习题九

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2024.3.27

Exercise 27

(1)我们构造拉格朗日乘子函数

$$L = x^2 + y^2 - \lambda(xy - 1)$$

求偏导

$$\begin{cases} \frac{\partial L}{\partial x} = 2x - \lambda y = 0 \\ \frac{\partial L}{\partial y} = 2y - \lambda x = 0 \\ \frac{\partial L}{\partial \lambda} = xy - 1 = 0 \end{cases}$$

我们解这个这个方程组可得

$$\begin{cases} x = 1 \\ y = 1 \\ \lambda = 2 \end{cases} or \begin{cases} x = -1 \\ y = -1 \\ \lambda = 2 \end{cases}$$

我们接下来求二阶偏导,直接写出矩阵

$$\begin{pmatrix} 2 & -\lambda \\ -\lambda & 2 \end{pmatrix}$$

其特征值为$2 - \lambda, 2 + \lambda$.我们没办法直接判断其是否正定.

我们对限制条件在临界点处进行微分,也就得到

$$xdy + ydx = 0$$

在这里就有

$$dx + dy = 0$$

进行参数表达

$$\begin{cases} dx = dt \\ dy = -dt \end{cases}$$

进而可以将矩阵化为

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -\lambda \\ -\lambda & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 4 + 2\lambda = 8$$

这个矩阵是正定的,因此在$(-1,-1)$和$(1,1)$处取得极小值2.

(2) 我们考虑求$g=\ln f=a\ln x+b\ln y+c\ln z$的极值,构建拉格朗日乘子函数

$$L=a\ln x+b\ln y+c\ln z-\lambda(x+y+z-1)$$

求一阶偏导数

$$\begin{cases}\frac{\partial L}{\partial x}=\frac{a}{x}-\lambda=0\\ \frac{\partial L}{\partial y}=\frac{b}{y}-\lambda=0\\ \frac{\partial L}{\partial z}=\frac{c}{z}-\lambda=0\\ \frac{\partial L}{\partial \lambda}=x+y+z-1=0\end{cases}$$

解得

$$\begin{cases}x=\frac{a}{a+b+c}\\ y=\frac{b}{a+b+c}\\ z=\frac{c}{a+b+c}\\ \lambda=a+b+c\end{cases}$$

我们求二阶导得到矩阵

$$\begin{pmatrix}-\frac{a}{x^2} & 0 & 0\\ 0 & -\frac{b}{y^2} & 0\\ 0 & 0 & -\frac{c}{z^2}\end{pmatrix}$$

这个矩阵已经是负定的了,因此我们直接可以判断$(\frac{a}{a+b+c},\frac{b}{a+b+c},\frac{c}{a+b+c})$是我们的极大值点,对应极大值

$$f_{\max}=\left(\frac{a}{a+b+c}\right)^a\left(\frac{b}{a+b+c}\right)^b\left(\frac{c}{a+b+c}\right)^c$$

(3)我们考虑$g=f^2=4x^2+y^2+z^2+4+4xy+4xz-8x+2yz-4y-4z$的条件极值,构建乘子函数

$$L=4x^2+y^2+z^2+4+4xy+4xz-8x+2yz-4y-4z-\lambda(y^2-xz-1)-\mu(z^2-xy-1)$$

求一阶偏导数

$$\frac{\partial L}{\partial x}=8x+4y+4z-8+\lambda z+\mu y=0$$
$$\frac{\partial L}{\partial y}=2y+4x+2z-4-2\lambda y+\mu x=0$$
$$\frac{\partial L}{\partial z}=2z+4x+2y-4+x\lambda-2\mu z=0$$
$$\frac{\partial L}{\partial \lambda}=y^2-xz-1=0$$
$$\frac{\partial L}{\partial \mu}=z^2-xy-1=0$$

可以解得

$$\begin{cases} x = 0 \\ y = 1 \\ z = 1 \\ \lambda = 0 \\ \mu = 0 \end{cases}$$

求二阶偏导数得到矩阵

$$\begin{pmatrix} 8 & 4+\mu & 4+\lambda \\ 4+\mu & 2-2\lambda & 2 \\ 4+\lambda & 2 & 2-2\mu \end{pmatrix} = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

显然这个矩阵有0特征值,因此我们要考虑将变量进行参数表达.

对两个约束关系式在临界点微分得到

$$\begin{cases} 2ydy - xdz - zdx = 0 \\ 2zdz - xdy - ydx = 0 \end{cases} \Rightarrow \begin{cases} 2dy - dx = 0 \\ 2dz - dx = 0 \end{cases}$$

不妨取

$$\begin{cases} dx = 2dt \\ dy = dt \\ dz = dt \end{cases}$$

从而有

$$\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 8 & 4 & 4 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 72$$

我们得到的矩阵是正定的,因此我们在临界点$(0,1,1)$处取到极小值0 $\square$

Exercise 28

不妨设内接点中的一点为(x,y,z),利用题目要求中各面均与坐标平面垂直以及该椭球的对称性可知其他对称点所在位置,这里为简便起见,利用对称性设$x,y,z>0$,我们不难得到体积

$$V = 2x \cdot 2y \cdot 2z = 8xyz$$

由于内接点的约束(设$a,b,c>0$)

$$\frac{x^2}{a^2} + \frac{y^2}{4a^2} + \frac{z^2}{b^2} = 1$$

这个问题利用三元均值不等式可以立刻得出结果,但是这里我们使用条件极值的方式求解.

首先构造拉格朗日乘子函数

$$L(x,y,z,\lambda) = 8xyz - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{4a^2} + \frac{z^2}{b^2} - 1\right)$$

求一阶偏导数

$$\frac{\partial L}{\partial x} = 8yz - \lambda\frac{2x}{a^2} = 0$$
$$\frac{\partial L}{\partial y} = 8xz - \lambda\frac{y}{2a^2} = 0$$
$$\frac{\partial L}{\partial z} = 8xy - \lambda\frac{2z}{b^2} = 0$$
$$\frac{\partial L}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{4a^2} + \frac{z^2}{b^2} - 1 = 0$$

解得

$$\begin{cases} x = \frac{\sqrt{3}}{3}a \\ y = 2\frac{\sqrt{3}}{3}a \\ z = \frac{\sqrt{3}}{3}b \\ \lambda = 8\frac{\sqrt{3}}{3}a^2b \end{cases}$$

之后我们继续考虑二阶偏导数,这里我们直接写出矩阵

$$\begin{pmatrix} \frac{-2\lambda}{a^2} & 8z & 8y \\ 8z & \frac{-\lambda}{2a^2} & 8x \\ 8y & 8x & \frac{-2\lambda}{b^2} \end{pmatrix}$$

我们在这个约束条件上作微分

$$\frac{2xdx}{a^2} + \frac{ydy}{2a^2} + \frac{2zdz}{b^2} = 0$$

代入我们解出的临界点,可化简得到

$$\frac{2}{a}dx + \frac{1}{a}dy + \frac{2}{b}dz = 0$$

进行参数表达

$$\begin{cases} dx = adt_1 \\ dy = 2adt_2 \\ dz = -bdt_1 - bdt_2 \end{cases}$$

于是可以写出

$$\begin{aligned}\begin{pmatrix} a & 0 & -b \\ 0 & 2a & -b \end{pmatrix}\begin{pmatrix} \frac{-2\lambda}{a^2} & 8z & 8y \\ 8z & \frac{-\lambda}{2a^2} & 8x \\ 8y & 8x & \frac{-2\lambda}{b^2} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 2a \\ -b & -b \end{pmatrix} &= \begin{pmatrix} -\frac{64\sqrt{3}}{3}a^2b & -\frac{32\sqrt{3}}{3}a^2b \\ -\frac{32\sqrt{3}}{3}a^2b & -\frac{64\sqrt{3}}{3}a^2b \end{pmatrix} \\ &= -\frac{32\sqrt{3}}{3}a^2b\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\end{aligned}$$

不难发现矩阵

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

的特征值是$1, 3$,故整个矩阵的特征值为

$$-\frac{32\sqrt{3}}{3}a^2b, -32\sqrt{3}a^2b$$

这个矩阵是一个负定的矩阵,从而我们求出的(x, y, z)是极大值点,代入可得

$$V = 8\frac{\sqrt{3}}{3}a \cdot 2\frac{\sqrt{3}}{3}a \cdot \frac{\sqrt{3}}{3}b = \frac{16\sqrt{3}a^2b}{9}$$

□

Exercise 29

我们设切点为(x_0, y_0, z_0),在该点处求梯度可得

$$\nabla F(x_0) = (\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, -\frac{2z_0}{c^2})$$

我们的切平面就满足

$$(\boldsymbol{x} - \boldsymbol{x}_0) \cdot (\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, -\frac{2z_0}{c^2}) = 0$$

$$\frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) - \frac{z_0}{c^2}(z - z_0) = 0$$

□

Exercise 30

题目这里的"曲线"应该改成"曲面".

在(x_0, y_0, z_0)处,我们固定φ不动,对θ求偏导可得

$$x_\theta = -a\sin\theta\sec\varphi$$

$$y_\theta = b\cos\theta\sec\varphi$$

$$z_\theta = 0$$

固定θ不动,求偏导可得

$$x_\varphi = a\cos\theta\frac{\sin\varphi}{\cos^2\varphi}$$

$$y_\varphi = b\sin\theta\frac{\sin\varphi}{\cos^2\varphi}$$

$$z_\varphi = c\frac{1}{\cos^2\varphi}$$

于是我们在(x_0, y_0, z_0)处的切平面就可以表示为

$$r(h, k) = (x_0, y_0, z_0) + h(-a\sin\theta\sec\varphi, b\cos\theta\sec\varphi, 0) + k(a\cos\theta\frac{\sin\varphi}{\cos^2\varphi}, b\sin\theta\frac{\sin\varphi}{\cos^2\varphi}, c\frac{1}{\cos^2\varphi})$$

□

Exercise 31

切线为

$$r'(\theta) = \left(-a\sin\frac{\theta}{2}\left(1+\frac{1}{2}\sec^2\frac{\theta}{2}\right), b\cos\frac{\theta}{2}, \frac{c}{2}\sec^2\frac{\theta}{2}\right)$$

$$s(\theta) = \int_0^\theta \sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2}\, d\theta = \frac{1}{2}\int_0^\theta \sqrt{b^2\cos^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2}\sec^2\frac{\theta}{2} + a^2\sin^2\sec^4\frac{\theta}{2} + c^2\sec^4\frac{\theta}{2}}\, d\theta$$

$$\boldsymbol{T} = \frac{dr/d\theta}{ds/d\theta} = \frac{r'(\theta)}{s'(\theta)} = \frac{\left(-a\sin\frac{\theta}{2}\left(2+\sec^2\frac{\theta}{2}\right), 2b\cos\frac{\theta}{2}, c\sec^2\frac{\theta}{2}\right)}{\sqrt{b^2\cos^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2}\sec^2\frac{\theta}{2} + a^2\sin^2\sec^4\frac{\theta}{2} + c^2\sec^4\frac{\theta}{2}}}$$

我们可以先求出r''

$$r''(\theta) = \left(-\frac{a}{2}\cos\frac{\theta}{2} + \frac{a}{4}\sec\frac{\theta}{2} - \frac{a}{2}\sec^3\frac{\theta}{2}, -\frac{b}{2}\sin\frac{\theta}{2}, \frac{c}{2}\sin\frac{\theta}{2}\sec^3\frac{\theta}{2}\right)$$

$$r' \times r'' = \left(\frac{3}{4}bc\sin\frac{\theta}{2}\sec^2\frac{\theta}{2}, \frac{3}{8}ac(-2\sec\frac{\theta}{2} + \sec^3\frac{\theta}{2}), \frac{3}{4}ab\sec^2\frac{\theta}{2}\right)$$

$$\boldsymbol{N} = \frac{r''}{s'(\theta)} \cdot \frac{\|r'\|^3}{\|r' \times r''\|} = \frac{\|r'\| r''}{\|r' \times r''\|}$$

代入可得(其中很多模长的计算代入范数并不能实际带来任何形式的简化,故保留模长的记号,同时$r''(\theta)$可参见上面未代入)

$$\boldsymbol{N} = \frac{\sqrt{b^2\cos^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2}\sec^2\frac{\theta}{2} + a^2\sin^2\sec^4\frac{\theta}{2} + c^2\sec^4\frac{\theta}{2}}}{\left\|\left(3bc\sin\frac{\theta}{2}\sec^2\frac{\theta}{2}, \frac{3}{2}ac(-2\sec\frac{\theta}{2} + \sec^3\frac{\theta}{2}), 3ab\sec^2\frac{\theta}{2}\right)\right\|} r''(\theta)$$

$$\boldsymbol{B} = \frac{r' \times r''}{\|r' \times r''\|}$$

代入得到

$$\boldsymbol{B} = \frac{\left(\frac{3}{4}bc\sin\frac{\theta}{2}\sec^2\frac{\theta}{2}, \frac{3}{8}ac(-2\sec\frac{\theta}{2} + \sec^3\frac{\theta}{2}), \frac{3}{4}ab\sec^2\frac{\theta}{2}\right)}{\left\|\left(\frac{3}{4}bc\sin\frac{\theta}{2}\sec^2\frac{\theta}{2}, \frac{3}{8}ac(-2\sec\frac{\theta}{2} + \sec^3\frac{\theta}{2}), \frac{3}{4}ab\sec^2\frac{\theta}{2}\right)\right\|}$$

□

Exercise 32

$$r'(t) = (-a\sin t, a\cos t, b)$$

从而有

$$s(t) = \int_0^t \sqrt{a^2+b^2}\, dt = \sqrt{a^2+b^2}t$$

$$r(s) = (a\cos\frac{1}{\sqrt{a^2+b^2}}s, a\sin\frac{1}{\sqrt{a^2+b^2}}s, b\frac{1}{\sqrt{a^2+b^2}}s)$$

$$\dot{r} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin\frac{1}{\sqrt{a^2+b^2}}s, a\cos\frac{1}{\sqrt{a^2+b^2}}s, b) = \boldsymbol{T}$$

$$\ddot{r} = \frac{1}{a^2+b^2}(-a\cos\frac{1}{\sqrt{a^2+b^2}}s, -a\sin\frac{1}{\sqrt{a^2+b^2}}s, 0)$$

$$\boldsymbol{N} = \frac{\ddot{r}}{\|\ddot{r}\|} = (-\frac{a}{|a|}\cos\frac{1}{\sqrt{a^2+b^2}}s, -\frac{a}{|a|}\sin\frac{1}{\sqrt{a^2+b^2}}s, 0)$$

$$\boldsymbol{B} = \boldsymbol{N}\times\boldsymbol{T} = -\frac{a}{|a|}\frac{1}{\sqrt{a^2+b^2}}(b\sin\frac{1}{\sqrt{a^2+b^2}}s, -b\cos\frac{1}{\sqrt{a^2+b^2}}s, a)$$

$$\kappa = \|\ddot{r}\| = \frac{|a|}{a^2+b^2}$$

$$\dddot{r} = \frac{1}{(a^2+b^2)^{\frac{3}{2}}}(a\sin\frac{1}{\sqrt{a^2+b^2}}s, -a\cos\frac{1}{\sqrt{a^2+b^2}}s, 0)$$

$$\tau = \frac{\dddot{r}\cdot(\dot{r}\times\ddot{r})}{\|\ddot{r}\|^2} = \frac{b}{a^2+b^2}$$

$\square$