Solutions

School of ~~Mathematics and~~ Statistics
Carleton University
Math. 1004A, Fall 2014
# TEST 2

STUDIO 56 calculator ONLY permitted, 1 or more blank sheets permitted for roughs

**Print Name :** ______________________

**Student Number:** ______________________

**Tutorial Section (A1, A2, A3, A4, or A5):** ______________________

## PART I: Multiple Choice Questions
(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** Let $f$ be a differentiable function with $f'(0) = 1/2$ whose differentiable inverse, $F$, satisfies $F(-1) = 0$. Find the value of $F'(-1)$.
   (a) 2, (b) $-2$, (c) $1/2$, (d) 0, e) none of these

   $F'(-1) = \dfrac{1}{f'(F(-1))} = \dfrac{1}{f'(0)} = 2 = (a).$

2. **[2 marks]** Calculate the derivative $f'(\sqrt{\pi})$ of $f$ where $f(x) = \tan(\sin(x^2))$.
   (a) $\sqrt{\pi}$, (b) 0, (c) $-2\sqrt{\pi}$, (d) 1, e) none of these

   $f'(x) = \sec^2(\sin(x^2)) \cdot 2x\cos(x^2)$
   $f'(\sqrt{\pi}) = \sec^2(0) \cdot 2\sqrt{\pi}\,(-1) = (c).$

3. **[2 marks]** Evaluate the following limit: $\lim\limits_{x\to\pi/2} \dfrac{\sin(x-\pi/2)}{\cos x}$.
   (a) 2, (b) 0, (c) 1, (d) $-1$, e) The limit does not exist

   L'Hospital: $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos(x-\pi/2)}{-\sin x} = \dfrac{\cos 0}{-\sin\frac{\pi}{2}} = (d)$

4. **[2 marks]** Let $(x-y)\sin(x+y) = 0$ define a differentiable curve in the $xy$-plane and that $y$ may also be defined as a differentiable function of $x$ near $x = \pi$. Calculate the slope of the tangent line to this curve at the point where $x = \pi, y = 0$.
   (a) 0, (b) $-1$, (c) 1, (d) 2, e) none of these.

   $(1-y')\sin(x+y) + (x-y)\cos(x+y)\cdot(1+y') = 0.$
   $x=\pi,\ y=0 \Rightarrow\ 0 + (\pi-0)\cos\pi\,(1+y') = 0 \quad \therefore\ y' = -1$

5. **[2 marks]** $\dfrac{d}{dx}\mathrm{Arcsin}(\tan(2x))$ at the point $x = 0$ is equal to 2.
   (a) TRUE, (b) FALSE,

   $\left.\dfrac{1}{\sqrt{1-\tan^2(2x)}} \cdot 2\sec^2(2x)\right|_{x=0} = \dfrac{2\cdot 1}{\sqrt{1+0}} = 2.$ (TR

## PART II: Show all work here and give details.
## No additional pages will be accepted

6. **[5+5 marks]** a) Evaluate the following limit using any method: $\lim\limits_{x\to 0} \dfrac{\sin x + \sin 2x}{3x}$.

   b) Evaluate the following limit using any method: $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x+1} - \sqrt{2x}}{\sqrt{x}}$.

a) ① Simplify:

$\dfrac{\sin x + \sin 2x}{3x} = \dfrac{1}{3}\dfrac{\sin x}{x} + \dfrac{2}{3}\cdot\dfrac{\sin 2x}{2x}$ ① ① ① ① $\therefore$ limit as $x \to 0$ gives

$\to \dfrac{1}{3}\cdot 1 + \dfrac{2}{3}\cdot 1 = \underline{\underline{1}}.$ ①

OR ② Use L'Hospital's Rule: $\lim\limits_{x\to 0} \dfrac{D(\sin x + \sin 2x)}{D(3x)} = \lim\limits_{x\to 0}\left(\dfrac{\cos x + 2\cos 2x}{3}\right)$ ① ① ①

$= \dfrac{\cos 0 + 2\cos 0}{3} = \underline{\underline{1}}$ ②

b). ① Simplify $\dfrac{\sqrt{2x+1} - \sqrt{2x}}{\sqrt{x}} = \dfrac{(2x+1) - 2x}{\underbrace{\sqrt{x}(\sqrt{2x+1} + \sqrt{2x})}_{②}} = \dfrac{1}{\sqrt{x}(\sqrt{2x+1} + \sqrt{2x})} \to 0$ ① ① ①

as $x \to \infty$.

OR ② $\dfrac{\sqrt{2x+1} - \sqrt{2x}}{\sqrt{x}} = \sqrt{\dfrac{2x+1}{x}} - \dfrac{\sqrt{2x}}{\sqrt{x}} = \sqrt{2+\dfrac{1}{x}} - \sqrt{\dfrac{2x}{x}} = \sqrt{2+\dfrac{1}{x}} - \sqrt{2}$

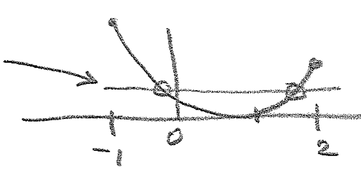
As $x \to \infty$, $\sqrt{2+\frac{1}{x}} - \sqrt{2} \to 0$.

Can't use L'Hospital's Rule directly as not of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

7. **[3+4+3 marks]** Let $f(x) = (x-1)^2$.

- Let the domain of $f$ be the interval $[-1, 2]$. Does $f$ have an inverse function? Explain.
- Show that $f$ has an inverse function, $F$, if its domain is the interval $[1, 2]$. What is the domain of $F$?
- Calculate the derivative $F'(x)$ of the previous inverse function, $F$.

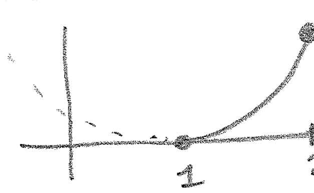
1. a) No, because it <u>fails</u> the <u>Horizontal line Test</u> (HLT).




(1) (1) (1)

or it isn't one-to-one on $[-1,2]$, Ex. $f(0) = f(2) = 1$ but $0 \neq 2$. (1)

b). On $[1,2]$ <u>$f$ satisfies the HLT</u> (1) $\therefore$ <u>it has an inverse</u> (1).




$\text{Dom } F = \text{Ran } f = [0, 1]$.
(1) (1)

(1)

c) Find $F(x)$ first. $y = (x-1)^2$ & interchanging $x$ & $y$
we set $x = (y-1)^2$ & solve for $y$, i.e., $\sqrt{x} = y - 1$ $\therefore$

$y = \boxed{F(x) = \sqrt{x} + 1} \Leftarrow$ (1)

$\therefore \boxed{F'(x) = \dfrac{1}{2\sqrt{x}}}$, for $1 < x < 2$. $\Leftarrow$ (1)

or $F(x) = \sqrt{x} + 1 \Rightarrow f'(F(x)) = 2(F(x) - 1) = 2\sqrt{x}$.

$\therefore \dfrac{1}{2\sqrt{x}} = \dfrac{1}{f'(F(x))} = F'(x)$