School of Mathematics and Statistics
Carleton University
Math. 1004A, Fall 2016

TEST 5

Solutions

Only calculators are permitted, ~~1 or more blank sheets~~ permitted for roughs

Print Name :

Student Number:

Tutorial Section (A1, A4, . . .):

Show all work here and give details.

1. [**5 marks**] Find the partial fraction decomposition of the rational function $\frac{5x-7}{x^2-3x+2}$ (do NOT integrate it).

$$\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} \leftarrow ① = \frac{A}{x-1} \leftarrow ① + \frac{B}{x-2} \leftarrow ①$$

(or something like this)

$$\Rightarrow 5x-7 = A(x-2)+B(x-1)$$

$$\Rightarrow x=1 \Rightarrow -2=-A \Rightarrow A=2 \qquad A=2 \leftarrow ①$$

$$x=2 \Rightarrow 3=B \qquad B=3 \leftarrow ①$$

2. [**5 marks**] Evaluate $I = \int \frac{x+4}{(x-1)(x+6)}\,dx$ using any method

Use partial fractions

$$\frac{x+4}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6} \Rightarrow x+4 = A(x+6)+B(x-1). \leftarrow ①$$

① ①

$$x=1 \Rightarrow 5=7A \Rightarrow A=5/7$$

$$x=-6 \Rightarrow -2=-7B \Rightarrow B=2/7 \leftarrow ①$$

$$\therefore I = \frac{5}{7}\ln|x-1| + \frac{2}{7}\ln|x+6| + C$$

$\left(\frac{1}{2}\right)$ $\left(\frac{1}{2}\right)$

3. [**5 marks**] Evaluate $I = \int_0^{\pi/2} \sin^3 x\cos^3 x\,dx$ using any method

$I = \int \sin^3 x\cos^3 x\,dx$

$$\sin^3 x\cos^3 x = \sin^2 x\cos^2 x\sin x\cos x$$

$$= (1-\cos^2 x)\cos^3 x\sin x \leftarrow ①$$

①

$$= (1-\sin^2 x)\sin^3 x\cos x. \leftarrow ①$$

Now let $\left(\begin{matrix} u=\sin x \\ du=\cos x\,dx\end{matrix}\right.$ or $\left.\begin{matrix} u=\cos x \\ du=-\sin x\,dx\end{matrix}\right)$ ①

Then $I = \int (1-u^2)u^3\,du \leftarrow ①$

$$= \frac{u^4}{4} - \frac{u^6}{6}$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} \leftarrow ①$$

or $\int_0^{\pi/2}(\ ) = \frac{1}{4} - \frac{1}{6} = \frac{2}{24} = \boxed{\frac{1}{12}} \leftarrow ①$

① → (or) $I = -\int (1-u^2)u^3\,du$

$$= \frac{u^6}{6} - \frac{u^4}{4}$$

$$① \rightarrow = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4}$$

or $\int_0^{\pi/2}(\dots) = (0-0) - \left(\frac{1}{6} - \frac{1}{4}\right)$

$$= \frac{1}{4} - \frac{1}{6}$$

$$① \rightarrow = \boxed{\frac{1}{12}}$$

4. [**5 marks**] Evaluate $I = \int_0^5 \sqrt{25-x^2}\,dx$ using any method

1 $\left\{ \begin{array}{l} \text{Let } x = 5\sin\theta \text{ or } 5\cos\theta \\ dx = 5\cos\theta\, d\theta \end{array} \right\} \sqrt{25-x^2} = 5\cos\theta \leftarrow 1$

Then $\int \sqrt{25-x^2}\,dx = \int (5\cos\theta)(5\cos\theta)\,d\theta = \underbrace{25\int \cos^2\theta\, d\theta}_{1}$

$= \frac{25}{2}\int (1+\cos 2\theta)\,d\theta \leftarrow 1/2$

$= \frac{25}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \leftarrow 1/2$

$\theta = \text{Arcsin}(x/5) \Rightarrow \quad I = \frac{25}{2}\left(\text{Arcsin}\left(\frac{x}{5}\right) + \frac{\sin\left(2\,\text{Arcsin}(x/5)\right)}{2}\right)\Big|_0^5$

$= \frac{25}{2}\left(\left(\frac{\pi}{2}+0\right) - (0-0)\right) = \frac{25\pi}{4} \leftarrow 1$

5. **[5 marks]** Evaluate $\int_0^\infty 2x\,e^{-x}\,dx$ using any method

By parts first

$\left.\underbrace{\begin{array}{lll} x & \overset{+}{\searrow} & e^{-x} \\ 1 & \overset{-}{\searrow} & -e^{-x} \\ 0 & & e^{-x} \end{array}}_{1}\right\}$

$\int_0^T 2xe^{-x}\,dx = 2\left[-xe^{-x} - e^{-x}\right]_0^T$

$= \left(\underbrace{-2Te^{-T} - 2e^{-T}}_{1}\right) + \underbrace{2}_{1}$

But

$\lim_{T\to\infty} (-2Te^{-T}) = 0 \leftarrow 1$ by l'Hosp. Rule

& so $\int_0^\infty 2xe^{-x}\,dx = 2 \leftarrow 1$

6. **[5 marks]** Evaluate $I = \int_0^1 4x\,\ln x\,dx$ using any method

$I = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 4x\ln x\,dx \leftarrow 1/2$

By parts first

$\left.\begin{array}{lll} \ln x & \overset{+}{\searrow} & 4x \\ \frac{1}{x} & \overset{-}{\leftarrow} & 2x^2 \end{array}\right\} \int 4x\ln x\,dx = 2x^2\ln x - 2\int x\,dx$

$= \underbrace{2x^2\ln x}_{1} - \underbrace{x^2}_{1}$

$\therefore \int_\varepsilon^1 4x\ln x\,dx = (2\cdot\ln 1 - 1^2) - (2\varepsilon^2\ln\varepsilon - \varepsilon^2)$

$= \boxed{-1 - 2\varepsilon^2\ln\varepsilon + \varepsilon^2}$ $\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

But $\varepsilon^2 \to 0$ as $\varepsilon\to 0$ & $\underbrace{\lim_{\varepsilon\to 0}(2\varepsilon^2\ln\varepsilon) = 0}_{1/2}$ by L'Hosp Rule.

$\therefore I = -1 \leftarrow 1/2$

END