Design and Analysis of Algorithms
Dynamic Programming (II)

1. Chain Matrix Multiplication

2. Optimal Binary Search Tree

Chain Matrix Multiplication (矩阵链相乘)

Motivation. Suppose we want to multiply several matrices. This will involve iteratively multiplying two matrices at a time.

- Matrix multiplication is not *commutative* (in general $A \times B \neq B \times A$), but it is *associative*:

$$A \times (B \times C) = (A \times B) \times C$$

- We can compute product of matrices in many different ways, depending on how we parenthesize it.

Are some of these better than others?

Complexity of $C_{ik} = A_{ij} \times B_{jk}$

- Each element in C requires j multiplications, totally ik elements $\Rightarrow$ overall complexity $\Theta(ijk)$

Example

Suppose we want to multiply four matrices, $A \times B \times C \times D$, of dimensions 50×20, 20×1, 1×10, and 10×100, respectively.

Parenthesize	Computation	Cost
$A \times ((B \times C) \times D)$	$20 \cdot 1 \cdot 10 + 20 \cdot 10 \cdot 100 + 50 \cdot 20 \cdot 100$	$120,200$
$(A \times (B \times C)) \times D$	$20 \cdot 1 \cdot 10 + 50 \cdot 20 \cdot 10 + 50 \cdot 10 \cdot 100$	$60,200$
$(A \times B) \times (C \times D)$	$50 \cdot 20 \cdot 1 + 1 \cdot 10 \cdot 100 + 50 \cdot 1 \cdot 100$	$7,000$

The order of multiplication order makes a big difference in the final complexity.

Natural greedy approach of always perform the cheapest matrix multiplication available may not always yield optimal solution

- see second parenthesization as a counterexample

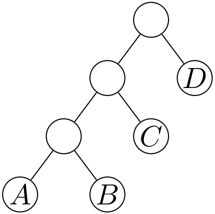
Brute Force Algorithm

Q. How many different parenthesization methods (add brackets) for $A_1 A_2 \dots A_n$?

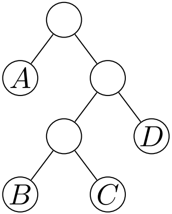
Observation. A particular parenthesiation can be represented naturally by a *full* binary tree

- individual matrices corresponds to leaves
- the root is the final product
- interior nodes are intermediate products

$((A \times B) \times C) \times D$



$A \times ((B \times C) \times D)$



Estimate the Size

The number of possible orders correspond to various full binary trees with n leaves.

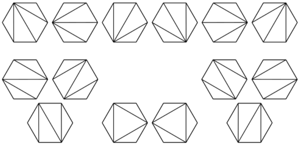
$$C(1) = 1, C(2) = 1, C(3) = C(1)C(2) + C(2)C(1)$$
$$C(4) = C(1)C(3) + C(2)C(2) + C(2)C(1)$$
$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i} = \frac{1}{n+1}\binom{2n}{n}$$

The formula is of convolution form, can be calculated via generate function.

- The result is known as Catalan number, which is exponential in n

Catalan Number

Catalan number (named after the Belgian mathematician Eugène Charles Catalan). First discovered by Euler when counting the number of different ways of dividing a convex polygon with $n+2$ sides into triangles.



$$
\begin{aligned}
C(n) =& \Omega\left(\frac{1}{n+1}\frac{(2n)!}{n!n!}\right) \quad \text{//Stirling formula} \\
=& \Omega\left(\frac{1}{n+1}\frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi 2n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi 2n}\left(\frac{n}{e}\right)^{n}}\right) = \Omega(4^n/(n^{3/2}\sqrt{\pi}))
\end{aligned}
$$

- Occur in various counting problems: number of parenthesis methods, full binary tree, monotonic lattice paths (often involving recursively-defined objects)

Dynamic Programming

The binary tree is suggestive: for a tree to be optimal, its subtrees must be also be optimal $\rightsquigarrow$ satisfy the optimal substructure

- subproblems corresponding to the subtrees: products of the form $A_i \times A_{i+1} \times \cdots A_j$

Optimized function:

$$C(i,j) = \text{minimum cost of multiplying } A_i \times A_{i+1} \times \cdots A_j$$

$$\text{the corresponding dimension is } m_{i-1}, m_i, \ldots, m_j$$

Iteration relation:

$$\underline{C(i,j)} = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ \underline{C(i,k)} + \underline{C(k+1,j)} + m_{i-1} m_k m_j \} & i < j \end{cases}$$

A_i	$\ldots$	A_k	A_{k+1}	$\ldots$	A_j
	$m_{i-1} \times m_k$			$m_k \times m_j$	

Recursive Approach

Algorithm 1: MChain(C, i, j) // subproblem $i - j$

$C(i,i) = 0$, $C(i,j) \leftarrow \infty$;
$s(i,j) \leftarrow \perp$ //record split position;
for $k \leftarrow i$ *to* $j - 1$ **do**
    $t \leftarrow$ MChain(C, i, k) + MChain$(C, k+1, j) + m_{i-1} m_k m_j$;
    if $t < C(i,j)$ **then** //find better solution
        $C(i,j) \leftarrow t$;
        $s(i,j) \leftarrow k$;
    end
end
return $C(i,j)$;

Complexity Analysis

Recurrence relation is:

$$T(n) = \begin{cases} O(1) & n = 1 \\ \sum_{k=1}^{n-1}(T(k) + T(n-k) + \underline{O(1)}) & n > 1 \end{cases}$$

- $O(1)$: sum and compare

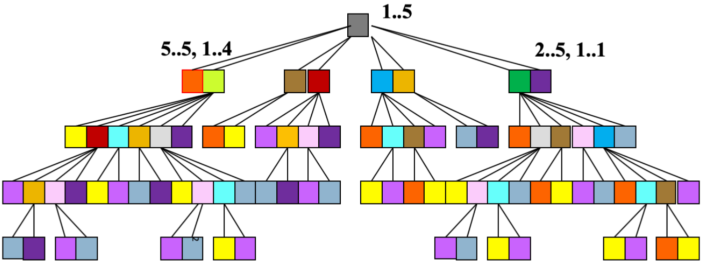
$$T(n) = \sum_{k=1}^{n-1} T(k) + \sum_{k=1}^{n-1} T(n-k) + O(n) = 2\sum_{k=1}^{n-1} T(k) + O(n)$$

Claim. $T(n) = \Omega(2^{n-1})$

- Induction basis: $n = 2$, $T(2) \geq c = c_1 2^{2-1}$, let $c_1 = c/2$.
- Induction step: $P(k < n) \Rightarrow P(n)$.

$$\begin{aligned} T(n) =& O(n) + c_1 2 \sum_{k=1}^{n-1} 2^{k-1} \quad \text{//induction premise} \\ \geq& O(n) + c_1 2(2^{n-1} - 1) = \Omega(2^{n-1}) \quad \text{//geometric series} \end{aligned}$$

Root of Inefficiency (Case $n = 5$)



different subproblems 15 vs. computing subproblems 81

> Those who cannot remember the past are condemned to repeat it.
>
> \- Dynamic Programming

Iterative Approach

size $= 1$: n different subproblems

- $C(i,i) = 0$ for $i \in [n]$ (no computation cost)

size $= 2$: $n-1$ different subproblems

- $C(1,2)$, $C(2,3)$, $C(3,4)$, ..., $C(n-1,n)$

...

size $= i$: $n-i+1$ different subproblems

...

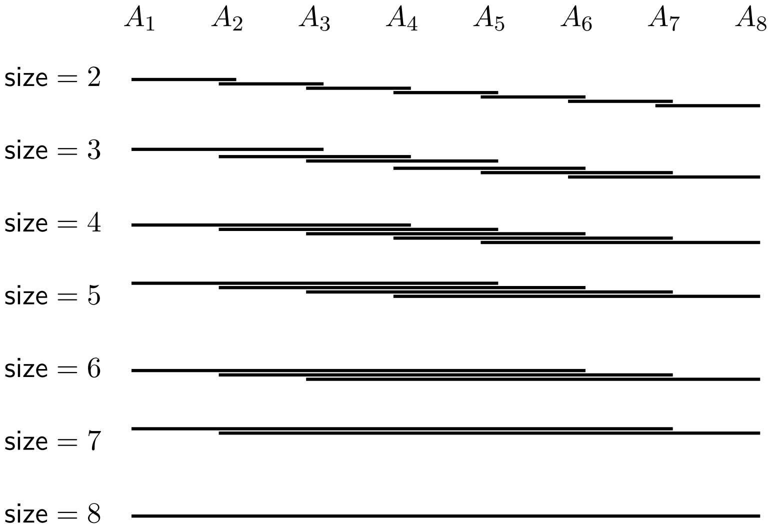
size $= n-1$: 2 different subproblems

- $C(1,n-1)$, $C(2,n)$

size $= n$: original problem

- $C(1,n)$

Demo of $n = 8$


A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8
size $= 2$
size $= 3$
size $= 4$
size $= 5$
size $= 6$
size $= 7$
size $= 8$

Algorithm 2: MatrixChain(C, n)

$C(i,i) \leftarrow 0$, $C(i,j)_{i \neq j} \leftarrow +\infty$;
for $\ell \leftarrow 2$ *to* n **do** //size of subproblem
  for $i = 1$ *to* $n - \ell + 1$ **do** //left boundary i
    $j \leftarrow i + \ell - 1$ //right boundary j;
    for $k \leftarrow i$ *to* $j - 1$ **do** //try all split position
      $t \leftarrow C(i,k) + C(k+1,j) + m_{i-1}m_k m_j$;
      if $t < C(i,j)$ **then**
        $C(i,j) \leftarrow t$, $s(i,j) = k$ //update
      end
    end
  end
end

Algorithm 3: Trace(s, i, j) //initially $i = 1$, $j = n$

if i=j **then return**;
output $k \leftarrow s(i,j)$, Trace(s, i, k), Trace$(s, k+1, j)$;

Complexity Analysis

According to the code: line $2, 3, 5$ constitute three-fold loop, length of each loop is $O(n)$

- 2: subproblem size
- 3: the left boundary of subproblem (the right boundary is also fixed)
- 5: try all split position to find the optimal break point

The cost in the inner loop is $O(1) \rightsquigarrow$ complexity $O(n^3)$

According to the memo

- there are totally n^2 elements in the memo, to determine the value of each element, try and comparison cost is $O(n) \rightsquigarrow$ complexity $O(n^3)$

Trace complexity: $n - 1$ (number of non-leaf nodes)

Example

Matrix chain. $A_1A_2A_3A_4A_5$, $A_1 : 30 \times 35$, $A_2 : 35 \times 15$, $A_3 : 15 \times 5$, $A_4 : 5 \times 10$, $A_5 : 10 \times 20$

$\ell = 2$	$C(1,2) = 15750$	$C(2,3) = 2625$	$C(3,4) = 750$	$C(4,5) = 1000$
$\ell = 3$	$C(1,3) = 7875$	$C(2,4) = 4375$	$C(3,5) = 2500$	
$\ell = 4$	$C(1,4) = 9375$	$C(2,5) = 7125$		
$\ell = 5$	$C(1,5) = 11875$			

$\ell = 2$	$s(1,2) = 1$	$s(2,3) = 2$	$s(3,4) = 3$	$s(4,5) = 4$
$\ell = 3$	$s(1,3) = 1$	$s(2,4) = 3$	$s(3,5) = 3$	
$\ell = 4$	$s(1,4) = 3$	$s(2,5) = 3$		
$\ell = 5$	$s(1,5) = 3$			

$$s(1,5) \Rightarrow (A_1A_2A_3)(A_4A_5)$$

$$s(1,3) \Rightarrow A_1(A_2A_3)$$

- optimal computation order: $(A_1(A_2A_3))(A_4A_5)$
- minimum multiplication: $C(1,5) = 11875$

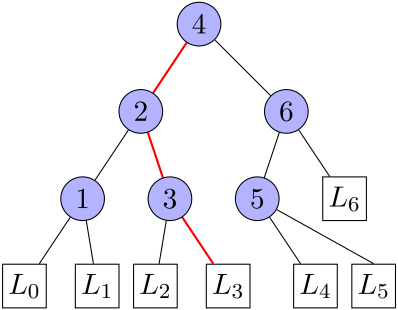
Binary Search Tree

Let S be an ordered set with elements $x_1 < x_2 < \cdots < x_n$. To admit efficient search, we store them on the nodes of a binary tree.

Search: If $x \in S$, output the index. Else, output the interval.

$x = 3.5$

1	2	3	4	5	6



x vs. root

- $x <$ root, enter left subtree;
- $x >$ root, enter right subtree;
- $x =$ root, halt and output x;

x reach leaf, halt, output $\perp$.

The Distribution of Search Element

When $x \xleftarrow{\mathsf{R}} S \Rightarrow$ balance binary tree is optimal

What if the distribution of x is not uniform?

Let $S = (x_1, \dots, x_n)$. Consider intervals (x_0, x_1), (x_1, x_2), $\dots$, (x_{n-1}, x_n), (x_n, x_{n+1}), where $x_0 = -\infty, x_{n+1} = +\infty$

- $\Pr[x = x_i] = b_i$, $\Pr[x \in (x_i, x_{i+1})] = a_i$

The distribution of x over $S \cup \bar{S}$ is

$$P = (a_0, b_1, a_1, b_2, a_2, \dots, b_n, a_n)$$

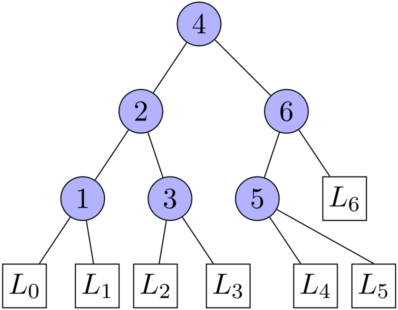
Example: $S = (1, 2, 3, 4, 5, 6)$. The distribution P of x is

$$(0.04, 0.1, 0.01, 0.2, 0.05, 0.2, 0.02, 0.1, 0.02, 0.1, 0.07, 0.05, 0.04)$$

$x = 1, 2, 3, 4, 5, 6$: 0.1, 0.2, 0.2, 0.1, 0.1, 0.05

x lies at interval: 0.04, 0.01, 0.05, 0.02, 0.02, 0.07, 0.04

Binary Search Tree 1



$$S = (1, 2, 3, 4, 5, 6)$$

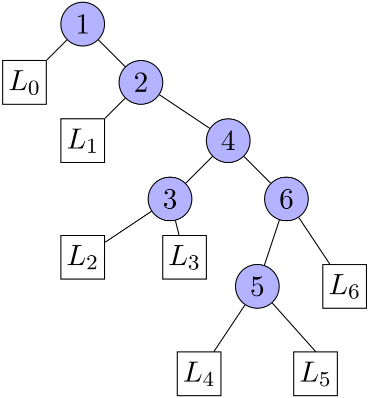
$$(0.1, 0.2, 0.2, 0.1, 0.1, 0.05)$$

$$(0.04, 0.01, 0.05, 0.02, 0.02, 0.07, 0, 04)$$

Average search times:

$$\begin{aligned} A(T_1) =& [1 \times 0.1 + 2 \times (0.2 + 0.05) + 3 \times (0.1 + 0.2 + 0.1)] \\ & + [3 \times (0.04 + 0.01 + 0.05 + 0.02 + 0.02 + 0.07) \\ & + 2 \times 0.04] \\ =& 1.8 + 0.71 = 2.51 \end{aligned}$$

Binary Search Tree 2



$$S = (1, 2, 3, 4, 5, 6)$$

$$(0.1, 0.2, 0.2, 0.1, 0.1, 0.05)$$

$$(0.04, 0.01, 0.05, 0.02, 0.02, 0.07, 0, 04)$$

Average search times:

$$\begin{aligned} A(T_2) = & [1 \times 0.1 + 2 \times 0.2 + 3 \times 0.1 + 4 \times (0.2 + 0.05) + 5 \times 0.1] \\ & + [1 \times 0.04 + 2 \times 0.01 + 4 \times (0.05 + 0.02 + 0.04) \\ & + 5 \times (0.02 + 0.07)] = 2.3 + 0.95 = 3.25 \end{aligned}$$

Formula of Average Search Time

Set S $(x_1, x_2, \ldots, x_n)$

Distribution $P = (a_0, b_1, a_1, b_2, \ldots, a_i, b_{i+1}, \ldots, b_n, a_n)$

- the depth of x_i in T is $d(x_i)$, $i = 1, 2, \ldots, n$.
  - depth is counted from 0
  - the k-level node requires $k + 1$ times compare
- the depth of interval I_j is $d(I_j)$, $j = 0, 1, \ldots, n$.

Average search time

$$A(T) = \sum_{i=1}^{n} b_i(1 + d(x_i)) + \sum_{j=0}^{n} a_j d(I_j)$$

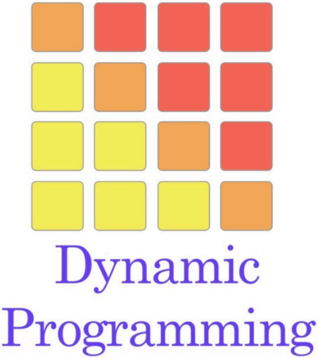
When the depth of all nodes increase by 1, the average search time increases by:

$$\sum_{i=1}^{n} b_i + \sum_{j=0}^{n} a_j$$

Modeling of Optimal Search Tree

Problem. Given set $S = (x_1, x_2, \ldots, x_n)$ and distribution of search element $P = (a_0, b_1, a_1, b_2, a_2, \ldots, b_n, a_n)$,

Goal. Find an optimal binary search tree (with minimal average search times)



Subproblems: defined by (i, j), i is the left boundary, j is the right boundary

- dataset: $S[i, j] = (x_i, x_{i+1}, \dots, x_j)$
- distribution: $P[i, j] = (a_{i-1}, b_i, a_i, b_{i+1}, \dots, b_j, a_j)$

Input instance: $S = (A, B, C, D, E)$

$P = (0.04, 0.1, 0.02, 0.3, 0.02, 0.1, 0.05, 0.2, 0.06, 0.1, 0.01)$

Subproblem: $(2, 4)$

- $S[2, 4] = (B, C, D)$
- $P[2, 4] = (0.02, 0.3, 0.02, 0.1, 0.05, 0.2, 0.06)$

Break Up to Subproblem

Using x_k as root, break up one problem into two subproblems:

- $S[i, k-1]$, $P[i, k-1]$
- $S[k+1, j]$, $P[k+1, j]$

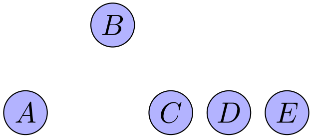
Example. Choose node B as root, break up the original problem into the following two subproblems:

Subproblem: $(1, 1)$

- $S[1, 1] = (A)$, $P[1, 1] = (0.04, 0.1, 0.02)$

Subproblem: $(3, 5)$

- $S[3, 5] = (C, D, E)$,
  $P[3, 5] = (0.02, 0.1, 0.05, 0.2, 0.06, 0.1, 0.01)$



Probability Sum of Subproblem

For subproblem $S[i, j]$ and $P[i, j]$, the probability sum in $P[i, j]$ (including elements and intervals) is:

$$w[i, j] = \sum_{s=i-1}^{j} a_s + \sum_{t=i}^{j} b_t$$

Example of subproblem $(2, 4)$

- $S[2, 4] = (B, C, D)$
- $P[2, 4] = (0.02, 0.3, 0.02, 0.1, 0.05, 0.2, 0.06)$
- $w[2, 4] = (0.3 + 0.1 + 0.2) + (0.02 + 0.02 + 0.05 + 0.06) = 0.75$

Optimized Function

Optimized function $\mathsf{OPT}(i, j)$: the optimal average compare times of subproblem (i, j) for $S[i, j]$, $P[i, j]$.

Parameterized optimized function. $\mathsf{OPT}_k(i, j)$: optimal compare times with x_k as root

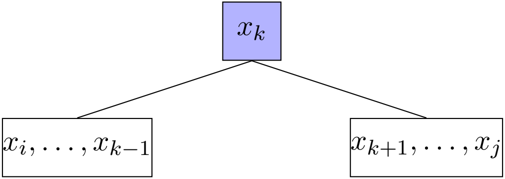
Initial values: $\mathsf{OPT}(i, i-1) = a_{i-1}$ for $i = 1, 2, \ldots, n, n+1$ corresponds to empty subproblem.

Example: $S = (A, B, C, D, E)$

1. choose A as root ($k = 1$), yield subproblem $(1, 0)$ and $(2, 5)$, $(1, 0)$ is an empty subproblem: corresponding to $S[1, 0]$, $w[1, 0] = a_0 = 0.04$
2. choose E as root ($k = 5$), yield subproblem $(1, 4)$ and $(6, 5)$, $(6, 5)$ is an empty subproblem: corresponding to $S[6, 5]$, $w[6, 5] = a_5 = 0.01$

Iterate relation for Optimized Function

$$\begin{aligned}
\mathsf{OPT}(i,j) &= \min_{i\le k\le j}\{\mathsf{OPT}_k(i,j)\}, 1 \le i \le j \le n \\
&= \min_{i\le k\le j}\{\mathsf{OPT}(i,k-1) + \mathsf{OPT}(k+1,j) + {\color{red}w[i,j]}\}
\end{aligned}$$



- the depth of all nodes in left subtree and right subtree increase by 1 (featured by a multiplicative factor)

$$w[i,k-1] + b_k + w[k+1,j] = w[i,j]$$

Proof of $\mathsf{OPT}_k(i, j)$

$$
\begin{aligned}
&\mathsf{OPT}_k(i,j) \\
&= (\mathsf{OPT}(i,k-1) + \underline{w[i,k-1]}) + (\mathsf{OPT}(k+1,j) + \underline{w[k+1,j]}) + b_k \\
&= (\mathsf{OPT}(i,k-1 + \mathsf{OPT}(k+1,j)) + (w[i,k-1] + b_k + w[k+1,j]) \\
&= (\mathsf{OPT}(i,k-1) + \mathsf{OPT}(k+1,j)) \\
&+ \left(\sum_{s=i-1}^{k-1} a_s + \sum_{t=i}^{k-1} b_t\right) + b_k + \left(\sum_{s=k}^{j} a_s + \sum_{t=k+1}^{j} b_t\right) \\
&= (\mathsf{OPT}(i,k-1) + \mathsf{OPT}(k+1,j)) + \sum_{s=i-1}^{j} a_s + \sum_{t=i}^{j} b_t \quad \text{//simplify} \\
&= \mathsf{OPT}(i,k-1) + \mathsf{OPT}(k+1,j) + w[i,j]
\end{aligned}
$$

Pseudocode

Computation order: the size of subtree grows from 1 to n

Algorithm 4: BinarySearchTree(S, P, n)

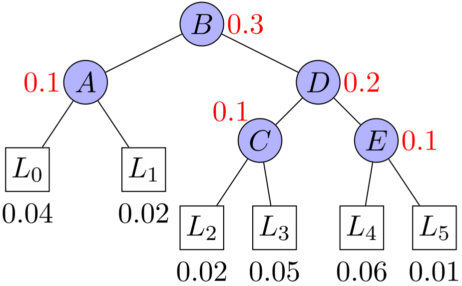
OPT$(i, i-1) \leftarrow a_{i-1}$ for all $i \in [1, n+1]$;
OPT$(i, j) \leftarrow 0$ for all $i \leq j$;
for $\ell \leftarrow 1$ *to* n **do** //size of subproblem
    for $i = 1$ *to* $n - \ell + 1$ **do** //left boundary i
        $j \leftarrow i + \ell - 1$ //right boundary j;
        for $k \leftarrow i$ *to* j **do** //try all split position
            $t \leftarrow$ OPT$(i, k-1) +$ OPT$(k+1, j) + b_k$;
            if $t <$ OPT(i, j) **then**
                OPT$(i, j) \leftarrow t$, $s(i, j) = k$ //update
            end
        end
    end
end

Demo

$$\mathsf{OPT}(i,j) = \min_{i\le k\le j}\{\mathsf{OPT}(i,k-1)+\mathsf{OPT}(k+1,j)+w[i,j]\}$$

$$\text{for } 1\le i\le j\le n$$

$$\mathsf{OPT}(i,i-1)=a_{i-1}, i=1,2,\ldots,n,n+1$$



choose B as root, $k=2$

$\mathsf{OPT}(1,1)=0.16$

$\mathsf{OPT}(3,5)=0.88$

$\mathsf{OPT}(3,3)=0.17$

$\mathsf{OPT}(5,5)=0.17$

$w[3,5]=0.54$

$$\begin{aligned}\mathsf{OPT}(1,5) =& 1+\min_{k\in[5]}\{\mathsf{OPT}(1,k-1),\mathsf{OPT}(k+1,5)\}\\ =& 1+(\mathsf{OPT}(1,1)+\mathsf{OPT}(3,5)) = 1+(0.16+0.88)=2.04\end{aligned}$$

Complexity Analysis

$$\mathsf{OPT}(i,j) = \min_{i \le k \le j} \{\mathsf{OPT}(i,k-1) + \mathsf{OPT}(k+1,j) + w[i,j]\}$$

$$\text{for } 1 \le i \le j \le n$$

$$\mathsf{OPT}(i,i-1) = a_{i-1}, i = 1,2,\ldots,n,n+1$$

The number of (i,j) combination is $O(n^2)$

For each $\mathsf{OPT}(i,j)$, computation requires computing k terms and finding min. The cost of each term computation is constant time.

- Time complexity: $T(n) = O(n^3)$
- Space complexity: $S(n) = O(n^2)$