



# THE SCHOEN-YAU POSITIVE MASS THEOREM

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# INTRODUCTION

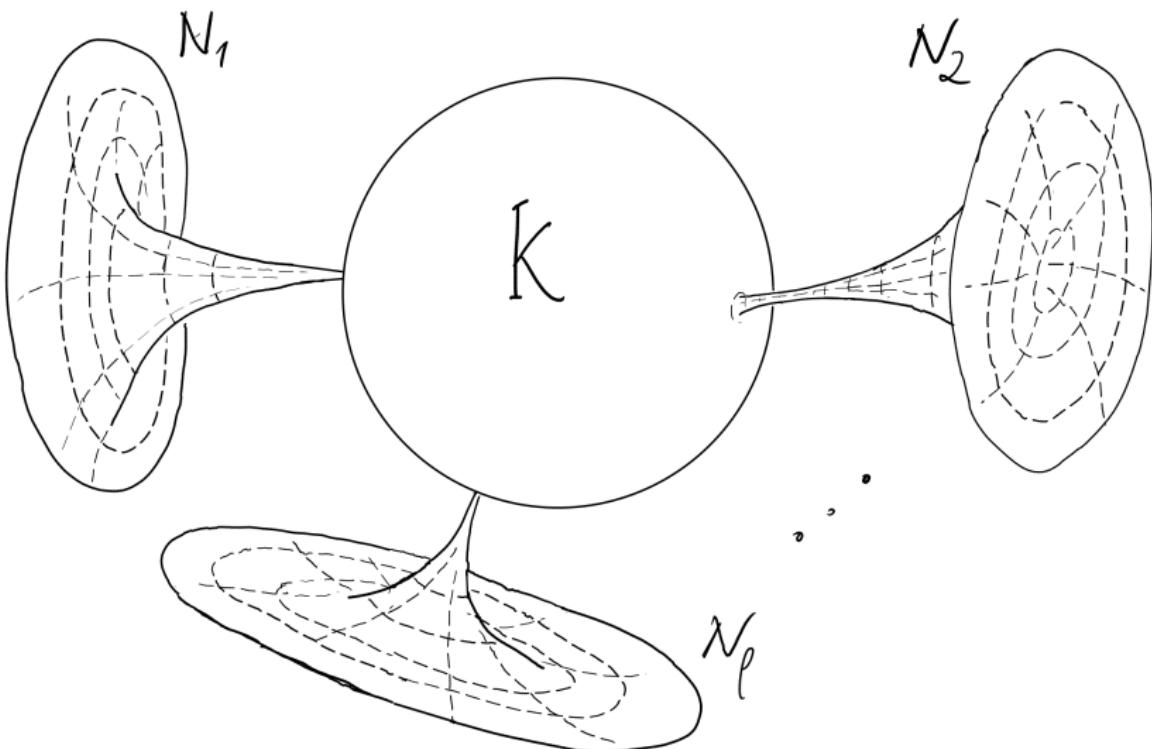
- Riemannian geometry and general relativity
- The ADM mass
- Positivity of mass
- Zero-mass flatness

# RIEMANNIAN GEOMETRY AND GENERAL RELATIVITY

- *Lorentzian 4-manifold*: manifold  $M$  of dimension 4 endowed with a metric  $\bar{g}$  of signature  $(-, +, +, +)$ .
- *Spacelike hypersurface*: 3-submanifold  $\iota: N \rightarrow M$  such that the metric  $g = \iota^* \bar{g}$  induced on  $N$  by the immersion  $\iota$  is Riemannian.
- *Ends*: given a spacelike hypersurface  $(N, g)$  such that there is a compact  $K \subset N$  for which  $N \setminus K = \bigsqcup_{k=1}^l N_k$  we get

$$N_k \simeq \mathbb{R}^3 \setminus B_\sigma(p)$$

for all  $k$ ,  $N_k$  is called an end of  $N$ .



# THE ADM MASS

## DEFINITION

The metric  $g$  is called *asymptotically flat* if on every  $N_k$  there exists a coordinate system  $\{x^1, x^2, x^3\}$  such that  $g = g_{ij}dx^i dx^j$ , where

$$\begin{aligned} g_{ij} &= \left(1 + \frac{M_k}{2r}\right)^4 \delta_{ij} + h_{ij}, \quad |h_{ij}| \leq \frac{c_1}{1+r^2} \\ |\partial h_{ij}| &\leq \frac{c_2}{1+r^3}, \quad |\partial^2 h_{ij}| \leq \frac{c_3}{1+r^4} \end{aligned} \tag{1}$$

The number  $M_k$  is the *total ADM mass* of the end  $N_k$ .

The original definition is given by

$$M_k = \frac{1}{16\pi} \lim_{r \rightarrow +\infty} \int_{S_r} \sum_{i,j=1}^3 (\partial_j g_{ij} - \partial_i g_{jj}) d\sigma_i \tag{2}$$

# THE POSITIVE MASS THEOREM

Schoen and Yau split this into two main theorems in their original article [1]. Here the theorem is proved in the case of maximal  $N$  in  $M$ .

## NON-NEGATIVITY OF MASS

Let  $(N, g)$  be an oriented Riemannian 3-manifold such that  $g$  is an asymptotically flat metric. If the scalar curvature of  $N$  is  $R \geq 0$ , then the ADM mass of each end satisfies  $M_k \geq 0$ .

- The proof is developed in three steps.
- The idea is to assume the mass of an end to be negative in order to deduce a contradiction.

- The second theorem focuses on the case of null mass and needs the following technical condition:

$$|\partial^3 h_{ij}| + |\partial^4 h_{ij}| + |\partial^5 h_{ij}| \leq \frac{c_4}{1+r^5} \quad (3)$$

### ZERO-MASS FLATNESS

Suppose that for some end  $N_k$  the condition (3) is verified and that  $M_k = 0$ . If  $R \geq 0$  on  $N$ , then the metric  $g$  is flat. Precisely,  $N$  is isometric to  $\mathbb{R}^3$  with the standard Euclidean metric.

- This proof is articulated in three steps, too.
- The argument is based on the fact that it is possible to reduce the situation to the previous case unless the Ricci tensor is identically zero.

# THE NON-NEGATIVITY OF MASS

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Fix an end  $N_k \simeq \mathbb{R}^3 \setminus B_{\sigma_0}(0)$  and denote by  $M$  its total ADM mass. The three steps are:

- assuming  $M < 0$  and  $R \geq 0$  to show that  $R > 0$  outside of a compact subset of  $N_k$ ;
- constructing a specific minimal surface embedded in  $N$ ;
- through the second variation formula and the local Gauss-Bonnet theorem, proving that such a surface cannot exist if  $R \geq 0$ .

# FIRST STEP

## LEMMA

If  $g$  is an asymptotically flat metric on  $N$  with  $R \geq 0$  and  $M < 0$ , then there exists an asymptotically flat metric  $\tilde{g}$  conformally equivalent to  $g$ , such that  $\tilde{R} \geq 0$  on  $N$ ,  $\tilde{R} > 0$  outside of a compact subset of  $N_k$  and  $\tilde{M} < 0$ .

- The assumption  $M < 0$  lets us define a function  $\psi: N \rightarrow \mathbb{R}$  such that, for  $\sigma > \sigma_0$ ,

$$\Delta\psi \leq 0 \quad \text{on } N, \quad \Delta\psi < 0 \quad \text{for } r > 2\sigma \text{ on } N_k \quad (4)$$

- By defining

$$\tilde{g} = \psi^4 g$$

we find a metric that satisfies the desired properties since

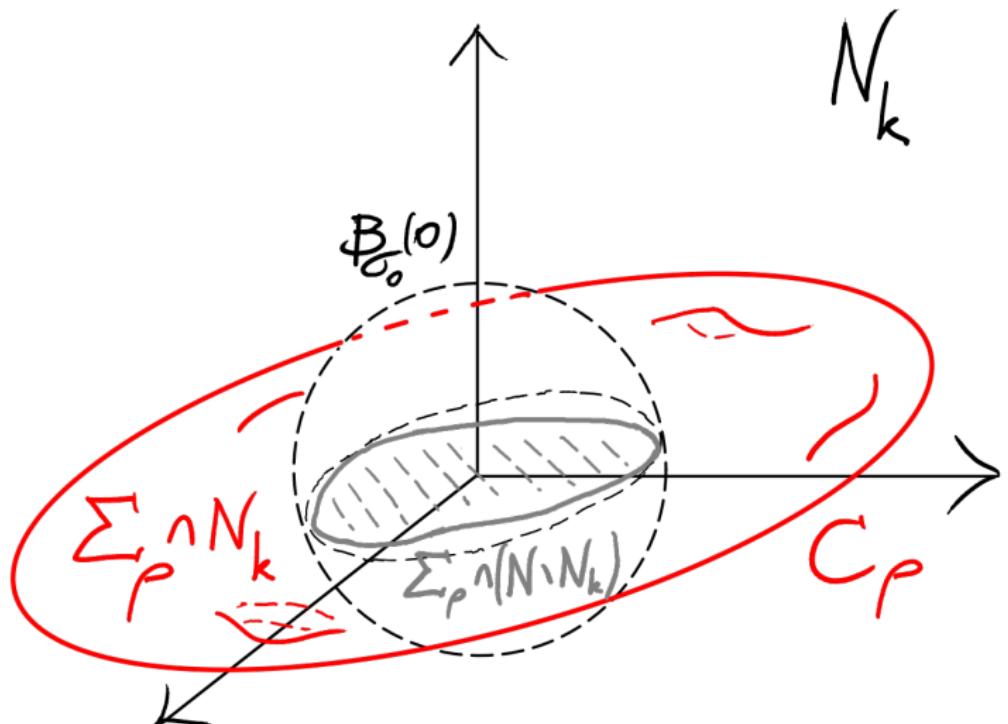
$$\tilde{R} = \psi^{-5} (R \psi - 8\Delta\psi)$$

## SECOND STEP

### PROPOSITION

With respect to the metric  $g$ , there exists a complete area-minimizing surface  $\Sigma$  properly embedded in  $N$ , such that  $\Sigma \cap (N \setminus N_k)$  is compact and that  $\Sigma \cap N_k$  lies between two parallel Euclidean 2-planes in  $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$ .

- We construct a sequence  $\{\Sigma_\rho\}$  of minimal surfaces with the circumference  $C_\rho$  as their boundary.
- The convergence of the sequence is a consequence of a regularity estimate on the surfaces.
- It can be shown that there is a compact  $H \subset N$  such that  $\Sigma_\rho \cap (N \setminus N_k) \subseteq H$ .
- With minimum and maximum arguments, it is possible to bound  $\Sigma_\rho \cap N_k$  along the vertical coordinate.



# THIRD STEP

## PROPOSITION

The surface  $\Sigma$  does not exist.

- Thanks to the minimality of  $\Sigma$ , the Gauss equation on curvature yields

$$\frac{1}{2}|\mathrm{II}|^2 = K_{12} - K$$

where  $\mathrm{II}$  is the second fundamental form of  $\Sigma$ ,  $K_{ij}$  is the  $ij$ -th component of the sectional curvature and  $K$  is the Gauss curvature.

- This allows for a suitable manipulation of the stability inequality

$$\int_{\Sigma} (\mathrm{Ric}(\nu) + |\mathrm{II}|^2) f^2 dS \leq \int_{\Sigma} |\mathrm{grad}_{\Sigma} f|^2 dS \quad (5)$$

- Hence it is possible to show that  $\int_{\Sigma} K > 0$  since  $R \geq 0$  and  $R > 0$  outside of a compact subset of  $\Sigma$ .
- Let  $\{D_\rho\}$  be an exhaustion of  $\Sigma$ , where for large  $\rho$ ,  $D_\rho$  is a topologic disk thanks to the properties mentioned in the second step. By the local Gauss-Bonnet theorem we get

$$\int_{D_\rho} K = 2\pi - \int_{\partial D_\rho} k_g$$

and it can be shown that

$$\int_{\partial D_{\rho_i}} k_g \geq 2\pi - o(1) \quad \text{for } i \rightarrow +\infty \tag{6}$$

- This implies  $\int_{\Sigma} K \leq 0$ , giving the desired contradiction.

# ZERO-MASS FLATNESS

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Suppose that for some end  $N_k$  the condition (3) is verified and that  $M_k = 0$ . If  $R \geq 0$  on  $N$ , then the metric  $g$  is flat. Precisely,  $N$  is isometric to  $\mathbb{R}^3$  with the standard Euclidean metric.

The three steps are:

- estimating the asymptotic behavior of the solution to a suitable elliptic problem;
- showing that, through a conformal transformation, if  $M = 0$  and  $R \geq 0$ , then  $R = 0$ ;
- proving that in this case, then  $N$  is Ricci-flat.

# FIRST STEP

The mentioned elliptic problem is

$$\Delta v - fv = h \quad \text{in } N \tag{7}$$

where  $f, h \in C^\infty(N)$  satisfy  $|f| + |\partial f| \leq c_5/(1+r)^5$ ,  $|h| + |\partial h| \leq c_6/(1+r)^5$  on  $N_k$ , and  $f$  satisfies  $(\int_N (f_-)^{3/2})^{2/3} \leq \varepsilon_0$ . The solution is given by

$$v = \frac{A}{r} + \omega, \quad \text{with} \quad r^2|\omega| + r^3|\partial\omega| + r^4|\partial^2\omega| \leq c_4$$

where  $\partial_{\hat{n}} v = 0$  on  $\partial N$  and  $A = -\frac{1}{4\pi} \int_N (fv + h)$ .

The estimates on the derivatives of  $\omega$  are a consequence of a Schauder estimate [3] on linear elliptic operators.

# SECOND STEP

## LEMMA

Suppose that  $g$  is an asymptotically flat metric on  $N$  for which (3) is satisfied. Let  $R$  be the scalar curvature of  $g$  and suppose that it satisfies

$\frac{1}{8} \left( \int_N (R_-)^{3/2} \right)^{2/3} \leq \varepsilon_0$ . Then there exists a unique positive function  $\varphi$  such that  $\partial_{\hat{n}} \varphi = 0$  on  $\partial N$  and the metric  $\bar{g} = \varphi^4 g$  is asymptotically flat, scalar-flat and so that it has mass  $\bar{M} = -(32\pi)^{-1} \int_N R \varphi$ .

- As in the first step of the first theorem,

$$\bar{R} = \varphi^{-5} (R \varphi - 8\Delta\varphi)$$

- Asking  $\bar{R} = 0$  translates in solving (7) for  $\varphi$ .
- The first theorem and this lemma imply that an asymptotically flat metric with  $M = 0$  and  $R \geq 0$  necessarily has  $R \equiv 0$ .

# THIRD STEP

## PROPOSITION

Let  $g$  be an asymptotically flat metric for which (3) such that  $M = 0$  and  $R = 0$  on  $N$ . Then the Ricci tensor relative to  $g$  vanishes identically.

- Let  $g_t = g + t \text{ric}$  be a new metric defined in a neighborhood of  $t = 0$  that is asymptotically flat thanks to (3), where  $\text{ric}$  is the Ricci tensor of  $g$ .
- Denoting by  $M(t)$  the mass of  $N_k$  relative to  $g_t$ , through the problem (7) it is possible to show that

$$\frac{d}{dt} M(t) \Big|_{t=0} = \frac{1}{32\pi} \int_N \|\text{ric}\|^2$$

- If  $\text{ric} \neq 0$ , then  $M'(0) > 0$ . Hence there is a  $t_0$  such that  $M(t_0) < 0$ .
- Therefore, by the lemma in the previous step, there exists a metric  $\tilde{g}_{t_0} = \varphi_{t_0}^4 g_{t_0}$  with negative mass, in contrast with the first theorem, and so necessarily  $\text{ric} \equiv 0$ .

# CONCLUSIONS

Finally,

- if the hypersurface  $N$  with induced asymptotically flat metric  $g$  has non-negative scalar curvature, the mass of each end is non-negative;
- since in three dimensions the Ricci tensor is zero if and only if the Riemann tensor is zero, we conclude that  $(N, g)$  is isometric to Euclidean space when an end has null mass.



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