习题十二

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Exercise 9

若无特殊说明,本题所有小问均采取极坐标变换

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

我们可以求得这个变换的雅可比行列式为r.

(1)

$$\begin{aligned} \iint_{1\le x^2+y^2\le 9} \sin\sqrt{x^2+y^2}\, d(x,y) &= \int_{[1,3]\times[0,2\pi]} r\sin r\, d(r,\theta) \\ &= \int_0^{2\pi} d\theta \int_1^3 r\sin r\, dr \\ &= 2\pi\cdot(-3\cos 3+\sin 3+3\cos 1-\sin 1) \end{aligned}$$

(2)这里改用变量代换

$$\begin{cases} x = 1 + r\cos\theta \\ y = 1 + r\sin\theta \end{cases}$$

可以发现雅可比行列式仍然为r,于是

$$\begin{aligned} \iint_{x^2+y^2\le 2x+2y} (3x+2y)\, d(x,t) &= \int_{[0,\sqrt{2}]\times[0,2\pi]} (5r+r^2(3\cos\theta+2\sin\theta))\, d(r,\theta) \\ &= \int_0^{\sqrt{2}} dr \int_0^{2\pi} (5r+r^2(3\cos\theta+2\sin\theta))\, d\theta \\ &= \int_0^{\sqrt{2}} dr 10\pi r \\ &= 10\pi \end{aligned}$$

(3)

$$\begin{aligned} \iint_{x^2+y^2\le a^2} \tan(x^2+y^2)\, d(x,y) &= \int_{[0,a]\times[0,2\pi]} \tan(r^2) r d(r,\theta) \\ &= \int_0^{2\pi} d\theta \int_0^a \tan(r^2)\, d(r^2) \\ &= 2\pi\cdot(-\frac{1}{2}\ln(\cos r^2))\Big|_0^a \\ &= -\pi\ln(\cos a^2) \end{aligned}$$

(4)

$$\begin{aligned}\iint_{x^2+y^2<1} \frac{1}{\sqrt{1-x^2-y^2}}\, d(x,y) &= \lim_{\delta\to 0+} \int_{[0,1-\delta]\times[0,2\pi]} \frac{rd(r,\theta)}{\sqrt{1-r^2}} \\ &= \int_0^{2\pi} d\theta \lim_{\delta\to 0+} \int_0^{1-\delta} \frac{rdr}{\sqrt{1-r^2}} \\ &= \int_0^{2\pi} d\theta \lim_{\delta\to 0+} (-\sqrt{1-r^2})\Big|_0^{1-\delta} \\ &= 2\pi\end{aligned}$$

□

Exercise 10

(1)利用对称性可知

$$\begin{aligned}\iiint_V (x^3y^2z + xy^3z^2 + x^2yz^3)\, d(x,y,z) &= \iiint_V x^2yz^3\, d(x,y,z) \\ &= \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} x^2yz^3\, dz \\ &= \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \frac{1}{4}x^2y(1-x^2-y^2)^2 \\ &= \int_{-1}^{1} \frac{1}{24}x^2(1-x^2)^3 \\ &= \frac{4}{945}\end{aligned}$$

(2)做变量替换

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

此时得到雅可比行列式为r,积分转化为

$$\begin{aligned}&\int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} dr \int_{r^2}^{\sqrt{1-r^2}} rr^3(\sin^3\theta + \cos^3\theta) + rz^3\, dz \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} dr \int_{r^2}^{\sqrt{1-r^2}} r^4(\sin^3\theta+\cos^3\theta) + \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} dr \int_{r^2}^{\sqrt{1-r^2}} rz^3\, dz \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} dr \int_{r^2}^{\sqrt{1-r^2}} rz^3\, dz \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} rdr \left(\frac{1}{4}z^4\right)\Big|_{r^2}^{\sqrt{1-r^2}} \\ &= \frac{25\sqrt{5}-47}{120}\pi\end{aligned}$$

(3)我们做球坐标变换

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\varphi \end{cases}$$

这个变换的雅可比行列式为$r^2\cos\varphi$,于是原积分转化为

$$\begin{aligned}\lim_{a\to+\infty}\int_{[1,a]\times[0,2\pi]\times[0,\pi]}\frac{r^2\cos\varphi}{r^4}\,d(r,\theta,\varphi)&=\int_0^{2\pi}\,d\theta\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos\varphi\,d\varphi\lim_{a\to+\infty}\int_1^a\frac{1}{r^2}\,dr\\&=4\pi\end{aligned}$$

□

Exercise 11

(1)转换为累次积分

$$\int_0^1\,dz\int_0^{6\sqrt{1-z^2}}\,dy\int_0^{4\sqrt{1-z^2}-\frac{2}{3}y}\,dx=\int_0^1\,dz\int_0^{6\sqrt{1-z^2}}\,dy\left(4\sqrt{1-z^2}-\frac{2}{3}y\right)=8$$

(2)转换为累次积分

$$\int_0^2 dy\int_{-y}^{1-y}(x+y)\,dx=1$$

□

Exercise 12

(1)

$$\begin{aligned}\int_\Gamma(2x+3y)ds&=\int_{AB}(2x+3y)ds+\int_{BC}(2x+3y)ds+\int_{CD}(2x+3y)ds+\int_{DA}(2x+3y)ds\\&=\int_0^2(0+3y)dy+\int_0^3(2x+6)dx+\int_2^0(6+3y)d(-y)+\int_3^0(2x+0)d(-x)\\&=60\end{aligned}$$

(2)记这里的螺旋线为r,则

$$r'(t)=(-2\sin t,2\cos t,1)$$

于是

$$\|r'(t)\|=\sqrt{5}$$

$$\int_\Gamma(x^2+y^2+2z^2)\,ds=\int_0^{3\pi}(4+2t^2)\sqrt{5}\,dt=12\sqrt{5}\pi+18\sqrt{5}\pi^3$$

(3)利用对称性,我们考虑第一象限内的部分,参数化这部分曲线为

$$r(\theta)=(a\cos\theta,b\sin\theta)$$

从而有

$$r'(\theta)=(-a\sin\theta,b\cos\theta)$$

$$\|r'(\theta)\|=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

于是我们的积分就表示为

$$
\begin{aligned}
4\int_0^{\frac{\pi}{2}}\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}{a^2\sin^\theta+b^2\cos^2\theta}d\theta &= 4\int_0^{\frac{\pi}{2}}\frac{\sqrt{b^2-(b^2-a^2)\sin^2\theta}}{a^2-(a^2-b^2)\sin^2\theta}d\theta \\
&= \frac{4b}{a^2}\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-\frac{b^2-a^2}{b^2}\sin^2\theta}}{1-\frac{a^2-b^2}{a^2}\sin^2\theta}d\theta \\
&= \frac{4b}{a^2}\int_0^1\frac{1-\frac{b^2-a^2}{b^2}x^2}{\left(\sqrt{1-x^2}\right)\left(\sqrt{1-\frac{b^2-a^2}{b^2}x^2}\right)\left(1-\frac{a^2-b^2}{a^2}x^2\right)}dx \\
&= \frac{4}{b}\int_0^1\frac{1}{\left(\sqrt{1-x^2}\right)\left(\sqrt{1-\frac{b^2-a^2}{b^2}x^2}\right)}dx \\
&+ \frac{4b}{a^2}\int_0^1\frac{1-\frac{a^2}{b^2}}{\left(\sqrt{1-x^2}\right)\left(\sqrt{1-\frac{b^2-a^2}{b^2}x^2}\right)\left(1-\frac{a^2-b^2}{a^2}x^2\right)}dx
\end{aligned}
$$

这是第二型椭圆积分与第三型椭圆积分的和,对于a,b没有解析形式的积分值. □

Exercise 13

(1)设

$$
\begin{cases}
x=\frac{a}{2}+\frac{a}{2}\cos\theta \\
y=\frac{a}{2}\sin\theta \\
z=z
\end{cases}
$$

其中z满足约束

$$-a\sqrt{\frac{7}{2}-\frac{1}{2}\cos\theta}\le z\le a\sqrt{\frac{7}{2}-\frac{1}{2}\cos\theta}$$

由于我们

$$r_\theta\times r_z=(\frac{a}{2}\cos\theta,0,0)$$

利用对称性,原积分转化为积分

$$
\begin{aligned}
\int_D\frac{a}{2}|\cos\theta|d(\theta,z) &= \int_0^{2\pi}d\theta\int_{-a\sqrt{\frac{7}{2}-\frac{1}{2}\cos\theta}}^{a\sqrt{\frac{7}{2}-\frac{1}{2}\cos\theta}}\frac{a}{2}|\cos\theta|\,dz \\
&= a^2\int_0^{\pi}|\cos\theta|\sqrt{14-2\cos\theta}d\theta
\end{aligned}
$$

推测这个函数也可以转化为椭圆积分的形式.

(2) 利用对称性原积分可化为

$$4\int_{[0,1]\times[0,1]}ududv+1\int_{[0,1]\times[0,1]}1dudv=3$$

(3)设

$$
\begin{cases}
x=a\cos\theta \\
y=a\sin\theta \\
z=a^2
\end{cases}
$$

其中a满足约束条件$0 \le a \le 1$

$$r_\theta \ = (-a\sin\theta, a\cos\theta, 0), r_a = (\cos\theta, \sin\theta, 2a)$$

$$r_\theta \times r_a = (2a^2\cos\theta, -2a^2\sin\theta, -a)$$

$$\|r_\theta \times r_a\| = a\sqrt{4a^2+1}$$

于是原积分化为

$$\int_0^{2\pi} d\theta \int_0^1 a^2\sqrt{4a^2+1}\sin\theta\, da = 0$$

□