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PRÉSENTÉE PAR : **HAKAN METIN**

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SUJET DE LA THÈSE :

EXPLOITATION DES SYMÉTRIES DYNAMIQUES POUR LA RÉSOLUTION DES PROBLÈMES SAT

SOUTENUE LE :

DEVANT LE JURY COMPOSÉ DE :

Rapporteurs :	Y. XXX	XXX
	Y. XXX	XXX
Examinateurs :	Y. XXX	XXX
	Y. XXX	XXX
	Y. XXX	XXX
	Y. XXX	XXX

Ah, la thèse.

Contents

SAT Theory NP complete Cite [COOK 71]

Used In
- Formal methods Hardware model checking;Software model checking; Termination analysis of term-rewrite systems ; Test pattern generation (testing of software hardware) ; etc
- AI planning, game n queen sudoku
- Bio info Haplotype inference ; Pedigree checking ; Analysis of Genetic; Regulatory Networks; etc.
- Design Automation: Equivalence checking; Delay computation; Fault diagnosis ; etc
- Security: Cryptanalysis ; Inversion attacks on hash functions; etc

Where can found
- Computationally hard problems: Graph coloring, Traveling salesperson
- Mathematical: van der Waerden numbers ; Quasigroup open problems
- Core engine for other solvers :0-1 ILP/Pseudo Boolean ;QBF ; #SAT ; SMT ; MAXSAT;
- Integrated into theorem provers :HOL ; Isabelle ;

- Integrated into widely used software: Eclipse provisioning system based on a Pseudo Boolean solver Suse 10.1 dependency manager based on a custom SAT solver

Interest in BMC [ ]Biere 99]

SAT solver Def

Input: Can encode any Boolean formula into Normal Form

Classical simplification: - Resolution - Subsumption

More complicated Hidden tautology BVA ... Loo Heule simplification slides

Algo CDCL

CHAPTER 1

INTRODUCTION

~~Nowadays, computers are powerfull and used in many applications in different domains. On of this domain is critical application, these applications are running in planes, cars and some software must be secure. Proving the correctness of these software leads to combinatorial explosion.~~

Over the years, computers scientist have developed many techniques to solve this kind of problems like *constraint programming* (CP) [26], *Propositional Satisfiability* (SAT) [5], *Satisfiability Modulo Theory* (SMT) [4].

In this thesis, we focus on *propositional satisfiability* that is used in many applications in different domains: *formal methods*: hardware model checking, software model checking, etc; *artificial inteligence*: planning; *games resolution*: sudoku, n-queens, *Bioinformatics* : Haplotype inference, *design automation* : equivalence checking

At its most basic, symmetry is some transformations of an object that leaves this object unchanged. In the case of satisfiability problems it maps a solution of a problem to another solution of the problem.

CHAPTER 2

PRELIMINARIES

2.1 SAT basics

Satisfiability problem

A *Boolean variable*, or *propositional variable*, is a variable that has two possible values : true or false (noted respectively $\top$ or $\bot$). A *literal* l is a propositional variable or its negation. For a given variable x, the positive literal is represented by x and the negative one by $\neg x$. A *clause* ω is a finite disjunction of literals represented equivalently by $\omega = \bigvee_{i=1}^{k} l_i$ or the set of its literals $\omega = \{l_i\}_{i \in [\![1,k]\!]}$. A clause with a single literal is called *unit clause*. A clause is a *tautology* if it is always true, a clause that contains a positive and negative value of a clause for example. A *conjunctive normal form (CNF) formula* φ is a finite conjunction of clauses. A CNF can be either noted $\varphi = \bigwedge_{i=1}^{k} \omega_i$ or $\varphi = \{\omega_i\}_{i \in [\![1,k]\!]}$. We denote $\mathcal{V}_\varphi$ ($\mathcal{L}_\varphi$) the set of variables (literals) used in φ (the index in $\mathcal{V}_\varphi$ and $\mathcal{L}_\varphi$ is usually omitted when clear from context).

For a given formula φ, an *assignment* of the variables of φ is a function $\alpha : \mathcal{V} \mapsto \{\top, \bot\}$. As usual, α is *total*, or *complete*, when all elements of $\mathcal{V}$ have an image by α, otherwise it is *partial*. By abuse of notation, an assignment is often represented by the set of its true literals. The set of all (possibly partial) assignments of $\mathcal{V}$ is noted $Ass(\mathcal{V})$.

The assignment α *satisfies* the clause ω, denoted $\alpha \models \omega$, if $\alpha \cap \omega \neq \emptyset$. Similarly, the assignment α satisfies the propositional formula φ, denoted $\alpha \models \varphi$, if α satisfies all the clauses of φ. Note that a formula may be satisfied by a partial assignment. In this case, unassigned variable are called *dont care*. A formula is said to be *satisfiable* (SAT) if there is at least one assignment that satisfies it; otherwise the formula is *unsatisfiable* (UNSAT).

An NP-complete problem

The SAT problems is the first NP-complete algorithm as proven by Stephen Cook in 1971 [6]. NP-completeness means that a SAT problem can be solved with a non deterministic machine in polynomial time (NP) and is also NP-hard. A problem is said NP-hard if everything in NP can be transformed into it in polynomial time. One of the most important unsolved problem in theoretical computer science is the P versus NP problem. This question is one of the seven millennium prize problems.

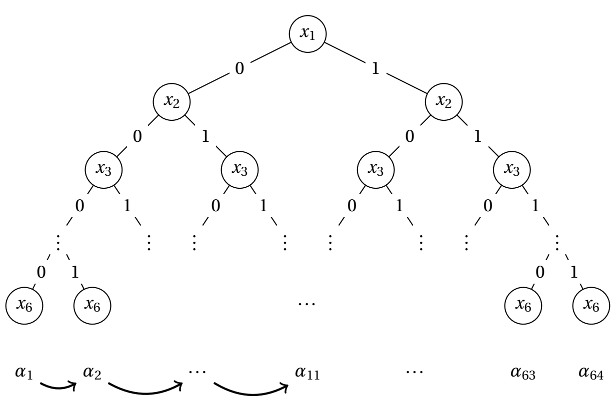
Some particular form of the SAT problems can be computed in linear time for 2-SAT [3] where each clause is in binary form i.e. size of two. Others particular form can be solved in polynomial time like Horn SAT [3] in which it suffice to apply *unit propagation* explained in section 2.1 until fix point. Xor-SATisfiability where each clause contains exclusive or belong also to the polynomial class.

Solving a SAT problem

Two kinds of algorithm exists to solve satisfiability problems. First, the *incomplete* algorithm which does not provide any guarantee that will eventually report either any satisfiable assignment or declare that formula is unsatisfiable. This kind of algorithm is out of scope of this thesis. Second, the *complete* algorithm, which provides a guarantee that if an assignment exists it will be reached or it will declare that formula is unsatisfiable. This section describes different *complete* algorithm to solve a propositional formula.

A naive algorithm

A naive approach to solve a SAT problem is to try all possible assignments. In total, for a propositional formula with n variables, it leads to 2^n assignments in the worth case. Figure 2.1 illustrate the search tree for a given problem with six variables. In this case α_{11} $(\neg x_1, \neg x_2, x_3, \neg x_4, x_5, \neg x_6)$ is found as a solution of the problem. In the general case, this algorithm is obviously intractable on real problems even for a formula with few variables.



$$\omega_1 = \{x_1, x_2, x_3\}$$
$$\omega_2 = \{x_4, x_5, x_6\}$$
$$\omega_3 = \{\neg x_1, \neg x_5\}$$
$$\omega_4 = \{\neg x_2, \neg x_4\}$$
$$\omega_5 = \{\neg x_3, \neg x_4\}$$
$$\omega_6 = \{\neg x_3, \neg x_6\}$$

Figure 2.1: All possible assignments for a problem with 6 variables

Davis Putnam Logemann Loveland (DPLL) algorithm

One of the first non memory intensive algorithm to solve the SAT problems is the Davis Putnam Logemann Loveland (DPLL) algorithm [8]. It explores a binary tree using depth first search as given in Algorithm 1. The construction of the tree is related to a *decision* literal (line 8) then, recursive call with each value are checked. When a leaf report UNSAT(line 5), other branches are explored. By recursive construction of the algorithm, when positive and negative value of a literal reach UNSAT, solver *backtracks* at most one level, this fact is called *chronological backtracking*. If all leaves report UNSAT the formula φ is unsatisfiable line 14. Finally, if any branch found a solution i.e. the problem is empty, formula is satisfiable corresponding assignment is returned (lines 10 and 12)

```
1  function DPLL(φ: CNF formula, α assignment)
2      returns an assignment if φ is SAT and UNSAT otherwise
3      φ, α ← unitPropagation(φ, α);
4      if {} ∈ φ then
5          return UNSAT;                       // This branch is UNSAT
6      if φ = {} then
7          return α;                           // φ is SAT
8      x ← assignDecisionLiteral();
9      if α ← DPLL(φ ∪ {x}, α) ≠ UNSAT then
10         return α
11     else if α ← DPLL(φ ∪ {¬x}, α) ≠ UNSAT then
12         return α
13     else
14         return UNSAT;                       // φ is UNSAT
```

Algorithm 1: The DPLL algorithm.

An important function in the DPLL algorithm is `unitPropagation` line 3 and it is detailed in Algorithm 2. It searches all unit clauses, then, to ensure satisfiability these literals must be true and added to the current assignment. Formula is then simplified as follow, clauses that contains this literal are already satisfied and can be deleted; negative literals are removed from the clause that belongs to him. This procedure ends when either no unit clause remains or an inconsistency was found (empty clause).

When DPLL algorithm is executed on the formula in Figure 2.1, after the decision of literal $\neg x_1$ and $\neg x_2$ unit propagation detects that x_3 must be true. This propagation prevents to explore assignments from α_1 to α_8. Moreover, application to unit propagation provokes more unit clauses and leads directly to a solution. An important part of efficiency of DPLL is due to choose the variable that divide the search tree made by the procedure `assignDecisionLiteral` . The objective of this function is to find a literal that will generate a maximum of unit propagation. Intuitively, decision literals can be viewed as

```
function unitPropagation(φ: CNF formula, α assignment)
    returns CNF formula and assignment α
  while {l} ∈ φ and {} ∉ φ do
      // Remove all clauses containing l, all literals ¬l
      φ ← φ|_l
      α ← α ∪ {l}
  return φ, α
```

Algorithm 2: Unit propagation

"guesses" and propagated literals can be viewed as "deductions". Finding a optimal variable is NP-Hard. Different heuristics exists to choose the decision variable, some of them will be presented in section 16.

Conflict Driven Clause Learning (CDCL) algorithm

The principal weakness of DPLL algorithm is to make same inconsistencies several times (principally due to chronological backtracking), incurring unnecessary CPU usage.
Conflict Driven Clause Learning (CDCL) algorithm 3 is another sound and complete algorithm to resolve a SAT problem and overcome principal weakness of DPLL.

Algorithm 3 gives an overview of CDCL, Like DPLL, it walks on a binary search tree. Initially, the current assignment is empty and decision level that indicated the depth of the search tree noted as dl is set to zero. Algorithm first applies unit propagation to the formula φ for the current assignment α (line 6). Note that it is exactly the same procedure as the one used for DPLL. An inconsistency or a *conflict* at level zero indicates that the formula is unsatisfiable, and the algorithm reports it (lines 8 and 9). When the conflict is occurring at a higher level, it reason was analyzed and a clause called *conflict clause* is deduced (line 10). This clause is *learnt* (line 12) (added to the formula). This clause is redundant from the current formula and so as it does not change the satisfiability of φ. It also avoids encountering a conflict with the same causes in the future. Working of this function will be presented thereafter. The analysis is completed by the computation of a *backjump,* solver unassign some literals and decrease the decision level (line 11). As the level can be much lower than the current assignment this is called *non chronological backtracking.* Finally, if no conflict appears, the algorithm chooses a new decision literal (lines 14 and 15). The above steps are repeated until the satisfiability status of the formula is determined.

Conflict Analysis

A conflict is an inconsistency discovered by the solver, a situation that requires for a variable to be set simultaneously to the $\top$and $\bot$value. Figure 2.2 shows an assignments that leads to a conflict. First the solver chose $\neg x_1$ as decision then $\neg x_6$ and then $\neg x_5$. This last one propagates x_4 which in turn propagates x_2 and x_3. On clause ω_1, x_3 needs to be $\top$and $\bot$in ω_5 so a conflict appears.

```
function CDCL(φ: CNF formula)
    returns ⊤ if φ is SAT and ⊥ otherwise
  dl ← 0;                                    // Current decision level
  α ← ∅;
  while not all variables are assigned do
    φ, α ← unitPropagation(φ|α, α);
    if {} ∈ φ then                           // A conflict occurs
      if dl = 0 then
        return ⊥;                            // φ is UNSAT
      ω ← analyzeConflict();
      dl ← backjumpAndRestartPolicies();
      φ ← φ ∪ {ω};
    else
      α ← α ∪ assignDecisionLiteral();
      dl ← dl + 1;
  return ⊤;                                  // φ is SAT
```

Algorithm 3: The CDCL algorithm.

$\omega_1 = \{x_1, x_2, x_3\}$	$\omega_1 = \{x_1, x_2, x_3\}$	$\omega_1 = \{x_1, x_2, x_3\}$	$\omega_1 = \{x_1, x_2, x_3\}$	$\omega_1 = \{x_1, x_2, x_3\}$
$\omega_2 = \{x_4, x_5, x_6\}$	$\omega_2 = \{x_4, x_5, x_6\}$	$\omega_2 = \{x_4, x_5, x_6\}$	$\omega_2 = \{x_4, x_5, x_6\}$	$\omega_2 = \{x_4, x_5, x_6\}$
$\omega_3 = \{\neg x_1, \neg x_5\}$	$\omega_3 = \{\neg x_1, \neg x_5\}$	$\omega_3 = \{\neg x_1, \neg x_5\}$	$\omega_3 = \{\neg x_1, \neg x_5\}$	$\omega_3 = \{\neg x_1, \neg x_5\}$
$\omega_4 = \{\neg x_2, \neg x_4\}$	$\omega_4 = \{\neg x_2, \neg x_4\}$	$\omega_4 = \{\neg x_2, \neg x_4\}$	$\omega_4 = \{\neg x_2, \neg x_4\}$	$\omega_4 = \{\neg x_2, \neg x_4\}$
$\omega_5 = \{\neg x_3, \neg x_4\}$	$\omega_5 = \{\neg x_3, \neg x_4\}$	$\omega_5 = \{\neg x_3, \neg x_4\}$	$\omega_5 = \{\neg x_3, \neg x_4\}$	$\omega_5 = \{\neg x_3, \neg x_4\}$
$\omega_6 = \{\neg x_3, \neg x_6\}$	$\omega_6 = \{\neg x_3, \neg x_6\}$	$\omega_6 = \{\neg x_3, \neg x_6\}$	$\omega_6 = \{\neg x_3, \neg x_6\}$	$\omega_6 = \{\neg x_3, \neg x_6\}$

$\alpha =$ D $\neg x_1$ → D $\neg x_6$ → D $\neg x_5$ → P x_4 → P $\neg x_2 \longrightarrow$ C

Figure 2.2: Decisions/Propagations that leads to a conflict

This series of decisions would provoke same propagation and leads to the same conflict. To avoid this situation, the solver needs to analyze the reason of the conflict with so called *implication graph*. Implication graph represents the current state of the solver proof system. It records every dependencies among variables and so it is updated either when a variable is assigned on decision/propagation or when a variable is unassigned. The implication graph is a directed acyclic graph (DAG) in which a vertex represents an assigned variable labeled as $l@dl(l)$ where l represents assigned literal and $dl(l)$ represents the decision level of the literal l. Root vertexes , that have no incoming edges, are literals chosen by decision heuristics and others are propagated literals. Incoming arcs labeled with a clause represents the *reason* of this propagation. This clause must be assertive i.e. all of its literals are false except one that are not yet assigned. Figure 2.3 shows implication graph of the previous example (fig. 2.2) until the conflict.

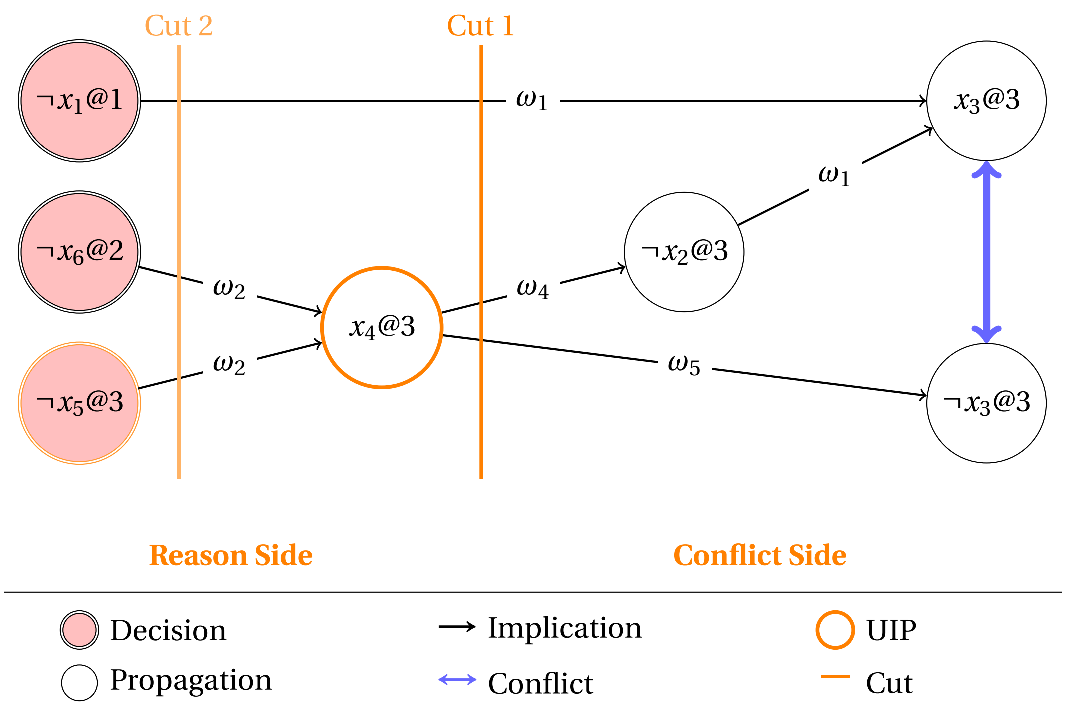


Figure 2.3: Implication graph

`analyzeConflict` procedure analyzes this graph to found the reason of the conflict. To do that, a search of *unique implication point* (UIP) is performed. UIP of a decision level in the implication graph is a variables which lies on every path from the decision to the conflict. Note that, there are many UIP for a given decision level. In such case, UIPs are ordered according to the distance with the contradiction. The first UIP is the closest to the conflict. It is well known that the first UIP provides the smallest set of assignment that is responsible for the contradiction [30]

An UIP divides the implication in two sides with a *cut*, the reason side contains decision variables that is responsible for the contradiction and the conflict side that contains the conflict. Note that, UIP is always is the reason side. Figure 2.3 depicts two cuts in the implication graph.

Once the reason side of a conflict is established, a conflict driven clause or conflict clause is produced. It purposes to avoid same contradiction. To build this clause, it suffices to negate literals that have an ongoing arc to the first cut that contains first UIP. In fig. 2.3, produced clause will be $\omega_l = \{x_1, \neg x_4\}$. Since the information of this clause is redundant with respect to the original formula. It can be added without any satisfiability restrictions. The conflict clause can be simplified using the implication graph [29].

`backjumpAndRestartPolicies` procedure is executed after producing the conflict clause. It will unassigned all decision until the first one that is responsible for the inconsistency. Add the conflict clause prune search space that contains no solution. This is the key point of the CDCL algorithm. The restart policy will be discussed in the next section. In our example Figure 2.3, the target decision level is one. The first UIP variable must be assertive, and will be propagated in the next loop of the solving algorithm. If a conflict implied only one level, the decision variable must be assigned to the opposite value at level zero. Roughly speaking, to ensure satisfiability of the formula, this literal must be true without any decision.

Heuristics

This sections gives an overview of different heuristics present in modern SAT solvers.

Decision heuristics. Variable used to divide problems have a huge impact on the overall solving time by the solver. Decision variable may impact the number of propagation and so the depth of the search tree.

Variable State Independent Decaying Sum (VSIDS) [25] is one of the decision heuristic and used nowadays in almost all solvers. Each variable has an activity and was increased by a multiplicative factor when it participate to the resolution of the conflict. A solver has thousands conflicts during the solving and so activity of variables are very volatile. Decision heuristics choose unassigned variable with the highest activity. The idea behind this heuristics is to solve "hard" part of problem at the top of the search tree. Hence, it is much more efficient when coupled with the restart heuristics.

Learning rate based branching (LRB [21]) is a most recent decision heuristics. It is a generalization of VSIDS and its goal is to optimize the *learning rate* (LR), defined as the ability to generate learnt clauses. The LRB of a variable is the weighted average (computed with *exponential recency weighted average* (ERWA)) value taken by its LR over the time. Unassigned variable with a highest LRB are chose as decision. The idea behind this heuristics is to keep variables that used to generate learnt clause in the search tree.

Restarts. Another important heuristic is *restart*. Basically the solver abandons it current assignment and so start from the top of the tree, while maintaining other information notably learnt clauses but also scores of variables in the decision heuristic. It prevents the solver to get stuck in "hard" (heavy tailing [15] part of the search space and can not escape due to backjump few levels after conflict resolution. Restart is best effort heuristics, hoping that, with more information, a better assignments was made. Hence, in practice, SAT solvers usually restarts after a certain number of conflict. Empirically a solver with restart has a better results [16] and is today used in almost all state of the art solvers.

Preprocessing / Inprocessing

In order to optimize resolution time by the solver, some transformation to simplify the original formula can be applied. This is done by *preprocessing* engine before the start of solving. When it is used at some point during the solving, usually after a restart, it is called *inprocessing*.

Simplification of the formula is made by removing clauses and/or variables.
Variable elimination simplification is based on *Resolution inference rule*. Suppose two clauses $\omega_1 = \{x_1, x_i, ..., x_j\}$ and $\omega_2 = \{\neg x_1, y_i, ..., y_j\}$. The resolution inference rule allows to derive clause $\omega_3 = \{x_i, ..., x_j, y_i, ..., y_j\}$ which is called the *resolvent* as it results from resolving two clauses on the literal x_1 and $\neg x_1$. Moreover, applying variable elimination until either an empty clause is derived (unsatisfiable formula) or no more application of the resolution are possible (satisfiable formula). This is a complete algorithm to solve a SAT problem. Its major issue is to explicitly generate all resolvent and can be exponential in CNF size. Hence, the memory of computer will be limiting factor.

Subsumption is a simple principle to remove clauses. Suppose two clauses ω_1 and ω_2 such that $\omega_1 \subset \omega_2$, then ω_2 can be safely removed from the original formula. *Self Subsuming resolution* is a principle that use resolution rule and subsumption. Resolvent clause subsumes the original one. Example $\omega_1 = \{x_1, \neg x_2, x_3\}$ and $\omega_2 = \{x_1, \neg x_2, x_3, x_4\}$, then resolvent clause will be $\omega_3 = \{x_1, x_3\}$ which subsumes ω_2. This principle is presents in `SatElite` [12] preprocessor engine and used in almost all modern SAT solvers.

Other simplification techniques exists such that *Gaussian elimination* which detect sub formula in a xor-SAT form and solve it in a polynomial complexity. Moreover, this technique can also be used as inprocessing [28].

Some techniques exploits the structure of the original formula and add relevant clauses to speed up the resolution time of the SAT solver. One of them use community structure of the formula to find good clauses to add into. A preprocessor engine doing that is `modprep` [2]. Usage of symmetries also add relevant clause in the formula and will be detailed in the next chapter.

Parallel SAT solving

With the emergence of multi core architectures and increasing power of computer, one way to optimize the solving of a SAT problem is the exploitation of these cores. Effectively, SAT problems are a good candidate for parallelism. *Portfolio* is a technique that launches several SAT solver in parallel with different heuristics (decisions, restarts, ...) that communicates or not between us. When one of them found a solution or found that none exists, the overall computation is finished. Another technique to make parallel SAT solver exists and called *divide and conquer* in which the search space was divided dynamically according to positive and negative value of the decision literal. Several solvers cooperate to found solution, each of them are assigned to sub formula induced by the division. Some specific techniques like load balancing and work stealing is applied to avoid a solver to be idle. A recent framework *PaInleSS* (a Framework for Parallel SAT Solving) can be used to easily create a new parallel SAT solver with different heuristics [19] [20]. Authors of this framework win the parallel tracks of SAT competition [1] in 2018.

[1] `http://www.satcompetition.org/`

CHAPTER 3

SYMMETRY AND SAT

Despite SAT solving is an NP-complete algorithm, it works well on many real industrial problems. This is principally due to capacity to cut off search space with learning clause. Another way to cut off search space is the exploitation of symmetry. Some instances exhibit symmetries and not taking them into account forces solvers to needlessly explore isomorphic search space. Hakan: symmetry leaves objects invariant

3.1 Group basics

As symmetries is a belongs to a branch of mathematics called theory group. This section gives us an overview of group theory.

Groups

A *group* is a structure $\langle G, * \rangle$, where G is a non-empty set and $*$ a binary operation such as the following axioms are satisfied:

- *associativity*: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
- *closure*: $\forall a, b \in G, a * b \in G.$
- *identity*: $\forall a \in G, \exists e$ such that $a * e = e * a = a$
- *inverse*: $\forall a \in G, \exists b \in G,$ commonly denoted a^{-1} such that $a * a^{-1} = a^{-1} * a = e$

Note that *commutativity* is not required i.e $a * b = b * a$, for $a, b \in G$. The group is *abelian* if it satisfies the commutativity rule. Moreover, the last definition leads to important properties which are: i) uniqueness of the identity element. To prove this property, assume $\langle G, * \rangle$ a group with two identity elements e and f then $e = e * f = f$. ii) uniqueness of the inverse element. To prove this property, suppose that an element x_1 has two inverses, denoted b

and c in groups $\langle G, * \rangle$, then

$$\begin{aligned} b &= b * e & \\ &= b * (a * c) & c \text{ is an inverse of } a, \text{so } e = a * c \\ &= (b * a) * c & \textit{associativity} \text{ rule} \\ &= e * c & b \text{ is an inverse of } a, \text{so } e = a * b \\ &= c & \textit{identity} \text{ rule} \end{aligned}$$

The structure $\langle G, * \rangle$ is denoted as G when clear from context that G is a group with a binary operation. In this thesis, we are interested only with the *finite* groups i.e with a finite number of elements. Given a group G, a *subgroup* is a non-empty subset of G which is also a group with the same binary operation. If H is a subgroup of G, we denote as $H \leq G$. A group has at least two subgroups: i) the subgroup composed by the identity element $\{e\}$, denoted *trivial* subgroup. All other subgroups are *nontrivial*; ii) the subgroup composed by itself, denoted *improper* subgroup. All other subgroups are *proper*.

Generators of a group

If every element in a group G can be expressed as a linear combination of a set of a group of elements S = $\{g_1, g_2, ..., g_n\}$ then we say G is generated by the S. we denote this as G = $\langle S \rangle$ = $\langle \{g_1, g_2, ..., g_n\} \rangle$

Permutation groups

A *permutation* is a bijection from a set X to itself.
Example: given a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$,

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_1 & x_4 & x_6 & x_5 \end{pmatrix}$$

g is a permutation that maps x_1 to x_2, x_2 to x_3, x_3 to x_1, x_4 to x_4, x_5 to x_6 and x_6 to x_5. Permutations are generally written in *cycle notation*, the self-mapped elements are omitted. So the permutation in cycle notation will be

$$g = (x_1 \;\; x_2 \;\; x_3)\,(x_5 \;\; x_6)$$

We say *support* of the permutation g noted $supp_g$ the elements that not mapped to themselves:

$$supp_g = \{x \in X \mid g.x \neq x\}$$

A variable x is *stable* by a permutation g if $x \notin supp_g$. A clause ω is *stabilized* by a permutation g if $\omega \cap supp_g = \emptyset$.

A set of permutations of a given set X form a group G, with the composition operation ($\circ$) and is called *permutation group*. The *symmetric group* is the set of all possible permutations of a set X and noted $\mathfrak{S}(X)$. So, a *permutation group* is a subgroup of $\mathfrak{S}(X)$.

A permutation group G induces an *equivalence relation* on the set of elements X being permuted. Two elements $x_1, x_2 \in X$ are equivalent if there exists a permutation $g \in G$ such that $g.x_1 = x_2$. The equivalence relation partition X into *equivalence classes* referred to as the *orbits* of X under G. The orbit of an element x under group G (or simply orbit of x when clear from the context) is the set. $[x]_G = \{g.x \mid g \in G\}$

3.2 Symmetries in SAT

The previous mathematical definition of group theory is applied to the CNF formula. The symmetric group of permutations of $\mathcal{V}$ (i.e. bijections from $\mathcal{V}$ to $\mathcal{V}$) is noted $\mathfrak{S}(\mathcal{V})$. The group $\mathfrak{S}(\mathcal{V})$ naturally acts on the set of literals: for $g \in \mathfrak{S}(\mathcal{V})$ and a literal $\ell \in \mathcal{L}$, $g.\ell = g(\ell)$ if ℓ it is a positive literal, $g.\ell = \neg g(\neg \ell)$ if ℓ is a negative literal. The group $\mathfrak{S}(\mathcal{V})$ also acts on assignments possibly partial of $\mathcal{V}$ as follows:

$$\forall g \in \mathfrak{S}(\mathcal{V}),\ \alpha \in Ass(\mathcal{V}),\ g.\alpha = \{g.\ell \mid \ell \in \alpha\}.$$

We say that $g \in \mathfrak{S}(\mathcal{V})$ is a symmetry of φ if following conditions holds:

- permutation fixes the formula, $g.\varphi = \varphi$
- g commutes with the negation: $g.\neg l = \neg g.l$

The set of symmetries of φ is noted $G_(\varphi)$ and is a subgroup of $\mathfrak{S}(\mathcal{V})$. Symmetries of a formula φ preserves the satisfaction, for every *complete* assignment α:

$$\alpha \models \varphi \Leftrightarrow g.\alpha \models \varphi$$

3.3 Symmetry detection in SAT

For the detection of symmetries in SAT, we fist introduce the graph automorphism notion. Given a colored graph $G = (V, E, \gamma)$, with vertex set $V \in [1, n]$, edge set E and γ a function that apply a mapping : $V \to C$ where C is a set of *colors*. An automorphism of G is a permutation from its vertices $g : V \to V$ such that:

- $\forall (u, v) \in E \implies (g.u, g.v) \in E$
- $\forall v \in V, \gamma(v) = \gamma(g.v)$

The graph automorphism problem is to find if a given graph has a non-trivial permutation group. The computational complexity of this algorithm is conjectured to be strictly between P and NP. Several tools exist to tackle this problem like `saucy3` [18], `bliss` [17], `nauty` [24], etc.

There exists different ways to encode a SAT problem, which leads to different symmetries in these problems. When a symmetry depends on the structure of the problem, we say *syntactic* symmetries. In contrast, symmetries were *semantic*, when it is not inherent to the encoding. To find symmetries in SAT problem, the formula is transformed into colored graphs and an automorphism tool is applied onto. Specifically, given a formula φ with m clauses over n variables, the graph is constructed as follows [5]:

- *clause nodes*: represent each of the m clauses by a node with color 0;
- *literal nodes*: represent each of the l literals by a node with color 1;
- *clauses edges*: connect a clause to its literal by linking the corresponding clause node and literal nodes;
- *boolean consistency edges*: connect each pair of literals that correspond to the same variables.

$\omega_1 = \{x_1, x_2, x_3\}$
$\omega_2 = \{x_4, x_5, x_6\}$
$\omega_3 = \{x_1, x_4\}$
$\omega_4 = \{x_2, x_5\}$
$\omega_5 = \{x_3, x_6\}$
$\omega_6 = \{\neg x_1, \neg x_2\}$
$\omega_7 = \{\neg x_1, \neg x_3\}$
$\omega_8 = \{\neg x_2, \neg x_3\}$
$\omega_9 = \{\neg x_4, \neg x_5\}$
$\omega_{10} = \{\neg x_4, \neg x_6\}$
$\omega_{11} = \{\neg x_5, \neg x_6\}$

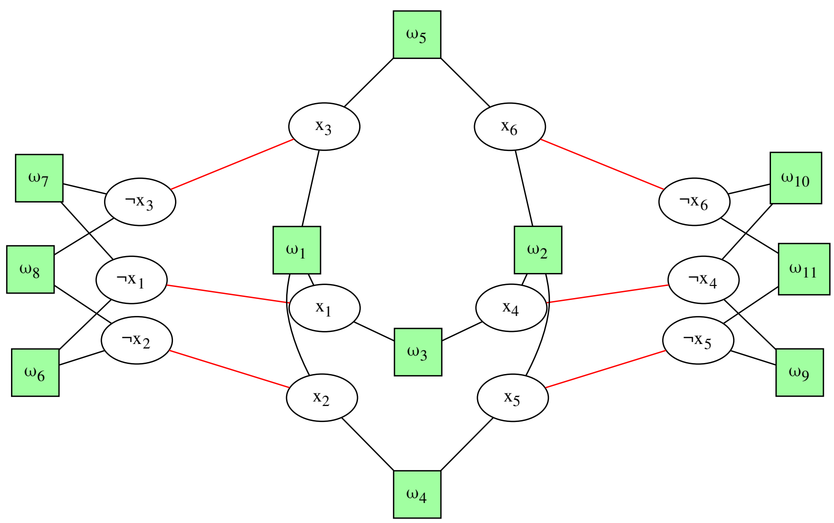


Figure 3.1: Example of constructed symmetry graph for a given CNF

Figure 3.1 shows the graph representation of a CNF. This problem has 6 variables and 11 clauses. So, the graph will have 12 + 11 = 33 vertexes where 12 represents literal vertexes (circle in the figure) and 11 represents the number of clause vertexes (square in the figure). The graph will also have 6 + 24 = 30 edges, 6 for boolean consistency (Hakan: XXX color in the figure) and 24 edges that rely clause vertexes to the literals.

An optimization of this graph is possible with the usage of binary clauses i.e. a clause with only two literals. The clause node can be omitted and we connect the two literals. As we cannot distinguish between the optimized edge and boolean consistency edges, we must check if the produced permutations are spurious. To do so, as we ensure the permutation commutes with the negation it suffices to check: $\forall x \in supp(g), g.\neg x = \neg g.x$. Roughly speaking, we check if the image of the negation of x is equals to the negation of the image of x, for each element x in the support of the permutation. This optimization allows to compute symmetries of the problem more efficiently. In the previous example, the graph has deleted 12 nodes and 12 edges. More generally, the graph removes as many nodes and edges as binary clauses on the formula. Figure 3.2 represents the optimized version the graph.

$$
\begin{aligned}
\omega_1 &= \{x_1, x_2, x_3\} \\
\omega_2 &= \{x_4, x_5, x_6\} \\
\omega_3 &= \{x_1, x_4\} \\
\omega_4 &= \{x_2, x_5\} \\
\omega_5 &= \{x_3, x_6\} \\
\omega_6 &= \{\neg x_1, \neg x_2\} \\
\omega_7 &= \{\neg x_1, \neg x_3\} \\
\omega_8 &= \{\neg x_2, \neg x_3\} \\
\omega_9 &= \{\neg x_4, \neg x_5\} \\
\omega_{10} &= \{\neg x_4, \neg x_6\} \\
\omega_{11} &= \{\neg x_5, \neg x_6\}
\end{aligned}
$$

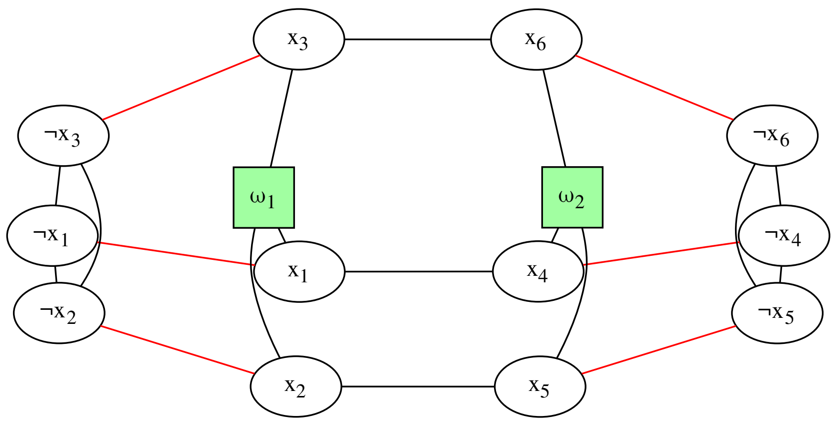


Figure 3.2: Example of constructed symmetry graph for a given CNF

After the construction of a such graph, a graph automorphism tools take it as input and give the set of generators as output.With the previous graph, the following generators are obtained:

$$
\begin{aligned}
g_1 &= (x_2 \;\; x_3)(x_5 \;\; x_6)(\neg x_2 \;\; \neg x_3)(\neg x_5 \;\; \neg x_6) \\
g_2 &= (x_1 \;\; x_2)(x_4 \;\; x_5)(\neg x_1 \;\; \neg x_2)(\neg x_4 \;\; \neg x_5) \\
g_3 &= (x_1 \;\; x_4)(x_2 \;\; x_5)(x_3 \;\; x_6)(\neg x_1 \;\; \neg x_4)(\neg x_2 \;\; \neg x_5)(\neg x_3 \;\; \neg x_6)
\end{aligned}
$$

These permutations form a permutation group and so induce an equivalence relation. Figure 3.3 shows graphical representation of an orbit, where each node represents a literal. Two literals are linked with an arc if it exists a permutation that maps one to the other. An orbit must be a *strongly connected component* (SCC). Some permutations have a special form like two-dimensional array as in this example. A further section (3.4) shows how to exploit this special form.

3.4 Usage of symmetries

Symmetry breaking aims at eliminating symmetry, either by *statically* posting symmetry breaking constraints that invalidate symmetric assignments, or by altering the search space *dynamically* to avoid symmetric search paths.

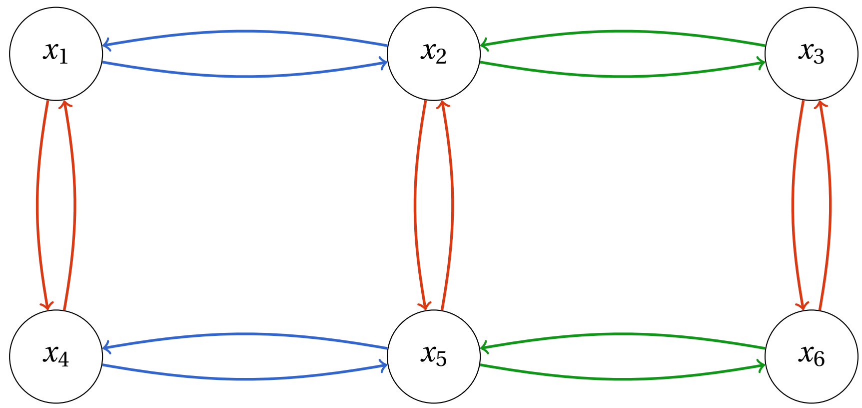


Figure 3.3: Graphical representation of an orbit

Static symmetry breaking

One way to exploit symmetry properties is to forbid equivalent assignments except one. Let introduce an ordering relation between the assignments.

Definition 1 (Assignments ordering)**.** *We assume a total order, $\prec$, on $\mathcal{V}$. Given two assignments $(\alpha,\beta) \in Ass(\mathcal{V})^2$, we say that α is strictly smaller than β, noted $\alpha < \beta$, if there exists a variable $v \in \mathcal{V}$ such that:*

- *for all $v' \prec v$, either $v' \in \alpha \cap \beta$ or $\neg v' \in \alpha \cap \beta$.*
- *$\neg v \in \alpha$ and $v \in \beta$* [1]*.*

In other words, the prefix of both assignment is equal according to the ordering relation $\prec$ and the next variable v has a different value, $\alpha(v) = \bot, \beta(v) = \top$, then $\alpha < \beta$. Note that $<$ coincides with the lexicographical order on *complete* assignments. Furthermore, the $<$ relation is monotonic as expressed in the following proposition:

Proposition 1 (Monotonicity of assignments ordering)**.** *Let $(\alpha,\alpha',\beta,\beta') \in Ass(\mathcal{V})^4$ be four assignments.*

$$\text{If } \alpha \subseteq \alpha' \text{ and } \beta \subseteq \beta', \text{ then } \alpha < \beta \implies \alpha' < \beta'$$

Proof. The proposition follows on directly from definition 1. □

Given a formula φ and its group of symmetry G, the *orbit of α under G* (or simply the *orbit of α* when G is clear from the context) is the set $[\alpha]_G = \{g.\alpha \mid g \in G\}$. The lexicographic leader (*lex-leader* for short) of an orbit $[\alpha]_G$ is defined by $min_<([\alpha]_G)$. This *lex-leader* is unique because the lexicographic order is a total order. The optimal approach to solve a symmetric SAT problem would be to explore only one assignment per orbit (for instance

[1]We could have chosen as well $v \in \alpha$ and $\neg v \in \beta$ without loss of generality.

each lex-leader). Figure 3.4 shows different orbits, each dot in an orbit (ellipse in the figure) is an assignment, and the lex-leader is the empty red one. To avoid exploring symmetry search space, *symmetry breaking predicates* (SBP) also called *lex-leader constraints* are added to the formula. These constraints are only true for the *lex-leader* [7] and prevent other assignments from being explored.

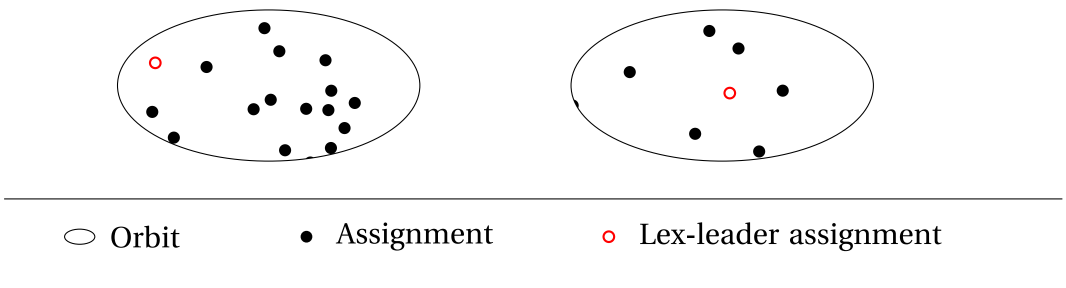


Figure 3.4: Show lex-leader per orbit

Lex-leader predicates for a permutation $g \in G_\varphi$ is defined as :

$$LL_g = \forall i : (\forall j < i : x_j = g.x_j) \Rightarrow x_i \preceq g.x_i$$

In other words, each assignment whose have a variable such that its image under g is smaller according to the ordering relation $\prec$, is pruned by LL_g. Conjunction of LL_g, for all permutations $g \in G_\varphi$ results a sound and complete symmetry breaking predicates also called *full symmetry breaking*. Only lex-leader assignment will be visited per orbit. Hence, finding the lex-leader of an orbit is computationally hard [23]. Conjunction of LL_g for some $G \subset G_\varphi$ results a symmetry breaking predicates that aims to visit at least one assignment per orbit and called is *partial symmetry breaking*. In both cases, the set of symmetry breaking predicates generated is denoted as ψ. Since a group may have a exponential number of permutations, all symmetry breaking predicates belongs to the group must be generated to ensure full symmetry breaking. These constraints will overload the solver and slow down its core principle (unit propagation). Hence, slow down overall time computation. Conversely, partial symmetry breaking adds few constraints and bring often considerable reduction of the search space. Generally, the set of generators produced by automorphism tool is chosen as a subgroup. Partial symmetry breaking gives a good trade off between the number of generated constraints and reduction of the search space.

Theorem 1 (Satisfiability preservation SBPs)**.** *Let φ be a formula and ψ the computed SBPs for the set of symmetries in G_φ:*

$$\varphi \textit{ and } \varphi \wedge \psi \textit{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under G_φ. Thus, g does not contradict β, which implies that $\beta \models \psi$. $\square$

Generation of lex-leader constraints proposed by Crawford et al. [7] is defined as follows:

$$LL_g = \forall i : (\forall j < i : x_j = g.x_j) \Rightarrow \neg x_i \vee g.x_i$$

Figure 3.5 shows an example of generated clauses for the permutation g_3 of the previous example and a lexicographic order. Last constraint present in the figure produce tautological clause, effectively variable x_1 or x_4 are present in both polarity. The constraints of other variables produce also tautological clauses.

Order	:	$x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \mid (\bot < \top)$
Permutation	:	$g_3 = (x_1 \ \ x_4)(x_2 \ \ x_5)(x_3 \ \ x_6)(\neg x_1 \ \ \neg x_4)(\neg x_2 \ \ \neg x_5)(\neg x_3 \ \ \neg x_6)$

Constraints	Generated SBP
$x_1 \preceq x_4$	$\neg x_1 \vee x_4$
$x_1 = x_4 \Rightarrow x_2 \preceq x_5$	$x_1 \vee x_4 \vee \neg x_2 \vee x_5$ $\neg x_1 \vee \neg x_4 \vee \neg x_2 \vee x_5$
$x_1 = x_4 \wedge x_2 = x_5 \Rightarrow x_3 \preceq x_6$	$x_1 \vee x_4 \vee x_2 \vee x_5 \vee \neg x_3 \vee x_6$ $\neg x_1 \vee \neg x_4 \vee x_2 \vee x_5 \vee \neg x_3 \vee x_6$ $x_1 \vee x_4 \vee \neg x_2 \vee \neg x_5 \vee \neg x_3 \vee x_6$ $\neg x_1 \vee \neg x_4 \vee \neg x_2 \vee \neg x_5 \vee \neg x_3 \vee x_6$
$x_1 = x_4 \wedge x_2 = x_5 \wedge x_3 = x_6 \Rightarrow x_4 \preceq x_1$	$x_1 \vee x_4 \vee x_2 \vee x_5 \vee x_3 \vee x_6 \vee \neg x_4 \vee x_1$ $\cdots$ $\neg x_1 \vee \neg x_4 \vee x_2 \vee x_5 \vee x_3 \vee x_6 \neg x_4 \vee x_1$ $\cdots$

Figure 3.5: Example of generated SBPs for one permutation

Moreover, the number of clauses generated per constraint increase exponentially with the number of variable present in the permutation. Hence, Aloul et al [1] propose a more compact representation of symmetry breaking predicates.

Let g a permutation, let $supp_g = \{x_1, \cdots, x_n\}$ the support of the permutation g be ordered such that $x_i \preceq x_j$ iff $i \leq j$ and let $\{y_0, \cdots, y_n\}$ be a set of auxiliary variables disjoint from $supp_g$. These auxiliary variables encode equality of literals in such y_0 is set as an unit clause and encodes the first equality. Following clauses encode a compact lex-leader for a permutation:

$$\begin{array}{ll|ll} \neg y_i \vee \neg x_{i-1} \vee \neg x_i \vee g.x_i & 1 \leq i \leq n & \neg y_i \vee \neg x_{i-1} \vee \neg y_{i+1} & 1 \leq i \leq n \\ \neg y_i \vee g.x_{i-1} \vee \neg x_i \vee g.x_i & 1 \leq i \leq n & \neg y_i \vee g.x_{i-1} \vee \neg y_{i+1} & 1 \leq i \leq n \end{array}$$

Figure 3.6 shows the compact encoding of generated clauses. This form grows linearly with the number of variables. Auxiliary variable encodes the equality of two literals allows to achieve this reduction. Three auxiliary variables are introduced in this example x_7, x_8, x_9 such that x_7 encode the equality of x_1 and x_4, x_8 equality of x_2 and x_5, and x_9 equality of x_3 and x_6.

Order	:	$x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \;\mid\; (\bot < \top)$
Permutation	:	$g_3 = (x_1 \;\; x_4)(x_2 \;\; x_5)(x_3 \;\; x_6)(\neg x_1 \;\; \neg x_4)(\neg x_2 \;\; \neg x_5)(\neg x_3 \;\; \neg x_6)$

Constraints	Generated SBP
$x_1 \preceq x_4$	$\neg x_1 \lor x_4$ x_7
$x_1 = x_4 \Rightarrow x_2 \preceq x_5$	$\neg x_7 \lor \neg x_1 \lor \neg x_2 \lor x_5$ $\neg x_7 \lor \neg x_1 \lor x_8$ $\neg x_7 \lor x_4 \lor \neg x_2 \lor x_5$ $\neg x_7 \lor x_4 \lor x_8$
$x_1 = x_4 \land x_2 = x_5 \Rightarrow x_3 \preceq x_6$	$\neg x_8 \lor \neg x_2 \lor \neg x_3 \lor x_6$ $\neg x_8 \lor \neg x_2 \lor x_9$ $\neg x_8 \lor x_5 \lor \neg x_3 \lor x_6$ $\neg x_8 \lor x_5 \lor \neg x_9$

Figure 3.6: Example of compact generated SBPs for one permutation

`Shatter` [1] is a tool fot partial symmetry breaking that computes symmetry with `saucy3` automorphism tool and generate a new formula with compact lex-leader encoding. It uses only generators given by the automorphism tool. Following table shows the number of symmetry breaking predicates clauses and the number of auxiliary variables added to the original formula.

battleship-12-12-unsat	936	144	1498	378
battleship-12-23-sat	1662	276	5464	1375
battleship-14-26-sat	2562	364	3688	929
battleship-14-27-sat	2653	378	7222	1814
battleship-16-16-unsat	2176	256	4388	1102
battleship-16-31-sat	3976	496	12094	3035
battleship-24-57-sat	16308	1368	40372	10113
chnl10_11	1122	220	2416	615
chnl10_12	1344	240	2736	696
chnl10_13	1586	260	3252	826
chnl11_12	1476	264	3204	813
chnl11_13	1742	286	3636	922
chnl11_20	4220	440	6760	1710
fpga10_15_uns_rcr	2130	300	4580	1160
fpga10_20_uns_rcr	3840	400	6768	1712
fpga11_12_uns_rcr	1476	264	3704	938
fpga11_13_uns_rcr	1742	286	4076	1032
fpga11_14_uns_rcr	2030	308	4740	1199
fpga11_15_uns_rcr	2340	330	5196	1314
fpga11_20_uns_rcr	4220	440	7864	1986
hole010	561	110	1054	269
hole015	1816	240	3280	828
hole020	4221	420	6478	1630
hole030	13981	930	21322	5346
hole040	32841	1640	44934	11254
hole050	63801	2550	81682	20446
Urq6_5	1756	180	109	0
Urq7_5	2194	240	143	0
Urq8_5	3252	327	200	0
x1_40	314	118	42	1
x1_80	634	238	80	0

An improvement of static symmetry breaking was made by Devriendt et al [10] with a tool called `BreakID`. It exploits some properties from the structure of generators. On some circumstance a linear number of constraints can break all group. The other tries to add a maximum of binary clauses that is useful because it can participate often to unit propagation and so to the conflict analysis.

Special form of the group

Some formula presents a specific type of symmetry called *row (column) interchangeability*, when a subset of variables is structured as a two-dimensional matrix. Each row (column) is interchangeable with the symmetries. This form of symmetry is common in different kind of problem like pigeon hole problems in which pigeons and holes are interchangeable or in the delivery system in which trucks of a fleet are interchangeable. Usage of row (column) interchangeability can significantly improve SAT performance. Effectively symmetries can be eliminated by the addition of only a linear number of symmetry-breaking constraints [14]. One condition must be satisfied to ensure this linear number of constraints: lexicographic order needs to respect the structure of the matrix. In practice, automorphism tools give only the set generators which contains no information on the structure of the group. Authors of `BreakID` [10] develop an algorithm to detect this specific structure and exploit it.

Binary lex-leader constraints

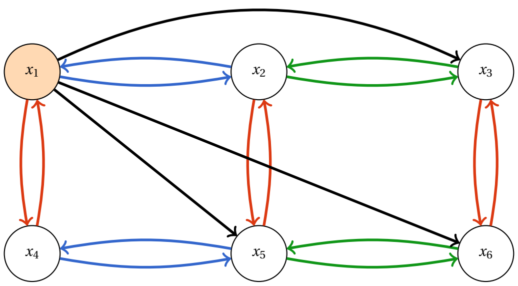
`BreakID` has another approach that aims to post many lex-leader constraints. The first constraint of symmetry breaking predicates must produce a binary clause. Building many binary clauses is possible without enumerating the whole symmetry group. It suffices to compute the orbit of the smallest variable according to the ordering relation. As the orbit can be seen as a strong connected component, it must exist a permutation that permutes the smallest variable with all other variables in the same orbit. Then, as many binary clauses as variables (without the smallest variable) in the orbit can be added to the formula. Constructing a sequence of subgroups that stabilize the smallest variable (i.e. not have the smallest variable in its support) results to new binary clauses. This sequence ends when trivial subgroup is reached and is called a *stabilizer chain*.

Figure 3.7 shows application of the generation of binary clauses. In the example, considered group has three permutations and its graphical representation is showing. Given the lexicographic order, the smallest variable is x_1 and all other variables are in its orbits. According to the ordering relation, five symmetry breaking predicates are generated with the formula $\neg x_1 \vee g.x_1$. Then, subgroup that stabilizes x_1 is computed and it remains only one permutation g_2. As it smallest variable is x_2, the constraint $\neg x_2 \vee x_3$ is generated. Stabilizer chain leads to trivial group and no more binary clauses are generated. In total, six binary clauses is generated without adding any auxiliary variables. Moreover, a property can be observed, when the smallest variable have the greatest value ($\top$in this case), all variables in the orbits must have the same value.

The size of the stabilizer chain is heavily dependent of the chose lexicographic order. More stabilizer discard permutations and more trivial subgroup is reached quickly and fewer binary clauses are generated. An incremental order is proposed to optimize the number of generates binary clauses. First, orbit of all variables is computed and the variable with fewest number of occurrence is choosing among the biggest orbit. The idea is that biggest orbit produces more clauses and the variable appearing in few permutations reduce the number of stabilized permutations. This procedure is applied until trivial group is reached. At the end, remaining variables are added to the order.

$$\text{Order} \quad : \quad x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \quad | \quad (\bot < \top)$$

$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(\neg x_2\ \ \neg x_3)(\neg x_5\ \ \neg x_6)$
$g_2 = (x_1\ \ x_2)(x_4\ \ x_5)(\neg x_1\ \ \neg x_2)(\neg x_4\ \ \neg x_5)$
$g_3 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)(\neg x_1\ \ \neg x_4)(\neg x_2\ \ \neg x_5)(\neg x_3\ \ \neg x_6)$

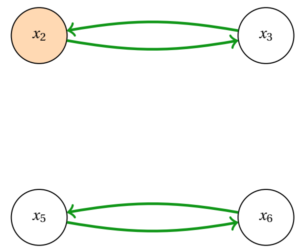


$\omega_1 = \{\neg x_1, x_2\}$ $\omega_4 = \{\neg x_1, x_5\}$

$\omega_2 = \{\neg x_1, x_3\}$ $\omega_5 = \{\neg x_1, x_6\}$

$\omega_3 = \{\neg x_1, x_4\}$

$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(\neg x_2\ \ \neg x_3)(\neg x_5\ \ \neg x_6)$



$\omega_6 = \{\neg x_2, x_3\}$

Figure 3.7: Caption

Evaluation of performance

Hakan: Faire une section pour An extremely important point is the chosen lexicographic order. Variable ordering may impact the number of generated constraints and so the performance of the underlying SAT solver. Different orders are studied in the literature. One of the simplest orders was the sorted variables according to their numbers. Some other order exists and exploit structural properties of the problem. In particular, the orbits of the variables in different ways. For example, the variables are chosen with their number of occurrences in the initial problem. This order is equivalent to put largest orbit first and so on, because each variable on the same orbit must have the same number of occurrences. Another example of exploiting structural property of the orbit is the usage of *stabilizer*. The order chooses a variable which maximize the number of stabilized permutations, removes the not stabilized ones and loop over until the empty set. The remaining variables are added to the order to get a complete order.

Conclusion

Static symmetry breaking acts as a preprocessor that augment the initial formula with symmetry breaking predicates. These constraints avoid exploration of symmetric search space. In the general case, the number of these clauses is often too large to be effectively handled by a SAT solver [23]. On the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be perfect and its effectiveness depends heavily on the heuristically chosen symmetries [5]. Some recent works brings some optimizations and solves more instances. Despite these optimizations and the good reduction of the search space with symmetries, some formula still intractable for a state-of-the-art SAT solver.

Static symmetry breaking acts as a preprocessor, a new formula is produced and solver takes it as input. No modification of the existing solver is required. Another disadvantage is that the solver is influenced by SBPs and explore the search space with a different manner and can affect performance negatively.

Dynamic symmetry breaking

Dynamic symmetry breaking approaches aims to exploits symmetries during the solving by altering search space dynamically. In the literature, different approaches of dynamic symmetry breaking are used to reduce the search space. This section presents some of them.

SymChaff

One of the first dy namic symmetry breaking approaches is *SymChaff* [27] and is applicable only on special groups where all couple of variables are symmetric section 3.4. The idea of this approach is to treat each orbit like a *symbolic variable*, i.e. instead of considering a single variable (branching), all symmetric variables are considered at the same time and so backtracked at the same time (k-branching). In this special case of groups, the number of orbits is easy to compute but the order in which they would be applied has a tremendous impact of the solver performance. In the general case, when we take any groups computing the number of orbits will be very difficult and this approach will be intractable.

Symmetry Propagation

A different approach can be used to reduce search space using symmetries is *symmetry propagation* [11]. The general idea of this approach is to propagate symmetrical literals of those already propagated. In other words, it accelerates the tree traversal by “transforming some guessing (decisions) to deductions (propagation)”. Indeed, problem that presents symmetries makes possible to deduce some value for the variables that would be guessed if symmetry properties were ignored. These deductions will reduce the overall tree traversal depth and hence eventually accelerate the solving process. To explain this approach, some definitions are required.

Definition 2 (Logical consequence)**.** *A formula ϕ is a logical consequence of a formula φ denoted by $\varphi \models \phi$ if for all assignment α satisfying φ, it also satisfies ϕ. Two formulas are* logically equivalent *if each is a logical consequence of the other.*

Proposition 2 (Symmetry propagation)**.** *Let φ be a formula, α an assignment and l a literal. If g is a symmetry (permutation) of $\varphi \cup \alpha$ and $\varphi \models \{l\}$, then $\varphi \cup \alpha \models g.\{l\}$ is also true.*

In other words, if a literal l was propagated by the solver and g is a *valid* symmetry for the sub problem $\varphi \cup \alpha$ (in which all satisfied clauses and false literals are removed), so , the solver can also propagate the symmetrical of l. The problem here is to determinate which symmetries are valid for the formula $\varphi \cup \alpha$.

Definition 3 (Active symmetry)**.** *A symmetry g is called active under a partial assignment α if $g.\alpha = \alpha$*

The definition 3 leads to the following proposition:

Proposition 3. *Let φ a formula and α a partial assignment. Let g a symmetry of φ, if g is active under the assignment α, then g is also a symmetry of $\varphi \cup \alpha$*

The previous proposition states that an active symmetry g for a partial assignment α still valid for the formula $\varphi \cup \alpha$. So when a literal l is propagated, and a symmetry g is active for a partial assignment α, the solver can also propagate $g.l$. Moreover, the group theory allows to compose permutations with the composition operator $\circ$ and the composition of two active symmetries is also an active symmetry so the solver can also propagate. $g^2.l, g^3.l, \ldots$

Devriendt et al improves the active symmetries in the SAT context, introducing *weakly active* symmetries.

Definition 4 (Weakly active symmetry)**.** *Let φ a formula and (δ, α, γ) a state of a CDCL solver in which δ is the set of decisions α is the current assignment and γ the reasons of the learned clauses. Then a symmetry g is weakly active if $g.\delta \subseteq \alpha$*

This definition leads to the following proposition:

Proposition 4. *Let φ be a formula, α an assignment. If there exists a subset $\delta \subseteq \alpha$ and a symmetry g of φ such that $g.\delta \subseteq \alpha$ and $\varphi \cup \delta \models \varphi \cup \alpha$, then g also is a symmetry of $\varphi \cup \alpha$.*

In other words, we can detect with a minimal effort, the symmetries of $\varphi \cup \alpha$ by keeping track of the set of variables δ, which are in a state-of-the-art complete SAT solving algorithms, the set of decision variables. Obviously, a weakly active symmetry can also propagate the symmetrical literals of a propagated one. Moreover, weakly active symmetries allow more propagation and so is more efficient. Note that if a weakly active symmetry wants to propagate a symmetrical literal which are already affected to the opposite value, this leads to a symmetry conflict and the solver backtrack to propagate the symmetrical value correctly.

Hakan: Mettre des tableaux, courbes etc ...
Hakan: Courbe VS static an no sbp
Hakan: Conclu SP, depend on the solver choice

Symmetry propagation gives good performances on many symmetric instances. The overall performance of the symmetry propagation is intrinsically related to the decision heuristics of the underlying SAT solver.

Note that, this approach don't discard any assignments like in the static approach where not lex-leader assignment were eliminated by symmetry breaking predicates.

Symmetry Explanation Learning

Another approach to exploit symmetry without removing any satisfiable assignment of the problem is *Symmetry Explanation Learning* [9] (SEL). Symmetries of a formula leaves this one invariant. Moreover all learned clauses are logical consequence of the problem, symmetric of these clauses are also valid. The idea of this approach is to learn useful symmetrical variant of learned clauses. A clause is said useful if it participates to the unit propagation or conflict analysis. Computing all symmetrical learn clauses will create a huge overhead and will be intractable on real problems.

Symmetry Explanation Learning uses the following fact: on the unit propagation, propagated literals has a reason clause which is assertive. Generally , symmetries permute only few literals in a clause and so symmetrical clauses may also be assertive and participate to unit propagation. These clauses are stored in different learning scheme and treated separately. Solver promotes these clause uniquely where they are effectively useful at the end of unit propagation. As unit propagation is done until fix point, it ensures that no duplicate clause is added in the problem. To limit memory impact, symmetrical clauses are removed when the propagated literal responsible of the computation is unaffected.

SEL provides some interesting properties. First, the authors proves that its propagation are a superset of the one provided by symmetry propagation. It also no need to track any status of symmetries as opposed to symmetry propagation. Like symmetry propagation no satisfying assignment are discarded. As disadvantage, SEL may flood the solver if the used set of symmetries is big and take time to compute symmetrical clauses.

Hakan: Perf of differents approaches

CHAPTER 4

SYMMSAT

This section presents our first contribution published in TACAS 2018 conference 4.

In the general case, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be totally computed. Even in more favorable situations, the size of the generated *sbp* is often too large to be effectively handled by a SAT solver [23]. On the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be that interesting and its effectiveness depends heavily on the heuristically chosen symmetries [5]. Besides, these approaches are preprocessors, so their combination with other techniques, such as *symmetry propagation* [11], can be very hard. Also, tuning their parameters during the solving turns out to be very difficult. For all these reasons, some classes of SAT problems cannot be solved yet despite the presence of symmetries. To handle these issues, we propose a new approach that reuses the principles of the static approaches, but operates dynamically: the symmetries are broken during the search process without any pre-generation of the *sbp*. It is a best effort approach that tries to eliminate, *dynamically*, the *non lex-leading* assignments with a minimal computation effort. To do so, we first introduce the notions of *reducer*, *inactive* and *active* permutation with respect to an assignment α and *effective symmetric breaking predicates* (*esbp*).

Definition 5 (Reducer, inactive and active permutation)**.** *A permutation g is a* reducer *of an assignment α if $g.\alpha < \alpha$ (hence α cannot be the lex-leader of its orbit. g reduces it and all its extensions). g is* inactive *on α when $\alpha < g.\alpha$ (so, g cannot reduce α and all the extensions). A symmetry is said to be* active *with respect to α when it is neither inactive nor a reducer of α.*

Proposition 5 restates this definition in terms of variables and is the basis of an efficient algorithm to keep track of the status of a permutation during the solving. Let us, first, recall that the *support* of a permutation g, $supp_g$, the set $\{v \in \mathcal{V} \mid g.v \neq v\}$.

Proposition 5. *Let $\alpha \in Ass(\mathcal{V})$ be an assignment, $g \in \mathfrak{S}$ a permutation and $supp_g \subseteq \mathcal{V}$ the support of g. We say that g is:*

1. a reducer of α *if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{v, \neg g^{-1}(v)\} \subseteq \alpha$;
2. inactive *on α if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{\neg v, g^{-1}(v)\} \subseteq \alpha$;
3. active *on α, otherwise.*

When g is a *reducer* of α we can define a predicate that contradicts α yet preserves the satisfiability of the formula. Such a predicate will be used to discard α, and all its extensions, from a further visit and hence pruning the search tree.

Definition 6 (Effective Symmetry Breaking Predicate)**.** *Let $\alpha \in Ass(\mathcal{V})$, and $g \in \mathfrak{S}\mathcal{V}$. We say that the formula ψ is an effective symmetry breaking predicate (esbp for short) for α under g if:*

$$\alpha \not\models \psi \textit{ and for all } \beta \in Ass(\mathcal{V}), \beta \not\models \psi \Rightarrow g.\beta < \beta$$

The next definition gives a way to obtain such an effective symmetry-breaking predicate from an assignment and a reducer.

Definition 7 (A construction of an *esbp*)**.** *Let φ be a formula. Let g be a symmetry of φ that reduces an assignment α. Let v be the variable whose existence is given by item 1. in Proposition 5. Let $U = \{v', \neg v' \mid v' \in \mathcal{V}_g$ and $v' \preceq v\}$. We define $\eta(\alpha, g)$ as $(U \cup g^{-1}.U) \setminus \alpha$.*

Example. Let us consider $\mathcal{V} = \{x_1, x_2, x_3, x_4, x_5\}$, $g = (x_1\, x_3)(x_2\, x_4)$, and a partial assignment $\alpha = \{x_1, x_2, x_3, \neg x_4\}$. Then, $g.\alpha = \{x_1, \neg x_2, x_3, x_4\}$ and $v = x_2$. So, $U = \{x_1, \neg x_1, x_2, \neg x_2\}$ and $g^{-1}.U = \{x_3, \neg x_3, x_4, \neg x_4\}$ and we can deduce than $\eta(\alpha, g) = (U \cup g^{-1}.U) \setminus \alpha = \{\neg x_1, \neg x_2, \neg x_3, x_4\}$.

Proposition 6. *$\eta(\alpha, g)$ is an effective symmetry-breaking predicate.*

Proof. It is immediate that $\alpha \not\models \eta(\alpha, g)$.

Let $\beta \in Ass(\mathcal{V})$ such that $\beta \wedge \eta(\alpha, g)$ is UNSAT. We denote a α' and β' as the restrictions of α and β to the variables in $\{v' \in \mathcal{V}_g \mid v' \preceq v\}$. Since $\beta \wedge \eta(\alpha, g)$ is UNSAT, $\alpha' = \beta'$. But $g.\alpha' < \alpha'$, and $g.\beta' < \beta'$. By monotonicity of $<$, we thus also have $g.\beta < \beta$. □

It is important to observe that the notion of *ebsp* is a refinement of the classical concept of *sbp* defined in [1]. In particular, like *sbp*, *esbp* preserve satisfiability.

Theorem 2 (Satisfiability preservation)**.** *Let φ be a formula and ψ an esp for some assignment α under $g \in S(\varphi)$. Then,*

$$\varphi \textit{ and } \varphi \wedge \psi \textit{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under $S(\varphi)$. Thus, g does not reduce β, which implies that $\beta \models \psi$.

□

Algorithm

This section describes how to augment the state-of-the-art CDCL algorithm with the aforementioned concepts to develop an efficient symmetry-guided SAT solving algorithm. The approach is implemented using a couple of components: (1) a *Conflict Driven Clauses Learning (CDCL) search engine*; (2) *a symmetry controller*. Roughly speaking, the first component performs the classical search activity on the SAT problem, while the second observes the engine and maintains the status of the symmetries. When the controller detects a situation where the engine is starting to explore a redundant part[1], it orders the engine to operate a backjump. The detection is performed thanks to *symmetry status tracking* and the backjump order is given by a simple injection of an *esbp* computed on the fly. Principle of CDCL is described in section 6, algorithm 4 explains how to extend it with a *symmetry controller* component which guides the behavior of CDCL algorithm depending on the status of symmetries.

The symmetry controller is initially given a set of symmetries G [2]. It observes the behavior of the SAT engine and updates its internal data according to the current assignment, to keep track of the status of the symmetries. This observation is *incremental*: whenever a literal is assigned or cancelled, the symmetry controller updates the status of all the symmetries. This corresponds to lines 6 and 17 of Algorithm 3. When the controller detects that the current assignment can not be a *lex-leader* (line 7), it generates the corresponding *esbp* (line 14).

In the remainder of this section, functions composing the symmetry controller are detailed.

Symmetries Status Tracking.

The `updateAssign`, `updateCancel` and `isNotLexLeader` functions (Algorithm 5) track the status of symmetries based on Proposition 5 ; there, resides th core of our algorithm.

[1] Isomorphic to a part that has been/will be explored.
[2] The generators of the group of symmetries.

```
1  function CDCLSym (φ: CNF formula, SymController: symmetry controller)
     returns ⊤ if φ is SAT and ⊥ otherwise
2    dl ← 0                                        // Current decision level
3    α ← ∅
4    while not all variables are assigned do
5      isConflict ← unitPropagation()
6      SymController.updateAssign(α)
7      isReduced ← SymController.isNotLexLeader(α)
8      if isConflict || isReduced then
9        if dl == 0 then
10         return ⊥                                // φ is UNSAT
11       if isConflict then
12         ω ← analyzeConflict()
13       else
14         ω ← SymController.generateEsbp(α)
15       φ ← φ ∪ {ω}
16       dl ← backjumpAndRestartPolicies()
17       SymController.updateCancel(α)
18     else
19       α ← α ∪ assignDecisionLiteral()
20       dl ← dl + 1
21   return ⊤                                      // φ is SAT
```

Algorithm 4: the CDCLSym SAT Solving Algorithm.

All these functions rely on the pt structure: a map of variables indexed by permutations. Initially, $pt[g] = \min_{\prec}(supp_g)$ for all $g \in G$ according to the ordering relation and all permutations are marked *active*.

For each permutation, g, the symmetry controller keeps track of the smallest variable $pt[g]$ in the support of g such that $pt[g]$ and $g^{-1}(pt[g])$ do not have the same value in the current assignment. If one of the two variables is not assigned, they are considered not to have the same value.

When new literals are assigned, only active symmetries need to have their $pt[g]$ updated (line 2). This update is done thanks to a while loop (lines 4 – 5).

When literals are cancelled, we need to update the status of symmetries for which some variable v before $pt[g]$, or $g^{-1}(v)$, becomes unassigned (lines 9 – 10). Symmetries that were inactive may be reactivated (line 11).

The current assignment is not a *lex-leader* if some symmetry g is a reducer. This is detected by comparing the value of $pt[g]$ with the value of $g^{-1}(pt[g])$ (line 16). The function `isNotLexLeader` also marks symmetries as *inactive* when appropriate (lines 18 – 19).

Generation of the *esbp*.

When the current assignment cannot be a *lex-leader*, some symmetry g is a reducer. The function `generateEsbp` computes the $\eta(\alpha, g)$ defined in Definition 7, which is an effective symmetry-breaking predicate by Proposition 6. This will prevent the SAT engine to explore further the current partial assignment.

Illustrative example

Let us illustrate the previous concepts and algorithms on a simple example. Let the ordering relation $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \;\mid \bot < \top$, and a set of symmetries $G = \{g_2 = (x_1\ \ x_2)(x_4\ \ x_5), g_3 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)\}$ (written in cycle notation with opposite cycles omitted). Their respective supports sorted according to ordering relation are, $supp_{g_2} = \{x_1, x_2, x_4, x_5\}$ and $supp_{g_3} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

On the assignment $\alpha = \emptyset$, both permutations are active and $pt[g_1] = pt[g_2] = x_1$. When the solver updates the assignment to $\alpha = \{x_4\}$, both permutations remain active and $pt[g_2] = pt[g_3] = x_1$. On the assignment $\alpha = \{x_4, x_1\}$, the symmetry controller updates $pt[g_3]$ to x_2, while $pt[g_2]$ remains unchanged. On the assignment $\alpha = \{x_4, x_1, \neg x_2\}$, $g_2.\alpha = \{x_5, x_2, \neg x_1\}$, which is smaller than α (because $x_1 \in \alpha$ and $\neg x_1 \in g.\alpha$): g_2 is a reducer of α. The symmetry controller then generates the corresponding *esbp* $\omega = \{\neg x_1, x_2\}$.

4.1 Implementation and Evaluation

In this section, we first highlight some details on our implementation of the symmetry controller. Then, we experimentally assess the performance of our algorithm against three other state-of-the-art tools.

cosy: an efficient implementation of the symmetry controller

We have implemented our method in a C++ library called cosy (1630 LoC). It implements a symmetry controller as described in the previous section, and can be interfaced with virtually any CDCL SAT solver. cosy is released under GPL v3 licence and is available at `https://github.com/lip6/cosy`.

Heuristics and Options.

Let us recall that finding the optimal ordering of variables (with respect to the exploitation of symmetries) is NP-hard [22], so the choice for this ordering is heuristic. cosy offers several possibilities to define this ordering:

- a naive ordering, where variables are ordered by the lexicographic order of their names; [redacted]

```
function updateAssign(α: assignment)
    foreach active g ∈ G do
        v ← pt[g];
        while {v, g⁻¹(v)} ⊆ α or {¬v, ¬g⁻¹(v)} ⊆ α do
            v ← next variable in 𝒱_g;
        pt[g] ← v
function updateCancel(α: assignment)
    foreach g ∈ G do
        u ← min{v ∈ 𝒱_g | {v, ¬v} ∩ α = ∅ or {g⁻¹(v), ¬g⁻¹(v)} ∩ α = ∅};
        if u ≤ pt[g] then
            mark g as active;
            pt[g] ← u;
function isNotLexLeader(α: assignment)
    foreach active g ∈ G do
        v ← pt[g];
        if {v, ¬g⁻¹(v)} ⊆ α then
            return ⊤;                         // g is a reducer
        if {¬v, g⁻¹(v)} ⊆ α then
            mark g as inactive ;              // g can't reduce α or its
             extentions
    return ⊥
function generateEsbp(α: assignment) returns ω: generated esbp
    ω ← {};
    g ← the reducer of α detected in isNotLexLeader;
    v ← min(𝒱_g);
    u ← pt[g];
    while u ≠ v do
        if v ∈ α then ω ← ω ∪ {¬v} else ω ← ω ∪ {v};
        if g⁻¹(v) ∈ α then ω ← ω ∪ {¬g⁻¹(v)} else ω ← ω ∪ {g⁻¹(v)};
        v ← next variable in 𝒱_g
    ω ← ω ∪ {¬v, g⁻¹(v)};
    return ω
```

Algorithm 5: the functions keeping track of the status of the symmetries and generating the *esbp*.

- an ordering based on occurrences, where variables are sorted according to the number of times they occur in the input formula. The lexicographic order of variables names is used for those having the same number of occurrences;

- an ordering based on symmetries, where variables belonging to the same orbit (under the given set of symmetries) are grouped together. Orbit are ordered by their numbers of occurrences.

The ordering of assignments we use in this paper orders positive literals before negative ones (thus, $\top < \bot$), but using the converse ordering does not change the overall method. However, it can impact the performance of the solver on some instances, so that it is an option of the library.

All the symmetries we used for the presentation of our approach are permutations of variables. Our method straightforwardly extends to permutations of literals, also known as *value permutations* [5].

Integration in `MiniSAT`.

We show how to integrate cosy to an existing solver, through example of `MiniSAT` [13].

First, we need an adapter that allows the communication between the solver and cosy (30 LoC). Then, we adapt Algorithm 3 to the different methods and functions of `MiniSAT`. In particular, the function `updateAssign` is moved into the `uncheckEnqueue` function of `MiniSAT` (2 LoC). The `updateCancel` function is moved to the `cancelUntil` function of `MiniSAT` that performs the backjumps (2 LoC). The `isNotLexLeader` and `generateEsbp` functions are integrated in the `propagate` function of `MiniSAT` (30 LoC). This is to keep track of the assignments as soon as they occur, then the *esbp* is produced as soon as an assignment is identified as not being *lex-leader*. Initialization issues are located in the main function of `MiniSAT`(15 LoC).

The integration of cosy increases `MiniSAT` code by 3%.

Evaluation

This section presents the evaluation of our approach. All experiments have been performed with our modified `MiniSAT` called `MiniSym`. The symmetries of the SAT problem instances have been computed by two different state-of-the-art tools `saucy3` [18] and `bliss` [17]. For a given group of symmetries, the first tool generates less permutations to represent the group than the second one, but it is slower than the other one.

We selected from the last six editions of the SAT contests [**?**], the CNF instances for which `bliss` finds at least 2% of the variables are involved in some symmetries that could be computed in at most 1000*s* of CPU time. We obtained a total of 1350 symmetric instances (discarding repetitions) out of 3700 instances in total.

All experiments have been conducted using the following conditions: each solver has been run once on each instance, with a time-out of 5000 seconds (including the execution time of the symmetries generation except for `MiniSAT`) and limited to 8GB of memory. Experiments were executed on a computer with an Intel Xeon X7460 2.66 GHz featuring 24 cores and 128GB of memory, running a Linux 4.4.13, along with g++ compiler version 6.3.

We compare `MiniSym` using the occurrence order, value symmetries, and without *lex-leader* forcing, against:

- `MiniSAT`, as the reference solver without symmetry handling [13];
- `Shatter`, a symmetry breaking preprocessor described in [1], coupled with the `MiniSAT` [redacted] SAT engine;
- `BreakID`, another symmetry breaking preprocessor, described in [**?**], also coupled with the `MiniSAT` SAT engine.

Each SATsolution was successfully checked against the initial CNF. For UNSATsituations, there is no way to provide an UNSATcertificate in presence of symmetries. Nevertheless, we checked our results were also computed by the other measured tools. Unfortunately, out of the 1350 benchmarked formulas, we have no proof or evidence for the 15 UNSATformulas computed by `MiniSym` only.

Results are presented Tables in 4.1, 4.2, and 4.3. We report the number of instances solved within the time and memory limits for each solver and category. We separate the UNSAT instances (Table 4.1) from the SAT ones (Table 4.2). Besides the reference with no symmetry (column `MiniSAT`), we have compared the performance of the three tools when using symmetries computed by `saucy3` (see Table 4.1a and Table 4.2a), and `bliss` (see Table 4.1b and Table 4.2b). Rows correspond to groups of instances: from each edition of the SAT contest, and when possible, we separated applicative instances (app$\langle x \rangle$ where $\langle x \rangle$ indicates the year) from hard combinatorial ones (hard$\langle x \rangle$). This separation was not possible for the editions 2015 and 2017 (all2015 and all2017). The total number of instances for each bench is indicated between parentheses. For each row, the cells corresponding to the tools solving the most instances (within time and memory limits) are typeset in bold and greyed out. Table 4.3 shows the cumulative and average PAR-2 times of the evaluated tools.

We observe that `MiniSym` with `saucy3` solves the most instances in only half of the UNSATcategories. However, with `bliss`, `MiniSym` solves the most instances in all but four of the UNSATcategories ; it then also solves the highest number of instances among its competitors. This shows the interest of our approach for UNSATinstances. Since symmetries are used to reduce the search space, we were expecting that it will bring the most performance gain for UNSATinstances.

The situation for SATinstances is more mitigated (Table 4.2), especially when using `saucy3`. [redacted] Again, this is not very surprising: our method may cut the exploration of a satisfying assignment because it is not a *lex-leader*. This delays the discovery of a satisfying assignment. The

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	18	19	**20**	17
app2014 (161)	23	23	22	**24**
app2013 (145)	6	8	8	**10**
app2012 (367)	115	115	**120**	**120**
hard2016 (128)	8	17	**50**	42
hard2014 (107)	9	24	**30**	29
hard2013 (121)	12	24	**48**	29
hard2012 (289)	86	84	88	**93**
all2017 (124)	8	14	**15**	14
all2015 (65)	9	8	8	**10**
TOTAL (no dup)	261	302	**371**	345

(a) With saucy3

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	18	**21**	18	19
app2014 (161)	23	21	20	**24**
app2013 (145)	6	7	10	**11**
app2012 (367)	115	106	114	**123**
hard2016 (128)	8	11	**79**	77
hard2014 (107)	9	45	40	**53**
hard2013 (121)	12	51	**56**	54
hard2012 (289)	86	69	90	**93**
all2017 (124)	8	14	**15**	**15**
all2015 (65)	**9**	7	8	8
TOTAL (no dup)	261	324	415	**439**

(b) With bliss

Table 4.1: comparison of different approaches on the UNSATinstances of the benchmarks of the six last editions of the SAT competition.

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	20	**22**	21	20
app2014 (161)	**24**	**24**	**24**	22
app2013 (145)	34	35	35	**43**
app2012 (367)	121	112	119	**126**
hard2016 (128)	**0**	**0**	**0**	**0**
hard2014 (107)	14	**17**	**17**	14
hard2013 (121)	23	23	**24**	22
hard2012 (289)	135	141	**143**	138
all2017 (124)	23	20	26	**27**
all2015 (65)	**7**	5	**7**	6
TOTAL (no dup)	325	323	**337**	335

(a) With saucy3

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	20	20	**22**	20
app2014 (161)	**24**	**24**	23	22
app2013 (145)	**34**	32	30	33
app2012 (367)	**121**	112	120	118
hard2016 (128)	**0**	**0**	**0**	**0**
hard2014 (107)	14	14	17	**18**
hard2013 (121)	23	24	**26**	25
hard2012 (289)	135	134	141	**142**
all2017 (124)	23	25	26	**29**
all2015 (65)	**7**	5	6	6
TOTAL (no dup)	325	316	334	**336**

(b) With bliss

Table 4.2: comparison of different approaches on the SATinstances of the benchmarks of the six last editions of the SAT competition.

Solver	PAR-2 sum	PAR-2 avg
MiniSAT	8 074 348	5 981
Shatter	7 770 434	5 756
BreakID	6 909 999	5 119
MiniSym	7 229 700	5 355

(a) With saucy3

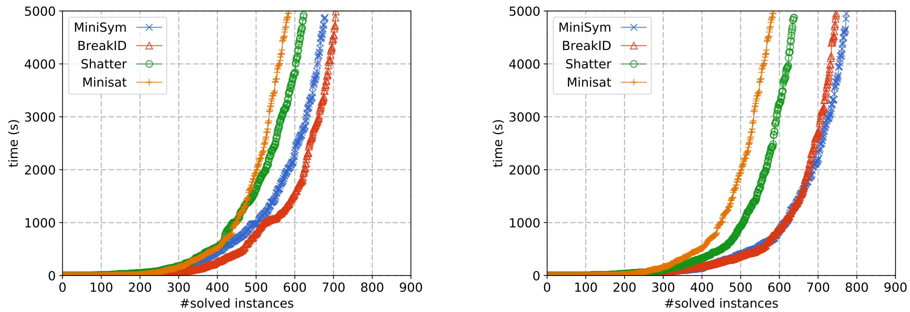
Solver	PAR-2 sum	PAR-2 avg
MiniSAT	8 074 348	5 981
Shatter	7 517 556	5 569
BreakID	6 444 954	4 774
MiniSym	6 245 448	4 626

(b) With bliss

Table 4.3: comparison of PAR-2 times (in seconds) of the benchmarks on the six last editions of the SAT competition.

other tools suffer less from such a delay, because they rely on symmetry breaking predicates generated in a pre-processing step. Also, when seeing the global results of MiniSAT, we can globally state that the use of symmetries in the case of satisfiable instances only offers a marginal improvement.

We observe that performances our tool are better with bliss than with saucy3 (see fig 4.1). We explain it as follows: saucy3 is known to compute fewer generators for the



(a) with saucy3 (b) with bliss

Figure 4.1: cactus plot total number of instances

group of symmetries than `bliss`. Since, the larger the symmetries set is, the earlier the detection of an *evidence* that an assignment is not a *lex-leader* will be, we generate less symmetry-breaking predicates (only the effective ones). This is shown in Table 4.4; `MiniSym` [redacted] generates an order of magnitude fewer predicates than `BreakID`.

Number of SBPs	`BreakID`	`MiniSym`
UNSAT(316)	12 088 433	1 579 623
SAT(312)	13 839 689	359 352

(a) With `saucy3`

Number of SBPs	`BreakID`	`MiniSym`
UNSAT(399)	2 576 349	913 339
SAT(320)	12 179 513	457 452

(b) With `bliss`

Table 4.4: Comparison of the number of generated SBPs each time `BreakID` and `MiniSym` both compute a verdict (number of verdicts between parentheses).

We also conducted experiments on highly symmetrical instances (all variables are involved in symmetries), whose results are presented in Table 4.5. The performance of `BreakID` on this benchmark is explained by a specific optimization for the *total symmetry groups* that are found in these examples, that is neither implemented in `Shatter` nor in `MiniSym`. However, the difference between `BreakID` and `MiniSym` is rather thin when using `bliss`. [redacted] Our tool still outperforms `Shatter` on this benchmark.

Benchmark	`MiniSAT`	`Shatter`	`BreakID`	`MiniSym`
battleship(6)	**5**	**5**	**5**	**5**
chnl(6)	4	**6**	**6**	**6**
clqcolor(10)	3	4	5	**6**
fpga(10)	6	**10**	**10**	**10**
hole(24)	10	12	**23**	11
hole shuffle(12)	1	2	**12**	3
urq(6)	1	2	**6**	2
xorchain(2)	1	1	**2**	**2**
TOTAL	31	42	**69**	45

(a) With `saucy3`

Benchmark	`MiniSAT`	`Shatter`	`BreakID`	`MiniSym`
battleship(6)	5	5	5	**6**
chnl(6)	4	**6**	**6**	**6**
clqcolor(10)	3	5	8	**10**
fpga(10)	6	**10**	**10**	**10**
hole(24)	10	**24**	**24**	23
hole shuffle(12)	1	3	**7**	4
urq(6)	1	2	**6**	5
xorchain(2)	1	1	**2**	**2**
TOTAL	31	56	**68**	66

(b) With `bliss`

Table 4.5: comparison of the tools on 99 highly symmetric UNSAT problems.

Bibliography

[1] F. Aloul, K. Sakallah, and I. Markov. Efficient symmetry breaking for boolean satisfiability. *IEEE Trans. Computers*, 55(5):549–558, 2006.

[2] C. Ansótegui, J. Giráldez-Cru, J. Levy, and L. Simon. Using community structure to detect relevant learnt clauses. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 238–254. Springer, 2015.

[3] B. Aspvall, M. F. Plass, and R. E. Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.

[4] C. Barrett and C. Tinelli. Satisfiability modulo theories. In *Handbook of Model Checking*, pages 305–343. Springer, 2018.

[5] A. Biere, M. Heule, and H. van Maaren. *Handbook of satisfiability*, volume 185. IOS press, 2009.

[6] S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158. ACM, 1971.

[7] J. Crawford, M. Ginsberg, E. Luks, and A. Roy. Symmetry-breaking predicates for search problems. *KR*, 96:148–159, 1996.

[8] M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397, July 1962.

[9] J. Devriendt, B. Bogaerts, and M. Bruynooghe. Symmetric explanation learning: Effective dynamic symmetry handling for sat. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 83–100. Springer, 2017.

[10] J. Devriendt, B. Bogaerts, M. Bruynooghe, and M. Denecker. Improved static symmetry breaking for sat. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 104–122. Springer, 2016.

[11] J. Devriendt, B. Bogaerts, B. de Cat, M. Denecker, and C. Mears. Symmetry propagation: Improved dynamic symmetry breaking in SAT. In *IEEE 24th International Conference on Tools with Artificial Intelligence, ICTAI 2012, Athens, Greece, November 7-9, 2012*, pages 49–56, 2012.

[12] N. Eén and A. Biere. Effective preprocessing in sat through variable and clause elimination. In *International conference on theory and applications of satisfiability testing*, pages 61–75. Springer, 2005.

[13] N. Eén and N. Sörensson. An extensible sat-solver. In *International conference on theory and applications of satisfiability testing*, pages 502–518. Springer, 2003.

[14] P. Flener, A. M. Frisch, B. Hnich, Z. Kiziltan, I. Miguel, J. Pearson, and T. Walsh. Breaking row and column symmetries in matrix models. In *International Conference on Principles and Practice of Constraint Programming*, pages 462–477. Springer, 2002.

[15] C. P. Gomes, B. Selman, and N. Crato. Heavy-tailed distributions in combinatorial search. In *International Conference on Principles and Practice of Constraint Programming*, pages 121–135. Springer, 1997.

[16] J. Huang et al. The effect of restarts on the efficiency of clause learning. In *IJCAI*, volume 7, pages 2318–2323, 2007.

[17] T. Junttila and P. Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In D. Applegate, G. S. Brodal, D. Panario, and R. Sedgewick, editors, *Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics*, pages 135–149. SIAM, 2007.

[18] H. Katebi, K. Sakallah, and I. Markov. Symmetry and satisfiability: An update. *Theory and Applications of Satisfiability Testing–SAT 2010*, pages 113–127, 2010.

[19] L. Le Frioux, S. Baarir, J. Sopena, and F. Kordon. Painless: a framework for parallel sat solving. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 233–250. Springer, 2017.

[20] L. Le Frioux, S. Baarir, J. Sopena, and F. Kordon. Modular and efficient divide-and-conquer sat solver on top of the painless framework. In *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 135–151. Springer, 2019.

[21] J. H. Liang, V. Ganesh, P. Poupart, and K. Czarnecki. Learning rate based branching heuristic for sat solvers. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 123–140. Springer, 2016.

[22] E. Luks and A. Roy. The complexity of symmetry-breaking formulas. *Annals of Mathematics and Artificial Intelligence*, 41(1):19–45, 2004.

[23] E. M. Luks and A. Roy. The complexity of symmetry-breaking formulas. *Ann. Math. Artif. Intell.*, 41(1):19–45, 2004.

[24] B. D. McKay. nauty user's guide (version 2.2). Technical report, Technical Report TR-CS-9002, Australian National University, 2003.

[25] M. W. Moskewicz, C. F. Madigan, Y. Zhao, L. Zhang, and S. Malik. Chaff: Engineering an efficient sat solver. In *Proceedings of the 38th annual Design Automation Conference*, pages 530–535. ACM, 2001.

[26] F. Rossi, P. Van Beek, and T. Walsh. *Handbook of constraint programming*. Elsevier, 2006.

[27] A. Sabharwal. Symchaff: A structure-aware satisfiability solver. In *AAAI*, volume 5, pages 467–474, 2005.

[28] M. Soos. Enhanced gaussian elimination in dpll-based sat solvers. In *POS@ SAT*, pages 2–14, 2010.

[29] N. Sörensson and A. Biere. Minimizing learned clauses. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 237–243. Springer, 2009.

[30] L. Zhang, C. F. Madigan, M. H. Moskewicz, and S. Malik. Efficient conflict driven learning in a boolean satisfiability solver. In *Proceedings of the 2001 IEEE/ACM international conference on Computer-aided design*, pages 279–285. IEEE Press, 2001.