Lab5

March 22, 2022

1 Lab 5. Extreme Eigenvalues and Spiked Model.

1.0.1 Due date: Friday 03/04 at 10:59 pm

[2]:
```
!pip install powerlaw
```

```
Defaulting to user installation because normal site-packages is not writeable
Collecting powerlaw
  Downloading powerlaw-1.5-py3-none-any.whl (24 kB)
Requirement already satisfied: matplotlib in /opt/conda/lib/python3.9/site-
packages (from powerlaw) (3.4.2)
Requirement already satisfied: numpy in /opt/conda/lib/python3.9/site-packages
(from powerlaw) (1.21.1)
Requirement already satisfied: mpmath in /opt/conda/lib/python3.9/site-packages
(from powerlaw) (1.2.1)
Requirement already satisfied: scipy in /opt/conda/lib/python3.9/site-packages
(from powerlaw) (1.7.0)
Requirement already satisfied: pillow>=6.2.0 in /opt/conda/lib/python3.9/site-
packages (from matplotlib->powerlaw) (8.3.1)
Requirement already satisfied: pyparsing>=2.2.1 in
/opt/conda/lib/python3.9/site-packages (from matplotlib->powerlaw) (2.4.7)
Requirement already satisfied: python-dateutil>=2.7 in
/opt/conda/lib/python3.9/site-packages (from matplotlib->powerlaw) (2.8.2)
Requirement already satisfied: cycler>=0.10 in /opt/conda/lib/python3.9/site-
packages (from matplotlib->powerlaw) (0.10.0)
Requirement already satisfied: kiwisolver>=1.0.1 in
/opt/conda/lib/python3.9/site-packages (from matplotlib->powerlaw) (1.3.1)
Requirement already satisfied: six in /opt/conda/lib/python3.9/site-packages
(from cycler>=0.10->matplotlib->powerlaw) (1.16.0)
Installing collected packages: powerlaw
Successfully installed powerlaw-1.5
```

[3]:
```
import matplotlib.pyplot as plt
import numpy as np
import powerlaw
import scipy.linalg
import scipy.sparse.linalg
import scipy.special
```

```python
from scipy import stats
pi = np.pi
```

1.1 Part 1. Extreme Eigenvalues of Covariance Matrix

Theorem (Bai and Yin, 1993). Let X_n denote a $p \times n$ random matrix whose entries are independent identically distributed random variables with mean 0 and variance 1. Define the $p \times p$ Wishart matrix

$$Y_n = \frac{1}{n} X_n X_n^T \in \mathbb{R}^{p \times p}.$$

Denote the largest and smallest eigenvalue of Y_n by $\lambda_+^{(n)}$ and $\lambda_-^{(n)}$,respectively. Additionally, we assume that the entries $(X_n)_{ij}$ of the (random noise) feature matrix have finite 4th moments, as $n, p \to \infty$ with $\gamma = \lim_{n\to\infty}(p/n) < 1$, we have

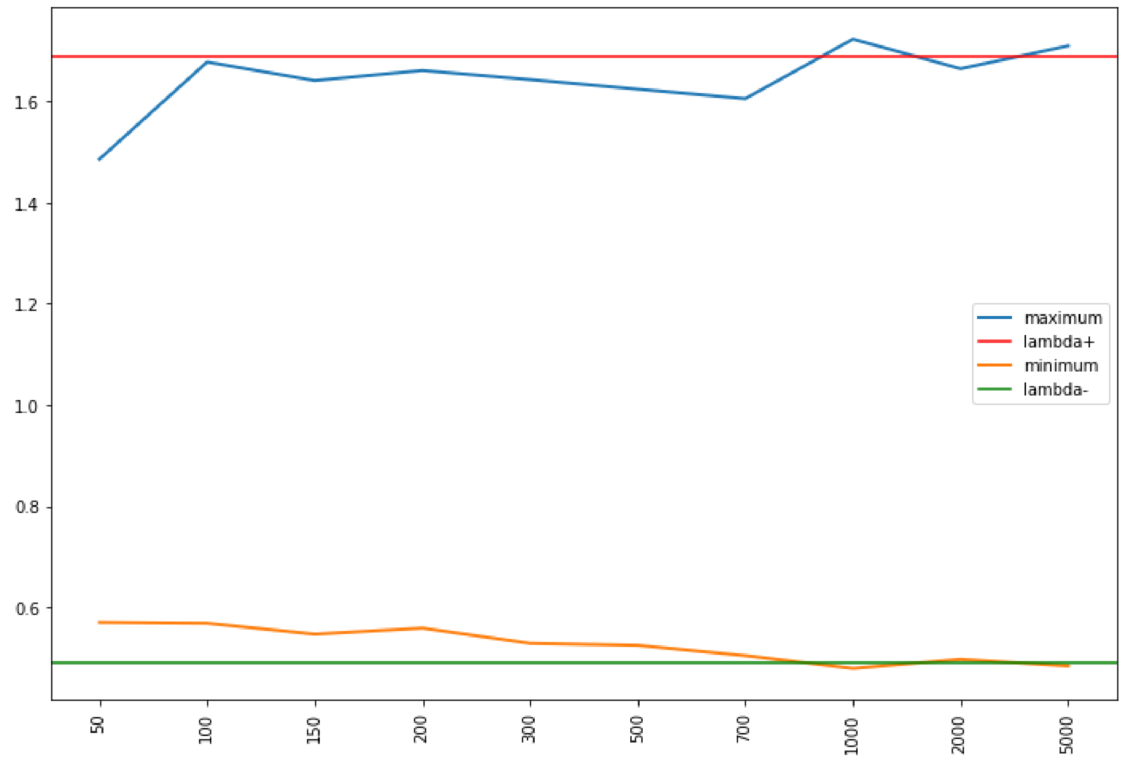
$$\lambda_+^{(n)} \xrightarrow{a.s.} (1+\sqrt{\gamma})^2, \quad \lambda_-^{(n)} \xrightarrow{a.s.} (1-\sqrt{\gamma})^2,$$

That is, the largest and smallest eigenvalues converge almost surely to the edges of the support of the Marcenko–Pastur density. * Remark. Actually, we have a more precise description, Tracy–Widom law, to characterize the behaviour of extreme eigenvalues.

[32]:
```python
gamma = 0.09
N = np.array([50, 100, 150, 200, 300, 500, 700, 1000, 2000, 5000])
P = gamma*N
P = P.astype(int)

eig_max = np.array([])
eig_min = np.array([])

for i in range(len(N)):
    X = np.random.normal(0,1,(P[i],N[i]))
    Y = X@(X.T)/N[i]
    eigvals, eigvecs = np.linalg.eigh(Y)
    eig_max = np.append(eig_max, np.max(eigvals))

    """
    TODO: 1. append the minimum eigenvalue to eig_min
    """
    ### BEGIN SOLUTION
    eig_min = np.append(eig_min, np.min(eigvals))
    ### END SOLUTION
```

[33]:
```python
plt.figure(figsize=(12,8))
plt.plot(np.arange(len(eig_max)), eig_max, lw=2, label='maximum')
plt.axhline(y=(1 + np.sqrt(gamma))**2, color='r', linestyle='-',␣
 ↪label='lambda+')
plt.plot(np.arange(len(eig_min)), eig_min, lw=2, label='minimum')
"""
TODO: 2. plot the horizontal line of (1 - sqrt(gamma))^2
```

```
"""
### BEGIN SOLUTION
plt.axhline(y=(1 - np.sqrt(gamma))**2, color='g', linestyle='-',␣
 ↪label='lambda-')
### END SOLUTION

plt.xticks(np.arange(len(eig_max)), N, rotation='vertical')
_ = plt.legend()
```



Existence of 4th Moments In previous theorem, we we assume that the entries $(X_n)_{ij}$ have finite 4th moments. Then we will have the convergence of the largest and smallest eigenvalue as $n, p \to \infty$ with $\gamma = \lim_{n\to\infty}(p/n) < 1$. The existence of 4th Moments does matter.

Let's consider the `powerlaw` distribution instead with probability density function

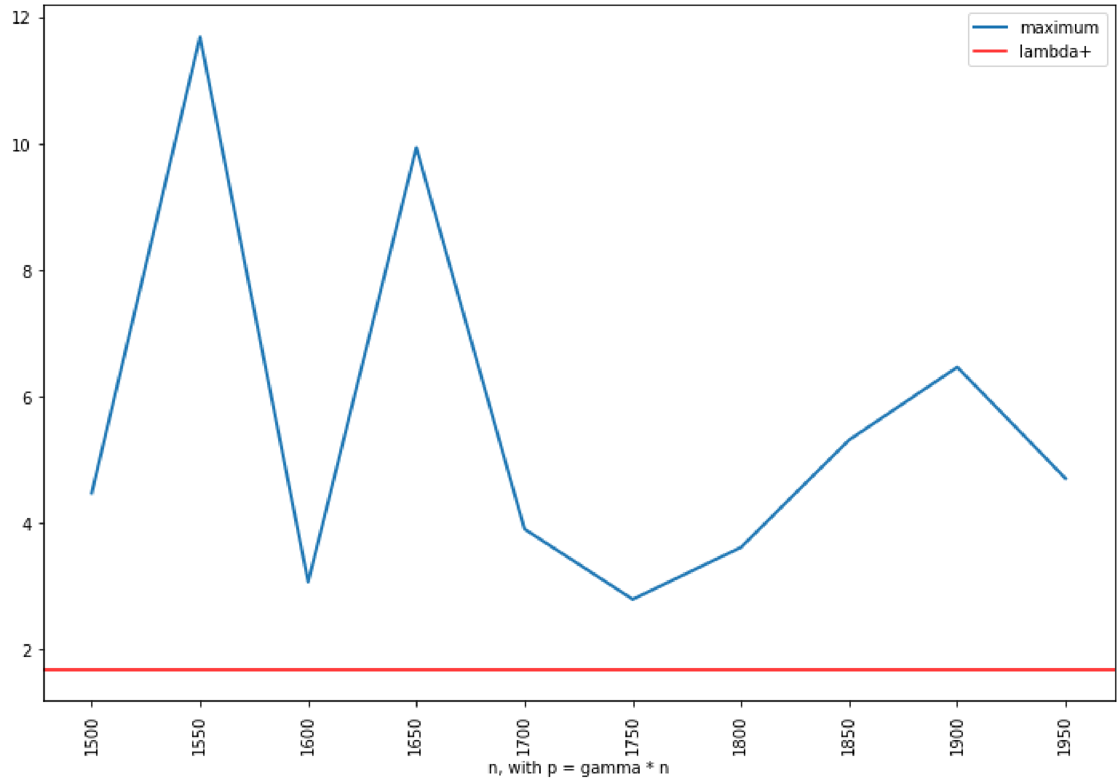
$$f(x) = 3 * x^{-4}\mathbf{1}_{\{x\geq 1\}}(x)\,,$$

where the first moment $\mathbb{E}[X] = \int xf(x)dx = 3/2$ and second moment $\mathbb{E}[X^2] = \int x^2 f(x)dx = 3$, while the fourth moment is not finite. The theorem doesn't apply anymore. That is, we don't expect the largest and the smallest eigenvalues to converge to the edge of the support of the Marcenko–Pastur density.

[23]:
```
gamma = 0.09
N = np.arange(1500, 2000, 50)
```

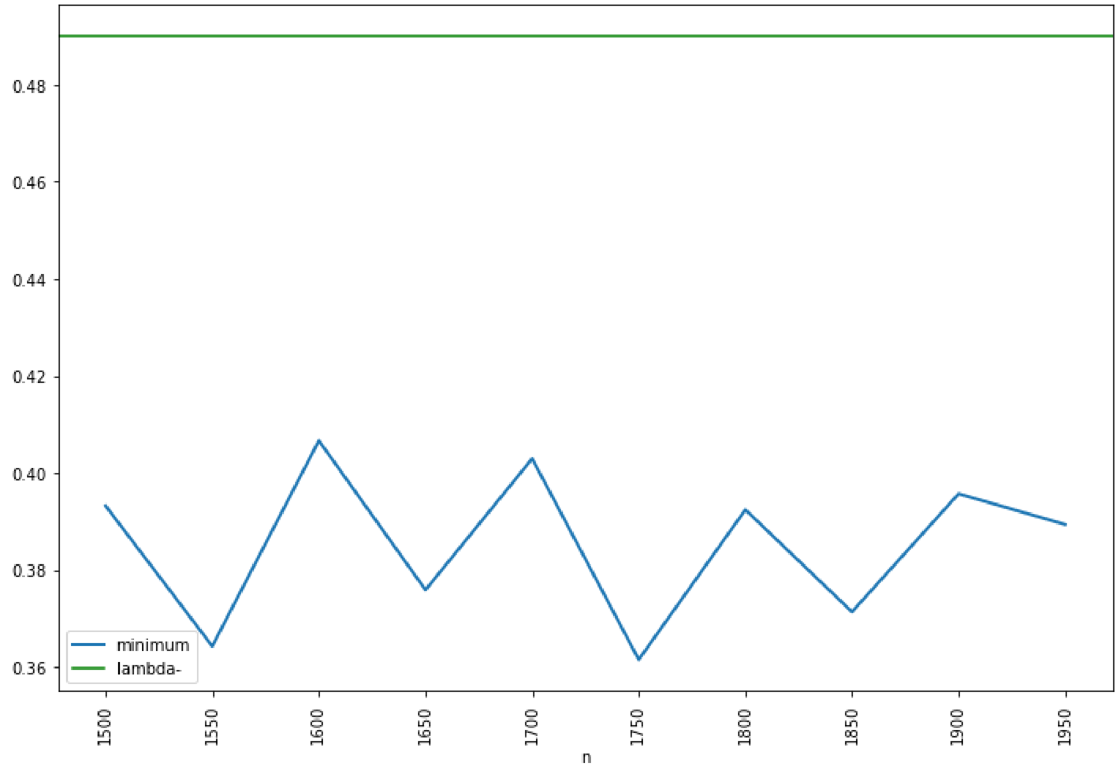
```python
P = gamma*N
P = P.astype(int)

beta = 4

# define powerlaw distribution
powerlaw_distribution = powerlaw.Power_Law(xmin=1., parameters=[beta],␣
↪discrete=False)

eig_max = np.array([])
eig_min = np.array([])

for i in range(len(N)):
    samples = powerlaw_distribution.generate_random(P[i]*N[i])
    samples = (samples - np.mean(samples))/np.sqrt(np.var(samples)) ␣
↪#standardise the data
    X = np.array(samples).reshape((P[i], N[i]))
    Y = X@(X.T)/N[i]
    eigvals, eigvecs = np.linalg.eigh(Y)
    eig_max = np.append(eig_max, np.max(eigvals))
    eig_min = np.append(eig_min, np.min(eigvals))
```

[30]:
```python
plt.figure(figsize=(12,8))
plt.plot(np.arange(len(eig_max)), eig_max, lw=2, label='maximum')
plt.axhline(y=(1 + np.sqrt(gamma))**2, color='r', linestyle='-',␣
↪label='lambda+')
plt.xticks(np.arange(len(eig_max)), N, rotation='vertical')
plt.xlabel('n')
_ = plt.legend()
```



[31]:

```
plt.figure(figsize=(12,8))
plt.plot(np.arange(len(eig_min)), eig_min, lw=2, label='minimum')
plt.axhline(y=(1 - np.sqrt(gamma))**2, color='g', linestyle='-',␣
 ↪label='lambda-')
plt.xticks(np.arange(len(eig_min)), N, rotation='vertical')
plt.xlabel('n')
_ = plt.legend()
```



1.2 Part 2. Spiked Covariance Models

Theorem. Let X_n denote a $p \times n$ random matrix whose entries are independent identically distributed random variables with mean 0 and variance 1. Let $C = I_n + D_r$ be a rank-r diagonal perturbation of the identity matrix with fixed r. Define the $p \times p$ `spiked Wishart matrix`

$$Y_n = \frac{1}{n} X_n C X_n^T \in \mathbb{R}^{p \times p}.$$

and let $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_p^{(n)}$ denote the eigenvalues of Y_n (viewed as random variables). Finally, consider the random measure

$$\mu_p(A) = \frac{1}{p} \# \left\{ \lambda_j^{(n)} \in A \right\}, \quad A \subset \mathbb{R}.$$

Assume that $p, n \to \infty$ so that the ratio $p/n \to \gamma \in (0, +\infty)$. Then $\mu_p \to \mu$ in distribution, where

$$\mu(A) = \begin{cases} (1 - \frac{1}{\gamma}) \mathbf{1}_{0 \in A} + \nu(A), & \text{if } \gamma > 1, \\ \nu(A), & \text{if } 0 \leq \gamma \leq 1, \end{cases}$$

and

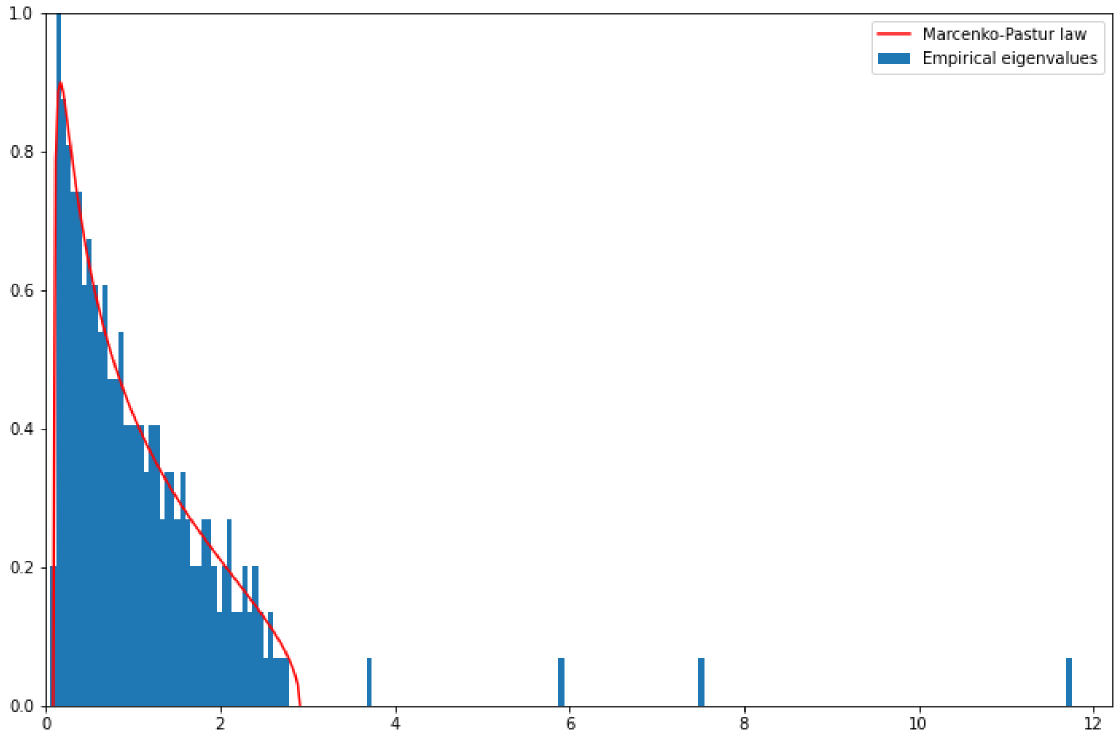
$$d\nu(x) = \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi\gamma x} \mathbf{1}_{x \in [\lambda_-, \lambda_+]} \, dx$$

with $\lambda_{\pm} = (1 \pm \sqrt{\lambda})^2$.

- Remark. As far as the limit histogram of eigenvalues is concerned, spiked models made no difference. The limit of the proportion of eigenvalues in a given interval tells us nothing about the extreme eigenvalues. In first part, the Theorem(Bai and Yin) proved that, in the non-spiked case ($C = I_n$), the largest eigenvalue sticks to the edges of the Marcenko–Pastur density, that is, all eigenvalues are inside the interval $[(1-\sqrt{\gamma})^2, (1+\sqrt{\gamma})^2]$. However, this fails for spiked covariance models($C = I_n + D_r$): they have `outlier eigenvalues`.

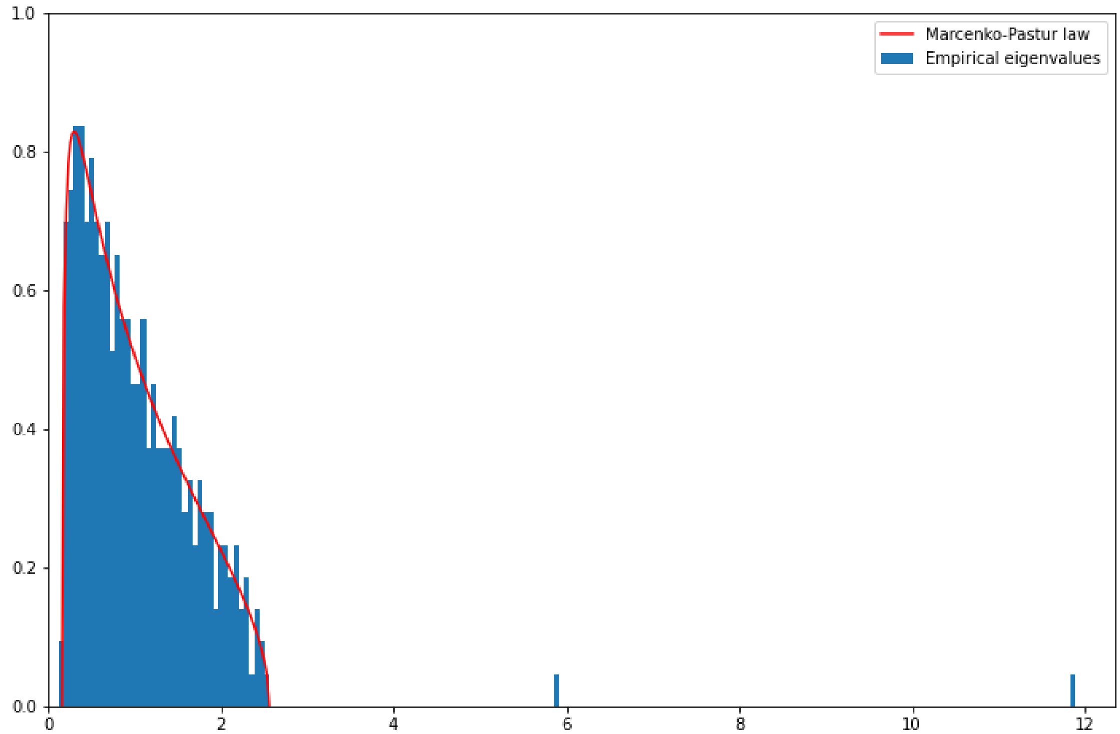
[34]:
```
r = 4
gamma = 0.5
n = 500
p = int(gamma*n)

X = np.array(np.random.randn(p,n))
eig_D = np.array([4.2, 8.7, 11.8, 20.9])
D = np.diag(np.concatenate([eig_D, np.zeros(n-len(eig_D))]))

"""
TODO: 3. Compute C = I_n + D and Y = XCX.T/n
"""
### BEGIN SOLUTION
C = np.eye(n) + D
Y = X@C@(X.T)/n
### END SOLUTION
eigen_Y = np.sort(np.linalg.eigh(Y)[0])
```

[37]:
```
edges = np.linspace(np.min(eigen_Y) - 0.1, np.max(eigen_Y) + 0.1, 200)

a = (1 - np.sqrt(gamma))**2
b = (1 + np.sqrt(gamma))**2

plt.figure(figsize=(12,8))
plt.hist(eigen_Y, bins=edges,weights=1/(p*(edges[1]-edges[0]))*np.
 ↪ones(p),label='Empirical eigenvalues')
edges_MP = np.linspace(a,b,100)
mu = np.sqrt((edges_MP-a)*(b-edges_MP) )/(2*pi*gamma)
plt.plot(edges_MP,mu/edges_MP,'r',label='Marcenko-Pastur law')
plt.gca().set_xlim(0,np.max(eigen_Y)+0.5)
plt.gca().set_ylim(0,1)
_ = plt.legend()
```



[39]:
```
"""
TODO: 4. Following the procedure above, try another example by setting gamma =␣
↪0.36, n = 1000
        and eig_D = np.array([13, 27])
"""
### BEGIN SOLUTION
gamma = 0.36
n = 1000
p = int(gamma*n)

eig_D = np.array([13, 27])
D = np.diag(np.concatenate([eig_D, np.zeros(n-len(eig_D))]))
C = np.eye(n) + D

X = np.array(np.random.randn(p,n))
Y = X@C@(X.T)/n
eigen_Y = np.sort(np.linalg.eigh(Y)[0])

edges = np.linspace(np.min(eigen_Y) - 0.1, np.max(eigen_Y) + 0.1, 200)
a = (1 - np.sqrt(gamma))**2
b = (1 + np.sqrt(gamma))**2

plt.figure(figsize=(12,8))
```

```
plt.hist(eigen_Y, bins=edges,weights=1/(p*(edges[1]-edges[0]))*np.
→ones(p),label='Empirical eigenvalues')
edges_MP = np.linspace(a,b,100)
mu = np.sqrt((edges_MP-a)*(b-edges_MP) )/(2*pi*gamma)
plt.plot(edges_MP,mu/edges_MP,'r',label='Marcenko-Pastur law')
plt.gca().set_xlim(0,np.max(eigen_Y)+0.5)
plt.gca().set_ylim(0,1)
_ = plt.legend()

### END SOLUTION
```



1.3 Submission Instructions

1.3.1 Download Code Portion

- Restart the kernel and run all the cells to make sure your code works.
- Save your notebook using File > Save and Checkpoint.
- Use File > Downland as > PDF via Latex.
- Download the PDF file and confirm that none of your work is missing or cut off.
- **DO NOT** simply take pictures using your phone.

1.3.2 Submitting

- Submit the assignment to Lab1 on Gradescope.

- **Make sure to assign only the pages with your implementation to the question.**