NP-hardness of ℓ_0 minimization problems: revision and extension to the non-negative setting

Vu Thi Hai Yen and Abdel Wedoud Mohamed

Master Datascience

March 25, 2020

Outline

Outline

ℓ_0 minimization problems

- In compressed sensing, we are interested in solving the following problem:

$$\min_{\mathbf{x} \text{ s.t. } \mathbf{y}=A\mathbf{x}} \|x\|_0 \tag{$\ell_0 LC$}$$

- However, when the data contains noise, we are interested in the following problems:

$$\min_{\mathbf{x} \text{ s.t. } \|\mathbf{y}-A\mathbf{x}\|_2 \leq \epsilon} \|x\|_0 \tag{$\ell_0 C$}$$

$$\min_{\mathbf{x} \text{ s.t. } \|x\|_0 \leq K} \|\mathbf{y} - A\mathbf{x}\|_2^2 \tag{$\ell_0 C'$}$$

- Penalized version:

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda \|x\|_0 \tag{$\ell_0 P$}$$

Complexity of ℓ_0 minimization problems

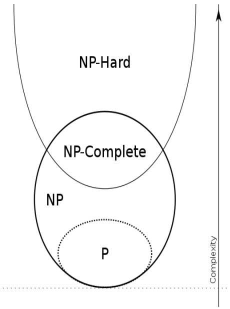
- This work aims to study the complexities of these problems.
- A naive approach could have a complexity of:

$$\sum_{i=1}^{S} \binom{n}{i} \geq \sum_{i=1}^{S} \left(\frac{n}{i}\right)^{i} \geq \left(\frac{n}{S}\right)^{S} \tag{1}$$

  which is exponential in S.
- Indeed, all the aforementioned problems are NP-hard.

Outline

Notions of complexity analysis



- **P**: Exist an algorithm that solves in polynomial time.
- **NP**: Exist an algorithm that **verifies** the solution in polynomial time.
  - **NP-complete**: Harder than any problem in NP.
- **NP-hard**: Harder than any problem in NP, but not necessarily in NP.
- Y is **harder** than X, or X is reduced to Y: Each instance of X can be transformed to an instance of Y in polynomial time, and solving Y also solves X.

Previous complexity analyses

- $(\ell_0 LC)$, $(\ell_0 C)$ and $(\ell_0 C')$ are NP-hard
  - Reduced from the X3C problem, which is known to be NP-complete.
- $(\ell_0 P)$ was claimed to be NP-hard: particular case of general analyses, however, they contain some defects:
  - Show that the class of problems:

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_q^q + \lambda \|\mathbf{x}\|_p^p \tag{ℓ_q-ℓ_p}$$

    are NP-hard using transformation from the case $\lambda = 1/2$. But this is not true for $p = 0$.
  - Show that the problem

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda \sum_{i=1}^{n} \phi(|x_i|) \tag{PLS}$$

    is NP-hard for ϕ satisfying some conditions. But these conditions are not satisfied for ℓ_0 function.

Outline

$\ell_0 P$ First Theorem

Theorem

The problem $(\ell_0 P)$ *is NP-hard for* $0 < \lambda < 3$

Proof.

- Construct $(\ell_0 P)$ instance from X3C instance: let C, a collection of triplets, be the instance for X3C, then by setting $A = (a_{i,j})_{ij}$ such that $a_{ij} = 1_{\{s_i \in c_j\}}$, and $y = 1 - vector$, we get the wanted result.
- Solution of $(\ell_0 P)$ from X3C : absurdity argument
- Treat cases with X3C admitting a solution : $\|\overline{x}\|_0 \neq \dfrac{m}{3}$ (1), $\|\overline{x}\|_0 = \dfrac{m}{3}$, $y \neq A\overline{x}$ (2) and $\|\overline{x}\|_0 = \dfrac{m}{3}$, $y = A\overline{x}$ (3). Conclude by absurdity argument.

$\square$

Third Theorem : non-negative ℓ_0 minimization problems

- The non-negative versions of $(\ell_0 C)$, $(\ell_0 C')$ and $(\ell_0 P)$:

$$\min_{\mathbf{x}\geq 0 \text{ s.t. } \|\mathbf{y}-A\mathbf{x}\|_2 \leq \epsilon} \|x\|_0 \qquad (\ell_0 C+)$$

$$\min_{\mathbf{x}\geq 0 \text{ s.t. } \|x\|_0 \leq K} \|\mathbf{y}-A\mathbf{x}\|_2^2 \qquad (\ell_0 C'+)$$

$$\min_{\mathbf{x}\geq 0} \|\mathbf{y}-A\mathbf{x}\|_2^2 + \lambda \|x\|_0 \qquad (\ell_0 P+)$$

Third Theorem : non-negative ℓ_0 minimization problems

Theorem

$(\ell_0 C+), (\ell_0 C'+)$ *are NP-hard.* $(\ell_0 P+)$ *is NP-hard for* $0 < \lambda < 3$.

Proof.

Transform $(\ell_0 P+)$ into an $(\ell_0 P)$ problem by setting :

$$\tilde{\mathbf{y}} = \mathbf{y}, \quad \tilde{A} = [A, -A], \quad \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix}.$$

Similar for other problems. $\square$

Second and fourth Theorems : First Generalisation

- Generalizes to ℓ_p norm:

$$\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|_p^p + \lambda \|x\|_0 \qquad (\ell_p\text{-}\ell_0 P)$$

$$\min_{\mathbf{x} \geq 0} \|\mathbf{y} - A\mathbf{x}\|_p^p + \lambda \|x\|_0 \qquad (\ell_p\text{-}\ell_0 P+)$$

Theorem

The problem (ℓ_p-$\ell_0 P$) is NP-hard for $p \geq 1$ and $0 < \lambda < 3$.

Theorem

The problem (ℓ_p-$\ell_0 P+$) is NP-hard for $p \geq 1$ and $0 < \lambda < 3$.

Proof.

Comes naturally by following the same steps as in theorems 1 and 3. □

Outline

Extending to all $\lambda > 0$

- If we can show that the exact cover by k-set (XkC) is NP-complete, then $(\ell_0 P)$ is NP-hard for all $0 < \lambda < k$.

- Thus, if the XkC problem is NP-complete for certains $k = k_1, k_2, \ldots$ s.t. $k_n \to +\infty$, then $(\ell_0 P)$ is NP-hard $\forall \lambda > 0$.

Extending to all $\lambda > 0$

Lemma

The exact cover by k-set (XkC) problem is NP-complete $\forall k = 3^n, n \in \mathbb{N}$.

Proof.

Proof by induction. If XkC is NP-complete, reduce it to the same problem with additional constraint that $|X|$ is divisible by $3k$, then reduce this problem to X(3k)C. $\square$

Theorem

The problems ($\ell_0 P$) and ($\ell_0 P+$) are NP-hard for $\lambda > 0$. *The problems (ℓ_p-$\ell_0 P$) and (ℓ_p-$\ell_0 P+$) are NP-hard for* $p \geq 1,\ \lambda > 0$.

Outline

First Analysis

- The equivalence between X3C and ($\ell_0 P$) is shown for particular instances of ($\ell_0 P$), that means it is at least as hard as X3C
- To have an idea of trivial lower boundaries, we should find them for X3C (But NP = P not answered yet)
- A greedy algorithm with ratio decaying very slowly though tending to 0 is already known for X3C

Quantile efficient algorithm

- Choose uniformly M distinct subsets S_i of $[1, ..., n] = [n]$, $1 \leq i \leq M$
- For i in range $[1, ..., M]$: Solve $(\ell_0 P)$ such that $x_j = 0$ if $j \in S_i$, and $\|x\|_0 = |S_i|$, and get the solution x_i^* and its associated value $f(x_i^*)$
- Return the x_i^* with the smallest value $f(x_i^*)$

Outline

Contributions

- The $(\ell_0 P)$ is defined for every λ (very implicit in the article)
- A proof for the validity of the results for $\lambda > 0$
- Quantile efficient algorithm

Perspectives

- A final general result: couldn't we even extend to cases that involve duality in general
- Heuristic algorithm that optimises efficiently with respect to the problem value

References I

Thi Thanh Nguyen, Charles Soussen, Jérôme Idier, and El-Hadi Djermoune.
NP-hardness of $\ell 0$ minimization problems: revision and extension to the non-negative setting.
In *13th International Conference on Sampling Theory and Applications, SampTA 2019*, 13th International Conference on Sampling Theory and Applications, SampTA 2019, Bordeaux, France, July 2019.