习题十三

徐海翁

2024.5.29

Exercise 14

(1)转化为

$$\begin{aligned}
&\int_{\overset{\frown}{AB}} y^2dy + \int_{\overset{\frown}{BC}}(2x+6)dx + \int_{\overset{\frown}{CD}}(9+y^2)dy + \int_{\overset{\frown}{DA}} 2xdx \\
&\qquad = \int_0^2 y^2dy + \int_0^3(2x+6)dx + \int_2^0(9+y^2)dy + \int_3^0 2xdx \\
&\qquad = \frac{8}{3} + 27 - \left(18+\frac{8}{3}\right) - 9 \\
&\qquad = 0
\end{aligned}$$

(2)代入螺旋线的参数表示,原积分转化为

$$\int_0^{3\pi}(4+2t^2)(-2\sin t) + (\sin(2\cos t) + e^{2\sin t})dt$$

(3)作代换

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

于是转化为

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{a\cos\theta b\cos\theta - b\sin\theta(-a\sin\theta)}{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta &= ab\int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta} \\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{a^2\cos^2\theta+b^2\sin^2\theta} \\
&= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d(\tan\theta)}{a^2+b^2\tan^2\theta} \\
&= 2ab\int_{-\infty}^{+\infty}\frac{dt}{a^2+b^2t^2} \\
&= 2\left(\arctan(1+\frac{b}{a}t)\right)\Big|_{-\infty}^{+\infty} \\
&= 2\pi
\end{aligned}$$

(4) 作代换

$$\begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

原积分化为

$$\int_{(0,0,0)}^{(1,1,1)} (1,2,3) \times (x,y,z) \cdot (dx,dy,dz) = \int_0^1 t\left[(1,2,3) \times (1,1,1) \cdot (1,1,1)\right] dt = 0$$

□

Exercise 15

(1)考虑参数化为

$$\begin{cases} x = z\cos\theta \\ y = z\sin\theta \\ z = z \end{cases}$$

则我们可将被积函数视作

$$P(x,y,z) = (y,0,z^2)$$

$$\begin{cases} r_z = (\cos\theta, \sin\theta, 1) \\ r_\theta = (-z\sin\theta, z\cos\theta, 0) \\ r_z \times r_\theta = (-z\cos\theta, -z\sin\theta, z) \end{cases}$$

$$\begin{aligned} I &= \int_E P(x,y,z) \cdot dS \\ &= \int_D (z\sin\theta, 0, z^2) \cdot (-z\cos\theta, -z\sin\theta, z) dz d\theta \\ &= \int_0^{2\pi} d\theta \int_1^2 z^3 - z^2 \sin\theta\cos\theta \\ &= \frac{15}{2}\pi \end{aligned}$$

(2)曲面为$x+y+z=1$的限制,不妨取

$$\begin{cases} x = u \\ y = v \\ z = 1-u-v \end{cases}, u \in [0,1], v \in [0,1-u]$$

被积函数为

$$P(x,y,z) = (x,0,0)$$

$$\begin{aligned} I &= \int_S P \cdot n dS \\ &= \int_D (u,0,0) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) d(u,v) \\ &= \int_0^1 du \int_0^{1-u} \frac{\sqrt{3}}{3} u dv \\ &= \frac{\sqrt{3}}{18} \end{aligned}$$

□

Exercise 16

利用对称性,我们只考虑第一象限的部分,我们考虑参数化

$$\begin{cases} x = a^3 \cos^3 \theta \\ y = a^3 \sin^3 \theta \end{cases} ,\theta \in \left[0, \frac{\pi}{2}\right]$$

不难有

$$\begin{cases} dx = -3a^3 \cos^2 \theta \sin \theta d\theta \\ dy = 3a^3 \sin^2 \theta \cos \theta d\theta \end{cases}$$

于是就有

$$S = 4 \int_0^{\frac{\pi}{2}} 3a^6 (\cos^2 \theta + \sin^2 \theta) \sin^2 \theta \cos^2 \theta d\theta = 3a^6 \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta = \frac{3a^6}{8}\pi$$

□