The equations of fluid motion

ATM2106

Last time

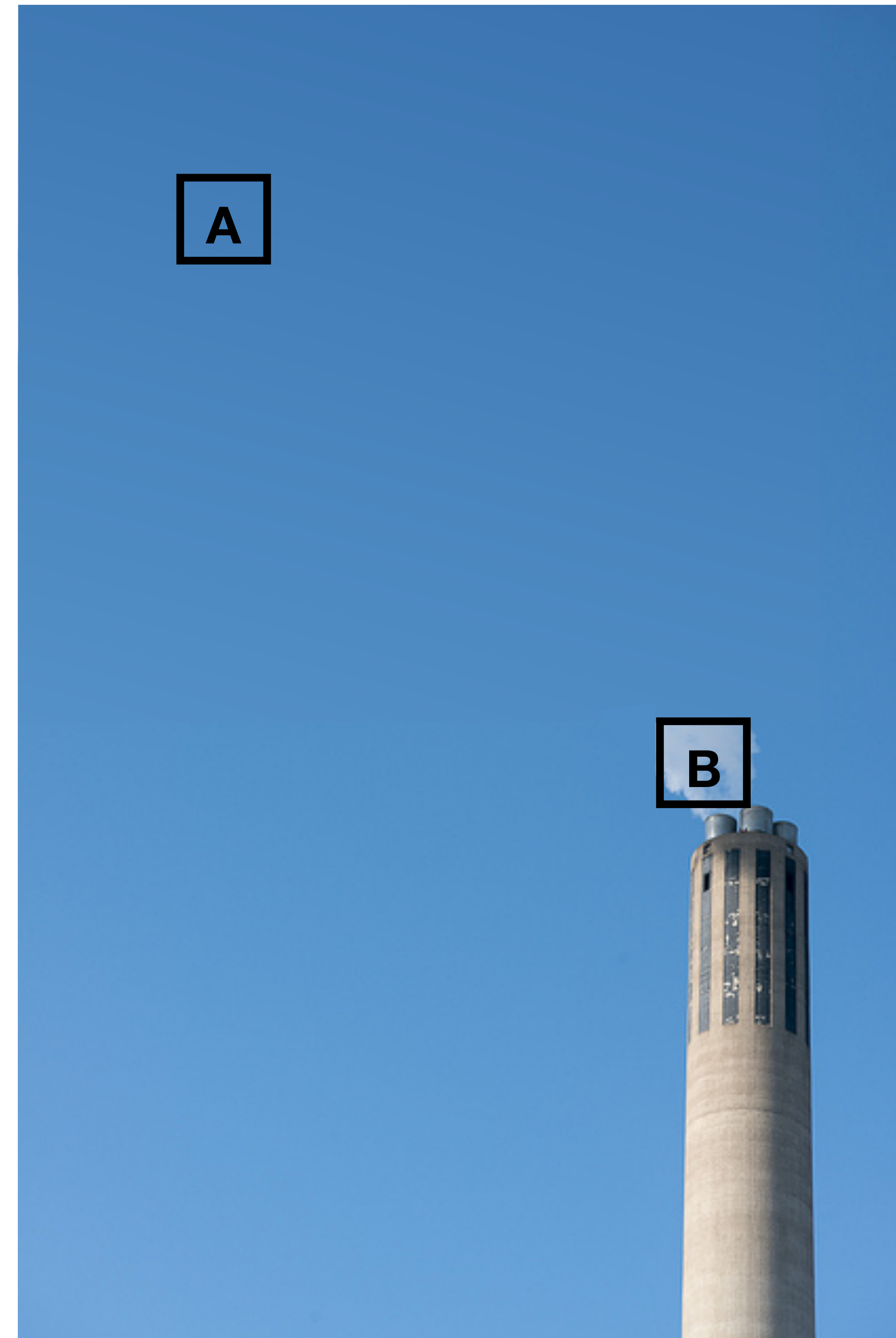
- Meridional structure of the atmosphere
    - temperature, potential temperature
    - specific humidity, saturated specific humidity, RH
    - zonal velocity, meridional overturning circulation

Today’s topic

- The equations of motion
  - Eulerian differentiation
  - Lagrangian differentiation
  -

1. Differentiation following the motion

- Two ways to look at the fluid
  - From a fixed position (A)
  - From a fixed parcel (B)
- How will the concentration, C = C(x, y, z, t), change in time? (Let’s assume no diffusion, or spread of the smoke.)



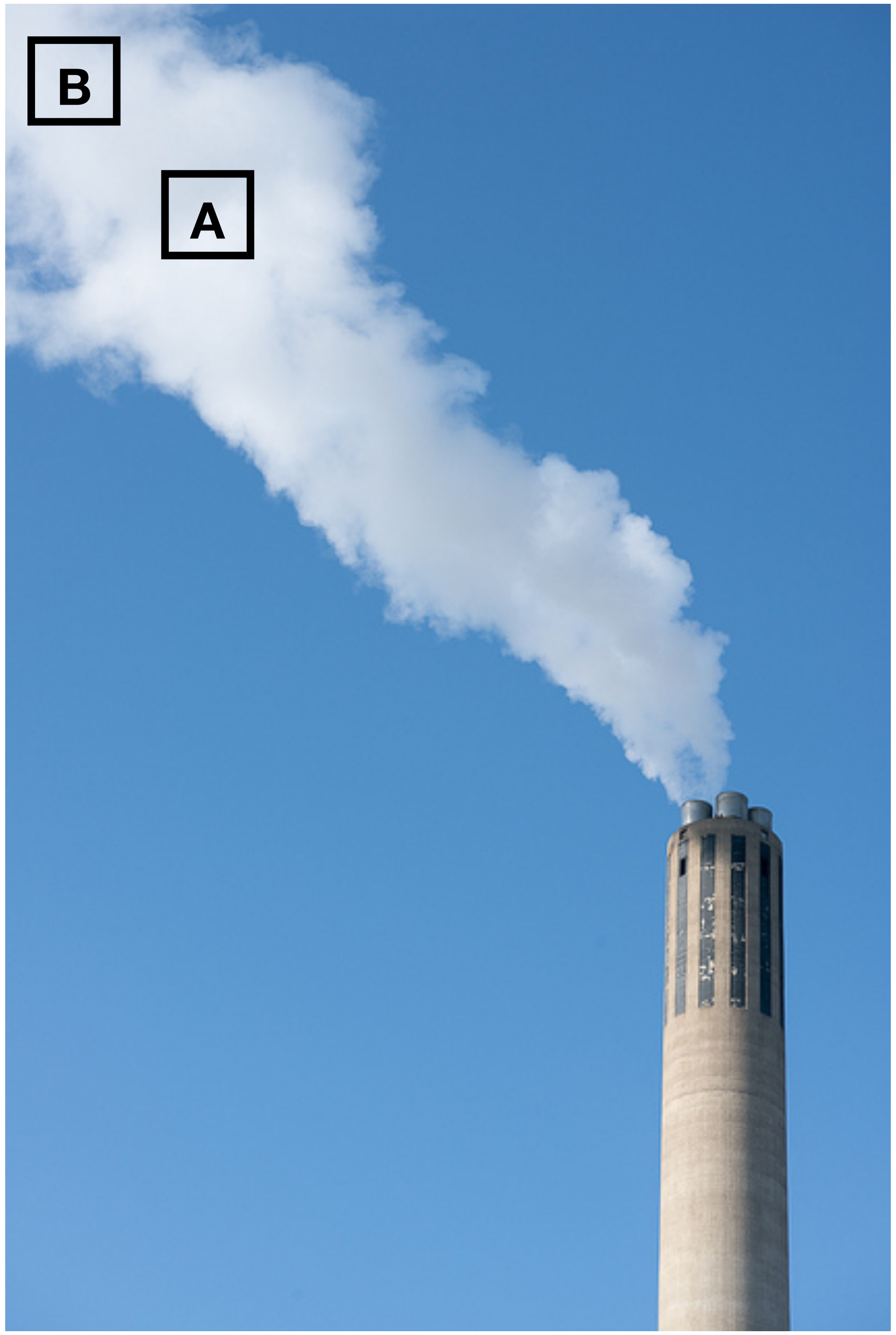
1. Differentiation following the motion

- After some time,
  - The fixed position, A, feels the increase of the concentration of the smoke

$$\left(\frac{\partial C}{\partial t}\right)_{fixed\ position} \neq 0$$

  - The fixed parcel, B, have the same smoke concentration.

$$\left(\frac{\partial C}{\partial t}\right)_{fixed\ parcel} = 0$$



1. Differentiation following the motion

- How do we mathematically express "differentiation following the motion"?
- We can write the changes in *C*, δC as follows:

$$\delta C = \frac{\partial C}{\partial t}\delta t + \frac{\partial C}{\partial x}\delta x + \frac{\partial C}{\partial y}\delta y + \frac{\partial C}{\partial z}\delta z$$

- We know $\delta x = u\delta t,\ \delta y = v\delta t,\ \delta z = w\delta t$, where *u, v, w* are the speeds in the *x, y* and *z* directions.
- Thus,

$$(\delta C)_{fixed\ particle} = \left(\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z}\right)\delta t$$

1. Differentiation following the motion

- Or

$$\left(\frac{\delta C}{\partial t}\right)_{fixed\ particle} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{DC}{Dt}$$

- So the symbol, $\frac{DC}{Dt}$ indicates the rate of change following the motion

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \qquad\qquad where\ \mathbf{u} = (u, v, w)$$

2. Equations of motion for a non-rotating fluid

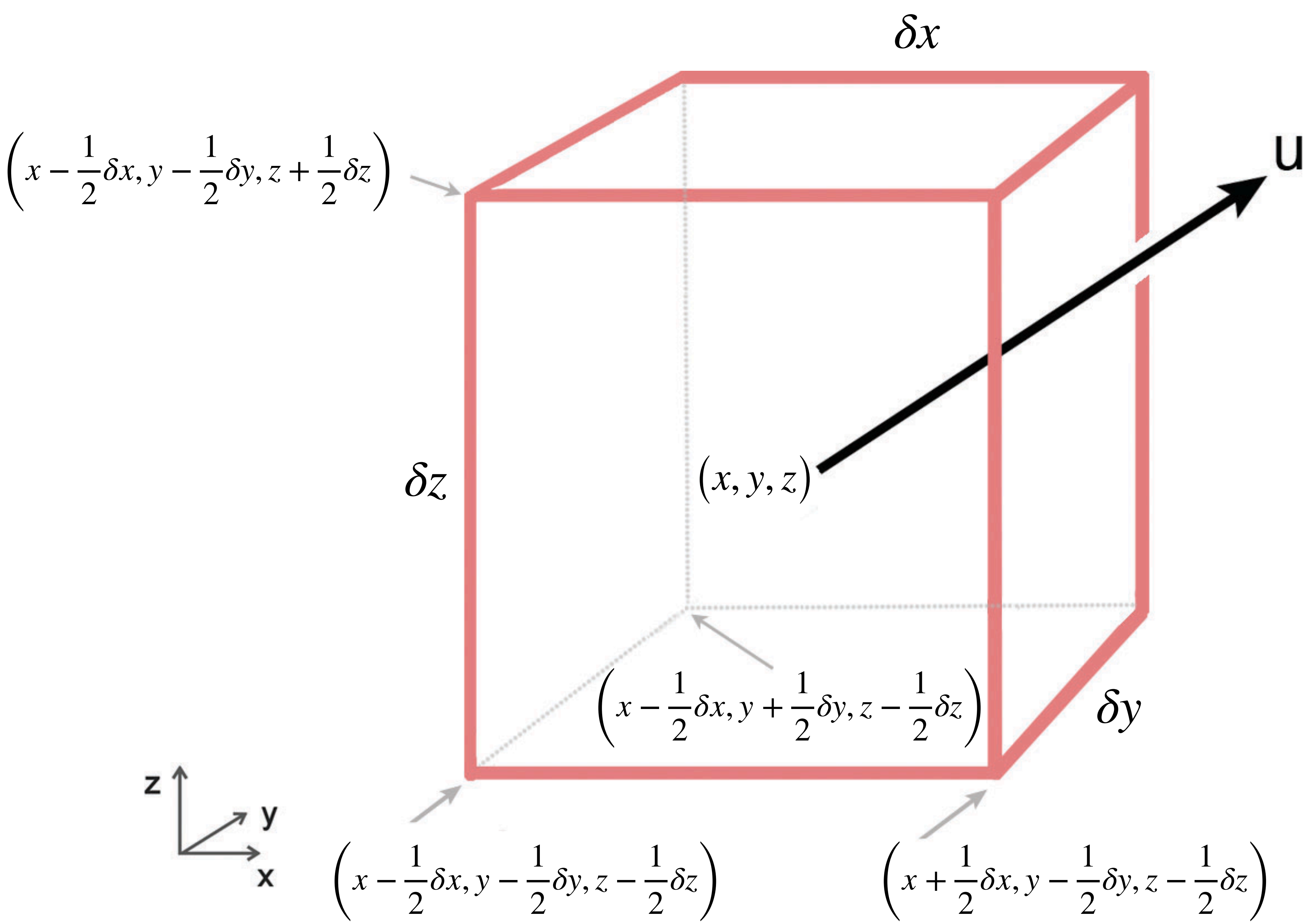
- At a given time, the state of the atmosphere or ocean can be defined by five key variables:

$$\mathbf{u} = (u, v, w);\ \ p\ \ and\ T$$

- To find out the values for these five variables, we need five independent equations
    1. Three equations that describe the laws of motion
    2. Conservation of mass
    3. The law of thermodynamics

2. Equations of motion for a non-rotating fluid

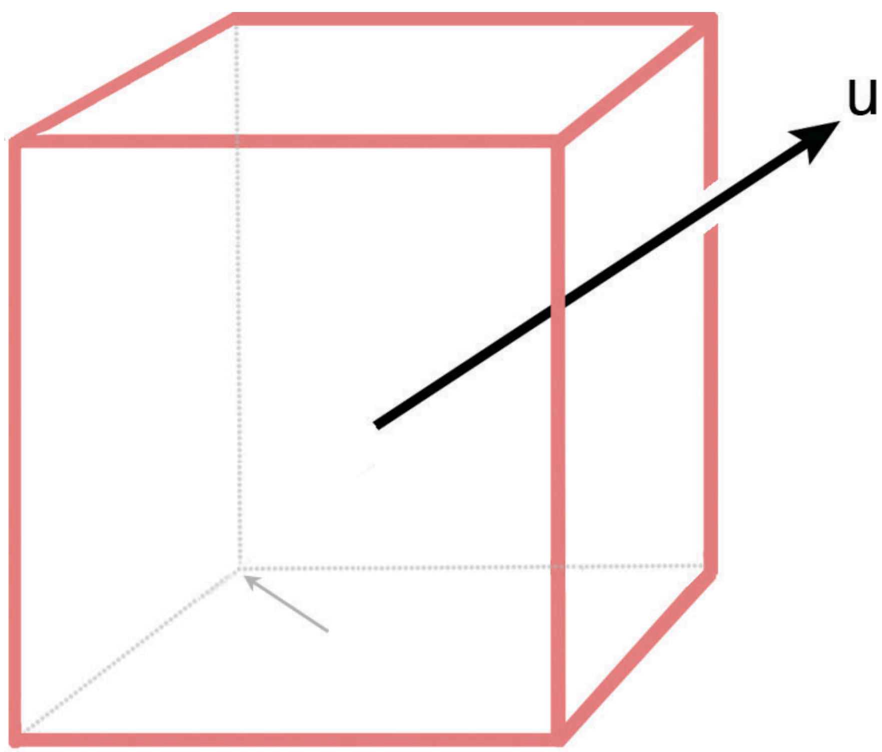
- The laws of motion



2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **The net force**
  - The mass of the parcel : $\delta M = \rho \delta x \delta y \delta z$
  - Using Newton's Law of Motion, we can write the net force, F as

$$F = \rho \delta x \delta y \delta z \frac{D\mathbf{u}}{Dt}$$

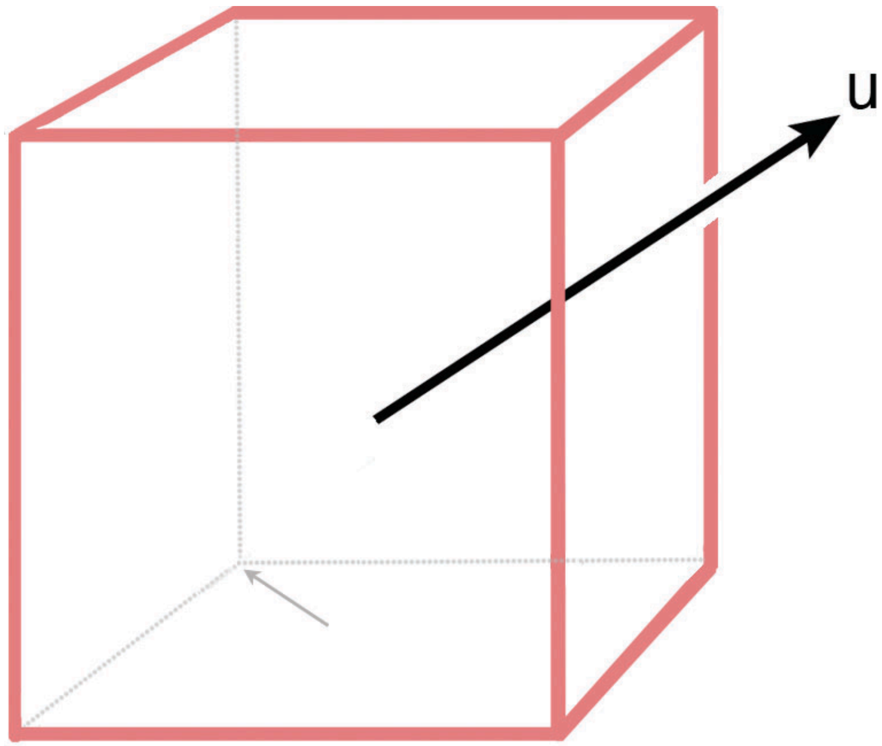


where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **gravity**

$$F_{gravity} = -g \ \hat{\mathbf{z}} \ \rho \delta x \delta y \delta z$$



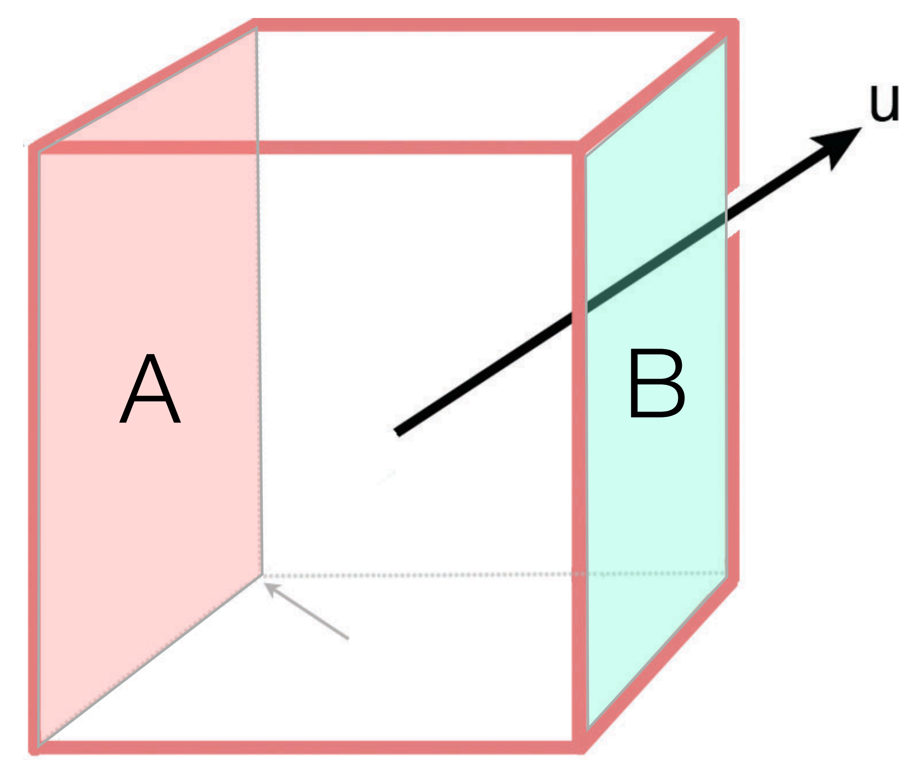
2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **pressure gradient**
  - $p = p(x, y, z)$ → pressure: the force acting on a unit area.
  - The force on A:

$$F(A) = p\left(x - \frac{\delta x}{2}, y, z\right)\delta y \delta z$$

  - The force on B:

$$F(B) = -p\left(x + \frac{\delta x}{2}, y, z\right)\delta y \delta z$$



2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **pressure gradient**
  - The force in the x-direction:

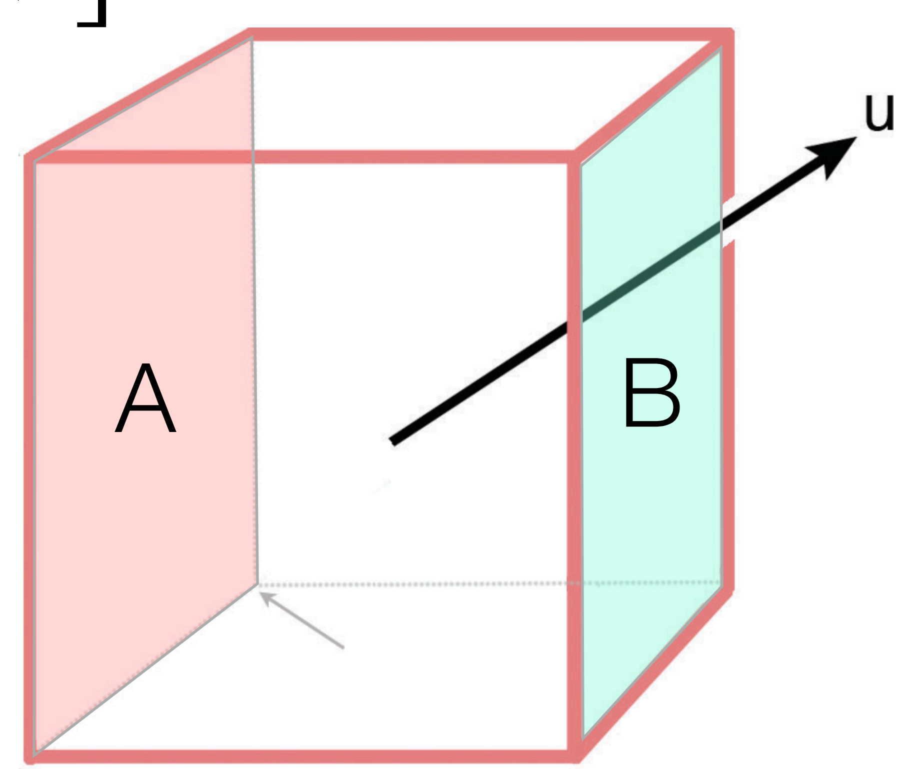
$$p(x,y,z) - \frac{\delta x}{2}\left(\frac{\partial p}{\partial x}\right)$$

$$p(x,y,z) + \frac{\delta x}{2}\left(\frac{\partial p}{\partial x}\right)$$

$$F_x = F(A) + F(B)$$

$$= \left[p\left(x - \frac{\delta x}{2}, y, z\right) - p\left(x + \frac{\delta x}{2}, y, z\right)\right]\delta y \delta z$$

$$= -\frac{\partial p}{\partial x}\delta x \delta y \delta z$$



2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **pressure gradient**
  - The pressure gradient force in the x-direction:

$$F_x = -\frac{\partial p}{\partial x}\delta x \delta y \delta z$$

  - The pressure gradient force in the y-direction:

$$F_y = -\frac{\partial p}{\partial y}\delta x \delta y \delta z$$

  - The pressure gradient force in the z-direction:

$$F_z = -\frac{\partial p}{\partial z}\delta x \delta y \delta z$$

2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **pressure gradient**
  - In total, the net vector pressure force is

$$\mathbf{F}_{pressure} = (F_x, F_y, F_z)$$

$$= -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)\delta x \delta y \delta z$$

$$= -\nabla p \delta x \delta y \delta z$$

2. Equations of motion for a non-rotating fluid

- Forces on a fluid parcel: **friction**
  - For typical atmospheric and oceanic flows, frictional effects are negligible except close to boundaries.
  - Conceptually, the frictional force on a fluid parcel can be expressed as

$$\mathbf{F}_{fric} = \rho \mathscr{F} \delta x \delta y \delta z$$

where $\mathscr{F}$ is the frictional force per unit mass.

2. Equations of motion for a non-rotating fluid

- **The equation of motion**
  - Net force = sum of all forces

$$\rho\ \delta x\ \delta y\ \delta z\ \frac{D\mathbf{u}}{Dt} = \mathbf{F}_{gravity} + \mathbf{F}_{pressure} + \mathbf{F}_{fric}$$

$$\frac{D\mathbf{u}}{Dt} = -g\hat{\mathbf{z}} - \frac{1}{\rho}\nabla p + \mathscr{F}$$

2. Equations of motion for a non-rotating fluid

- **The equation of motion**
  - The equation of motion for each coordinate:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \mathscr{F}_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \mathscr{F}_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \mathscr{F}_w$$

2. Equations of motion for a non-rotating fluid

- **The equation of motion**
  - Recall the hydrostatic balance:

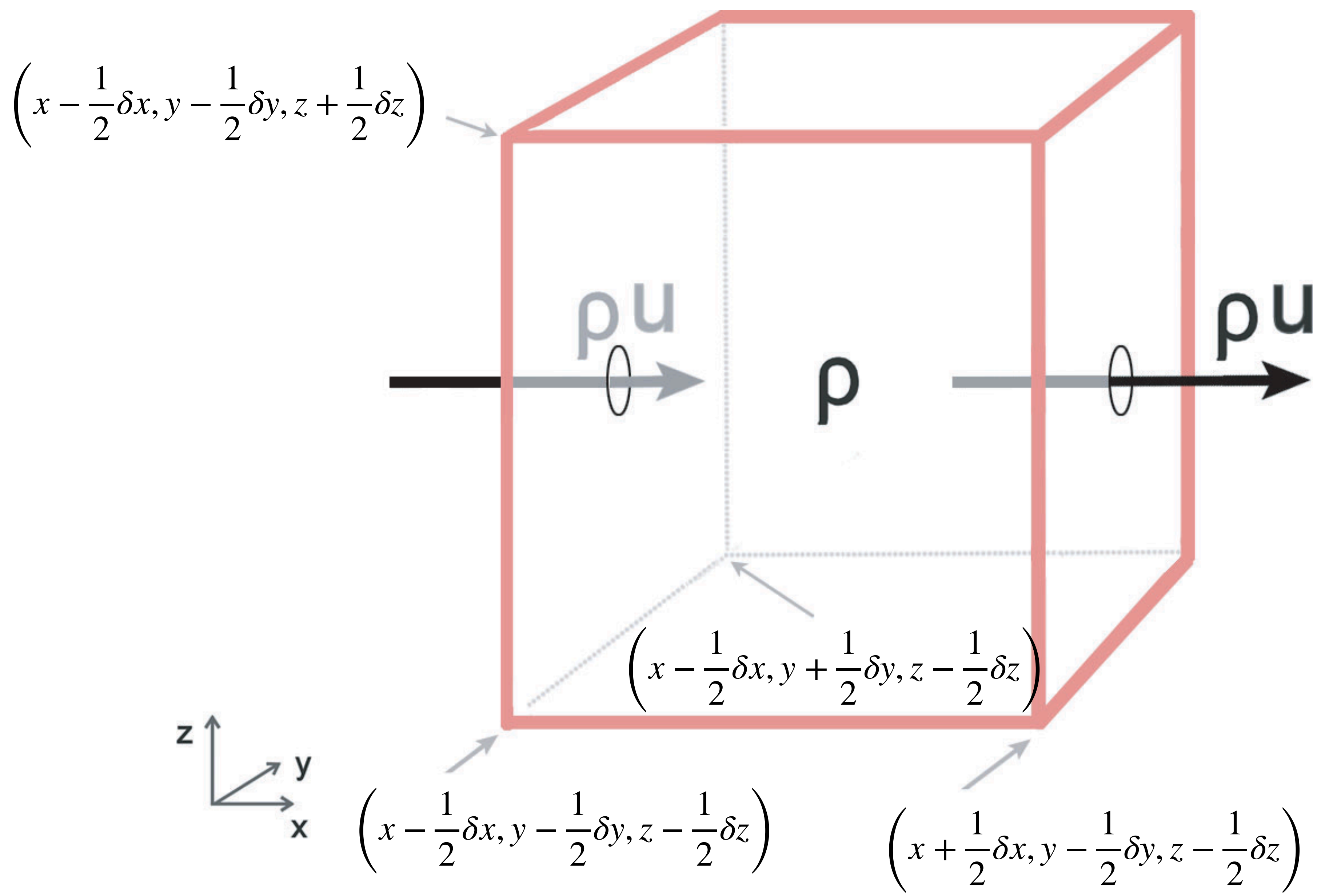
$$\frac{\partial p}{\partial z} + \rho g = 0$$

  - If friction and vertical acceleration (Dw/Dt) are negligible, the equation of motion in the z-direction becomes the equation of hydrostatic balance.

$$\cancel{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = \cancel{\mathcal{F}_w}$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass:**
  - The change of the mass = net mass flux

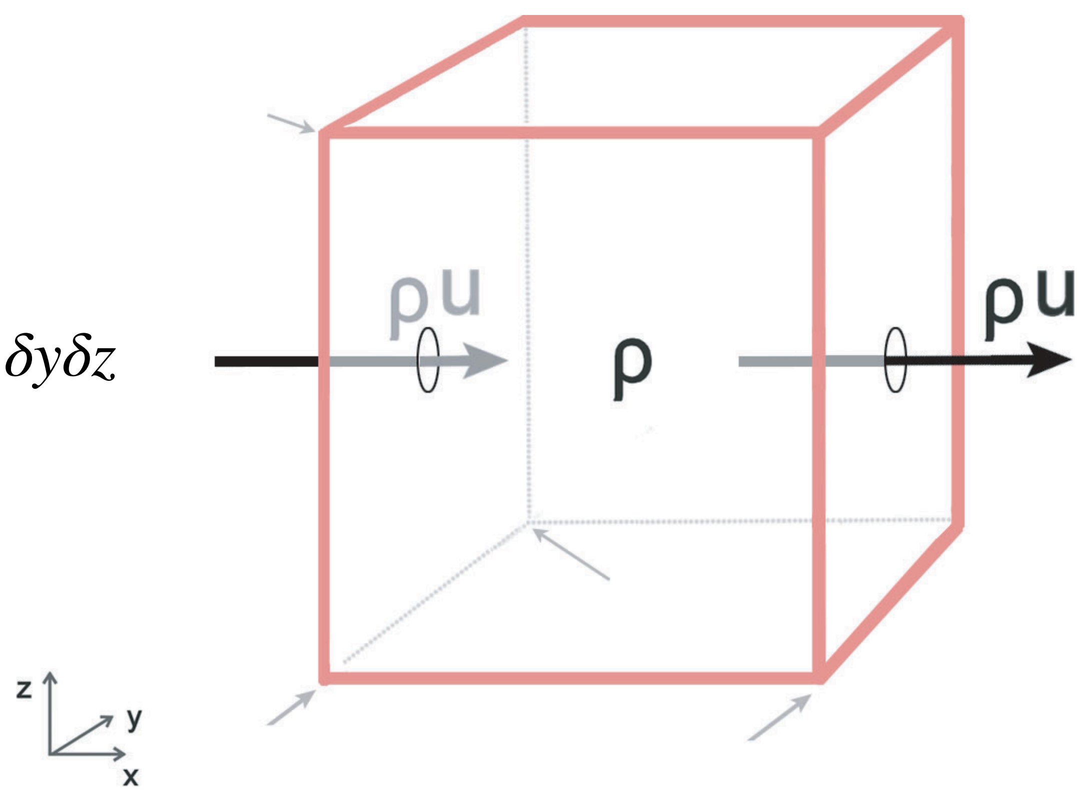


2. Equations of motion for a non-rotating fluid

- **Conservation of mass**

$$\frac{\partial}{\partial t}\left(\rho\delta x\delta y\delta z\right) = \frac{\partial \rho}{\partial t}\delta x\delta y\delta z = \text{Net mass flux into the volume}$$

$$[\rho u]\left(x - \frac{\delta x}{2}, y, z\right)\delta y\delta z$$



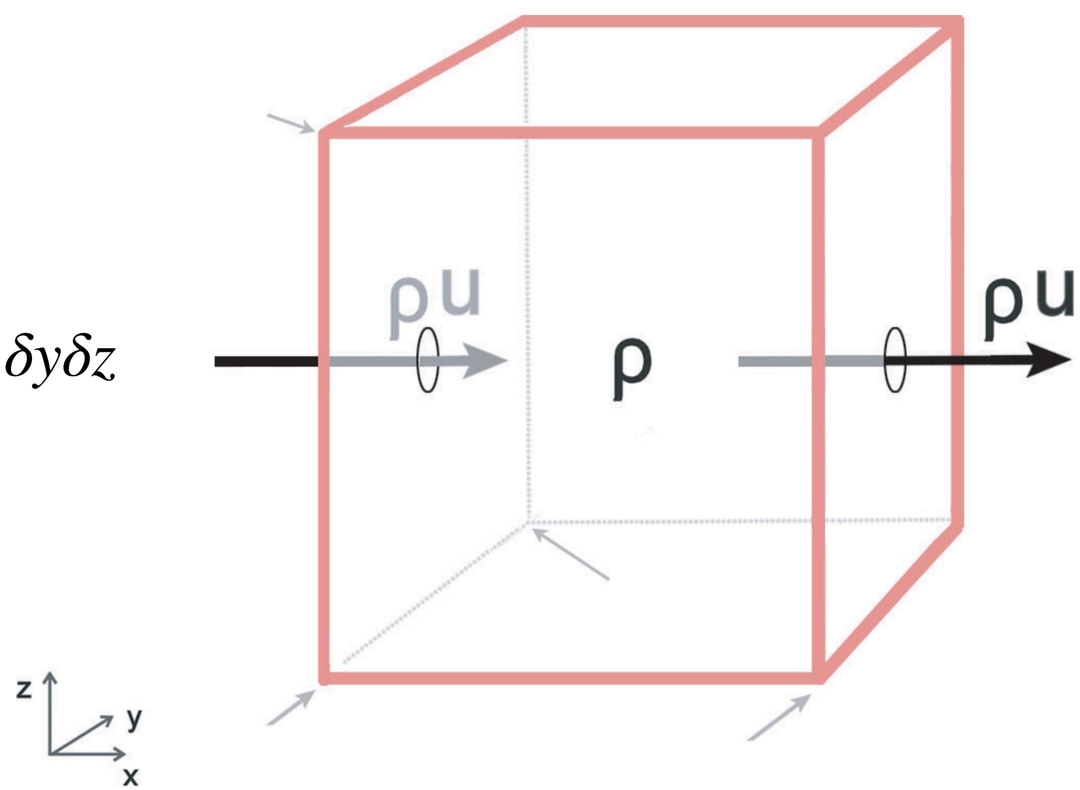
$$[\rho u]\left(x + \frac{\delta x}{2}, y, z\right)\delta y\delta z$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass**

$$\frac{\partial}{\partial t}\left(\rho\delta x\delta y\delta z\right) = \frac{\partial\rho}{\partial t}\delta x\delta y\delta z = \text{ Net mass flux into the volume}$$

$$[\rho u]\left(x - \frac{\delta x}{2}, y, z\right)\delta y\delta z$$



$$[\rho u]\left(x + \frac{\delta x}{2}, y, z\right)\delta y\delta z$$

$$[\rho u]\left(x - \frac{\delta x}{2}, y, z\right)\delta y\delta z - [\rho u]\left(x + \frac{\delta x}{2}, y, z\right)\delta y\delta z = -\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass**

$$\frac{\partial}{\partial t}\left(\rho\delta x\delta y\delta z\right) = \frac{\partial \rho}{\partial t}\delta x\delta y\delta z = \text{ Net mass flux into the volume}$$

Net mass flux into the volume :

$$-\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z - \frac{\partial}{\partial y}(\rho v)\delta x\delta y\delta z - \frac{\partial}{\partial z}(\rho w)\delta x\delta y\delta z$$

The equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass**

The equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

2. Equations of motion for a non-rotating fluid

- **Conservation of mass :**
  - For compressible flow, the hydrostatic assumption allows us to write the unit volume as $\delta x\ \delta y\ \delta p$
  - Then the mass of the fluid parcel becomes

$$\delta M = \rho\ \delta x\ \delta y\ \delta z\ = -\frac{1}{g}\delta x\ \delta y\ \delta p$$

  - The mass is conserved in pressure coordinates, and

$$\nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

2. Equations of motion for a non-rotating fluid

- **Thermodynamic equation**
  - The first law of thermodynamics we dealt with for the dry adiabatic lapse rate is $\delta Q = c_p dT - \frac{dp}{\rho}$
  - If we consider the first law of thermodynamics applied to a moving parcel of fluid,

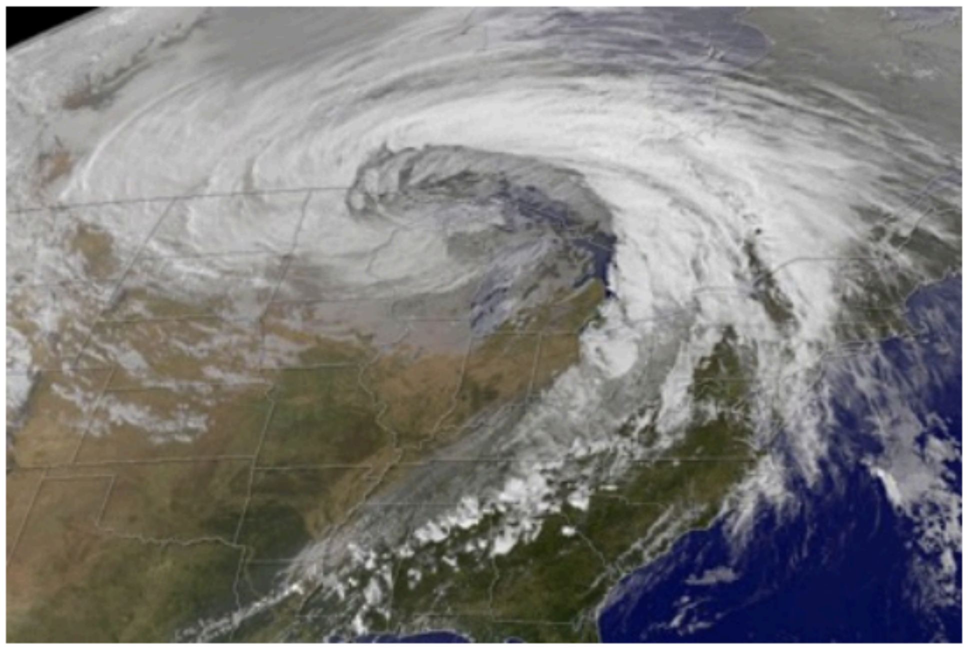
$$\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

Diabatic heating rate (e.g. latent heating or cooling)

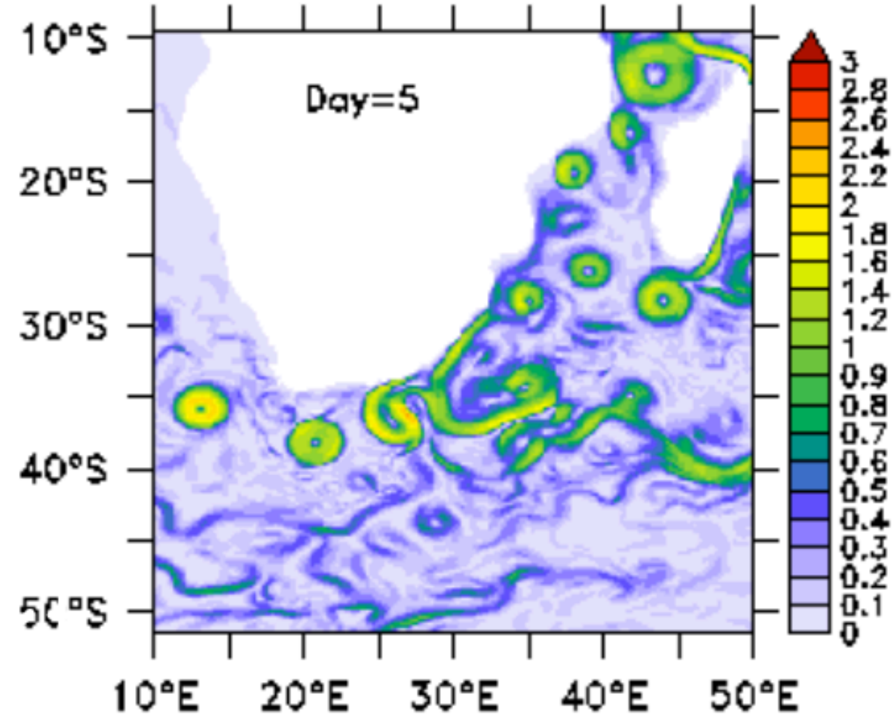
Temperature changes from the heat and/or expansion or compression

3. Equations of motion for a rotating fluid

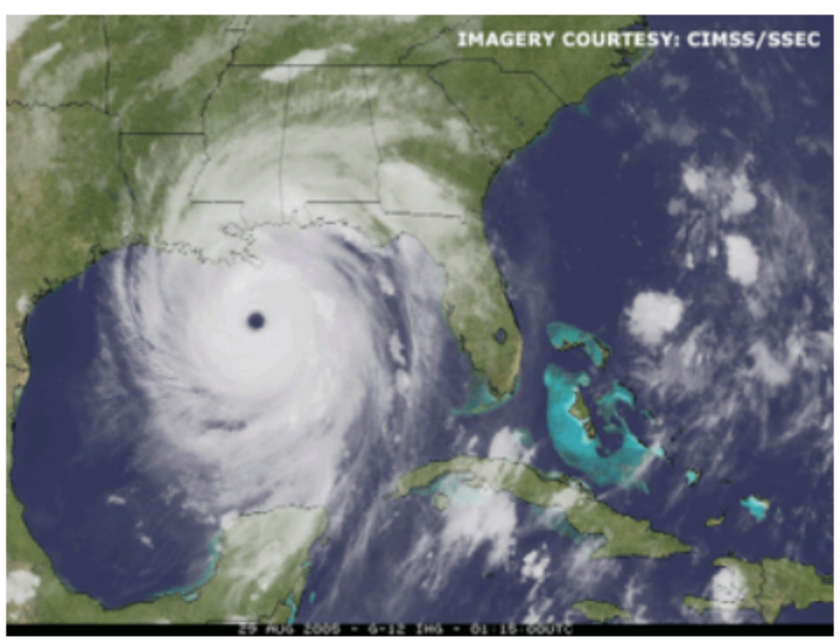
Extratropical cyclone 2010



Ocean eddies in GFDL model



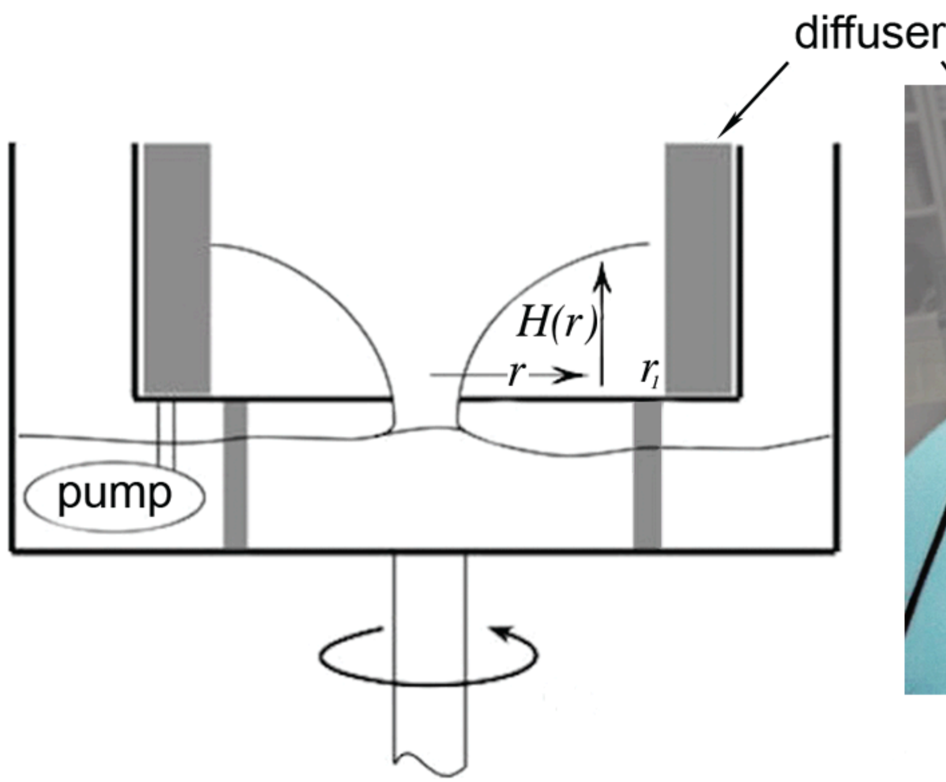
Hurricane Katrina 2005

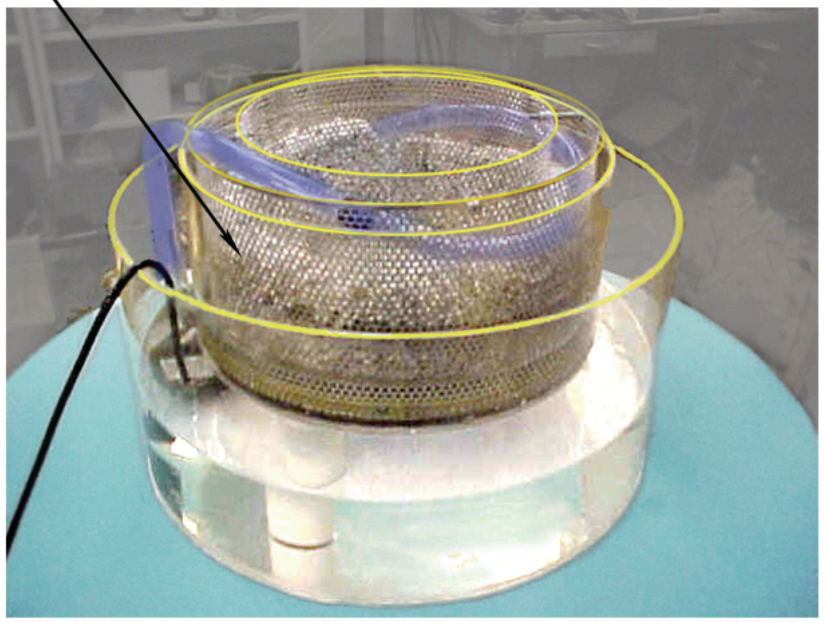


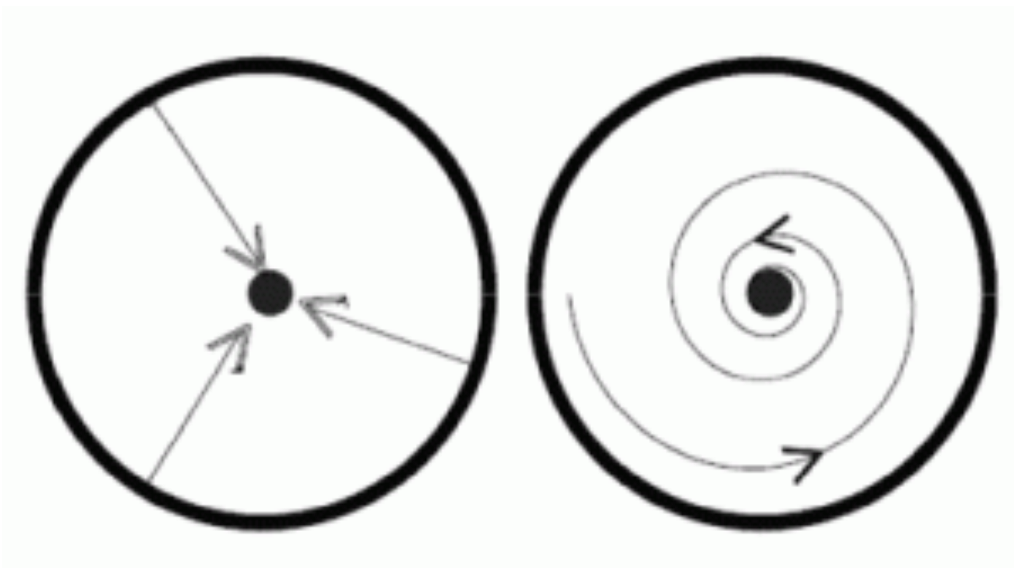
Tornado

3. Equations of motion for a rotating fluid

- Balanced vortex

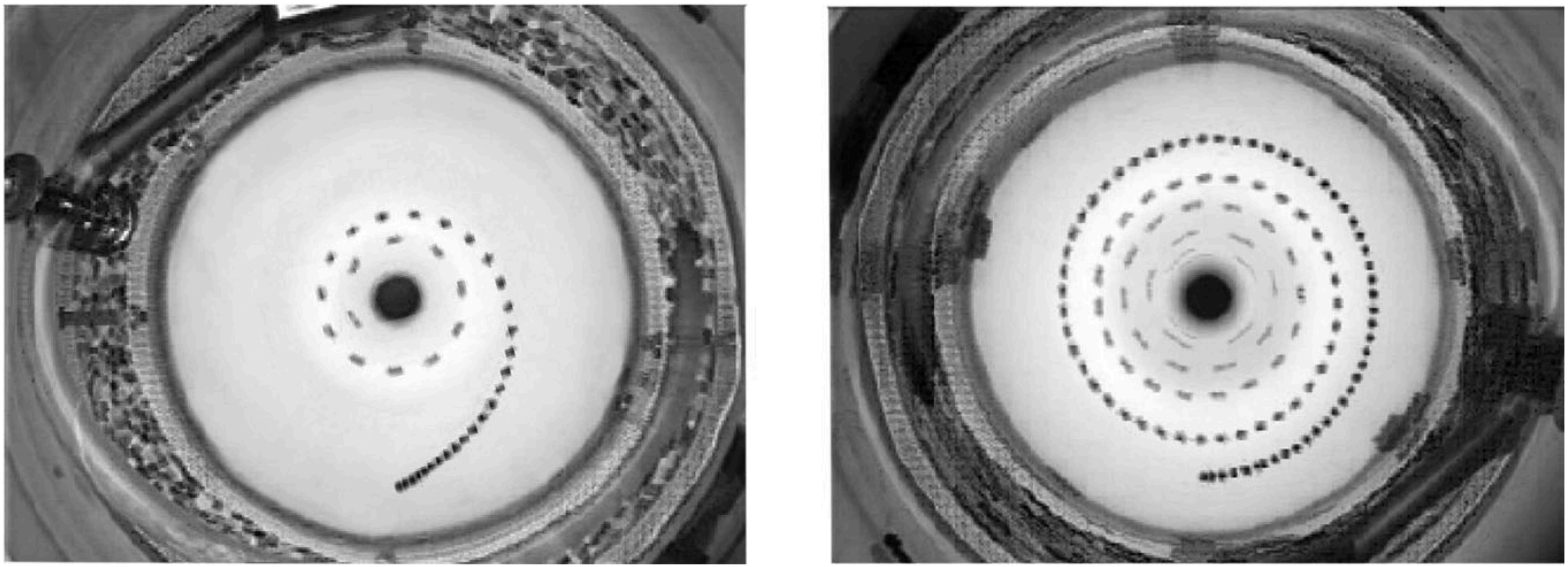






Flow patterns (left) in the absence of rotation and (right) when the apparatus is rotating in an anticlockwise direction.

3. Equations of motion for a rotating fluid



Trajectories of particles in the radial inflow experiment viewed in the rotating frame. The positions are plotted every 1/30 s. On the left Ω= 5 rpm (revolutions per minute). On the right Ω = 10 rpm. Note how the pitch of the particle trajectory increases as Ω increases, and how in both cases the speed of the particles increases as the radius decreases.

3. Equations of motion for a rotating fluid

PHYSICS

The Bath-Tub Vortex in the Southern Hemisphere

IT has long been thought that water draining from a tank would rotate counter-clockwise in the northern hemisphere and clockwise in the southern hemisphere, provided other influences were kept small compared with the influence of the rotation of the Earth. This idea has only recently been tested, by Shapiro in Watertown, Massachusetts, as part of a film on vorticity[2–4], and later by Binnie in Cambridge, England[1]. Shapiro and Binnie both acquired confidence, after surmounting difficulties in their early experiments, that the counter-clockwise rotations observed in their later experiments were due to the rotation of the Earth.

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Magnetic latitude (deg.)

Fig. 1. Plot of spread-F variation with sunspot number versus magnetic latitude

range of 40–120. It is found that the slopes of these linear portions vary widely from latitude to latitude, giving both positive and negative values. Positive slopes are obtained for the stations which show positive correlation of mean percentage occurrence of spread-F with sunspot numbers,

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