MATH1104E, Linear Algebra for Engineering or Science,
Fall 2014, TEST # 3, Prof. Steven Wang

Name(print)	Signature	Student Number
	Solutions	

Total Pages: 4

Total Marks: 40

INSTRUCTION:
Write your solution in the space provided below the question. If necessary, continue onto the back of the sheet, but remind your marker to look there. Show all your work. Calculator is NOT allowed.

1. [12 marks]

(a) **[6 marks]** Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$. Justify your answers.

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 6 \\ 1 & 2 & 2 & 4 \end{bmatrix} \underset{R_3 - R_1}{\overset{R_2 - 2R_1}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix} \underset{R_3 - R_2}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\Rightarrow \operatorname{Col}(A) \neq \mathbb{R}^3)$$

No, it does not span $\mathbb{R}^3$

(b) **[6 marks]** Determine whether or not the set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \right\}$ is linearly independent. Justify your answers.

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix} \overset{R_2 + R_1}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 8 \\ 0 & -1 & 1 \end{bmatrix} \overset{\frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \underset{R_3 + R_2}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

Yes, it is a linearly independent set.

2. [**12 marks**] Let $A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) [**6 marks**] Find the null space of A and column space of A. Express your answers in terms of spans of vectors.

$$\begin{aligned} x_1 \quad + \tfrac{4}{5}x_3 + 2x_4 &= 0 \\ x_2 + \tfrac{1}{5}x_3 - x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{cases} x_1 = -\tfrac{4}{5}s - 2t \\ x_2 = -\tfrac{1}{5}s + t \\ x_3 = s \\ x_4 = t \end{cases} \qquad s, t \in \mathbb{R}$$

$$Null(A) = Span\left\{ \begin{bmatrix} -\frac{4}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$Col(A) = Span\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix} \right\} \text{ or } Span\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$$

(b) [**3 marks**] Specify the basis of the null space of A and column space of A, respectively.

A basis for $Null(A)$ is $\left\{ \begin{bmatrix} -\frac{4}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for $Col(A)$ is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$

(c) [**3 marks**] Specify the $rank(A)$ and $Nullity(A)$, and verify the Rank Theorem.

$rank(A) = 2$

$Nullity(A) = 2$

The Rank Theorem: $rank(A) + nullity(A) = \#$ of Columns of A

$$2 + 2 = 4.$$

3. **[10 marks]** Let T be the linear transformation given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ x+y+2z \end{bmatrix}$.

(a) **[4 marks]** Find the standard matrix of T.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \left(\text{or } A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \right)$$

(c) **[6 marks]** Justify that T is indeed a linear transformation.

For any $u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$, $a, b \in \mathbb{R}$.

$$T(au+bv) = T\left(a\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} ax_1+bx_2 \\ ay_1+by_2 \\ az_1+bz_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (ax_1+bx_2)+(az_1+bz_2) \\ (ax_1+bx_2)+(ay_1+by_2)+2(az_1+bz_2) \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1+z_1)+b(x_2+z_2) \\ a(x_1+y_1+2z_1)+b(x_2+y_2+2z_2) \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1+z_1) \\ a(x_1+y_1+2z_1) \end{bmatrix} + \begin{bmatrix} b(x_2+z_2) \\ b(x_2+y_2+2z_2) \end{bmatrix}$$

$$= a\cdot T(u) + b\cdot T(v).$$

4. [**6 marks**] Truth or False questions (1 mark each).

(a) If $S = \{u, v\}$ is a linear independent set, so is the set $\{u, v, u+v\}$. T F

(b) An $n \times n$ matrix A is invertible if and only if the set of all the column vectors is a linearly independent set of $\mathbb{R}^n$. T F

(c) The coordinate vector of $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ relative to $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. T F

(d) Given a vector space V, then any set of linearly independent vectors $\{v_1, v_2, \cdots, v_p\} \in V$ is a basis of the subspace $span\{v_1, v_2, \cdots, v_p\}$ of V. T F

(e) If S is a spanning set of a vector space V, then S is a basis of V. T F

(f) An $n \times n$ matrix A is invertible if and only if $rank(A) = n$. T F

5. [**Bonus: 2 marks**] Multiple choice questions (1 mark each).

(i) The coordinate vector of $\vec{v} = (2, 3)$ relative to the basis $\mathcal{B} = \{(1,0), (1,1)\}$ in vector space $\mathbb{R}^2$ is

(a) $(2,3)$ **(b)** $(-2,3)$ **(c)** $(-1,3)$ **(d)** $(1,3)$ **(e)** $(0,0)$

(ii) Which of the following is true?

(a) Any spanning set of a subspace S of $\mathbb{R}^n$ is a basis for S.

(b) If a matrix A can be reduced to a reduced row echelon form R, then $col(A) = col(R)$.

(c) If a matrix A can be reduced to a reduced row echelon form R, then $row(A) = row(R)$.

(d) The dimension of $Null(A)$ is the number of variables in the equation $AX = 0$.