The equations of fluid motion with rotation

ATM2106

Last time

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathscr{F}$$

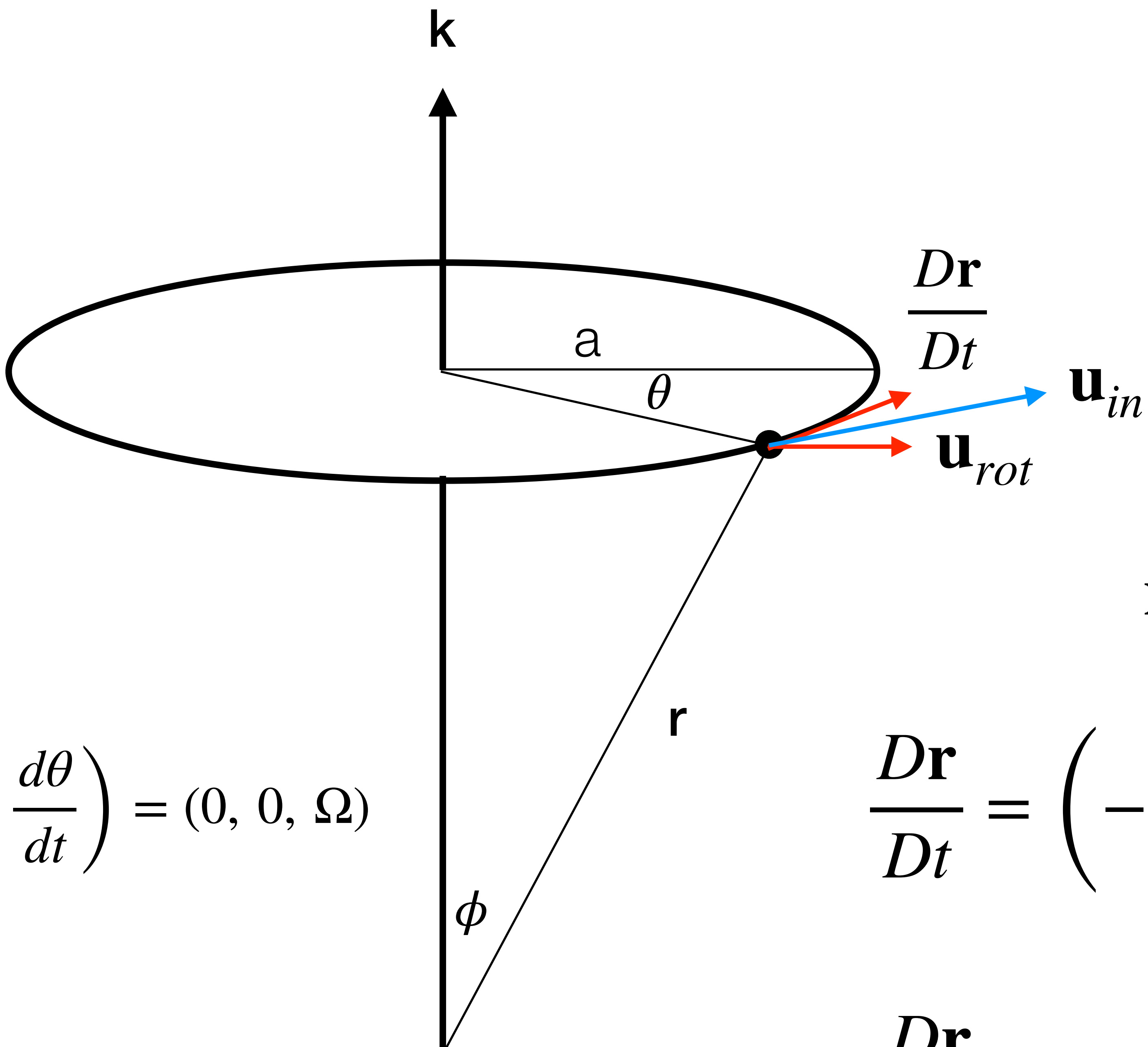
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{DQ}{Dt} = c_p\frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

Today's topic

- The equations of motion with rotation

1. Equations of motion for a rotating fluid



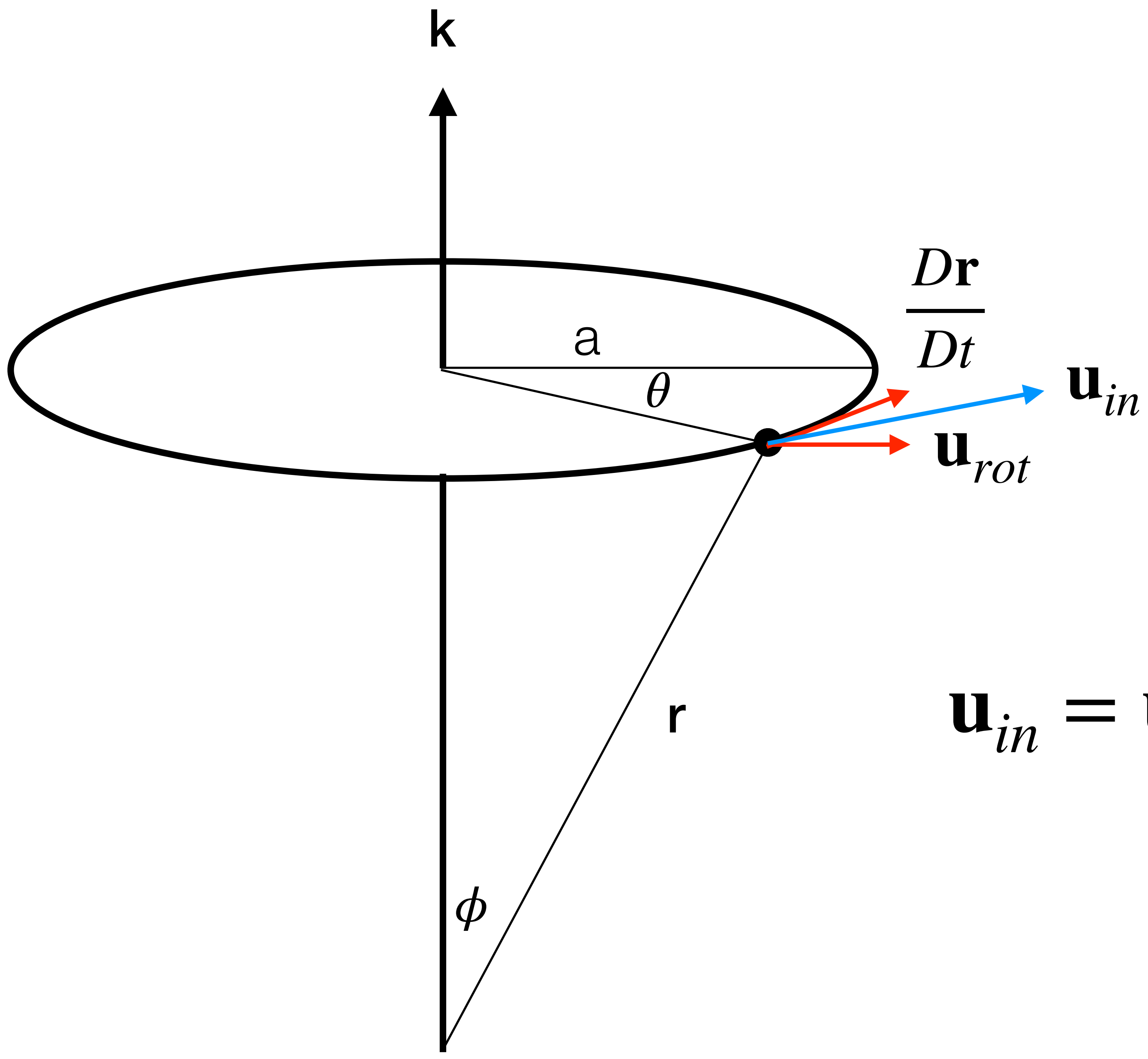
$$\boldsymbol{\Omega} = \left(0,\ 0,\ \frac{d\theta}{dt}\right) = (0,\ 0,\ \Omega)$$

$$\mathbf{r} = \left(a\cos\theta,\ a\sin\theta,\ h\right)$$

$$\frac{D\mathbf{r}}{Dt} = \left(-a\sin\theta\frac{d\theta}{dt},\ a\cos\theta\frac{d\theta}{dt},\ 0\right)$$

$$\frac{D\mathbf{r}}{Dt} = \Omega \times \mathbf{r}$$

1. Equations of motion for a rotating fluid



$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$

1. Equations of motion for a rotating fluid

A vector, $\mathbf{A}$, can be written with any three independent unit vectors.

$$\mathbf{A} = A_i\ \hat{\mathbf{i}} + A_j\ \hat{\mathbf{j}} + A_k\ \hat{\mathbf{k}} \longrightarrow \text{in the absolute frame}$$

$$= A_x\ \hat{\mathbf{x}} + A_y\ \hat{\mathbf{y}} + A_z\ \hat{\mathbf{z}} \longrightarrow \text{in the rotating frame}$$

Then, let's find out the Lagrangian differentiation of $\mathbf{A}$
with respect to the absolute (inertial) frame

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{in} = \frac{D}{Dt}\left(A_i\ \hat{\mathbf{i}}\right) + \frac{D}{Dt}\left(A_j\ \hat{\mathbf{j}}\right) + \frac{D}{Dt}\left(A_k\ \hat{\mathbf{k}}\right)$$

$$= \hat{\mathbf{i}}\frac{DA_i}{Dt} + \hat{\mathbf{j}}\frac{DA_j}{Dt} + \hat{\mathbf{k}}\frac{DA_k}{Dt}$$

1. Equations of motion for a rotating fluid

$\left(\frac{D\mathbf{A}}{Dt}\right)_{in}$ can also be written as

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{in} = \frac{D}{Dt}\left(A_x\,\hat{\mathbf{x}}\right) + \frac{D}{Dt}\left(A_y\,\hat{\mathbf{y}}\right) + \frac{D}{Dt}\left(A_z\,\hat{\mathbf{z}}\right)$$

$$\frac{D\mathbf{r}}{Dt} = \boldsymbol{\Omega} \times \mathbf{r}$$

$$= \hat{\mathbf{x}}\frac{DA_x}{Dt} + \hat{\mathbf{y}}\frac{DA_y}{Dt} + \hat{\mathbf{z}}\frac{DA_z}{Dt} + A_x\frac{D\hat{\mathbf{x}}}{Dt} + A_y\frac{D\hat{\mathbf{y}}}{Dt} + A_z\frac{D\hat{\mathbf{z}}}{Dt}$$

$$= \hat{\mathbf{x}}\frac{DA_x}{Dt} + \hat{\mathbf{y}}\frac{DA_y}{Dt} + \hat{\mathbf{z}}\frac{DA_z}{Dt} + \boldsymbol{\Omega} \times \left(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right)$$

$$= \left(\frac{D\mathbf{A}}{Dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{A}$$

1. Equations of motion for a rotating fluid

$$\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in} = \left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{u}_{in}$$

$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}$$

$$= \left(\frac{D\left(\mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}\right)}{Dt}\right)_{rot} + \mathbf{\Omega} \times \left(\mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}\right)$$

$$= \left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot} + 2\mathbf{\Omega} \times \mathbf{u}_{rot} + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}\right)$$

1. Equations of motion for a rotating fluid

Momentum equation we did last time can be applied to the parcel in the absolute frame

$$\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathscr{F}$$

$$\downarrow$$

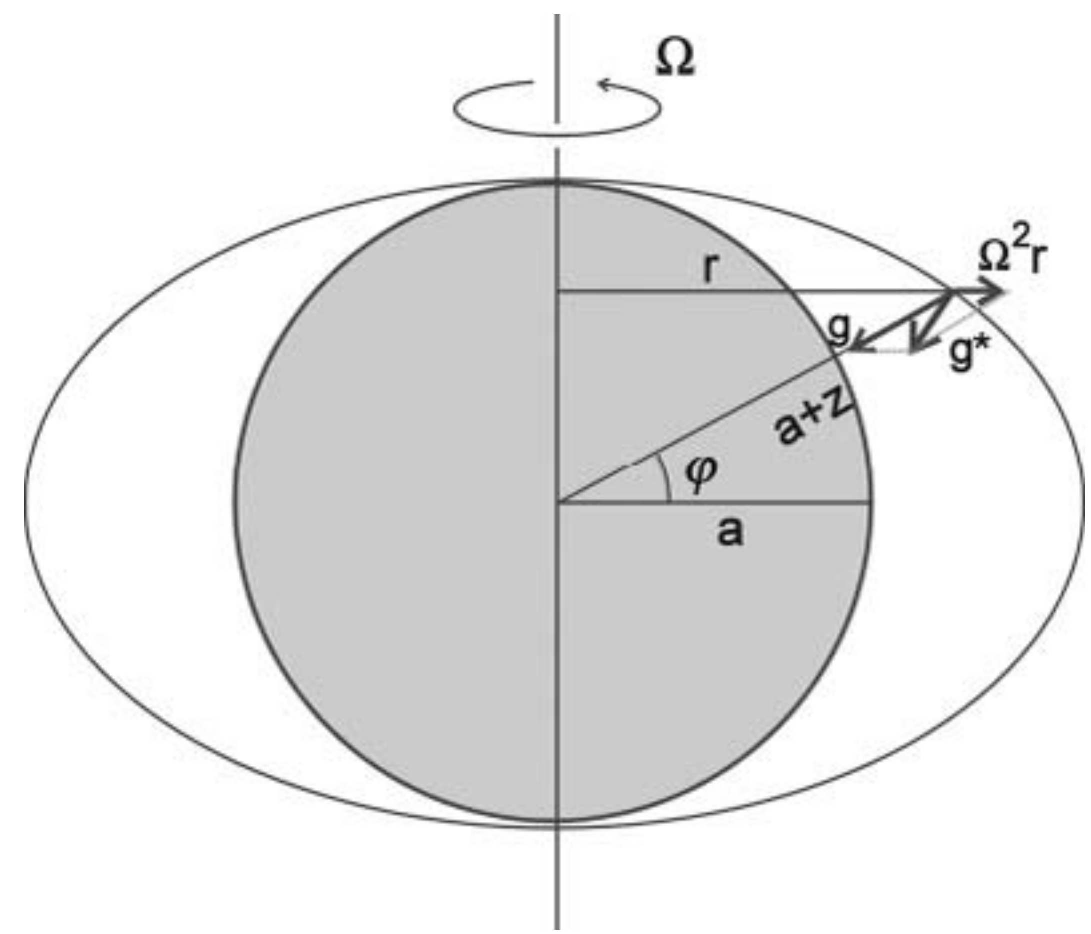
$$\left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \underbrace{-\,2\boldsymbol{\Omega}\times\mathbf{u}_{rot}}_{\text{Coriolis acceleration}}\;\underbrace{-\,\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)}_{\text{Centrifugal acceleration}} + \mathscr{F}$$

Centrifugal acceleration

Coriolis acceleration

1. Equations of motion for a rotating fluid

- Centrifugal acceleration: $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$
  - Modifies gravity acceleration
  - Effective gravity : g*
  - $\mathbf{\Omega}$ ~ 7.27 x 10^{-5} s^{-1} makes centrifugal acceleration small.
  - Geoid : The surface perpendicular to the effective gravity
  - We may use this surface to make centrifugal acceleration to disappear.

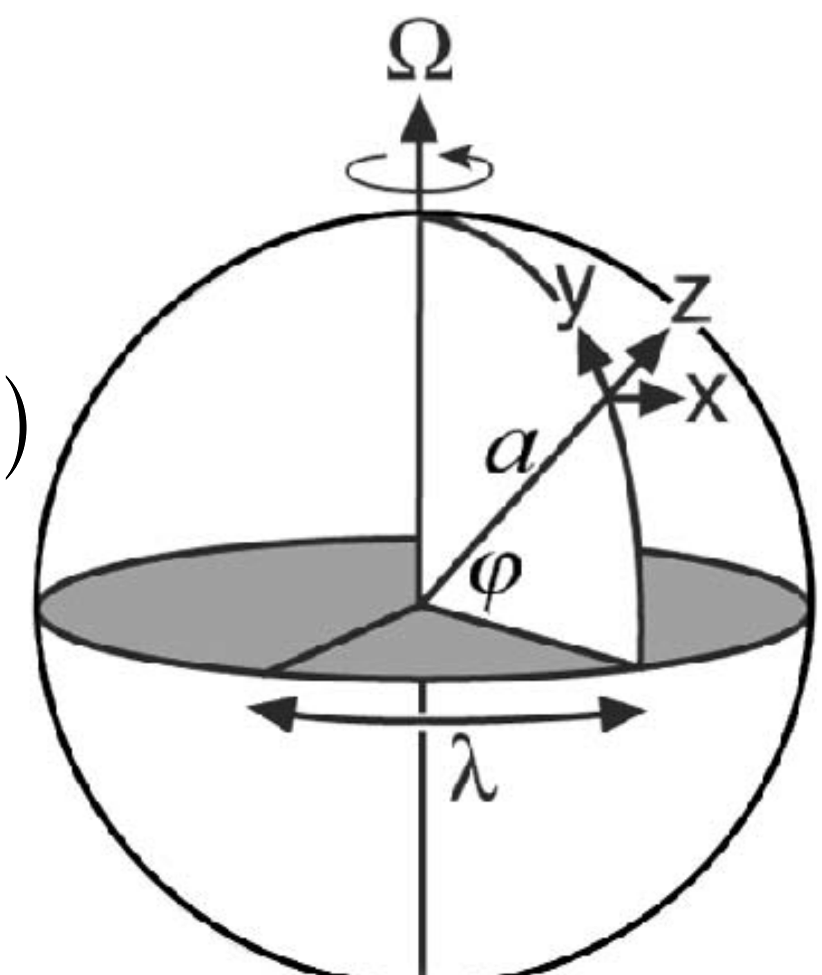


1. Equations of motion for a rotating fluid

- Coriolis force: $-2\,\mathbf{\Omega} \times \mathbf{u}$
  - $\mathbf{\Omega} = \left(0,\ \Omega\cos\phi,\ \Omega\sin\phi\right)$
  - $\mathbf{\Omega} \times \mathbf{u} = \left(\Omega\cos\phi\ w - \Omega\sin\phi\ v,\ \Omega\sin\phi\ u,\ -\Omega\cos\phi\ u\right)$
  - w is smaller than other terms.
  - Ωu is smaller than g.
  - $-2\mathbf{\Omega} \times \mathbf{u} \approx -\left(-2\Omega\sin\phi\ v,\ 2\Omega\sin\phi\ u,\ 0\right)$

$$= f\hat{\mathbf{z}} \times \mathbf{u}$$

$$(f = 2\Omega\sin\phi)$$



https://youtu.be/RrWKSOvqV-0

1. Equations of motion for a rotating fluid

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g^*\hat{\mathbf{z}} + f\hat{\mathbf{z}} \times \mathbf{u} = \mathscr{F}$$

- Hydrostatic approximation
- Vertical component of the frictional force is negligible compared with gravity

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = \mathscr{F}_x$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = \mathscr{F}_y$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass**

The equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

- https://youtu.be/RrWKSOvqV-0

2. Equations of motion for a non-rotating fluid

- **Conservation of mass :**
  - For compressible flow, the hydrostatic assumption allows us to write the unit volume as $\delta x \; \delta y \; \delta p$
  - Then the mass of the fluid parcel becomes

$$\delta M = \rho \; \delta x \; \delta y \; \delta z \; = -\frac{1}{g}\delta x \; \delta y \; \delta p$$

  - The mass is conserved in pressure coordinates, and

$$\nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

2. Equations of motion for a non-rotating fluid

- **Thermodynamic equation**
  - The first law of thermodynamics we dealt with for the dry adiabatic lapse rate is $\delta Q = c_p dT - \frac{dp}{\rho}$
  - If we consider the first law of thermodynamics applied to a moving parcel of fluid,

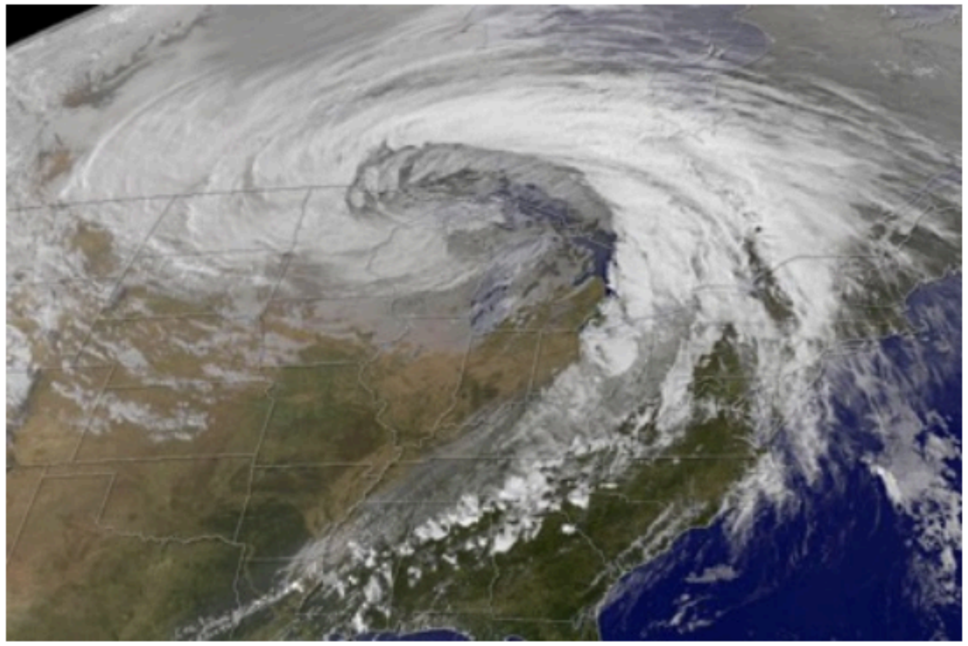
$$\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

Diabatic heating rate (e.g. latent heating or cooling)

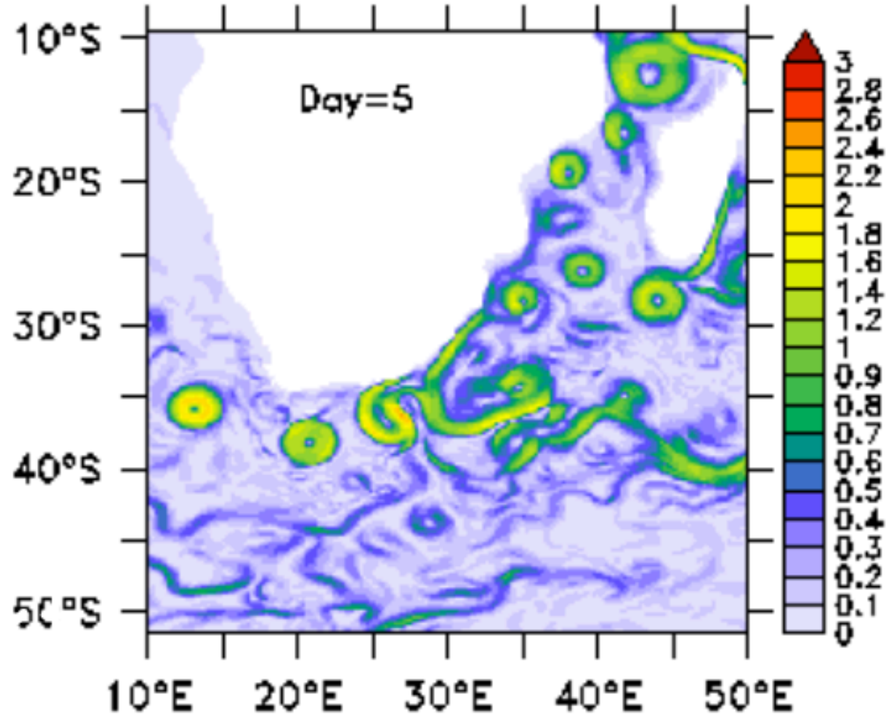
Temperature changes from the heat and/or expansion or compression

3. Equations of motion for a rotating fluid

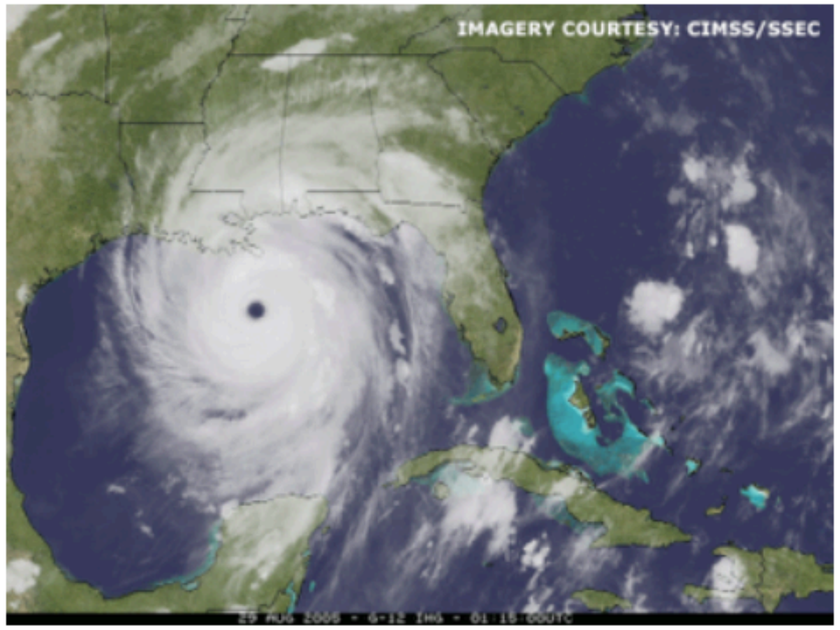
Extratropical cyclone 2010



Ocean eddies in GFDL model



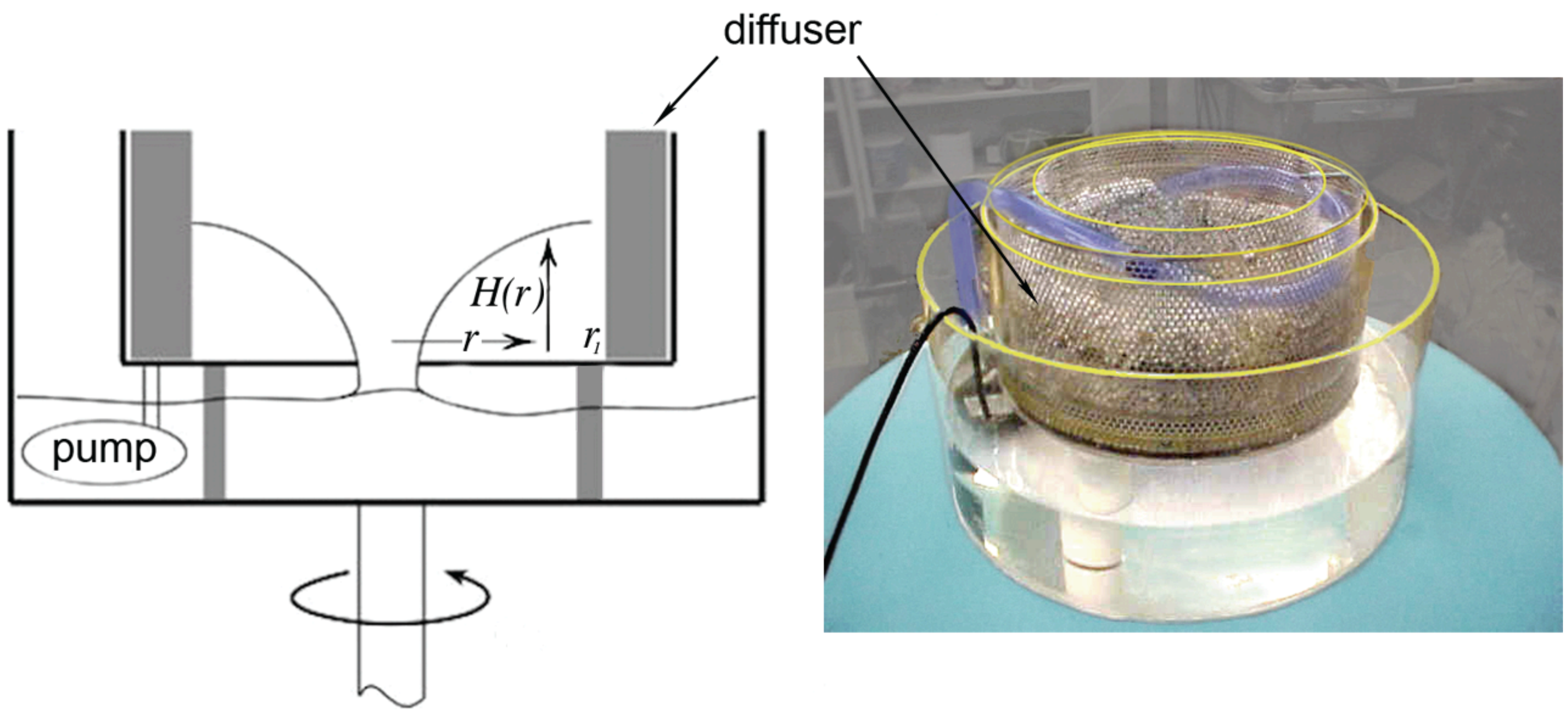
Hurricane Katrina 2005

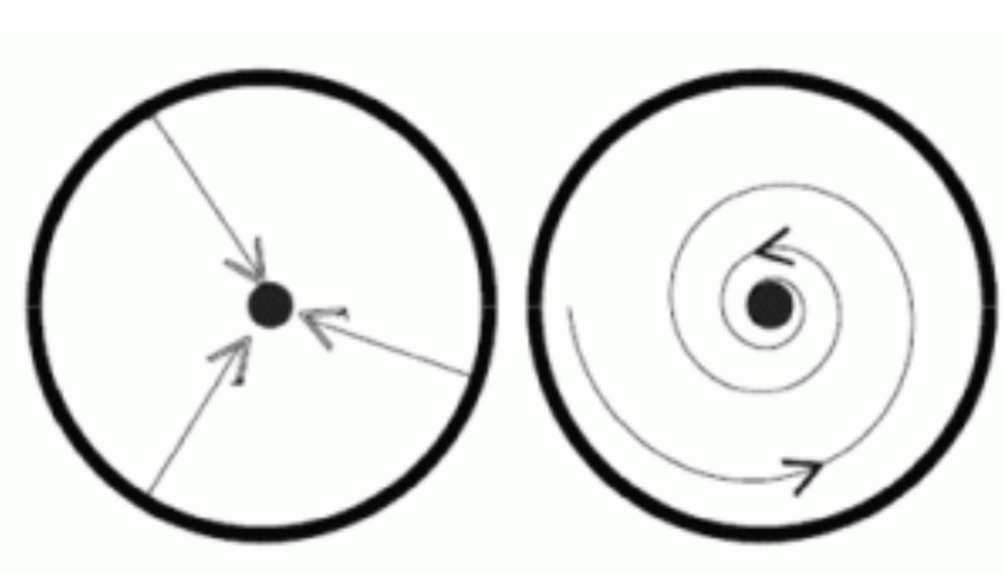


Tornado

3. Equations of motion for a rotating fluid

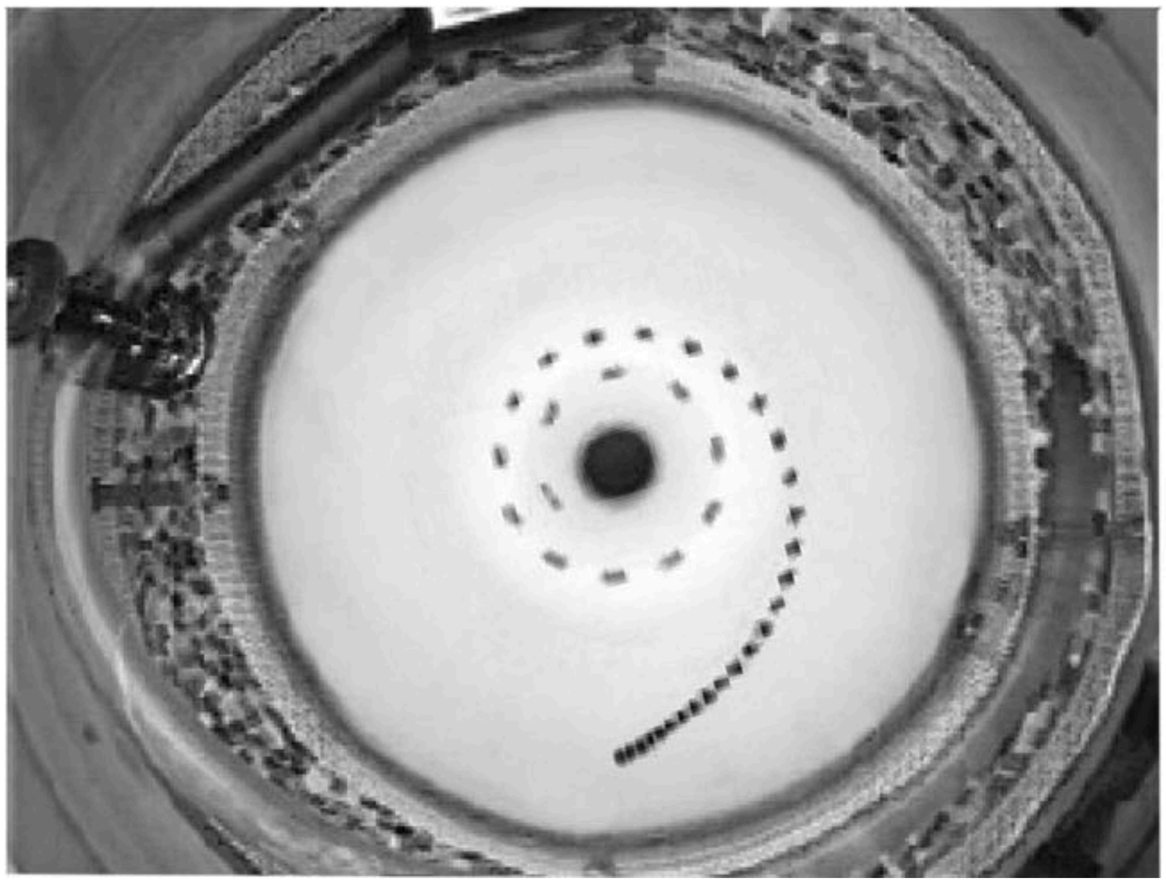
- Balanced vortex

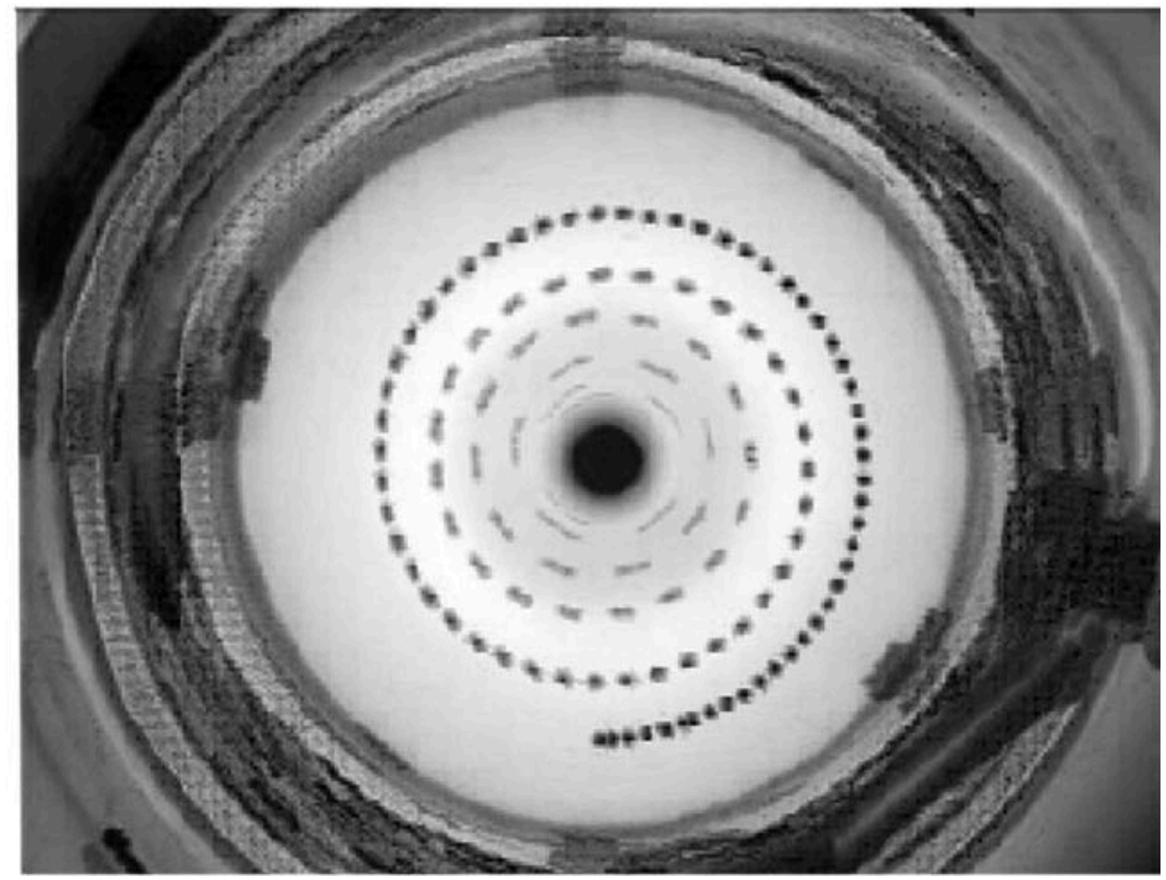




Flow patterns (left) in the absence of rotation and (right) when the apparatus is rotating in an anticlockwise direction.

3. Equations of motion for a rotating fluid





Trajectories of particles in the radial inflow experiment viewed in the rotating frame. The positions are plotted every 1/30 s. On the left Ω= 5 rpm (revolutions per minute). On the right Ω = 10 rpm. Note how the pitch of the particle trajectory increases as Ω increases, and how in both cases the speed of the particles increases as the radius decreases.

3. Equations of motion for a rotating fluid

Magnetic latitude (deg.)

Fig. 1. Plot of spread-F variation with sunspot number versus magnetic latitude

range of 40–120. It is found that the slopes of these linear portions vary widely from latitude to latitude, giving both positive and negative values. Positive slopes are obtained for the stations which show positive correlation of mean percentage occurrence of spread-F with sunspot numbers,

One of us (V. R. J. K.) thanks the Indian Council of Scientific and Industrial Research for financial support of this investigation.

V. R. Jagath Kumar
B. Ramachandra Rao

Radio Physics and Electronics Laboratory,
Andhra University,
Waltair.

Wells, H. W., *J. Geophys. Res.*, **59**, 66 (1954).
Singleton, D. G., *Austral. J. Phys.*, **10**, 60 (1957).
Kotadia, K. M., *Proc. Ind. Acad. Sci.*, A, **50**, 259 (1959).
Lyon, A. J., Skinner, N. J., and Wright, R. W. H., *J. Atmos. Terr. Phys.*, **19**, 145 (1960).
Rangaswamy, S., and Kapasi, K. B., *J. Atmos. Terr. Phys.*, **25**, 721 (1963).

PHYSICS

The Bath-Tub Vortex in the Southern Hemisphere

It has long been thought that water draining from a tank would rotate counter-clockwise in the northern hemisphere and clockwise in the southern hemisphere, provided other influences were kept small compared with the influence of the rotation of the Earth. This idea has only recently been tested, by Shapiro in Watertown, Massachusetts, as part of a film on vorticity[2–4], and later by Binnie in Cambridge, England[1]. Shapiro and Binnie both acquired confidence, after surmounting difficulties in their early experiments, that the counter-clockwise rota-

PHYSICS

The Bath-Tub Vortex in the Southern Hemisphere

It has long been thought that water draining from a tank would rotate counter-clockwise in the northern hemisphere and clockwise in the southern hemisphere, provided other influences were kept small compared with the influence of the rotation of the Earth. This idea has only recently been tested, by Shapiro in Watertown, Massachusetts, as part of a film on vorticity[2–4], and later by Binnie in Cambridge, England[1]. Shapiro and Binnie both acquired confidence, after surmounting difficulties in their early experiments, that the counter-clockwise rotations observed in their later experiments were due to the rotation of the Earth.