Optimalization Methods

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Contents

1 Linear Programming

Definition: Let us consider the following notation

- Coefficient matrix: A
- the capacity vector (r.h.s vector): b
- profit function (objective function), coefficients: c
- Then the problem is written as follows:

$$Ax \leq b$$
$$x \geq 0$$
$$z = c * x \rightarrow max$$

1.1 Simplex Method

1.1.1 Production model, Primal Simplex Method

	P_1	P_2	P_3	P_4	P_5	Capacity
Res_1	1	2	1	3	0	24
Res_2	0	1	1	5	1	43
Res_3	1	0	0	2	2	18
Profit	19	23	15	42	33	

- P_i is the product
- Res_i is the resource
- The production will consume the capacities of the resources, and these capacities must not be exceeded
- For the production we will gain a certain amount of profit, this gain is the linear combination of the *production plan* and the vector of coefficients of the profit. e.g $x(0, 8, 0, 1, 0)$, introducing vector $c(19, 23, 15, 42, 33)$ for the profit the gained profit will be $c * x = 226$
- The goal is to find a production plan where all constraints are satisfied and the gained profit is as large as possible.

Definiton:

- $\underline{x}$ is a *solution*, if substituting it, the constraints are fulfilled.
- $\underline{x}$ is a *feasible solution*, if it is a solution, and moreover $\underline{x} \geq 0$.
- $\underline{x}$ is an *optimal solution*, if it is a feasible solution, and the best one among all feasible solutions.
- Examples:
  - $x(-1, 1, 1, 1, 1)$ solution, but not feasible
  - $x(10, 10, 0, 0, 0)$ feasible, but not solution
  - $x(5, 1, 1, 0, 0)$ feasible solution

Primal simplex method 1st Phase: Algorithm to find the optimal solution

- **0th Step**: Linear Programming

$$x_1 + 2x_2 + x_3 + 3x_4 \leq 24$$
$$x_2 + x_3 + 5x_4 + x_5 \leq 43$$
$$x_1 + 2x_4 + 2x_5 \leq 18$$
$$x_i \geq 0, (1 \leq i \leq 5)$$
$$19x_1 + 23x_2 + 15x_3 + 43x_4 + 33x_5 = z \to max$$

- **1st Step**: All constraints formed to =, introducing slack variables

$$x_1 + 2x_2 + x_3 + 3x_4 + s_1 = 24$$
$$x_2 + x_3 + 5x_4 + x_5 + s_2 = 43$$
$$x_1 + 2x_4 + 2x_5 + s_3 = 18$$
$$x \geq 0, s \geq 0$$
$$19x_1 + 23x_2 + 15x_3 + 43x_4 + 33x_5 = z \to max$$

Simplex tableau: Instead of system equation, we can use a brief form to handle the data.

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
u_1	24	1	2	1	3	0	1	0	0
u_2	43	0	1	1	5	1	0	1	0
u_3	18	1	0	0	2	2	0	0	1
z	0	-19	-23	-15	-42	-33	0	0	0

- B means *basis* (maximal linearly independent set of vectors)
- $B = \{u_1, u_2, u_3\}$, the basis is composed (initially) from the three unit vectors.
- The corresponding *feasible solution* is $x_B = (0, 0, 0, 0, 0|24, 43, 18)$.
- Last row is the *reduced cost*

Theorem: Exactly one of the following options happens:

a) There is no negative entry in the last row $\to$ the corresponding basic solution is optimal

b) There is negative entry in the last row, such that there is no positive value in its column. $\to$ there is no optimal solution, as the objective value is not bounded from above.

c) Otherwise we can perform a basis transformation so that the objective does not decrease.

Primal Simplex Method 2nd phase: Perform the basis transformation

- Choose a column, where the reduced cost is negative $\to$ the objective value will grow
- Use the minimum rule $\to$ keep the $primal feasibility$(all the components of $\underline{x}$ are non-negative)
- *Dual feasibility*: Primal feasibility + the reduced cost is ≥ 0
- The simplex method is finite.

The steps in use:

- Choose a column for the vector, that will enter the basis. With the minimum rule choose the vector that will leave the basis. We take the $min(\frac{x_Bj}{a_ij})$ where a_i is the chosen vector and j is the row number.
- The minimum rule gives us where will the chosen vector enter, and also gives a *pivot number*.
- With the pivot number we perform the *basis transformation*
  - The row of the pivot is divided by the pivot value.
  - Add/substract the pivot number multiplied by λ, thus the values of the pivot's number column will be 0. $\lambda = \frac{a_ij}{pivot}$
- Do the steps above until there is no negative value in the last row. $\rightarrow$ dual feasibility + objective growing $\rightarrow$ optimality

Example:

- Choose a_2 vector for enter the basis. Use the minimum rule for choosing pivot value $min(\frac{24}{2}, \frac{43}{1}, \frac{18}{0}) = a_2[0]$.
- Perform the basis transformation with the pivot value of a_2:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	1/2	1	1/2	3/2	0	1/2	0	0
u_2	31	-1/2	0	1/2	7/2	1	-1/2	1	0
u_3	18	1	0	0	2	2	0	0	1
z	276	-15/2	0	-7/2	-15/2	-33	23/2	0	0

- Choose a_5 enter the basis, by the minimum rule u_3 leaves the basis, after performing the basis transformation:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_2	12	1/2	1	1/2	3/2	0	1/2	0	0
u_2	22	-1	0	1/2	5/2	0	-1/2	1	-1/2
a_5	9	1/2	0	0	1	1	0	0	1/2
z	573	9	0	-7/2	51/2	0	23/2	0	33/2

- Only one negative entry remained in the last row $\rightarrow$ choose a_3 enter the basis, by the minimum rule a_2 leaves the basis, after the basis transformation:

B	x_B	a_1	a_2	a_3	a_4	a_5	u_1	u_2	u_3
a_3	24	1	2	1	3	0	1	0	0
u_2	10	-3/2	-1	0	1	0	-1	1	-1/2
a_5	9	1/2	0	0	1	1	0	0	1/2
z	657	25/2	7	0	36	0	15	0	33/2

- There is no negative entry in the last row $\rightarrow$ optimal solution. $x_B = (0, 0, 24, 0, 9|0, 10, 0)$ while $Z_{opt} = 657$
- We produce 24 and 9 units from P_3 and P_5. 10 units remain from Res_2. The value of the objective function is 657.