Tugas 2 Aljabar Linear 2022

Hezkiel Bram Setiawan
M0521030
dibuat menggunakan LaTeX
Source code: github.com/haizk

Fakultas Matematika dan Ilmu Pengetahuan Alam
Universitas Sebelas Maret

Linear Transformations

1. **Exercise 7.1.3**
   In each case, assume that T is a linear transformation.

   f) If $T : \mathbf{P}_2 \to \mathbb{R}$ and $T(x+2) = 1$, $T(1) = 5$, $T(x^2+x) = 0$, find $T(2-x+3x^2)$.

   Diketahui:

   $T : \mathbf{P}_2 \to \mathbb{R}$

   $T(x+2) = 1$

   $T(1) = 5$

   $T(x^2 + x) = 0$

   Ditanya:

   - Nilai $T(2 - x + 3x^2)$?

   Jawab:

$$\begin{aligned} T(2 - x + 3x^2) &= T(3x^2 - x + 2) \\ &= T(3x^2 + 3x - 4x - 8 + 10) \\ &= T(3(x^2 + x)) - T(4(x + 2)) + T(10(1)) \\ &= 3T(x^2 + x) - 4T(x + 2) + 10T(1) \\ &= 3(0) - 4(1) + 10(5) \\ &= 46 \end{aligned}$$

$\therefore\ T(2 - x + 3x^2) = 46$

2. **Exercise 7.1.4**
   In each case, find a linear transformation with the given properties and compute $T(v)$.
   c) $T : \mathbf{P}_2 \to \mathbf{P}_3$; $T(x^2) = x^3$, $T(x+1) = 0$, $T(x-1) = x$; $v = x^2 + x + 1$

   Diketahui:

   $T : \mathbf{P}_2 \to \mathbf{P}_3$

   $T(x^2) = x^3$

   $T(x+1) = 0$

   $T(x-1) = x$

   $v = x^2 + x + 1$

   Ditanya:

   - Nilai $T(v)$?

   Jawab:

$$\begin{aligned} T(v) &= T(x^2 + x + 1) \\ &= T(x^2) + T(x+1) \\ &= x^3 + 0 \\ &= x^3 \end{aligned}$$

   $\therefore\ T(v) = x^3$

3. **Exercise 7.2.1**
   For each matrix A, find a basis for the kernel and image T_A, and find the rank and nullity of T_A.
   a) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$

   Diketahui:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

   Ditanya:

   - Basis dari $\mathrm{Ker}(T_A)$ dan $\mathrm{Im}(T_A)$?
   - Rank dan nullity dari T_A?

Jawab:

$$A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} R_2 \to R_2 - 2R_1 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -5 \\ -1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_1 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -5 \\ 0 & 3 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_1 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -5 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_4 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 3 & 3 \\ 0 & -5 & -5 \end{bmatrix}$$

$$R_3 \to R_3 + 3R_2 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -5 & -5 \end{bmatrix}$$

$$R_4 \to R_4 - 5R_2 \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis dari $\mathrm{Im}(T_A)$ adalah $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$ dan rank T_A adalah 2. Rank didapat dari nilai $\dim(\mathrm{Im}(T_A))$.

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix} R_2 \to R_2 - 3B_1 \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -1 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_1 \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -1 \\ 0 & -5 & 3 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to -\frac{R_2}{5} \begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Untuk mencari kernel, kita harus menyelesaikan:

$$\begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Misal $x_3 = t, x_4 = s,$ maka $x_1 = -\frac{3s}{5} - \frac{t}{5}, x_2 = -\frac{s}{5} + \frac{3t}{5}$, didapat:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{3s}{5} - \frac{t}{5} \\ -\frac{s}{5} + \frac{3t}{5} \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} s$$

$\therefore$ Basis dari $\mathrm{Ker}(T_A)$ adalah $\left\{ \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$ dan nullity T_A adalah 2.

Nullity didapat dari nilai $\dim(\mathrm{Ker}(T_A))$.

c) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$

Diketahui:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$$

Ditanya:

- Basis dari $\mathrm{Ker}(T_A)$ dan $\mathrm{Im}(T_A)$?
- Rank dan nullity dari T_A?

Jawab:

$$A^T = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 2 & 1 & -1 & 2 \\ -1 & 2 & 5 & -2 \end{bmatrix} R_2 \to R_2 - 2R_1 \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -5 & -9 & 2 \\ -1 & 2 & 5 & -2 \end{bmatrix}$$

$$R_3 \to R_3 + R_1 \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -5 & -9 & 2 \\ 0 & 5 & 9 & -2 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -5 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis dari $\mathrm{Im}(T_A)$ adalah $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ -9 \\ 2 \end{bmatrix} \right\}$ dan rank T_A adalah 2. Rank didapat dari nilai $\dim(\mathrm{Im}(T_A))$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix} R_2 \to R_2 - 3R_1 \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{bmatrix}$$

$$R_3 \to R_3 - 4R_1 \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & -9 & 9 \\ 0 & 2 & -2 \end{bmatrix}$$

$$R_2 \to -\frac{R_2}{5} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -9 & 9 \\ 0 & 2 & -2 \end{bmatrix}$$

$$R_3 \to R_3 - 9R_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_2 \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Untuk mencari kernel, kita harus menyelesaikan:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Misal $x_3 = t,$ maka $x_1 = -t, x_2 = t$, didapat:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t$$

$\therefore$ Basis dari Ker(T_A) adalah $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ dan nullity T_A adalah 1. Nullity didapat dari nilai dim(Ker(T_A)).

4. **Exercise 7.2.2**
In each case, (i) find a basis of Ker(T), and (ii) find a basis of Im(T). You may assume that T is linear.

f) $T : \mathbf{M}_{22} \to \mathbb{R}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$

Diketahui:

$T : \mathbf{M}_{22} \to \mathbb{R}$

$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$

T linear

Ditanya:

- Basis dari Ker(T) dan Im(T)?

Jawab:

$$\begin{aligned} \mathrm{Ker}(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22} \middle| a = -d \right\} \\ &= \left\{ \begin{bmatrix} -d & b \\ c & d \end{bmatrix} \middle| b, c, d \in \mathbb{R} \right\} \\ &= \left\{ d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \middle| b, c, d \in \mathbb{R} \right\} \\ &= span \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \end{aligned}$$

Terlihat set $\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ ialah linearly independent.

$\therefore \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ adalah basis dari Ker(T).

Untuk setiap $a \in \mathbb{R}$, kita mempunyai:

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathbf{M}_{22} \text{ dan } T \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) = a$$

Oleh karena itu, T surjektif dan mengimplikasi bahwa Im(T) $= \mathbb{R}$.

$\therefore \{1\}$ adalah basis dari Im(T).

5. **Exercise 7.2.14**
Consider $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a+c=b+d \right\}$

a) Consider $S : \mathbf{M}_{22} \to \mathbb{R}$ with $S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+c-b-d$. Show that S is linear and onto and that V is a subspace of $\mathbf{M}_{22}$. Compute $\dim(V)$.

Diketahui:

$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a+c=b+d \right\}$

$S : \mathbf{M}_{22} \to \mathbb{R}$

$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+c-b-d$

Ditanya:

- Buktikan S linear dan onto, serta V adalah subspace dari $\mathbf{M}_{22}$.
- Nilai $\dim(V)$?

Jawab:

$$\begin{aligned} S(\mathbf{X}_1 + \mathbf{X}_2) &= S\left\{ \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right\} \\ &= S \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \\ &= (a_1+a_2)+(c_1+c_2)-(b_1+b_2)_(d_1+d_2) \\ &= (a_1+c_1-b_1-d_1)+(a_2+c_2-b_2-d_2) \\ &= S\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + S\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\ &= S(\mathbf{X}_1) + S(\mathbf{X}_2) \end{aligned} \tag{1}$$

$$\begin{aligned} S(r\mathbf{X}_1) &= S\left(r \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right) \\ &= S\begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix} = ra_1 + rc_1 - rb_1 - rd_1 \\ &= r(a_1+c_1-b_1-d_1) = rS(\mathbf{X}_1) \end{aligned} \tag{2}$$

$\therefore$ S adalah linear berdasarkan persamaan (1) dan (2). ■

Misal $r \in \mathbb{R}$. Kita mendapatkan:

$$S\begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} = r$$

$\therefore$ S adalah onto. ■

$$
\begin{aligned}
V &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a + c = b + d \right\} \\
0 + 0 = 0 + 0 &\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R} \\
&\Rightarrow 0 \in \mathbb{R} \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X}_1, \mathbf{X}_2 \in V &\Rightarrow \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in V \wedge \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in V, a_i + c_i = b_i + d_i, i = 1, 2 \\
&\Rightarrow \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in V, \sum_{i=1}^{2} (a_i + c_i) = \sum_{i=1}^{2} (b_i + d_i) \\
&\Rightarrow \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in V, \sum_{i=1}^{2} (a_i + c_i) = \sum_{i=1}^{2} (b_i + d_i) \\
&\Rightarrow \mathbf{X}_1 + \mathbf{X}_2 \in \mathbb{R} \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{X}_1 \in V, r \in \mathbb{R} &\Rightarrow \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \mathbb{R}, a_1 + c_1 = b_1 + d_1, a \in \mathbb{R} \\
&\Rightarrow \begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix} \in V, ra_1 + rc_1 = rb_1 + rd_1, r \in \mathbb{R} \\
&\Rightarrow r\mathbf{X}_1 \in V, r \in \mathbb{R} \qquad (5)
\end{aligned}
$$

$\therefore$ V adalah subspace dari $\mathbf{M}_{22}$ berdasarkan persamaan (3), (4), dan (5). ■

$$
\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V &\Rightarrow a + c = b + d \\
&\Leftrightarrow a = b + d - c \\
&\Leftrightarrow \begin{bmatrix} b + d - c & b \\ c & d \end{bmatrix} \in V \\
&\Rightarrow \left\{ b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; b, c, d \in \mathbb{R} \right\} = V \\
&\Rightarrow span \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = V
\end{aligned}
$$

Terlihat set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ialah linearly independent.

$\therefore$ Karena set tersebut merupakan basis dari subset V, $\dim(V) = 3$.

b) Consider $T : V \to \mathbb{R}$ with $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + c$. Show that T is linear and onto, and use this information to compute $\dim(\mathrm{Ker}(T))$.

Diketahui:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a + c = b + d \right\}$$

$$T : V \to \mathbb{R}$$

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + c$$

Ditanya:

- Buktikan T linear dan onto.
- Nilai $\dim(\mathrm{Ker}(T))$?

Jawab:

$$\begin{aligned} T(\mathbf{X}_1 + \mathbf{X}_2) &= T \left\{ \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right\} \\ &= T \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\ &= (a_1 + a_2) + (c_1 + c_2) \\ &= (a_1 + c_1) + (a_2 + c_2) \\ &= T \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + T \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\ &= T(\mathbf{X}_1) + T(\mathbf{X}_2) \end{aligned} \tag{1}$$

$$\begin{aligned} T(r\mathbf{X}_1) &= T \left(r \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right) \\ &= T \begin{bmatrix} ra_1 & rb_1 \\ rc_1 & rd_1 \end{bmatrix} = ra_1 + rc_1 \\ &= r(a_1 + c_1) = rS(\mathbf{X}_1) \end{aligned} \tag{2}$$

$\therefore$ T adalah linear berdasarkan persamaan (1) dan (2). ■

Misal $r \in \mathbb{R}$. Kita mendapatkan:

$$T \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} = r$$

$\therefore$ $T : V \to \mathbb{R}$ adalah onto. ■

Dengan demikian, kita mendapatkan $T(V) = \mathbb{R}$, dan menghasilkan nilai $\dim(\mathrm{Im}(T)) = 1$

$$
\begin{aligned}
\dim(V) &= \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) \\
\Rightarrow \dim(\mathrm{Ker}(T)) &= \dim(V) - \dim(\mathrm{Im}(T)) \\
\Rightarrow \dim(\mathrm{Ker}(T)) &= 3 - 1 = 2
\end{aligned}
$$

$\therefore$ Nilai $\dim(\mathrm{Ker}(T))$ adalah 2.

Orthogonality

Orthogonal Complements and Projections

1. **Exercise 8.1.1**
   In each case, use the Gram-Schmidt algorithm to convert the given basis B of V into an orthogonal basis.
   c) $V = \mathbb{R}^3$, $B = \{(1, -1, 1),\ (1, 0, 1),\ (1, 1, 2)\}$
      Diketahui:
      $V = \mathbb{R}^3$
      $B = (1, -1, 1), (1, 0, 1), (1, 1, 2)$
      $\mathbf{x}_1 = (1, -1, 1)$
      $\mathbf{x}_2 = (1, 0, 1)$
      $\mathbf{x}_3 = (1, 1, 2)$
      Ditanya:
      - Orthogonal basis dari basis B of V?

      Jawab:

$$
\begin{aligned}
\mathbf{f}_1 &= \mathbf{x}_1 \\
&= (1, -1, 1) \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{f}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\
&= (1, 0, 1) - \frac{(1, 0, 1)(1, -1, 1)}{\|(1, -1, 1)\|^2}(1, -1, 1) \\
&= (1, 0, 1) - \frac{2}{3}(1, -1, 1) \\
&= (\frac{1}{3}, \frac{2}{3}, \frac{1}{3}) = \frac{1}{3}(1, 2, 1) \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{f}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&= (1,1,2) - \frac{(1,1,2)(1,-1,1)}{\|(1,-1,1)\|^2}(1,-1,1) - \frac{(1,1,2)\cdot\frac{1}{3}(1,2,1)}{\|\frac{1}{3}(1,2,1)\|^2}\cdot\frac{1}{3}(1,2,1) \\
&= (1,1,2) - \frac{2}{3}(1,-1,1) - \frac{5}{3}(1,2,1) \\
&= (-\frac{4}{3}, -\frac{5}{3}, -\frac{1}{3}) = \frac{1}{3}(-4,-5,-1) \qquad (3)
\end{aligned}
$$

$\therefore$ Orthogonal basisnya adalah $\{(1,-1,1), \frac{1}{3}(1,2,1), \frac{1}{3}(-4,-5,-1)\}$.

2. **Exercise 8.1.2**
   In each case, write $\mathbf{x}$ as the sum of a vector in U and a vector in $U^{\perp}$.

   f) $\mathbf{x} = (a,b,c,d)$,
   $U =$ span $\{(1,-1,2,0),\ (-1,1,1,1)\}$

   Diketahui:

   $\mathbf{x} = (a,b,c,d)$

   $U = \text{span}(1,-1,2,0),(-1,1,1,1)$

   Ditanya:

   - Tulis $\mathbf{x}$ sebagai sum dari vector in U dan vector in $U^{\perp}$.

   Jawab:

   Misal $\mathbf{e}_1 = (1,-1,2,0)$ dan $\mathbf{e}_2 = (-1,1,1,1)$ maka:

$$
\begin{aligned}
\mathbf{x}_1 &= \text{proj}_U\mathbf{x} \\
&= \frac{a-b+2c}{6}(1,-1,2,0) + \frac{-a+b+c+d}{4}(-1,1,1,1) \\
&= (\frac{5a-5b+c-3d}{12}, \frac{-5a+5b-c+3d}{12}, \\
&\frac{a-b+11c+3d}{12}, \frac{-3a+3b+3c+3d}{12}) \\
\mathbf{x}_2 &= \mathbf{x} - \mathbf{x}_1 \\
&= (\frac{7a+5b-c+3d}{12}, \frac{5a+7b+c-3d}{12} \\
&\frac{-a-b+c-3d}{12}, \frac{3a-3b-3c+9d}{12})
\end{aligned}
$$

$\therefore\ \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$

3. **Exercise 8.1.3**
   Let $\mathbf{x} = (1, -2, 1, 6)$ in $\mathbb{R}^4$, and let $U = \text{ span } \{(2, 1, 3, -4),\ (1, 2, 0, 1)\}$.
   a) Compute $\text{proj}_U\ \mathbf{x}$.
   b) Show that $\{(1, 0, 2, -3),\ (4, 7, 1, 2)\}$ is another orthogonal basis of U.
   c) Use the basis in part (b) to compute $\text{proj}_U\ \mathbf{x}$.

   Diketahui:

   $\mathbf{x} = (1, -2, 1, 6) \in \mathbb{R}^4$

   $U = \text{ span } \{(2, 1, 3, -4),\ (1, 2, 0, 1)\}$

   Ditanya:
   - Nilai $\text{proj}_U\mathbf{x}$?
   - Tunjukkan bahwa $\{(1, 0, 2, -3), (4, 7, 1, 2)\}$ adalah contoh lain orthogonal basis U.
   - Gunakan basis di atas untuk menghitung nilai $\text{proj}_U\mathbf{x}$.

   Jawab:

$$
\begin{aligned}
\text{proj}_U\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&= \frac{(1, -2, 1, 6)(2, 1, 3, -4)}{\|(2, 1, 3, -4)\|^2}(2, 1, 3, -4) \\
&+ \frac{(1, -2, 1, 6)(1, 2, 0, 1)}{\|(1, 2, 0, 1)\|^2}(1, 2, 0, 1) \\
&= -\frac{21}{30}(2, 1, 3, -4) + \frac{3}{6}(1, 2, 0, 1) \\
&= \frac{3}{10}(-3, 1, -7, 11)
\end{aligned}
$$

   $\therefore\ \text{proj}_U\mathbf{x} = \frac{3}{10}(-3, 1, -7, 11)$.

$$
\begin{aligned}
U &= \text{span}\{(2, 1, 3, -4), (1, 2, 0, 1)\} \\
&= \alpha(2, 1, 3, -4) + \beta(1, 2, 0, 1) \\
&= (2\alpha + \beta, \alpha + 2\beta, 3\alpha, -4\alpha + \beta) \\
\mathrm{f}_1 &= (1, 0, 2, -3); \alpha = \frac{2}{3}, \beta = -\frac{1}{3} \\
\mathrm{f}_2 &= (4, 7, 1, 2); \alpha = \frac{1}{3}, \beta = \frac{10}{3}
\end{aligned}
$$

$$
\mathrm{f}_1 \cdot \mathrm{f}_2 = 4 + 0 + 2 - 6 = 0
$$

   $\therefore\ \{(1, 0, 2, -3), (4, 7, 1, 2)\}$ juga span $\{(2, 1, 3, -4),\ (1, 2, 0, 1)\}$

$$\begin{aligned}
\text{proj}_U\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x}\cdot\mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&= \frac{(1,-2,1,6)(1,0,2,-3)}{\|(1,0,2,-3)\|^2}(1,0,2,-3) \\
&+ \frac{(1,-2,1,6)(4,7,1,2)}{\|(4,7,1,2)\|^2}(4,7,1,2) \\
&= -\frac{15}{14}(1,0,2,-3) + \frac{3}{70}(4,7,1,2) \\
&= \frac{3}{10}(-3,1,-7,11)
\end{aligned}$$

$\therefore$ Nilai $\text{proj}_U\mathbf{x}$ tetap sama walaupun menggunakan basis yang berbeda.

4. **Exercise 8.1.4**
   In each case, use the Gram-Schmidt algorithm to find an orthogonal basis of the subspace U, and find the vector in U closest to $\mathbf{x}$.

   c) $U =$ span $\{(1,0,1,0),\ (1,1,1,0),\ (1,1,0,0)\}$,
   $\mathbf{x} = (2,0,-1,3)$

   Diketahui:

   $U = \text{span}\{(1,0,1,0),(1,1,1,0),(1,1,0,0)\}$

   $\mathbf{x} = (2,0,-1,3)$

   Ditanya:

   - Orthogonal basis dari subspace U?
   - Vector dalam U yang terdekat dengan $\mathbf{x}$?

   Jawab:

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{x}_1 \\
&= (1,0,1,0)
\end{aligned} \tag{1}$$

$$\begin{aligned}
\mathbf{f}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2\cdot\mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\
&= (1,1,1,0) - \frac{(1,1,1,0)(1,0,1,0)}{\|(1,0,1,0)\|^2}(1,0,1,0) \\
&= (1,1,1,0) - \frac{2}{3}(1,0,1,0) \\
&= (0,1,0,0)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\mathbf{f}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\
&= (1,1,0,0) - \frac{(1,1,0,0)(1,0,1,0)}{\|(1,0,1,0)\|^2}(1,0,1,0) \\
&- \frac{(1,1,0,0)(0,1,0,0)}{\|(0,1,0,0)\|^2}(0,1,0,0) \\
&= (1,1,0,0) - \frac{1}{2}(1,0,1,0) - (0,1,0,0) \\
&= \frac{1}{2}(1,0,-1,0) \qquad (3)
\end{aligned}$$

$\therefore$ $\{(1,0,1,0),(0,1,0,0),(\frac{1}{2}(1,0,-1,0))\}$ adalah orthogonal basis dari subspace U.

$$\begin{aligned}
\mathrm{proj}_U &= \frac{(\mathbf{x}) \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 \\
&= \frac{(2,0,-1,3) \cdot (1,0,1,0)}{\|(1,0,1,0)\|^2}(1,0,1,0) + \frac{(2,0,-1,3) \cdot (0,1,0,0)}{\|(0,1,0,0)\|^2}(0,1,0,0) \\
&+ \frac{(2,0,-1,3) \cdot \frac{1}{2}(1,0,-1,0)}{\|\frac{1}{2}(1,0,-1,0)\|^2}\frac{1}{2}(1,0,-1,0) \\
&= \frac{1}{2}(1,0,1,0) + 0 + \frac{3}{2}(1,0,-1,0) \\
&= \frac{1}{2}(4,0,-2,0) = (2,0,-1,0)
\end{aligned}$$

$\therefore$ $(2,0,-1,0)$ adalah vector dalam U yang terdekat dengan $\mathbf{x}$.

5. **Exercise 8.1.5**
   Let $U = \text{ span } \{\mathbf{v}_1, \mathbf{v}_2, \ \ldots \ , \mathbf{v}_k\}$; $\mathbf{v}_i$ in $\mathbb{R}^n$, and let A be the $k \times n$ matrix with the $\mathbf{v}_i$ as rows.

   a) Show that $U^\perp = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n, A\mathbf{x}^T = 0\}$.
   b) Use part (a) to find $U^\perp$ if
   $U = \text{ span } \{(1,-1,2,1), \ (1,0,-1,1)\}$.

   Diketahui:

   $U = \text{ span } \{\mathbf{v}_1, \mathbf{v}_2, \ \ldots \ , \mathbf{v}_k\}$

   A $k \times n$ matriks dengan $\mathbf{v}_i$ sebagai baris.

   Ditanya:

   - Tunjukkan bahwa $U^\perp = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n, A\mathbf{x}^T = 0\}$.
   - Nilai $U^\perp$ jika $U = \text{ span } \{(1,-1,2,1), \ (1,0,-1,1)\}$?

Jawab:

$U^{\perp}$ mempunyai semua element seperti $v_1x = 0, v_2x = 0, \ldots, v_kx = 0$

A adalah matriks dengan v_i sebagai baris.

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_k \end{bmatrix} \mathbf{x}^T$$

Maka:

$$A\mathbf{x}^T = \begin{bmatrix} v_1x \\ v_2x \\ \cdots \\ v_kx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

$\therefore A\mathbf{x}^T = 0.$

$U = \text{ span } \{(1, -1, 2, 1), (1, 0, -1, 1)\}$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$B_2 \to B_2 - B_1$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{x}_2 - 3\mathbf{x}_3 &= 0 \\ \mathbf{x}_2 &= 3\mathbf{x}_3 \\ \mathbf{x}_1 - \mathbf{x}_2 + 2\mathbf{x}_3 + \mathbf{x}_4 &= 0 \\ \mathbf{x}_1 - 3\mathbf{x}_3 + \mathbf{4} &= 0 \\ \mathbf{x}_1 &= \mathbf{x}_3 - \mathbf{x}_4 \end{aligned}$$

Maka:

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_3 - \mathbf{x}_4 \\ 3\mathbf{x}_3 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{bmatrix} = \mathbf{x}_3 \begin{bmatrix} 1 & 3 & 1 & 0 \end{bmatrix} + \mathbf{x}_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore\ U^{\perp} =$ span $\{(1,3,1,0),(-1,0,0,1)\}$.

Orthogonal Diagonalization

6. **Exercise 8.2.1**
   Normalize the rows to make each of the following matrices orthogonal.

   c) $A = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}$

   Diketahui:

   $$A = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}$$

   Ditanya:

   - Normalize A.

   Jawab:

   $$\|(1,2)\| = \sqrt{1+4} = \sqrt{5}$$
   $$\|(-4,2)\| = \sqrt{16+4} = 2\sqrt{5}$$

   $$\therefore\ A \sim \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{-4\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{-2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

7. **Exercise 8.2.5**
   For each matrix A, find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

   g) $A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

   Diketahui:

   $$A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Ditanya:

- Orthogonal matriks P, di mana $P^{-1}AP$ adalah diagonal?

Jawab:

Mencari eigenvalues dan eigenvectors

$$\begin{bmatrix} 5-\lambda & 3 & 0 & 0 \\ \lambda^4 - 24\lambda^3 + 204\lambda^2 - 704\lambda = 0 \\ \lambda_1 = 6; \lambda_2 = 2; \lambda_3 = 8; \lambda_4 = 8 \end{bmatrix}$$

Untuk $\lambda_1 = 6$

$$\begin{bmatrix} -1 & 3 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Null space dari matriks adalah: $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Untuk $\lambda_2 = 2$

Null space dari matriks adalah: $\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$

Untuk $\lambda_3 = 8$

Null space dari matriks adalah: $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$

Untuk $\lambda_4 = 8$

Null space dari matriks adalah: $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$

$\therefore$ P matriks adalah:

$$P = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Dari diagonal matriks, D adalah:

$$\lambda_1 = 6; \lambda_2; \lambda_3 = 8; \lambda_4 = 8$$

$$P^{-1}AP = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} = D$$

8. **Exercise 8.2.7**

Consider $A = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$. Show that $c_A(x) = (x-b)(x-a)(x+a)$ and find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Diketahui:

$$A = \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$$

Ditanya:

- Tunjukkan $c_A(x) = (x-b)(x-a)(x+a)$.
- Orthogonal matriks P, di mana $P^{-1}AP$ adalah diagonal?

Jawab:

Mencari karakteristik polinomial, eigenvalues, dan eigenvectors

$$\begin{aligned} \det(\mathbf{x}I - A) &= \begin{vmatrix} \mathbf{x} & 0 & -a \\ 0 & \mathbf{x}-b & 0 \\ -a & 0 & \mathbf{x} \end{vmatrix} \\ &= (\mathbf{x}-b)\begin{vmatrix} \mathbf{x} & -a \\ -a & \mathbf{x} \end{vmatrix} \\ &= (\mathbf{x}-b)(\mathbf{x}^2 - a^2) \\ &= (\mathbf{x}-b)(\mathbf{x}-a)(\mathbf{x}+a) = c_A(x) \end{aligned}$$

$\therefore\ \lambda_1 = b; \lambda_2 = a; \lambda_3 = -a$

$$(\lambda_1 I - A)\mathbf{x}_1 = 0 \Leftrightarrow \begin{bmatrix} b & 0 & -a \\ 0 & b-b & 0 \\ -a & 0 & b \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^{(1)} \\ \mathbf{x}_1^{(2)} \\ \mathbf{x}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} b\mathbf{x}_1^{(1)} - a\mathbf{x}_1^{(3)} = 0 &\Rightarrow \mathbf{x}_1^{(3)} = \frac{b}{a}\mathbf{x}_1^{(3)} \\ -a\mathbf{x}_1^{(1)} + b\mathbf{x}_1^{(3)} = 0 &\Rightarrow \mathbf{x}_1^{(3)} = \frac{a}{b}\mathbf{x}_1^{(1)} \end{aligned}$$

Terapkan $a = \pm b$, ambil $a = -b$

$\mathbf{x}_1^{(1)} = 1$ dan $\mathbf{x}_1^{(2)} = 1$

Kita mendapat $\mathbf{x}_1^{(3)} = 1$ dan $\mathbf{x}_1 = (1, 1, -1)$

$$(\lambda_2 I - A)\mathbf{x}_2 = 0 \Leftrightarrow \begin{bmatrix} a & 0 & -a \\ 0 & a-b & 0 \\ -a & 0 & a \end{bmatrix} \begin{bmatrix} \mathbf{x}_2^{(1)} \\ \mathbf{x}_2^{(2)} \\ \mathbf{x}_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a\mathbf{x}_2^{(1)} - a\mathbf{x}_2^{(3)} &= 0 \Rightarrow \mathbf{x}_2^{(1)} = \mathbf{x}_2^{(3)} \\ (a-b)\mathbf{x}_2^{(2)} &= 0 \Rightarrow -2\mathbf{x}_2^{(2)} = 0 \Rightarrow \mathbf{x}_2^{(2)} \\ a\mathbf{x}_2^{(1)} - a\mathbf{x}_2^{(3)} &= 0 \Rightarrow \mathbf{x}_2^{(1)} = \mathbf{x}_2^{(3)} \end{aligned}$$

Terapkan $\mathbf{x}_2^{(3)} = 1$

Kita mendapat $\mathbf{x}_2^{(1)} = 1$ dan $\mathbf{x}_2 = (1, 0, 1)$

$$(\lambda_3 I - A)\mathbf{x}_3 = 0 \Leftrightarrow \begin{bmatrix} -a & 0 & -a \\ 0 & -a-b & 0 \\ -a & 0 & -a \end{bmatrix} \begin{bmatrix} \mathbf{x}_3^{(1)} \\ \mathbf{x}_3^{(2)} \\ \mathbf{x}_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a\mathbf{x}_3^{(1)} - a\mathbf{x}_3^{(3)} &= 0 \Rightarrow \mathbf{x}_3^{(1)} = -\mathbf{x}_3^{(3)} \\ (-a-b)\mathbf{x}_3^{(2)} &= 0 \Rightarrow a = -b; \mathbf{x}_3^{(2)} \in \mathbb{R} \\ -c\mathbf{x}_3^{(2)} - k\mathbf{x}_3^{(3)} &= 0 \Rightarrow \mathbf{x}_3^{(1)} = -\mathbf{x}_3^{(3)} \end{aligned}$$

Terapkan $\mathbf{x}_3^{(1)} = 1$ dan $\mathbf{x}_3^{(2)} = -2$

Kita mendapat $\mathbf{x}_3^{(3)} = 1$ dan $\mathbf{x}_3 = (1, -2, -1)$

Bukti vector orthogonal:

$$\begin{aligned} \mathbf{x}_1\mathbf{x}_2 &= (1, 1, -1)(1, 0, 1) \\ &= 1 - 1 = 0 \\ \mathbf{x}_1\mathbf{x}_3 &= (1, 1, -1)(1, -2, -1) \\ &= 1 - 2 + 1 = 0 \\ \mathbf{x}_2\mathbf{x}_3 &= (1, 0, 1)(1, -2, -1) \\ &= 1 + 0 - 1 = 0 \end{aligned}$$

Mencari norms dari eigenvectors:

$$\begin{aligned} \|\mathbf{x}_1\| &= \|(1, 1, -1)\| \\ &= \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3} \\ \|\mathbf{x}_2\| &= \|(1, 0, -1)\| \\ &= \sqrt{1^2 + 0 + 1^2} = \sqrt{2} \\ \|\mathbf{x}_3\| &= \|(1, -2, -1)\| \\ \sqrt{1^2 + (-2)^2 + (-1)^2} &= \sqrt{6} \end{aligned}$$

$\therefore$ Matriks P adalah:

$$P = \begin{bmatrix} \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} & \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} & \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} \end{bmatrix}$$
$$= \frac{\sqrt{6}}{6} \begin{bmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & = 2 \\ -\sqrt{2} & \sqrt{3} & -1 \end{bmatrix}$$

Positive Definite Matrices

9. **Exercise 8.3.1**
Find the Cholesky decomposition of each of the following matrices.

d) $\begin{bmatrix} 20 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix}$

Diketahui:

$$A = \begin{bmatrix} 20 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix}$$

Ditanya:

- Cholesky decomposition A?

Jawab:

$$\det({}^{(1)}A) = 20 > 0; \det({}^{(2)}A) = 24 > 0; \det({}^{(3)}A) = 10 > 0$$
$$A = \begin{bmatrix} 20 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix}$$
$$\mathbf{R}_2 \to \mathbf{R}_2 - \frac{\mathbf{R}_1}{5} \begin{bmatrix} 20 & 4 & 5 \\ 0 & \frac{6}{5} & 2 \\ 5 & 3 & 5 \end{bmatrix}$$
$$\mathbf{R}_3 \to \mathbf{R}_3 - \frac{\mathbf{R}_1}{4} \begin{bmatrix} 20 & 4 & 5 \\ 0 & \frac{6}{5} & 2 \\ 0 & 2 & \frac{15}{4} \end{bmatrix}$$
$$\mathbf{R}_3 \to \mathbf{R}_3 - \frac{5\mathbf{R}_2}{3} \begin{bmatrix} 20 & 4 & 5 \\ 0 & \frac{6}{5} & 2 \\ 0 & 0 & \frac{5}{12} \end{bmatrix} = V_1$$

$$\therefore V = \begin{bmatrix} 2\sqrt{5} & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{2} \\ 0 & \frac{\sqrt{30}}{5} & \frac{2\sqrt{30}}{6} \\ 0 & 0 & \frac{\sqrt{60}}{12} \end{bmatrix}$$

QR-Factorization

10. **Exercise 8.4.1**

    In each case find the QR-factorization of A.

    d) $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

    Diketahui:

    $$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

    Ditanya:

    - QR-factorization dari A?

    Jawab:

    $$\begin{aligned} \mathbf{x}_1 &= \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \mathbf{x}_2 &= \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}^T \\ \mathbf{x}_3 &= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \end{aligned}$$

    Gunakan Gram-Schmidt algorithm:

    $$\begin{aligned} \mathbf{f}_1 &= \mathbf{x}_1 = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \mathbf{f}_1 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 \\ &= \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}^T - 0 = \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix} \\ \mathbf{f}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T - \frac{-1}{3}\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T - \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}^T \\ &= \frac{2}{3}\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T \end{aligned}$$

    Menormalisasi:

    $$\begin{aligned} \mathbf{Q}_1 &= \frac{1}{\|\mathbf{f}_1\|}\mathbf{f}_1 = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T \\ \mathbf{Q}_2 &= \frac{1}{\|\mathbf{f}_2\|}\mathbf{f}_2 = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}^T \\ \mathbf{Q}_3 &= \frac{1}{\|\mathbf{f}_3\|}\mathbf{f}_3 = \frac{\sqrt{3}}{3}\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T \end{aligned}$$

$$\therefore\ \mathbf{Q} = \tfrac{\sqrt{3}}{3} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore\ R = \tfrac{\sqrt{3}}{3} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Computing Eigenvalues

11. **Exercise 8.5.1**

In each case, find the exact eigenvalues and determine corresponding eigenvectors. Then start with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and compute $\mathbf{x}_4$ and r_3 using the power method.

a) $A = \begin{bmatrix} 2 & -4 \\ -3 & 3 \end{bmatrix}$

Diketahui:

$$A = \begin{bmatrix} 2 & -4 \\ -3 & 3 \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ditanya:

- Eigenvalues, eigenvectors, $\mathbf{x}_4$, dan r_3?

Jawab:

$$\begin{aligned} (A - \mathbf{x}I) &= \begin{bmatrix} 2-\lambda & -4 \\ -3 & \\ 3-\lambda & \end{bmatrix} \\ &= (2-\lambda)(3-\lambda) - (4)(3) \\ &= \lambda^2 - 5\lambda - 6 \\ &= (\lambda + 1)(\lambda - 6) = 0 \end{aligned}$$

$$\therefore \lambda_1 = -1 \vee \lambda_2 = 6$$

$\lambda_1 = -1$

$$\begin{aligned}
\begin{bmatrix} 3 & -4 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
3\mathbf{x}_1 - 4\mathbf{x}_2 &= 0 \\
-3\mathbf{x}_1 + 4\mathbf{x}_2 &= 0 \\
\mathbf{x}_1 &= \frac{4\mathbf{x}_2}{3} \\
\mathbf{x}_2 &= \mathbf{x}_2 \\
\therefore \text{Eigenvector} &= \mathbf{x}_2 \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix}
\end{aligned}$$

$\lambda_2 = 6$

$$\begin{aligned}
\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
4\mathbf{x}_1 + 4\mathbf{x}_2 &= 0 \\
3\mathbf{x}_1 + 3\mathbf{x}_2 &= 0 \\
\mathbf{x}_1 &= -\mathbf{x}_2 \\
\mathbf{x}_2 &= \mathbf{x}_2 \\
\therefore \text{Eigenvector} &= \mathbf{x}_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{x}_1 = A\mathbf{x}_0 &= \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\
\mathbf{x}_2 = A\mathbf{x}_1 &= \begin{bmatrix} -4 \\ 6 \end{bmatrix} \\
\mathbf{x}_3 = A\mathbf{x}_2 &= \begin{bmatrix} -32 \\ 30 \end{bmatrix} \\
\therefore \mathbf{x}_4 = A\mathbf{x}_3 &= \begin{bmatrix} -184 \\ 186 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore r_3 &= \frac{\mathbf{x}_k \cdot \mathbf{x}_{k+1}}{\|\mathbf{x}_k\|^2} \\
&= \frac{\mathbf{x}_3 \cdot \mathbf{x}_4}{\|\mathbf{x}_3\|^2} \\
&= \frac{2867}{481}
\end{aligned}$$

b) $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$

Diketahui:

$$A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ditanya:

- Eigenvalues, eigenvectors, $\mathbf{x}_4$, dan r_3?

Jawab:

$$\begin{aligned}
(A - \mathbf{x}I) &= \begin{bmatrix} 5-\lambda & 2 \\ -3 & \\ -2-\lambda & \end{bmatrix} \\
&= (5-\lambda)(2-\lambda) - (-3)(2) \\
&= \lambda^2 - 3\lambda - 4 \\
&= (\lambda+1)(\lambda-4) = 0 \\
\therefore \lambda_1 = -1 \vee \lambda_2 = 4
\end{aligned}$$

$\lambda_1 = -1$

$$\begin{aligned}
&\begin{bmatrix} 6 & 2 \\ -3 & 3 \end{bmatrix} \\
\mathbf{R}_1 \to \frac{\mathbf{R}_1}{6} &\begin{bmatrix} 1 & \frac{1}{3} \\ -3 & -1 \end{bmatrix} \\
\mathbf{R}_2 \to \mathbf{R}_2 - 3\mathbf{R}_1 &\begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix} \\
\mathbf{x}_1 &= -\frac{\mathbf{x}_2}{3} \\
\mathbf{x}_2 &= \mathbf{x}_2 \\
\therefore \text{Eigenvector} &= \mathbf{x}_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}
\end{aligned}$$

$\lambda_2 = 4$

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} \\
\mathbf{R}_2 \to \mathbf{R}_2 + 3\mathbf{R}_1 &\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\
\mathbf{x}_1 &= -2\mathbf{x}_2 \\
\mathbf{x}_2 &= \mathbf{x}_1 \\
\therefore \text{Eigenvector } &= \mathbf{x}_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{x}_1 = A\mathbf{x}_0 &= \begin{bmatrix} 7 \\ -5 \end{bmatrix} \\
\mathbf{x}_2 = A\mathbf{x}_1 &= \begin{bmatrix} 25 \\ -11 \end{bmatrix} \\
\mathbf{x}_3 = A\mathbf{x}_2 &= \begin{bmatrix} 103 \\ -53 \end{bmatrix} \\
\therefore \mathbf{x}_4 = A\mathbf{x}_3 &= \begin{bmatrix} 409 \\ -203 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore r_3 &= \frac{\mathbf{x}_k \cdot \mathbf{x}_{k+1}}{\|\mathbf{x}_k\|^2} \\
&= \frac{\mathbf{x}_3 \cdot \mathbf{x}_4}{\|\mathbf{x}_3\|^2} \\
&= \frac{52886}{13418}
\end{aligned}$$

The Singular Value Decomposition

12. **Exercise 8.6.8**
Let $A^{-1} = A = A^T$ where A is $n \times n$. Given any orthogonal $n \times n$ matrix U, find an orthogonal matrix V such that $A = U\Sigma_A V^T$ is an SVD for A.
If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ do this for:

    a) $U = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Diketahui:

$A^{-1} = A = A^T$

$A = U\Sigma_A V^T$ adalah SVD dari A

$U = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Ditanya:

- Matriks orthogonal V?

Jawab:

$$\begin{aligned}
A^T \cdot A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
(A - \mathbf{x}I) &= \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \\
&(1-\lambda)^2 = 0 \\
&\lambda_1 = 1 \vee \lambda_2 = 1
\end{aligned}$$

$\lambda = 1$

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\mathbf{x}_1 &= \mathbf{x}_1 \\
\mathbf{x}_2 &= \mathbf{x}_2 \\
\mathbf{x}_1 &\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{x}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

$\therefore$ Eigenvalues adalah $\lambda_1 = 1, \lambda_2 = 1$ dan eigenvectors adalah $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned}
\sigma_1 &= 1; \sigma_2 = 1 \\
\Sigma_A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
V^{T-1} &= A^{-1} \vee \Sigma_A \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} \\
V^T &= \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \\
\therefore V &= \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}
\end{aligned}$$

13. **Exercise 8.6.9**
Find an SVD for the following matrices:

b) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$

Diketahui:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

Ditanya:

- SVD dari A?

Jawab:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

$$(A - \mathbf{x}I) = \begin{bmatrix} 3-\lambda & 3 & 3 \\ 3 & 5-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{bmatrix}$$

$$\lambda_1 = 0 \vee \lambda_2 = 4 \vee \lambda_3 = 9$$

$\lambda_1 = 0$

$$\begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 4$

$$\begin{bmatrix} -1 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_3 = 9$

$$\begin{bmatrix} -6 & 3 & 3 \\ 3 & -4 & 1 \\ 3 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigenvalues adalah $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = 9$ dan eigenvectors adalah $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$$\begin{aligned}
\sigma_1 &= \sqrt{9} = 3 \\
\sigma_2 &= \sqrt{4} = 2 \\
\Sigma_A &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
q_1 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}, q_2 &= \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, q_3 = \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \\
\therefore Q &= \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{bmatrix} \\
P_1 = \begin{bmatrix} \frac{\sqrt{3}}{9} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{9} \end{bmatrix}, P_2 &= \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, P_3 = \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \\
\therefore P &= \begin{bmatrix} \frac{\sqrt{3}}{9} & 0 & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{9} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{bmatrix} \\
\therefore A &= P\Sigma_A Q^T \\
&= \begin{bmatrix} \frac{\sqrt{3}}{9} & 0 & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{9} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -6\sqrt{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \end{bmatrix}
\end{aligned}$$

Complex Matrices

14. **Exercise 8.7.2**

    In each case, determine whether the two vectors are orthogonal.

    b) $(i, -i, 2+i),\ (i, i, 2-i)$

    Diketahui:

    $V = (i, -i, 2+i)$

    $U = (i, i, 2-i)$

    Ditanya:

    - Apakah kedua vector orthogonal?

    Jawab:

    Vector orthogonal apabila $V \cdot U = 0$

$$\begin{aligned}(i, -i, 2+i) \cdot (i, i, 2-i) &= i^2 - i^2 + (2+i)(2-i) \\ &= -2i^2 \neq 0\end{aligned}$$

    $\therefore$ Kedua vector tersebut tidak orthogonal karena $V \cdot U \neq 0$.

15. **Exercise 8.7.8**

    In each case, find a unitary matrix U such that $U^H AU$ is diagonal.

    a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 1-i & 2 \end{bmatrix}$

    Diketahui:

    $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 1-i & 2 \end{bmatrix}$

    $U^H AU$ diagonal

    Ditanya:

    - Matriks unitary U?

    Jawab:

$$(\mathbf{x}I - A) = \begin{bmatrix} \mathbf{x}-1 & 0 & 0 \\ 0 & \mathbf{x}-1 & -1-i \\ 0 & -1-i & x-2 \end{bmatrix}$$

$$(\mathbf{x}-1)(\mathbf{x}^2 - 3\mathbf{x}) = (\mathbf{x}-1)(\mathbf{x}-3)$$

$$\mathbf{x} = 1 \vee \mathbf{x} = 0 \vee \mathbf{x} = 3$$

    $\therefore$ Eigenvalues adalah $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

$\lambda_1 = 0$

$$\begin{bmatrix} =1 & 0 & 0 \\ 0 & -1 & -1-i \\ 0 & -1+i & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_1 = \mathbf{x}_1$$

$$\mathbf{x}_2 = 0$$

$$\mathbf{x}_3 = 0$$

$$\mathbf{x}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1-i \\ 0 & -1+i & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1+i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_1 = 0$$

$$\mathbf{x}_2 = \mathbf{x}_2$$

$$\mathbf{x}_3 = 1 - i(\mathbf{x}_2)$$

$$\mathbf{x}_2 \begin{bmatrix} 0 & 1 & 1-i \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}+i\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}-i\sqrt{3}}{3} \end{bmatrix} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+i & 1 \\ 0 & -1 & 1-i \end{bmatrix} ; U \text{ orthogonal}$$

$$U^H A U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$