The Schoen-Yau Positive Mass Theorem

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Introduction

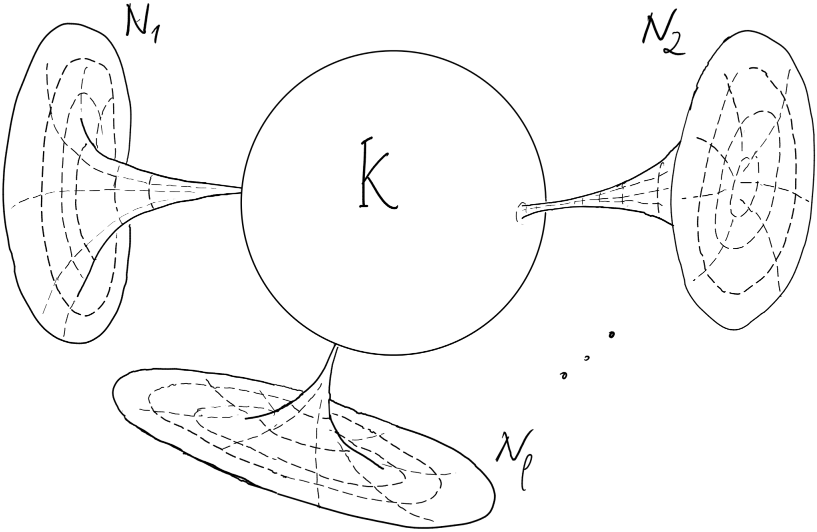
- Riemannian geometry and general relativity
- The ADM mass
- Positivity of mass
- Zero-mass flatness

Riemannian Geometry and General Relativity

- *Lorentzian* 4-manifold: manifold M of dimension 4 endowed with a metric $\bar{g}$ of signature $(-,+,+,+)$.
- *Spacelike hypersurface*: 3-submanifold $\iota \colon N \to M$ such that the metric $g = \iota^*\bar{g}$ induced on N by the immersion ι is Riemannian.
- *Ends*: given a spacelike hypersurface (N, g) such that there is a compact $K \subset N$ for which $N \setminus K = \bigsqcup_{k=1}^{l} N_k$ we get

$$N_k \simeq \mathbb{R}^3 \setminus B_\sigma(p)$$

for all k, N_k is called an end of N.



N_1
N_2
K
N_ℓ

The ADM Mass

Definition

The metric g is called *asymptotically flat* if on every N_k there exists a coordinate system $\{x^1, x^2, x^3\}$ such that $g = g_{ij}dx^i dx^j$, where

$$g_{ij} = \left(1 + \frac{M_k}{2r}\right)^4 \delta_{ij} + h_{ij}, \quad |h_{ij}| \leq \frac{c_1}{1+r^2}$$
$$|\partial h_{ij}| \leq \frac{c_2}{1+r^3}, \quad |\partial^2 h_{ij}| \leq \frac{c_3}{1+r^4} \tag{1}$$

The number M_k is the *total ADM mass* of the end N_k.

The original definition is given by

$$M_k = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{S_r} \sum_{i,j=1}^{3} (\partial_j g_{ij} - \partial_i g_{jj}) d\sigma_i \tag{2}$$

The Positive Mass Theorem

Schoen and Yau split this into two main theorems in their original article [1]. Here the theorem is proved in the case of maximal N in M.

Non-negativity of Mass

Let (N, g) be an oriented Riemannian 3-manifold such that g is an asymptotically flat metric. If the scalar curvature of N is $\mathrm{R} \geq 0$, then the ADM mass of each end satisfies $M_k \geq 0$.

- The proof is developed in three steps.
- The idea is to assume the mass of an end to be negative in order to deduce a contradiction.

- The second theorem focuses on the case of null mass and needs the following technical condition:

$$\left|\partial^3 h_{ij}\right| + \left|\partial^4 h_{ij}\right| + \left|\partial^5 h_{ij}\right| \leq \frac{c_4}{1+r^5} \tag{3}$$

Zero-mass Flatness

Suppose that for some end N_k the condition (3) is verified and that $M_k = 0$. If $\mathrm{R} \geq 0$ on N, then the metric g is flat. Precisely, N is isometric to $\mathbb{R}^3$ with the standard Euclidean metric.

- This proof is articulated in three steps, too.
- The argument is based on the fact that it is possible to reduce the situation to the previous case unless the Ricci tensor is identically zero.

The Non-negativity of Mass

Non-negativity of Mass

Let (N, g) be an oriented Riemannian 3-manifold such that g is an asymptotically flat metric. If the scalar curvature of N is $\mathrm{R} \geq 0$, then the ADM mass of each end satisfies $M_k \geq 0$.

Fix an end $N_k \simeq \mathbb{R}^3 \setminus B_{\sigma_0}(0)$ and denote by M its total ADM mass. The three steps are:

- assuming $M < 0$ and $R \geq 0$ to show that $\mathrm{R} > 0$ outside of a compact subset of N_k;
- constructing a specific minimal surface embedded in N;
- through the second variation formula and the local Gauss-Bonnet theorem, proving that such a surface cannot exist if $\mathrm{R} \geq 0$.

First Step

Lemma

If g is an asymptotically flat metric on N with $\mathrm{R} \geq 0$ and $M < 0$, then there exists an asymptotically flat metric $\tilde{g}$ conformally equivalent to g, such that $\widetilde{\mathrm{R}} \geq 0$ on N, $\widetilde{\mathrm{R}} > 0$ outside of a compact subset of N_k and $\widetilde{M} < 0$.

- The assumption $M < 0$ lets us define a function $\psi\colon N \to \mathbb{R}$ such that, for $\sigma > \sigma_0$,

$$\Delta\psi \leq 0 \quad \text{on } N, \quad \Delta\psi < 0 \quad \text{for } r > 2\sigma \text{ on } N_k \tag{4}$$

- By defining

$$\tilde{g} = \psi^4 g$$

we find a metric that satisfies the desired properties since

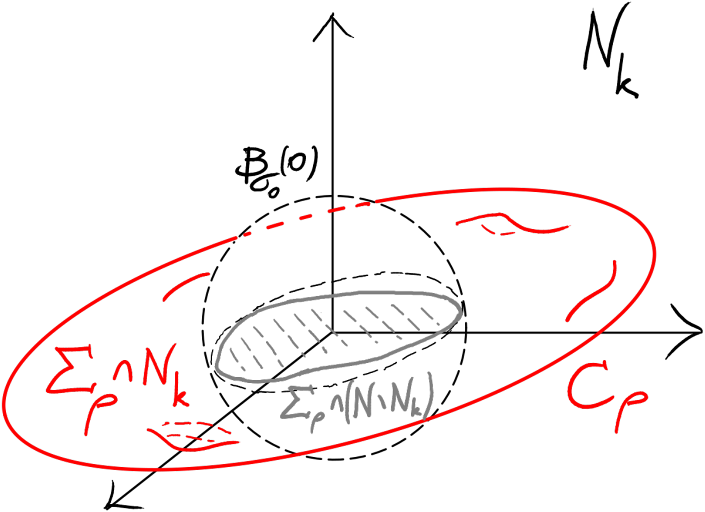
$$\widetilde{\mathrm{R}} = \psi^{-5}\left(\mathrm{R}\,\psi - 8\Delta\psi\right)$$

Second Step

Proposition

With respect to the metric g, there exists a complete area-minimizing surface Σ properly embedded in N, such that $\Sigma \cap (N \setminus N_k)$ is compact and that $\Sigma \cap N_k$ lies between two parallel Euclidean 2-planes in $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$.

- We construct a sequence $\{\Sigma_\rho\}$ of minimal surfaces with the circumference C_ρ as their boundary.
- The convergence of the sequence is a consequence of a regularity estimate on the surfaces.
- It can be shown that there is a compact $H \subset N$ such that $\Sigma_\rho \cap (N \setminus N_k) \subseteq H$.
- With minimum and maximum arguments, it is possible to bound $\Sigma_\rho \cap N_k$ along the vertical coordinate.



N_k
$B_{\sigma_0}(0)$
$\Sigma_p \cap N_k$
$\Sigma_p \cap (N \setminus N_k)$
C_p

Third Step

Proposition

The surface Σ does not exist.

- Thanks to the minimality of Σ, the Gauss equation on curvature yields

$$\frac{1}{2}|\mathrm{II}|^2 = K_{12} - K$$

where II is the second fundamental form of Σ, K_{ij} is the ij-th component of the sectional curvature and K is the Gauss curvature.

- This allows for a suitable manipulation of the stability inequality

$$\int_\Sigma (\mathrm{Ric}(\nu) + |\mathrm{II}|^2) f^2 dS \leq \int_\Sigma |\mathrm{grad}_\Sigma f|^2 dS \tag{5}$$

- Hence it is possible to show that $\int_\Sigma K > 0$ since $\mathrm{R} \geq 0$ and $\mathrm{R} > 0$ outside of a compact subset of Σ.

- Let $\{D_\rho\}$ be an exhaustion of Σ, where for large ρ, D_ρ is a topologic disk thanks to the properties mentioned in the second step. By the local Gauss-Bonnet theorem we get

$$\int_{D_\rho} K = 2\pi - \int_{\partial D_\rho} k_g$$

and it can be shown that

$$\int_{\partial D_{\rho_i}} k_g \geq 2\pi - o(1) \quad \text{for } i \to +\infty \tag{6}$$

- This implies $\int_\Sigma K \leq 0$, giving the desired contradiction.

Zero-mass Flatness

Zero-mass Flatness

Suppose that for some end N_k the condition (3) is verified and that $M_k = 0$. If $\mathrm{R} \geq 0$ on N, then the metric g is flat. Precisely, N is isometric to $\mathbb{R}^3$ with the standard Euclidean metric.

The three steps are:

- estimating the asymptotic behavior of the solution to a suitable elliptic problem;
- showing that, through a conformal transformation, if $M = 0$ and $\mathrm{R} \geq 0$, then $\mathrm{R} = 0$;
- proving that in this case, then N is Ricci-flat.

First Step

The mentioned elliptic problem is

$$\Delta v - fv = h \quad \text{in } N \tag{7}$$

where $f, h \in \mathcal{C}^\infty(N)$ satisfy $|f| + |\partial f| \le c_5/(1+r)^5$, $|h| + |\partial h| \le c_6/(1+r)^5$ on N_k, and f satisfies $\left(\int_N (f_-)^{3/2}\right)^{2/3} \le \varepsilon_0$. The solution is given by

$$v = \frac{A}{r} + \omega, \quad \text{with} \quad r^2|\omega| + r^3|\partial\omega| + r^4\left|\partial^2\omega\right| \le c_4$$

where $\partial_{\hat{n}} v = 0$ on ∂N and $A = -\frac{1}{4\pi}\int_N (fv + h)$.
The estimates on the derivatives of ω are a consequence of a Schauder estimate [3] on linear elliptic operators.

Second Step

Lemma

Suppose that g is an asymptotically flat metric on N for which (3) is satisfied. Let R be the scalar curvature of g and suppose that it satisfies $\frac{1}{8}\left(\int_N (\mathrm{R}_-)^{3/2}\right)^{2/3} \leq \varepsilon_0$. Then there exists a unique positive function φ such that $\partial_{\hat{n}}\varphi = 0$ on ∂N and the metric $\bar{g} = \varphi^4 g$ is asymptotically flat, scalar-flat and so that it has mass $\bar{M} = -(32\pi)^{-1}\int_N \mathrm{R}\,\varphi$.

- As in the first step of the first theorem,

$$\bar{\mathrm{R}} = \varphi^{-5}\left(\mathrm{R}\,\varphi - 8\Delta\varphi\right)$$

- Asking $\bar{\mathrm{R}} = 0$ translates in solving (7) for φ.
- The first theorem and this lemma imply that an asymptotically flat metric with $M = 0$ and $\mathrm{R} \geq 0$ necessarily has $\mathrm{R} \equiv 0$.

Third Step

Proposition

Let g be an asymptotically flat metric for which (3) such that $M = 0$ and $\mathrm{R} = 0$ on N. Then the Ricci tensor relative to g vanishes identically.

- Let $g_t = g + t\,\mathrm{ric}$ be a new metric defined in a neighborhood of $t = 0$ that is asymptotically flat thanks to (3), where ric is the Ricci tensor of g.
- Denoting by $M(t)$ the mass of N_k relative to g_t, through the problem (7) it is possible to show that

$$\frac{d}{dt} M(t)\bigg|_{t=0} = \frac{1}{32\pi} \int_N \|\mathrm{ric}\|^2$$

- If $\mathrm{ric} \neq 0$, then $M'(0) > 0$. Hence there is a t_0 such that $M(t_0) < 0$.
- Therefore, by the lemma in the previous step, there exists a metric $\tilde{g}_{t_0} = \varphi_{t_0}^4 g_{t_0}$ with negative mass, in contrast with the first theorem, and so necessarily $\mathrm{ric} \equiv 0$.

Conclusions

Finally,

- if the hypersurface N with induced asymptotically flat metric g has non-negative scalar curvature, the mass of each end is non-negative;
- since in three dimensions the Ricci tensor is zero if and only if the Riemann tensor is zero, we conclude that (N, g) is isometric to Euclidean space when an end has null mass.

Thank you for listening!

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