

Design and Analysis of Algorithm

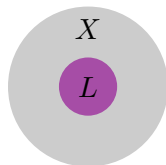
Basics of Complexity Theory

- 1 Decision Problem
- 2 Deterministic Computation
- 3 Several Important Complexity Classes
 - \mathcal{P} vs. \mathcal{NP}
 - \mathcal{NP} -complete
- 4 Randomized Computation
 - \mathcal{BPP}

Decision Problem

Decision Problem: recognition of a set of strings $L \subseteq X$

- X : a set of strings
- x : a string in X (each string corresponds to an instance)
- L : language (a subset of X satisfying some property)

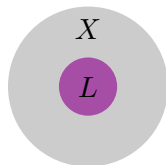


Task: Decide membership — if $x \in L$

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Example

- $X = \mathbb{N}$
- L are Primes = $\{2, 3, 5, 7, 11, 13, \dots\}$
- decide if x is a prime.

Motivation for Complexity Theory

We always want to know if a given problem can be *efficiently* solved by an algorithm.

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We always want to know if a given problem can be *efficiently* solved by an algorithm.

- ① Precisely model algorithms
 - What is computation?
 - What is computable?
- ② Precisely define what does it means for efficient.

Turing Machine

1936, London Mathematical Society: On computable numbers, with an application to the Entscheidungsproblem.

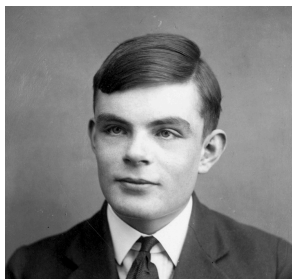
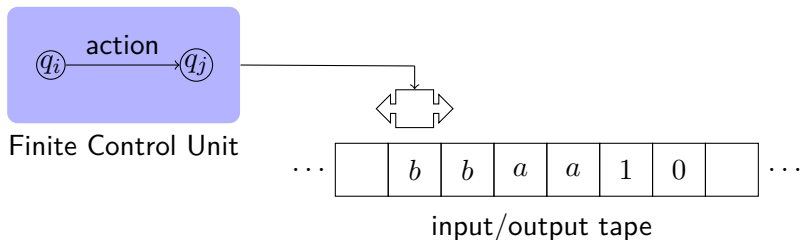


Figure: Alan Turing

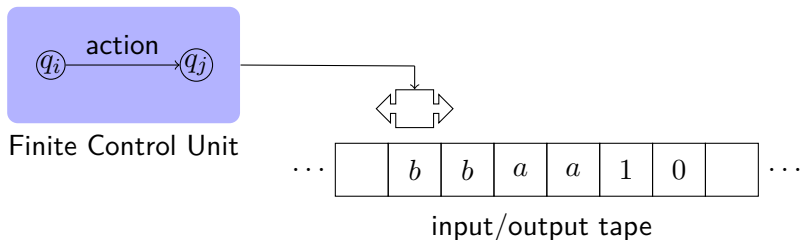
Turing Machine

Turing machine: automatic machine that has a tape (divided into infinite cells), a control unit and a read/write head.



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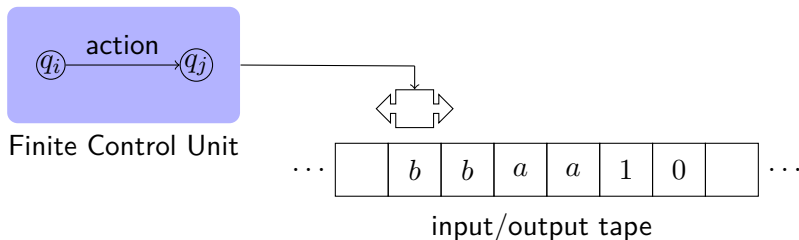
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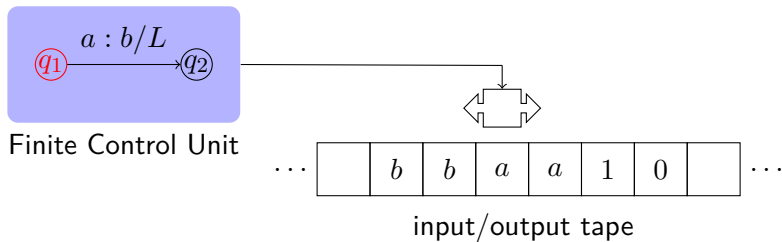
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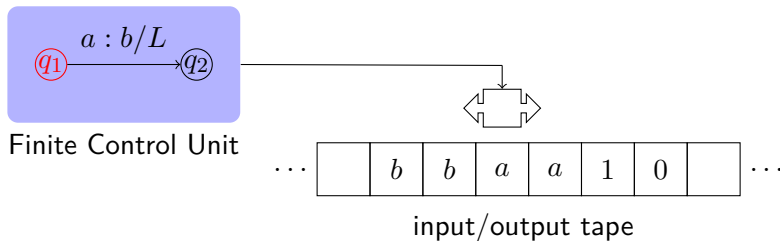


- At the beginning, the tape contains the input in several cells. Other places are empty.
- During computation, the control unit monitor current state and the head value, can do the following operations:
 - ① wipe off old value and write new values
 - ② change the current state
 - ③ move head left or right

An Example

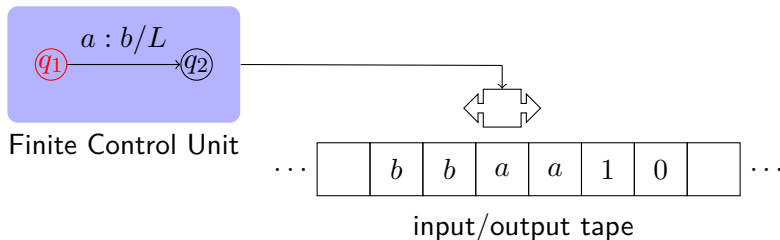


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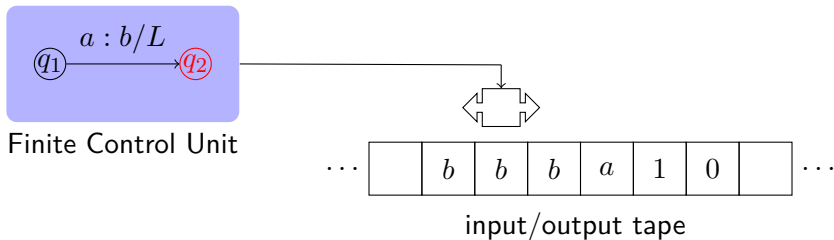


- $a \rightarrow b$; move left; current state $q_1 \rightarrow q_2$

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Mimic how human being solve a problem

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Mimic how human being solve a problem

- TM has a finite number of states (memory)
- TM is provided a tape, which contains infinite cells (paper)
- a symbol can be scanned from a cell or printed to a cell (reading and writing)

Definition 1 (Turing Machine)

TM consists $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$

- Q : a finite set of states
- Σ : input alphabets
- Γ : working alphabets (including \perp , $\Sigma \subseteq \Gamma$)
- q_0 : the initial state of Q ;
- $q_{\text{acc}}, q_{\text{rej}}$: accept and reject state of Q
- δ : transition function

$$\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

Running Time of TM

Definition 2

We denote the running time of TM by $t_M(n)$, which is the maximum steps that TM runs on all inputs of length n

Polynomial Time

$$\bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$$

The Extended Church-Turing Thesis

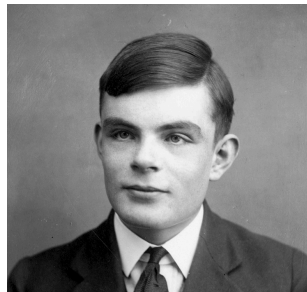


Figure: Alonzo Church & Alan Turing

Everyone's intuition of **Efficient** Algorithms = **Polynomial-Time**
deterministic TMs

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NDTM doesn't really correspond to any real-world physical model, it's just a theoretical construction.

Non-determinism doesn't give TM any power to recognize more languages.

- Any NDTM can be simulated by a TM (with potentially exponential time overhead) by trying all branches of the NDTM machine “in parallel” by using BFS.

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Why TMs are so powerful?

- TM has a working tape (好记性不如烂笔头)
- TM itself can be treated as data! TM can take another TM as its input.

Universal TM



Universal TM



||

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Time Complexity Hierarchy: \mathcal{P} and \mathcal{NP}

We have introduced

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Next, we introduce two important sets of problems, characterized by time complexity by DTM and NDTM:

$$\mathcal{P} \text{ and } \mathcal{NP}$$

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Example of \mathcal{P} Languages

- $L = \{\text{even integers}\}$, M just need to check if the last bit is 0.
- $L = \text{PRIME}$, M is the AKS primality test algorithm.

Definition 4 (\mathcal{NP} Languages - Conventional)

$L \in \mathcal{NP}$ if there exists a non-deterministic poly-time TM M :

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Alert

\mathcal{NP} means non-deterministic poly-time, not **non-poly-time**!

Modern Definition

Definition 5 (\mathcal{NP} Complexity - Modern)

$L \in \mathcal{NP}$ if there exists a deterministic poly-time TM M :

$$x \in L \iff \exists w \in \{0, 1\}^{\text{poly}(n)} \text{ s.t. } M(x, w) = 1$$

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Equivalence between traditional and modern definitions

- Even though M is a deterministic machine, its second argument w captures the nondeterminism in the definition.

Examples of \mathcal{NP} Language - Composites

$L = \text{COMPOSITE}$

- instance x is an integer
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- M just need to check if w divides x , which could be done in polynomial time.

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In fact, COMPOSITE also belong to \mathcal{P} (think why?)

Examples of \mathcal{NP} Language - SAT and 3-SAT

SAT: Given a CNF formula Φ , check if it has a satisfying truth assignment.

3-SAT: SAT where each clause contains exactly 3 literals

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Example of 3-SAT

- instance $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- witness: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$

Examples of \mathcal{NP} Language - Hamilton Path

Hamilton Graph: Given an undirected graph $G = (V, E)$, does there exist a simple path that visits every node?

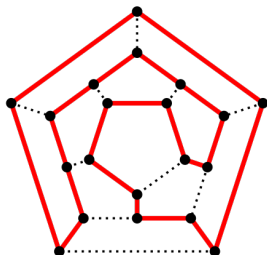


Figure: Hamiltonian Graph (a path traverses through each vertex exactly once)

witness: a path

M check if the path contains each node in V exactly once

\mathcal{P} vs. \mathcal{NP}

As per definition, $\mathcal{P} \subseteq \mathcal{NP}$. Because $L \in \mathcal{P} \Rightarrow L \in \mathcal{NP}$:

- $M'(x, w)$ can always sets $w = \perp$ and decide whether $x \in L$ using M .
- Alternatively, “short” M can be viewed as a witness for $x \in L$. Think about why the description of M is short?

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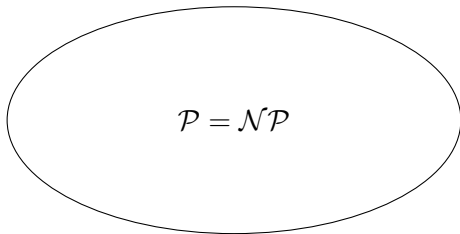
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1971: Cook, Edmonds, Levin, Yablonski, Gödel

Perhaps the most prominent question in TCS:

$$\mathcal{P} = ? \mathcal{NP}$$

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If $\mathcal{P} = \mathcal{NP}$

The foundation of modern cryptography collapse!

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In principle, every aspect of life could be efficiently and globally optimized ...

... life as we know it would be different!

The Consequence of $\mathcal{P} = \mathcal{NP}$

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Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$. To efficiently find a pre-image x of y , the idea is to determine x bit-by-bit. $f(x_1 || \cdots || x_n) = y$.

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Define $L = \{(y, x) | \exists w \text{ s.t. } y = f(x || w)\}$ (clearly $L \in \mathcal{NP}$ and thus also belong to \mathcal{P} by assumption), we define Invert as:

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Algorithm 4: Invert(y)

```
1:  $x = \epsilon$ ;  
2: for  $i \leftarrow 1$  to  $n$  do  
3:   if  $(y, 0) \in L$  then  $x = x || 0$ ;  
4:   else  $x = x || 1$ ;  
5: end  
6: return  $x$ 
```

The Reverse Direction

OWF exists $\Rightarrow \mathcal{P} \neq \mathcal{NP}$

- We have many candidates of OWFs, but they require assumptions.

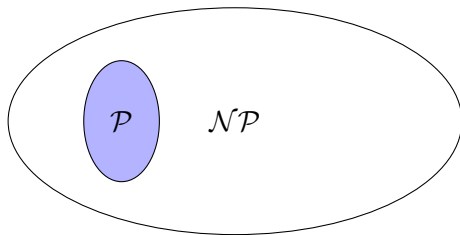
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Warning

OWFs do not exist *does not imply* $\mathcal{P} = \mathcal{NP}$



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Conjecture: $\underbrace{\text{No poly-time algorithm}}_{\text{intractable}}$ for 3-SAT

Motivation of Reduction

\mathcal{NP} is the set of many problems.

How to figure out the relations among them?

A central approach is finding reductions

Polynomial Time Reducibility

Language L' is *poly-time reducible* or *reduces* to language L , written as $L' \leq_p L$, if there is a deterministic poly-time function $\mathcal{R} : L' \rightarrow L$ so that:

$$x \in L' \iff \mathcal{R}(x) \in L$$

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We should pay attention to:

- the direction of \mathcal{R}
- the time complexity of \mathcal{R}

Definition 6 (\mathcal{NP} -Hard)

L is said to be \mathcal{NP} -hard if for every \mathcal{NP} -language L' , there is a deterministic poly-time algorithm (a reduction) \mathcal{R} :

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- We can interpret that the languages in \mathcal{NP} is not harder than that in \mathcal{NP} -hard.

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L is said to be \mathcal{NP} -hard if for every \mathcal{NP} -language L' , there is a deterministic poly-time algorithm (a reduction) \mathcal{R} :

$$x \in L' \iff \mathcal{R}(x) \in L.$$

- We can interpret that the languages in \mathcal{NP} is not harder than that in \mathcal{NP} -hard.

Fact: languages in \mathcal{NP} -hard may **not** fall in \mathcal{NP} .

Definition 7 (\mathcal{NP} -Complete)

L is \mathcal{NP} -complete if it is \mathcal{NP} -hard, and is itself in \mathcal{NP} .

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- We can solve all problems in \mathcal{NP} if we find an efficient algorithm for any problems in \mathcal{NP} -complete.

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- This theorem essentially states that if $\mathcal{P} \cap \mathcal{NPC}$ is non-empty iff $\mathcal{P} = \mathcal{NP}$.

\mathcal{P} vs. \mathcal{NP} revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.

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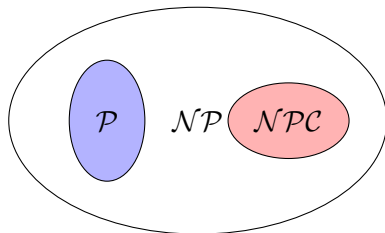


Figure: $\mathcal{P} \neq \mathcal{NP}$

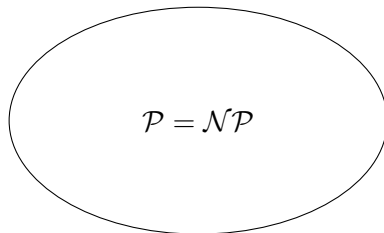


Figure: $\mathcal{P} = \mathcal{NP}$

\mathcal{P} vs. \mathcal{NP} revisited

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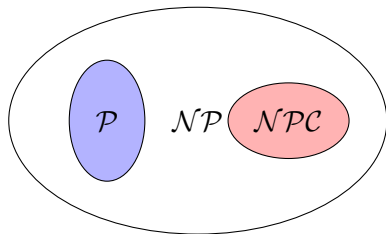


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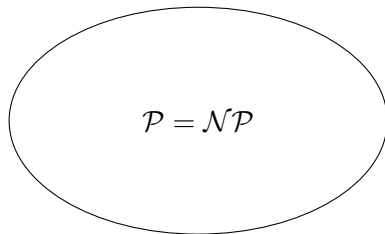
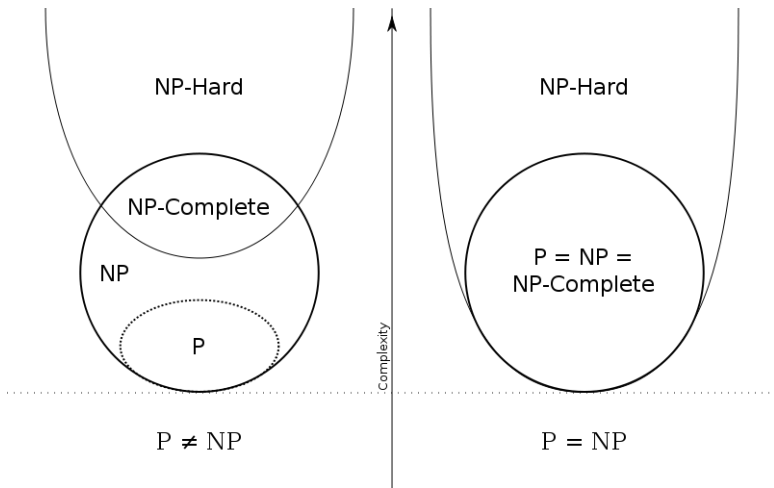


Figure: $\mathcal{P} = \mathcal{NP}$

Why we believe $\mathcal{P} \neq \mathcal{NP}$? Because some problems appear significantly harder.



Motivation of Randomized Algorithm

TM models deterministic algorithms.

TM does not seem to capture one aspect of reality — the ability to make random choices during computation

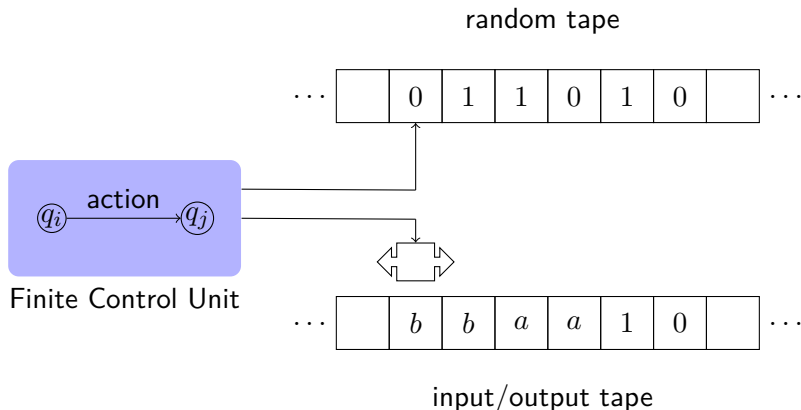
- Most programming languages provide a built-in RNG.

It makes sense to consider algorithms that can toss a coin, a.k.a. use a source of random bits. Such algorithms have been implicitly studied for a long time.

- estimate facts about a large sample by taking a small sample
- simulate real-world systems that are themselves probabilistic, such as nuclear fission and the stock market
- differential equations

Probabilistic Turing Machine

Probabilistic Polynomial-time TM models probabilistic algorithm.



PTM vs. NDTM

NDTM is a TM with two transition functions. PTM is syntactically similar.

The difference is in how we interpret the working of TM.

- In a PTM, each transition is taken with probability $1/2$, a computation that runs for time t gives rise 2^t branches in the graph of all computations, each of which is taken with probability $1/2^t$. $\Pr[M(x) = 1]$ is simply the *fraction* of branches that end with M outputting a 1.
- In a NDTM, $M(x) = 1$ iff there exists a branch that outputs 1

On a conceptual level, PTM and NDTM are very different

- PTM like TM and unlike NDTM, is intended to model realistic computation devices.

Bounded-Error Probabilistic Polynomial Time

Definition 9 (\mathcal{BPP} Complexity)

$L \in \mathcal{BPP}$ iff there exists a probabilistic polynomial time TM M such that:

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Bounded-error Probabilistic Polynomial Time (weak version)

- A typical choices is $\alpha = 2/3$, $\beta = 1/3$. In this case, the class of decision problems solvable by a probabilistic TM in polynomial time with an error probability e bounded away from $1/3$ for all instances

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In practice, an error probability of $1/3$ might not be acceptable.

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- It does not even have to be constant: ϵ could be as high as $1/2 - n^{-c}$ on one hand, or as small as 2^{-n^c} on the other hand, where c is any positive constant, and n is the length of input.
- The idea is if the algorithm is run many times, the chance that the majority of the runs are wrong drops off exponentially as a consequence of the Chernoff bound.

Reduce the Error (2/2)

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Chernoff Bounds (Lower Tail): Let $X = \sum_{i=1}^n X_i$, $\Pr[X_i] = p$, $\mu = \mathbb{E}(X) = np$.

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Do the Majority Vote, i.e., set $(1 - \delta)\mu = n/2$ and thus $\delta = 1 - 1/2p$, we obtain:

$$\Pr[X \leq n/2] \leq e^{-n \frac{(1-2p)^2}{8p}}$$

About BPP

BPP is one of the largest practical classes of problems, since problems in BPP have efficient probabilistic algorithms that can be run quickly on real modern machines.

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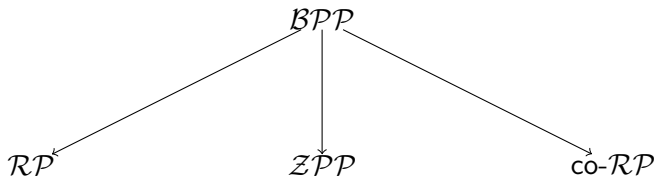
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For a long time, one of the most famous problems that was known to be in BPP but not known to be in \mathcal{P} was the PRIME.

[Agrawal, Kayal, Saxena 2002]: gave a deterministic polynomial-time algorithm for PRIME, thus showing that it is in \mathcal{P} .

One-sided and Zero-sided Error

two-sided error



$x \in L \Rightarrow \Pr[M(x) = 1] \geq 2/3$
 $x \notin L \Rightarrow M(x) = 0$

always correct
no error

$x \in L \Rightarrow M(x) = 1$
 $x \notin L \Rightarrow \Pr[M(x) = 1] \leq 1/3$

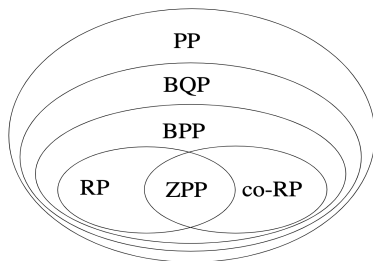
ZPP: probabilistic polynomial-time TM always returns correct YES or NO answer, or halts with low probability, a.k.a. running time is polynomial **in expectation** for every input

- **BPP**: Monte Carlo algorithms (probabilistic) likely to be correct in strict polynomial running time
- **ZPP**: Las Vegas algorithms (probabilistic) are always correct in expected polynomial running time

BPP in Relation to Other Probabilistic Complexity Classes

BQP (bounded-error quantum polynomial time): the class of decision problems solvable by a quantum TM in polynomial time with bounded error

- It is the quantum analogue of BPP



Limits of BPP

Consensus: $\mathcal{P} \subseteq \underline{\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}} \subseteq BPP \subseteq \mathcal{NP}$

$\mathcal{P} \subseteq BPP$

- An important example of a problem in BPP still not known to be in \mathcal{P} is **polynomial identity testing** — determining whether a polynomial is identically equal to the zero polynomial, when you have access to the value of the polynomial for any given input, but not to the coefficients.

$BPP \subseteq \mathcal{NP}$

- Adleman's theorem: $BPP \subseteq P/\text{poly}$ (polynomial-size Boolean circuits)
- Karp-Levin theorem: $\mathcal{NP} \subseteq P/\text{poly} \Rightarrow PH = \sum_2^P$

Thus, $\mathcal{NP} \subseteq BPP$ will imply collapse of PH, which is unlikely to be true. In other words, \nexists bounded-error probabilistic algorithms for \mathcal{NPC} problems.