

Exploitation of dynamic symmetries for solving SAT problems

Doctorat de Sorbonne Université

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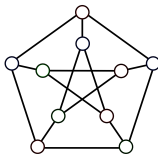
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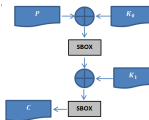


Motivation

Graph coloring



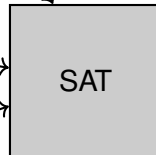
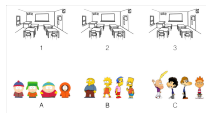
Cryptanalysis



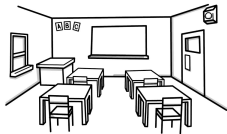
Hardware model checking



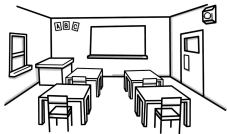
Planning



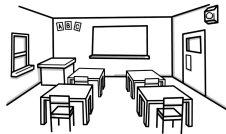
SAT: an example (1/2)



1



2



3



A



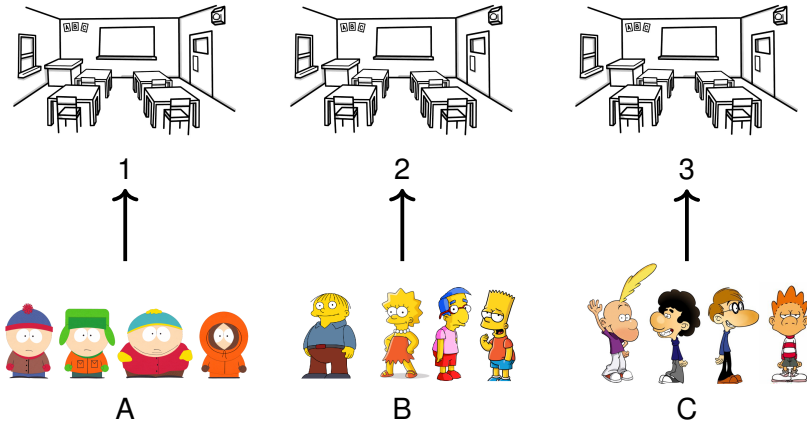
B



C

Is it possible to attribute each group to a unique classroom?

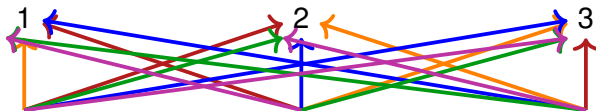
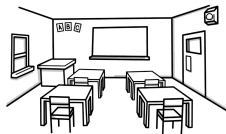
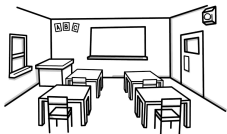
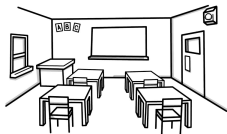
SAT: an example (1/2)



Is it possible to attribute each group to a unique classroom?

YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$

SAT: an example (1/2)



A



B



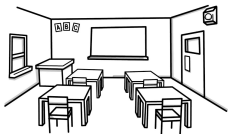
C

Is it possible to attribute each group to a unique classroom?

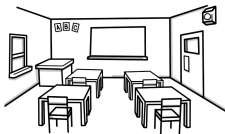
YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$

Many solutions $\alpha = (A, 2), (B, 3), (C, 1); \dots$

SAT: an example (2/2)



1



2



A



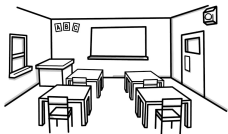
B



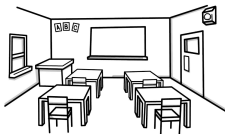
C

Is it possible to attribute each group to a unique classroom?

SAT: an example (2/2)



1



2



A



B



C

Is it possible to attribute each group to a unique classroom?

No! UNSAT

Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

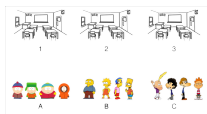
$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$

$(x_1 \vee x_2 \vee x_3) \wedge$
 $(x_4 \vee x_5 \vee x_6) \wedge$
 $(x_7 \vee x_8 \vee x_9) \wedge$

$(\neg x_1 \vee \neg x_4) \wedge$
 $(\neg x_1 \vee \neg x_7) \wedge$
 $(\neg x_4 \vee \neg x_7) \wedge$

$(\neg x_2 \vee \neg x_5) \wedge$
 $(\neg x_2 \vee \neg x_8) \wedge$
 $(\neg x_5 \vee \neg x_8) \wedge$

$(\neg x_3 \vee \neg x_6) \wedge$
 $(\neg x_3 \vee \neg x_9) \wedge$
 $(\neg x_6 \vee \neg x_9)$



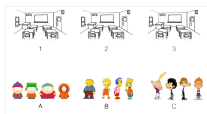
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



$(x_1 \vee x_2 \vee x_3) \wedge$
 $(x_4 \vee x_5 \vee x_6) \wedge$
 $(x_7 \vee x_8 \vee x_9) \wedge$

Clause

$(\neg x_1 \vee \neg x_4) \wedge$
 $(\neg x_1 \vee \neg x_7) \wedge$
 $(\neg x_4 \vee \neg x_7) \wedge$

$(\neg x_2 \vee \neg x_5) \wedge$
 $(\neg x_2 \vee \neg x_8) \wedge$
 $(\neg x_5 \vee \neg x_8) \wedge$

$(\neg x_3 \vee \neg x_6) \wedge$
 $(\neg x_3 \vee \neg x_9) \wedge$
 $(\neg x_6 \vee \neg x_9)$

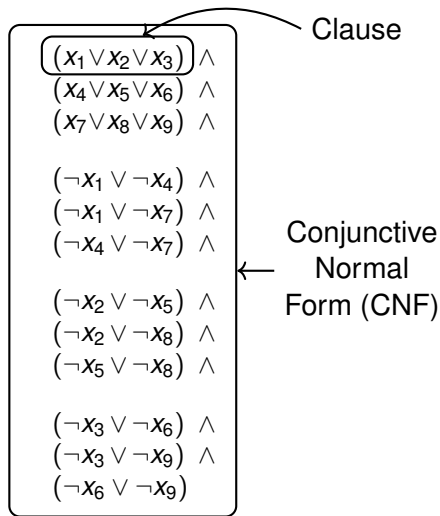
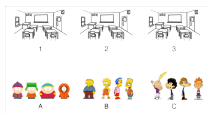
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



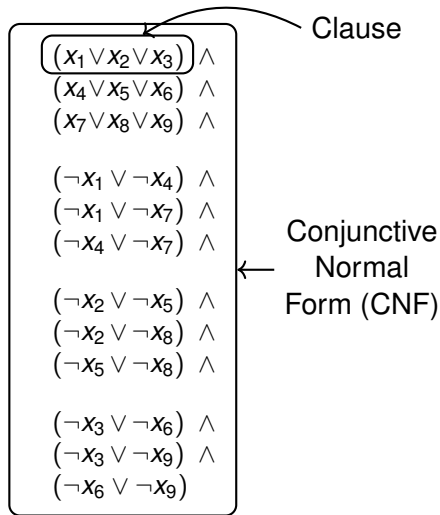
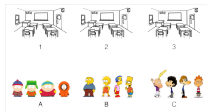
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

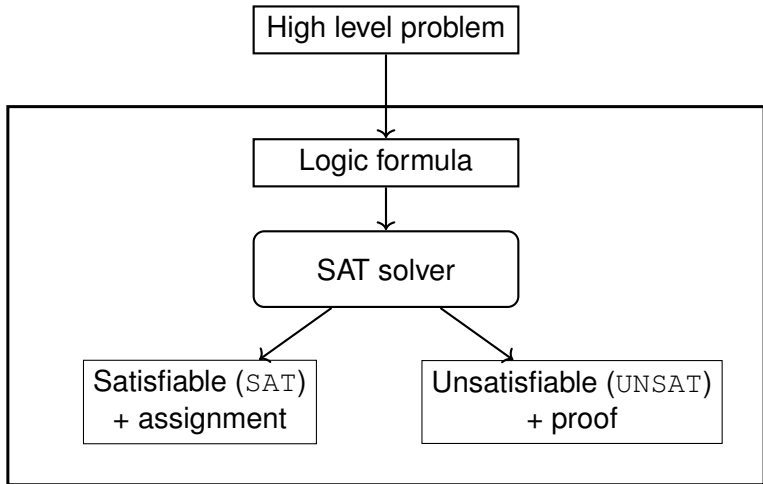
$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



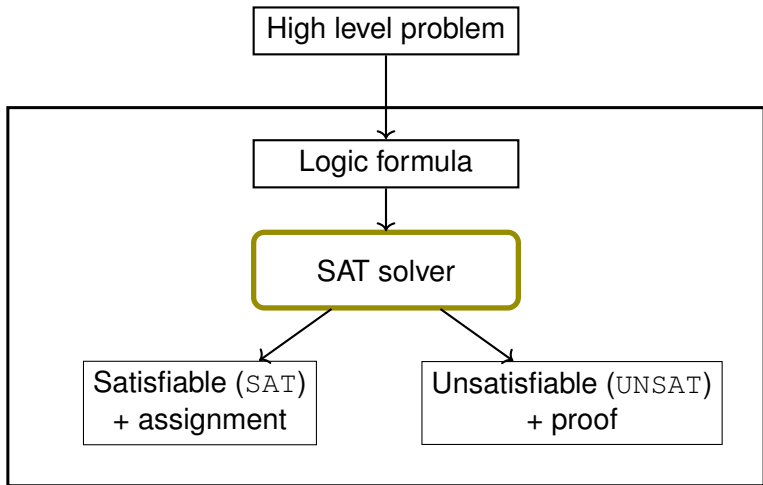
Clause represented as a set:

$$(x_1 \vee x_2 \vee x_3) \rightarrow \{x_1, x_2, x_3\}$$

SAT design



SAT design



SAT Solving

Solving SAT formula is known to be **NP-complete** [Coo71]

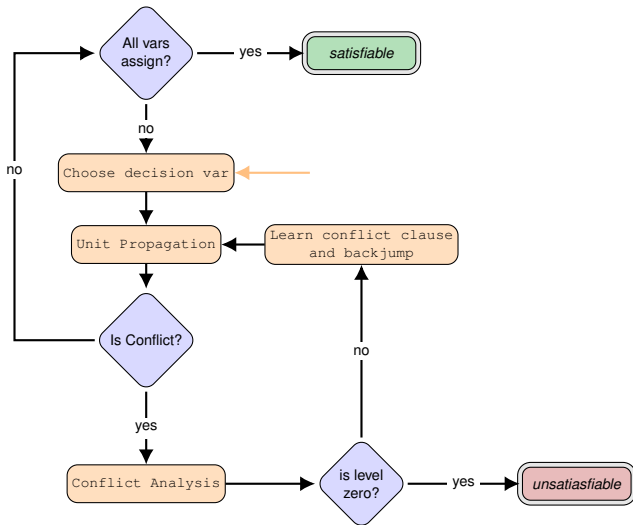
Good performance in practice:

- Handle large problem (million variables and clauses)
- International SAT competition each year on academic and industrial problems

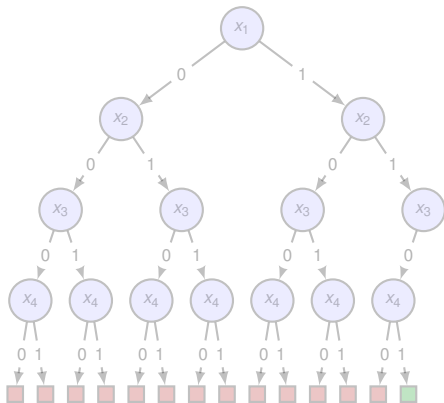
Enumerative algorithms:

- Davis, Putnam, Logemann, and Loveland (DPLL) [DLL62]
 - Boolean Constraint Propagation (BCP)
- Conflict Driven Clause Learning (CDCL) [MSS99]
 - Derived from DPLL
 - Clause learning

CDCL in detail



CDCL in action



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

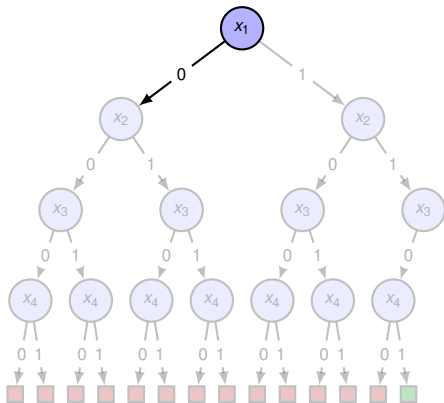
$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{\mathbf{x}_1, \neg x_4\}$$

$$\omega_3 = \{\mathbf{x}_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

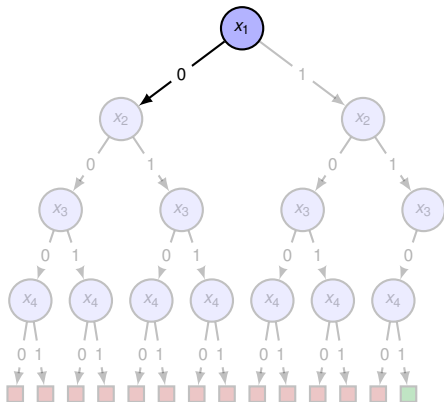
$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg x_1\}$$

CDCL in action

Conflict Analysis



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{\mathbf{x}_1, \neg \mathbf{x}_4\}$$

$$\omega_3 = \{\mathbf{x}_1, \mathbf{x}_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

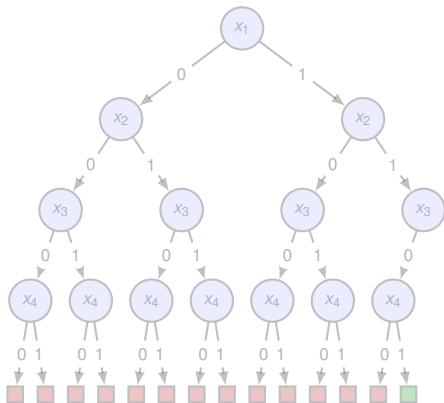
$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg \mathbf{x}_1\}$$

CDCL in action

Learn conflict clause
and backjump



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

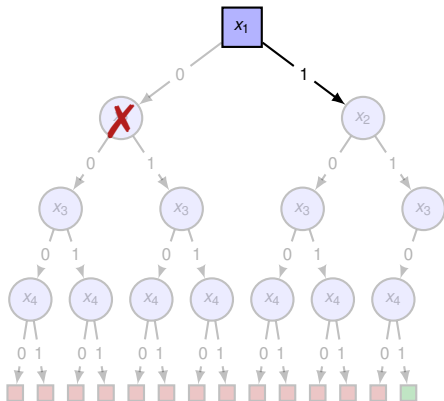
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{\}$$

CDCL in action

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

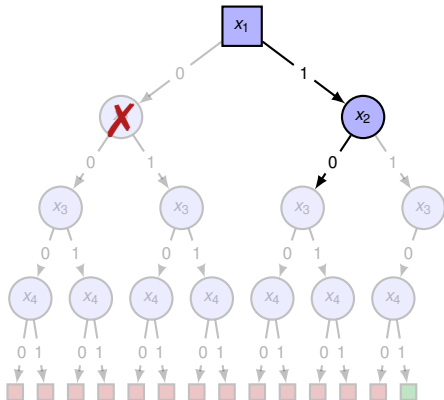
$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

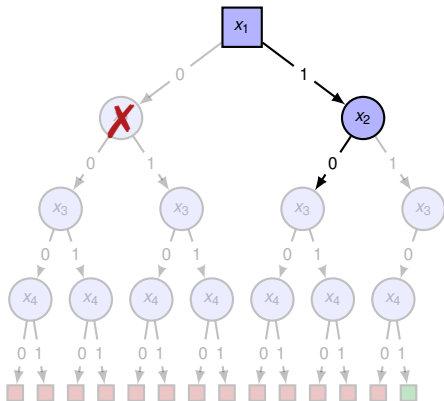
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

Conflict Analysis



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

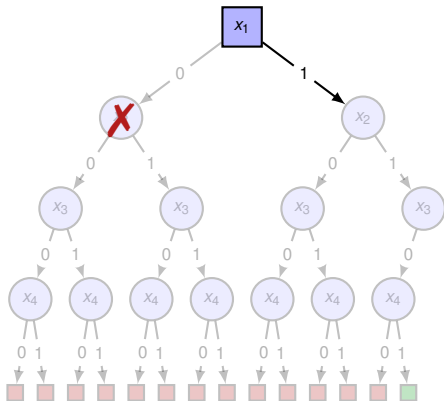
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

Learn conflict clause
and backjump



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

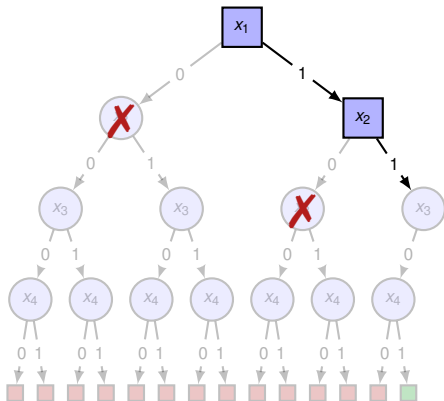
$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1\}$$

CDCL in action

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

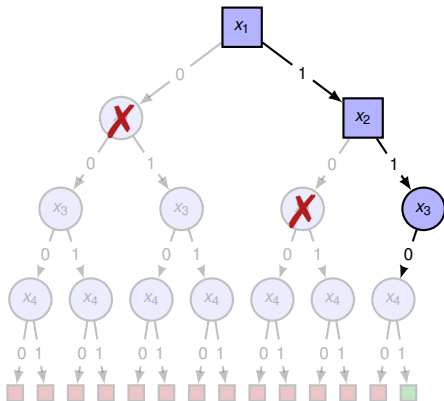
$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1, x_2\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

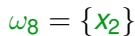
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

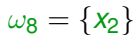
$$\alpha = \{x_1, x_2, \neg x_3\}$$

Unit Propagation



9/35

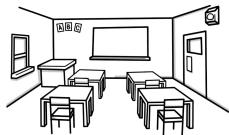
satisfiable



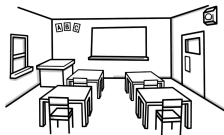
9/35

SAT and symmetries

Presence of symmetries



1



2



A

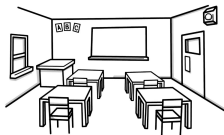
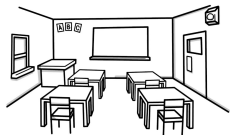


B



C

Presence of symmetries



A

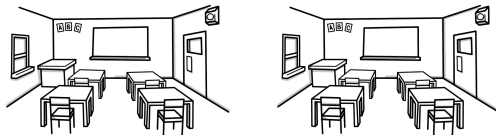


B



C

Presence of symmetries



1 ← → 2



A

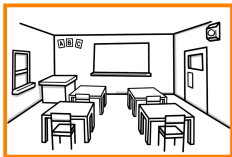


B

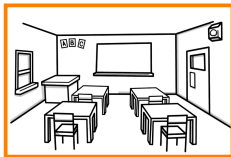


C

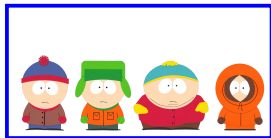
Presence of symmetries



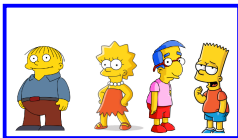
1



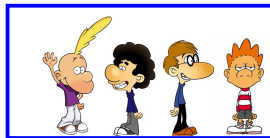
2



A

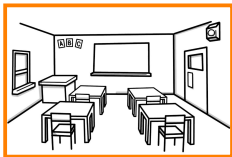


B

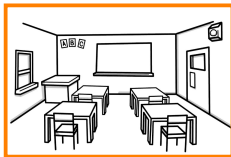


C

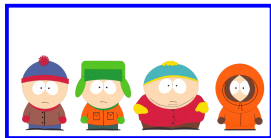
Presence of symmetries



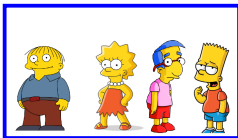
2



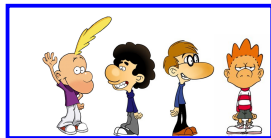
1



A



C

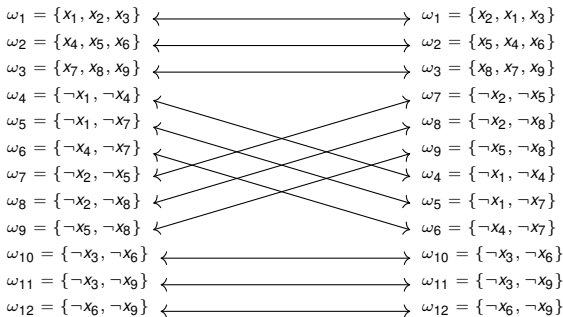


B

Symmetry (Syntactic)

A symmetry (permutation) g is a bijective function (on variables) that leaves the formula invariant

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_2 & x_1 & x_3 & x_5 & x_4 & x_6 & x_8 & x_7 & x_9 \end{pmatrix} \rightarrow (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$



Equi-satisfiability:

$$\alpha \models \varphi \Leftrightarrow g.\alpha \models \varphi$$

Computing symmetries of a SAT problem

CNF formula

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9) \\ & \wedge (\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7) \\ & \wedge (\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8) \\ & \wedge (\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9) \end{aligned}$$

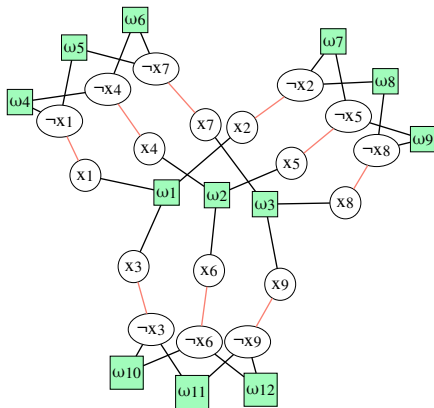
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colored graph



Computing symmetries of a SAT problem

CNF formula

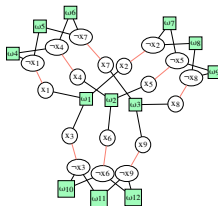


colored graph



graph automorphism

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9) \\ & \wedge (\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7) \\ & \wedge (\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8) \\ & \wedge (\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9) \end{aligned}$$



(bliss, saucy, ...)

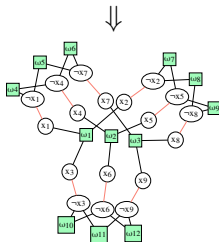
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colored graph



⇓
graph automorphism

⇓
(bliss, saucy, ...)

⇓
set of symmetries

⇓

$$\begin{aligned} g_1 &= (x_2 \ x_3)(x_5 \ x_6)(x_8 \ x_9) \\ g_2 &= (x_4 \ x_7)(x_5 \ x_8)(x_6 \ x_9) \\ g_3 &= (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8) \\ g_4 &= (x_1 \ x_4)(x_2 \ x_5)(x_3 \ x_6) \end{aligned}$$

The set of symmetries of a formula is a group noted G

Exploitation of symmetries

Static symmetry breaking

Orbit

Orbit of an assignment α for a group G :

$$G.\alpha = \{g.\alpha \mid g \in G\}$$

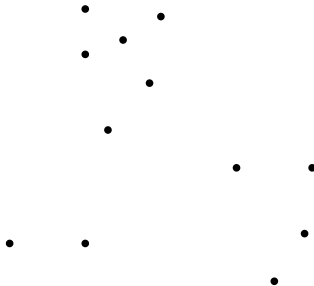
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Example:

- full assignment



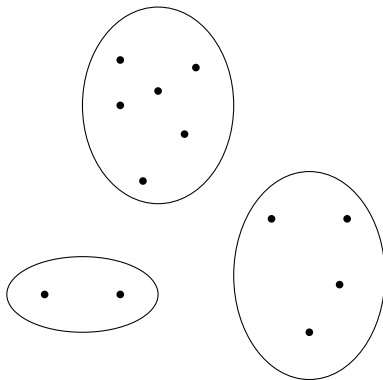
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Example:

- full assignment
- orbit



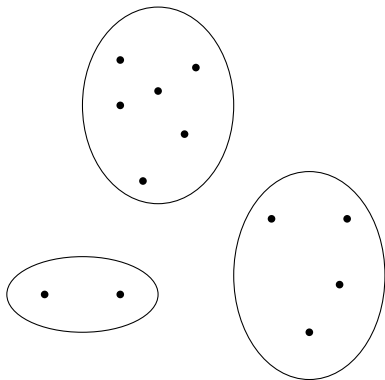
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Example:

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Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

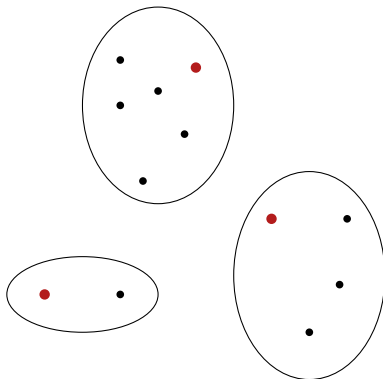
Orbit

Orbit of an assignment α for a group G :

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Example:

- full assignment
- orbit
- representative



Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

Compare assignment

Define an ordering relation to compare assignment (\prec)

- Total ordering on variables
- Minimum value: $F < T$ or $T < F$

Allow only minimal (maximal) value

Forbids other assignment in each orbit

→ Add Symmetry breaking predicates (SBP)

Example:

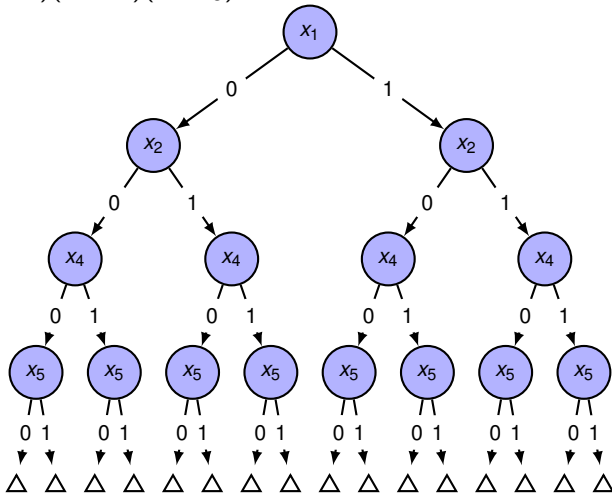
$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; F < T$$

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	F	T	F	F	F	F	F	F
				\prec				
$g.\alpha$	F	T	F	F	F	F	F	F

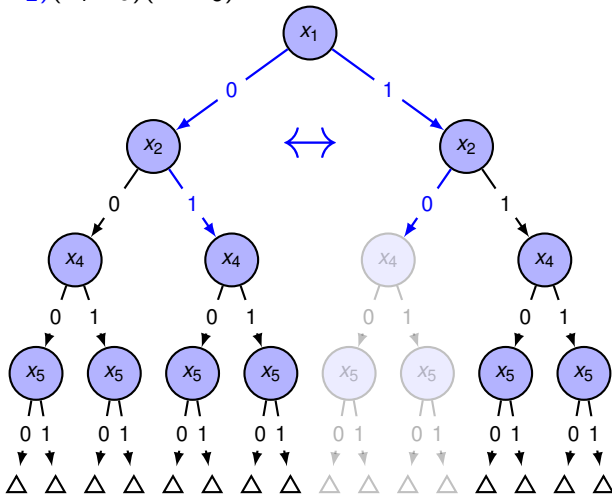
Using symmetries to prune the search space

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

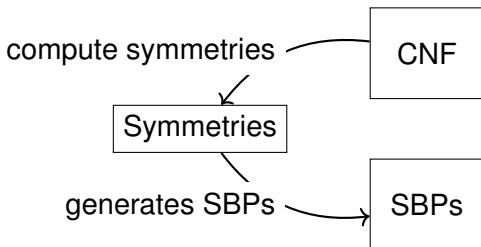


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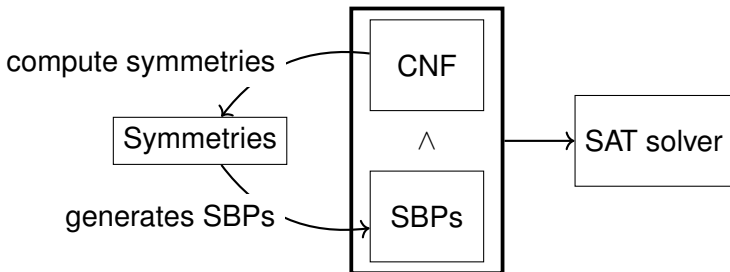
Static symmetry breaking



State-of-the-art approaches:

- Shatter [ASM06]
- BreakID [DBBD16]
- ...

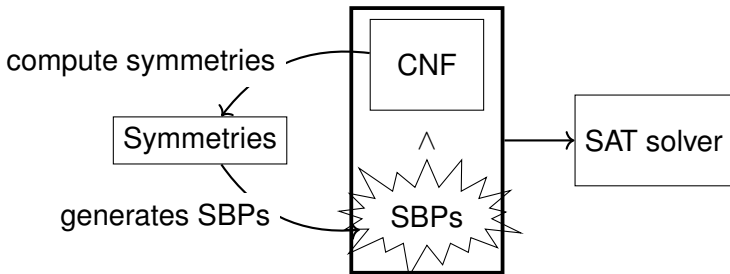
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Static symmetry breaking



State-of-the-art approaches:

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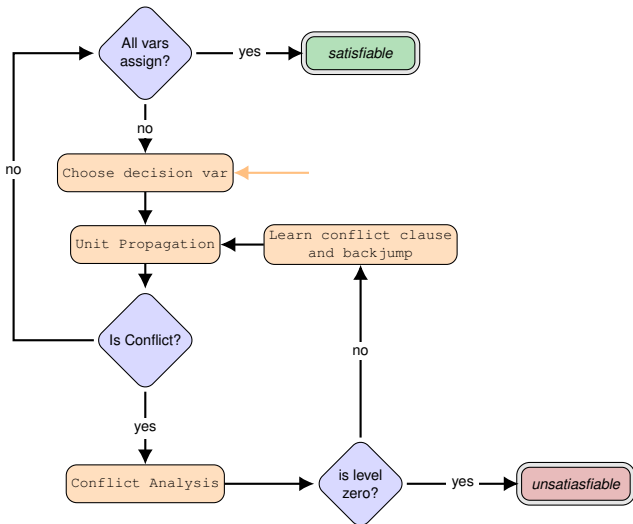
The solver can "explode" instead of being helped

CDCL[sym] Introducing Effective Symmetry Breaking in SAT Solving

TACAS'18 [MBCK18]

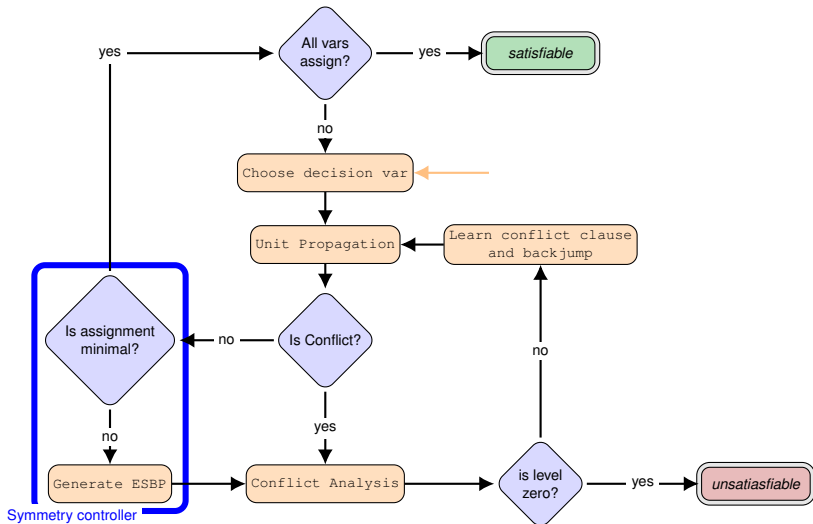
CDCL[Sym]

Compute and inject SBP **opportunistically**, during the solving



CDCL[Sym]

Compute and inject SBP **opportunistically**, during the solving



Symmetry status

- reducer: $g.\alpha \prec \alpha$
- inactive: $\alpha \prec g.\alpha$
- active: *not enough information*

Efficient implementation of symmetry status

Keep track the smallest unassigned variable x :

- ① $\alpha(g.x) \leq \alpha(x)$, then g is `reducer` \Rightarrow Effective SBP (ESBP)
- ② $\alpha(x) \leq \alpha(g.x)$, then g is `inactive` $\Rightarrow g$ cannot reduce α
- ③ $\alpha(g.x)$ or $\alpha(x)$ is unassigned then g is `active`

Update whenever variables are assigned / unassigned

CDCL[Sym] Implementation

- Packaged as a library **cosy**¹
 - Lightweight
 - Fast update and low memory
 - Follows symmetry status
-
- Works with any enumerative SAT solver
 - Can be integrated easily
- e.g. +3% LOC on MiniSAT.

¹<https://github.com/lip6/cosy>

Experiments

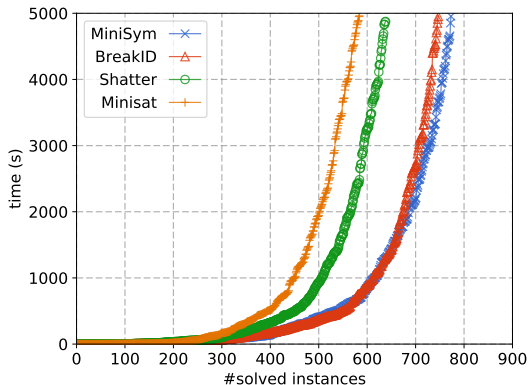
Benchmark:

- from SAT contests 2012 – 2017
- filter: `bliss` finds symmetries in 1000s
- 36 % of instances, 1 350/3 700

Setup:

- four tools
 - MiniSat (no symmetry, baseline)
 - MiniSat + BreakID (SOTA SAT solver using symmetries)
 - MiniSat + Shatter (SOTA SAT solver using symmetries)
 - **MiniSym** = MiniSat + CDCL[Sym] (our approach)
- 5000s timeout, 8GB memory
- includes time to compute symmetries (except for MiniSat)

Experimental results



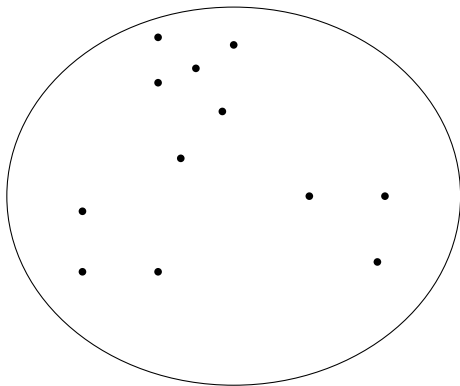
Solver	PAR-2	ALL	SAT	UNSAT
MiniSAT	2243h	586	325	261
Shatter	2088h	640	316	324
BreakID	1790h	749	334	415
MiniSym	1735h	775	336	439

Exploitation of symmetries

Dynamic symmetry breaking

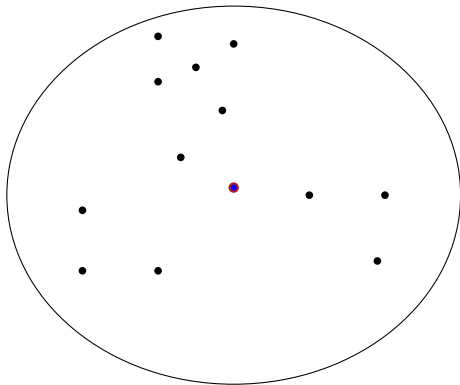
Learn symmetrical clauses

○ formula
• clause



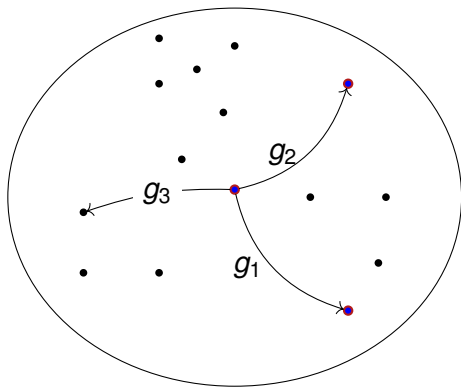
Learn symmetrical clauses

- formula
- clause
- learnt clause



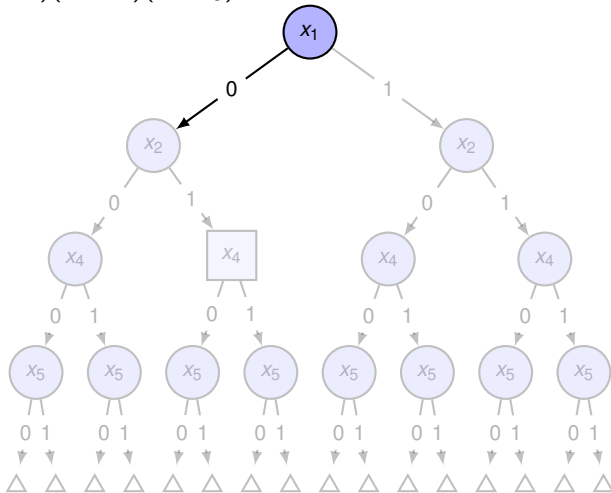
Learn symmetrical clauses

- formula
- clause
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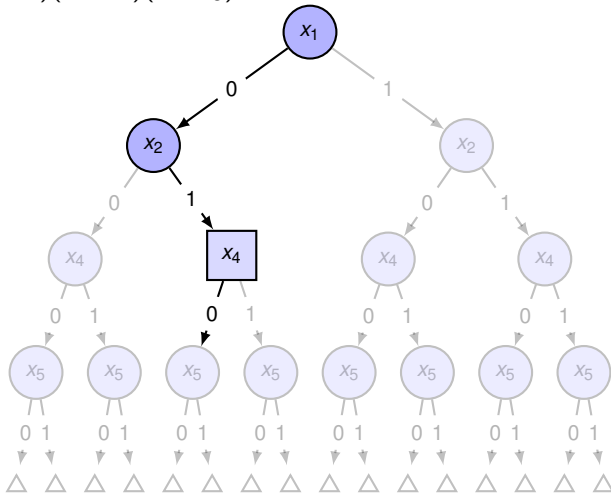
Using symmetries to accelerate the tree traversal

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$



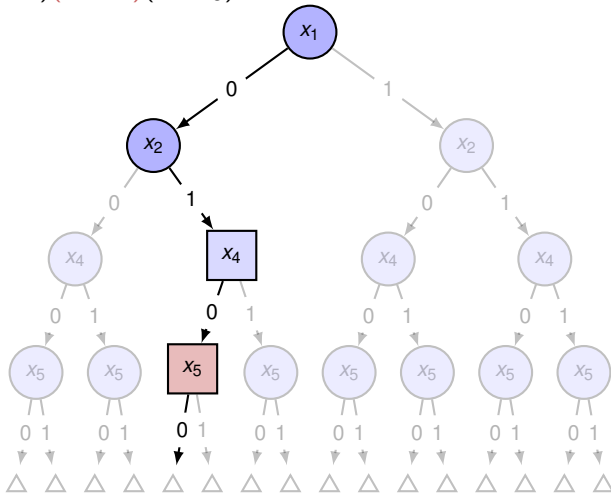
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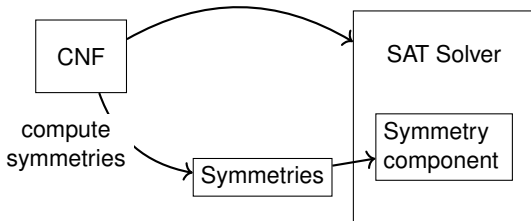
Using symmetries to accelerate the tree traversal

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$



Use symmetries to deduce symmetrical facts.

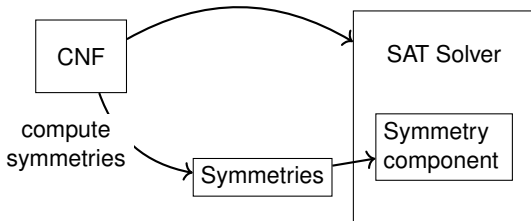
Dynamic Symmetry Breaking



State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC⁺12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Learning (SEL) [DBB17]

Dynamic Symmetry Breaking



State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC⁺12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Learning (SEL) [DBB17]

Cannot handle some instances solved by static approach

Composing Symmetry Propagation and Effective Symmetry Breaking for SAT Solving

NFM'19 [MBK19]

ESBP + SP

Compose the symmetry propagation and the ESBP

prune the decision tree while accelerating its traversal

Problems:

- ESBP breaks symmetries (incrementally)
- SP considers the manipulated symmetries valid all time

In a hybrid approach, SP must be able to identify
valid symmetries

Local symmetries

Formula $\leftarrow (\{g_1, g_2, g_3\})$

$$\omega_1 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_2 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_3 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_4 \leftarrow (\{g_1, g_2, g_3\})$$

Macro level

\rightarrow

Micro level

Local symmetries

Formula $\leftarrow (\{g_1, g_2, g_3\})$

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ω_5

Macro level

\rightarrow

Micro level

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Macro level

\rightarrow

Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

Local symmetries

Formula $\leftarrow (\{g_1, g_2, g_3\})$

$$\omega_1 \leftarrow (\{g_1, g_2, g_3\})$$

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$$\omega_5 \leftarrow (\{g_1, g_3\})$$

Macro level

\rightarrow

Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

- Inductive construction of the valid symmetries
- During the solving
- At a minimal cost

Experimental results

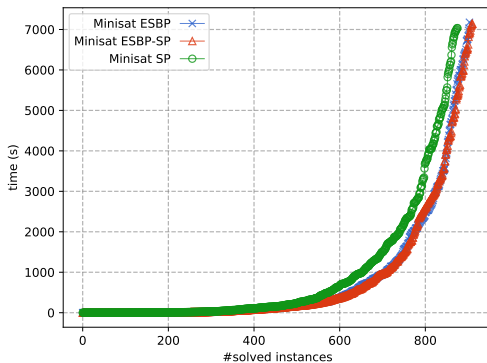
Benchmark:

- from SAT contests 2012 – 2018
- retain only instances for which `bliss` finds significant symmetries in 1000s
- 1400 symmetric instances (out of 4000)

Setup:

- three tools
 - MiniSat SP (Minisat with Symmetry Propagation)
 - MiniSat ESBP (Minisat with CDCL[Sym])
 - **Minisat ESBP-SP** (our approach)
- 7200s timeout

Experimental results



Solver	PAR-2	ALL	SAT	UNSAT
SP	1674h00	876	406	470
ESBP	1578h30	904	416	488
ESBP-SP	1570h15	911	420	491

Conclusion

- A new dynamic symmetry breaking approach
 - Generation of SBP on the fly
 - Package as a library cosy usable with any CDCL solver
 - Overcomes drawbacks of the existing approaches
- A new hybrid approach (ESBP-SP)
 - Take advantage of static and dynamic approach
 - Introduce local symmetries

Perspectives

- Combination of CDCL[Sym] with other dynamic symmetry breaking approach
- Combination with parallel SAT solver
- Exploitation of partial symmetries

Perspectives

- Combination of CDCL[Sym] with other dynamic symmetry breaking approach
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Thanks !



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IEEE Transactions on Computers, 48(5):506–521, 1999.



Ashish Sabharwal.

Symchaff: A structure-aware satisfiability solver.

In *AAAI*, volume 5, pages 467–474, 2005.

CDCL in action TODO



$$\omega_1 = \{x_1, x_2, x_3\}$$

$$\omega_2 = \{x_4, x_5, x_6\}$$

$$\omega_3 = \{\neg x_1, \neg x_5\}$$

$$\omega_4 = \{\neg x_2, \neg x_4\}$$

$$\omega_5 = \{\neg x_3, \neg x_4\}$$

$$\omega_6 = \{\neg x_3, \neg x_6\}$$

CDCL in action TODO



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3\}$$

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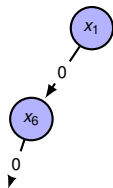
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CDCL in action TODO



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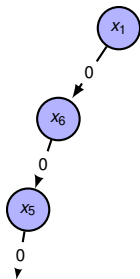
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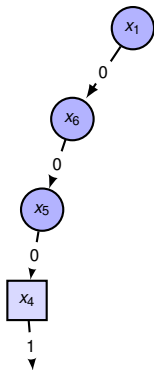
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CDCL in action TODO



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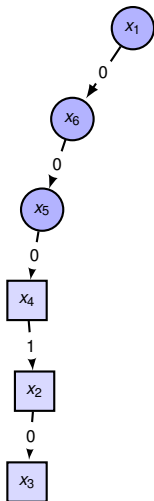
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CDCL in action TODO



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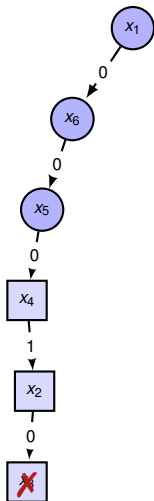
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CDCL in action TODO



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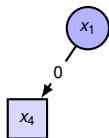
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CDCL in action TODO



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$$\omega_4 = \{\neg x_2, \neg x_4\}$$

$$\omega_5 = \{\neg x_3, \neg x_4\}$$

$$\omega_6 = \{\neg x_3, \neg x_6\}$$

$$\omega_7 = \{x_1, \neg x_4\}$$

Weakly active symmetries

Logical consequence

When ω is satisfied in all satisfying assignments of φ , we say that ω is a logical consequence of φ , and we denote this by $\varphi \vdash \omega$.

Weakly active symmetries

Logical consequence

When ω is satisfied in all satisfying assignments of φ , we say that ω is a logical consequence of φ , and we denote this by $\varphi \vdash \omega$.

Weakly active symmetries

Let a subset $\delta \subseteq \alpha$, a symmetry σ of φ such that $\varphi \cup \delta \vdash \varphi \cup \alpha \wedge \sigma.\delta \subseteq \alpha$ then σ is weakly active symmetry.

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Symmetry propagation

Let σ a weakly active symmetry, then

$$\varphi \cup \alpha \vdash \{I\} \Leftrightarrow \varphi \cup \alpha \vdash \sigma.\{I\}$$

Local symmetries

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Local Symmetries

Let φ be a formula. We define $L_{\omega, \varphi}$, the set of *local symmetries* for a clause ω , and with respect to a formula φ , as follows:

$$L_{\omega, \varphi} = \{\sigma \in \mathfrak{S} \mid \varphi \vdash \sigma.\omega\}$$

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We can state that:

$$\bigcap_{\omega \in \varphi} L_{\omega, \varphi} \subseteq G.$$

Computing local symmetries

Formula can be decomposed as : $\varphi = \varphi_o \cup \varphi_e \cup \varphi_d$ where

- φ_o is the set of the original clauses
- φ_e is the set of ESBPs
- φ_d is the set of deduced clauses.

Local symmetries

- $\omega \in \varphi_o, L_{\omega, \varphi} \supseteq G$
- $\omega \in \varphi_e, L_{\omega, \varphi} \supseteq \text{Stab}(\omega) = \{\sigma \in G \mid \omega = \sigma.\omega\}$
- $\omega \in \varphi_d, L_{\omega, \varphi} \supseteq \left(\bigcap_{\omega' \in \varphi_1} L_{\omega', \varphi} \right) \cup \text{Stab}(\omega)$

where φ_1 is the set of clauses that derives ω .

Generates symmetry breaking predicates (SBP)

- Define lexicographic order
 - Define total order on variables
 - Define minimal value
- Forbid non minimal assignment for each orbit

Example:

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; \textcolor{red}{F} < \textcolor{green}{T}$$

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

x_1	x_2	x_3	x_4	x_5	\dots	lex-leader	SBP

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O_1	$\textcolor{red}{F}$	$\textcolor{green}{T}$	—	—	—	\dots	✓	

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O_2	F	F	—	F	T	\dots	✓	

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	T	F	—	—	—	\dots	✗	
O_2	F	F	—	F	T	\dots	✓	$\rightarrow x_1 \vee x_2 \vee \neg x_4 \vee x_5$
	F	F	—	T	F	\dots	✗	

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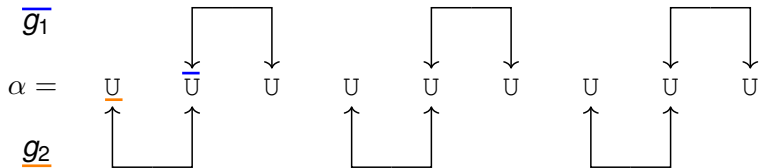
	x_1	x_2	x_3	x_4	x_5	\cdots	lex-leader	SBP
O_1	F	T	—	—	—	\cdots	✓	$\rightarrow \neg x_1 \vee x_2$
	T	F	—	—	—	\cdots	✗	
O_2	F	F	—	F	T	\cdots	✓	$\rightarrow x_1 \vee x_2 \vee \neg x_4 \vee x_5$
	F	F	—	T	F	\cdots	✗	
\dots								

Example

$$g_1 = (x_2 \ x_3)(x_5 \ x_6)(x_8 \ x_9)$$

$$g_2 = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

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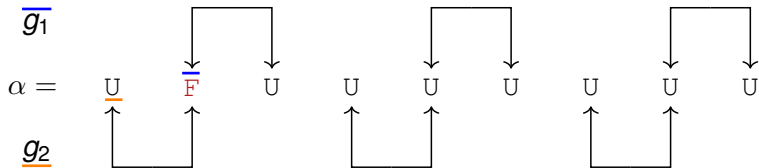


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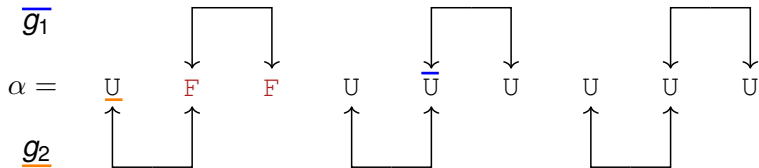


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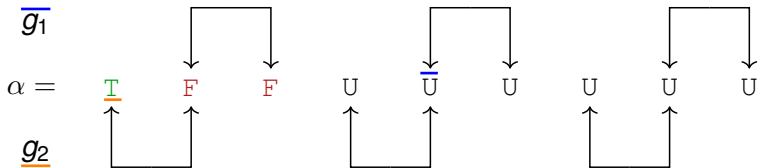


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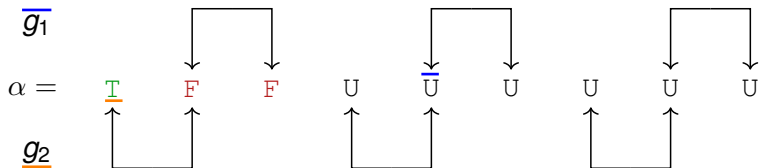


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g_2 generates ESBP $\omega = \{\neg x_1, x_2\}$

Example

- 1 reducer: $\alpha(g.x) \leq \alpha(x)$
- 2 inactive: $\alpha(x) \leq \alpha(g.x)$
- 3 active: $\alpha(g.x)$ or $\alpha(x)$ is unassigned

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 ; \text{ F } < \text{ T }$$

$$g_1 = \begin{array}{ccc} (x_2 & x_3) & (x_5 & x_6) & (x_8 & x_9) \end{array} \left| \begin{array}{l} x = x_2 \\ g.x = x_3 \\ \text{active} \end{array} \right.$$

↑

$$g_2 = \begin{array}{ccc} (x_1 & x_2) & (x_4 & x_5) & (x_7 & x_8) \end{array} \left| \begin{array}{l} x = x_1 \\ g.x = x_2 \\ \text{active} \end{array} \right.$$

↑

...

$$\alpha = \{ \quad \quad \quad \}$$

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...

$$\alpha = \{ \neg x_2 \quad \}$$

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active

$$g_2 = \begin{array}{cc|cc} (x_1 & x_2) & (x_4 & x_5) & (x_7 & x_8) \\ \uparrow & & & & & \end{array} \quad \left| \quad x = x_1 \quad g.x = x_2 \right.$$

reducer

...

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

Example

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...

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

$$g_2 \text{ generates } \omega = \{\neg x_1, x_2\}$$

Encoding the problem

$(A, 1)(A, 2)(A, 3)$

$(B, 1)(B, 2)(B, 3)$

$(C, 1)(C, 2)(C, 3)$

$x_1 \vee x_2 \vee x_3$

$x_4 \vee x_5 \vee x_6$

$x_7 \vee x_8 \vee x_9$

$\neg(A, 1)\neg(B, 1)$

$\neg(A, 1)\neg(C, 1)$

$\neg(B, 1)\neg(C, 1)$

$\neg x_1 \vee \neg x_4$

$\neg x_1 \vee \neg x_7$

$\neg x_4 \vee \neg x_7$

$\neg(A, 2)\neg(B, 2)$

$\neg(A, 2)\neg(C, 2)$

$\neg(B, 2)\neg(C, 2)$

$\neg x_2 \vee \neg x_5$

$\neg x_2 \vee \neg x_8$

$\neg x_5 \vee \neg x_8$

$\neg(A, 3)\neg(B, 3)$

$\neg(A, 3)\neg(C, 3)$

$\neg(B, 3)\neg(C, 3)$

$\neg x_3 \vee \neg x_6$

$\neg x_3 \vee \neg x_9$

$\neg x_6 \vee \neg x_9$