No. 1

Date 16.9.23

RG flows & bifurcations

1608.06638

Motivation

RG-flow $\frac{d\lambda^i}{dt} = -\beta^i(\lambda)$, $t \equiv -\ln\mu$.

⇝ energy scale をパラメタとして, 1-parameter motion in the space of $\{\lambda^i\} \in \mathcal{T}$ ← theory space.

⇓

RG-flow = dynamical system on $\mathcal{T}$

Fixed-point = CFT, $\mathcal{T}^*$ (とりあえず point のみ考える)

→ mathematics of dynamical system helps?

In particular, consider strongest version of $\mathcal{C}$-theorem. ← (たくさんある c-theorem, D=2,3,4)

"RG-flow* = gradient flow"

↑ ∃ C-function $\mathcal{C} : \mathcal{T} \to \mathbb{R}$ ← (CFT以外でも定義できると思う)

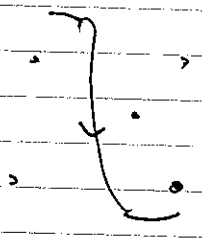
$\beta^i = \nabla^i \mathcal{C}$

(「βが grad-flow である」ということだけから言えることがいくつかある)

↓

For gradient flows we can use Bott-Morse theory

→ Necessary cond. on the topology of $\mathcal{T}$, ← structure of F.P.s in $\mathcal{T}$, …

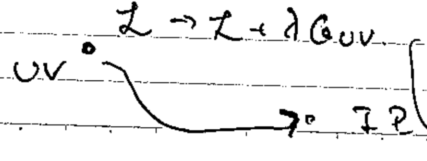


(いくつFPがあるか)
(どういうflowがあるか)
… etc.

One conjecture:

$\mathcal{C}$が連続でない.
Cが [illegible] なめらかでない.

μ-theorem: A gradient RG-flow breaks when $\mu_{UV} < \mu_{IR}$. $\mu = \#$(spin-0 relevant ops) @ CFT.



($\mu_{UV} < \mu_{IR}$ では. IR で新たな deformation が生じる)

"$\mu_{UV} < \mu_{IR}$" happens because some irrelevant ops. <u>cross</u> through marginality.

("dangerously irrelevant op.")

<u>Q1</u> How often does "marginality crossing" occur?

- examples from $d=4$. $\mathcal{N}=1$. theories (1503.01474)
- lower end of the conformal window (この論文)
- AdS/CFT?

<u>Q2</u> What happens at "marginality crossing"?

- use bifurcation theory

<u>Bifurcation of RG-flow</u>

couplings λ^i

$$\frac{d\lambda^i}{dt} = -\beta^i(\lambda)$$

<u>VS</u>

parameter $x \in X$.

controllable (do <u>NOT</u> run)

<u>e.g.</u> $d = 4 - \varepsilon$.

N_c. N_f. ほとんど走らない coupling.

$$\boxed{\text{bifurcation : change of FPs as parameters } \{x\} \text{ vary}}$$

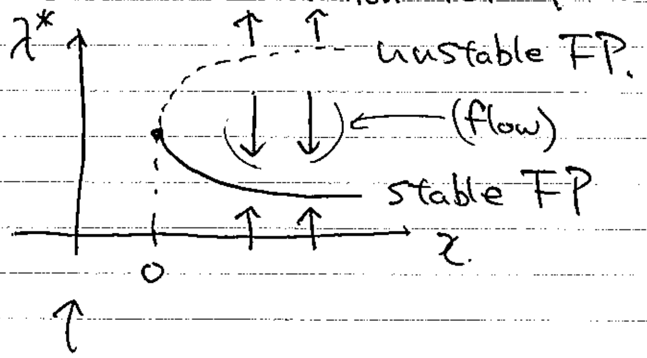
Examples

1) Saddle - node bifurcation.

$\frac{d\lambda}{dt} = \lambda^2 - x$ | coupling : λ
| parameter : x

$x > 0 \rightarrow \beta = 0$ @ $\lambda = \pm\sqrt{x}$

$x < 0 \rightarrow$ no solution for $\beta = 0$

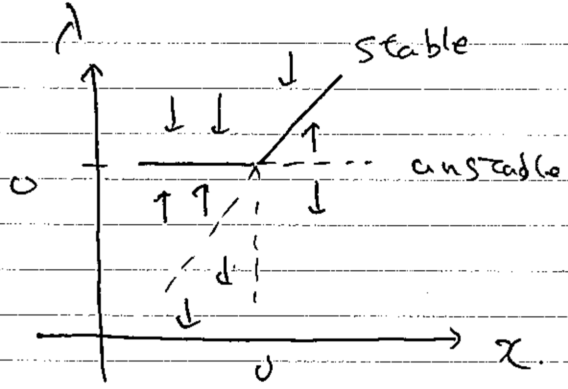


bifurcation diagram

(x=0 で FPが なくなる)

2) transcritical bifurcation.

$\frac{d\lambda}{dt} = x\lambda - \lambda^2$

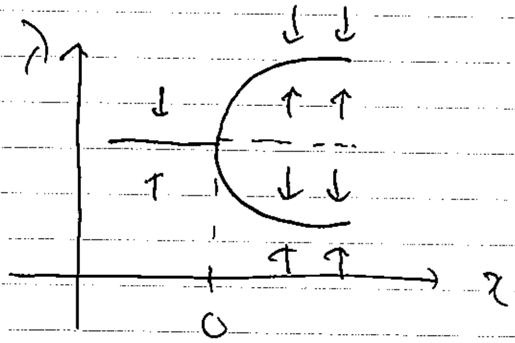


(x=0 で stable (unstable が かわる)

(くまで)

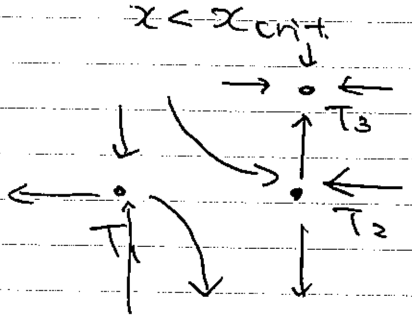
3) Pitchfolk bifurcation

$$\frac{d\lambda}{dt} = x\lambda - \lambda^3$$

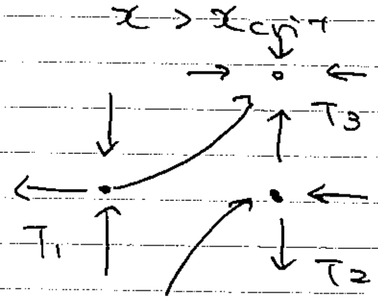


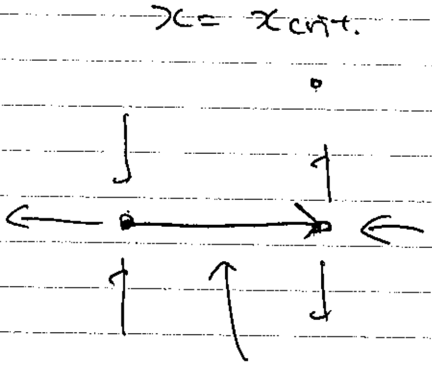
4) Heteroclinic bifurcation.

2-couplings. 1-parameter



⟹





new flow from T_1 to T_2.

(人せには T_2 につく。)

Stability & Unfolding of bifurcation

（理論の小さな摂動で
flowがどう変わるか）

Consider $\begin{cases} 1\text{-parameter } x \\ 1\text{-coupling } \lambda \end{cases}$ system,

with $\frac{d\lambda}{dt} = \beta(\lambda : x)$

→ Saddle-node : $\beta = 0$ かつ $\partial_\lambda \beta = 0$. @ bifurcation pt

generic cond. satisfied @ $x = x_{crit}$

"codim-1" bifurcation.

$\begin{cases} \text{transcritical} : \beta = 0 \text{ かつ } \partial_\lambda \beta = 0 \text{ かつ } \partial_x \beta = 0 \quad @ \cdots \\ \text{pitchfork} : \beta = 0 \text{ かつ } \partial_\lambda \beta = 0 \text{ かつ } \partial_\lambda^2 \beta = 0 \text{ かつ } \partial_x \beta = 0. \quad @ \cdots \end{cases}$

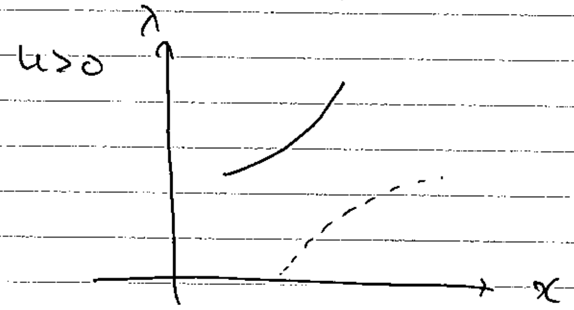
"higher codim" bifurcation.

⇒ $\boxed{\text{unsatisfied w/o "fine-tuning"!}}$

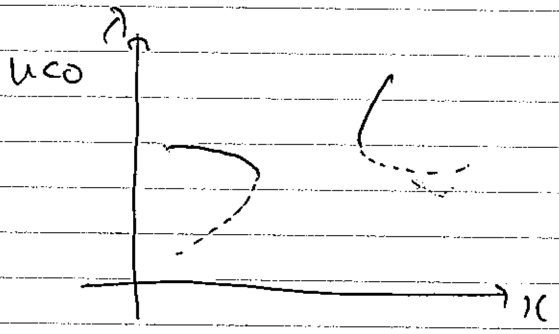
Example transcrit. bifurcation.

$$\beta(\lambda : x) = x\lambda - \lambda^2.$$

$$\rightarrow \beta(\lambda : x) = u + x\lambda - \lambda^2.$$



no bifurcation.



two saddle-node bifurcation.

pitchfolk bifurcation

$$\beta(\lambda : x) \to \beta(\lambda : x) = \underbrace{\underset{\sim}{u} + x\lambda + \underset{\sim}{v}\lambda^2}_{\text{unfolding parameter}} - \lambda^3$$

pitchfolk → saddle-nodes

(symmetry $\lambda \to -\lambda$ forbids u & v.)

(スキームをかえても大丈夫.

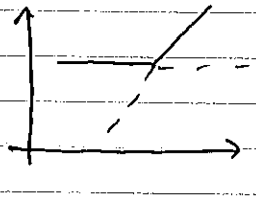
$$\lambda \to \lambda'(\lambda)$$

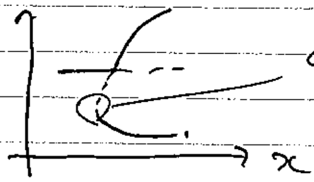
$$\frac{d\lambda'}{dt} = \frac{d\lambda}{dt}\frac{d\lambda'}{d\lambda} = \frac{d\lambda'}{d\lambda}\beta(\lambda(\lambda')) \equiv \beta'(\lambda')$$

$\beta = 0$, $\partial_\lambda \beta = 0$, $\partial_\lambda^2 \beta = 0$, … と要求しているので. 変な形にはならない).

Higher-order correction

Example : Transcrit. bifurcation





additional saddle-node bifurcation

$\beta = x\lambda - \lambda^2$ $\qquad$ $\beta = x\lambda - \lambda^2 - \lambda^3$

(こわれるようなこともあるかも)

Operator dimension @ $x = x_{crit}$

$J = \{\partial_i \beta_j\}$: matrix

dim{λi}=の primary op.を考えている

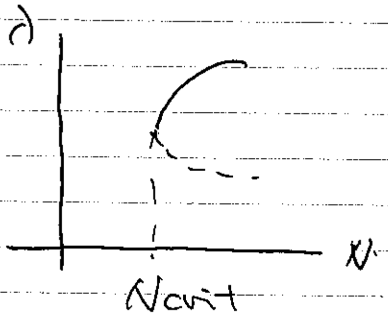
eigenvalues of J @ $\lambda = \lambda_*$ ↔ values of $d - \boxed{\Delta_i}$

stable FP. ⟷ $\forall$(eigenvalue of J) < 0

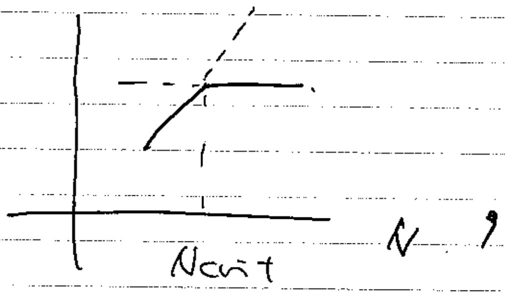
~~marginal op.~~ ⟷ zero eigenvalue of J

⇒ thus

If the loss of conformality @ lower end of conformal window is due to



or



at least one irrelevant op. crosses the marginality. (unstable F.P. と接しているから)

(とりあえず pitchfork は単に考えていない)

transcrit bifurcation

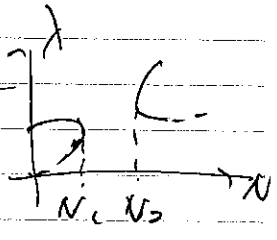
$\dot{\lambda} = u + x\lambda - \lambda^2$. $x = N - N_{crit}$, u: unfolding param.

$\Delta - d \sim \pm\sqrt{4u + x^2}$

from Ncrit

$$\longrightarrow \begin{cases} \Delta - d \sim \sqrt{N - N_{crit}}^{\,(1/2)} & u < 0 \\ \Delta - \Delta_0 \sim (N - N_{crit})^2 & u > 0 \\ \Delta - d \sim N - N_{crit} \end{cases}$$

何かの値 (定数)



2つある

Behavior of $\Delta(N)$ near $N = N_{crit}$

⟺ topology of the bifurcation diag.

"N_f" は. $2N_f$ コの "minimal" な spinorが いると思う.

Application to QED_3

$G = \mathbb{R}$ (monopole を許さない) ③d parity-invariant QED.

↑ ゲージ群

w/ massless Fermions,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \sum_{a}^{N_f} \overline{\Psi}_a i \gamma^{\mu} D_{\mu} \Psi^a$$

$$\boxed{+ \mathcal{L}_m + \mathcal{L}_{4f}}$$ (一部)

→ ~~[illegible]~~ 0 にとっておく $\left(SU(2N_f) \text{ flavor symmetry} \right)$

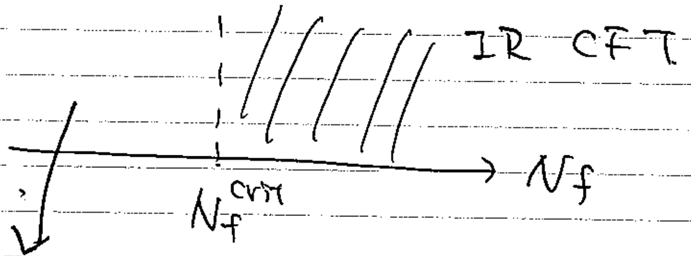
$\gamma_{\mu=0,1,2}$: 4 D Dirac mat.

Ψ^a : 4 - component spinor. (ψ_a, ψ_{a+N_f})

IR - phase

$[e^2] = 1$ ⇝ asymptotically free

large N_f ⇝ IR FP



- Confinement
- dynamical mass gen.
- χSB. $SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1)$

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Q ① What happens @ $N = N_f^{crit}$?

② Value of N_f^{crit}?

Couplings : $$\mathcal{L}_{4\text{-Fermi}} = \underbrace{\frac{g}{N_f}(\bar{\Psi}_a \Psi^a)^2 + \frac{g'}{N_f}(\bar{\Psi}_a \gamma_\mu \gamma_3 \Psi^a)^2}_{\text{preserve only } SU(N_f)\times SU(N_f)\times U(1)} + \underbrace{\frac{\lambda}{N_f}(\bar{\Psi}_a \gamma_3 \Psi_a)^2 + \frac{\lambda'}{N_f}(\bar{\Psi}_a \gamma_\mu \Psi^a)^2}_{\text{preserve } SU(2N_f)}$$

(Fierz 変換でこの4つのみ独立)

$$\mathcal{L}_{mass} = \underbrace{m(\bar{\psi}_a \psi^a - \bar{\psi}_{a+N_f}\psi^{a+N_f})}_{\text{preserve only } SU(N_f)\times SU(N_f)\times U(1),\ P} + \underbrace{\tilde{m}\bar{\psi}_c\psi^c}_{\text{breaks } P}$$

⇒ consider 3-couplings system $\{e^2, \lambda, \lambda'\}$

(conf. phase では $SU(2N_f)$ は破れない)

approach

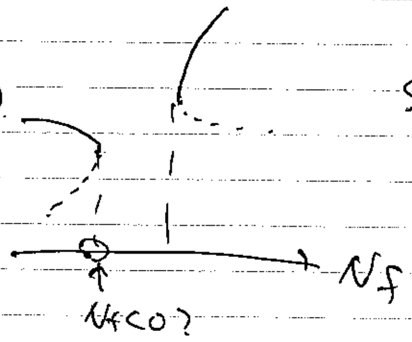
(1) large N_f. RG-flow eqn. (1-loop)

$$\begin{cases} \dot{e}^2 = e^2 - N_f e^4 & \text{— ①} \\ \left.\begin{array}{l} \dot{\lambda} = -\lambda - \lambda^2 + 4e^2\lambda + 18e^2\lambda' + 9N_f e^4 \\ \dot{\lambda}' = -\lambda' + \lambda'^2 + \frac{2}{3}e^2\lambda \end{array}\right\} & \text{— ②} \end{cases}$$

① ⇒ $e_*^2 = \frac{1}{N_f}$, ② becomes 2-coupling system $\{\lambda, \lambda'\}$

$$\Rightarrow \begin{cases} \mathring{\lambda} = \boxed{9N_f e^4} - \lambda - \lambda^2 + \cdots \\ \mathring{\lambda}' = -\lambda' + \cdots \end{cases}$$

unfolding param. of transcrit. bifurcation.

⇒ 

Saddle - node bifurcation

2) ε - expansion around $d=4$.

$d=4-\varepsilon$, 1-loop

→ Anom. dim. of $\mathcal{O}_3$, $\mathcal{O}_4$ — $(\bar{\Psi}\gamma_3\Psi)^2$, $(\bar{\Psi}\gamma_\mu\Psi)^2$.

$d-\Delta_{4\text{-Ferm}} = \cdots$

$\sim 0.54\,(\underline{N_f^{crit} - N_f})$. $N_f^{crit} \simeq 2.7$.

transcrit. bifurcation? ↑(1 = 2 : 2N−1N → transcrit.)

→ recent ε - expansion analysis
prefers "saddle - node bifurcation"