

也即

$$\frac{\partial f(tx,ty)}{\partial(tx)}x+\frac{\partial f(tx,ty)}{\partial(ty)}y=nt^{n-1}f(x,y)$$

取$t=1$即可得到

$$\frac{\partial f}{\partial x}x+\frac{\partial f}{\partial y}y=nf$$

$\square$

**Exercise 13**

(1) 首先利用极坐标的关系可以得到

$$\begin{cases}\theta=\arctan\frac{y}{x}\\ r=\sqrt{x^2+y^2}\end{cases}$$

从而有

$$\frac{\partial g}{\partial x}=\frac{\partial g}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x}=f'_r\frac{x}{\sqrt{x^2+y^2}}-f'_\theta\frac{y}{x^2+y^2}$$

同理可得

$$\frac{\partial g}{\partial y}=f'_r\frac{y}{\sqrt{x^2+y^2}}+f'_\theta\frac{x}{x^2+y^2}$$

接下来求二阶偏导数

$$\frac{\partial^2 g}{\partial x\partial x}=f'_{rr}\frac{x^2}{x^2+y^2}-f'_{r\theta}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}-f'_{\theta r}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+f'_{\theta\theta}\frac{y^2}{(x^2+y^2)^2}+f'_r\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}+f'_\theta\frac{2xy}{(x^2+y^2)^2}$$

同理有

$$\frac{\partial^2 g}{\partial y\partial y}=f'_{rr}\frac{y^2}{x^2+y^2}+f'_{r\theta}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+f'_{\theta r}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+f'_{\theta\theta}\frac{x^2}{(x^2+y^2)^2}+f'_r\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}-f'_\theta\frac{2xy}{(x^2+y^2)^2}$$

当$f(r,\theta)=\frac{1}{r}$时,我们有

$$\frac{\partial^2 g}{\partial x\partial x}=f'_{rr}\frac{x^2}{x^2+y^2}+f'_r\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 g}{\partial y\partial y}=f'_{rr}\frac{y^2}{x^2+y^2}+f'_r\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

将两式相加得到

$$\frac{\partial^2 g}{\partial x\partial x}+\frac{\partial^2 g}{\partial y\partial y}=f'_{rr}+f'_r\frac{1}{\sqrt{x^2+y^2}}=\frac{2}{r^3}-\frac{1}{r^2}\frac{1}{\sqrt{x^2+y^2}}$$

(2)

$$\frac{\partial^2 g}{\partial x\partial y}=f'_{rr}\frac{xy}{x^2+y^2}-f'_{r\theta}\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}+f'_{\theta r}\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}-f'_{\theta\theta}\frac{xy}{(x^2+y^2)^2}-f'_r\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+f'_\theta\frac{y^2-x^2}{(x^2+y^2)^2}$$

$\square$

**Exercise 14**

(1)首先先求出一阶偏导数

$$f'_x(x,y) = y\cos(xy) + ye^{xy}$$

$$f'_y(x,y) = x\cos(xy) + xe^{xy}$$

在这基础上求二阶偏导数

$$f'_{xx}(x,y) = -y^2\sin(xy) + y^2e^{xy}$$

$$f'_{xy}(x,y) = \cos(xy) + xy\cos(xy) + e^{xy} + xye^{xy}$$

$$f'_{xy}(x,y) = \cos(xy) + xy\cos(xy) + e^{xy} + xye^{xy}$$

$$f'_{yy}(x,y) = -x^2\sin(xy) + x^2e^{xy}$$

(2)首先先求出一阶偏导数

$$f'_x(x,y) = \frac{1}{x}$$

$$f'_y(x,y) = -\frac{1}{y}$$

在这基础上求二阶偏导数

$$f'_{xx}(x,y) = -\frac{1}{x^2}$$

$$f'_{xy}(x,y) = 0$$

$$f'_{xy}(x,y) = 0$$

$$f'_{yy}(x,y) = \frac{1}{y^2}$$

(3)首先先求出一阶偏导数

$$f'_x(x,y) = -\frac{y}{x^2+y^2}$$

$$f'_y(x,y) = \frac{x}{x^2+y^2}$$

在这基础上求二阶偏导数

$$f'_{xx}(x,y) = \frac{2xy}{(x^2+y^2)^2}$$

$$f'_{xy}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$f'_{xy}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$f'_{yy}(x,y) = -\frac{2xy}{(x^2+y^2)^2}$$

□

**Exercise 15**

$$u'_t = \frac{\partial f(x+ct)}{\partial (x+ct)}\frac{\partial (x+ct)}{\partial t} + \frac{\partial g(x-ct)}{\partial (x-ct)}\frac{\partial (x-ct)}{\partial t} = c\frac{\partial f(x+ct)}{\partial (x+ct)} - c\frac{\partial g(x-ct)}{\partial (x-ct)}$$

同理可以求得二阶偏导数

$$u'_{tt} = c^2\frac{\partial^2 f(x+ct)}{\partial (x+ct)^2} + c^2\frac{\partial^2 g(x-ct)}{\partial (x-ct)^2}$$

同理对于

$$u'_x = \frac{\partial f(x+ct)}{\partial (x+ct)}\frac{\partial (x+ct)}{\partial x} + \frac{\partial g(x-ct)}{\partial (x-ct)}\frac{\partial (x-ct)}{\partial x} = \frac{\partial f(x+ct)}{\partial (x+ct)} + \frac{\partial g(x-ct)}{\partial (x-ct)}$$

从而可以求得二阶偏导数

$$u'_x = \frac{\partial^2 f(x+ct)}{\partial (x+ct)^2} + \frac{\partial^2 g(x-ct)}{\partial (x-ct)^2}$$

故

$$u'_{tt} - c^2 u'_{xx} = 0$$

由上面的式子可列出方程组

$$\begin{cases} \varphi(x) = f(x) + g(x) \\ \psi(x) = cf'(x) - cg'(x) \end{cases}$$

解得

$$\begin{cases} f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int \psi(x)\,dx + C_1 \\ g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int \psi(x)\,dx + C_2 \end{cases}$$

当$\varphi(x) = \cos\pi x, \psi(x) = 0$时,代入可得

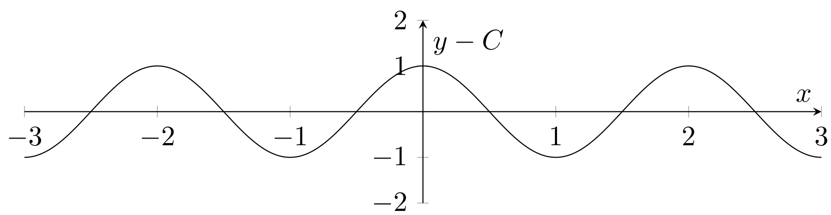
$$\begin{cases} f(x) = \frac{1}{2}\cos\pi x + C_1 \\ g(x) = \frac{1}{2}\cos\pi x + C_2 \end{cases}$$

将上述的$f,g$代入$u(x,t)$,同时取$c=\pi$,同时利用和差化积公式,可得

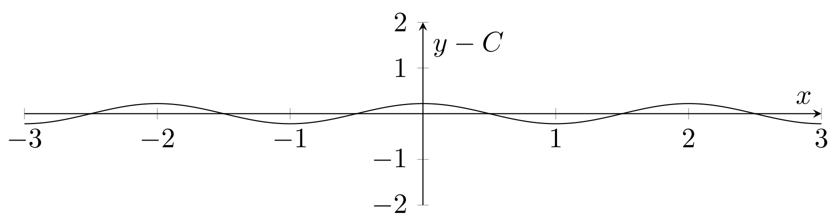
$$u(x,t)=\cos(\pi^2 t)\cos(\pi x)+C$$

以下是草图,注意纵轴的标注为$y-C$,因为这里$C$可以取任意的值

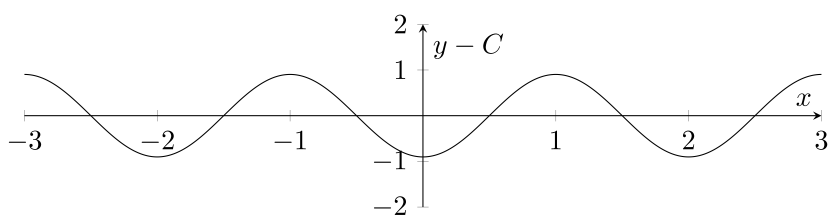
当$t=0$时,此时振幅为1,$x=0$为$1+C$



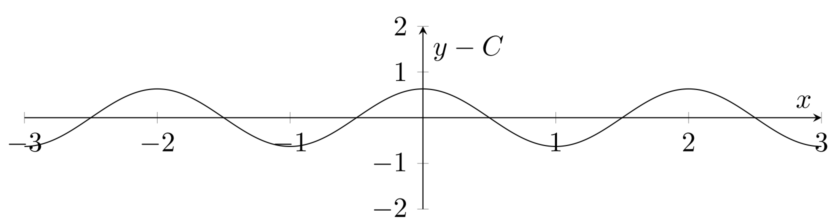

当$t=0.5$时,此时振幅为$\cos(0.5\pi^2)$,$x=0$为$\cos(0.5\pi^2)+C$




当$t=1$时,此时振幅为$-\cos(\pi^2)$,$x=0$为$\cos(\pi^2)+C$




当$t=2$时,此时振幅为$\cos(2\pi^2)$,$x=0$为$\cos(2\pi^2)+C$



$\square$

**Exercise 16**

$$u'_r = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = u'_x \cos\varphi\cos\theta + u'_y \sin\varphi\cos\theta + u'_z \sin\theta$$

$$u'_\varphi = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \varphi} = -u'_x r\sin\varphi\cos\theta + u'_y r\cos\varphi\sin\theta$$

$$u'_\theta = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = -u'_x r\cos\varphi\sin\theta - u'_y r\sin\varphi\sin\theta + u'_z r\cos\theta$$

由于二阶导数部分的项数过多,我们采用矩阵形式表示,

$$u'_{rr} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} \begin{pmatrix} u'_{xx} & u'_{xy} & u'_{xz} \\ u'_{yx} & u'_{yy} & u'_{yz} \\ u'_{zx} & u'_{zy} & u'_{zz} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial r} \end{pmatrix} + \begin{pmatrix} u'_x & u'_y & u'_z \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial r^2} \\ \frac{\partial^2 y}{\partial r^2} \\ \frac{\partial^2 z}{\partial r^2} \end{pmatrix}$$

$$u'_{r\varphi} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} \begin{pmatrix} u'_{xx} & u'_{xy} & u'_{xz} \\ u'_{yx} & u'_{yy} & u'_{yz} \\ u'_{zx} & u'_{zy} & u'_{zz} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \varphi} \end{pmatrix} + \begin{pmatrix} u'_x & u'_y & u'_z \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial r\partial\varphi} \\ \frac{\partial^2 y}{\partial r\partial\varphi} \\ \frac{\partial^2 z}{\partial r\partial\varphi} \end{pmatrix}$$

$$u'_{\theta\theta} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix} \begin{pmatrix} u'_{xx} & u'_{xy} & u'_{xz} \\ u'_{yx} & u'_{yy} & u'_{yz} \\ u'_{zx} & u'_{zy} & u'_{zz} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \theta} \end{pmatrix} + \begin{pmatrix} u'_x & u'_y & u'_z \end{pmatrix} \begin{pmatrix} \frac{\partial^2 x}{\partial\theta\partial\theta} \\ \frac{\partial^2 y}{\partial\theta\partial\theta} \\ \frac{\partial^2 z}{\partial\theta\partial\theta} \end{pmatrix}$$

将已知的偏导数的值代入

$$u'_{rr} = \begin{pmatrix} \cos\varphi\cos\theta & \sin\varphi\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} u'_{xx} & u'_{xy} & u'_{xz} \\ u'_{yx} & u'_{yy} & u'_{yz} \\ u'_{zx} & u'_{zy} & u'_{zz} \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{aligned} u'_{r\varphi} = {}& \begin{pmatrix} \cos\varphi\cos\theta & \sin\varphi\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} u'_{xx} & u'_{xy} & u'_{xz} \\ u'_{yx} & u'_{yy} & u'_{yz} \\ u'_{zx} & u'_{zy} & u'_{zz} \end{pmatrix} \begin{pmatrix} -r\sin\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} u'_x & u'_y & u'_z \end{pmatrix} \begin{pmatrix} -\sin\varphi\cos\theta \\ \cos\varphi\cos\theta \\ 0 \end{pmatrix} \end{aligned}$$

$$u'_{\theta\theta}=\begin{pmatrix}-r\cos\varphi\sin\theta & -r\sin\varphi\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}u'_{xx} & u'_{xy} & u'_{xz}\\ u'_{yx} & u'_{yy} & u'_{yz}\\ u'_{zx} & u'_{zy} & u'_{zz}\end{pmatrix}\begin{pmatrix}-r\cos\varphi\sin\theta\\ -r\sin\varphi\sin\theta\\ r\cos\theta\end{pmatrix}$$
$$+\begin{pmatrix}u'_x & u'_y & u'_z\end{pmatrix}\begin{pmatrix}-r\cos\varphi\cos\theta\\ -r\sin\varphi\cos\theta\\ -r\sin\theta\end{pmatrix}$$

□

**Exercise 17**

(1) 首先我们先来导出$e^{x^2}$的$n$阶导数公式,这里我们利用了$Fa\,di\,Bruno's\,formula$,可以证明有

$$P_n(x)=\frac{n!}{(2x)^n}\sum_{k=\left[\frac{n+1}{2}\right]}^{n}C_n^{n-k}x^{2k}$$

从而我们可以有$(e^{x^2})^{(n)}=\frac{n!}{(2x)^n}\sum_{k=\left[\frac{n+1}{2}\right]}^{n}C_n^{n-k}x^{2k}e^{x^2}$ 由于该函数是任意阶可微的,故任意阶偏导数都可以交换次序,因此我们不妨设对$x$求了$p\geq 0$阶偏导,对$y$求了$n-p$阶偏导,那么我们的结果有

$$\frac{\partial^n f}{\partial x^p\partial y^{n-p}}=\frac{p!}{(2x)^p}\sum_{k=\left[\frac{p+1}{2}\right]}^{p}C_p^{p-k}x^{2k}e^{x^2+y}$$

从而我们可以得到泰勒公式[记$x-x_0=h,y-y_0=k$]

$$f(x,y)=f(x_0,y_0)+\sum_{j=1}^{n}\frac{1}{j!}\sum_{p=0}^{n}C_n^p\frac{p!}{(2x)^p}\sum_{k=\left[\frac{p+1}{2}\right]}^{p}C_p^{p-k}x^{2k}e^{x^2+y}h^pk^{n-p}$$

(2)由于该函数是任意阶可微的,故任意阶偏导数都可以交换次序,因此我们不妨设对$x$求了$p\geq 0$阶偏导,对$y$求了$n-p$阶偏导,那么我们的结果有

$$\frac{\partial^n f}{\partial x^p\partial y^{n-p}}=2^{n-p}\sin(x+2y+\frac{n\pi}{2})$$

我们可以将泰勒公式写作以下形式[记$x-x_0=h,y-y_0=k$]

$$f(x,y)=f(x_0,y_0)+\sum_{j=1}^{n}\frac{1}{j!}\sum_{p=0}^{n}C_n^p2^{n-p}\sin(x+2y+\frac{n\pi}{2})h^pk^{n-p}+o(\Delta x^n)$$

(3)展开成为无穷级数讨论[先处理$\frac{1}{t^2+1}$,之后再考虑$\arctan t$] $\square$

**Exercise 18**

用极坐标的关系可以得到

$$\begin{cases}\theta = \arctan \frac{y}{x} \\ r = \sqrt{x^2+y^2}\end{cases}$$

可以得到

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}, \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}, \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

利用链式法则有

$$f_x = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f}{\partial \theta}\frac{-y}{x^2+y^2} + \frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial f}{\partial \theta}\frac{x}{x^2+y^2} + \frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}}$$

代入我们条件的式子可以得到

$$\sqrt{x^2+y^2}\frac{\partial f}{\partial r} = 0$$

如果我们考虑原点以外的点,那么就有

$$\frac{\partial f}{\partial r} = 0$$

故对于任何一个给定的$\theta_0$,$\varphi(r) = f(\theta_0, r)$的导数为零,利用一元微积分中的有限增量公式可知此函数为常数,因此$f$在极坐标下只是$\theta$的函数. $\square$

**Exercise 19**

(1)

对式子两侧微分可以得到

$$\begin{aligned}&[\cos(x+y-z) - \cos(x-y+z) - \cos(-x+y+z)]dz \\ =&[\cos(x+y-z) + \cos(x-y+z) - \cos(-x+y+z)]dx \\ +&[\cos(x+y-z) - \cos(x-y+z) + \cos(-x+y+z)]dy\end{aligned}$$

从而我们得到

$$\frac{\partial z}{\partial x} = \frac{\cos(x+y-z) + \cos(x-y+z) - \cos(-x+y+z)}{\cos(x+y-z) - \cos(x-y+z) - \cos(-x+y+z)}$$

$$\frac{\partial z}{\partial y}=\frac{\cos(x+y-z)-\cos(x-y+z)+\cos(-x+y+z)}{\cos(x+y-z)-\cos(x-y+z)-\cos(-x+y+z)}$$

在一阶微分式的基础上再次进行微分,可以得到

$$F\cdot(dz)^2+[\cos(x+y-z)-\cos(x-y+z)-\cos(-x+y+z)]d^2z=-F\cdot(dx)^2-F\cdot(dy)^2$$

由于$F(x,y,z)=0$,故$d^2z=0$,那么也就有

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}=0$$

(2)我们对方程两侧微分

$$\frac{dx}{\cos^2(x-z)}+\frac{dy}{\cos^2(y-z)}=\left(\frac{1}{\cos^2(x-z)}+\frac{1}{\cos^2(y-z)}-e^z\right)dz$$

从而有

$$\frac{\partial z}{\partial x}=\frac{-\cos^2(y-z)}{\cos^2(x-z)\cos^2(y-z)e^z-\cos^2(y-z)-\cos^2(x-z)}$$

$$\frac{\partial z}{\partial y}=\frac{-\cos^2(x-z)}{\cos^2(x-z)\cos^2(y-z)e^z-\cos^2(y-z)-\cos^2(x-z)}$$

在一阶微分式的基础上继续微分,即可有

$$\left[e^z+\frac{2\tan(x-z)}{2\cos^2(x-z)}+\frac{\tan(y-z)}{\cos^2(y-z)}\right](dz)^2-2\left[\frac{\tan(x-z)(dx)^2}{\cos^2(x-z)}+\frac{\tan(y-z)(dy)^2}{\cos^2(y-z)}\right]$$
$$=\left(\frac{1}{\cos^2(x-z)}+\frac{1}{\cos^2(y-z)}-e^z\right)d^2z$$

由于各类三角函数写起来较为麻烦,我们这里记$\cos(x-z):=c_x,\cos(y-z):=c_y$,$\tan(x-z):=t_x,\tan(y-z):=t_y$

那么有

$$\frac{\partial^2 z}{\partial x^2}=\frac{e^z c_y^2c_x^2+2t_xc_y^2+2t_xc_y^2}{(c_x^2c_y^2e^z-c_x^2-c_y^2)^3}c_y^4-\frac{2t_xc_y^2}{c_x^2c_y^2e^z-c_x^2-c_y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{e^z c_y^2c_x^2+2t_xc_y^2+2t_xc_y^2}{(c_x^2c_y^2e^z-c_x^2-c_y^2)^3}c_x^4-\frac{2t_yc_x^2}{c_x^2c_y^2e^z-c_x^2-c_y^2}$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{e^z c_y^2c_x^2+2t_xc_y^2+2t_xc_y^2}{(c_x^2c_y^2e^z-c_x^2-c_y^2)^3}c_x^2c_y^2$$

(3)我们对方程两侧进行微分

$$xdx + ydy + zdz - \frac{dx}{\sqrt{x^2-1}} = 0$$

我们可以得到

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2-1}} - x, \frac{\partial z}{\partial y} = -y$$

我们在前面的一阶微分式的基础上再次进行微分,可以得到

$$(dx)^2 + (dy)^2 + (dz)^2 + zd^2z = -\frac{x}{(x^2-1)^{\frac{3}{2}}}(dx)^2$$

我们将前面求得的$dz$代入,即可得到各二阶偏导数

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{z}\left[1 + \frac{x}{(x^2-1)^{\frac{3}{2}}} + \frac{1}{z^2}\left(\frac{1}{\sqrt{x^2-1}} - x\right)^2\right]$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{z}\left[1 + \frac{y^2}{z^2}\right]$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z}\left[\frac{y}{z^2}\left(\frac{1}{\sqrt{x^2-1}} - x\right)\right]$$

□

**Exercise 20**

(1)根据上课时所讲的方法,我们对两个方程两侧进行微分,可以得到

$$\begin{cases} du + 2dv + 3dx + 4dy + 5dz = 0 \\ du + 2vdv + 3x^2dx + 4y^3dy + 5z^4dz = 0 \end{cases}$$

可以解得

$$du = \frac{3v-3x^2}{v-1} + \frac{4v-4y^3}{v-1}dy + \frac{5v-5z^4}{v-1}dz$$

由于我们的两个方程对应的函数显然是连续可微的,因而有

$$\frac{\partial u}{\partial x} = \frac{3v-3x^2}{v-1}$$

我们在之前第一次微分得到的式子的基础上再做一次微分,注意此时$x, y, z$是自变量,因此

$$\begin{cases} d^2u + 2d^2v = 0 \\ d^2u + 2(dv)^2 + 2vd^2v + 6x(dx)^2 + 12y^2(dy)^2 + 20z^3(dz)^2 = 0 \end{cases}$$

消元并移项得到

$$d^2u=-\frac{2}{1-v}(dv)^2-\frac{6x}{1-v}(dx)^2-\frac{12y^2}{1-v}(dy)^2-\frac{20}{1-v}z^3(dz)^2$$

其中的$dv$我们利用第一次的微分式可以得到

$$dv=-\frac{3x^2-3}{2v-2}dx-\frac{4y^3-4}{2v-2}dy-\frac{5z^4-5}{2v-2}dz$$

由于全部展开项数比较多,我们只保留我们要求的项,可以求得

$$\frac{\partial^2 u}{\partial x^2}=\frac{6x}{v-1}+\frac{2}{v-1}\left(\frac{3x^2-3}{2v-2}\right)^2=\frac{6x}{v-1}+\frac{9(x^2-1)^2}{2(v-1)^3}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{2}{v-1}\cdot\frac{3x^2-3}{2v-2}\frac{4y^3-4}{2v-2}=\frac{6(x^2-1)(y^3-1)}{(v-1)^3}$$

(2)由于一阶微分的形式不变性,这里我们的

$$\frac{\partial u}{\partial x}=\frac{3v-3x^2}{v-1}$$

的结果保持不变

但是二阶微分的处理发生了不同,注意我们这里以$x,y,v$为自变量,因而对一阶微分式子在此进行微分可以得到

$$\begin{cases}d^2u+5d^2z=0\\ d^2u+2(dv)^2+6x(dx)^2+12y^2(dy)^2+20z^3(dz)^2+5z^4d^2z=0\end{cases}$$

消元移项可以得到

$$d^2u=\frac{2}{z^4-1}(dv)^2+\frac{6x}{z^4-1}(dx)^2+\frac{12y^2}{z^4-1}(dy)^2+\frac{20z^3}{z^4-1}(dz)^2$$

其中的$dz$可以利用前面的一阶微分式进行消元得到

$$dz=-\frac{2v-2}{5z^4-5}dv-\frac{3x^2-3}{5z^4-5}dx-\frac{4y^3-4}{5z^4-5}dy$$

我们可以求得

$$\frac{\partial^2 u}{\partial x^2}=\frac{6x}{z^4-1}+\frac{20z^3}{z^4-1}\left(\frac{3x^2-3}{5z^4-5}\right)^2=\frac{6x}{z^4-1}+\frac{36(x^2-1)^2z^3}{5(z^4-1)^3}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{20z^3}{z^4-1}\cdot\frac{3x^2-3}{5z^4-5}\frac{4y^3-4}{5z^4-5}=\frac{48(x^2-1)(y^3-1)z^3}{5(z^4-1)^3}$$

□