RG-flow and Bifurcation

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1608.06638

(A) Motivation

RG-flow = dynamical system

$$\frac{d\lambda^i}{dt} = \beta^i(\lambda).$$

"time" = energy scale

"velocity" = β function

1-parameter motion in the space of couplings $\{\lambda^i\} \in \mathcal{T}$

Fixed point T_* = CFT.

Strongest version of $\mathcal{C}$-theorem;

RG-flow = gradient flow for some C-function

$\mathcal{C} : \mathcal{T} \to \mathbb{R}$ (smooth).

- For gradient flows, we can use Bott-Morse theory to analyse the topology of fixed points in $\mathcal{T}$.
- One conjecture is as follows. S. Gukov. arXiv:1503.01474

$\boxed{\mathcal{M}\text{-theorem}}$: A gradient flow "breaks" at the point along the flow where irrelevant operators cross through marginality.

(i.e. dangerously irrelevant operators).

→ Q1. How often does "marginality crossing" occurs?

- $d=4$ $\mathcal{N}=1$ theories ← 1503.01474.
- lower end of conformal window ← this paper
- AdS/CFT: $S = \int d^{d+1}x\sqrt{-g}\left(\frac{1}{4}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi^i + V(\phi_i)\right)$.

adjust $V(\phi_i)$.

Q2. What happens at "marginality crossing"?

→ Phase transition

Answer this question for <u>lower end of conformal window</u> by using $\boxed{\text{bifurcation theory}}$.

Plan

Ⓐ Motivation

Ⓑ Bifurcation of RG-flow

Ⓒ Application to 3d $O(N)$ model

Ⓓ " QED_3

Ⓔ " QCD_4.

Ⓑ Bifurcation of RG-flows

Bifurcation = change of fixed points as parameters of the system $x \in X$ varies.

$$\frac{d\lambda^i}{dt} = \beta^i(\underset{\sim}{\lambda} ; \underset{\sim}{x})$$

couplings

parameters $x \in X$
- controllable (do not run)

$d = 4 - \underset{\sim}{\varepsilon}$, $\underset{\sim}{N_c}$, $\underset{\sim}{N_f}$

$\boxed{\text{Examples}}$

1) Saddle-node bifurcation.

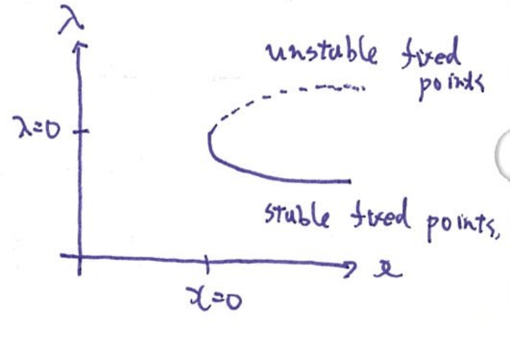
$$\frac{d\lambda}{dt} = \lambda^2 - x.$$

1 coupling λ
1 parameter x.

$x > 0 \rightsquigarrow \beta = 0$ @ 迷 $\lambda = \pm\sqrt{x}$.

$x < 0 \rightsquigarrow$ no solution for $\beta = 0$

<u>Bifurcation diagram</u>

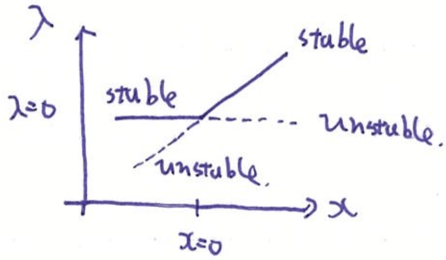


2) transcritical bifurcation

$$\frac{d\lambda}{dt} = x\lambda - \lambda^2$$

1 coupling λ.

1 parameter x.

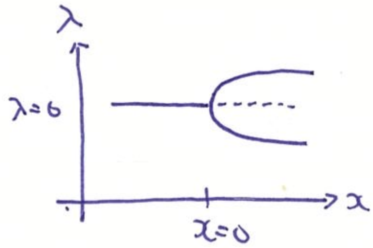


bifurcation diagram.

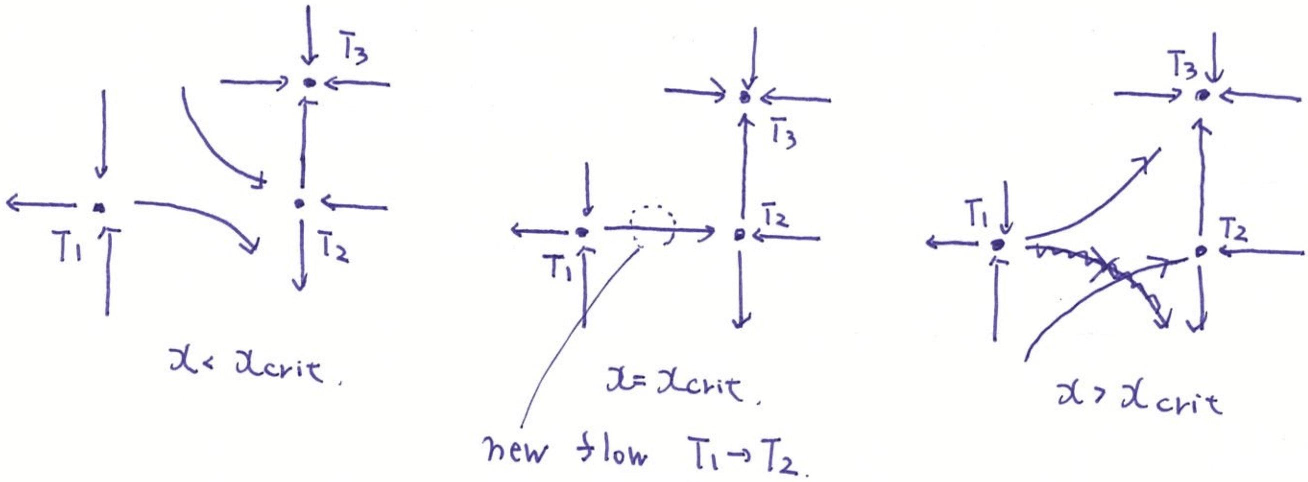
3) pitchfolk bifurcation

$$\frac{d\lambda}{dt} = x\lambda - \lambda^3.$$

1 coupling λ. 1 parameter x.



4) Heteroclinic bifurcation.



2 couplings λ_1, λ_2

1 parameter x.

5), 6), 7), --- and many other examples.

$\boxed{\text{Stability \& Unfolding of bifurcation.}}$

Consider 1 coupling λ, 1 parameter x system.

- Saddle-node ; $\beta = \partial_\lambda \beta = 0$ @ bifurcation point on (λ, x)-plane.

  $\longrightarrow$ generically have solution @ $\lambda = \lambda_*$, $x = x_{crit}$.

called as "<u>codim-1</u>" bifurcation

- transcritical $\beta = \partial_x \beta = \partial_\lambda \beta = 0$.
- pitchfolk $\beta = \partial_\lambda \beta = \partial_x \beta = \partial_\lambda^2 \beta = 0$

$\}$ multiple conditions

$\rightsquigarrow$ have no solutions on (λ, x) without "fine-tuning"

called as "higher-codim" bifurcation.

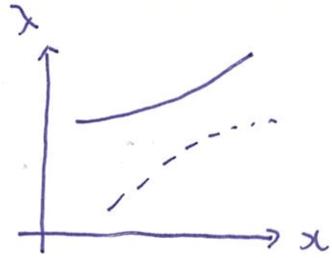
generic	vs.	non-generic.
saddle-node		transcritical

Example :

① transcritical

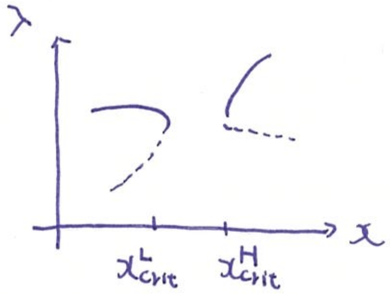
$$\beta(\lambda : x) = x\lambda - \lambda^2 \to \beta'(x;\lambda) = \underbrace{u}_{\text{unfolding parameter.}} + x\lambda - \lambda^2$$

$u > 0$



no bifurcation.

$u < 0$



two saddle-node bifurcations.

② pitchfolk

$$\beta(\lambda : x) = x\lambda - \lambda^3 \to \beta'(\lambda; x) = u + x\lambda + v\lambda^2 - \lambda^3.$$

unfolding parameters.

- pitchfolk $\to$ 3 saddle-nodes or 1 saddle-node depending on (u, v)

- symmetry $\lambda \to -\lambda$ forbids u & v.

$\Rightarrow$ transcritical, pitchfolk is <u>unstable</u>; saddle-node is <u>stable</u> under perturbations.
(without symmetry).

Higher-order correction

Example : transcritical.

$$\beta(x;\lambda) = x\lambda - \lambda^2 \rightarrow \beta'(\lambda:x) = x\lambda - \lambda^2 - \lambda^3.$$

additional saddle-node bifurcation.

Operator dimensions

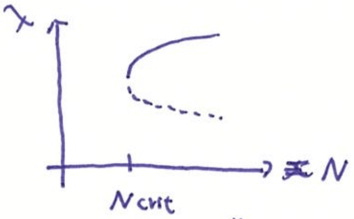
$J_{ij} = (\partial_i \beta_j)$: matrix. ← couplings $\{\lambda^i\}$

- eigenvalues of J @ $\lambda = \lambda_*$ $\leftrightarrow$ values of $d - \Delta_i$
- $\forall$ eigenvalue of $J < 0$ $\leftrightarrow$ stable fixed point λ_*
- zero eigenvalue of J $\leftrightarrow$ $\exists$ marginal operator $\mathcal{O}_i$

  ↑

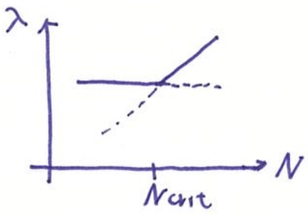
$\left\{\begin{array}{l} \text{saddle-node bifurcation} \\ \text{transcritical} \quad " \\ \text{pitchfolk} \quad " \end{array}\right.$ @ $x = x_{crit}$, $\lambda = \lambda_*$.

$\Rightarrow$ Th If the loss of conformality at the lower end of conformal window is due to.



"merger & annihilation"

or



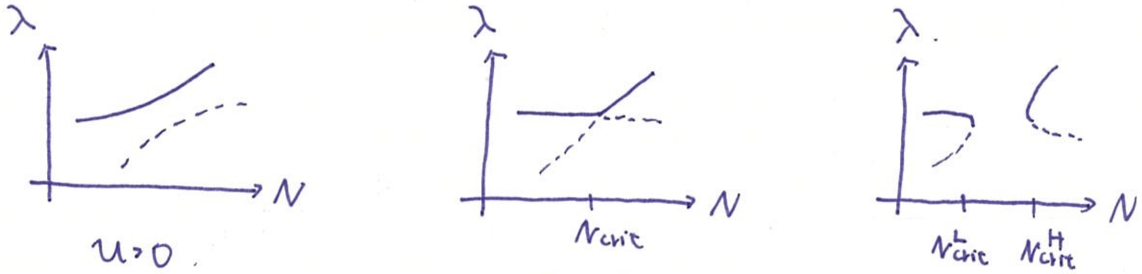
"go through each other"

$\Rightarrow$ At least one irrelevant operator becomes marginal at the transition point $N = N_{crit}$. $\lrcorner$

Example : transcritical bifurcation.

$$\dot{\lambda} = u + x\lambda - \lambda^2$$

unfolding parameter (u) $N - N_{crit}$ (x).



$u > 0$. $u = 0$. $u < 0$.

Operator dimension : $\Delta - d \sim \pm\sqrt{4u + x^2}$

$$\longrightarrow \begin{cases} \Delta - d \sim \sqrt{N - N_{crit}^{(L,H)}} & \text{for } u < 0. \\ \Delta - \Delta_0 \sim (N - N_{crit})^2 & \text{for } u > 0. \\ \Delta - d \sim N - N_{crit} & \text{for } u = 0 \end{cases}$$

$$\boxed{\begin{array}{l} \text{Scaling behavior of } \Delta(N) \text{ near } N = N_{crit} \\ \qquad \Leftrightarrow \text{topology of the bifurcation diagram.} \end{array}}$$

Ⓒ Application to 3d O(N) model

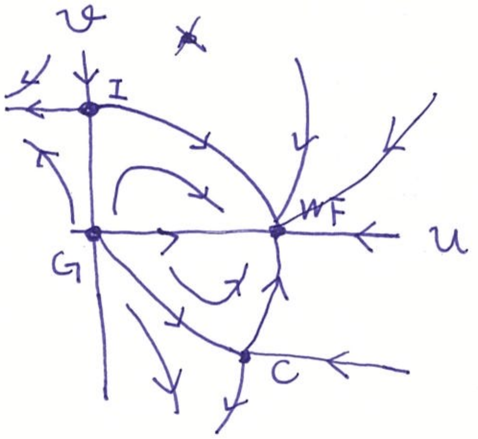
3d O(N) model.

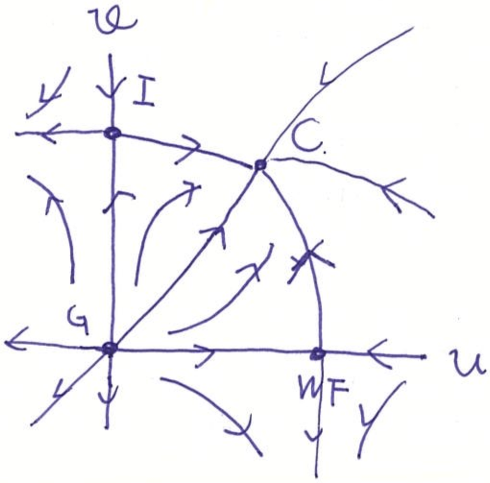
$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{N} (\partial_m \phi_i)^2 + \frac{1}{4!}\sum_{i,j=1}^{N} (u_0 + v_0\,\delta_{ij})\,\phi_i^2\,\phi_j^2.$$

real scalars.

$\begin{cases} u_0 \text{ preserves O(N) symmetry} \\ v_0 \text{ breaks} \quad \text{"} \end{cases}$

RG-flow





$N < \underset{\sim}{N_{crit}}$.

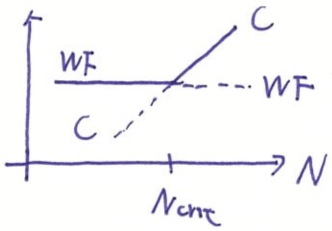
G (Gaussian) : unstable
WF (Wilson-Fischer) : stable
I (Ising), C (Cubic) : saddle

$N > \underset{\sim}{N_{crit}}$

G : unstable
C : stable
WF, I : saddle

Q. What happens @ $N = N_{crit}$?

— Ans transcritical bifurcation.



However, this bifurcation generically does not occur in
1-parameter (N) system ....

Arguement for transcritical bifurcation.

- Perturbative RG-flow in 3d $O(N)$ model

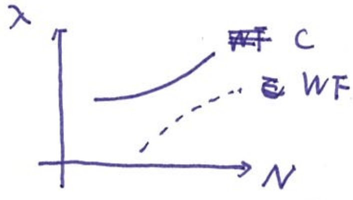
1-loop → transcritical bifurcation

2-loops → 〃

3-loops → 〃

• unfold the transcritical bifurcation.

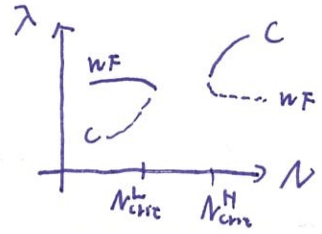
A)



no bifurcation

or

B)



two saddle-node bifurcations

A) does not happen.

∵ numerical study

$N=2$ $\quad \det(J)|_{WF} > 0, \quad \det(J)|_C < 0.$

$N=4$ $\quad \det(J)|_{WF} < 0, \quad \det(J)|_C > 0.$

$\rightsquigarrow$ bifurcation indeed happens @ $2 < N_{crit} < \cancel{3}\ 4$.

B) does not happen

∵ symmetry

WF $\quad O(N)$ symmetry

C $\quad S_N \ltimes (\mathbb{Z}_2)^N$ symmetry

different symmetry!

$\Rightarrow$ $\boxed{\text{Transcritical bifurcation happens in 3d } O(N) \text{ model @ } N = N_{crit}.}$

($\rightsquigarrow$ $\Delta - 3 \sim N - N_{crit}$)

Ⓓ Application to QED_3

$G = \mathbb{R}$, 3d parity-invariant QED

no monopoles,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \sum_{a=1}^{N_f} \bar{\Psi}_a i \gamma^\mu D_\mu \Psi^a + \mathcal{L}_{mass} + \mathcal{L}_{4\text{-Ferm}}$$

($\gamma_{0,1,2}$: 4×4 4d Dirac matrix

$\Psi_{a=1,\cdots,N_f}$: 4-component spinor ($\Psi_a \to \psi_a, \psi_{a+N_f}$ $\quad a=1,\cdots,N_f$))

2-component spinor.

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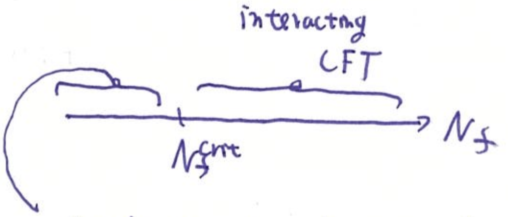
$\mathcal{L}_{mass} = \mathcal{L}_{4\text{-Fermi}} = 0 \rightsquigarrow$ Flavor symmetry $\boxed{SU(2N_f)}$

$\boxed{\text{IR phase}}$ (massless QED_3).

$[e^2] = 1 \rightsquigarrow$ asymptotically free

large N_f analysis $\rightsquigarrow$ IR fixed point.



confinement, dynamical mass generation, chiral symmetry breaking

$SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1)$.

$\boxed{\text{Couplings}}$

- Consider P-invariant, $SU(2N_f)$-invariant couplings.
  in $\mathcal{L}_{mass}$ & $\mathcal{L}_{4\text{-Fermi}}$

- $\mathcal{L}_{mass} = 0$

  $\because$ $$\mathcal{L}_{mass} = \underbrace{m(\bar{\psi}_a \psi^a - \bar{\psi}_{a+N_f}\psi^{a+N_f})}_{\substack{a=1,\cdots,N_f \\ \text{P-inv, but breaks} \\ SU(2N_f) \to SU(N_f)\times SU(N_f)\times U(1)}} + \underbrace{\tilde{m}\,\bar{\psi}_i \psi^i}_{\substack{i=1\cdots 2N_f \\ SU(2N_f)\text{-inv, but} \\ \text{breaks P.}}}$$

- $$\mathcal{L}_{4\text{-Fermi}} = \frac{\lambda}{N_f}(\bar{\Psi}_a \gamma_{35} \Psi^a)^2 + \frac{\lambda'}{N_f}(\bar{\Psi}_a \gamma^\mu \Psi^a)^2.$$

  (※ additional two couplings when we only require P & $SU(N_f)\times SU(N_f)\times U(1)$.)

$\Rightarrow$ 3-couplings system $\{e^2, \lambda, \lambda'\}$

for massless QED_3 @ $N_f > N_f^{crit}$.

$\boxed{\text{Bifurcation analysis}}$

Q. What happens @ $N_f = N_f^{crit}$?

— Ans Saddle-node bifurcation !

A) Large-N_f (1-loop) RG flow equation

Kaveh, Herbut 0411594.

$$\begin{cases} \dot{e}^2 = e^2 - N_f e^4 + \cdots & \text{①} \\ \dot{\lambda} = -\lambda - \lambda^2 + 4e^2\lambda + 18e^2\lambda' + \boxed{9N_f e^4} \\ \dot{\lambda}' = -\lambda' + \lambda'^2 + \frac{2}{3}e^2\lambda + \cdots \end{cases} \quad \text{②}$$

① $\rightarrow$ $e_*^2 = \frac{1}{N_f}$: parameter,

2-coupling system $\{\lambda, \lambda'\}$

with 1-parameter $\{N_f\}$.

② $\rightarrow$ $\boxed{\begin{matrix} \dot{\lambda} = \text{----} \\ \dot{\lambda}' = \text{----} \end{matrix}}$ + $9N_f e_*^4$

unfolding parameter

transcritical bifurcation

∘ Saddle-node bifurcation !

$\Delta_{4\text{-Fermi}} - d \sim \sqrt{N_f - N_f^{crit}}$.

B) ε-expansion around $d=4$

1-loop, ε-expansion, $d = 4-\varepsilon$.

Di Pietro, Komargodski, Shamir, Stamou 1508.06278.

⇝ Anomalous dimension of $(\bar{\Psi}_a \gamma_{35} \Psi^a)^2$ & $(\bar{\Psi}_a \gamma^\mu \Psi^a)^2$

$$d - \Delta_{4\text{-Fermi}} = -\frac{1}{2N_f}\left(4N_f + 1 \pm 2\sqrt{N_f^2 + N_f + 25}\right)$$

$$\approx 0.54\,(N_f^{crit} - N_f) \quad \text{w/ } N_f^{crit} \sim 2.7.$$

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transcritical bifurcation ?

→ Recent ε-expansion analysis prefers saddle-node bifurcation.

Giombi, Klebanov, Tarnopolsky 1508.06354
Herbut 1605.09482.

$\Rightarrow$ $\boxed{\text{Saddle-node bifurcation is likely to happens at } N_f = N_f^{crit} \text{ in } QED_3.}$

$$\Delta_{4\text{-Ferm}} - d \sim \sqrt{N_f - N_f^{crit}}.$$

Ⓔ. Application to QCD_4

two parameters (N_c, N_f).

$\boxed{\text{IR phase}}$

2-loops β-function for $\alpha \equiv \left(\frac{g}{4\pi}\right)^2$

$$\beta_\alpha = \gamma\alpha - b_1\alpha^2 - b_2\alpha^3$$

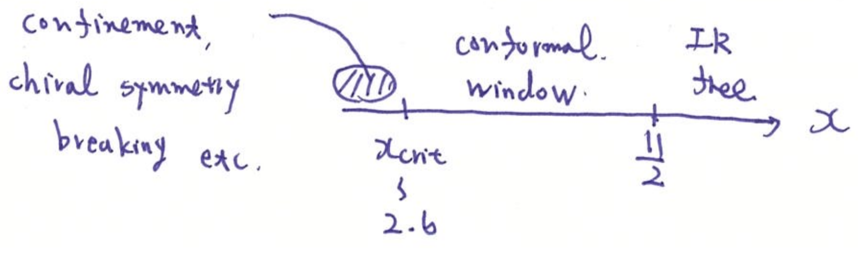
$$x \equiv \frac{N_f}{N_c}.$$

$$\begin{cases} \gamma = 0. \\ b_1 = \frac{2N_c}{3}(11 - 2x). \\ b_2 = \frac{2N_c^2}{3}\left(34 - 13x + \frac{3x}{N_c^2}\right) \end{cases}$$

$\longrightarrow$ $b_1 > 0$: UV, free-fixed point $(\alpha = 0)$.
$(x < \frac{11}{2})$. & asymptotically free

$$\begin{cases} ① \ b_2 < 0 : \text{Banks-Zaks fixed point @} \\ \qquad \alpha_* = -\frac{b_1}{b_2} = \frac{1}{N_c}\,\frac{11 - 2x}{13x - 34 - \frac{3x}{N_c^2}} \\ ② \ b_2 > 0 : \text{strong coupling behavior.} \end{cases}$$

In summary



$\boxed{\text{Bifurcation analysis.}}$

Q. What happens at $x = x_{crit}$?

– Ans Saddle-node bifurcation.

$$\mathcal{L} = -\frac{1}{4g^2} \mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu} + i \bar{\psi}_j \gamma^\mu D_\mu \psi^j + \mathcal{L}_{4\text{-Fermi}}$$

$$j = 1, \cdots, N_f.$$

※ • Consider P-invariant, $SU(N_f)_L \times SU(N_f)_R$-invariant couplings in $\mathcal{L}_{4\text{-Fermi}}$

$$\rightsquigarrow \mathcal{L}_{4\text{-Fermi}} = \frac{\lambda_1}{4\pi^2\Lambda^2} \mathcal{O}_1 + \frac{\lambda_2}{4\pi^2\Lambda^2} \mathcal{O}_2 + \frac{\lambda_3}{4\pi^2\Lambda^2} \mathcal{O}_3 + \frac{\lambda_4}{4\pi^2\Lambda^2} \mathcal{O}_4$$

$$\mathcal{O}_1 = (\bar{\psi}_i \gamma^\mu \psi^j)(\bar{\psi}_j \gamma_\mu \psi^i) + (\bar{\psi}_i \gamma^\mu \gamma_5 \psi^j)(\bar{\psi}_j \gamma_\mu \gamma_5 \psi^i)$$

etc …

Then, we have to consider 5-coupling system $\{g^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. difficult!

Veneziano limit ; $N_c, N_f \to \infty$, $x \equiv \frac{N_f}{N_c}$: fixed.

λ_3 & λ_4 decouples and the system becomes

3-couplings $\{g^2, \lambda_1, \lambda_2\}$ with 1-parameter $\{x\}$.

RG-eq

$\left(\begin{array}{l} \alpha \to \frac{1}{N_c}\alpha \\ (\lambda_1, \lambda_2) \to \frac{1}{N_c}(\lambda_1, \lambda_2) \end{array} \right)$

$$\begin{cases} \dot{\alpha} = -\frac{2}{3}(11-2x)\alpha^2 - \frac{2}{3}(34-13x)\alpha^3 + 2x\alpha^2\lambda_1 & \cdots ① \\ \dot{\lambda}_1 = 2\lambda_1 + (1+x)\lambda_1^2 + \frac{x}{4}\lambda_1^2 - \frac{3}{4}\alpha^2 & \cdots ② \\ \dot{\lambda}_2 = 2\lambda_2 - 2\lambda_2^2 + 2x\lambda_1\lambda_2 - 6\alpha\lambda_2 - \frac{9}{2}\alpha^2 & \cdots ③ \end{cases}$$

① $\longrightarrow \alpha_* = \dfrac{11-2x-3x\lambda_1}{13x-34} \sim \dfrac{11-2x}{13x-34}$ $(\lambda_1 \ll 1)$.

fixed point when $2.6 < x < \frac{11}{2}$.

② & ③ $\xrightarrow[\alpha=\alpha_*]{}$ 2-couplings system $\{\lambda_1, \lambda_2\}$

with 1-parameter $\{x\}$.

Result : saddle-node bifurcation @ $x_{crit} \approx 4$.

$$\boxed{\text{QCD}_4 \text{ is likely to exhibit saddle-node bifurcation @ } N_f = N_f^{crit}(N_c)}$$

Consequence of ∃ saddle-node bifurcation

① $\Delta_{4\text{-Ferm}} - 4 \sim \sqrt{N_f - N_f^{crit}}$ $(N_f^{crit} \sim 4N_c)$. $N_f \downarrow N_f^{crit}$.

② $m_{dyn} \sim \Lambda \exp\left(-\dfrac{c}{\sqrt{N_f^{crit} - N_f}}\right)$ $N_f \nearrow N_f^{crit}$.

└ dynamical mass of fermion @ $N_f < N_f^{crit}$.

called "Miransky scaling" Kaplan, Lee, Son, Stephanov. 0905.4752.