

Design and Analysis of Algorithm

Complexity Analysis

- 1 Notions of Algorithm and Time Complexity
- 2 Pseudocode of Algorithm
- 3 Asymptotic Order of Function
- 4 Important Function Class
- 5 Survey of Common Running Times

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Problem and Solution

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- Parameters that specify the problem (set, variable, function, sequences, etc.), include descriptions of domain and relation among them
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Instance. An assignment of parameters \rightarrow an instance of problem

Definition 1 (Algorithm)

An algorithm \mathcal{A} is a finite sequence of well-defined, computer implementable instructions that solve a class of problems

- algorithms are always unambiguous
- specifications for performing calculations, data processing, automated reasoning, and other tasks.

Algorithm \mathcal{A} for Problem P

- take any instance of P as \mathcal{A} 's input, computation of each step is deterministic
- \mathcal{A} halts in finite steps
- always output the correct solution

Basic Computer Steps and Input Size

An insightful analysis is based on the right simplifications.

Basic computer steps. capture abstract atomic operation

- Example. compare, add, multiplication, swap, assign ...

This is the first important simplification!

Input size. capture the scale of instance: proportional to the length of instance encoding string

- Example. number of array, number of scheduling tasks, number of vertices and edges

Examples of Input Size and Basic Computer Steps (1/2)

Sorting. array $a[n]$

- n : the number of elements in the array
- element compares and movement

Searching. search x in array $a[n]$

- n : the number of elements in the array
- element compares between x and $a[i]$

Integer multiplication. $a \times b$

- the binary length of a and b , a.k.a. $m = \log_2 a$, $n = \log_2 b$
- bit-wise multiplication – $a \times b$ requires $\#mn$ bit-wise multiplication

Examples of Input Size and Basic Computer Steps (2/2)

Matrix multiplication. $\mathbf{A}_{n_1 \times n_2} \cdot \mathbf{B}_{n_2 \times n_3}$

- dimensions of \mathbf{A} and \mathbf{B} , a.k.a. n_1, n_2, n_3
- point-wise multiplication – $\mathbf{A} \cdot \mathbf{B}$ requires $n_1 n_2 n_3$ -times point-wise multiplication
- $n_1 = n_2 = n_3 = n \leadsto n^3$

Graph visit. $G = (V, E)$

- number of vertices and edges
- assignment of flag variable

Measurement of Algorithm's Efficiency

Express running time by counting the number of basic computer steps as a function of the size of the input.

uncluttered, machine-independent characterization

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For different inputs of the same instance size, the number of basic computer steps might vary \Rightarrow functions could be different

Choose which one?

Three Types of Analyses

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Best-case. Minimum running time for **all inputs** of size n

Example. Insertion sort only requires n compares when the input is sorted already.

Average-case. **Expected** running time for a **random input** of size n

Example. expected number of element compares of Quicksort is
 $\sim n \log n$.

About Worst-Case

Algorithm. Some exponential-time algorithms are used widely in practice because the worst-case instances seem to be rare.

- Linux `grep` command

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Cryptography. Require hard instance to be efficiently samplable

Good news to Algorithms = Bad news to Cryptography

Win-Win flavor

Formula of $A(n)$

$A(n)$: average-case complexity

- Let X be the set of all inputs of size n , $\Pr[x \in X] = p(x)$
- $t(x)$: the number of basic operations that \mathcal{A} performs on input x

$$A(n) = \sum_{x \in X} p(x)t(x)$$

In many cases, we assume the input distribution is a uniform distribution.

Example of Search

Search Problem

Input. Array $a[n]$ with ascending order, search x

Output. $j \in [0, \dots, n]$

- if $x \in a[n]$, then j is the first index such that $a[j] = x$
- else, $j = 0$

Basic operation. element compare between x and $a[i]$

Sequential Search Algorithm

Algorithm 1: Search($a[n], x$)

```
1:  $flag \leftarrow 0$ ;  
2: for  $j = 1$  to  $n$  do  
3:   if  $a[j] = x$  then  
4:      $flag = 1$ ;  
5:     break;  
6:   end  
7: end  
8: if  $flag = 0$  then  $j = 0$  ;  
9: return  $j$ ;
```

Example. 1, 2, 3, 4, 5

- $x = 4$: 4 compares
- $x = 2.5$: 5 compares

Worst-case complexity

There are $2n + 1$ types different inputs:

- Case inside: $x = a[1], x = a[2], \dots, x = a[n]$
- Case outside: $x < a[1], a[1] < x < a[2], \dots, a[n] < x$

Worse-case input. $x \notin A \vee x = A[n]$, requires n compares

Worse-case complexity. $T(n) = n$

Average-case complexity

Assume $\Pr[x \in A] = p$, and distributes on each position with equal probability.

$$\begin{aligned} T(n) &= \sum_{i=1}^n i \cdot \frac{p}{n} + (1-p)n \quad // \text{sum of arithmetic sequence} \\ &= \frac{p(n+1)}{2} + (1-p)n \end{aligned}$$

When $p = 1/2$

$$T(n) = \frac{n+1}{4} + \frac{n}{2} \approx \frac{3n}{4}$$

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Pseudocode of Algorithm

Definition 2 (Pseudocode)

An informal high-level description of the operating principle of algorithms: uses the structural conventions of programming language, but is intended for human reading rather than machine reading.

Instruction	Symbol
Assignment	\leftarrow or $:=$
Branch statement	if...then...[else...]
Loop structure	while, for, repeat until
Transfer statement	goto
Return statement	return
Function call	Func()
Comment	// or /* */

Example: Euclid Algorithm for Greatest Common Divisor

Algorithm 2: EuclidGCD(n, m)

Input: $n, m \in \mathbb{Z}^+, n \geq m$

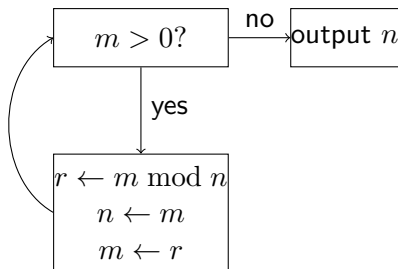
Output: GCD(n, m)

```
1: while  $m > 0$  do  
2:    $r \leftarrow n \bmod m$ ;  
3:    $n \leftarrow m$ ;  
4:    $m \leftarrow r$ ;  
5: end  
6: return  $n$ 
```

Demo: $n = 36$, $m = 15$

while	n	m	r
1st loop	36	15	6
2nd loop	15	6	3
3rd loop	6	3	0
	3	0	0

output 3



Example of Insertion Sort

Algorithm 3: Algorithm InsertSort($A[n]$)

Input: array $A[n]$

Output: $A[n]$ in ascending order

```
1: for  $j \leftarrow 2$  to  $n$  do
2:    $x \leftarrow A[j]$ ;
3:    $i \leftarrow j - 1$  //insert  $A[j]$  to  $A[1...j - 1]$ ;
4:   while  $i > 0$  and  $x < A[i]$  do
5:      $A[i + 1] \leftarrow A[i]$ ;
6:      $i \leftarrow i - 1$ ;
7:   end
8:    $A[i + 1] \leftarrow x$ ;
9: end
```

Demo of Insertion Sort

2	4	1	5	3
---	---	---	---	---

$j = 3, x = A[3] = 1$

$i = 2, A[2] = 4$

 $i > 0, x < A[2] \checkmark$

2	4	4	5	3
---	---	---	---	---

$A[3] = 4, i = 1, x = 1$

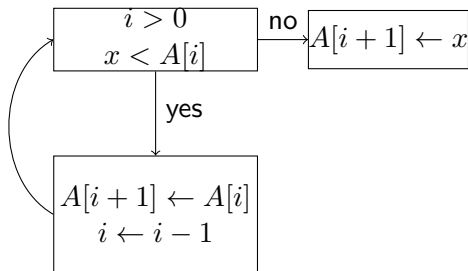
$i > 0, x < A[1] \checkmark$

2	2	4	5	3
---	---	---	---	---

$A[2] = 2, i = 0, x = 1$

$i > 0 \times$

1	2	4	5	3
---	---	---	---	---



Example of Binary Merge Sort

Algorithm 4: Algorithm MergeSort(A, l, r)

Input: array $A[l, r]$

Output: $A[l, r]$ in ascending order

```
1: if  $p < r$  then  
2:    $m \leftarrow \lfloor (l + r) / 2 \rfloor$ ;  
3:   MergeSort( $A, l, m$ );  
4:   MergeSort( $A, m + 1, r$ );  
5:   Merge( $A, l, m, r$ );  
6: end
```

MergeSort is a recursive algorithm

- call itself from within its own code

Pseudocode of Algorithm \mathcal{A}

Algorithm 5: Algorithm \mathcal{A}

Input: Array $P[0, \dots, n] \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}$

Output: y

- 1: $y \leftarrow P[0]$; $power \leftarrow 1$;
 - 2: **for** $i \leftarrow 1$ **to** n **do**
 - 3: $power \leftarrow power \times x$;
 - 4: $y \leftarrow y + P[i] \times power$;
 - 5: **end**
 - 6: **return** y ;
-

What do 3-4 compute?

for $i \in [n]$ \longrightarrow $\boxed{\begin{array}{l} power \leftarrow power \times x \\ y \leftarrow y + P[i] \times power \end{array}}$

loop	$power$	y
0	1	$P[0]$
1	x	$P[0] + P[1] \times x$
2	x^2	$P[0] + P[1] \times x + P[2] \times x^2$
3	x^3	$P[0] + P[1] \times x + P[2] \times x^2 + P[3] \times x^3$
		\dots

Input $P[0, \dots, n]$ is the coefficients of n -degree polynomial $P(x)$

- \mathcal{A} compute $P(x) = \sum_{i=0}^n P[i]x^i$

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Motivation

We use functions over \mathbb{N} to capture how the running time or space requirements of algorithms grow as the input size increases.

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How to compare them? How to classify them?

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How to compare them? How to classify them?

The first simplification leads to another. Now, second simplification comes into play, consider **the order of function** rather than its concrete form.

Big- O Notations

Paul Bachmann and Edmund Landau invented a family of notations known as **Big- O notation** to describe the limiting behavior of a function when the input tends towards infinity.

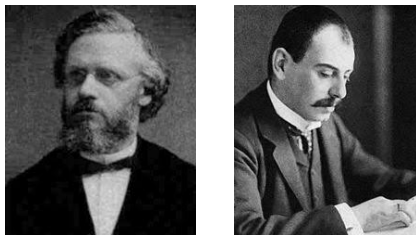


Figure: Paul Bachmann & Edmund Landau

- also known as Bachmann-Landau or asymptotic notation
- mathematical notation \sim describe running times

Big-O Notation

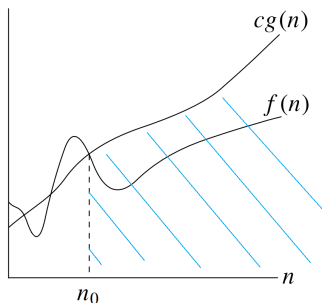
Definition 3 (Big-O)

$\exists c > 0, \exists n_0$, such that $\forall n \geq n_0$:

$$f(n) \leq cg(n)$$

f is bounded above by g (up to constant factor) asymptotically

$$f(n) = O(g(n))$$



Limit definition

$$\lim_{n \rightarrow \infty} \sup \frac{f(n)}{g(n)} < \infty$$

Some Remarks

Big- O notation characterizes functions according to their growth rates: different functions with the same growth rate may be represented using the same O notation.

- letter O is used because the growth rate of a function is also referred to as the order of the function.
- there are many (c, n_0) , it suffices to find one tuple
- for finite values $n \leq n_0$, the inequality may not hold
- constant functions can be written as $O(1)$

More about Big O

$f(n) = O(g(n))$: the order of $f(n)$ is less than that of $g(n)$

Typical usage. Insertion sort makes $O(n^2)$ compares to sort n elements.

Example 1. $f(n) = n^2 + n$

- $f(n) = O(n^2) \leftarrow$ choose $c = 2, n_0 = 1$
- $f(n) = O(n^3) \leftarrow$ choose $c = 1, n_0 = 2$

Example 2. $f(n) = 32n^2 + 17n + 1$

- $f(n) = O(n^2) \leftarrow$ choose $c = 50, n_0 = 1$
- $f(n)$ is also $O(n^3)$
- $f(n)$ is neither $O(n)$ nor $O(n \log n)$

Big- O notation only provides an upper bound on the growth rate of the function.

Associated with big- O notation are several related notations, using the symbols o , Ω , ω , and Θ , to describe other kinds of bounds on asymptotic growth rates.

Big Omega Notation

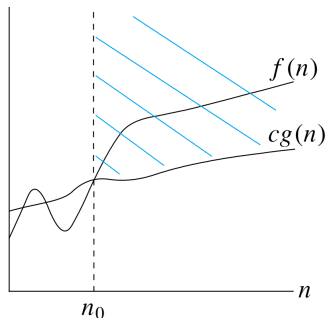
Definition 4 (Big- Ω)

$\exists c > 0, \exists n_0, \forall n \geq n_0$:

$$f(n) \geq cg(n)$$

f is bounded below by g asymptotically

$$f(n) = \Omega(g(n))$$



Limit definition

$$\liminf_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0$$

Example of Big Omega

$f(n) = \Omega(g(n))$: the order of $f(n)$ is greater than $g(n)$.

Typical usage. Any compare-based sorting algorithm requires $\Omega(n \log n)$ compares in the worst case.

Meaningless statement. Any compare-based sorting algorithm requires **at least** $O(n \log n)$ compares in the worst case. ($O(\cdot)$ cannot give lower bound)

Example. $f(n) = n^2 + n$

- $f(n) = \Omega(n^2) \leftarrow c = 1, n_0 = 1$
- $f(n) = \Omega(100n) \leftarrow c = 1/100, n_0 = 1$

Big O and Ω notations are originally used as a tight upper-bound (resp. lower-bound) on the growth of an algorithm's effort

But, according to the definitions

$g(n)$ could be a **loose** upper-bound (resp. lower-bound).

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But, according to the definitions

$g(n)$ could be a **loose** upper-bound (resp. lower-bound).

To make the role as a tight upper-bound more clear, small o and ω notations are used to describe an upper-bound/lower-bound that cannot be tight.

Small O Notation

Definition 5 (Small- o)

$\forall c > 0, \exists n_0$, such that $\forall n \geq n_0$:

$$f(n) < cg(n)$$

f is dominated by g asymptotically:

$$f(n) = o(g(n))$$

Limit definition

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

More about Small O

$f(n) = o(g(n))$: the order of $f(n)$ is strictly smaller than that of $g(n)$

Typical usage. $\log n = o(n)$

Example. $f(n) = n^2 + n$, $f(n) = o(n^3)$

- $c \geq 1$: obviously holds, since $n^2 + n < cn^3$, choose $n_0 = 2$
- $0 < c < 1$: choose $n_0 > \lceil 2/c \rceil$, because

$$\begin{aligned} cn &\geq cn_0 \geq 2 \\ n^2 + n &< 2n^2 < cn^3 \end{aligned}$$

Small Omega Notation

Definition 6

$\forall k > 0, \exists n_0, \forall n \geq n_0:$

$$f(n) > c \cdot g(n)$$

f dominates g asymptotically

$$f(n) = \omega(g(n))$$

Limit definition:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$$

Example of Small Omega

$f(n) = \omega(g(n))$: the order of $f(n)$ is strictly larger than that of $g(n)$

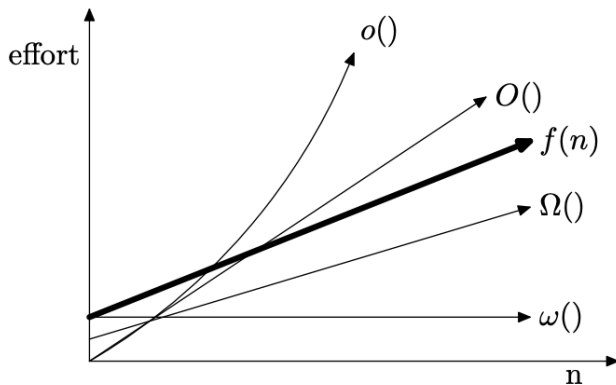
Typical usage. $n = \omega(\log n)$

Example. $f(n) = n^2 + n$, $f(n) = \omega(n)$

- $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = \infty$
- $f(n) \neq \omega(n^2)$: choose $c = 2$, there does not exist n_0 such that $\forall n \geq n_0$

$$cn^2 = 2n^2 < n^2 + n$$

Visualize the Relationships between these notations



Comparisons

Notation	$? c > 0$	$? n_0$	$f(n) ? c \cdot g(n)$
O	\exists	\exists	\leq
o	\forall	\exists	$<$
Ω	\exists	\exists	\geq
ω	\forall	\exists	$>$

While o and ω are not often used to described algorithms

- We define a combination of O and Ω : Θ , which means $g(n)$ is both a tight upper-bound and a tight lower-bound

Big Theta Notation: Aims to a Tight Bound

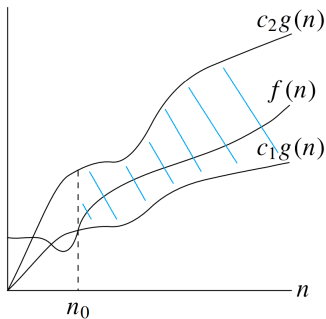
Definition 7 (Big- Θ)

$\exists c_1 > 0, \exists c_2 > 0, \exists n_0$, such that $\forall n > n_0$

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

f is bounded both above and below by g asymptotically

$$f(n) = \Theta(g(n))$$



Limit definition

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = c$$

Proof of Equivalence

Proof. Definition of the limit $\Rightarrow \forall \varepsilon > 0, \exists n_0, \forall n \geq n_0$:

$$\begin{aligned} |f(n)/g(n) - c| &< \varepsilon \\ c - \varepsilon &< f(n)/g(n) < c + \varepsilon \end{aligned}$$

Proof of Equivalence

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choose $\varepsilon = c/2 \Rightarrow c/2 < f(n)/g(n) < 3c/2 < 2c$

- $\forall n \geq n_0, f(n) \leq 2cg(n) \Rightarrow f(n) = O(g(n))$
- $\forall n \geq n_0, f(n) \geq (c/2)g(n) \Rightarrow f(n) = \Omega(g(n)).$

Proof of Equivalence

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- $\forall n \geq n_0, f(n) \leq 2cg(n) \Rightarrow f(n) = O(g(n))$
- $\forall n \geq n_0, f(n) \geq (c/2)g(n) \Rightarrow f(n) = \Omega(g(n)).$

This proves $f(n) = \Theta(g(n))$

More about Big Theta

$f(n) = \Theta(g(n))$: $f(n) = O(g(n)) \wedge f(n) = \Omega(g(n))$, $f(n)$ and $g(n)$ have the same order

Typical usage. Mergesort makes $\Theta(n \log n)$ compares to sort n elements.

Example 1. $f(n) = n^2 + n$, $g(n) = 100n^2$
 $f(n) = \Theta(g(n))$

Example 2. $f(n) = 32n^2 + 17n + 1$

- $f(n)$ is $\Theta(n^2)$ \leftarrow choose $c_1 = 32$, $c_2 = 50$, $n_0 = 1$
- $f(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$

Example of Primality Test

Algorithm 6: PrimalityTest(n)

Input: odd integer $n > 2$

Output: true or false

```
1:  $s \leftarrow \lfloor n^{1/2} \rfloor$ ;  
2: for  $j \leftarrow 2$  to  $s$  do  
3:   if  $j$  divides  $n$  then return false;  
4: end  
5: return true;
```

If $n^{1/2}$ is computable in $O(1)$ -time, the basic operation is divide

$$W(n) = O(n^{1/2}) \quad \checkmark \quad W(n) = \Theta(n^{1/2}) \quad \times$$

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Consider inputs of the form $3m$, then $n^{1/2}$ is not the lower bound

Example of Primality Test

Algorithm 8: PrimalityTest(n)

Input: odd integer $n > 2$

Output: true or false

```
1:  $s \leftarrow \lfloor n^{1/2} \rfloor$ ;  
2: for  $j \leftarrow 2$  to  $s$  do  
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If $n^{1/2}$ is computable in $O(1)$ -time, the basic operation is divide

$$W(n) = O(n^{1/2}) \quad \checkmark \quad W(n) = \Theta(n^{1/2}) \quad \times$$

Consider inputs of the form $3m$, then $n^{1/2}$ is not the lower bound

Remark. Typically, we use λ (length of binary representation of n) as input of $W(n)$. In that case, the above two statements are both correct. a.k.a. $W(\lambda) = \Theta(2^{\lambda/2})$

Big O notation with multiple variables

Upper bounds. $f(m, n)$ is $O(g(m, n))$ if $\exists c > 0$, $m_0 \geq 0$ and $n_0 \geq 0$ such that $\forall n \geq n_0$ and $m \geq m_0$, $f(m, n) \leq c \cdot g(m, n)$

Example. $f(m, n) = 32mn^2 + 17mn + 32n^3$

- $f(m, n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$
- $f(m, n)$ is neither $O(n^3)$ nor $O(mn^2)$

Typical usage. Breadth-first search takes $O(m + n)$ time to find the shortest path from s to t in a digraph

Properties of Big-O Notations (1/2)

Transitivity. The order of functions are transitive.

- $f = O(g) \wedge g = O(h) \Rightarrow f = O(h)$
- $f = \Omega(g) \wedge g = \Omega(h) \Rightarrow f = \Omega(h)$
- $f = \Theta(g) \wedge g = \Theta(h) \Rightarrow f = \Theta(h)$
- $f = o(g) \wedge g = o(h) \Rightarrow f = o(h)$
- $f = \omega(g) \wedge g = \omega(h) \Rightarrow f = \omega(h)$

Properties of Big-O Notations (2/2)

Product

- $f_1 = O(g_1) \wedge f_2 = O(g_2) \Rightarrow f_1 f_2 = O(g_1 g_2)$
- $f \cdot O(g) = O(fg)$

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Sum

- $f_1 = O(g_1) \wedge f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(\max(g_1, g_2))$
- This implies $f_1 = O(g) \wedge f_2 = O(g) \Rightarrow f_1 + f_2 \in O(g)$, which means that $O(g)$ is a convex cone.

Properties of Big-O Notations (2/2)

Product

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- $f \cdot O(g) = O(fg)$

Sum

- $f_1 = O(g_1) \wedge f_2 = O(g_2) \Rightarrow f_1 + f_2 = O(\max(g_1, g_2))$
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This property extends to a finite composition of f_i

- **Application.** For an algorithm, if the running time of its each step is upper bounded by $h(n)$, and the algorithm only consists of constant steps, then the overall complexity is $O(h(n))$.

Properties of Big-O Notations (2/2)

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Multiplication by a constant. Let $k > 0$ be a constant. Then:

- $O(kg) = O(g)$, if $k \neq 0$.
- $f = O(g) \Rightarrow kf = O(g)$ (multiplicative constants can be omitted)

- 1 Notions of Algorithm and Time Complexity
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Important Function Class

We list important function class in ascending order

- constant: $O(1)$
 - double logarithmic: $\log \log n$
 - logarithmic: $\log n$
 - polylogarithmic: $(\log^n)^c$, $c > 1$
 - fractional power: n^c , $0 < c < 1$
-
- linear: $O(n)$
 - loglinear or quasilinear: $n \log n$
 - polynomial: n^c , $c > 1$ (quadratic: n^2 , cubic n^3)
-
- exponential: c^n , $c > 1$
 - factorial: $n!$

Asymptotic Bounds for some Common Functions (1/3)

Technical tool. Limit Definitions of $O, \Omega, \Theta, o, \omega$

Polynomials. Let $f(n) = a_0 + a_1n + \cdots + a_dn^d$, then $f(n) = \Theta(n^d)$.

Proof.

$$\lim_{n \rightarrow \infty} \frac{a_0 + a_1n + \cdots + a_dn^d}{n^d} = a_d > 0$$

Example. Let $f(n) = n^2/2 - 3n$, $f(n) = \Theta(n^2)$.

Asymptotic Bounds for some Common Functions (2/3)

Logarithms. $\Theta(\log_a n) \sim \Theta(\log_b n)$ for any constants $a, b > 0$

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Logarithms vs. Polynomials. $\forall d > 1, \log n = o(n^d)$.

Proof.

- Both $\lim_{n \rightarrow \infty} \ln n = \infty$ and $\lim_{n \rightarrow \infty} n^d = \infty$ and are differentiable:
- Apply L'Hôpital (Bernoulli) rule once

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\ln n}{n^d} &= \lim_{n \rightarrow \infty} \frac{1/n}{dn^{d-1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{dn^d} = 0\end{aligned}$$

Asymptotic Bounds for some Common Functions (3/3)

Exponentials vs. Polynomials. $\forall c > 1$ and $\forall d > 0$, $n^d = o(c^n)$.

Asymptotic Bounds for some Common Functions (3/3)

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Proof. W.L.O.G, choose d as a positive integer,

- Both $\lim_{n \rightarrow \infty} n^d = \infty$ and $\lim_{n \rightarrow \infty} c^n = \infty$ and are differentiable.
- Apply L'Hôpital (Bernoulli) rule repeatedly until the numerator is constant

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n^d}{c^n} &= \lim_{n \rightarrow \infty} \frac{dn^{d-1}}{c^n \ln c} = \lim_{n \rightarrow \infty} \frac{d(d-1)n^{d-2}}{c^n (\ln c)^2} \\ &= \dots = \lim_{n \rightarrow \infty} \frac{d!}{c^n (\ln c)^d} = 0\end{aligned}$$

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Stirling Formula (named after James Stirling, though it was first stated by Abraham de Moivre)

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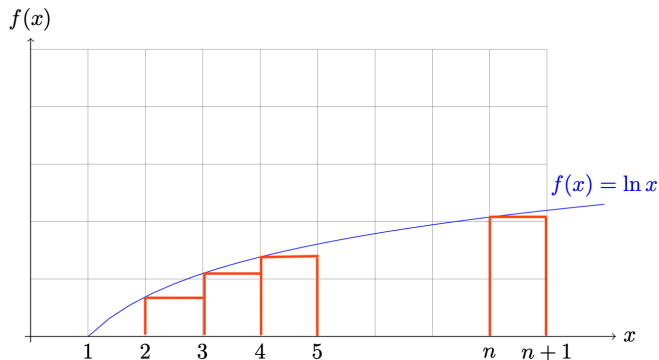
Simple form:

$$\ln n! = n \ln n - n + O(\ln n)$$

- $n! = o(n^n)$
- $n! = \omega(2^n)$
- $\ln(n!) = \Theta(n \ln n)$ (integral method)

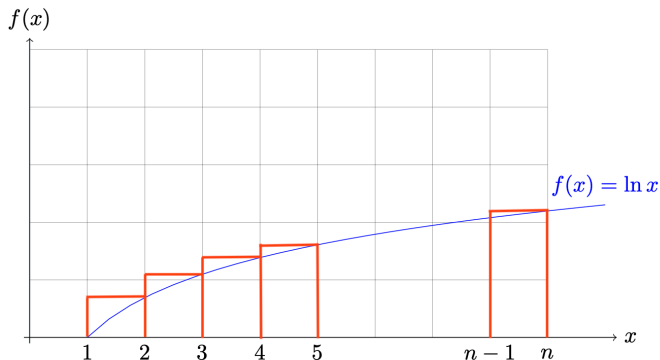
Proof of the Upper Bound

$$\begin{aligned}\ln n! &= \sum_{k=1}^n \ln k \leq \int_2^{n+1} \ln x dx \\ &= (x \ln x - x)_2^{n+1} \\ &= O(n \ln n)\end{aligned}$$



Proof of the Lower Bound

$$\begin{aligned}\ln n! &= \sum_{k=1}^n \ln k \geq \int_1^n \ln x dx \\ &= (x \ln x - x)_1^n \\ &= n \ln n - n + 1 = \Omega(n \ln n)\end{aligned}$$



Application: Estimate the Size of Search Space

Recall the ROI optimization problem: the number of different investment schemes: m coins on n projects

$$\begin{aligned} C_{m+n-1}^m &= \frac{(m+n-1)!}{m!(n-1)!} \\ &= \frac{\sqrt{2\pi(m+n-1)}(m+n-1)^{m+n-1} \left(1 + \Theta\left(\frac{1}{m+n-1}\right)\right)}{\sqrt{2\pi n} m^m \left(1 + \Theta\left(\frac{1}{m}\right)\right) \sqrt{2\pi(n-1)} (n-1)^{n-1} \left(1 + \Theta\left(\frac{1}{n-1}\right)\right)} \\ &= \Theta((1+\varepsilon)^{m+n-1}) \end{aligned}$$

Rounding Function

Rounding a number means replacing it with a different number that is *approximately equal* to the original, but has a *shorter, simpler* representation

- round down (or take the floor)

$y = \text{floor}(x) = \lfloor x \rfloor$: y is the largest integer that does not exceed x

- round up (or take the ceiling)

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Example. $\lfloor 2.6 \rfloor = 2$, $\lceil 2.6 \rceil = 3$, $\lfloor 2 \rfloor = \lceil 2 \rceil = 2$

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Application. When performing binary search in $A[n]$, the index of median is $\lfloor n/2 \rfloor$, the subproblem is of size $\lfloor n/2 \rfloor$.

Properties of Rounding Function

Proposition 1. $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

Proof. We proof this by considering two cases:

- ① x is an integer: obvious
- ② $\exists n \in \mathbb{Z}$ such that $n < x < n + 1$, definition of rounding function $\Rightarrow \lfloor x \rfloor = n, \lceil x \rceil = n + 1$

Proposition 2. Let $n, a, b \in \mathbb{Z}$, we have:

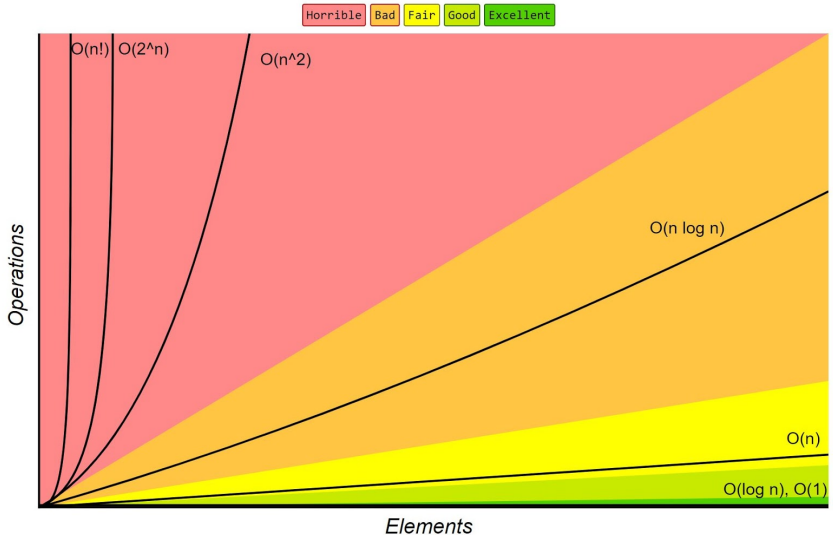
$$\lfloor x + n \rfloor = \lfloor x \rfloor + n, \lceil x + n \rceil = \lceil x \rceil + n$$

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = n$$

$$\left\lceil \frac{\lceil \frac{n}{a} \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil, \left\lfloor \frac{\lfloor \frac{n}{a} \rfloor}{b} \right\rfloor = \left\lfloor \frac{n}{ab} \right\rfloor$$

Running Times

Big-O Complexity Chart



- 1 Notions of Algorithm and Time Complexity
- 2 Pseudocode of Algorithm
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Common Running Times (1/4)

Constant time. $T(n) = O(1)$

- Determine if a binary number is even or odd
- Hash map (key-value) access or random access of array $A[i]$

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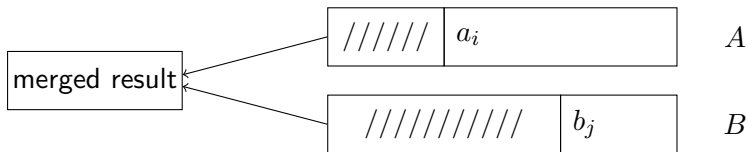
Fractional power. $T(n) = n^{1/2}$

- Primality test

Common Running Times (2/4)

Linear time. $T(n) = O(n)$: running time is proportional to input size

- Compute maximum if n numbers a_1, \dots, a_n
- Merge: combine two sorted lists $A = a_1, \dots, a_n$ with $B = b_1, \dots, b_n$ into sorted whole



After each compare, the length of output list increases by at least 1. When one list is empty, the rest part of another list is directly merged to the result list.

- Upper bound: $2n - 1$ vs. Lower bound: n

Common Running Times (3/4)

Loglinear time. $T(n) = O(n \log n)$ (arises in divide-and-conquer algorithms)

- Mergesort and heapsort are sorting algorithms that perform $O(n \log n)$ compares
- FFT

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- Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \dots, (x_n, y_n)$, find the pair that is closest. $O(n^2)$
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Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

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Cubic time. Enumerate all triples of elements

- Plain Matrix multiplication: $\mathbf{A}_{n \times n} \times \mathbf{B}_{n \times n}$: each $c_{i,j}$ requires $O(n)$ multiplications, totally n^2 elements in $\mathbf{C}_{n \times n}$

Common Running Times (4/4)

Polynomial time. $T(n) = O(n^k)$

- Independent set of size k : Given a graph of n nodes, are there k nodes such that no two are joined by an edge?
- $O(n^k)$ solution, enumerate all subsets of k nodes then check
 - check if S_k is an independent set takes $O(k^2)$ time
 - $\#(S_k) = C_n^k \leq n^k/k!$
- $O(k^2 n^k/k!) = O(n^k)$ (poly-time for $k = 17$, but not practical)

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Exponential time. $T(n) = O(c^n)$

- Independent set: Given a graph, what is the maximum cardinality of an independent set?
- Enumerate all subsets and check: $O(n^2 2^n)$

About Polynomial Running Time

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor c .

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We say that an algorithm is **efficient** if has a polynomial running time.

Exceptions. Some poly-time algorithms do have high constants and/or exponents, and/or \leadsto useless in practice.

Question. Which would you prefer $20n^{100}$ vs. $n^{1+0.02 \ln n}$

Summary of This Lecture (1/2)

Introduce abstract definition of algorithm

How to capture algorithm's complexity?

- First simplification: functions that express number of basic computer steps of input size,

How to compare functions?

- Second simplification: Big- O notations (five standard asymptotic notations) capture order of functions. We study the definitions, typical usages, examples, properties
-

Big- O notations lets us focus on the big picture.

- Helpful analog: $O(\leq)$, $\Omega(\geq)$, $\Theta(=)$, $o(\ll)$, $\omega(\gg)$

Summary of This Lecture (1/2)

Study important running time functions and classical algorithm examples.

Notation abuses. $O(g(n))$ is a set of functions, but computer scientists often write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

Bottom line. OK to abuse notation; not OK to misuse it.

Don't misunderstand this cavalier attitude towards **constants**. Programmers are *very* interested in constants and would gladly stay up nights in order to gain 5% efficiency improvement.



Figure: Theoretical breakthrough is toooooooo hard!