Short, forward rate models and the forward measure

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Introduction

The purpose of short rate models and forward rate models are used to give an expression of $r(t)$ and $f(t,T)$ and then provide a bond price at time t: $P(t,T)$. To price options on bonds, we set up $P(t,T)$ as numeraire and change measure to the forward measure. The forward measure is used to generate the price of options under bonds market by the Black Scholes framework.

1. Short rate models

Suppose we have a short rate satisfies the SDE:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t \tag{1.1}$$

and let

$$P(t,T) = F(t, r(t); T) \tag{1.2}$$

such that F is a smooth function in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ and the boundary condition is $F(T, r(T); T) = 1$ for any value of $r(T)$.

If there exists a self-financing portfolio (ϕ_t^0, ϕ_t^1) where ϕ_t^0 is the amount invests in bank account and ϕ_t^1 is the amount invests in a bond matures at T_2. We want to use the self-financing portfolio to replicate the bond with maturity T_1 at $T_3 < T_1 < T_2$. By existence of the self-financing portfolio, we have

$$P(T_3, T_1) = \phi_{T_3}^0 e^{\int_0^{T_3} r(s)ds} + \phi_{T_3}^1 P(T_3, T_2) \tag{1.3}$$

as the portfolio can replicate the bond price at time T_3.

If there is no arbitrage, at any time $t \leq T_3$, the value of the portfolio should be the same as the price of bond mature at T_1. That is for all $t \leq T_3$:

$$dP(t, T_1) = r(t)\phi_t^0 S_t^0 dt + \phi_t^1 dP(t, T_2) \tag{1.4}$$

where $S_t^0 := e^{\int_0^t r(s)ds}$.

If we apply Itô formula to $P(t, T_1) = F(t, r(t); T_1)$:

$$
\begin{aligned}
dF(t,r(t);T_1) &= \frac{\partial F^{(1)}}{\partial t}dt + \frac{\partial F^{(1)}}{\partial r}dr(t) + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2 dt \\
&= r(t)\phi_t^0 S_t^0 dt + \phi_t^1 \frac{\partial F^{(2)}}{\partial t}dt + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}dr(t) + \frac{1}{2}\phi_t^1 \frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2 dt \\
&= \left\{ r(t)\phi_t^0 S_t^0 + \phi_t^1 \frac{\partial F^{(2)}}{\partial t} + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\mu + \phi_t^1 \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2 \right\} dt + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\sigma dW_t
\end{aligned} \tag{1.5}
$$

where denote $F^{(1)} := F(t,r(t);T_1)$ and $F^{(2)} := F(t,r(t);T_2)$. Then if equate dW_t and dt term:

$$
\begin{gathered}
\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2}\sigma^2 = r(t)\phi_t^0 S_t^0 + \phi_t^1 \frac{\partial F^{(2)}}{\partial t} + \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\mu + \phi_t^1 \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2}\sigma^2 \\
\sigma \frac{\partial F^{(1)}}{\partial r} = \phi_t^1 \frac{\partial F^{(2)}}{\partial r}\sigma
\end{gathered} \tag{1.6}
$$

And thus

$$
\begin{aligned}
\phi_t^1 &= \frac{\partial F^{(1)}/\partial r}{\partial F^{(2)}/\partial r} \\
r\phi_t^0 S_t^0 &= r(F^{(1)} - \frac{\partial F^{(1)}/\partial r}{\partial F^{(2)}/\partial r}F^{(2)})
\end{aligned} \tag{1.7}
$$

When substitude (1.7) into (1.6), we have

$$
\frac{1}{\partial F^{(1)}/\partial r}(\frac{\partial F^{(1)}}{\partial t} + \frac{\partial F^{(1)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial r^2} - rF^{(1)}) = \frac{1}{\partial F^{(2)}/\partial r}(\frac{\partial F^{(2)}}{\partial t} + \frac{\partial F^{(2)}}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial r^2} - rF^{(2)}) \tag{1.8}
$$

The expression (1.8) holds for any $T_1, T_2 < T$. Then it means that there exists a process $\lambda(t,r(t))$ such that

$$
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF = \lambda\sigma\frac{\partial F}{\partial r} \tag{1.9}
$$

Since here $\lambda(t,r(t))$ is a process, we do not have determined price $P(t,T)$ of the bonds under measure P.

Now suppose we have a risk-neutral measure P^* such that P^* is equivalent to P. If two probability measures are equivalent, they agree on the probability that some events happened with probability 0 and some other events happened with probability 1. The following proposition proved that under risk-neutral measure P^*, we can have solution to (1.9) and thus the bond price $P(t,T)$ is determined under P^*.

Proposition 1. Let P^* be equivalent to P such that

$$
\frac{dP^*}{dP} = exp\left\{ -\int_0^T \lambda(s,r(s))dW_s - \frac{1}{2}\int_0^T \lambda^2(s,r(s))ds \right\} \tag{1.10}
$$

and then $F(t,r(t),T) = \mathbb{E}_{P^*}(e^{-\int_t^T r(s)ds}|\mathcal{F}_t)$, the bond with maturity at T has a price at time t is equal to the expected discounted value between time t and T under risk neutral measure P^*. The price is a solution to (1.9) with boundary condition $F(T,r(T);T) = 1$ and under P^*, $dr(t) = (\mu - \lambda\sigma)dt + \sigma d\tilde{W}_t$ with $\tilde{W}(\mathcal{F}_t)$ being a P^* Brownian motion.

Proof. The proof of Proposition 1 requires the Girsanov theorem.

Suppose P^* is equivalent to P such that

$$\frac{dP^*}{dP} = exp\left\{ -\int_0^T \lambda(s,r)dW_s - \frac{1}{2}\int_0^T \lambda^2(s,r)ds \right\}$$

And the Novikov condition $\mathbb{E}(exp\frac{1}{2}\int_0^T \lambda^2(s,r(s))) < \infty$ is satisfied. Then by the Girsanov theorem, we have

$$\tilde{W_t} = W_t + \int_0^t \lambda(s,r(s))ds$$

is $\mathcal{F}_t$ Brownian motion with respect to P^*.

If we apply It$\hat{o}$ formula to the discounted process $e^{-\int_0^t r(s)ds}F(t,r(t),T)$ and get the expression of the discounted process:

$$\begin{aligned} e^{-\int_0^t r(s)ds}F(t,r(t),T) &= F(0,r(0);T) + \int_0^t e^{\int_0^s r(u)du}(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF)ds + \int_0^t e^{-\int_0^s r(u)du}\frac{\partial F}{\partial r}\sigma dW_s \\ &= F(0,r(0);T) + \int_0^t e^{\int_0^s r(u)du}(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r})ds + \int_0^t e^{-\int_0^s r(u)du}\frac{\partial F}{\partial r}\sigma d\tilde{W}_s \end{aligned}$$

We get the last equality from changing the brownian motion under P to brownian motion under P^* by subtracting the $\lambda\sigma\frac{\partial F}{\partial r}$ in the second term.

And we have second equation such that under risk neutral measure

$$e^{-\int_0^t r(s)ds}F(t,r(t);T) = e^{-\int_0^t r(s)ds}\mathbb{E}_{P^*}(e^{-\int_t^T r(s)ds}|\mathcal{F}_t) = \mathbb{E}_{P^*}(e^{-\int_0^T r(s)ds}|\mathcal{F}_t)$$

If apply Martingale Representation Theorem, the second term would be zero, which proves that the

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2 - rF - \lambda\sigma\frac{\partial F}{\partial r} = 0$$

and thus it proved that there is determinisitic bond prices for any time t under measure P^*. □

2. Forward rate models

One of the drawback in short rate models is the difficulty of capturing term structure at initial time. Alternatively, we could use forward rate models and connect to short rate by the relation $r(t) = f(t,t)$. Denote

$$P(t,T) = exp\{-\int_t^T f(t,s)ds\}$$

Suppose under P^*, we have

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}_t \text{ , T} \geq 0 \tag{2.1}$$

with

$$f(0,T) = \hat{f}(0,T)$$

Let $X_t := -\int_t^T f(t,s)ds$, then $P(t,T) = e^{X_t}$. And if we express dX_t:

$$\begin{aligned}
dX_t &= f(t,t)dt - \int_t^T df(t,s)ds \\
&= f(t,t)dt - \int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)d\tilde{W}_t ds \\
&= (f(t,t) - \int_t^T \alpha(t,s)ds)dt - (\int_t^T \sigma(t,s)ds)d\tilde{W}_t
\end{aligned} \tag{2.2}$$

where the last equality is from stochastic Fubini theorem. Then:

$$\begin{aligned}
dlnP(t,T) &= \frac{dP(t,T)}{P(t,T)} \\
&= dX_t + \frac{1}{2}d < X >_t \\
&= (f(t,t) - \int_t^T \alpha(t,s)ds)dt - (\int_t^T \sigma(t,s)ds)dW_t + \frac{1}{2}(\int_t^T \sigma(t,s)ds)^2 dt \\
&= \{f(t,t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}(\int_t^T \sigma(t,s)ds)^2\}dt - (\int_t^T \sigma(t,s)ds)d\tilde{W}_t
\end{aligned} \tag{2.3}$$

Remind that $P(t,T) = \mathbb{E}_{P^*}(e^{-\int_t^T r(s)ds}|\mathcal{F}_t)$ under measure P^*. And suppose there exists an adapted process $(q_t)_{0\leq t\leq T}$ such that $\forall t \in [0,T]$, $Z_t = exp\{\int_0^t q(s)dW_s - \frac{1}{2}\int_0^t q^2(s)ds\}$. Then we could show that $\forall 0 \leq t \leq u$, $\frac{dP(t,u)}{P(t,u)} = (r(t) - \sigma_t^u q((t))dt + \sigma_t^u dW_t$, where $(\sigma_t^u)_{0\leq t\leq u}$ is an adapted process. And thus $\frac{dP(t,u)}{P(t,u)} = r(t)dt + \sigma_t^u d\tilde{W}_t$ where $\tilde{W}_t$ is brownian motion under P^* by applying Girsanov Theorem. Note that we have $r(t) = f(t,t)$ and thus $-\int_t^T \alpha(t,s)ds + \frac{1}{2}(\int_t^T \sigma(t,s)ds)^2 = 0$. We could solve for $\alpha(t,T) = (\int_t^T \sigma(t,s)ds)\sigma(t,T)$ and write (2.1) as

$$df(t,T) = \sigma(t,T)(\int_t^T \sigma(t,s)ds)dt + \sigma(t,T)d\tilde{W}_t \tag{2.4}$$

The (2.4) depends on $\sigma(t,s)$ only and we eliminated the drift term.

3. The forward measure

The forward measure is used when pricing the options in the bond market. When P^* is the risk neutral probability measure, it is a probability such that $(\tilde{P}(t,T))_{0\leq t\leq T}$ are martingales for any T. If we take the value of the bond with maturity at T as numeraire and under the numeraire, the bond with another maturity date $\tilde{T} > T$: $U_{T,\tilde{T}}(t) := \frac{P(t,\tilde{T})}{P(t,T)}$ are martingales for any $\tilde{T} > T$ under the forward measure P^T. Also, let $(S_t)_{0\leq t\leq T}$ be an asset and P^S be the probability that makes $(\frac{V_t}{S_t})_{0\leq t\leq T}$ a martingale where $(V_t)_{0\leq t\leq T}$ is a self-financing portfolio. Then the price of a call option with maturity T of the asset S and strike K could be proved as

$$\Pi(t;S) = S_t\mathbb{P}^S(S_T \geq K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^T(S_T \geq K|\mathcal{F}_t)$$

and if S is another bond with maturity $\tilde{T} > T$ the option with maturity at T on this bond (with maturity at $\tilde{T}$) can be priced by:

$$
\begin{aligned}
\Pi(t;S) &= P(t,\tilde{T})\mathbb{P}^{\tilde{T}}(P(T,\tilde{T}) \geq K|\mathcal{F}_t) - KP(t,T)\mathbb{P}^{T}(P(T,\tilde{T}) \geq K|\mathcal{F}_t) \\
&= P(t,\tilde{T})\mathbb{P}^{\tilde{T}}(\frac{P(T,T)}{P(T,\tilde{T})} \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)\mathbb{P}^{T}(\frac{P(T,\tilde{T})}{P(T,T)} \geq K|\mathcal{F}_t)
\end{aligned}
\tag{3.1}
$$

where $P(T,T) = 1$ and define $U(t,T,\tilde{T} := \frac{P(t,T)}{P(t,\tilde{T})}) = exp\{-A(t,\tilde{T}) + A(t,T) + (B(t,\tilde{T}) - B(t,T))r_t\}$. Here the last equality comes from the affine structures. Now if under measure P^*, we can get

$$
\frac{dU(t)}{U(t)} = dlnU(t) = dexp\{-A(t,\tilde{T}) + A(t,T) + (B(t,\tilde{T}) - B(t,T))r_t\} = (...)dt + (B(t,\tilde{T}) - B(t,T))\sigma_t d\tilde{W}_t
$$

where $\tilde{W}_t$ is a brownian motion under P^*

And if we apply Girsanov theorem and under $P^{\tilde{T}}$ and P^T, we have:

$$
\begin{aligned}
dU(t) &= U(t)(B(t,\tilde{T}) - B(t,T))\sigma_t dW_t^{\tilde{T}} \\
dU^{-1}(t) &= -U^{-1}(t)(B(t,\tilde{T}) - B(t,T))\sigma_t dW_t^{T}
\end{aligned}
\tag{3.2}
$$

where $W_t^{\tilde{T}}$ is a brownian motion under $P^{\tilde{T}}$ and W_t^T is a brownian motion under P^T.

By (3.2), we could solve for

$$
\begin{aligned}
U(T) &= \frac{P(t,T)}{P(t,\tilde{T})}exp\{-\int_t^T \sigma_{\tilde{T},T}(s)dW_s^{\tilde{T}} - \frac{1}{2}\int_t^T \sigma^2_{\tilde{T},T}(s)ds\} \\
U^{-1}(T) &= \frac{P(t,\tilde{T})}{P(t,T)}exp\{\int_t^T \sigma_{\tilde{T},T}(s)dW_s^{T} - \frac{1}{2}\int_t^T \sigma^2_{\tilde{T},T}(s)ds\}
\end{aligned}
$$

with $\sigma_{\tilde{T},T}(t) = -(B(t,\tilde{T}) - B(t,T))\sigma_t$. When σ_t is deterministic, the $logU(T)|\mathcal{F}_t$ and $logU^{-1}(T)|\mathcal{F}_t$ are gaussian with respect to P^T and $P^{\tilde{T}}$ with variance $\Sigma^2_{t,T,\tilde{T}}$. Then standardised $logU(T)|\mathcal{F}_t$ and $logU^{-1}(T)|\mathcal{F}_t$ by subtracting their expected value and divided by their standard deviation, to have standard normal distributions under P^T and $P^{\tilde{T}}$. Finally, we can get the pricing formula:

$$
\begin{aligned}
\Pi(t;S) &= P(t,\tilde{T})\mathbb{P}^{\tilde{T}}(U(T) \leq \frac{1}{K}|\mathcal{F}_t) - KP(t,T)\mathbb{P}^{T}(U^{-1}(T) \geq K|\mathcal{F}_t) \\
&= P(t,\tilde{T})\Phi(d_+) - KP(t,T)\Phi(d_-)
\end{aligned}
\tag{3.3}
$$

where $d_\pm = \frac{log(\frac{P(t,\tilde{T})}{KP(t,T)}) \pm \frac{1}{2}\Sigma^2_{t,T,\tilde{T}}}{\Sigma_{t,T,\tilde{T}}}$. The $\Phi(.)$ in (3.3) is standard normal cdf function. And the last equality comes from the fact that the $\mathbb{P}^{\tilde{T}}(.)$ and $\mathbb{P}^T(.)$ is standard normal cdf after standardising $logU(T)|\mathcal{F}_t$ and $logU^{-1}(T)|\mathcal{F}_t$.