

Exploitation of dynamic symmetries for solving SAT problems

Doctorat de Sorbonne Université

Hakan METIN

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Rapporteurs:

PASCAL FONTAINE
LAURE PETRUCCI

Professeur, Université de Liège
Professeur, Université Paris 13

Examineurs:

BART BOGAERTS
JEAN-MICHEL COUVREUR
EMMANUELLE ENCRENAZ

Assistant Professor, Vrije Universiteit Brussel
Professeur, Université d'Orléans
Maître de conférences, Sorbonne Université

Directeurs:

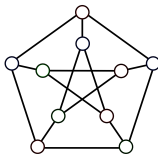
SOUHEIB BAARIR
FABRICE KORDON

Maître de conférences, Université Paris Nanterre
Professeur, Sorbonne Université

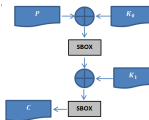


Motivation

Graph coloring



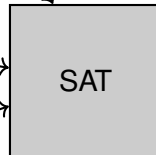
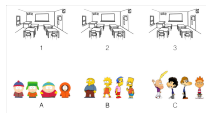
Cryptanalysis



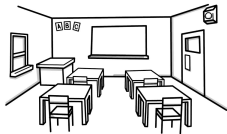
Hardware model
checking



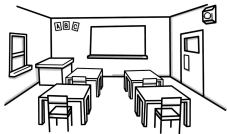
Planning



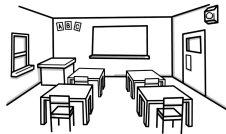
SAT: an example (1/2)



1



2



3



A



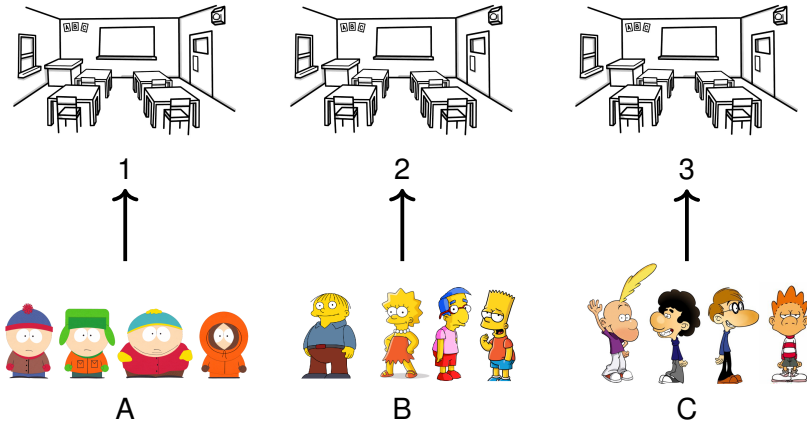
B



C

Is it possible to attribute each group to a unique classroom?

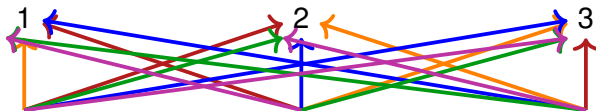
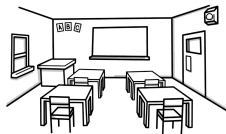
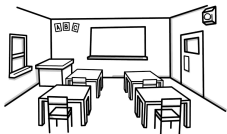
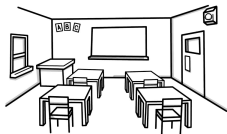
SAT: an example (1/2)



Is it possible to attribute each group to a unique classroom?

YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$

SAT: an example (1/2)



A



B



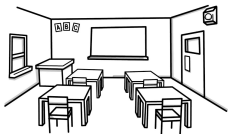
C

Is it possible to attribute each group to a unique classroom?

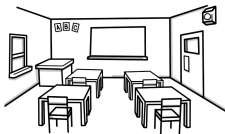
YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$

Many solutions $\alpha = (A, 2), (B, 3), (C, 1); \dots$

SAT: an example (2/2)



1



2



A



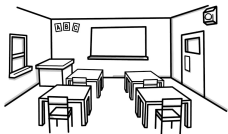
B



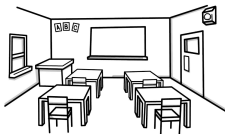
C

Is it possible to attribute each group to a unique classroom?

SAT: an example (2/2)



1



2



A



B



C

Is it possible to attribute each group to a unique classroom?

No! UNSAT

Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

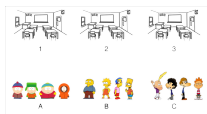
$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$

$(x_1 \vee x_2 \vee x_3) \wedge$
 $(x_4 \vee x_5 \vee x_6) \wedge$
 $(x_7 \vee x_8 \vee x_9) \wedge$

$(\neg x_1 \vee \neg x_4) \wedge$
 $(\neg x_1 \vee \neg x_7) \wedge$
 $(\neg x_4 \vee \neg x_7) \wedge$

$(\neg x_2 \vee \neg x_5) \wedge$
 $(\neg x_2 \vee \neg x_8) \wedge$
 $(\neg x_5 \vee \neg x_8) \wedge$

$(\neg x_3 \vee \neg x_6) \wedge$
 $(\neg x_3 \vee \neg x_9) \wedge$
 $(\neg x_6 \vee \neg x_9)$



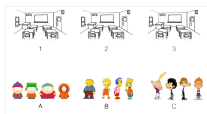
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



$(x_1 \vee x_2 \vee x_3) \wedge$
 $(x_4 \vee x_5 \vee x_6) \wedge$
 $(x_7 \vee x_8 \vee x_9) \wedge$

Clause

$(\neg x_1 \vee \neg x_4) \wedge$
 $(\neg x_1 \vee \neg x_7) \wedge$
 $(\neg x_4 \vee \neg x_7) \wedge$

$(\neg x_2 \vee \neg x_5) \wedge$
 $(\neg x_2 \vee \neg x_8) \wedge$
 $(\neg x_5 \vee \neg x_8) \wedge$

$(\neg x_3 \vee \neg x_6) \wedge$
 $(\neg x_3 \vee \neg x_9) \wedge$
 $(\neg x_6 \vee \neg x_9)$

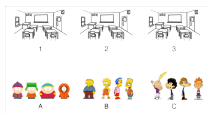
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



$(x_1 \vee x_2 \vee x_3) \wedge$
 $(x_4 \vee x_5 \vee x_6) \wedge$
 $(x_7 \vee x_8 \vee x_9) \wedge$

$(\neg x_1 \vee \neg x_4) \wedge$
 $(\neg x_1 \vee \neg x_7) \wedge$
 $(\neg x_4 \vee \neg x_7) \wedge$

$(\neg x_2 \vee \neg x_5) \wedge$
 $(\neg x_2 \vee \neg x_8) \wedge$
 $(\neg x_5 \vee \neg x_8) \wedge$

$(\neg x_3 \vee \neg x_6) \wedge$
 $(\neg x_3 \vee \neg x_9) \wedge$
 $(\neg x_6 \vee \neg x_9)$

Clause

Conjunctive
Normal
Form (CNF)

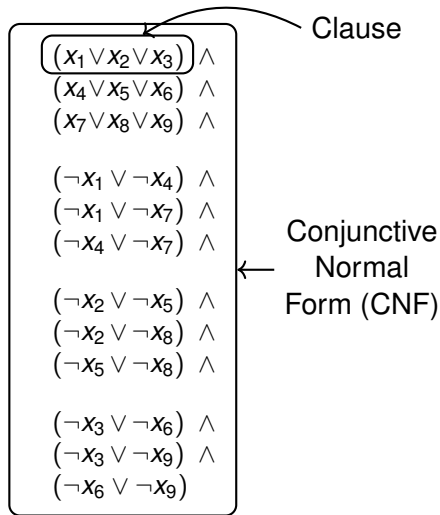
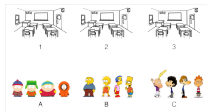
Encoding the problem

$(A, 1)(A, 2)(A, 3)$
 $(B, 1)(B, 2)(B, 3)$
 $(C, 1)(C, 2)(C, 3)$

$\neg(A, 1)\neg(B, 1)$
 $\neg(A, 1)\neg(C, 1)$
 $\neg(B, 1)\neg(C, 1)$

$\neg(A, 2)\neg(B, 2)$
 $\neg(A, 2)\neg(C, 2)$
 $\neg(B, 2)\neg(C, 2)$

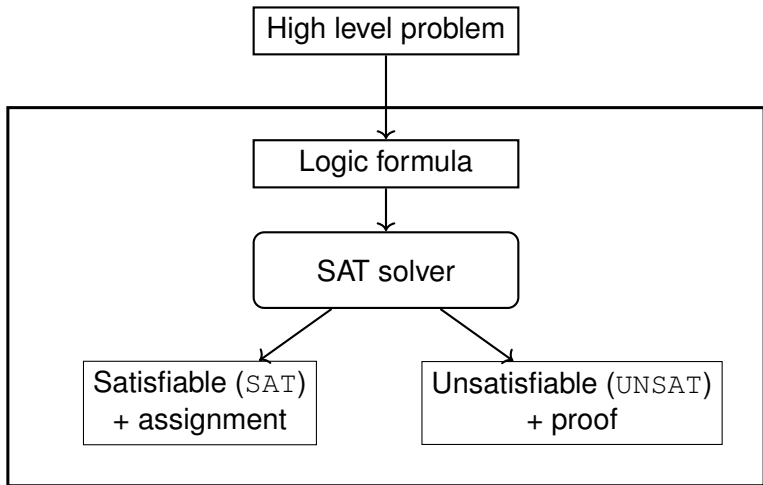
$\neg(A, 3)\neg(B, 3)$
 $\neg(A, 3)\neg(C, 3)$
 $\neg(B, 3)\neg(C, 3)$



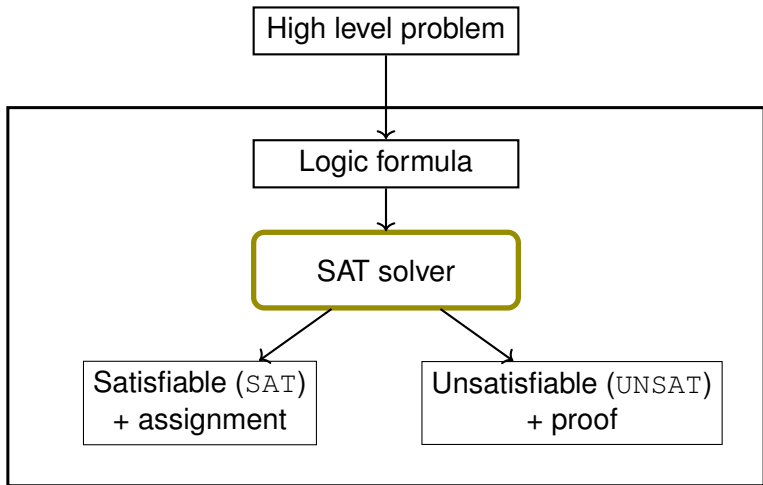
Clause represented as a set:

$$(x_1 \vee x_2 \vee x_3) \rightarrow \{x_1, x_2, x_3\}$$

SAT design



SAT design



SAT Solving

Solving SAT formula is known to be **NP-complete** [Coo71]

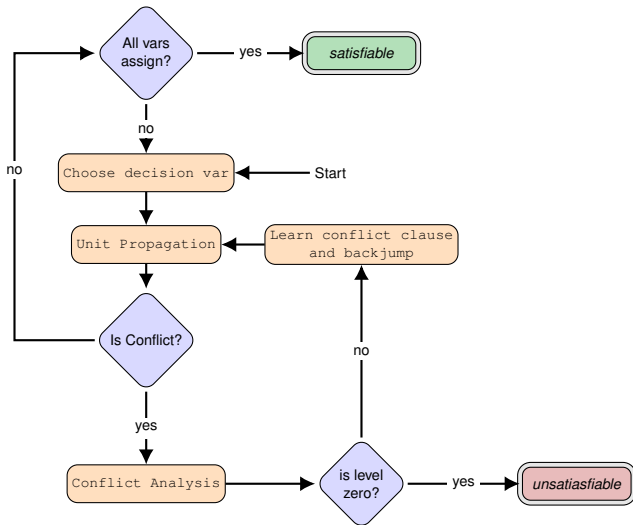
Good performance in practice:

- Handle large problem (million variables and clauses)
- International SAT competition each year on academic and industrial problems

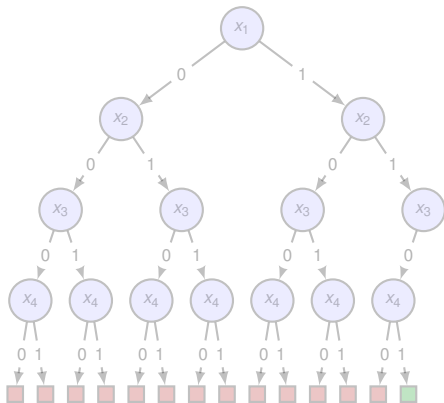
Enumerative algorithms:

- Davis, Putnam, Logemann, and Loveland (DPLL) [DLL62]
 - Boolean Constraint Propagation (BCP)
- Conflict Driven Clause Learning (CDCL) [MSS99]
 - Derived from DPLL
 - Clause learning

CDCL in detail



CDCL in action



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

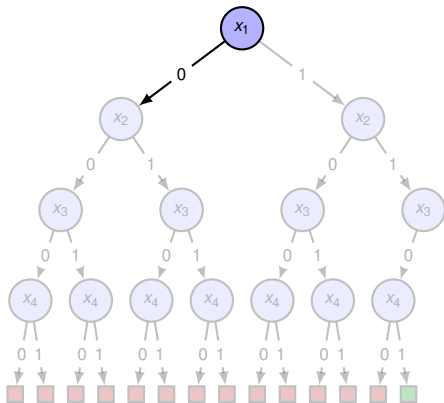
$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{\mathbf{x}_1, \neg x_4\}$$

$$\omega_3 = \{\mathbf{x}_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

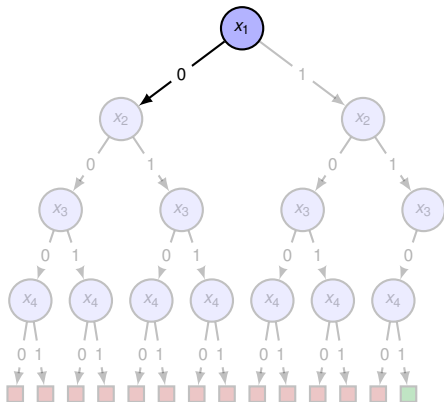
$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg x_1\}$$

CDCL in action

Conflict Analysis



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{\mathbf{x}_1, \neg \mathbf{x}_4\}$$

$$\omega_3 = \{\mathbf{x}_1, \mathbf{x}_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

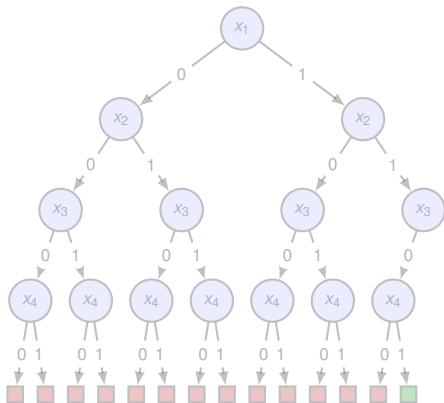
$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg \mathbf{x}_1\}$$

CDCL in action

Learn conflict clause
and backjump



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

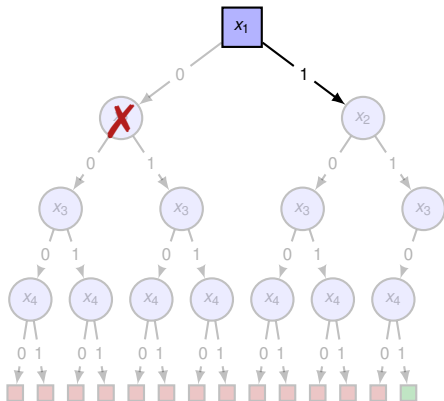
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{\}$$

CDCL in action

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

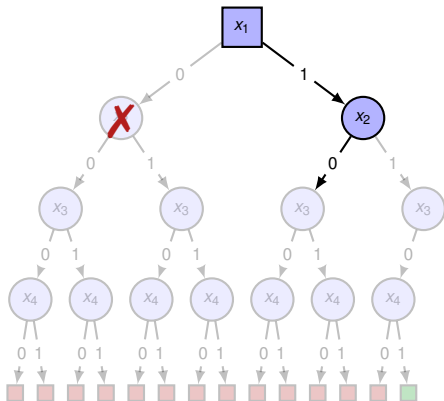
$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

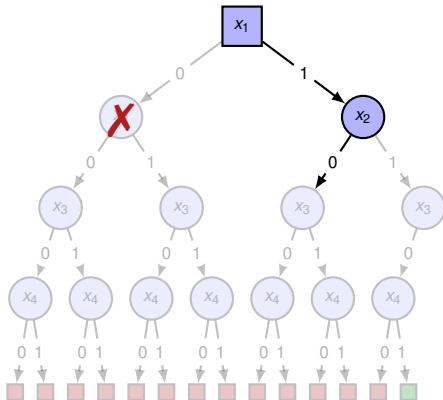
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

Conflict Analysis



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

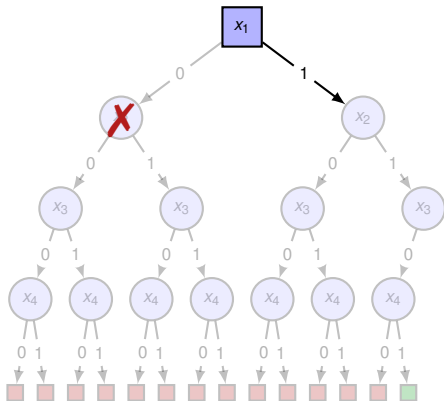
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

Learn conflict clause
and backjump



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

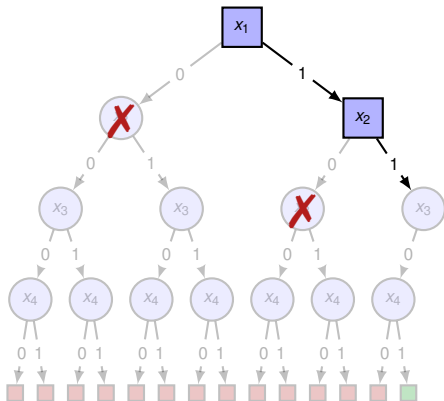
$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1\}$$

CDCL in action

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

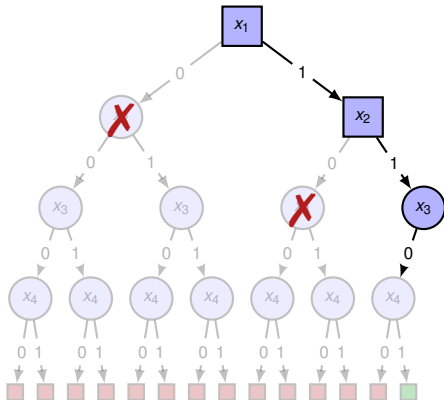
$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1, x_2\}$$

CDCL in action

Choose decision var

Unit Propagation



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

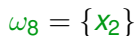
$$\omega_6 = \{x_3, x_4\}$$

$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1, x_2, \neg x_3\}$$

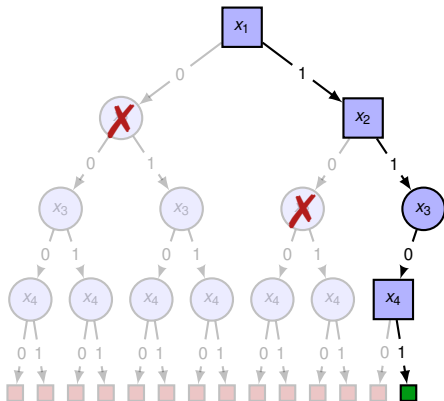
Unit Propagation



9/35

CDCL in action

satisfiable



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$

$$\omega_2 = \{x_1, \neg x_4\}$$

$$\omega_3 = \{x_1, x_4\}$$

$$\omega_4 = \{x_2, \neg x_4\}$$

$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

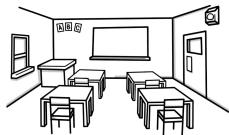
$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

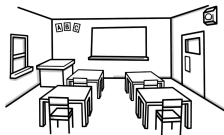
$$\alpha = \{x_1, x_2, \neg x_3, x_4\}$$

SAT and symmetries

Presence of symmetries



1



2



A

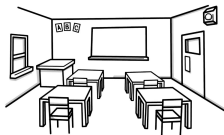
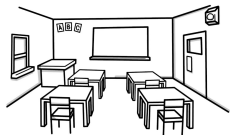


B



C

Presence of symmetries



A

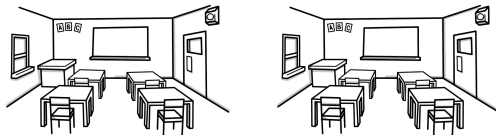


B



C

Presence of symmetries



1 ← → 2



A

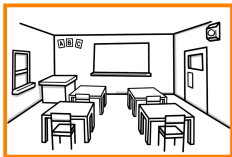


B

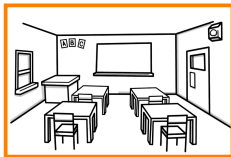


C

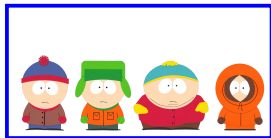
Presence of symmetries



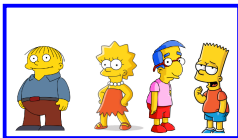
1



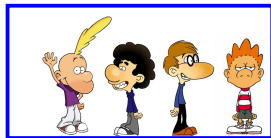
2



A

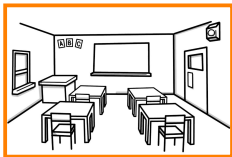


B

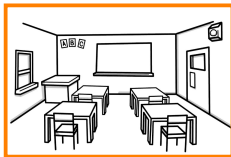


C

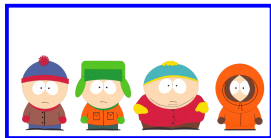
Presence of symmetries



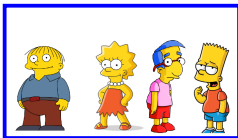
2



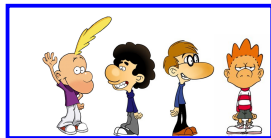
1



A



C



B

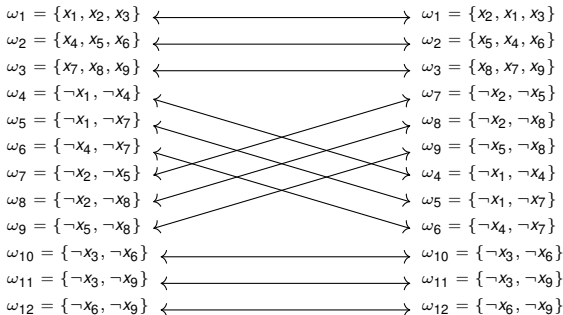
Symmetry (Syntactic)

A symmetry (permutation) g is a bijective function (on variables) that leaves the formula φ invariant

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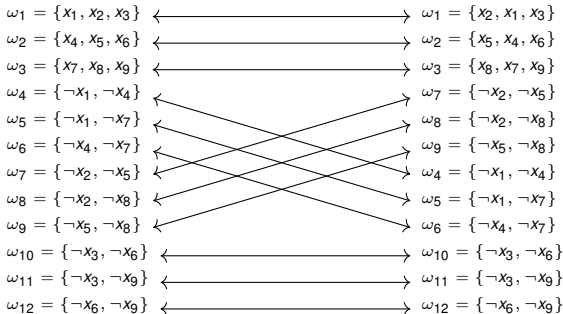
$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_2 & x_1 & x_3 & x_5 & x_4 & x_6 & x_8 & x_7 & x_9 \end{pmatrix} \rightarrow (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$



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Equi-satisfiability:

$$\alpha \models \varphi \Leftrightarrow g.\alpha \models \varphi$$

Computing symmetries of a SAT problem

CNF formula

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9) \\ & \wedge (\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7) \\ & \wedge (\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8) \\ & \wedge (\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9) \end{aligned}$$

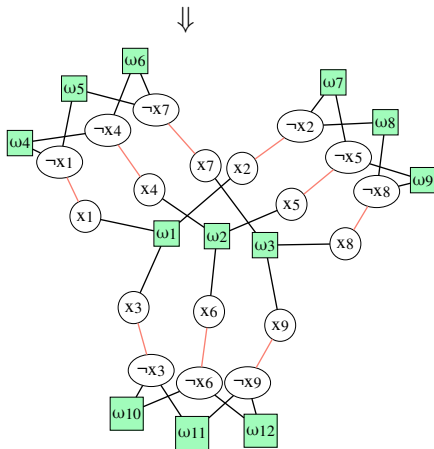
Computing symmetries of a SAT problem

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colored graph



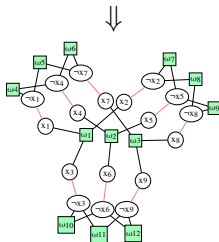
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colored graph



graph automorphism



(bliss, saucy, ...)

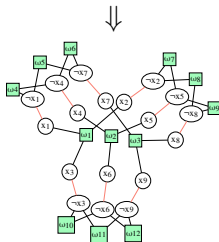
Computing symmetries of a SAT problem

CNF formula

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9) \\ & \wedge (\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7) \\ & \wedge (\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8) \\ & \wedge (\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9) \end{aligned}$$



colored graph



⇓
graph automorphism

⇓
(bliss, saucy, ...)

⇓
set of symmetries

⇓

$$\begin{aligned} g_1 &= (x_2 \ x_3)(x_5 \ x_6)(x_8 \ x_9) \\ g_2 &= (x_4 \ x_7)(x_5 \ x_8)(x_6 \ x_9) \\ g_3 &= (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8) \\ g_4 &= (x_1 \ x_4)(x_2 \ x_5)(x_3 \ x_6) \end{aligned}$$

The set of symmetries of a formula is a group noted G

Exploitation of symmetries

Static symmetry breaking

Orbit

Orbit of an assignment α for a group G :

$$G.\alpha = \{g.\alpha \mid g \in G\}$$

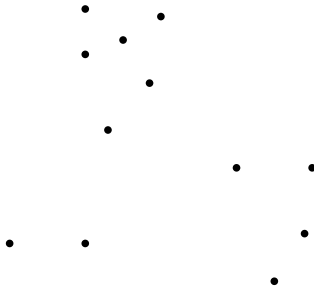
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Example:

- full assignment



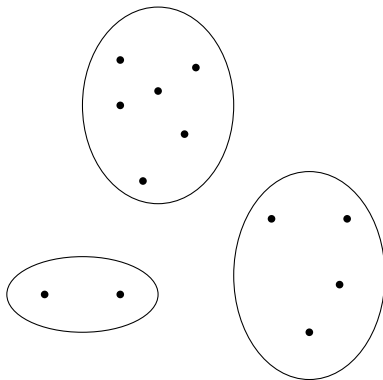
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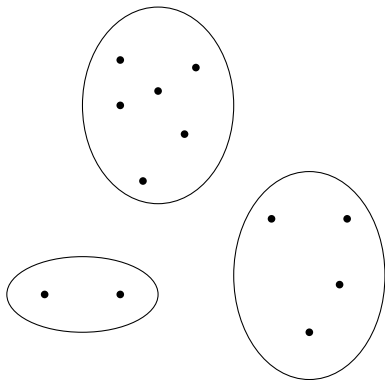
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Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

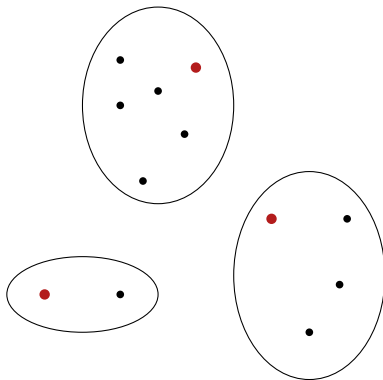
Orbit

Orbit of an assignment α for a group G :

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Example:

- full assignment
- orbit
- representative



Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

Compare assignment

Define an ordering relation to compare assignment (\prec)

- Total ordering on variables
- Minimum value: $F < T$ or $T < F$

Allow only minimal (maximal) value

Forbids other assignment in each orbit

→ Add Symmetry breaking predicates (SBP)

Example:

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; F < T$$

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	T	F	F	F	F	F	F	F

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$g.\alpha$	F	T	F	F	F	F	F	F
α	T	F	F	F	F	F	F	F

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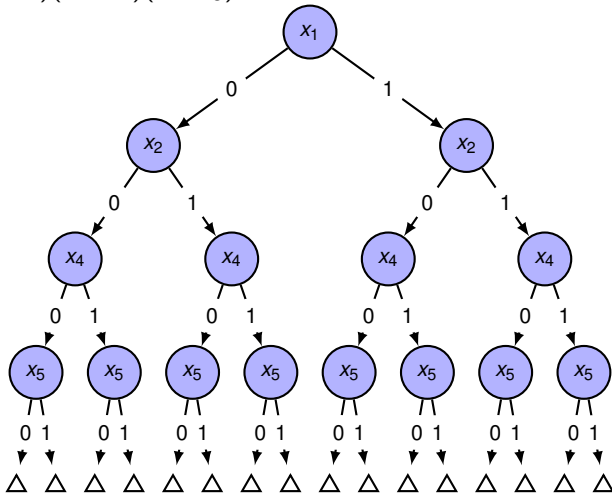
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$g.\alpha$	F	T	F	F	F	F	F	F
	\prec							
α	T	F	F	F	F	F	F	F

Generate SBP $\omega = \{\neg x_1, x_2\}$

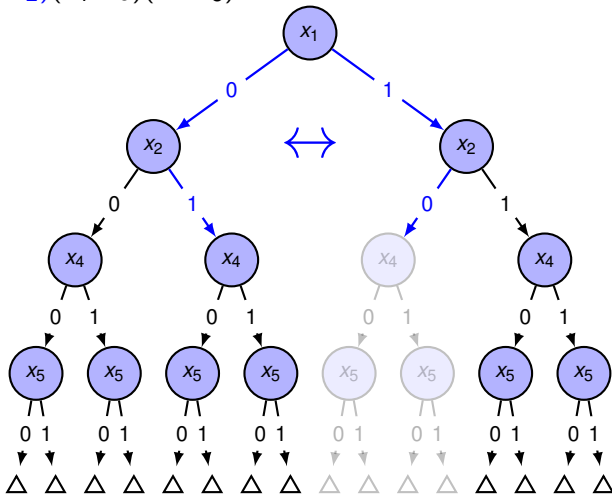
Using symmetries to prune the search space

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

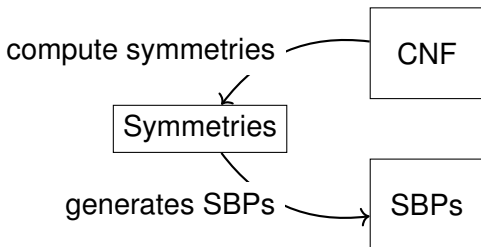


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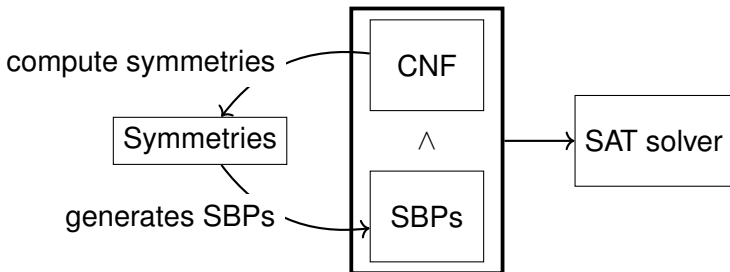
Static symmetry breaking



State-of-the-art approaches:

- Shatter [ASM06]
- BreakID [DBBD16]
- ...

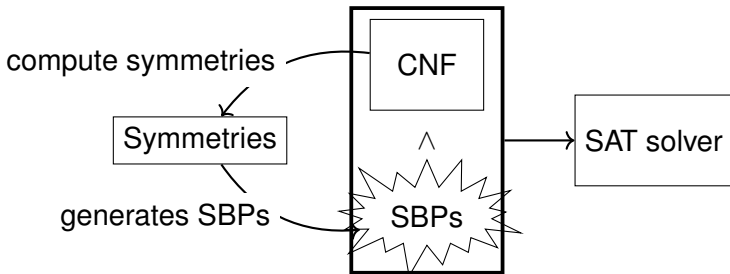
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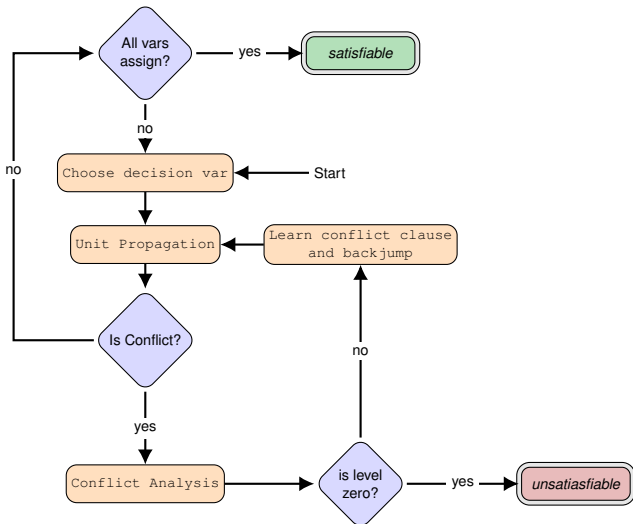
The solver can "explode" instead of being helped

CDCL[sym] Introducing Effective Symmetry Breaking in SAT Solving

TACAS'18 [MBCK18]

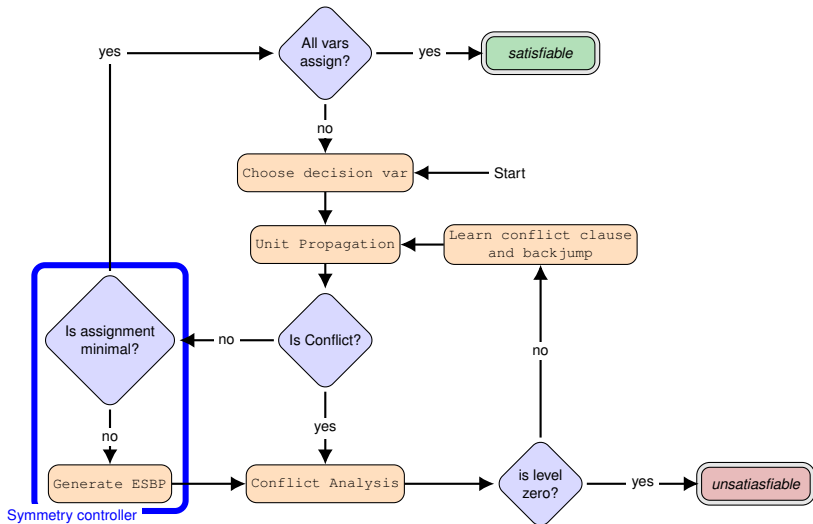
CDCL[Sym]

Compute and inject SBP **opportunistically**, during the solving



CDCL[Sym]

Compute and inject SBP **opportunistically**, during the solving



Symmetry status

- reducer: $g.\alpha \prec \alpha$
- inactive: $\alpha \prec g.\alpha$
- active: *not enough information*

Efficient implementation of symmetry status

Keep track the smallest unassigned variable x :

- ① $\alpha(g.x) \leq \alpha(x)$, then g is reducer \Rightarrow Effective SBP (ESBP)
- ② $\alpha(x) \leq \alpha(g.x)$, then g is inactive $\Rightarrow g$ cannot reduce α
- ③ $\alpha(g.x)$ or $\alpha(x)$ is unassigned then g is active

Update whenever variables are assigned / unassigned

CDCL[Sym] Implementation

- Packaged as a library **cosy**¹
 - Lightweight
 - Fast update and low memory
 - Follows symmetry status
-
- Works with any enumerative SAT solver
 - Can be integrated easily
- e.g. +3% LOC on MiniSAT.

¹<https://github.com/lip6/cosy>

Experiments

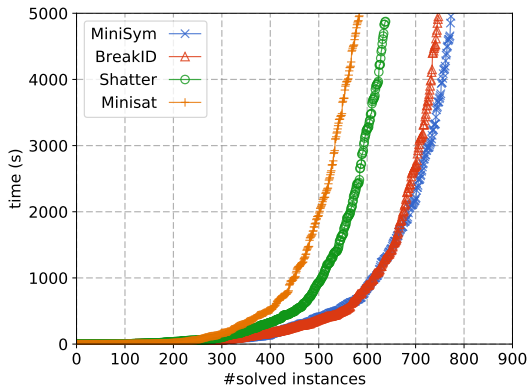
Benchmark:

- from SAT contests 2012 – 2017
- filter: `bliss` finds symmetries in 1000 seconds
- 36 % of instances, 1 350/3 700

Setup:

- four tools
 - MiniSat (no symmetry, baseline)
 - MiniSat + BreakID (SOTA SAT solver using symmetries)
 - MiniSat + Shatter (SOTA SAT solver using symmetries)
 - **MiniSym** = MiniSat + CDCL[Sym] (our approach)
- 5000 seconds timeout, 8GB memory
- includes time to compute symmetries (except for MiniSat)

Experimental results



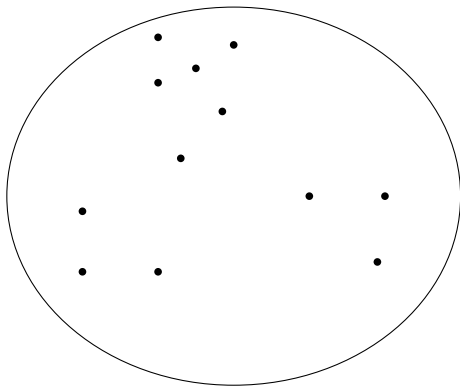
Solver	PAR-2	ALL	SAT	UNSAT
MiniSAT	2243h	586	325	261
Shatter	2088h	640	316	324
BreakID	1790h	749	334	415
MiniSym	1735h	775	336	439

Exploitation of symmetries

Dynamic symmetry breaking

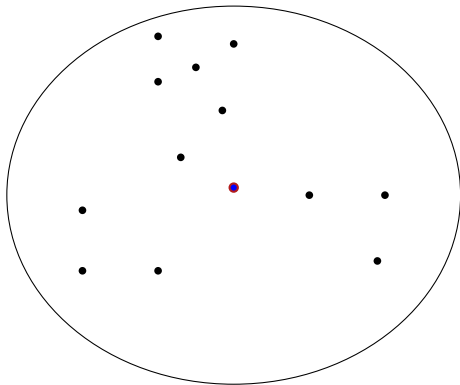
Learn symmetrical clauses

○ formula
• clause



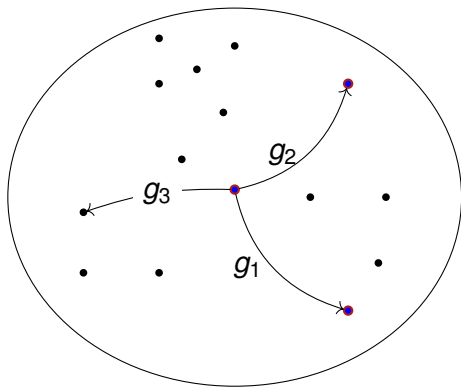
Learn symmetrical clauses

- formula
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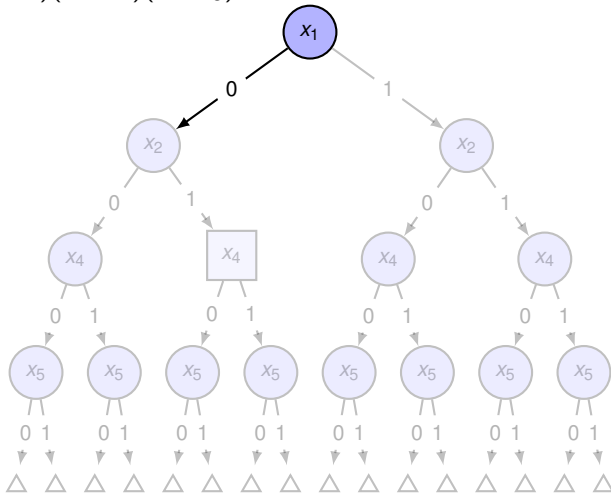
Learn symmetrical clauses

- formula
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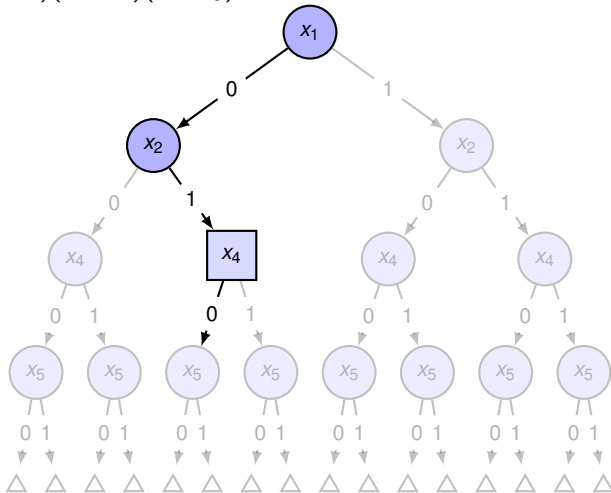
Using symmetries to accelerate the tree traversal

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$



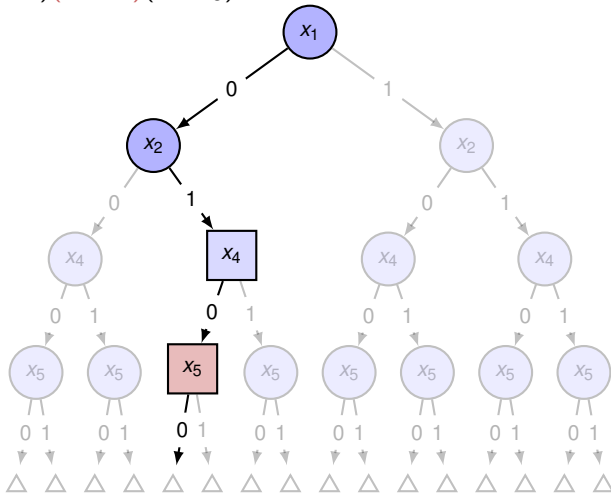
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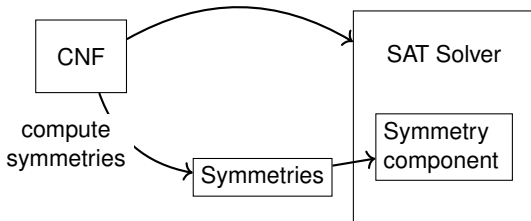
Using symmetries to accelerate the tree traversal

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Use symmetries to deduce symmetrical facts.

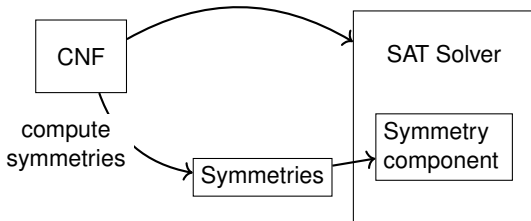
Dynamic Symmetry Breaking



State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC⁺12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Learning (SEL) [DBB17]

Dynamic Symmetry Breaking



State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC⁺12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Learning (SEL) [DBB17]

Cannot handle some instances solved by static approach

Composing Symmetry Propagation and Effective Symmetry Breaking for SAT Solving

NFM'19 [MBK19]

ESBP + SP

Compose the symmetry propagation and the ESBP

prune the decision tree while accelerating its traversal

Problems:

- ESBP breaks symmetries (incrementally)
- SP considers the manipulated symmetries valid all time

In a hybrid approach, SP must be able to identify
valid symmetries

Local symmetries

Formula $\leftarrow (\{g_1, g_2, g_3\})$

$$\omega_1 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_2 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_3 \leftarrow (\{g_1, g_2, g_3\})$$

$$\omega_4 \leftarrow (\{g_1, g_2, g_3\})$$

Macro level

\rightarrow

Micro level

Local symmetries

Formula $\leftarrow (\{g_1, g_2, g_3\})$

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$$\omega_5$$

Macro level

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Macro level

\rightarrow

Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

Local symmetries

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$$\omega_1 \leftarrow (\{g_1, g_2, g_3\})$$

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$$\omega_5 \leftarrow (\{g_1, g_3\})$$

Macro level

\rightarrow

Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

- Inductive construction of the valid symmetries
- During the solving
- At a minimal cost

Experimental results

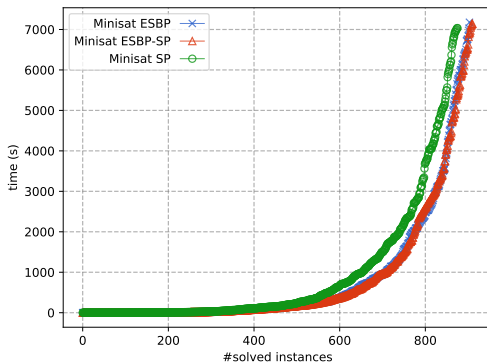
Benchmark:

- from SAT contests 2012 – 2018
- retain only instances for which `bliss` finds significant symmetries in 1000 seconds
- 1400 symmetric instances (out of 4000)

Setup:

- three tools
 - MiniSat SP (Minisat with Symmetry Propagation)
 - MiniSat ESBP (Minisat with CDCL[Sym])
 - **Minisat ESBP-SP** (our approach)
- 7200 seconds timeout

Experimental results



Solver	PAR-2	ALL	SAT	UNSAT
SP	1674h00	876	406	470
ESBP	1578h30	904	416	488
ESBP-SP	1570h15	911	420	491

Conclusion

- A new dynamic symmetry breaking approach
 - Generation of SBP on the fly
 - Package as a library cosy usable with any CDCL solver
 - Overcomes drawbacks of the existing approaches
- A new hybrid approach (ESBP-SP)
 - Take advantage of static and dynamic approach
 - Introduce local symmetries

Perspectives

- Combination of CDCL[Sym] with other dynamic symmetry breaking approach
- Combination with parallel SAT solver
- Exploitation of partial symmetries

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- Combination of CDCL[Sym] with other dynamic symmetry breaking approach
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Thanks !



Fadi A. Aloul, Karem A. Sakallah, and Igor L. Markov.
Efficient symmetry breaking for boolean satisfiability.
IEEE Trans. Computers, 55(5):549–558, 2006.



Belaid Benhamou, Tarek Nabhani, Richard Ostrowski, and Mohamed Reda Saidi.

Enhancing clause learning by symmetry in sat solvers.

In *2010 22nd IEEE International Conference on Tools with Artificial Intelligence*, volume 1, pages 329–335. IEEE, 2010.



Stephen A Cook.

The complexity of theorem-proving procedures.

In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158. ACM, 1971.



Jo Devriendt, Bart Bogaerts, and Maurice Bruynooghe.

Symmetric explanation learning: Effective dynamic symmetry handling for sat.

In *International Conference on Theory and Applications of Satisfiability Testing*, pages 83–100. Springer, 2017.



Jo Devriendt, Bart Bogaerts, Maurice Bruynooghe, and Marc Denecker.

Improved static symmetry breaking for sat.

In *International Conference on Theory and Applications of Satisfiability Testing*, pages 104–122. Springer, 2016.



Jo Devriendt, Bart Bogaerts, Broes de Cat, Marc Denecker, and Christopher Mears.

Symmetry propagation: Improved dynamic symmetry breaking in SAT.

In *IEEE 24th International Conference on Tools with Artificial Intelligence, ICTAI 2012, Athens, Greece, November 7-9, 2012*, pages 49–56, 2012.



Martin Davis, George Logemann, and Donald Loveland.

A machine program for theorem-proving.

Commun. ACM, 5(7):394–397, July 1962.



Hakan Metin, Souheib Baarir, Maximilien Colange, and Fabrice Kordon.

Cdclsym: Introducing effective symmetry breaking in sat solving.

In *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 99–114. Springer, 2018.



Hakan Metin, Souheib Baarir, and Fabrice Kordon.

Composing symmetry propagation and effective symmetry breaking for sat solving.

In *NASA Formal Methods Symposium*, pages 316–332. Springer, 2019.



Joao P Marques-Silva and Karem A Sakallah.

Grasp: A search algorithm for propositional satisfiability.

IEEE Transactions on Computers, 48(5):506–521, 1999.



Ashish Sabharwal.

Symchaff: A structure-aware satisfiability solver.

In *AAAI*, volume 5, pages 467–474, 2005.

CDCL in action TODO



$$\omega_1 = \{x_1, x_2, x_3\}$$

$$\omega_2 = \{x_4, x_5, x_6\}$$

$$\omega_3 = \{\neg x_1, \neg x_5\}$$

$$\omega_4 = \{\neg x_2, \neg x_4\}$$

$$\omega_5 = \{\neg x_3, \neg x_4\}$$

$$\omega_6 = \{\neg x_3, \neg x_6\}$$

CDCL in action TODO



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3\}$$

$$\omega_2 = \{x_4, x_5, x_6\}$$

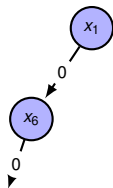
$$\omega_3 = \{\neg \mathbf{x}_1, \neg x_5\}$$

$$\omega_4 = \{\neg x_2, \neg x_4\}$$

$$\omega_5 = \{\neg x_3, \neg x_4\}$$

$$\omega_6 = \{\neg x_3, \neg x_6\}$$

CDCL in action TODO



$$\omega_1 = \{\mathbf{x}_1, x_2, x_3\}$$

$$\omega_2 = \{x_4, x_5, \mathbf{x}_6\}$$

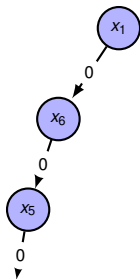
$$\omega_3 = \{\neg \mathbf{x}_1, \neg x_5\}$$

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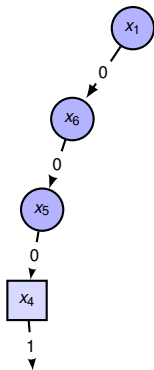
$$\omega_3 = \{\neg x_1, \neg x_5\}$$

$$\omega_4 = \{\neg x_2, \neg x_4\}$$

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$$\omega_6 = \{\neg x_3, \neg x_6\}$$

CDCL in action TODO



$$\omega_1 = \{\textcolor{red}{x}_1, x_2, x_3\}$$

$$\omega_2 = \{\textcolor{green}{x}_4, \textcolor{red}{x}_5, \textcolor{red}{x}_6\}$$

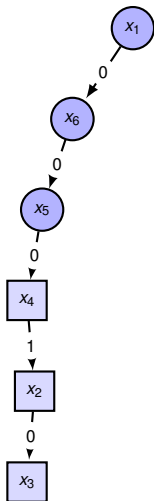
$$\omega_3 = \{\neg x_1, \neg x_5\}$$

$$\omega_4 = \{\neg \textcolor{brown}{x}_2, \neg \textcolor{brown}{x}_4\}$$

$$\omega_5 = \{\neg \textcolor{brown}{x}_3, \neg \textcolor{brown}{x}_4\}$$

$$\omega_6 = \{\neg x_3, \neg \textcolor{green}{x}_6\}$$

CDCL in action TODO



$$\omega_1 = \{x_1, x_2, x_3\}$$

$$\omega_2 = \{x_4, x_5, x_6\}$$

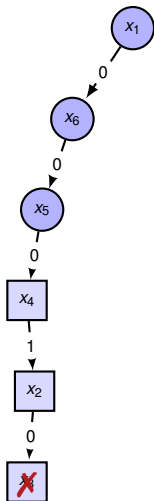
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CDCL in action TODO



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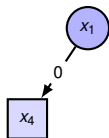
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$$\omega_7 = \{x_1, \neg x_4\}$$

Weakly active symmetries

Logical consequence

When ω is satisfied in all satisfying assignments of φ , we say that ω is a logical consequence of φ , and we denote this by $\varphi \vdash \omega$.

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Symmetry propagation

Let σ a weakly active symmetry, then

$$\varphi \cup \alpha \vdash \{I\} \Leftrightarrow \varphi \cup \alpha \vdash \sigma.\{I\}$$

Local symmetries

Logical consequence

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Local Symmetries

Let φ be a formula. We define $L_{\omega, \varphi}$, the set of *local symmetries* for a clause ω , and with respect to a formula φ , as follows:

$$L_{\omega, \varphi} = \{\sigma \in \mathfrak{S} \mid \varphi \vdash \sigma.\omega\}$$

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We can state that:

$$\bigcap_{\omega \in \varphi} L_{\omega, \varphi} \subseteq G.$$

Computing local symmetries

Formula can be decomposed as : $\varphi = \varphi_o \cup \varphi_e \cup \varphi_d$ where

- φ_o is the set of the original clauses
- φ_e is the set of ESBPs
- φ_d is the set of deduced clauses.

Local symmetries

- $\omega \in \varphi_o, L_{\omega, \varphi} \supseteq G$
- $\omega \in \varphi_e, L_{\omega, \varphi} \supseteq \text{Stab}(\omega) = \{\sigma \in G \mid \omega = \sigma.\omega\}$
- $\omega \in \varphi_d, L_{\omega, \varphi} \supseteq \left(\bigcap_{\omega' \in \varphi_1} L_{\omega', \varphi} \right) \cup \text{Stab}(\omega)$

where φ_1 is the set of clauses that derives ω .

Generates symmetry breaking predicates (SBP)

- Define lexicographic order
 - Define total order on variables
 - Define minimal value
- Forbid non minimal assignment for each orbit

Example:

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; \textcolor{red}{F} < \textcolor{green}{T}$$

$$g = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

x_1	x_2	x_3	x_4	x_5	\dots	lex-leader	SBP

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O_1	$\textcolor{red}{F}$	$\textcolor{green}{T}$	-	-	-	\dots	✓	

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O_1	F	T	—	—	—	\dots	✓	$\rightarrow \neg x_1 \vee x_2$
	T	F	—	—	—	\dots	✗	

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O_1	$\textcolor{red}{F}$	$\textcolor{green}{T}$	—	—	—	\dots	✓	$\rightarrow \neg x_1 \vee x_2$
	$\textcolor{green}{T}$	$\textcolor{red}{F}$	—	—	—	\dots	✗	
O_2	$\textcolor{red}{F}$	$\textcolor{red}{F}$	—	$\textcolor{red}{F}$	$\textcolor{green}{T}$	\dots	✓	

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	x_1	x_2	x_3	x_4	x_5	\dots	lex-leader	SBP
O_1	F	T	—	—	—	\dots	✓	$\rightarrow \neg x_1 \vee x_2$
	T	F	—	—	—	\dots	✗	
O_2	F	F	—	F	T	\dots	✓	$\rightarrow x_1 \vee x_2 \vee \neg x_4 \vee x_5$
	F	F	—	T	F	\dots	✗	

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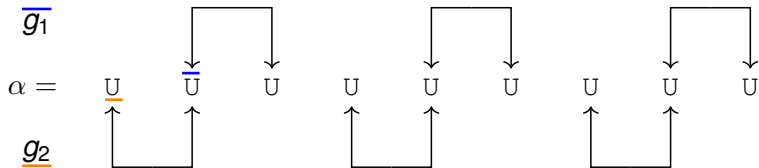
	x_1	x_2	x_3	x_4	x_5	\cdots	lex-leader	SBP
O_1	F	T	—	—	—	\cdots	✓	$\rightarrow \neg x_1 \vee x_2$
	T	F	—	—	—	\cdots	✗	
O_2	F	F	—	F	T	\cdots	✓	$\rightarrow x_1 \vee x_2 \vee \neg x_4 \vee x_5$
	F	F	—	T	F	\cdots	✗	
\dots								

Example

$$g_1 = (x_2 \ x_3)(x_5 \ x_6)(x_8 \ x_9)$$

$$g_2 = (x_1 \ x_2)(x_4 \ x_5)(x_7 \ x_8)$$

$$\textcolor{red}{F} < \textcolor{green}{T} \quad x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 \leq x_9$$

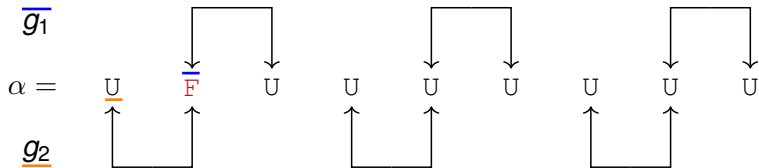


Example

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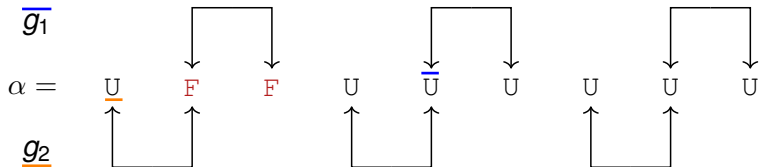


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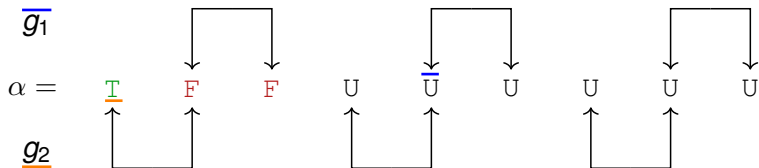


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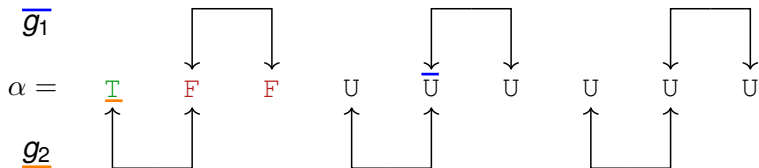


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g_2 generates ESBP $\omega = \{\neg x_1, x_2\}$

Example

- 1 reducer: $\alpha(g.x) \leq \alpha(x)$
- 2 inactive: $\alpha(x) \leq \alpha(g.x)$
- 3 active: $\alpha(g.x)$ or $\alpha(x)$ is unassigned

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 ; \text{ F } < \text{ T }$$

$$g_1 = \begin{array}{ccc} (x_2 & x_3) & (x_5 & x_6) & (x_8 & x_9) \end{array} \left| \begin{array}{l} x = x_2 \\ g.x = x_3 \\ \text{active} \end{array} \right.$$

↑

$$g_2 = \begin{array}{ccc} (x_1 & x_2) & (x_4 & x_5) & (x_7 & x_8) \end{array} \left| \begin{array}{l} x = x_1 \\ g.x = x_2 \\ \text{active} \end{array} \right.$$

↑

...

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...

$$\alpha = \{ \neg x_2 \quad \}$$

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active

$$g_2 = \begin{array}{cc|cc} (x_1 & x_2) & (x_4 & x_5) & (x_7 & x_8) \\ \uparrow & & & & & \end{array} \quad \left| \quad x = x_1 \quad g.x = x_2 \right.$$

reducer

...

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

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...

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

$$g_2 \text{ generates } \omega = \{\neg x_1, x_2\}$$

Encoding the problem

$(A, 1)(A, 2)(A, 3)$

$(B, 1)(B, 2)(B, 3)$

$(C, 1)(C, 2)(C, 3)$

$x_1 \vee x_2 \vee x_3$

$x_4 \vee x_5 \vee x_6$

$x_7 \vee x_8 \vee x_9$

$\neg(A, 1)\neg(B, 1)$

$\neg(A, 1)\neg(C, 1)$

$\neg(B, 1)\neg(C, 1)$

$\neg x_1 \vee \neg x_4$

$\neg x_1 \vee \neg x_7$

$\neg x_4 \vee \neg x_7$

$\neg(A, 2)\neg(B, 2)$

$\neg(A, 2)\neg(C, 2)$

$\neg(B, 2)\neg(C, 2)$

$\neg x_2 \vee \neg x_5$

$\neg x_2 \vee \neg x_8$

$\neg x_5 \vee \neg x_8$

$\neg(A, 3)\neg(B, 3)$

$\neg(A, 3)\neg(C, 3)$

$\neg(B, 3)\neg(C, 3)$

$\neg x_3 \vee \neg x_6$

$\neg x_3 \vee \neg x_9$

$\neg x_6 \vee \neg x_9$