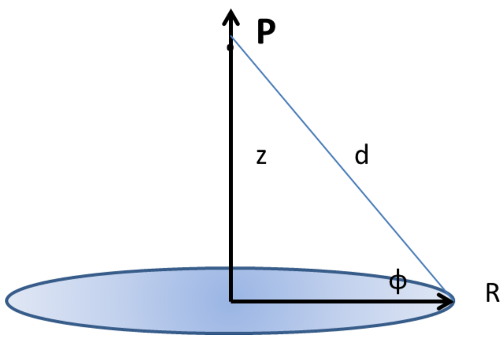
B1

A charged disk has a surface charge density function of $\sigma(r) = br^6$ C/m^2, where b is a constant.

Derive the equation for the electric field at a point P, z above the disk on the axis, assuming the disk has radius R. You can also assume that z >> R



In this case, we are going to do two integrations, one varying with angle around the vertical axis, and the other varying with radius.

A small area increment dA at a distance r from the centre of the disk has an area

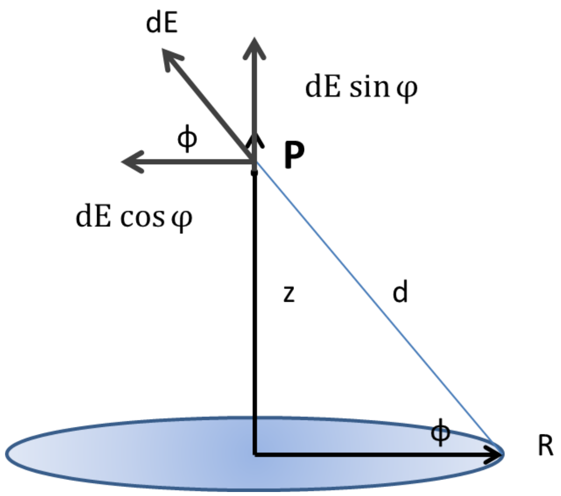
$$dA = drd\theta$$

The charge on this area is

$$dq = \sigma(r)drd\theta$$

The electric field produced at point P is thus

$$dE = \frac{\sigma(r)drd\theta}{4\pi\varepsilon_0 d^2}$$



Notice that the horizontal components are going to cancel out because of symmetry, so the vertical components are the ones which contribute. The total electric field is thus

$$E = \int \sin\varphi \, dE = \int_{r=0}^{r=R} \int_{0}^{2\pi} \frac{\sin\varphi \, \sigma(r) dr d\theta}{4\pi\varepsilon_0 d^2}$$

We need to express the angle ϕ and distance d in terms of r and z.

$$\sin\varphi = \frac{opp}{hyp} = \frac{z}{d}$$

$$E_z = \int \sin\varphi \, dE = \int_{r=0}^{r=R} \int_{0}^{2\pi} \frac{z\sigma(r) dr d\theta}{4\pi\varepsilon_0 d^3}$$

And now we express d as

$$d = \sqrt{z^2 + r^2}$$

$$E_z = \int_{r=0}^{r=R} \int_{0}^{2\pi} \frac{z\sigma(r) dr d\theta}{4\pi\varepsilon_0 (z^2 + r^2)^{\frac{3}{2}}}$$

We can simplify this, because we can integrate with respect to dθ easily, as there is no dependence on θ in the integrand

$$E_z = 2\pi \int_{r=0}^{r=R} \frac{z\sigma(r)dr}{4\pi\varepsilon_0(z^2 + r^2)^{\frac{3}{2}}}$$

$$E_z = \frac{z}{2\varepsilon_0} \int_{r=0}^{r=R} \frac{br^6 dr}{(z^2 + r^2)^{\frac{3}{2}}}$$

Now this would require integrating by parts, except that we are also given that z >> r

$$E_z = \frac{z}{2\varepsilon_0} \int_{r=0}^{r=R} \frac{br^6 dr}{z^3}$$

$$E_z = \frac{1}{2\varepsilon_0 z^2} \int_0^R br^6 dr$$

$$E_z = \frac{b}{14\varepsilon_0 z^2} R^7$$

Now we also need to replace the constant b with an expression for the total charge Q

The total charge on the disk is the sum of all the charge increments dq

$$Q = \int_0^{2\pi} \int_0^R \sigma(r) dr d\theta$$

$$Q = \int_0^{2\pi} \int_0^R br^6 dr d\theta$$

$$Q = 2\pi b \int_0^R r^6 dr$$

$$Q = 2\pi b \frac{R^7}{7}$$

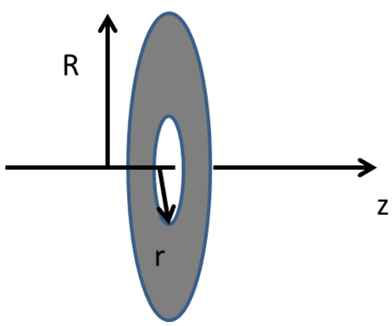
$$b = \frac{7Q}{2\pi R^7}$$

$$E_z = \frac{b}{14\varepsilon_0 z^2} R^7 = \frac{1}{14\varepsilon_0 z^2} R^7 \frac{7Q}{2\pi R^7} = \frac{1}{4\varepsilon_0 z^2} \frac{Q}{\pi}$$

Notice that this is the same as for a point charge. Not surprising, as we are at a large distance from the disk.

B2

An electron accelerating plate in an electron microscope is a conducting disk of radius R, with a hole of radius r in it along the axis.



The electric field along the axis of a charged disk of radius R is given by

$$E(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right]$$

(a) Show that the electric field along the z axis is:

$$E(z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{\sqrt{z^2 + r^2}} - \frac{1}{\sqrt{z^2 + R^2}}\right]$$

You can consider the system as the superposition of a smaller negatively charged disk on top of the larger positively charged disk. The charge densities must be equal in magnitude, but opposite in sign.

$$E(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right] - \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + r^2}}\right]$$

$$E(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right] - \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + r^2}}\right]$$

$$E(z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{\sqrt{z^2 + r^2}} - \frac{1}{\sqrt{z^2 + R^2}}\right]$$

(b) For cases where z << r, show that this becomes

$$E(z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{r} - \frac{1}{R}\right]$$

Since z << r , it is also << R, so the square root terms simplify to

$$\frac{1}{\sqrt{z^2 + r^2}} - \frac{1}{\sqrt{z^2 + R^2}} = \frac{1}{r} - \frac{1}{R}$$

(c) For a disk with radius 1.20 cm and hole radius 1.00 mm, calculate the Electric field at a distance 7.00 mm from the disk. Assume that the total charge on the plate is +1.4×10^{-6} C.

For this case, you cannot use the approximation from part (b) as z >>r

Use the full expression or simplify to

$$E(z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}}\right]$$

The charge on the plate is +1.41 μC. The area of the plate is

$$A = \pi(R^2 - r^2)$$

Hence charge density is

$$\sigma = \frac{q}{\pi(R^2 - r^2)}$$

$$E(z) = \frac{q}{\pi(R^2 - r^2)}\frac{z}{2\varepsilon_0}\left[\frac{1}{\sqrt{z^2 + R^2}} - \frac{1}{\sqrt{z^2 + r^2}}\right]$$

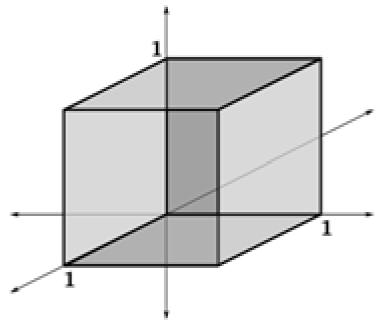
$$E(z) = \frac{+1.41\times10^{-6}}{\pi(R^2 - r^2)}\frac{(7.00\times10^{-3})}{2\times8.85\times10^{-12}F/m}\left[\frac{1}{(7.00\times10^{-3}m)^2} - \frac{1}{\sqrt{(7.00\times10^{-3}m)^2 + (1.20\times10^{-2}m)^2}}\right]$$

$$E(z) = 1.25\times10^5 V/m$$

B3

The electric field is given by the equation $\vec{E} = (x^2+1)\boldsymbol{i} + 9.0y\boldsymbol{j} + z^2\boldsymbol{k}\ V/m$

Now consider the unit cube with the origin at (0,0,0), each side 1.0 m long, and each face with area 1.0 m^2



a) Calculate the net flux of electric field E through each face of the cube (12 marks)
b) Calculate the net flux passing through the cube (4 marks)
c) Use Gauss' Law to determine the charge enclosed in the cube (2 marks)
d) If the unit cube was made up of a dielectric material with κ = 80.4 (water), what would the charge enclosed by, assuming the net flux was the same? (2 marks)

Solution

Calculate the flux through the 6 sides of the cube, with the area vectors **i, j, k,- i, -j**, and **-k.**

Face defined by **i** (the plane defined by x = 1.0)

$$\Phi_i = \vec{E}.\vec{A} = (2\boldsymbol{i} + 9.0y\boldsymbol{j} + z^2\boldsymbol{j}).\boldsymbol{i} = 2.0\ V.m$$

Face defined by **–i** (the plane defined by x = 0)

$$\Phi_{-i} = \vec{E}.\vec{A} = (1\boldsymbol{i} + 9.0y\boldsymbol{j} + z^2\boldsymbol{j}).-\boldsymbol{i} = -1.0\ V.m$$

Face defined by **j (**the plane defined by y = 1)

$$\Phi_j = \vec{E}.\vec{A} = \left((x^2+1)\boldsymbol{i} + 9.0\boldsymbol{j} + z^2\boldsymbol{j}\right).\boldsymbol{j} = 9.0\ Nm^2/C$$

Face defined by **–j (**the plane defined by y = 0)

$$\Phi_{-j} = \vec{E}.\vec{A} = \left((x^2+1)\boldsymbol{i} + 0\boldsymbol{j} + z^2\boldsymbol{k}\right).-\boldsymbol{j} = 0\ Nm^2/C$$

Face defined by **+k** (the plane where z = 1)

$$\Phi_z = \vec{E}.\vec{A} = \left((x^2+1)\boldsymbol{i} + 9.0y\boldsymbol{j} + z^2\boldsymbol{k}\right).\boldsymbol{k} = 1.0\ Nm^2/C$$

Face defined by **–k** (the plane where z = 0)

$$\Phi_{-z} = \vec{E}.\vec{A} = \left((x^2+1)\boldsymbol{i} + 9.0y\boldsymbol{j} + 0\boldsymbol{k}\right).-\boldsymbol{k} = \mathbf{0}\ Nm^2/C$$

(b) Net flux
Summing up the 6 fluxes we get

$$\Phi = 2.0 - 1.0 + 9.0 + 0 + 1.0 + 0 \ = 11\ Nm^2/C$$

(c) Use Gauss' Law to calculate the charge enclosed. (2 marks)

$$\Phi = \frac{q}{\varepsilon_0}$$

$$q = \Phi\varepsilon_0 = 11\ V.m \times 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2} = 9.7 \times 10^{-11}C$$

d) Use Gauss' Law to calculate the charge enclosed. (2 marks)

$$\Phi = \frac{q}{\kappa\varepsilon_0}$$

$$q = \kappa\Phi\varepsilon_0 = 11\ V.m \times 80.4\ \times 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2} = 7.8 \times 10^{-9}C$$