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SUJET DE LA THÈSE :

EXPLOITATION DES SYMÉTRIES DYNAMIQUES POUR LA RÉSOLUTION DES PROBLÈMES SAT

SOUTENUE LE :

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TODO AND REMOVE

SAT Theory NP complete Cite [COOK 71]

Used In

- Formal methods Hardware model checking; Software model checking; Termination analysis of term-rewrite systems ; Test pattern generation (testing of software hardware) ; etc
- AI planning, game n queen sudou
- Bio info Haplotype inference ; Pedigree checking ; Analysis of Genetic; Regulatory Networks; etc.
- Design Automation: Equivalence checking; Delay computation; Fault diagnosis; Noise analysis ; etc
- Security: Cryptanalysis ; Inversion attacks on hash functions; etc

Where can found

- Computationally hard problems: Graph coloring, Traveling salesperson
- Mathematical: van der Waerden numbers ; Quasigroup open problems
- Core engine for other solvers :0-1 ILP/Pseudo Boolean ; QBF ; #SAT ; SMT ; MAXSAT;
- Integrated into theorem provers :HOL ; Isabelle ;
- Integrated into widely used software: Eclipse provisioning system based on a Pseudo Boolean solver Suse 10.1 dependency manager based on a custom SAT solver

Interest in BMC [Biere 99]

SAT solver Def

Input: Can encode any Boolean formula into Normal Form

Classical simplification: - Resolution - Subsumption

More complicated Hidden tautology BVA ... Loo Heule simplification slides

Algo CDCL

INTRODUCTION

Nowadays, computers are powerfull and used in many applications in different domains. On of this domain is critical application, these applications are running in planes, cars and some software must be secure. Proving the correctness of these software leads to combinatorial explosion.

Over the years, computers scientist have developed many techniques to solve this kind of problems like *constraint programming* (CP) [7], *Propositional Satisfiability* (SAT) [2], *Solver Modulo Theory* (SMT) [1].

In this thesis, we focus on *propositional satisfiability* that is used in many applications in different domains: *formal methods*: hardware model checking, software model checking, etc; *artificial intelligence*: planning; *games resolution*: sudoku, n-queens, *Bioinformatics* : Haplotype inference, *design automation* : equivalence checking

At its most basic, symmetry is some transformations of an object that leaves this object unchanged. In the case of satisfiability problems it maps a solution of a problem to another solution of the problem.

PRELIMINARIES

3.1 SAT basics

Satisfiability problem

A *Boolean variable*, or *propositional variable*, is a variable that has two possible values : true or false (noted \top or \perp , respectively). A *literal* l is a propositional variable or its negation. For a given variable x , the positive literal is represented by x and the negative one by $\neg x$. A *clause* ω is a finite disjunction of literals represented equivalently by $\omega = \bigvee_{i=1}^k l_i$ or the set of its literals $\omega = \{l_i\}_{i \in [1,k]}$. A clause with a single literal is called *unit clause*. A *conjunctive normal form (CNF) formula* φ is a finite conjunction of clauses. A CNF can be either noted $\varphi = \bigwedge_{i=1}^k \omega_i$ or $\varphi = \{\omega_i\}_{i \in [1,k]}$. We denote \mathcal{V}_φ (\mathcal{L}_φ) the set of variables (literals) used in φ (the index in \mathcal{V}_φ and \mathcal{L}_φ is usually omitted when clear from context).

For a given formula φ , an *assignment* of the variables of φ is a function $\alpha : \mathcal{V} \mapsto \{\top, \perp\}$. As usual, α is *total*, or *complete*, when all elements of \mathcal{V} have an image by α , otherwise it is *partial*. By abuse of notation, an assignment is often represented by the set of its true literals. The set of all (possibly partial) assignments of \mathcal{V} is noted $\text{Ass}(\mathcal{V})$.

The assignment α *satisfies* the clause ω , denoted $\alpha \models \omega$, if $\alpha \cap \omega \neq \emptyset$. Similarly, the assignment α satisfies the propositional formula φ , denoted $\alpha \models \varphi$, if α satisfies all the clauses of φ . Note that a formula may be satisfied by a partial assignment. A formula is said to be *satisfiable* (SAT) if there is at least one assignment that satisfies it; otherwise the formula is *unsatisfiable* (UNSAT).

Algorithm

The state of the art sound and complete algorithm to resolve a SAT problem is Conflict-Driven Clause learning (CDCL) algorithm 1. This algorithm is inspired by Davis Putnam Logemann Loveland [3].

The CDCL algorithm walks a binary search tree. It first applies unit propagation to the formula φ for the current assignment α (line 5). A conflict at level 0 indicates that the formula is not satisfiable, and the algorithm reports it (lines 7-8). If a conflict is detected, it is analyzed, which provides a *conflict clause* explaining the reason for the conflict (line 9). The analysis is completed by the computation of a backjump point to which the algorithm backtracks (line 10). This clause is learnt (line 11), as it does not change the satisfiability of φ , and avoids encountering a conflict with the same causes in the future. Finally, if no conflict appears, the algorithm chooses a new decision literal (line 13-14). The above steps are repeated until the satisfiability status of the formula is determined.

```

1 function CDCL( $\varphi$ : CNF formula)
2   returns  $\top$  if  $\varphi$  is SAT and  $\perp$  otherwise
3    $dl \leftarrow 0$ ;                                     // Current decision level
4   while not all variables are assigned do
5      $isConflict \leftarrow \text{unitPropagation}()$ ;
6     if  $isConflict$  then
7       if  $dl = 0$  then
8         return  $\perp$ ;                                     //  $\varphi$  is UNSAT
9        $\omega \leftarrow \text{analyzeConflict}()$ ;
10       $dl \leftarrow \text{backjumpAndRestartPolicies}()$ ;
11       $\varphi \leftarrow \varphi \cup \{\omega\}$ ;
12    else
13       $\text{assignDecisionLiteral}()$ ;
14       $dl \leftarrow dl + 1$ ;
15  return  $\top$ ;                                     //  $\varphi$  is SAT

```

Algorithm 1: The CDCL algorithm.

3.2 Groups basics

Groups

Group

A *group* is a structure $\langle G, * \rangle$, where G is a non empty set and $*$ a binary operation such the following axioms are satisfied:

- *associativity*: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
- *closure*: $\forall a, b \in G, a * b \in G$.
- *identity*: $\forall a \in G, \exists e$ such that $a * e = e * a = a$
- *inverse*: $\forall a \in G, \exists b \in G$, commonly denoted a^{-1} such that $a * a^{-1} = e$

Hakan: TODO ? commutativity implies abelian group Moreover, the last definition leads to important properties such that:

- **uniqueness of the identity element**

Proof. Let $\langle G, * \rangle$ a group with two identity elements e and f then $e = e * f = f$ □

- **uniqueness of the inverse element**

Proof. Let suppose that an element x_1 has two inverses, denoted b and c in group $\langle G, * \rangle$

$$\begin{aligned}
 b &= b * e \\
 &= b * (a * c) \quad c \text{ is an inverse of } a, \text{ so } e = a * c \\
 &= (b * a) * c \quad \text{associativity rule} \\
 &= e * c \quad b \text{ is an inverse of } a, \text{ so } e = a * b \\
 &= c \quad \text{identity rule}
 \end{aligned}
 \quad \square$$

The structure $\langle G, * \rangle$ is denoted as G when clear from context that G is a group with a binary operation.

Given a group G , a *subgroup* is a non empty subset of G which is also a group with the same binary operation. If H is a subgroup of G , we denote as $H \leq G$. So, $\{e\}$ is a *trivial* subgroup. All other subgroups are *nontrivial* and G itself is an *improper* subgroup. All other subgroups are *proper*.

Generators of a group

If every elements in a group G can be expressed as a linear combination of a set of group of elements $S = \{g_1, g_2, \dots, g_n\}$ then we say G is generated by the S . we denote this as $G = \langle S \rangle = \langle \{g_1, g_2, \dots, g_n\} \rangle$

Permutation groups

A *permutation* is a bijection from a set X to itself. Permutations are generally written in *cycle notation* : given a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $g = (x_1 \ x_2 \ x_3) (x_5 \ x_6)$ is a permutation that maps x_1 to x_2 , x_2 to x_3 , x_3 to x_1 , x_5 to x_6 and x_6 to x_5 . All other elements maps themselves. We say *support* of the permutation g noted $supp(g)$ the elements that not mapped to themselves, $supp(g) = \{x \in X \mid g(x) \neq x\}$.

The set of permutations of a given set X form a group G , with the composition operation (\circ) and called *permutation group*. The set of **all** permutations of a set X is the *symmetric group* of X and noted $\mathfrak{S}(X)$. A set of permutations P is a set of *generators* of a group G if each permutation of G can be expressed as a composition of permutations in P .

A permutation group G induces a *equivalence relation* on the set of element X being permuted. Two elements $x_1, x_2 \in X$ are equivalent if there exists a permutation $g \in G$ such that $gx_1 = x_2$. Then equivalence relation partitions X into *equivalence classes* referred to as the *orbits* of X under G . **Hakan:** Add an example

Hakan: Orbit of the permutation

SYMMETRY IN SAT

This chapter presents the computation and usage of symmetry in SAT problems.

4.1 Symmetry detection in SAT

For the detection of symmetries in SAT, we first introduce the graph automorphism notion. Given a colored graph $G = (V, E, \gamma)$, with vertex set $V \in [1, n]$, edge set E and γ a function that apply a mapping $: V \rightarrow C$ where C is a set of *colors*. An automorphism of G is a permutation from its vertices $g : V \rightarrow V$ such that:

- $\forall (u, v) \in E \implies (g(u), g(v)) \in E$
- $\forall v \in V, \gamma(v) = \gamma(g(v))$

The graph automorphism problem is to find if a given graph has a non trivial permutation group. The computational complexity of this algorithm is conjectured to be strictly between P and NP. Several tools exist to tackle this problem like saucy3 [5], bliss [4], nauty [6], etc.

Hakan: *syntax semantic symmetries*

To find symmetries in SAT problem, the formula is transformed into colored graph and an automorphism tool is applied onto. Specifically, given a formula φ with m clauses and l literals over n variables, the graph is constructed as follows:

- *clause nodes*: represent each of the m clauses by a node with color 0;
- *literals nodes*: represent each of the l literals by a node with color 1;

- *clauses edges*: connect each clause node to the node of the literals that appear in clause
- *boolean consistency edges*: connect each pair of literals that correspond to the same variables.

Hakan: optimisation du graph

Hakan: example of graph

Symmetry Group of a Formula

The group of permutations of \mathcal{V} (i.e. bijections from \mathcal{V} to \mathcal{V}) is noted $\mathfrak{S}(\mathcal{V})$. The group $\mathfrak{S}(\mathcal{V})$ naturally acts on the set of literals: for $g \in \mathfrak{S}(\mathcal{V})$ and a literal $\ell \in \mathcal{L}$, $g.\ell = g(\ell)$ if ℓ is a positive literal, $g.\ell = \neg g(\neg\ell)$ if ℓ is a negative literal. The group $\mathfrak{S}(\mathcal{V})$ also acts on (partial) assignments of \mathcal{V} as follows: for $g \in \mathfrak{S}(\mathcal{V})$, $\alpha \in \text{Ass}(\mathcal{V})$, $g.\alpha = \{g.\ell \mid \ell \in \alpha\}$. Let φ be a formula, and $g \in \mathfrak{S}(\mathcal{V})$. We say that $g \in \mathfrak{S}(\mathcal{V})$ is a symmetry of φ if for every *complete* assignment α , $\alpha \models \varphi$ if and only if $g.\alpha \models \varphi$. The set of symmetries of φ is noted $S(\varphi) \subseteq \mathfrak{S}(\mathcal{V})$.

Let G be a subgroup of $\mathfrak{S}(\mathcal{V})$. The *orbit of α under G* (or simply the *orbit of α* when G is clear from the context) is the set $[\alpha]_G = \{g.\alpha \mid g \in G\}$. The lexicographic leader (*lex-leader* for short) of an orbit $[\alpha]_G$ is defined by $\min_{<}([\alpha]_G)$. This *lex-leader* is unique because the lexicographic order is a total order.

4.2 Usage of symmetries

Since the set of permutations is obtained from the formula

We say that g is a symmetry of φ if $\alpha \models \varphi$ and $g.\alpha \models \varphi$

The optimal approach to solve a symmetric SAT problem would be to explore only one assignment per orbit (for instance each lex-leader). However, finding the lex-leader of an orbit is computationally hard. Hakan: CITATION

Symmetry breaking aims at eliminating symmetry, either by *statically* posting symmetry breaking constraints that invalidate symmetric assignments, or by altering the search space *dynamically* to avoid symmetric search paths.

Symmetry breaking aims to eliminate symmetries. Two kind of practice exists: the first one is *static symmetry breaking*: some constraint

Lexicographic leader

Compact CNF encoding of Lexicographic leader

Some special groups

CHAPTER 5

CONCLUSION

This is conclusion.

BIBLIOGRAPHY

- [1] C. Barrett and C. Tinelli. Satisfiability modulo theories. In *Handbook of Model Checking*, pages 305–343. Springer, 2018.
- [2] A. Biere, M. Heule, and H. van Maaren. *Handbook of satisfiability*, volume 185. IOS press, 2009.
- [3] M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397, July 1962.
- [4] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In D. Applegate, G. S. Brodal, D. Panario, and R. Sedgewick, editors, *Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics*, pages 135–149. SIAM, 2007.
- [5] H. Katebi, K. Sakallah, and I. Markov. Symmetry and satisfiability: An update. *Theory and Applications of Satisfiability Testing–SAT 2010*, pages 113–127, 2010.
- [6] B. D. McKay. nauty user’s guide (version 2.2). Technical report, Technical Report TR-CS-9002, Australian National University, 2003.
- [7] F. Rossi, P. Van Beek, and T. Walsh. *Handbook of constraint programming*. Elsevier, 2006.