CALIFORNIA INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

ME 96
Analysis and Control of an Inverted Pendulum

Revised: March 2003

1 Introduction

While all design methods for developing control systems require a working model of the system to be controlled, the construction of these models is often ignored in control systems courses. Only after a model has been developed, can a control system be designed. This experiment is designed to illustrate the procedure involved in developing a control system for an electro-mechanical system. This includes developing an experimental model for the system, verifying this model, and finally developing a controller.

This experiment consists of three steps. The first step is to determine the parameters which appear in the mathematical model (and can be measured easily). This will involve simple mass and length measurements. The next step involves comparing the time response of several second order linear ordinary differential equations to determine the two remaining system parameters. The data required for these comparisons will be taken in the first week. The last step is to design controllers for the rotary pendulum by using the steps outlined below.

1.1 Notation

The following convention is used in the equations presented here: scalar variables are slanted (m), vector quantities are bold face lower case letters ($\mathbf{x}$), and matrix quantities are bold face capital letters ($\mathbf{M}$). So a matrix function that depends on vector of scalars could be $\mathbf{M}(\boldsymbol{\theta_1}, \boldsymbol{\theta_2})$.

1.2 Mathematical Model

There are several methods which can be used to generate the equations of motion of the rotary pendulum. While the most familiar are probably Newton's equations, these are not always the most convenient. We will rely on Lagrangian analysis for our derivation. This method relies on the energy properties of mechanical systems to compute the equations of motion, and is usually simpler when non-Euclidean coordinates are used, or when the system involves constraints. The Lagrangian, L, is defined as

$$L = T - V, \tag{1}$$

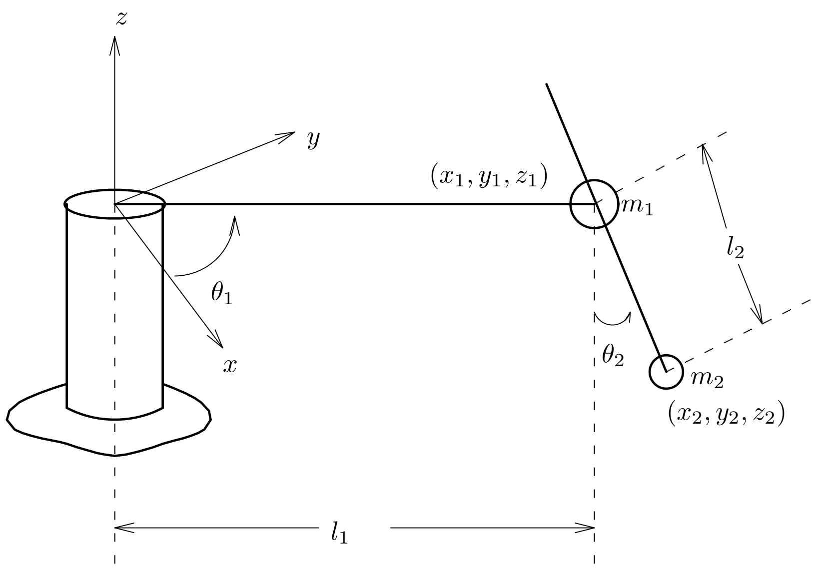

Figure 1: A model for the rotary inverted pendulum.

where T is the kinetic energy and V is the potential energy of the system in generalized coordinates. The equations of motion are then given by

$$Q_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \qquad i = 1, \ldots, m \tag{2}$$

where the Q_i are called the generalized forces.

The first step in using Lagrange's equations is to develop expressions for the kinetic and potential energy of the system in terms of the generalized coordinates. Choosing the coordinates θ_1 and θ_2 and using the parameterization of the rotary pendulum shown in Figure 1, the velocities of the two masses can be derived from their positions. The position (x, y, and z components) of each of the two masses is given by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 - l_2 \sin\theta_1 \sin\theta_2 \\ l_1 \sin\theta_1 + l_2 \cos\theta_1 \sin\theta_2 \\ -l_2 \cos\theta_2 \end{bmatrix} \tag{3}$$

These two expressions for the positions, (x_1, y_1, z_1) for m_1 and (x_2, y_2, z_2) for m_2, can be differentiated with respect to time and the standard relationship for kinetic energy of each mass,

$$T_i = \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2), \tag{4}$$

then used to obtain the following expression for the total kinetic energy of the system:

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left((l_1^2 + l_2^2 \sin{\theta_2}^2) \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2 \right) \tag{5}$$

The potential energy of the system, V, is given by

$$V = m_2 g l_2 (1 - \cos\theta_2) \tag{6}$$

We also need expressions for the generalized forces (the Q_i). For a kinematic chain parameterized by the joint angles, each Q_i is equal to the i^{th} joint torque. There are torques applied at each of the joints shown in Figure 1. At joint 1 (θ_1) there are two torques acting, one due to damping (we assume the damping to be viscous damping), with the corresponding damping coefficient b_1, and one due to the torque applied by the motor, τ. At joint 2 (θ_2) there is a single torque due to damping (again we assume viscous damping), with corresponding damping coefficient b_2. The assumption of viscous damping is not always the most accurate one, often other forms of friction are present, two of the most common are coulomb (dry friction) and stiction. The only way to determine what type of friction is present is through experimentation, and for our case viscous damping is a good choice for the model. Thus the generalized forces for our model of the rotary pendulum are given as

$$Q_1 = \tau - b_1\dot{\theta}_1 \quad \text{and} \quad Q_2 = -b_2\dot{\theta}_2 \tag{7}$$

Let $\theta = (\theta_1, \theta_2)$ be the two joint angles for the rotary pendulum, then equations of motion are given by Lagrange's equations as

$$\begin{bmatrix} \tau \\ 0 \end{bmatrix} = \mathbf{M}(\theta) \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \mathbf{C}(\theta, \dot{\theta}) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \mathbf{N}(\theta, \dot{\theta}), \tag{8}$$

where $\mathbf{M}(\theta)$, $\mathbf{C}(\theta, \dot{\theta})$, and $\mathbf{N}(\theta, \dot{\theta})$ are given by the following expressions:

$$\mathbf{M}(\theta) = \begin{bmatrix} l_1^2(m_1 + m_2) + l_2^2 m_2 \sin\theta_2{}^2 & l_1 l_2 m_2 \cos\theta_2 \\ l_1 l_2 m_2 \cos\theta_2 & l_2^2 m_2 \end{bmatrix}, \tag{9}$$

$$\mathbf{C}(\theta, \dot{\theta}) = \begin{bmatrix} l_2^2 m_2 \dot{\theta}_2 \cos\theta_2 \sin\theta_2 & l_2^2 m_2 \dot{\theta}_1 \cos\theta_2 \sin\theta_2 - l_1 l_2 m_2 \dot{\theta}_2 \sin\theta_2 \\ -l_2^2 m_2 \dot{\theta}_1 \cos\theta_2 \sin\theta_2 & 0 \end{bmatrix}, \tag{10}$$

$$\mathbf{N}(\theta, \dot{\theta}) = \begin{bmatrix} b_1\dot{\theta}_1 \\ b_2\dot{\theta}_2 + g l_2 m_2 \sin\theta_2 \end{bmatrix} \tag{11}$$

The matrix $\mathbf{M}(\theta)$ contains the moments of inertia of the rotary pendulum for each position of θ_1 and θ_2. The matrix $\mathbf{C}(\theta, \dot{\theta})$ is called the *Coriolis matrix,* it includes the Coriolis and the centrifugal terms. The vector $\mathbf{N}(\theta, \dot{\theta})$ includes all of the gravity and damping effects. In the actual pendulum that you will use for this lab, b_2 and b_1 are nearly negligible.

The equations of motion for the rotary inverted pendulum are a set of two coupled second order nonlinear differential equations. One way to approach the stablization problem is by linearizing about the configuration of interest. The linearization about the pendulum up configuration ($\theta_2 = \pi$) is given by

$$\begin{bmatrix} \tau \\ 0 \end{bmatrix} = \begin{bmatrix} l_1^2(m_1 + m_2) & -l_1 l_2 m_2 \\ -l_1 l_2 m_2 & l_2^2 m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} b_1\dot{\theta}_1 \\ b_2\dot{\theta}_2 - g l_2 m_2 \theta_2 \end{bmatrix} \tag{12}$$

This set of equations can be re-written in state space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{13}$$

with the input u as our torque τ and with the state vector $\mathbf{x}$ given as:

$$\mathbf{x}^{\mathbf{T}} = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}. \tag{14}$$

Using state space notation equation (12) becomes:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm_2}{l_1 m_1} & \frac{-b_1}{l_1^2 m_1} & \frac{-b_2}{l_1 l_2 m_1} \\ 0 & \frac{g(m_1+m_2)}{l_2 m_1} & \frac{-b_1}{l_1 l_2 m_1} & \frac{-b_2(m_1+m_2)}{l_2^2 m_1 m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{l_1^2 m_1} \\ \frac{1}{l_1 l_2 m_1} \end{bmatrix} \tau \tag{15}$$

So, in the pendulum up case, we have

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm_2}{l_1 m_1} & \frac{-b_1}{l_1^2 m_1} & \frac{-b_2}{l_1 l_2 m_1} \\ 0 & \frac{g(m_1+m_2)}{l_2 m_1} & \frac{-b_1}{l_1 l_2 m_1} & \frac{-b_2(m_1+m_2)}{l_2^2 m_1 m_2} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{l_1^2 m_1} \\ \frac{1}{l_1 l_2 m_1} \end{bmatrix} \tag{16}$$

For your lab report you will need to work out the linearization about the pendulum–down case for the model–verification of the experiment (i.e. find the matrices A and B when θ_2 near zero).

2 Experiments

The experimental set-up consists of a 486 based PC which controls an output voltage to an amplifier which drives a DC motor. The motor is connected to the carriage of the rotary pendulum by a timing belt. The voltage output by the PC is the control input that you will use to stabilize the pendulum in its inverted position. Filtered signals for the angles and angular velocities are computed real-time by the PC. This allows a controller based on these quantities to be designed. Such a controller is called a *proportional-derivative* (PD) controller. A slightly more complicated problem is to design a controller based on position information alone. For the PD controller, the torque is given by:

$$\tau = -K_m(K_{p1}\theta_1 + K_{p2}\theta_2 + K_{d1}\dot{\theta}_1 + K_{d2}\dot{\theta}_2) \tag{17}$$

where the gains K_{p1}, K_{p2}, K_{d1}, and K_{d2} are input by the user and K_m is the overall gain from the computer to the output torque.

The experiment has three parts, the first of which you must perform during the first week of the lab. Without the first section results, you cannot do the rest of the experiment.

The first step is to make measurements of the physical parameters that appear in the model, including the value of K_m and all of the parameters in the state–space matrices $\mathbf{A}$ and $\mathbf{B}$. Most of the parameters can be measured directly, but m_1 and K_m take a little more work. One method of determining them is explained below. The second step is to verify the model by comparing the actual response of the experimental setup to the response predicted by the model. The final step is to design a controller, based on this model, to stabilize the inverted pendulum.

2.1 System Identification

Most of the parameters that appear in the state–space matrices **A** and **B** can be directly measured with a scale or a ruler. Some, such as the mass $m-1$, are not easily measured without dismantling the system, and still others, such as the damping constants b_1 and b_2, are impossible to measure directly, and must be determined experimentally. Determining system parameters from experimental data is called *system identification*.

Here, we demonstrate how to find the unknowns m_1, b_1, and K_m. A similar procedure may be used to find the damping constant b_2.

If the pendulum rod l_2 and mass m_2 are removed from the rotary pendulum system, the equations of motion (12) reduce to the following second–order differential equation:

$$m_1 l_1^2 \ddot{\theta}_1 + b_1 \dot{\theta}_1 = \tau \tag{18}$$

By choosing τ in the right way, we can turn equation (18) into a damped harmonic oscillator, which satisfies the differential equation:

$$m_1 l_1^2 \ddot{\theta}_1 + b_1 \dot{\theta}_1 = -K_m K_p \theta_1, \tag{19}$$

with K_m the overall gain from the computer output voltage to input torque and K_p the "spring constant" we apply with the computer. We can measure the value of l_1, and we specify the value K_1, but we cannot measure m_1, b_1, or K_m, so we must determine them experimentally. Recall that a standard second–order system of the form:

$$\ddot{x} + 2\zeta\dot{x} + {\omega_0}^2 x = 0 \tag{20}$$

has the general solution:

$$x(t) = e^{-\zeta t}(A \sin wt + B \cos wt) \tag{21}$$

where $\omega^2 = {\omega_0}^2 - \zeta^2$. Note that the second–order system (20) has natural frequency ω, and damping constant ζ.

By comparing our system (19) to the standard second-order system (20), we may relate the values m_1, b_1, and K_m to the frequency ω and damping constant ζ, which we may easily measure experimentally by taking step responses.

With the pendulum removed and the controller off, move the base angle about 45°. Turn on the rotary pendulum program's data capture routine (the TA will explain how to do this). Turn on the controller. Try this for several values of $K_p 1$, and select a couple that give good damped oscillations. Once you save your data, you can determine the frequencies and damping constants for the step responses (using for example the same computer program you used for the beam labs).

With three unknowns (m_1, b_1, K_m) and only two measurements (ω, ζ), the system is still underdetermined. In order to determine all three quantities, we must modify the experimental apparatus somehow and gather more data. One method is to add a known mass to the end of link 1. If we add a known mass m_0, the equation of motion becomes:

$$(m_1 + m_0)l_1^2\ddot{\theta}_1 + b_1\dot{\theta}_1 + K_m K_p \theta_1 = 0 \tag{22}$$

By measuring the natural frequency ω' and damping constant ζ' of the modified system, we may now solve for all three unknowns. Note that because we now have 4 measurements (ω, ζ, ω', ζ') to determine the 3 unknowns, the solution is not unique. Try to come up with a solution that is the least sensitive to experimental errors. (Between ω and ζ, which is the more repeatable measurement?). Also note that even though changing K_p1 changes the natural frequency of the system, it is not possible to solve for all three parameters by using two different values of K_p1 without changing the mass. (Why?)

The number you obtain for b_1 might not be very repeatable (it is very small) but the value of m_1 should be pretty good.

2.2 Verification of System Model

After working out the values of the system parameters, and the linearization about the down position, attach the pendulum to the system. You should now be able to predict the behavior of the system in response to a particular initial condition. You will have to use the capture routine of the rotary pendulum program to collect data from the experiment. Matlab provides a simple way to solve systems of ordinary differential equations. You should be able to predict the experimental behavior using Matlab simulations.

Try a couple of initial conditions and make sure that your model predicts the system behavior. Also try turning on the controller with a nonzero value of K_p1, to provide a restoring force as in the previous section, and compare this response to the response predicted by your model.

2.3 Applying a Controller to the System

Using the model developed in the previous sections, a controller can now be designed which stablizes the pendulum in the inverted position and maintains the base joint about a zero position.

Recall that solutions of the matrix equation

$$\dot{x} = \mathbf{A}\mathbf{x} \tag{23}$$

are made up of sums of exponentials of the form $\exp st$ (with vector coefficients), where s is an eigenvalue of the matrix $\mathbf{A}$. Solutions are *stable* if they decay in time, so solutions of (23) are stable if all of the eigenvalues of $\mathbf{A}$ have negative real parts.

The open–loop (no controller) eigenvalues of the linearized pendulum–up system, given by equation (13), are eigenvalues of $\mathbf{A}$, or roots of the characteristic polynomial

$$D_{ol}(s) = \det(s\mathbf{I} - \mathbf{A}), \tag{24}$$

where $\mathbf{I}$ is the identity matrix. For the rotary inverted pendulum, if the damping coefficients b_1 and b_2 are assumed to be zero, the roots of $D_{ol}(s)$ are located at:

$$s = 0, 0, -\sqrt{\frac{g(m_1 + m_2)}{l_2 m_1}}, +\sqrt{\frac{g(m_1 + m_2)}{l_2 m_1}}. \tag{25}$$

This is clearly an unstable system since one of the eigenvalues has a non-negative real part. We can stabilize the system by adding feedback control. Specifically, we set the control input u in equation (6) to

$$u = -\mathbf{K}\mathbf{x}, \tag{26}$$

where $\mathbf{K}$ is a matrix of gain given by:

$$\mathbf{K} = K_m \left[\begin{array}{cccc} K_{p1} & K_{p2} & K_{d1} & K_{d2} \end{array} \right]. \tag{27}$$

A controller such as this, which uses all the information contained in the state vector $\mathbf{x}$, is said to use full state feedback. Substituting (26) into (13) gives the closed–loop (controller on) system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}, \tag{28}$$

whose eigenvalues are roots of the closed–loop characteristic polynomial:

$$D_{cl}(s) = \det\left(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})\right). \tag{29}$$

By choosing arbitrarily different values for the gains in matrix $\mathbf{K}$, we can adjust the eigenvalues of the closed-loop system. In fact, if the system is *controllable* (which our system is), then we may choose the gains $\mathbf{K}$ to locate the closed-loop eigenvalues *wherever we choose* (the only restriction is that complex eigenvalues must occur in conjugate pairs).

In other words, controllability means that we can choose the characteristic polynomial of the closed-loop system arbitrarily, by choosing the elements of $\mathbf{K}$. This method of designing controllers is called *pole placement*.

Design several different controllers for the system by choosing different characteristic polynomials for the closed loop system and calculating the required entries in $\mathbf{K}$. Remember that there is a limit to how large the input torque can be (saturation) and to how quickly it can change (bandwidth limitation); i.e., if the roots of the characteristic polynomial are too far in the left half plane, the motor will not be able to apply the input torque τ that your controller calculates.

Show the results of your controller design to the teaching assistant before you attempt to use them on the real system. Now test out your controller designs (with the help of the TA).

3 Report

3.1 Results

The results of the laboratory report should contain the following information.

- Measurements of the parameters which appear in the matrices **A** and **B**, as well as the value of K_m. Include plots comparing the step response of the actual system to the response of the second–order model that approximates it, and include a table containing each value you determined for m_1, b_1, b_2, and K_m. You should also include uncertainty values for each of the system parameters.
- Comparisons of your measured model to the actual model. These show how well your model predicts the behavior of the system.
- Controllers designed via pole placement which stabilize the system. This should include some criteria for your choice of eigenvalue locations, as well as some measure of the performance of the system.

3.2 Discussion

The discussion section should include (but not be limited to) answers to the following questions.

- What are the major discrepancies between the theory and the experiments?
- What is the effect of the nonlinearities associated with the rotary inverted pendulum on the results of your analysis?
- Is it always possible to apply the linearized model instead of the nonlinear model?

4 Advanced Experiment (optional)

Design a controller for the double inverted pendulum. This will involve working out the state space equations of motion for the double inverted pendulum.

5 References

For an explanation of Lagrange's Equations without derivation see [1], for a more advanced treatment see [2]. [3] uses the linear inverted pendulum for many examples, beginning with a development of the equations of motion, and ending with a design for a linear controller which tracks a desired trajectory. [3] also provides information on basic control theory including the state space formulation.

[1] S. S. Rao. *Mechanical Vibrations.* Addison-Wesley, 1990.

[2] R. M. Rosenberg. *Analytical Dynamics of Discrete Systems.* Plenum Press, 1977.

[3] K. Ogata. *Modern Control Engineering.* Prentice Hall, 1990.

Prelab

- Derivation of equations for finding the parameters K_m, m_1, b_1, and b_2 in terms of experimentally measured quantities.
- Derivation of the linearized–pendulum down equations of motion (in state space form).