École doctorale EDITE de Paris (ED130)
Informatique, Télécommunication et Électronique

Thèse de doctorat de l'Université Sorbonne Université

Spécialité : **Ingénierie / Systèmes Informatiques**

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Pour obtenir le grade de :

Docteur de l'Université Sorbonne Université

Sujet de la thèse :

Exploitation des symétries dynamiques pour la résolution des problèmes SAT

Soutenue le :

Devant le jury composé de :

Rapporteurs :	Y. XXX	XXX
	Y. XXX	XXX
Examinateurs :	Y. XXX	XXX
	Y. XXX	XXX
	Y. XXX	XXX
	Y. XXX	XXX

Ah la these

Contents

SAT Theory NP complete Cite [COOK 71]

Used In
- Formal methods Hardware model checking;Software model checking; Termination analysis of term-rewrite systems ; Test pattern generation (testing of software hardware) ; etc
- AI planning, game n queen sudoku
- Bio info Haplotype inference ; Pedigree checking ; Analysis of Genetic; Regulatory Networks; etc.
- Design Automation: Equivalence checking; Delay computation; Fault diagnosis ; etc
- Security: Cryptanalysis ; Inversion attacks on hash functions; etc

Where can found
- Computationally hard problems: Graph coloring, Traveling salesperson
- Mathematical: van der Waerden numbers ; Quasigroup open problems
- Core engine for other solvers :0-1 ILP/Pseudo Boolean ;QBF ; #SAT ; SMT ; MAXSAT;
- Integrated into theorem provers :HOL ; Isabelle ;

- Integrated into widely used software: Eclipse provisioning system based on a Pseudo Boolean solver Suse 10.1 dependency manager based on a custom SAT solver

Interest in BMC [ ]Biere 99]

SAT solver Def

Input: Can encode any Boolean formula into Normal Form

Classical simplification: - Resolution - Subsumption

More complicated Hidden tautology BVA ... Loo Heule simplification slides

Algo CDCL

CHAPTER 1

INTRODUCTION

~~Nowadays, computers are powerfull and used in many applications in different domains. On of this domain is critical application, these applications are running in planes, cars and some software must be secure. Proving the correctness of these software leads to combinatorial explosion.~~

Over the years, computers scientist have developed many techniques to solve this kind of problems like *constraint programming* (CP) [15], *Propositional Satisfiability* (SAT) [3], *Satisfiability Modulo Theory* (SMT) [2].

In this thesis, we focus on *propositional satisfiability* that is used in many applications in different domains: *formal methods*: hardware model checking, software model checking, etc; *artificial inteligence*: planning; *games resolution*: sudoku, n-queens, *Bioinformatics* : Haplotype inference, *design automation* : equivalence checking

At its most basic, symmetry is some transformations of an object that leaves this object unchanged. In the case of satisfiability problems it maps a solution of a problem to another solution of the problem.

CHAPTER 2

PRELIMINARIES

2.1 SAT basics

Satisfiability problem

A *Boolean variable*, or *propositional variable*, is a variable that has two possible values : true or false (noted respectively $\top$ or $\bot$). A *literal* l is a propositional variable or its negation. For a given variable x, the positive literal is represented by x and the negative one by $\neg x$. A *clause* ω is a finite disjunction of literals represented equivalently by $\omega = \bigvee_{i=1}^{k} l_i$ or the set of its literals $\omega = \{l_i\}_{i \in [\![1,k]\!]}$. A clause with a single literal is called *unit clause*. A *conjunctive normal form (CNF) formula* φ is a finite conjunction of clauses. A CNF can be either noted $\varphi = \bigwedge_{i=1}^{k} \omega_i$ or $\varphi = \{\omega_i\}_{i \in [\![1,k]\!]}$. We denote $\mathcal{V}_\varphi$ ($\mathcal{L}_\varphi$) the set of variables (literals) used in φ (the index in $\mathcal{V}_\varphi$ and $\mathcal{L}_\varphi$ is usually omitted when clear from context).

For a given formula φ, an *assignment* of the variables of φ is a function $\alpha : \mathcal{V} \mapsto \{\top, \bot\}$. As usual, α is *total*, or *complete*, when all elements of $\mathcal{V}$ have an image by α, otherwise it is *partial*. By abuse of notation, an assignment is often represented by the set of its true literals. The set of all (possibly partial) assignments of $\mathcal{V}$ is noted $Ass(\mathcal{V})$.

The assignment α *satisfies* the clause ω, denoted $\alpha \models \omega$, if $\alpha \cap \omega \neq \emptyset$. Similarly, the assignment α satisfies the propositional formula φ, denoted $\alpha \models \varphi$, if α satisfies all the clauses of φ. Note that a formula may be satisfied by a partial assignment. In this case, unassigned variable are called *dont care*. A formula is said to be *satisfiable* (SAT) if there is at least one assignment that satisfies it; otherwise the formula is *unsatisfiable* (UNSAT).

An NP-complete problem

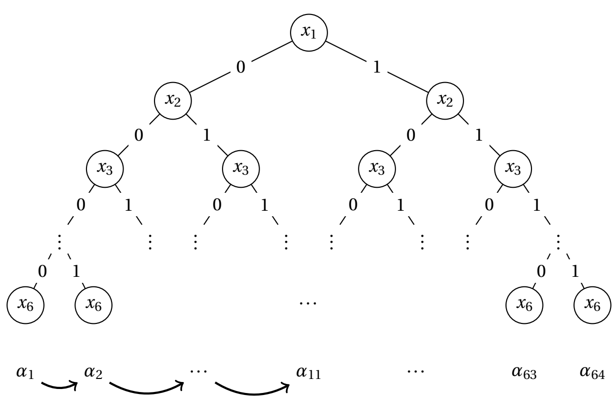
The SAT problems is the first NP-complete algorithm as proven by Stephen Cook in 1971 [4]. It is also proved by Leonid Levin in 1973 [18], for this purpose, it was known as Cook-Levin theorem. The proof of to show that can be found in [17]. Any NP problem can be reduced to a SAT problem in polynomial time and so open one of the most important unsolved problem in theoretical computer science is the P versus NP problem. This question is one of the seven millennium prize problems.

Solving a SAT problem

Two kinds of algorithm exists to solve satisfiability problems. First, the *incomplete* algorithm which does not provide any guarantee that will eventually report either any satisfiable assignment or declare that formula is unsatisfiable. This kind of algorithm is out of scope of this thesis. Second, the *complete* algorithm, which provides a guarantee that if an assignment exists it will be reached or it will declare that formula is unsatisfiable. This section describes different *complete* algorithm to solve a propositional formula.

A naive algorithm

A naive approach to solve a SAT problem is to try all possible assignments. In total, for a propositional formula with n variables, it leads to 2^n assignments in the worth case. Figure 2.1 illustrate the search tree for a given problem with six variables. In this case α_{11} $(\neg x_1, \neg x_2, x_3, \neg x_4, x_5, \neg x_6)$ is found as a solution of the problem. In the general case, this algorithm is obviously intractable on real problems even for a formula with few variables.



$\omega_1 = \{x_1, x_2, x_3\}$

$\omega_2 = \{x_4, x_5, x_6\}$

$\omega_3 = \{\neg x_1, \neg x_5\}$

$\omega_4 = \{\neg x_2, \neg x_4\}$

$\omega_5 = \{\neg x_3, \neg x_4\}$

$\omega_6 = \{\neg x_3, \neg x_6\}$

Figure 2.1: All possible assignments for a problem with 6 variables

Davis Putnam Logemann Loveland (DPLL) algorithm

One of the first non memory intensive algorithm to solve the SAT problems is the Davis Putnam Logemann Loveland (DPLL) algorithm [6]. It explores a binary tree using depth first search as given in Algorithm 1. The construction of the tree is related to a *decision* literal (line 8), then, recursive call with each value are checked. When a leaf report UNSAT(line 5), other branches are explored. By recursive construction of the algorithm, when positive and negative value of a literal reach UNSAT, solver backtracks at most one level and is called *chronological backtracking*. If all leaves report UNSATthe formula φ is unsatisfiable. And so if any branch found a solution i.e. the problem is empty, formula is satisfiable corresponding assignment is returned (lines 10 and 12)

```
1  function DPLL(φ: CNF formula, α assignment)
2      returns an assignment if φ is SAT and UNSAT otherwise
3      φ, α ← unitPropagation(φ, α);
4      if {} ∈ φ then
5          return UNSAT;                          // This branch is UNSAT
6      if φ = {} then
7          return α ;                             // φ is SAT
8      x ← assignDecisionLiteral();
9      if α ← DPLL(φ ∪ {x}, α) ≠ UNSAT then
10         return α
11     else if α ← DPLL(φ ∪ {¬x}, α) ≠ UNSAT then
12         return α
13     else
14         return UNSAT;                          // φ is UNSAT
```

Algorithm 1: The DPLL algorithm.

An important function in the DPLL algorithm is `unitPropagation` line 3 and it is detailed in Algorithm 2. It searches all unit clauses to ensure satisfiability, negative literal is removed from the clause that belongs to him. Then, given formula is either simplified or leads to an inconsistency (empty clause). Literal is also added to the current assignment.

```
1  function unitPropagation(φ: CNF formula, α assignment)
2      returns CNF formula and assignment α
3      while {l} ∈ φ and {} ∉ φ do
           // Remove all clauses containing l, all literals ¬l
4          φ ← φ |_l
5          α ← α ∪ {l}
6      return φ, α
```

Algorithm 2: Unit propagation

When DPLL algorithm is executed on the formula in Figure 2.1, unit propagation prevents to explore assignments from α_1 to α_8. Moreover, application to unit propagation until fix point leads directly to the solution.

Conflict Driven Clause Learning (CDCL) algorithm

Another sound and complete algorithm to resolve a SAT problem is Conflict-Driven Clause learning (CDCL) Algorithm 3 and is inspired DPLL. This algorithm introduces two principles. The first one is *no chronological backtracking*

The CDCL algorithm walks a binary search tree. It first applies unit propagation to the formula φ for the current assignment α (line 5). A conflict at level 0 indicates that the formula is not satisfiable, and the algorithm reports it (lines 7 and 8). If a conflict is detected, it is analyzed, which provides a *conflict clause* explaining the reason for the conflict (line 9). Different heuristics exists about the computation of conflict clause, on recent solvers the most used heuristic is the first Unique Implication Point (1^{th} UIP) [19]. The analysis is completed by the computation of a backjump point to which the algorithm backtracks (line 10). Restart is an important things in SAT solver, it allows solver to explore a new search space with the learned clauses. It is also finely intertwined with the decision heuristics. If the solver is working on "hard" part of the problem it will reconsider the decision variables and solve this part part at first. But if we restart too often the solver doesn't have to discover new things.

This clause is learnt (line 11), as it does not change the satisfiability of φ, and avoids encountering a conflict with the same causes in the future. Finally, if no conflict appears, the algorithm chooses a new decision literal (lines 13 and 14). The choice of the decision literal affect the performance of solver. The first most used heuristic is Variable State Independent Decaying Sum (VSIDS) [14]. The idea behind this heuristic is that the "hard" parts of the search space will be treated first. To do that, each variable has an activity and wa increase if it participate to the resolution of the conflict. The second most used heuristics is Learning rate based branching (LRB [11]) The above steps are repeated until the satisfiability status of the formula is determined.

Heuristics

Decision heuristics. Variable used to divide problems have a huge impact on the overall solving time by the solver. Decision variable may impact the number of propagation and so the depth of the search tree As propagated one can be seen as *deduction* and the

Hakan: Peut être mettre les heuristiques dans des paragraphes

Hakan: Literal Block Distance LBD

2.2 Groups basics

Symmetries is related to a branch of mathematics called group theory. This section give us an overview of group theory.

```
function CDCL(φ: CNF formula)
    returns ⊤ if φ is SAT and ⊥ otherwise
  dl ← 0;                                   // Current decision level
  while not all variables are assigned do
    isConflict ← unitPropagation();
    if isConflict then
      if dl = 0 then
        return ⊥;                           // φ is UNSAT
      ω ← analyzeConflict();
      dl ← backjumpAndRestartPolicies();
      φ ← φ ∪ {ω};
    else
      assignDecisionLiteral();
      dl ← dl + 1;
  return ⊤;                                 // φ is SAT
```

Algorithm 3: The CDCL algorithm.

Groups

A *group* is a structure $\langle G, *\rangle$, where G is a non empty set and $*$ a binary operation such the following axioms are satisfied:

- *associativity*: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
- *closure*: $\forall a, b \in G, a * b \in G$.
- *identity*: $\forall a \in G, \exists e$ such that $a * e = e * a = a$
- *inverse*: $\forall a \in G, \exists b \in G$, commonly denoted a^{-1} such that $a * a^{-1} = a^{-1} * a = e$

Note that *commutativity* is not required i.e $a * b = b * a$, for $a, b \in G$. The group is *abelian* if it satisfies the commutativity rule. Moreover, the last definition leads to important properties which are: i) uniqueness of the identity element. To prove this property, assume $\langle G, *\rangle$ a group with two identity elements e and f then $e = e * f = f$. ii) uniqueness of the inverse element. To prove this property, suppose that an element x_1 has two inverses, denoted b and c in group $\langle G, *\rangle$, then

$$
\begin{array}{lcll}
b & = & b * e & \\
 & = & b * (a * c) & c \text{ is an inverse of } a, \text{so } e = a * c \\
 & = & (b * a) * c & \textit{associativity} \text{ rule} \\
 & = & e * c & b \text{ is an inverse of } a, \text{so } e = a * b \\
 & = & c & \textit{identity} \text{ rule}
\end{array}
$$

The structure $\langle G, *\rangle$ is denoted as G when clear from context that G is a group with a binary operation. In this thesis, we interested only with the *finite* groups i.e with a finite number of elements.

Given a group G, a *subgroup* is a non empty subset of G which is also a group with the same binary operation. If H is a subgroup of G, we denote as $H \leq G$. A group has at least two subgroups: i) the subgroup composed by identity element $\{e\}$, denoted *trivial* subgroup. All other subgroups are *nontrivial*; ii) the subgroup composed by itself, denoted *improper* subgroup. All other subgroups are *proper*.

Generators of a group

If every elements in a group G can be expressed as a linear combination of a set of group of elements S = $\{g_1, g_2, ..., g_n\}$ then we say G is generated by the S. we denote this as G = $\langle S \rangle$ = $\langle \{g_1, g_2, ..., g_n\} \rangle$

Permutation groups

A *permutation* is a bijection from a set X to itself.

Example: given a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_1 & x_4 & x_6 & x_5 \end{pmatrix}$
g is a permutation that maps x_1 to x_2, x_2 to x_3, x_3 to x_1, x_4 to x_4, x_5 to x_6 and x_6 to x_5.

Permutations are generally written in *cycle notation* and the self mapped elements are omitted. So the permutation in cycle notation will be : $g = (x_1\ x_2\ x_3)\ (x_5\ x_6)$. We say *support* of the permutation g noted $supp(g)$ the elements that not mapped to themselves, $supp(g) = \{x \in X \mid g.x \neq x\}$. A variable x is *stabilized* by a permutation g if $x \notin Supp(g)$. A clause ω is *stabilized* by a permutation g if $\omega \cap Supp(g) = \emptyset$. Hakan: Maybe stabilisation as definition ?

The set of permutations of a given set X form a group G, with the composition operation $(\circ)$ and called *permutation group*. The *symmetric group* id the set of all possible permutations of a set X and noted $\mathfrak{S}(X)$. So, a *permutation group* is a subgroup of $\mathfrak{S}(X)$.

A permutation group G induces a *equivalence relation* on the set of element X being permuted. Two elements $x_1, x_2 \in X$ are equivalent if there exists a permutation $g \in G$ such that $gx_1 = x_2$. Then equivalence relation partitions X into *equivalence classes* referred to as the *orbits* of X under G. The orbit of an element x under group G (or simply orbit of x when clear from the context) is the set $[x]_G = \{g.x \mid g \in G\}$

CHAPTER 3

SYMMETRY AND SAT

The group of permutations of $\mathcal{V}$ (i.e. bijections from $\mathcal{V}$ to $\mathcal{V}$) is noted $\mathfrak{S}(\mathcal{V})$. The group $\mathfrak{S}(\mathcal{V})$ naturally acts on the set of literals: for $g \in \mathfrak{S}(\mathcal{V})$ and a literal $\ell \in \mathcal{L}$, $g.\ell = g(\ell)$ if ℓ is a positive literal, $g.\ell = \neg g(\neg \ell)$ if ℓ is a negative literal. The group $\mathfrak{S}(\mathcal{V})$ also acts on (partial) assignments of $\mathcal{V}$ as follows: for $g \in \mathfrak{S}(\mathcal{V})$, $\alpha \in Ass(\mathcal{V})$, $g.\alpha = \{g.\ell \mid \ell \in \alpha\}$. Let φ be a formula, and $g \in \mathfrak{S}(\mathcal{V})$. We say that $g \in \mathfrak{S}(\mathcal{V})$ is a symmetry of φ if for every *complete* assignment α, $\alpha \models \varphi$ if and only if $g.\alpha \models \varphi$. The set of symmetries of φ is noted $S(\varphi) \subseteq \mathfrak{S}(\mathcal{V})$.

The previous mathematical definitions of group theory is applied to the CNF formula. So, the group of permutations of $\mathcal{V}$ (i.e. bijections from $\mathcal{V}$ to $\mathcal{V}$) is noted $\mathfrak{S}(\mathcal{V})$. We say that $g \in \mathfrak{S}(\mathcal{V})$ is a symmetry of φ if following conditions holds:

- permutation fixes the formula, $g.\varphi = \varphi$
- g commutes with the negation: $g.\neg l = \neg g.l$

The set of symmetries of φ is noted $S(\varphi) \subseteq \mathfrak{S}(\mathcal{V})$. The sets of symmetries of a formula φ preserves the satisfaction, for every *complete* assignment α, $\alpha \models \varphi \leftrightarrow g.\alpha. \models \varphi$ for $g \in S(\varphi)$. The group $S(\varphi)$ also acts on (partial) assignments of $\mathcal{V}$ as follows: for $g \in S(\varphi)$, $\alpha \in Ass(\mathcal{V})$, $g.\alpha = \{g.\ell \mid \ell \in \alpha\}$, and acts also on clauses as follow g.$\omega = \{g.l \mid l \in \omega\}$.

The next section presents how to compute the set of *generators* of a given formula.

3.1 Symmetry detection in SAT

For the detection of symmetries in SAT, we fist introduce the graph automorphism notion. Given a colored graph $G = (V, E, \gamma)$, with vertex set $V \in [1, n]$, edge set E and γ a function that apply a mapping : $V \rightarrow C$ where C is a set of *colors*. An automorphism of G is a permutation from its vertices $g : V \rightarrow V$ such that:

- $\forall (u, v) \in E \implies (g.u, g.v) \in E$
- $\forall v \in V, \gamma(v) = \gamma(g.v)$

The graph automorphism problem is to find if a given graph has a non trivial permutation group. The computational complexity of this algorithm is conjectured to be strictly between P and NP. Several tools exists to tackle this problem like `saucy3` [10], `bliss` [9], `nauty` [13], etc.

There exists different ways to encode a SAT problems, which leads to different symmetries in these problem. When a symmetry depends on the structure of the problem, we say *syntactic* symmetries. In contrast, symmetries were *semantic*, when it is not inherent to the encoding. To find symmetries in SAT problem, the formula is transformed into colored graph and an automorphism tool is applied onto. Specifically, given a formula φ with m clauses over n variables, the graph is constructed as follows:

- *clause nodes*: represent each of the m clauses by a node with color 0;
- *literals nodes*: represent each of the l literals by a node with color 1;
- *clauses edges*: connect each clause node to the node of the literals that appear in clause;
- *boolean consistency edges*: connect each pair of literals that correspond to the same variables.

Hakan: Explication du graph + informations num nodes num edges. Probleme reel battleship

The battleship problems place one ship of size *** and two ships of size* in grid 3x4
1 2 3
4 5 6
7 8 9
10 11 12

one ship per row.

Produced graph contains 12 * 2 = 24 + 21 = 45 nodes and 24 + 36 = 60 edges

p cnf 12 21
−1 −2 0
−1 −3 0
−2 −3 0
−4 −5 0
−4 −6 0
−5 −6 0
−7 −8 0
−7 −9 0
−8 −9 0
−10 −11 0
−10 −12 0
−11 −12 0
1 4 7 10 0
2 5 8 11 0
3 6 9 12 0
1 6 8 10 0
2 4 9 11 0
3 5 7 12 0
1 5 9 10 0
2 6 7 11 0
3 4 8 12 0

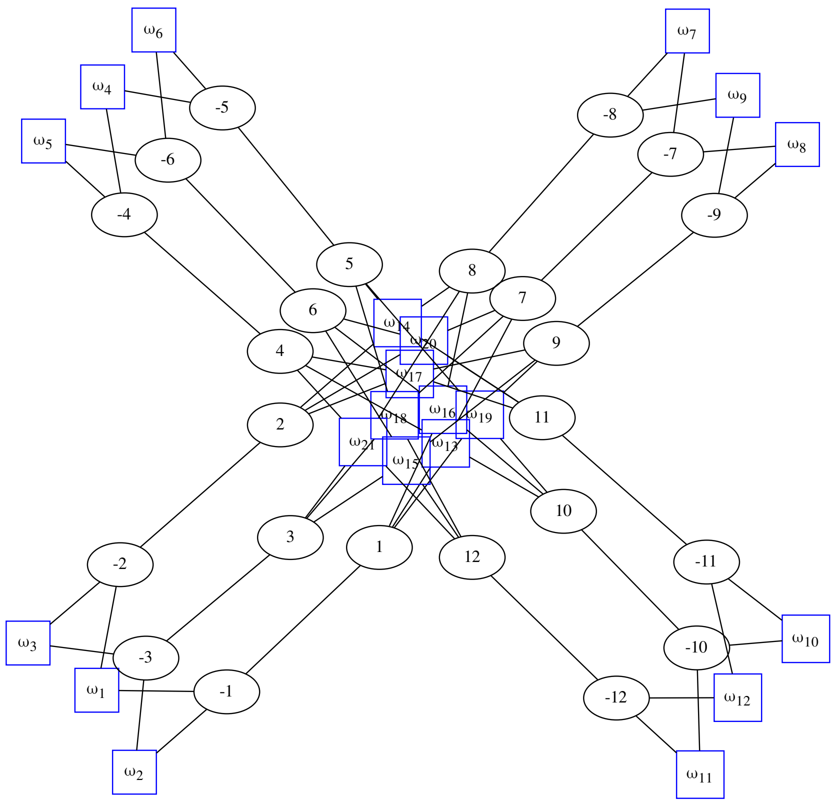


Figure 3.1: Example of constructed symmetry graph for a given CNF

An optimization of this graph is possible with the usage of binary clauses i.e. a clause with only two literals. The clause node can be omitted and we connect the two literals. As we cannot distinguish between the optimized edge and boolean consistency edges, we must check if the produced permutations are spurious. To do so, as we ensure the permutation commutes with the negation it suffice to check: $\forall x \in Supp(g), g.\neg x = \neg g.x$. Roughly speaking, we check if the image of the negation of x is equals to the negation of the image of x, for each element x in the support of the permutation. This optimization allows to compute symmetries of the problem more efficiently. In the previous example, the graph has deleted 12 nodes and 12 edges. More generally, the graph removes as many nodes and edges as binary clauses on the formula.

```
p cnf 12 21
−1 −2 0
−1 −3 0
−2 −3 0
−4 −5 0
−4 −6 0
−5 −6 0
−7 −8 0
−7 −9 0
−8 −9 0
−10 −11 0
−10 −12 0
−11 −12 0
1 4 7 10 0
2 5 8 11 0
3 6 9 12 0
1 6 8 10 0
2 4 9 11 0
3 5 7 12 0
1 5 9 10 0
2 6 7 11 0
3 4 8 12 0
```

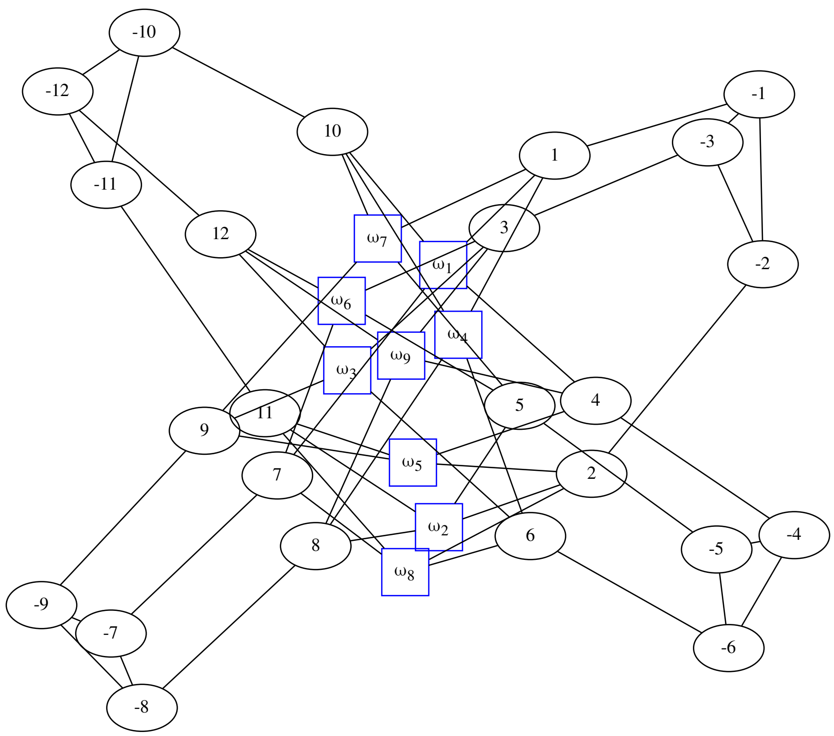


Figure 3.2: Example of constructed symmetry graph for a given CNF

When these graph is given to an automorphism tool like `bliss`, the following *generators* are obtained:

- g_0: (2 3)(5 6)(8 9)(11 12)(-2 -3)(-5 -6)(-8 -9)(-11 -12)
- g_1: (4 5 6)(7 9 8)(-4 -5 -6)(-7 -9 -8)
- g_2: (4 7)(5 8)(6 9)(-4 -7)(-5 -8)(-6 -9)
- g_3: (1 2)(5 6)(7 9)(10 11)(-1 -2)(-5 -6)(-7 -9)(-10 -11)
- g_4: (1 10)(2 11)(3 12)(-1 -10)(-2 -11)(-3 -12)

The visualization of the orbits of literals on the problem could be seen in figure 3.3, where each node represents a literal. Two literals are linked with an arc if it exists a permutation that maps the literal to the second one. By definition of the orbits, each literal belongs to a strongly connected components (SCC).

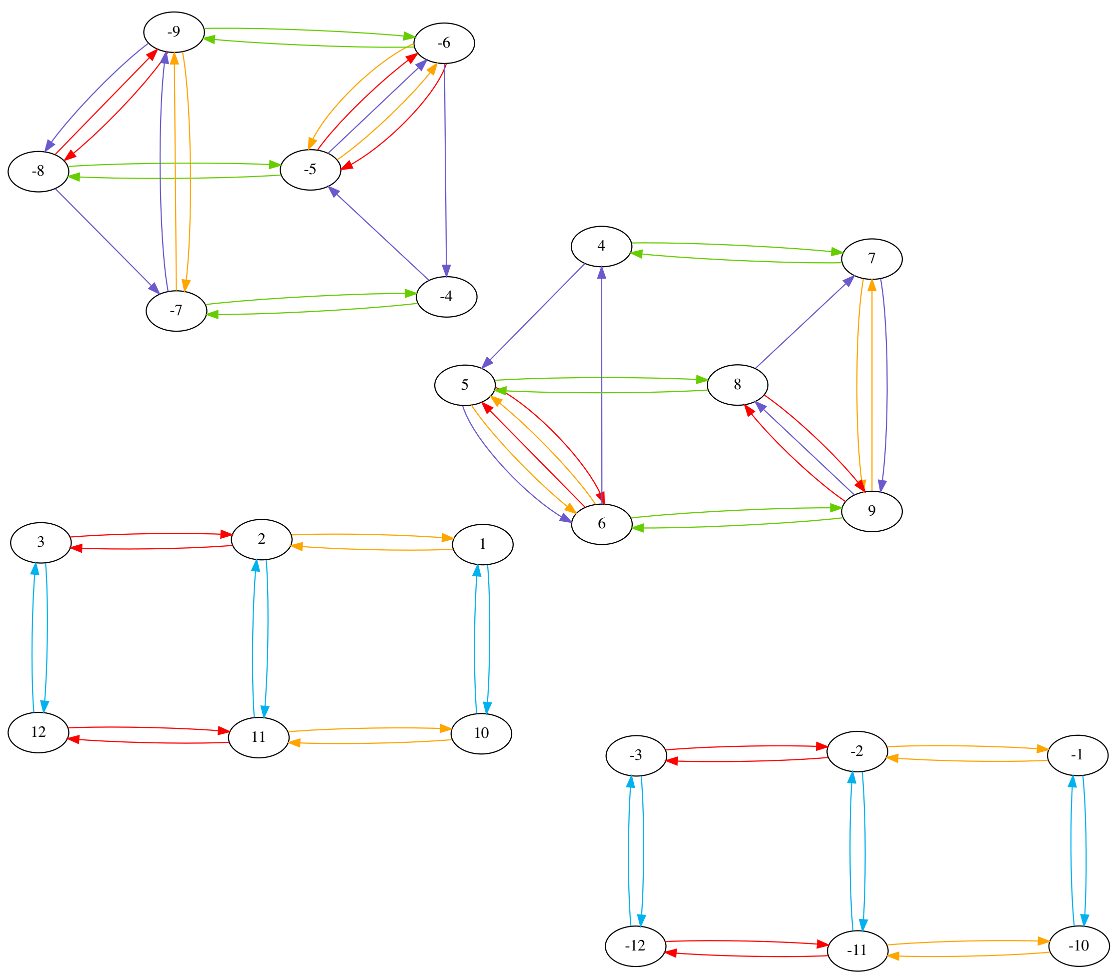


Figure 3.3: Orbits

3.2 Usage of symmetries

Symmetry breaking aims at eliminating symmetry, either by *statically* posting symmetry breaking constraints that invalidate symmetric assignments, or by altering the search space *dynamically* to avoid symmetric search paths.

In order to exploit the symmetry properties of a SAT problem, we need to introduce an ordering relation between the assignments.

Definition 1 (Assignments ordering)**.** *We assume a total order,* $\prec$*, on* $\mathcal{V}$*. Given two assignments* $(\alpha, \beta) \in Ass(\mathcal{V})^2$*, we say that* α *is strictly smaller than* β*, noted* $\alpha < \beta$*, if there exists a variable* $v \in \mathcal{V}$ *such that:*

- *for all* $v' \prec v$*, either* $v' \in \alpha \cap \beta$ *or* $\neg v' \in \alpha \cap \beta$*.*
- $\neg v \in \alpha$ *and* $v \in \beta$*.*[1]

Note that $<$ coincides with the lexicographical order on *complete* assignments. Furthermore, the $<$ relation is monotonic as expressed in the following proposition:

Proposition 1 (Monotonicity of assignments ordering)**.** *Let* $(\alpha, \alpha', \beta, \beta') \in Ass(\mathcal{V})^4$ *be four assignments.*

$$\text{If } \alpha \subseteq \alpha' \text{ and } \beta \subseteq \beta', \text{ then } \alpha < \beta \implies \alpha' < \beta'$$

Proof. The proposition follows on directly from Definition 1. □

Let φ a formula, and G the group from the formula. The *orbit of* α *under* G (or simply the *orbit of* α when G is clear from the context) is the set $[\alpha]_G = \{g.\alpha \mid g \in G\}$. The lexicographic leader (*lex-leader* for short) of an orbit $[\alpha]_G$ is defined by $min_<([\alpha]_G)$. This *lex-leader* is unique because the lexicographic order is a total order. The optimal approach to solve a symmetric SAT problem would be to explore only one assignment per orbit (for instance each lex-leader). However, finding the lex-leader of an orbit is computationally hard [12]. To avoid exploring symmetry search space, *symmetry breaking predicates* are added to the formula. These constraints are true only for the *lex-leader* [5].

Theorem 1 (Satisfiability preservation SBPs)**.** *Let* φ *be a formula and* ψ *the computed SBPs for the set of symmetries* $S(\varphi)$

$$\varphi \text{ and } \varphi \wedge \psi \text{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under $S(\varphi)$. Thus, g does not contradict β, which implies that $\beta \models \psi$. □

Hakan: Refaire la figure avec plus de point

It exists two kind of symmetry breaking, the first one is *full symmetry breaking* and it aims to visit exactly one assignment per orbit (generally the lex-leader). This approach is in practice infeasible in the most case due to the exponential number of permutations in a group. The

[1] We could have chosen as well $v \in \alpha$ and $\neg v \in \beta$ without loss of generality.

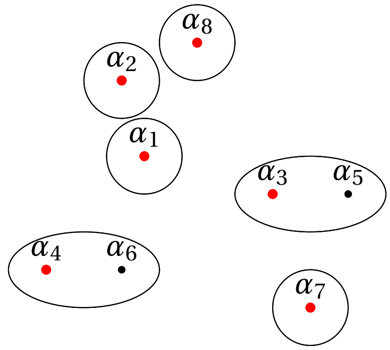


Figure 3.4: One lex-leader per assignment

second one is *partial symmetry breaking* and it aims to visit at least one assignment per orbit. This approach is easy to set up and bring us to considerable reduction of the search spaces. The used set of permutations is generally the set of generators of a formula given by the automorphism tool.

An extremely important point is the chosen lexicographic order. Variable ordering may impact the number of generated constraint and so the performance of the underlying SAT solver. Different orders are studied in the literature. One of the simplest order was the sorted variables according to their numbers. Some others orders exists and exploit structural properties of the problem. In particular, the orbits of the variables in different ways. For example, the variables are chosen with their number of occurrences in the initial problem. This order, is equivalent to put largest orbit first and so on, because each variable on the same orbit must have the same number of occurrences. Another example of exploiting structural property of the orbit is the usage of *stabilizer*. The order choose a variable which maximize the number of stabilized permutations, removes the not stabilized ones and loop over until the empty set. The remaining variables are added to the order to get a complete order.

Static symmetry breaking

The exploitation of symmetries statically is called *static symmetry breaking*. It acts like a preprocessor which add *symmetry breaking predicates* (SBPs) at the original formula and solve the augmented problem. This approach gives good performances in practice.

Hakan: Generation des clauses de SBPs
Hakan: Parler des groupes speciaux (totaux)
Hakan: Generation des clause binaires breakid lie aux orbits
Hakan: Parler de BreakID all in one tout au dessus

Hakan: Put a complete exemple

Hakan: Mettre des courbes
Hakan: Tableau sur le nombre de sbp genere

Hakan: Shatter, BreakID

Hakan: Conclu static

In the general case, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be totally computed. The underlying SAT solver can Even in more favorable situations, the size of the generated *sbp* is often too large to be effectively handled by a SAT solver [12]. On the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be that interesting and its effectiveness depends heavily on the heuristically chosen symmetries [3].

Hakan: Drawbacks de la methode pourquoi on fait du dynamique

Dynamic symmetry breaking

Another disadvantage is that the solver is influenced by SBPs and explore the search space with a different manner and can affect performance negatively Hakan: FIND A REF

In the literature, different approach of dynamic symmetry breaking are used to reduce the search space of the sat solver in different ways. Some of them *inject* constraints to allow only one representative assignment of each orbit like the static approach and others accelerate the propagation of variable using symmetrical properties. In this section, we present different approach to use symmetrical properties of a problem dynamically.

SymChaff

One of the first dynamic symmetry breaking approach is *SymChaff* [16] and is applicable only on special groups where all couple of variables are symmetric. The idea of this approach is to treat each orbit like a *symbolic variable*, i.e. instead of considering a single variable, all symmetric variables are considered at the same time and so backtracked at the same time. In this special case of groups the number of orbits is easy to compute but the order in which they would be applied has a tremendously impact of the solver performance. In the general case, when we take any groups computing the number of orbit will be very difficult and this approach will be intractable.

Symmetry Propagation

A different approach can be used to reduce search space using symmetries is *symmetry propagation* [8]. The general idea of this approach is to propagate symmetrical literals of those already propagated. In other words, it accelerate the tree traversal by “transforming some guessing (decisions) to deductions (propagation)”. Indeed, problem that presents symmetries makes possible to deduce some value for the variables that would be guessed if symmetry properties were ignored. These deductions will reduce the overall tree traversal depth and hence eventually accelerate the solving process.

To explain this approach, some definitions are required.

Hakan: Mettre logical conqequence quand on explique CDCL + Expliquer CDCL plus en detail avec les learnt et les raisons

Definition 2 (Logical consequence)**.** *A formula ϕ is a logical consequence of a formula φ denoted by $\varphi \models \phi$ if for all assignment α satisfying φ, it satisfies also ϕ. Two formulas are* logically equivalent *if each is a logical consequence of the other.*

Proposition 2 (Symmetry propagation)**.** *Let φ be a formula, α an assignment and l a literal. If g is a symmetry (permutation) of $\varphi \cup \alpha$ and $\varphi \models \{l\}$, then $\varphi \cup \alpha \models g.\{l\}$ is also true.*

In other words, if a literal l was propagated by the solver and g is a *valid* symmetry for the sub problem $\varphi \cup \alpha$ (in which all satisfied clauses and false literals are removed), so , the solver can also propagate the symmetrical of l. The problem here is to determinate which symmetries are valid for the formula $\varphi \cup \alpha$.

Definition 3 (Active symmetry)**.** *A symmetry g is called active under a partial assignment α if $g.\alpha = \alpha$*

The Definition 3 leads to the following proposition:

Proposition 3. *Let φ a formula and α a partial assignment. Let g a symmetry of φ, if g is active under the assignment α, then g is also a symmetry of $\varphi \cup \alpha$*

The previous proposition states that an active symmetry g for a partial assignment α still valid for the formula $\varphi \cup \alpha$. So when a literal l is propagated, and a symmetry g is active for a partial assignment α, the solver can also propagate $g.l$. Moreover, the group theory allow to compose permutations with the composition operator $\circ$ and the composition of two active symmetries is also an active symmetries so the solver can also propagate $g^2.l, g^3.l, \ldots$

Hakan: Peut etre expliquer les sym conflicts

The authors of symmetry propagation improve the active symmetries, introducing *weakly active* symmetries.

Definition 4 (Weakly active symmetry)**.** *Let φ a formula and (δ, α, γ) a state of a CDCL solver in which δ is the set of decisions, α is the current assignment and γ the reasons of the learned clauses. Then a symmetry g is weakly active if $g.\delta \subseteq \alpha$*

This definition leads to the following proposition:

Proposition 4. *Let φ be a formula, α an assignment. If there exists a subset $\delta \subseteq \alpha$ and a symmetry g of φ such that $g.\delta \subseteq \alpha$ and $\varphi \cup \delta \models \varphi \cup \alpha$, then g is also a symmetry of $\varphi \cup \alpha$.*

Hakan: Proof

In other words, we can detect with a minimal effort, the symmetries of $\varphi \cup \alpha$ by keeping track of the set of variables δ, which are in a state-of-the-art complete SAT solving algorithms, the set of decision variables. Obviously, a weakly active symmetry can also propagate the symmetrical literals of a propagated one. Moreover, weakly active symmetries allows more propagation and so is more efficient. Note that if a weakly active symmetry want to propagate a symmetrical literal which are already affected to the opposite value, this leads to a symmetry conflict and the solver backtracks to propagate the symmetrical value correctly.

Hakan: Mettre des tableaux, courbes etc ... Hakan: Courbe VS static an no sbp
Hakan: Conclu SP, depend on the solver choice

Symmetry propagation gives good performances on many symmetric instances. The overall performance of the symmetry propagation is intrinsically related to the decision heuristics of the underlying SAT solv1er.

One optimization of symmetry propagation is the following proposition, as seen in Section Hakan: DETERMINE SECTION SAT LEARNING each propagated clause has a reason which is an assertive clause. If the symmetrical clause is also an assertive one, this clause can be added in the formula without any requirements (even if the permutation is not weakly active). The added symmetrical clause will participate also to unit propagation and propagate the symmetrical literal.

Symmetry Explanation Learning

Another approach to exploit symmetry without removing any satisfiable assignment of the problem is *Symmetry Explanation Learning* [7]. This approach aim to learn useful symmetrical variant of clauses only where they are used by the solver. A useful clause is a clause that participate to the unit propagation or conflict. A naive approach to ensure that is to compute the symmetrical clause of each learned clauses and check if it is useful. In addition, due to size of a group a clause can have many symmetrical ones. the naive approach will have an important overhead and will be intractable on reel problems. SEL have different optimization based on the following facts. First, On the unit propagation, propagated literals has a reason clause which are assertive, and in the general case, symmetries permutes only few literals of the clause so symmetrical clauses can also be assertive and so useful. Secondly, avoiding to add identical clauses to the problem, symmetrical clauses ares stored in different learning scheme and used when classical unit propagation is finished. This ensure that no duplicate clause is added in the problem.

CHAPTER 4

SYMMSAT

In the general case, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be totally computed. Even in more favorable situations, the size of the generated *sbp* is often too large to be effectively handled by a SAT solver [12]. On the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be that interesting and its effectiveness depends heavily on the heuristically chosen symmetries [3]. Besides, these approaches are preprocessors, so their combination with other techniques, such as *symmetry propagation* [8], can be very hard. Also, tuning their parameters during the solving turns out to be very difficult. For all these reasons, some classes of SAT problems cannot be solved yet despite exhibiting symmetries. To handle these issues, we propose a new approach that reuses the principles of the static approaches, but operates dynamically: the symmetries are broken during the search process without any pre-generation of the *sbp*. It is a best effort approach that tries to eliminate, *dynamically*, the *non lex-leading* assignments with a minimal computation effort. To do so, we first introduce the notions of *reducer, inactive* and *active* permutation with respect to an assignment α and *effective symmetric breaking predicates* (*esbp*).

Definition 5 (Reducer, inactive and active permutation)**.** *A permutation g is a* reducer *of an assignment α if $g.\alpha < \alpha$ (hence α cannot be the lex-leader of its orbit. g reduces it and all its extensions). g is* inactive *on α when $\alpha < g.\alpha$ (so, g cannot reduce α and all the extensions). A symmetry is said to be* active *with respect to α when it is neither inactive nor a reducer of α.*

Proposition 5 restates this definition in terms of variables and is the basis of an efficient algorithm to keep track of the status of a permutation during the solving. Let us, first, recall that the *support*, $\mathcal{V}_g$, of a permutation g is the set $\{v \in \mathcal{V} \mid g(v) \neq v\}$.

Proposition 5. *Let $\alpha \in Ass(\mathcal{V})$ be an assignment, $g \in \mathfrak{S}\mathcal{V}$ a permutation and $\mathcal{V}_g \subseteq \mathcal{V}$ the support of g. We say that g is:*

1. a reducer of α *if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{v, \neg g^{-1}(v)\} \subseteq \alpha$;
2. inactive *on α if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{\neg v, g^{-1}(v)\} \subseteq \alpha$;
3. active *on α, otherwise.*

When g is a *reducer* of α we can define a predicate that contradicts α yet preserves the satisfiability of the formula. Such a predicate will be used to discard α, and all its extensions, from a further visit and hence pruning the search tree.

Definition 6 (Effective Symmetry Breaking Predicate)**.** *Let $\alpha \in Ass(\mathcal{V})$, and $g \in \mathfrak{S}\mathcal{V}$. We say that the formula ψ is an effective symmetry breaking predicate (esbp for short) for α under g if:*

$$\alpha \not\models \psi \textit{ and for all } \beta \in Ass(\mathcal{V}), \beta \not\models \psi \Rightarrow g.\beta < \beta$$

The next definition gives a way to obtain such an effective symmetry-breaking predicate from an assignment and a reducer.

Definition 7 (A construction of an *esbp*)**.** *Let φ be a formula. Let g be a symmetry of φ that reduces an assignment α. Let v be the variable whose existence is given by item 1. in Proposition 5. Let $U = \{v', \neg v' \mid v' \in \mathcal{V}_g$ and $v' \preceq v\}$. We define $\eta(\alpha, g)$ as $(U \cup g^{-1}.U) \setminus \alpha$.*

Example. Let us consider $\mathcal{V} = \{x_1, x_2, x_3, x_4, x_5\}$, $g = (x_1\, x_3)(x_2\, x_4)$, and a partial assignment $\alpha = \{x_1, x_2, x_3, \neg x_4\}$. Then, $g.\alpha = \{x_1, \neg x_2, x_3, x_4\}$ and $v = x_2$. So, $U = \{x_1, \neg x_1, x_2, \neg x_2\}$ and $g^{-1}.U = \{x_3, \neg x_3, x_4, \neg x_4\}$ and we can deduce than $\eta(\alpha, g) = (U \cup g^{-1}.U) \setminus \alpha = \{\neg x_1, \neg x_2, \neg x_3, x_4\}$.

Proposition 6. *$\eta(\alpha, g)$ is an effective symmetry-breaking predicate.*

Proof. It is immediate that $\alpha \not\models \eta(\alpha, g)$.

Let $\beta \in Ass(\mathcal{V})$ such that $\beta \wedge \eta(\alpha, g)$ is UNSAT. We denote a α' and β' as the restrictions of α and β to the variables in $\{v' \in \mathcal{V}_g \mid v' \preceq v\}$. Since $\beta \wedge \eta(\alpha, g)$ is UNSAT, $\alpha' = \beta'$. But $g.\alpha' < \alpha'$, and $g.\beta' < \beta'$. By monotonicity of $<$, we thus also have $g.\beta < \beta$. □

It is important to observe that the notion of *ebsp* is a refinement of the classical concept of *sbp* defined in [1]. In particular, like *sbp*, *esbp* preserve satisfiability.

Theorem 2 (Satisfiability preservation). *Let φ be a formula and ψ an esp for some assignment α under $g \in S(\varphi)$. Then,*

$$\varphi \textit{ and } \varphi \wedge \psi \textit{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under $S(\varphi)$. Thus, g does not reduce β, which implies that $\beta \models \psi$.

□

Algorithm

This section describes how to augment the state-of-the-art CDCL algorithm with the aforementioned concepts to develop an efficient symmetry-guided SAT solving algorithm. The approach is implemented using a couple of components: (1) a *Conflict Driven Clauses Learning (CDCL) search engine*; (2) *a symmetry controller*. Roughly speaking, the first component performs the classical search activity on the SAT problem, while the second observes the engine and maintains the status of the symmetries. When the controller detects a situation where the engine is starting to explore a redundant part[1], it orders the engine to operate a backjump. The detection is performed thanks to *symmetry status tracking* and the backjump order is given by a simple injection of an *esbp* computed on the fly. We first recall how the CDCL algorithm works. We then explain how to extend it with a *symmetry controller* component which guides the behavior of CDCL algorithm depending on the status of symmetries.

Conflict-Driven Clause Learning (CDCL) algorithm was already presented in Algorithm 3

epicted in Algorithm **??** (in black) is the same Algorithm . The parts in red (grey in B&W printings) should be ignored for the moment.

Hakan: Change Font !!

The main advantage of such an approach is to cope with the heavy (and potentially blocking) pre-generation phase of the static-based approaches, but also offers opportunities to combine with other dynamic-based approaches, like the *symmetry propagation* technique [8]. It also gives more flexibility for adjusting some parameters on the fly. Moreover, the overhead for non symmetric formulas is reduced to the computation time of the graph automorphism.

The extensive evaluation of our approach on the symmetric formulas of the last six SAT contests shows that it outperforms the state-of-the-art techniques, in particular on unsatisfiable instances, which are the hardest class of the problem.

Experiments

[1] Isomorphic to a part that has been/will be explored.

```
function CDCLSym(φ: CNF formula, SymController: symmetry controller)
  returns ⊤ if φ is SAT and ⊥ otherwise
    dl ← 0                                        // Current decision level
    while not all variables are assigned do
        isConflict ← unitPropagation()
        SymController.updateAssign(currentAssignment())
        isReduced ←
         SymController.isNotLexLeader(currentAssignment())
        if isConflict || isReduced then
            if dl == 0 then
                return ⊥                          // φ is UNSAT
            if isConflict then
                ω ← analyzeConflict()
            else
                ω ← SymController.generateEsbp(currentAssignment())
            addLearntClause(ω)
            dl ← backjumpAndRestartPolicies()
            SymController.updateCancel(currentAssignment())
        else
            assignDecisionLiteral()
            dl ← dl + 1
    return ⊤                                      // φ is SAT
```

Algorithm 4: the CDCLSym SAT Solving Algorithm.

Bibliography

[1] F. Aloul, K. Sakallah, and I. Markov. Efficient symmetry breaking for boolean satisfiability. *IEEE Trans. Computers*, 55(5):549–558, 2006.

[2] C. Barrett and C. Tinelli. Satisfiability modulo theories. In *Handbook of Model Checking*, pages 305–343. Springer, 2018.

[3] A. Biere, M. Heule, and H. van Maaren. *Handbook of satisfiability*, volume 185. IOS press, 2009.

[4] S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158. ACM, 1971.

[5] J. Crawford and M. Ginsberg. Symmetry-breaking predicates for search problems. 1996.

[6] M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397, July 1962.

[7] J. Devriendt, B. Bogaerts, and M. Bruynooghe. Symmetric explanation learning: Effective dynamic symmetry handling for sat. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 83–100. Springer, 2017.

[8] J. Devriendt, B. Bogaerts, B. de Cat, M. Denecker, and C. Mears. Symmetry propagation: Improved dynamic symmetry breaking in SAT. In *IEEE 24th International Conference on Tools with Artificial Intelligence, ICTAI 2012, Athens, Greece, November 7-9, 2012*, pages 49–56, 2012.

[9] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In D. Applegate, G. S. Brodal, D. Panario, and R. Sedgewick, editors, *Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics*, pages 135–149. SIAM, 2007.

[10] H. Katebi, K. Sakallah, and I. Markov. Symmetry and satisfiability: An update. *Theory and Applications of Satisfiability Testing–SAT 2010*, pages 113–127, 2010.

[11] J. H. Liang, V. Ganesh, P. Poupart, and K. Czarnecki. Learning rate based branching heuristic for sat solvers. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 123–140. Springer, 2016.

[12] E. M. Luks and A. Roy. The complexity of symmetry-breaking formulas. *Ann. Math. Artif. Intell.*, 41(1):19–45, 2004.

[13] B. D. McKay. nauty user's guide (version 2.2). Technical report, Technical Report TR-CS-9002, Australian National University, 2003.

[14] M. W. Moskewicz, C. F. Madigan, Y. Zhao, L. Zhang, and S. Malik. Chaff: Engineering an efficient sat solver. In *Proceedings of the 38th annual Design Automation Conference*, pages 530–535. ACM, 2001.

[15] F. Rossi, P. Van Beek, and T. Walsh. *Handbook of constraint programming*. Elsevier, 2006.

[16] A. Sabharwal. Symchaff: A structure-aware satisfiability solver. In *AAAI*, volume 5, pages 467–474, 2005.

[17] M. Sipser. *Introduction to the Theory of Computation*, volume 2. Thomson Course Technology Boston, 2006.

[18] B. A. Trakhtenbrot. A survey of russian approaches to perebor (brute-force searches) algorithms. *Annals of the History of Computing*, 6(4):384–400, Oct 1984.

[19] L. Zhang, C. F. Madigan, M. H. Moskewicz, and S. Malik. Efficient conflict driven learning in a boolean satisfiability solver. In *Proceedings of the 2001 IEEE/ACM international conference on Computer-aided design*, pages 279–285. IEEE Press, 2001.