习题二

徐海翁

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Exercise 6

不妨令$g(x) = \max\{f(x), 0\}, h(x) = \max\{-f(x), 0\}, G(x) = \int_a^x g(x)\,dx, H(x) = \int_a^x h(x)\,dx$,我们可以验证

$$f(x) = g(x) - h(x)$$

和

$$G'(x) = g(x) \geq 0, H'(x) = h(x) \geq 0$$

从而有

$$F(x) = \int_a^x f(x)\,dx = \int_a^x g(x)\,dx - \int_a^x h(x)\,dx = G(x) - H(x)$$

因此我们证明了想要的结论 □

Exercise 7

将原等式移项得到

$$2f(x) = 2f(a) + \int_a^x f(x)\,dx \tag{1}$$

由于$f(x) \in C[a, b]$,故$\int_a^x f(x)\,dx$在$[a, b]$上可导,从而有$f(x)$在$[a, b]$上可导.

对(1)左右两边求导得到

$$2f'(x) = f(x)$$

即有

$$(\ln(f(x)))' = \frac{1}{2}$$

从而
$$\ln(f(x)) = \frac{1}{2}x + C$$
即
$$f(x) = e^{\frac{x}{2}+C}$$
带入$x = a$时 $f(x) = f(a)$ 则得到
$$f(x) = f(a)e^{\frac{x-a}{2}}$$

□

Exercise 8

$$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{x^2} \sin^2 t\,dt &= \sin^2(x^2)(x^2)' - \sin^2(-x^2)(-x^2)' \\ &= \sin^2(x^2)2x + \sin^2(-x^2)2x \\ &= 4x\sin^2(x^2)\end{aligned}$$

□

Exercise 9

由条件有
$$f(x) = f(x) - f(a) = \int_a^x f'(t)\,dt$$
左右两式均取平方有
$$f^2(x) = \left(\int_a^x f'(t)\,dt\right)^2$$
在两边从a到b积分,得
$$\int_a^b f^2(x)\,dx = \int_a^b \left(\int_a^x f'(t)\,dt\right)^2 dx \tag{2}$$
我们先考虑右式中内层积分,利用积分形式的柯西不等式
$$\begin{aligned}\int_a^x f'(t)\,dt &= \int_a^x (f'(t)\cdot 1)\,dt \\ &\le \left(\int_a^x [f'(t)]^2\,dt\right)^{\frac{1}{2}} \cdot \left(\int_a^x 1\,dt\right)^{\frac{1}{2}} \end{aligned}\tag{3}$$

将(3)中结果代回(2)式,(2)式右侧可化为

$$\begin{aligned}\int_a^b \left(\int_a^x f'(t)\,dt\right)^2 dx &= \int_a^b \left(\left(\int_a^x [f'(t)]^2\,dt\right)\cdot\left(\int_a^x 1\,dt\right)\right) dx \\ &= \int_a^b \left(\left(\int_a^x [f'(t)]^2\,dt\right)(x-a)\right) dx\end{aligned} \tag{4}$$

由于变上限积分$\left(\int_a^x [f'(t)]^2\,dt\right) \geq 0$且$b-a \geq x-a$

$$\int_a^b \left(\left(\int_a^x [f'(t)]^2\,dt\right)(x-a)\right) dx \leq (b-a)\int_a^b \left(\int_a^x [f'(t)]^2\,dt\right) dx$$

由于变上限积分$\left(\int_a^x [f'(t)]^2\,dt\right)$在$[a,b]$上单调递增,从而

$$\int_a^x [f'(t)]^2\,dt \leq \int_a^b [f'(t)]^2\,dt$$

上式中的右侧是一个与常量,我们将其代回(4)式,得到

$$\begin{aligned}(b-a)\int_a^b \left(\int_a^x [f'(t)]^2\,dt\right) dx &\leq (b-a)\int_a^b \left(\int_a^b [f'(t)]^2\,dt\right) dx \\ &= (b-a)^2\int_a^b [f'(t)]^2\,dt \\ &= (b-a)^2\int_a^b [f'(x)]^2\,dx\end{aligned} \tag{5}$$

整理上述式子(2)(3)(4)(5),我们证明了

$$\int_a^b f^2(x)\,dx \leq (b-a)^2\int_a^b [f'(x)]^2\,dx$$

□

Exercise 10

(1)令$g(x)=1$,由第二积分中值定理可得,$\exists\theta\in[0,1]$

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 f(x)g(x)\,dx \\ &= f(0)\int_0^\theta g(x)\,dx + f(1)\int_\theta^1 g(x)\,dx \\ &= \theta f(0) + (1-\theta)f(1)\end{aligned}$$

(2)由(1)中结论可知,存在$p \in [0,1]$使得

$$\int_0^1 f(x)\,dx = pf(0) + (1-p)f(1)$$

且不难利用$f(x)$严格单调下降证明$p \neq 0$.

同样由于$f(x)$严格单调下降,$\int_0^1 f(x)\,dx - f(1) > 0$

我们考虑函数$\varphi(\theta) = 1 + \dfrac{\int_0^1 f(x)\,dx - f(1)}{\theta}$,由前面的符号判断我们可知其在$(0,1]$上单调递减.

这个函数在$(0,1]$是连续函数,并且利用$\varphi(p) = f(0) < c$[这是(1)中式子的变形],$\lim_{\theta\to 0+}\varphi(\theta) = +\infty > c$以及微分中值定理可知,$\exists\theta \in (0,p) \subset [0,1]$使得

$$\int_0^1 f(x)\,dx = \theta c + (1-\theta)f(1)$$

$\square$

Exercise 11

不妨记$I = \int_0^a u(x)\,dx + \int_0^B v(y)\,dy$,利用$v(u(x)) \equiv x$的性质可得

$$\begin{aligned}
I &= \int_0^a u(x)\,dx + \int_0^{v(B)} v(u(x))du(x) \\
&= \int_0^a u(x)\,dx + \int_0^{v(B)} xdu(x) \\
&= \int_0^a u(x)\,dx + (xu(x))|_0^{v(B)} - \int_0^{v(B)} u(x)dx \\
&= \int_0^a u(x)\,dx + Bv(B) - \int_0^{v(B)} u(x)dx \\
&= Bv(B) - \int_a^{v(B)} u(x)\,dx
\end{aligned}$$

由条件$0 \leq B \leq A$和单调性可知$v(B) \leq a$

其中$a = v(B)$时,上式取等号;

$a > v(B)$时, 由于$u(x)$单调性 $u(x) \geq 0$,故

$$\int_a^{v(B)} u(x)\,dx \leq 0$$

从而

$$I = Bv(B) - \int_a^{v(B)} u(x)\,dx \geq Bv(B) \geq aB$$

综上,待求证的式子成立 $\square$