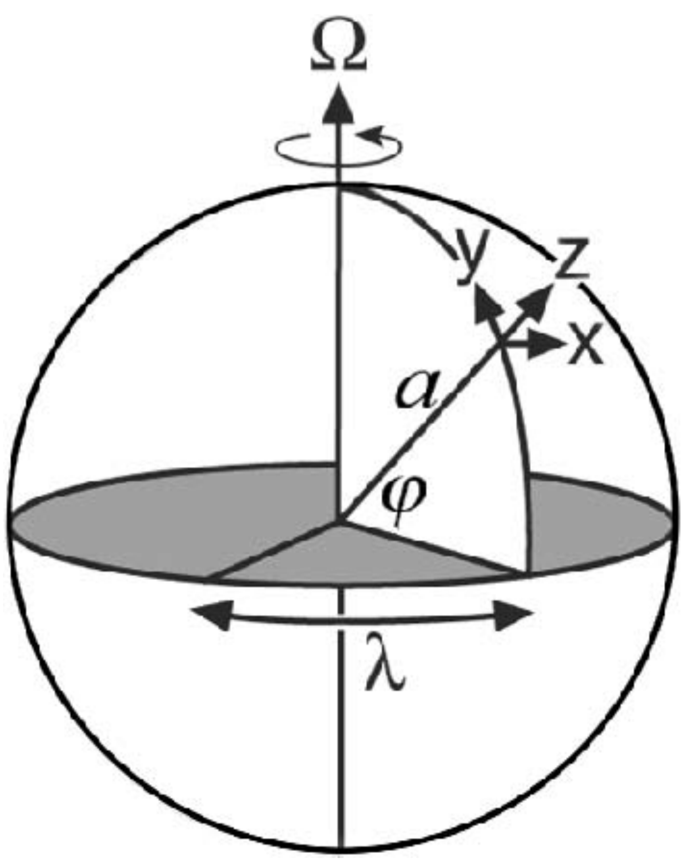
Balanced flows

ATM2106

Last time

$$\left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} \underbrace{- 2\boldsymbol{\Omega} \times \mathbf{u}_{rot}}_{\text{Coriolis acceleration}} \underbrace{- \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \mathbf{r}\right)}_{\text{Centrifugal acceleration}} + \mathscr{F}$$



Northern hemisphere: Ω is upward

→ Coriolis force acting to the right

Southern hemisphere: Ω is downward

→ Coriolis force acting to the left

Today’s topic

- Geostrophic motion
- The thermal wind equation
- Subgeostrophic flow

1. Geostrophic motion

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = \mathscr{F}_x$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = \mathscr{F}_y$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

Are all of these terms important every time?

1. Geostrophic motion

- First, we consider flows where friction can be neglected.
- Then, let's guess how big each term can be.

U/T $\quad U^2/L$ $\quad fU$

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

$1/fT$ $\quad U/(fL)$ $\quad 1$

$$R_o = \frac{U}{fL} \longrightarrow$$ **Rossby number**

1. Geostrophic motion

$$\underset{\overset{\uparrow}{U/T}}{}\quad \underset{\overset{\uparrow}{U^2/L}}{}\quad \underset{\overset{\uparrow}{fU}}{}$$

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

$$\underset{1/fT}{\downarrow}\quad \underset{U/(fL)}{\downarrow}\quad \underset{1}{\downarrow}$$

$$R_o = \frac{U}{fL} \rightarrow$$ **Rossby number**

- First, we consider flows where friction can be neglected.
- Then, let's guess how big each term can be.
- **For typical large-scale flows in the atmosphere:**

  - U ~ 10 m s^{-1} (horizontal velocity scale)
  - W ~ 1 cm s-1 (vertical velocity scale)
  - L ~ 10^6 m (length scale)
  - T ~ 10^5 s (time scale)
  - f ~ 10^{-4} s^{-1}
  - $1/\rho\ \partial p/\partial x$ ~ 10^{-3}

1. Geostrophic motion

$$R_o = \frac{U}{fL}$$ $\longrightarrow$ Rossby number ~ 10^{-1}

$$\frac{1}{\rho}\frac{\partial p}{\partial x} - fv = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + fu = 0$$

Geostrophic balance

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

Geostrophic wind

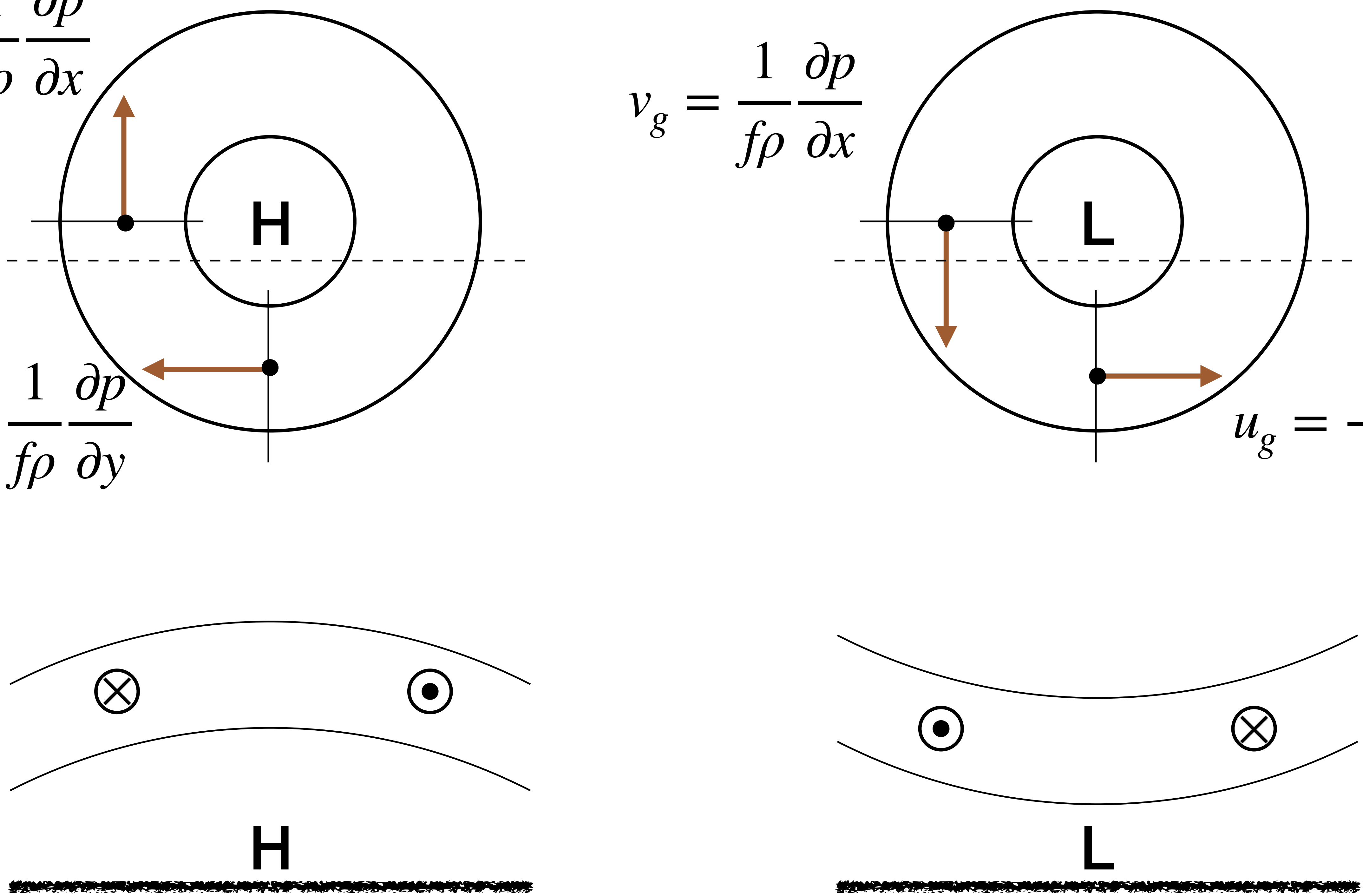
1. Geostrophic motion

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$

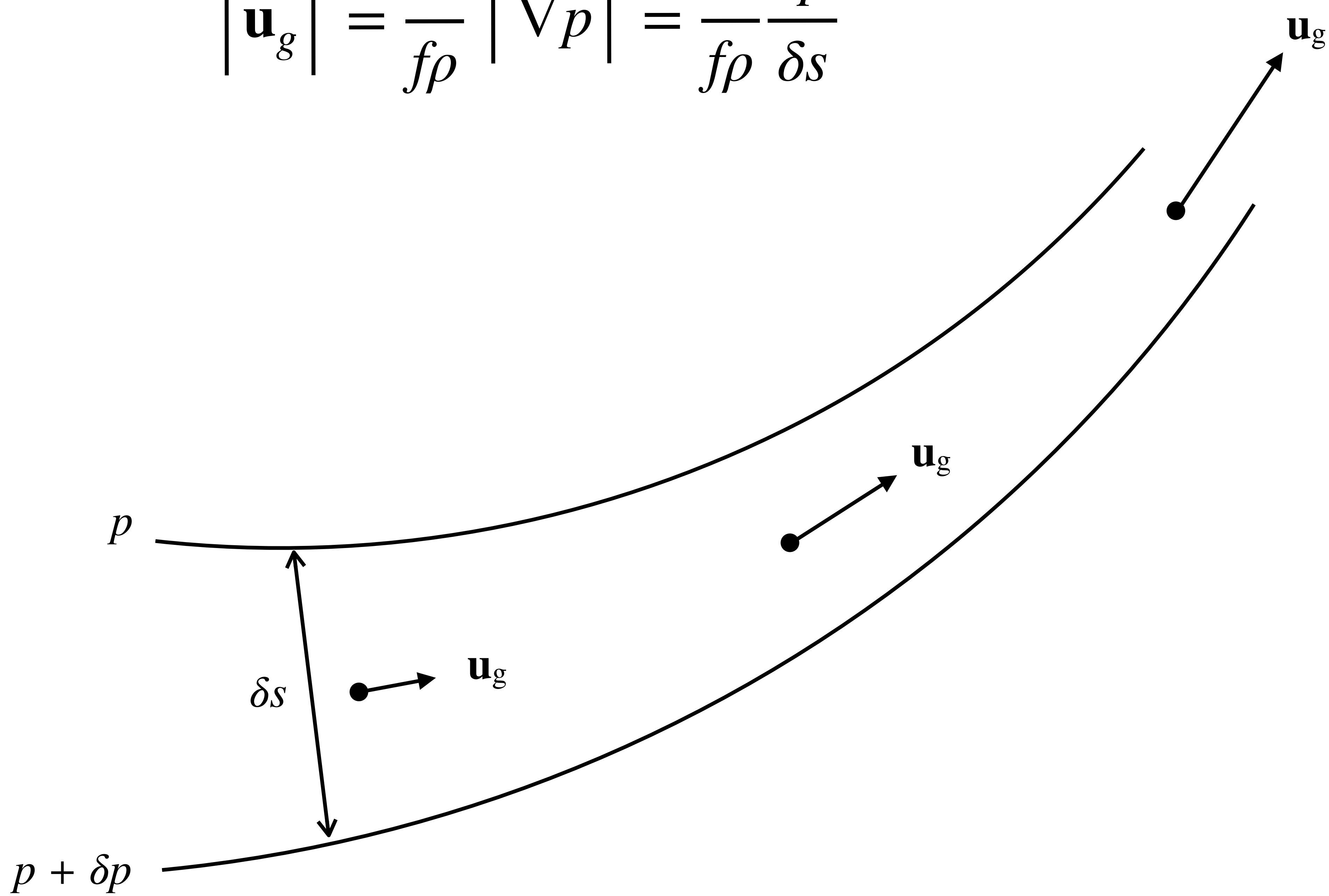
$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y}$$



1. Geostrophic motion

$$\left|\mathbf{u}_g\right| = \frac{1}{f\rho}\left|\nabla p\right| = \frac{1}{f\rho}\frac{\delta p}{\delta s}$$



1. Geostrophic motion : Nondivergent flow

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = -\frac{1}{f\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{f\rho}\frac{\partial^2 p}{\partial x \partial y} = 0$$

Non divergent flow:

any change in u_g will be compensated by the change in v_g

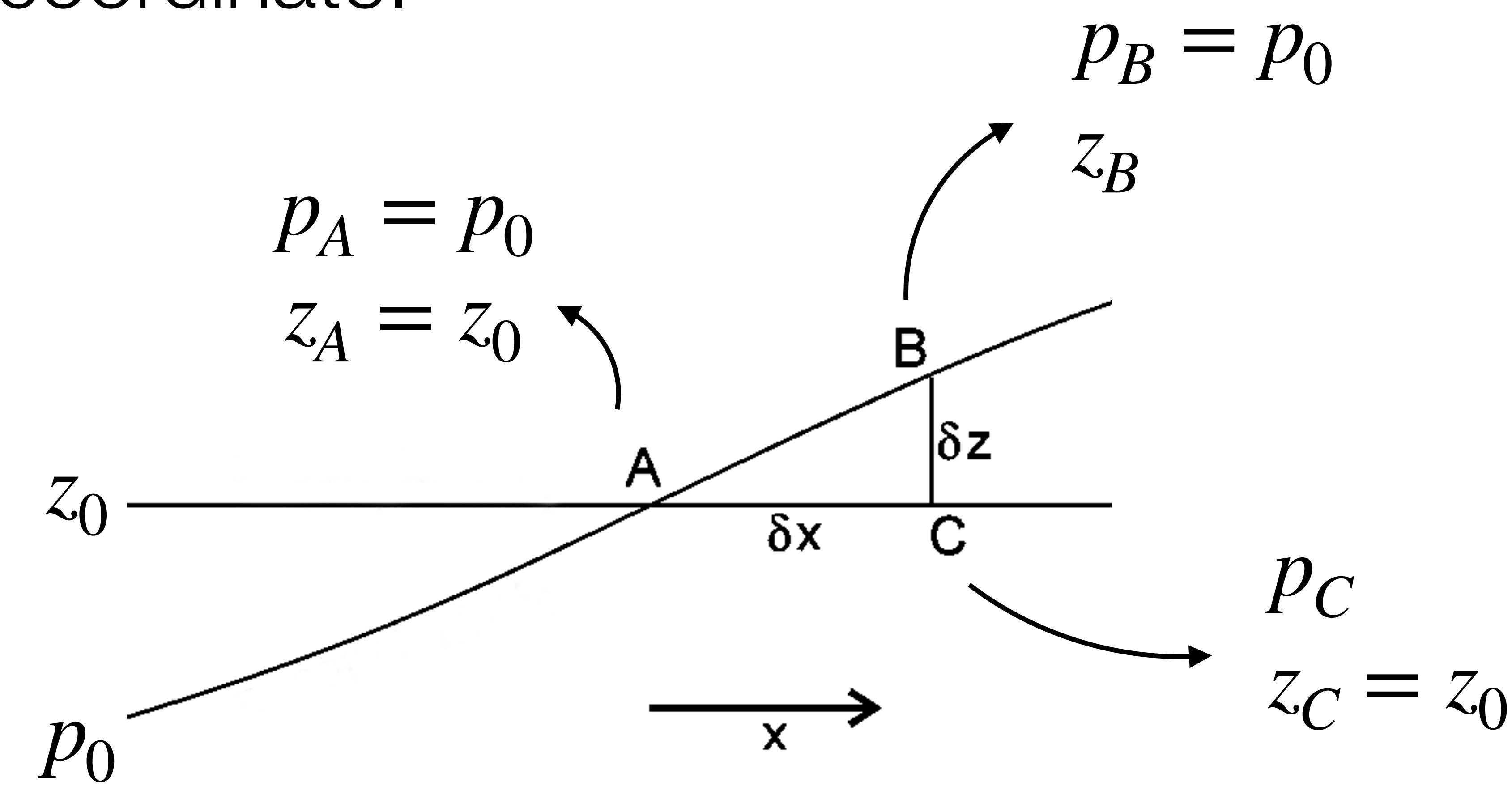
→ $\frac{\partial w_g}{\partial z} = 0$ (or $\frac{\partial \omega_g}{\partial p} = 0$)

→ if w_g=0 on a flat bottom boundary, then w_g=0 everywhere!

→ in this case, the geostrophic flows is horizontal.

1. Geostrophic motion : pressure coordinates

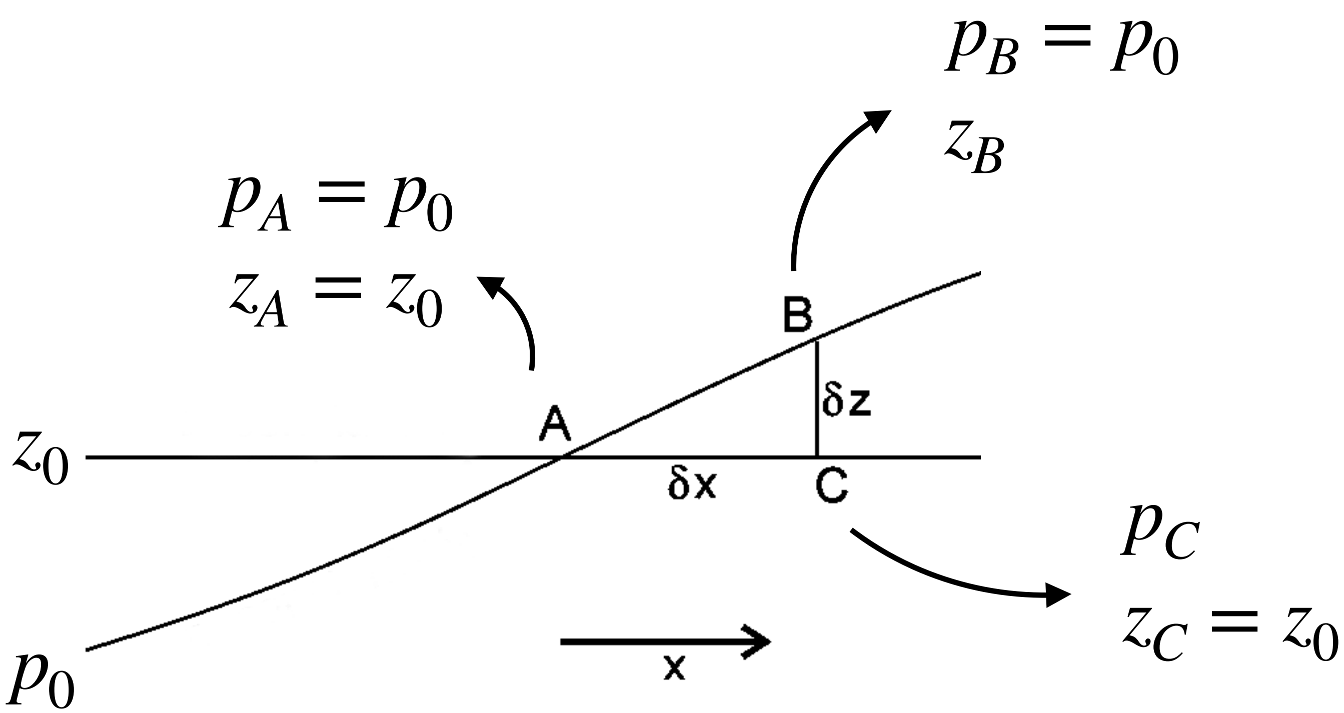
- It is convenient to look at the geostrophic wind on the pressure coordinate.



Pressure gradient between C and A : $\left(\dfrac{\partial p}{\partial x}\right)_z = \dfrac{p_C - p_0}{\delta x}$

The gradient of height between B and A : $\left(\dfrac{\partial z}{\partial x}\right)_p = \dfrac{z_B - z_0}{\delta x}$

1. Geostrophic motion : pressure coordinates



$$\frac{p_C - p_0}{z_B - z_0} = \frac{p_C - p_B}{z_B - z_C} = -\frac{\partial p}{\partial z} = g\rho$$

$$\downarrow$$

$$p_C - p_0 = g\rho(z_B - z_0)$$

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho\left(\frac{\partial z}{\partial x}\right)_p \qquad \left(\frac{\partial p}{\partial y}\right)_z = g\rho\left(\frac{\partial z}{\partial y}\right)_p$$

1. Geostrophic motion : pressure coordinates

$$\left(\frac{\partial p}{\partial x}\right)_z = g\rho \boxed{\left(\frac{\partial z}{\partial x}\right)_p}$$

$$\left(\frac{\partial p}{\partial y}\right)_z = g\rho \boxed{\left(\frac{\partial z}{\partial y}\right)_p}$$

$$\longrightarrow$$

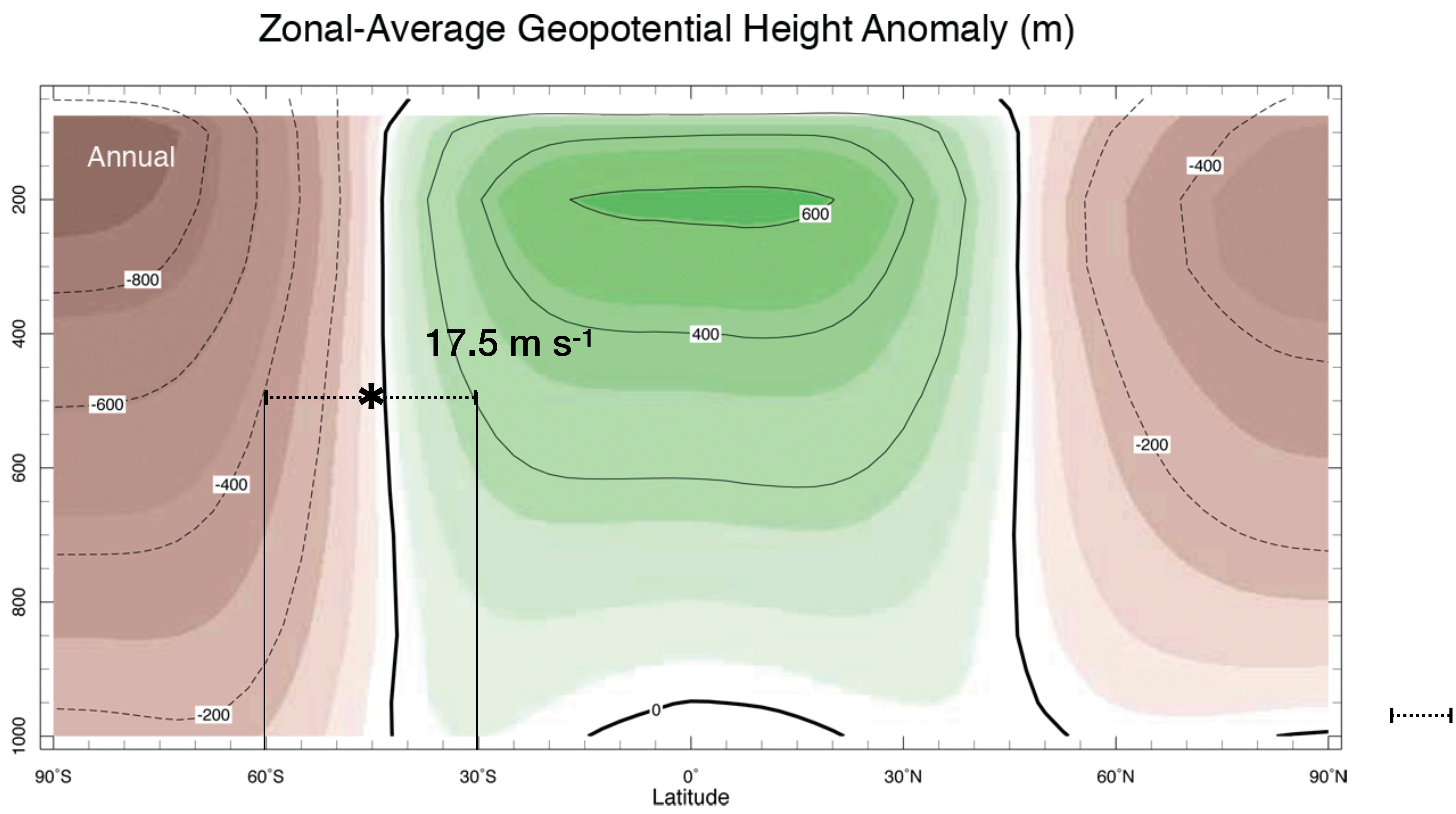
$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

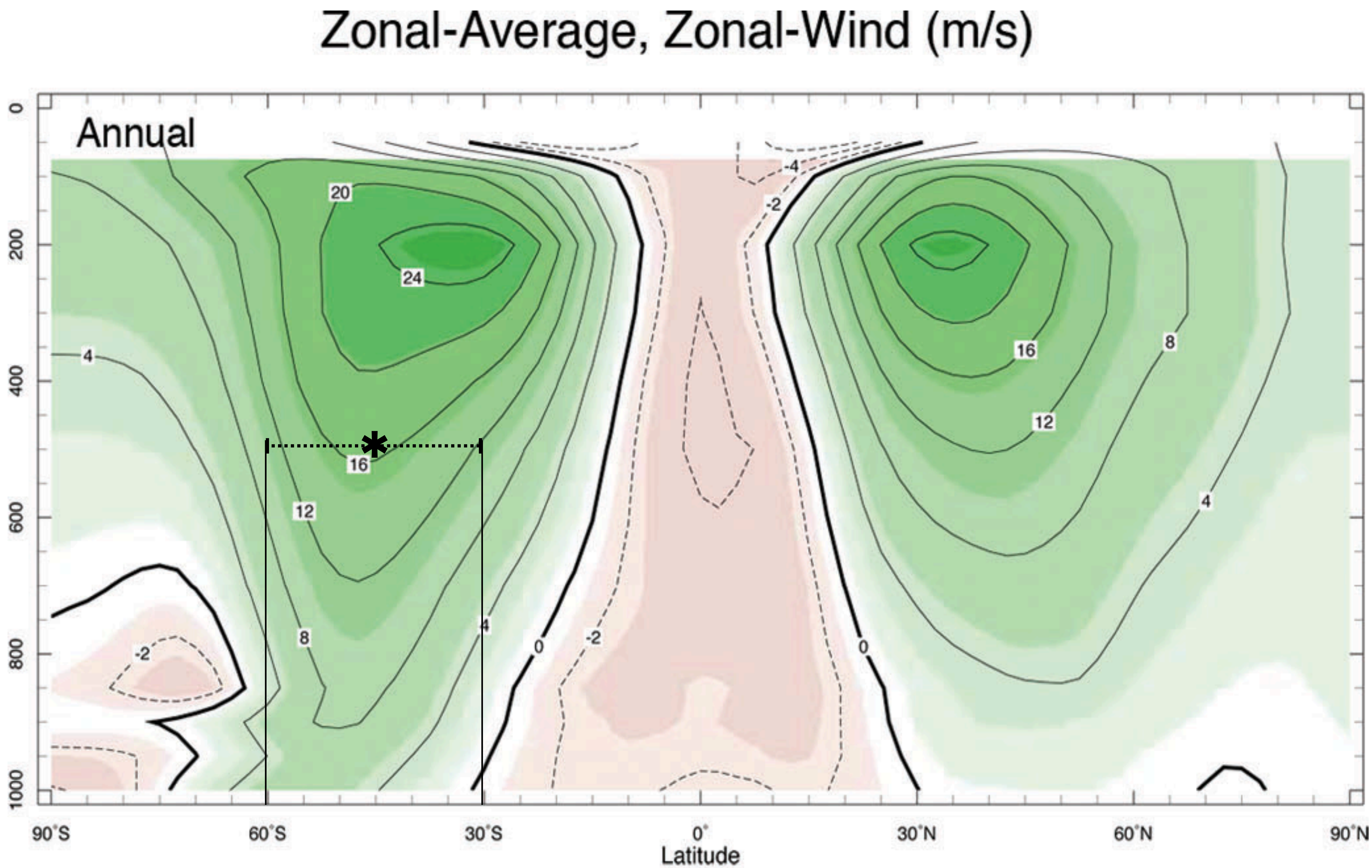
Lateral gradient of Geopotential height

No ρ !

1. Geostrophic motion : pressure coordinates

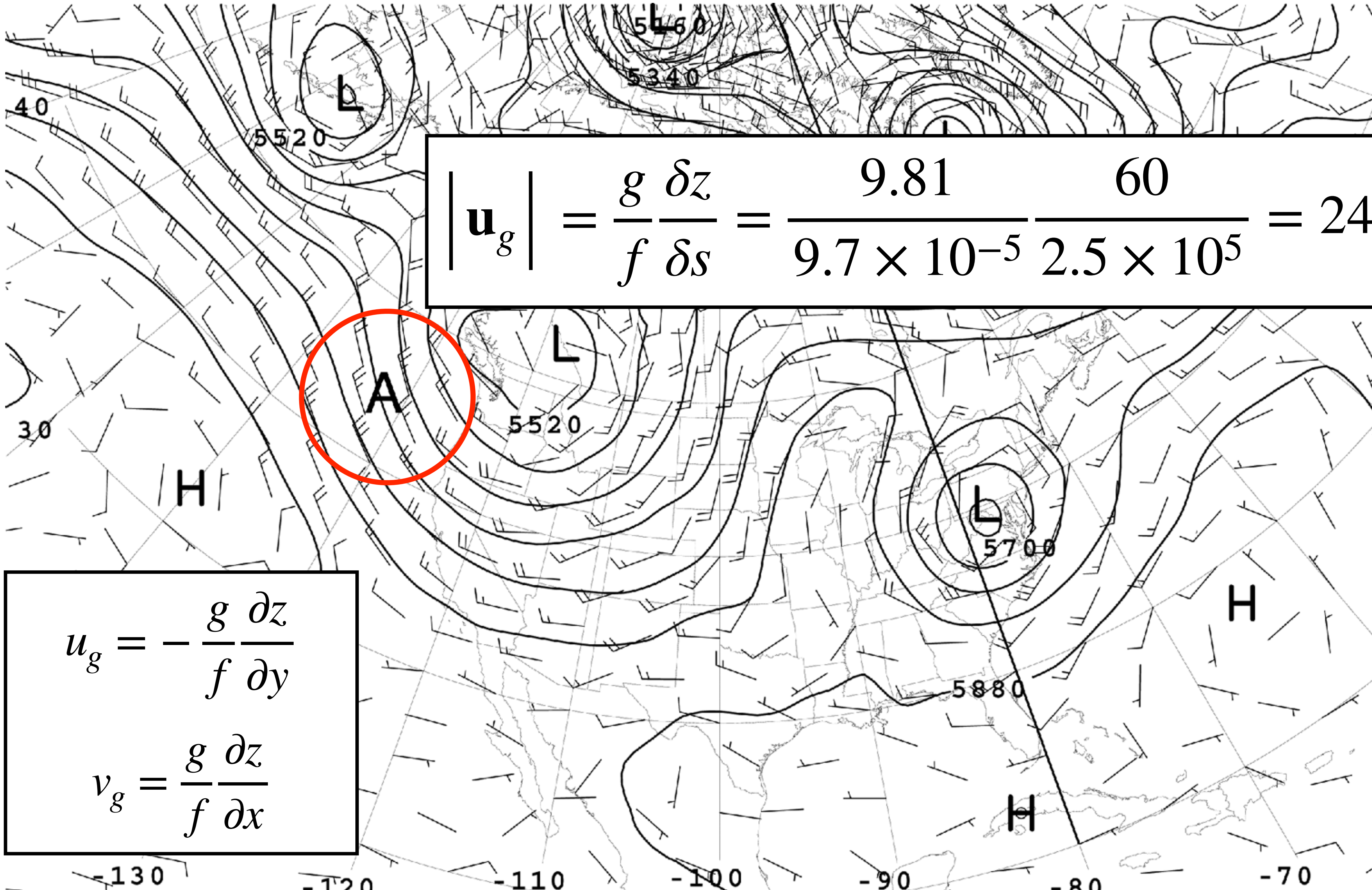


1. Geostrophic motion : pressure coordinates



1. Geostrophic motion : pressure coordinates

The wind speed near A is approximately 25 m s^{-1}



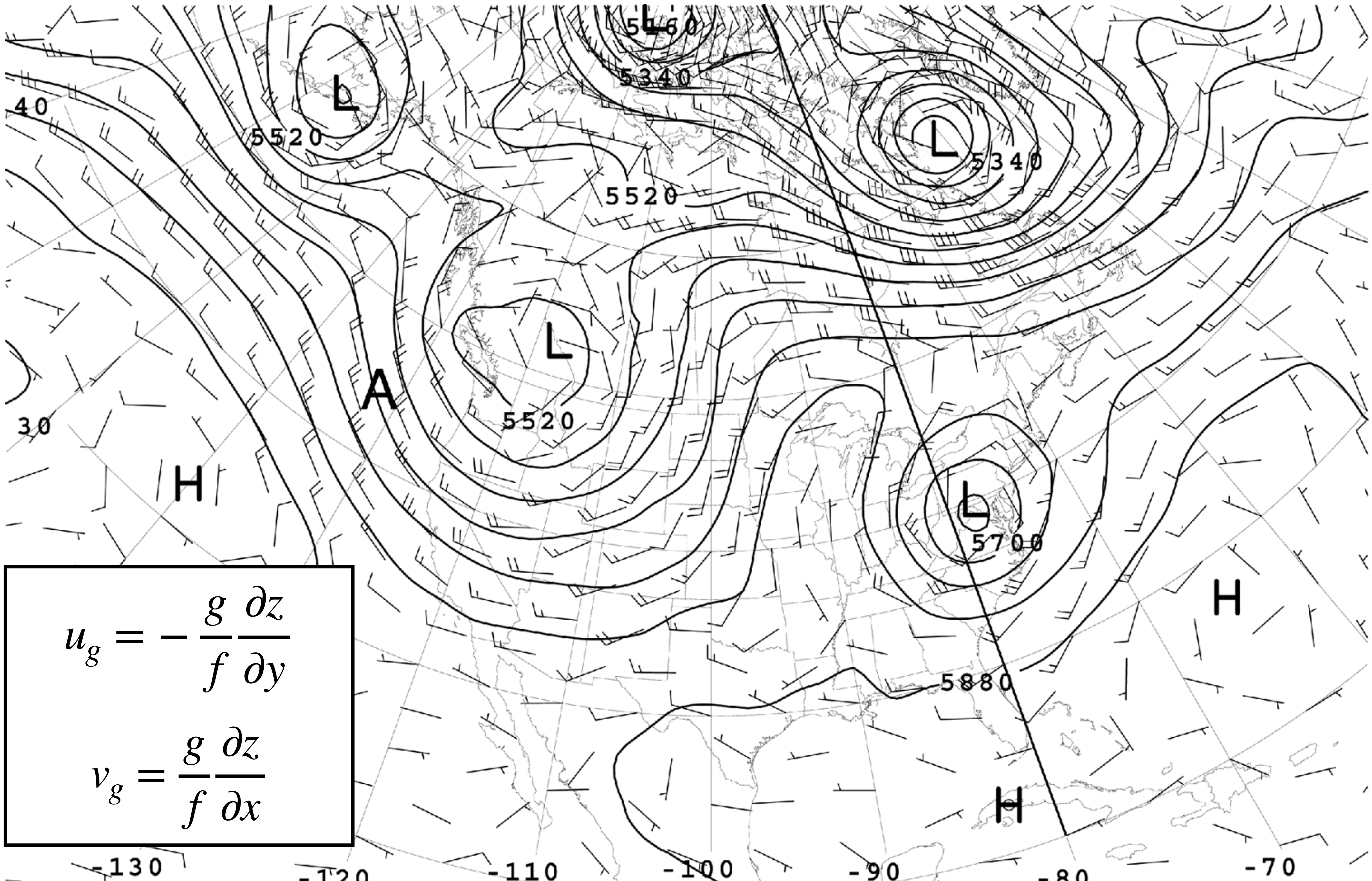
$$\left|\mathbf{u}_g\right| = \frac{g}{f}\frac{\delta z}{\delta s} = \frac{9.81}{9.7\times10^{-5}}\frac{60}{2.5\times10^{5}} = 24\ \mathrm{m\ s^{-1}}$$

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

1. Geostrophic motion : pressure coordinates

The strength of the geostrophic wind relies on the lateral gradient of geostrophic height.



$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

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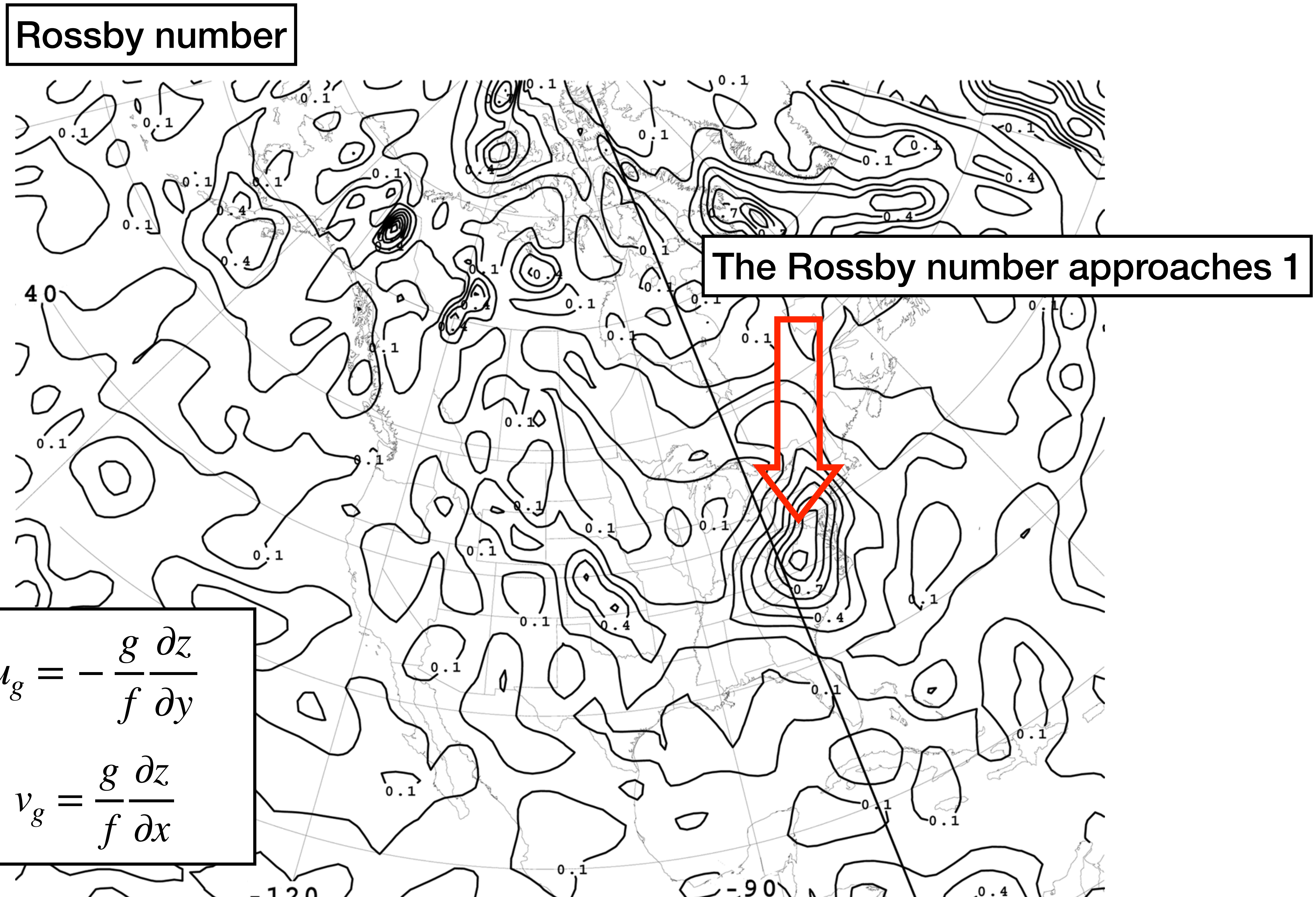
1. Geostrophic motion : pressure coordinates



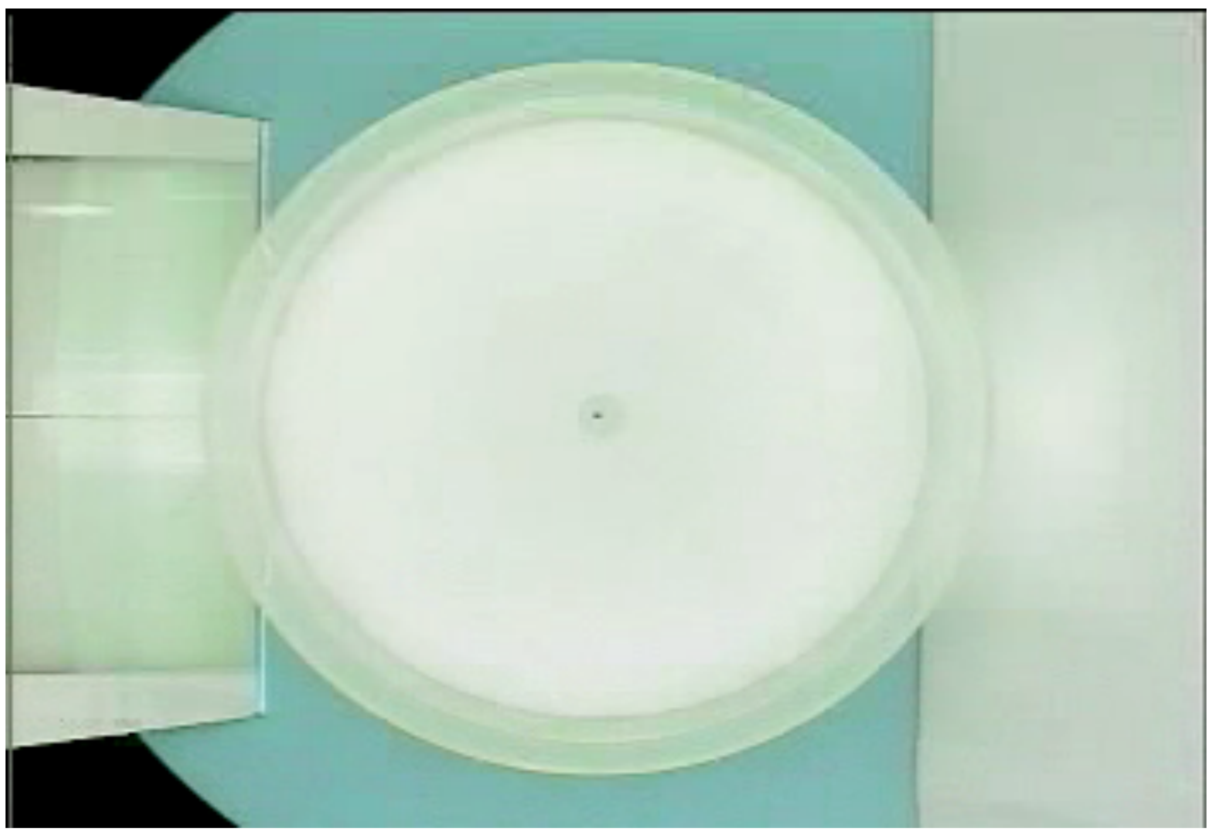
$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

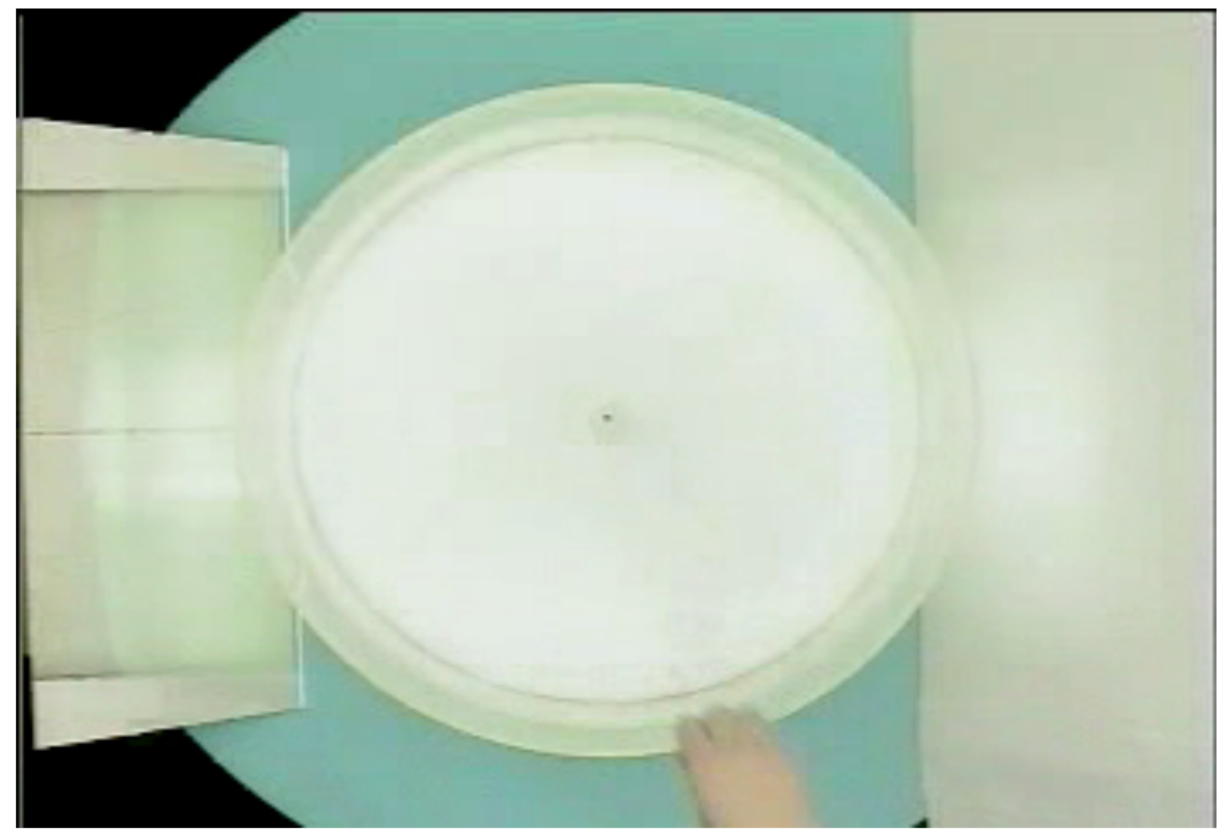
2. The Taylor-Proudman theorem

- Geostrophic motion is nondivergent.
- In case of water with a flat bottom, $w_g = 0$ and the flow has only horizontal components.
- Water moves as a column! (Taylor column)
- In fact, we already saw the effect of Taylor column in our first class hour.

2. The Taylor-Proudman theorem



Dissipation

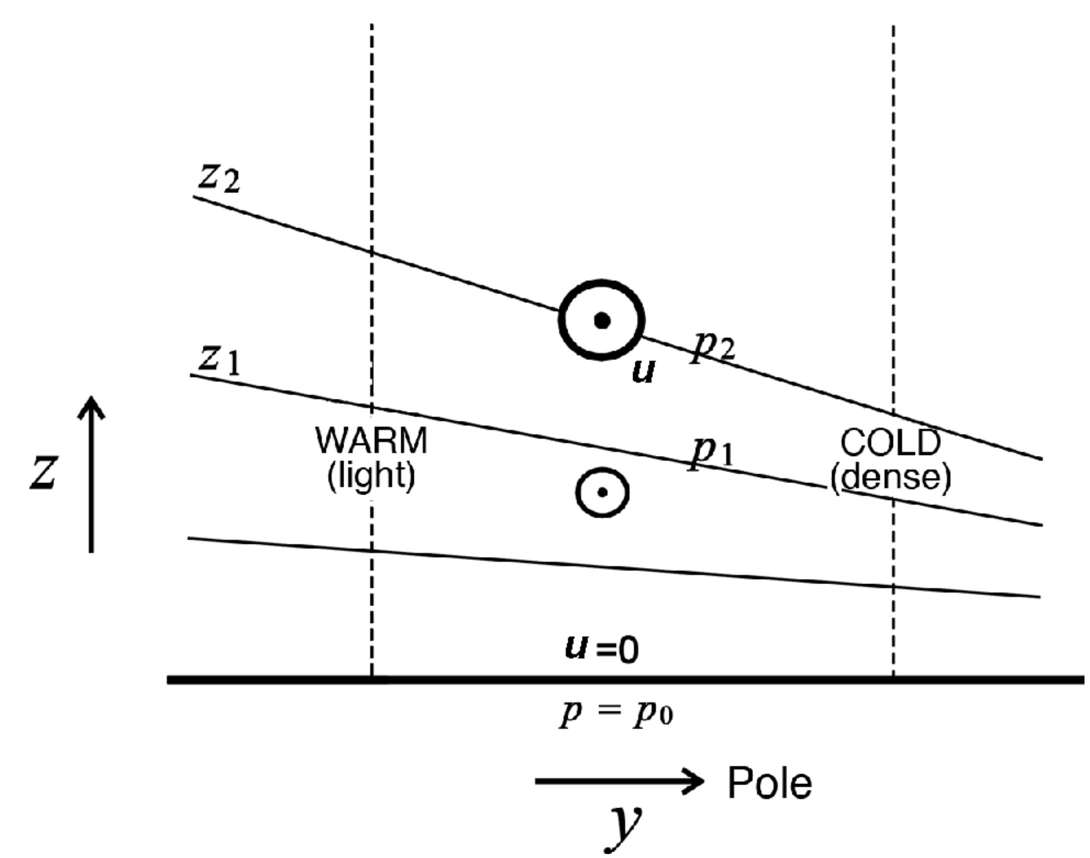


Streaks of dye falling vertically

Flows of vertical columns

3. Thermal wind equation

- The slopes of isobaric surfaces increase with height.
- According to the geostrophic relation, the geostrophic flow will increase with height.



3. Thermal wind equation

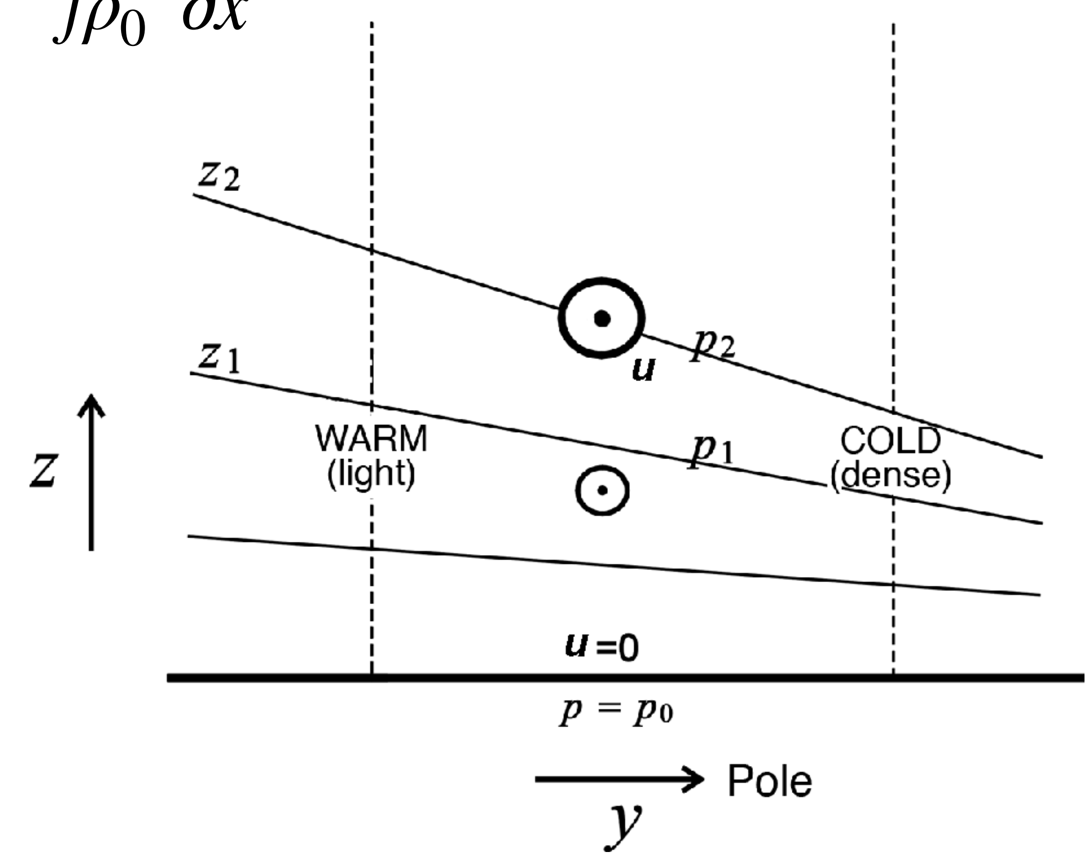
- Consider $\left(\frac{\partial u_g}{\partial z}, \frac{\partial u_g}{\partial z}\right)$ with $\rho = \rho_0 + \rho'$

$$\frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial y}\right) = -\frac{1}{f\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = \frac{g}{f\rho_0}\frac{\partial \rho'}{\partial y}$$

$$\frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) = \frac{1}{f\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = -\frac{g}{f\rho_0}\frac{\partial \rho'}{\partial x}$$

or

$$\frac{\partial \mathbf{u}_g}{\partial z} = -\frac{g}{f\rho_0}\hat{\mathbf{z}} \times \nabla \rho'$$



3. Thermal wind equation

- In case of water, $\rho' \approx -\alpha T'$

Thermal expansion coefficient

$$\frac{\partial \mathbf{u}_g}{\partial z} = \frac{\alpha g}{f}\hat{\mathbf{z}} \times \nabla T'$$

- For the air, we can use the pressure coordinate.

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f}\frac{\partial^2 z}{\partial p \partial y} = -\frac{g}{f}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial p}\right) = \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) = \frac{R}{fp}\frac{\partial T}{\partial y}$$

$$\frac{\partial v_g}{\partial p} = \frac{g}{f}\frac{\partial^2 z}{\partial p \partial x} = \frac{g}{f}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial p}\right) = -\frac{1}{f}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) = -\frac{R}{fp}\frac{\partial T}{\partial x}$$

hydrostatic balance

ideal gas law

3. Thermal wind equation

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  α: Thermal expansion coefficient

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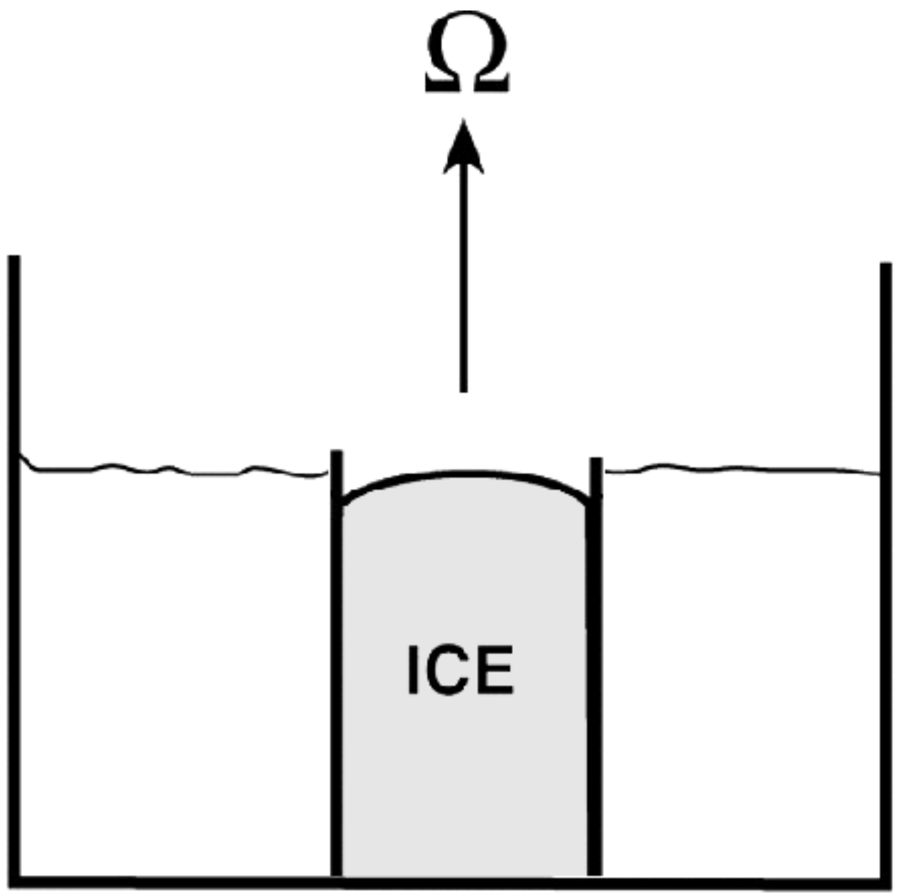
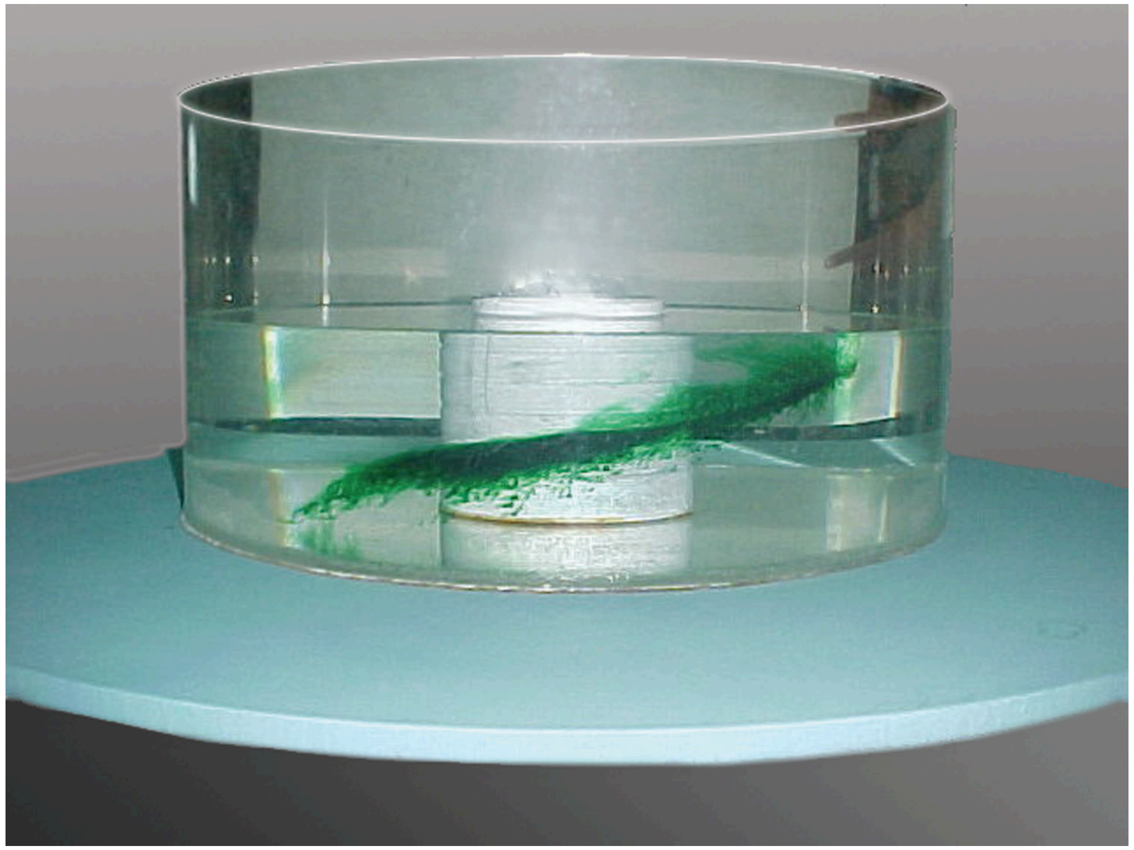
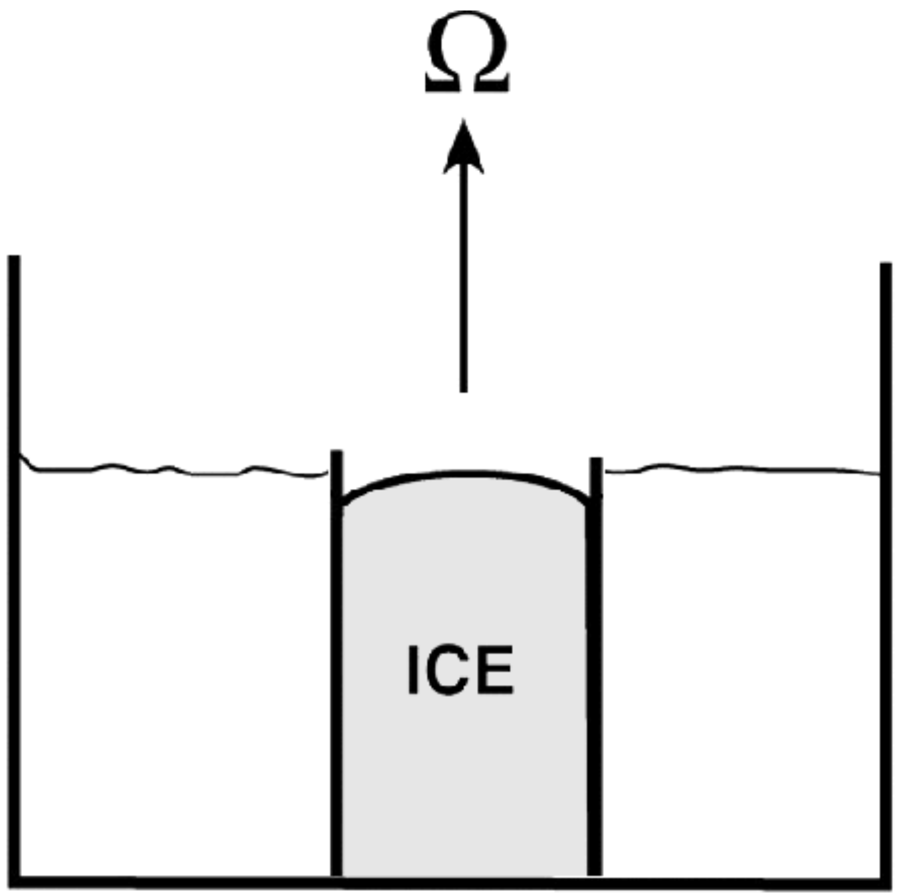
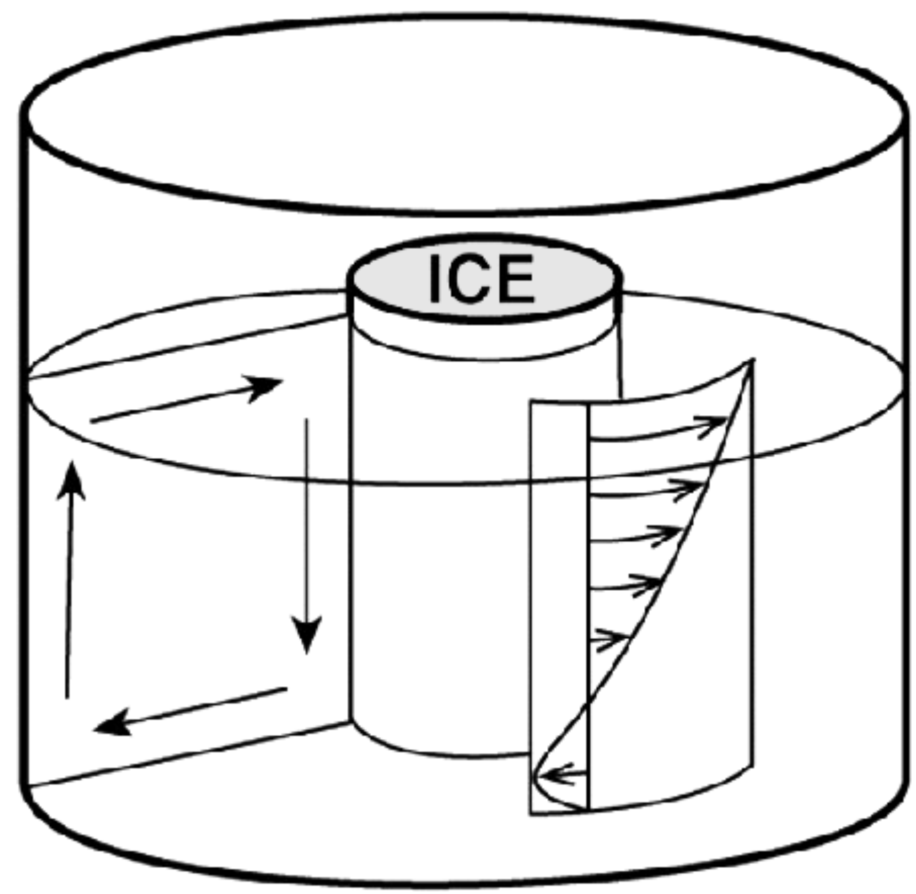
3. Thermal wind equation





3. Thermal wind equation





3. Thermal wind equation

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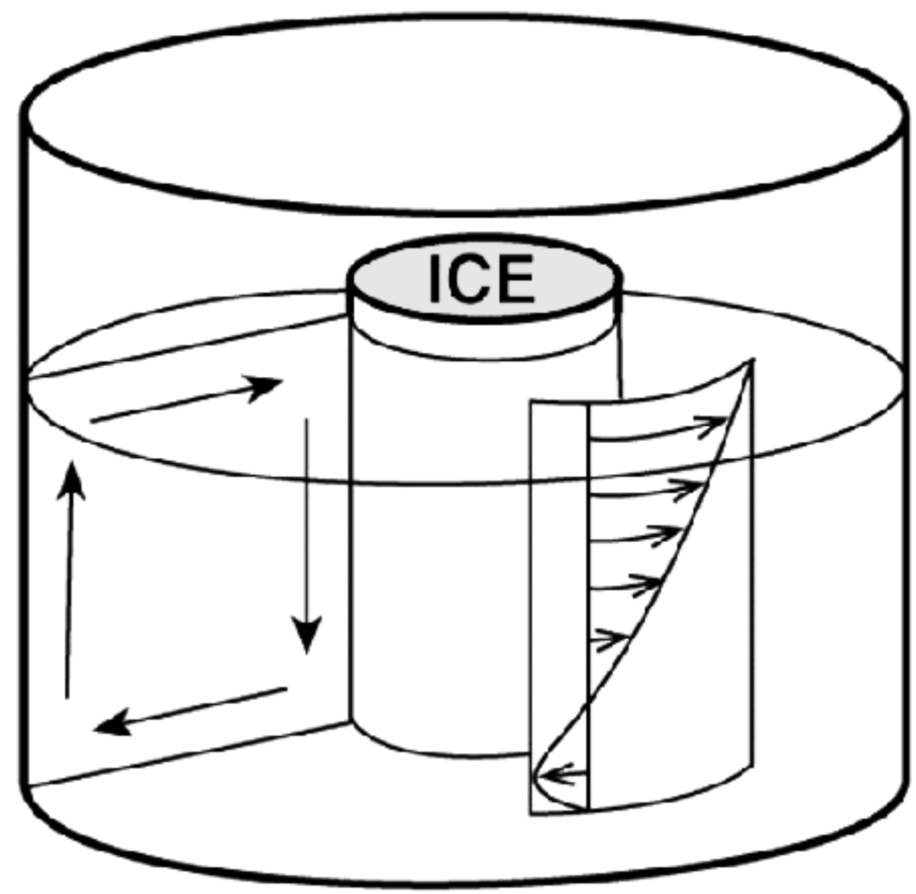
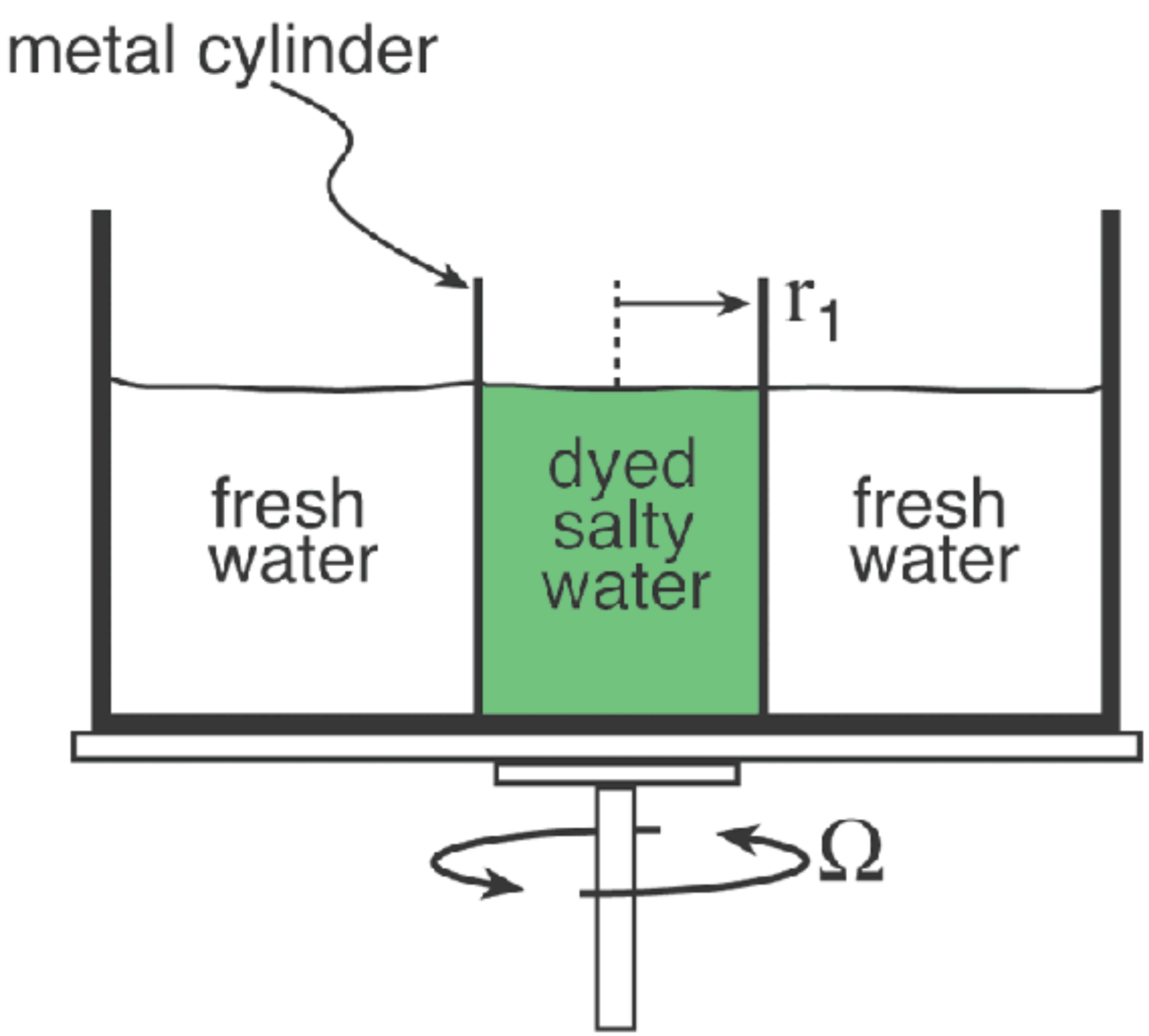


3. Thermal wind equation



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