The equations of fluid motion with rotation

ATM2106

Last time

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathscr{F}$$

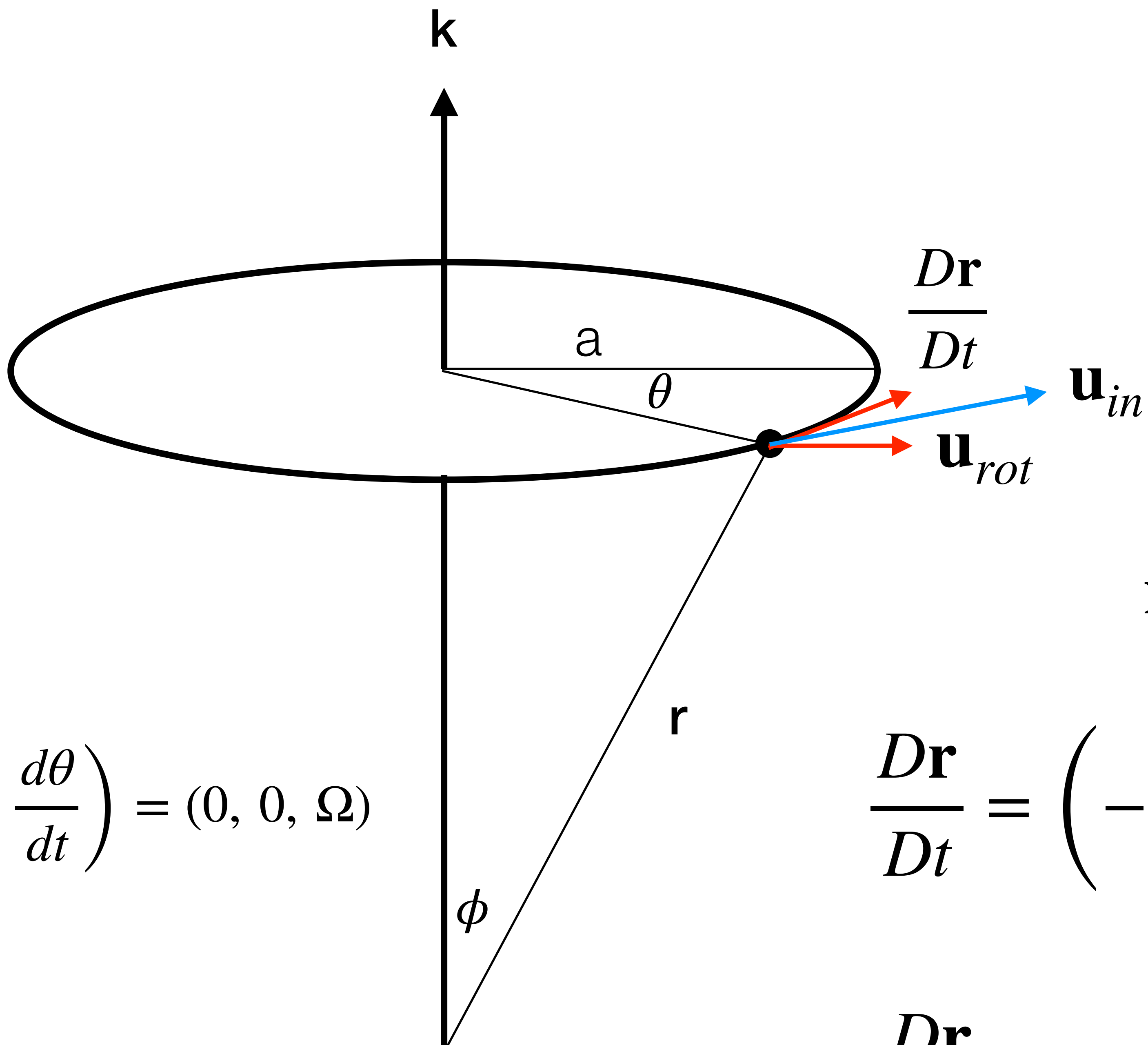
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{DQ}{Dt} = c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

Today's topic

- The equations of motion with rotation

1. Equations of motion for a rotating fluid



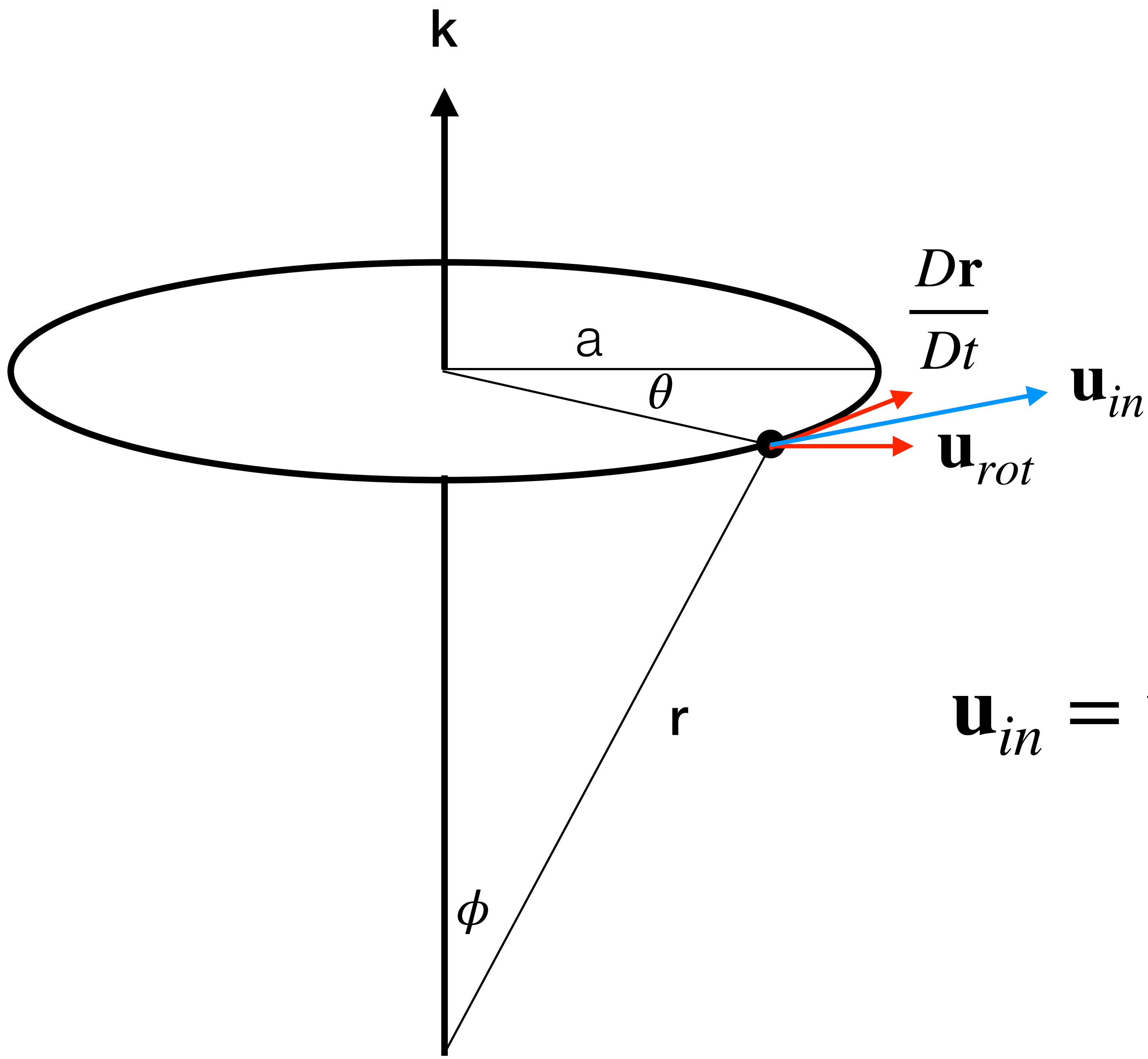
$$\mathbf{r} = \left(a\cos\theta,\ a\sin\theta,\ h\right)$$

$$\boldsymbol{\Omega} = \left(0,\ 0,\ \frac{d\theta}{dt}\right) = (0,\ 0,\ \Omega)$$

$$\frac{D\mathbf{r}}{Dt} = \left(-a\sin\theta\frac{d\theta}{dt},\ a\cos\theta\frac{d\theta}{dt},\ 0\right)$$

$$\frac{D\mathbf{r}}{Dt} = \Omega \times \mathbf{r}$$

1. Equations of motion for a rotating fluid



$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \mathbf{\Omega} \times \mathbf{r}$$

1. Equations of motion for a rotating fluid

A vector, $\mathbf{A}$, can be written with any three independent unit vectors.

$$\mathbf{A} = A_i\ \hat{\mathbf{i}} + A_j\ \hat{\mathbf{j}} + A_k\ \hat{\mathbf{k}} \longrightarrow \text{in the absolute frame}$$

$$= A_x\ \hat{\mathbf{x}} + A_y\ \hat{\mathbf{y}} + A_z\ \hat{\mathbf{z}} \longrightarrow \text{in the rotating frame}$$

Then, let's find out the Lagrangian differentiation of $\mathbf{A}$
with respect to the absolute (inertial) frame

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{in} = \frac{D}{Dt}\left(A_i\ \hat{\mathbf{i}}\right) + \frac{D}{Dt}\left(A_j\ \hat{\mathbf{j}}\right) + \frac{D}{Dt}\left(A_k\ \hat{\mathbf{k}}\right)$$

$$= \hat{\mathbf{i}}\frac{DA_i}{Dt} + \hat{\mathbf{j}}\frac{DA_j}{Dt} + \hat{\mathbf{k}}\frac{DA_k}{Dt}$$

1. Equations of motion for a rotating fluid

$\left(\dfrac{D\mathbf{A}}{Dt}\right)_{in}$ can also be written as

$$\frac{D\mathbf{r}}{Dt} = \mathbf{\Omega} \times \mathbf{r}$$

$$\left(\frac{D\mathbf{A}}{Dt}\right)_{in} = \frac{D}{Dt}\left(A_x\,\hat{\mathbf{x}}\right) + \frac{D}{Dt}\left(A_y\,\hat{\mathbf{y}}\right) + \frac{D}{Dt}\left(A_z\,\hat{\mathbf{z}}\right)$$

$$= \hat{\mathbf{x}}\frac{DA_x}{Dt} + \hat{\mathbf{y}}\frac{DA_y}{Dt} + \hat{\mathbf{z}}\frac{DA_z}{Dt} + A_x\frac{D\hat{\mathbf{x}}}{Dt} + A_y\frac{D\hat{\mathbf{y}}}{Dt} + A_z\frac{D\hat{\mathbf{z}}}{Dt}$$

$$= \hat{\mathbf{x}}\frac{DA_x}{Dt} + \hat{\mathbf{y}}\frac{DA_y}{Dt} + \hat{\mathbf{z}}\frac{DA_z}{Dt} + \mathbf{\Omega} \times \left(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right)$$

$$= \left(\frac{D\mathbf{A}}{Dt}\right)_{rot} + \mathbf{\Omega} \times \mathbf{A}$$

1. Equations of motion for a rotating fluid

$$\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in} = \left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{rot} + \boldsymbol{\Omega}\times\mathbf{u}_{in} \qquad \mathbf{u}_{in} = \mathbf{u}_{rot} + \boldsymbol{\Omega}\times\mathbf{r}$$

$$= \left(\frac{D\left(\mathbf{u}_{rot} + \boldsymbol{\Omega}\times\mathbf{r}\right)}{Dt}\right)_{rot} + \boldsymbol{\Omega}\times\left(\mathbf{u}_{rot} + \boldsymbol{\Omega}\times\mathbf{r}\right)$$

$$= \left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot} + 2\boldsymbol{\Omega}\times\mathbf{u}_{rot} + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)$$

1. Equations of motion for a rotating fluid

Momentum equation we did last time can be applied to the parcel in the absolute frame

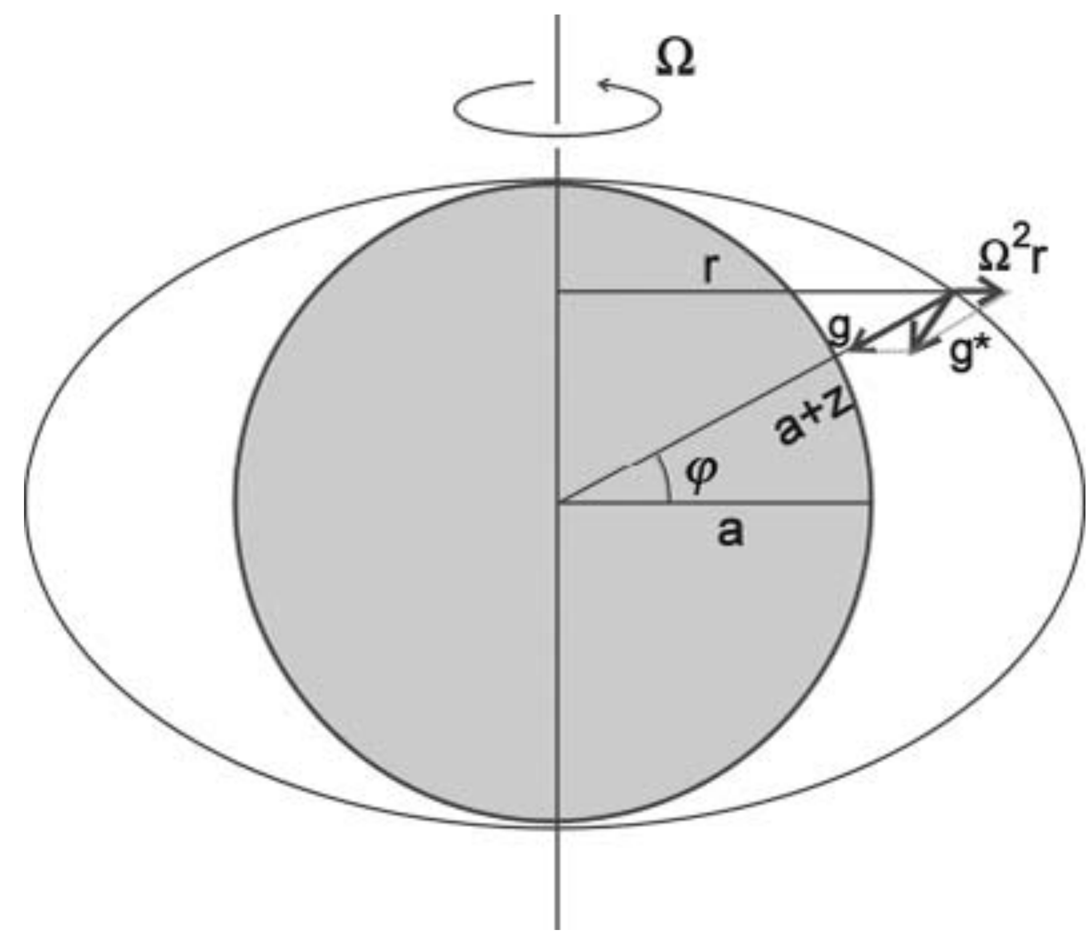
$$\left(\frac{D\mathbf{u}_{in}}{Dt}\right)_{in} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \mathscr{F}$$

$$\downarrow$$

$$\left(\frac{D\mathbf{u}_{rot}}{Dt}\right)_{rot} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{z}} = \underbrace{-\,2\boldsymbol{\Omega}\times\mathbf{u}_{rot}}_{\text{Coriolis acceleration}} \underbrace{-\,\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\mathbf{r}\right)}_{\text{Centrifugal acceleration}} + \mathscr{F}$$

1. Equations of motion for a rotating fluid

- Centrifugal acceleration: $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$
  - Modifies gravity acceleration
  - Effective gravity : g*
  - $\mathbf{\Omega} \sim 7.27 \times 10^{-5}\ s^{-1}$ makes centrifugal acceleration small.
  - Geoid : The surface perpendicular to the effective gravity
  - We may use this surface to make centrifugal acceleration to disappear.

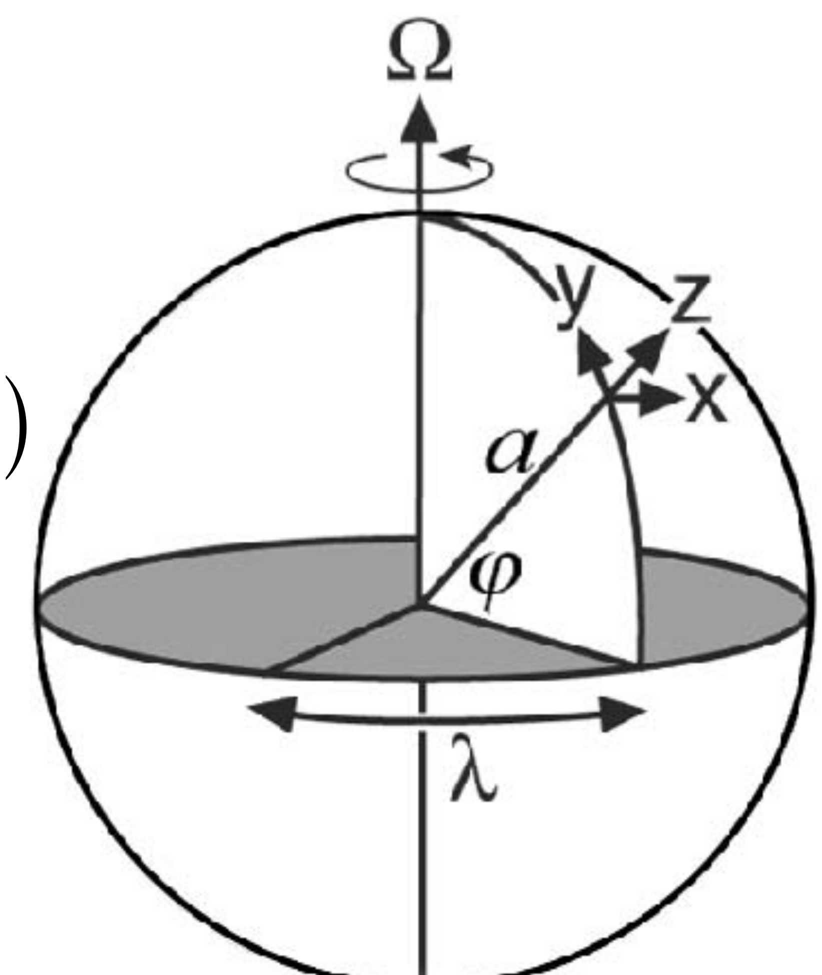


1. Equations of motion for a rotating fluid

- Coriolis force: $-2\,\mathbf{\Omega} \times \mathbf{u}$
    - $\mathbf{\Omega} = \left(0,\ \Omega\cos\phi,\ \Omega\sin\phi\right)$
    - $\mathbf{\Omega} \times \mathbf{u} = \left(\Omega\cos\phi\ w - \Omega\sin\phi\ v,\ \Omega\sin\phi\ u,\ -\Omega\cos\phi\ u\right)$
    - w is smaller than other terms.
    - Ωu is smaller than g.
    - $-2\mathbf{\Omega} \times \mathbf{u} \approx -\left(-2\Omega\sin\phi\ v,\ 2\Omega\sin\phi\ u,\ 0\right)$

$$= f\hat{\mathbf{z}} \times \mathbf{u}$$

$$(f = 2\Omega\sin\phi)$$



https://youtu.be/RrWKSOvqV-0

1. Equations of motion for a rotating fluid

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g^*\hat{\mathbf{z}} + f\hat{\mathbf{z}} \times \mathbf{u} = \mathscr{F}$$

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} - fv = \mathscr{F}_x$$

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} + fu = \mathscr{F}_y$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

- Hydrostatic approximation
- Vertical component of the frictional force is negligible compared with gravity