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EXPLOITATION DES SYMÉTRIES DYNAMIQUES POUR LA RÉSOLUTION DES PROBLÈMES SAT

SOUTENUE LE :

DEVANT LE JURY COMPOSÉ DE :

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Ah, la thèse.

CONTENTS

CHAPTER 1

INTRODUCTION

Propositional logic deals with propositions which can be true or false. The logical connection allows the proposals to be connected to each other. In this thesis, we focus on solving the problem of Boolean satisfiability (SAT).

Given a propositional formula (generally the constraints of an encoded problem), SAT solving consists in deciding whether the formula is satisfiable (i.e, all constraints can be satisfiable) or unsatisfiable (i.e, there is no way to satisfy all constraints). This computation is made by a SAT solver that answer SAT when the formula is satisfiable and UNSAT otherwise.

SAT problem is the first problem that was be proven NP-complete in 1971 [10], this meaning that every NP problem can be solved by transforming it into SAT. Solving this problem in polynomial time is equivalent to the P versus NP, one of the seven millennium prize problem.

Despite this complexity, SAT solver becomes more and more powerful. Over the last decades, it can handle more and more complicated problems in different domains: like *formal methods*: bounded model checking (BMC) [8]; *artificial intelligence*: planning decision [26]; *Bioinformatics* : Haplotype inference [33].

In a recent work, a SAT solver solved the problem Pythagorean triple, an old mathematical problem, with a proof of 200 TB [20]. This success comes from the introduction of sophisticated heuristics and optimization of the solving algorithm called Conflict Driven Clause Learning (CDCL) algorithm. It is based on the first non memory intensive algorithm named by its authors Davis, Putnam, Logemann, and Loveland (DPLL) [12].

Some problem have a huge search space and some instances cannot be handle. An example of problem can be the vehicle routing problem (VRP). It concerns the service of delivery company, in which given a fleet of vehicles, based in a depot, must make rounds between several customers who have requested each a certain amount of goods. All clients visited

by a vehicle refers to the tour of the vehicle. The goal is to find the tour that minimize the delivery cost (monetary, distance, time, ...). Finding the optimal solution for VRP problem is NP-Hard [43].

On an instance of this problem, renaming the set of identical vehicle will give us exactly the same problem. This is called a *symmetry*. In general, a symmetry is a transformation that leave an object (or some aspect of the object) unchanged. Symmetries are typically defined as a *syntactical* property of a problem when its presence is inherent to the encoding of the problem. A permutation of variable preserve the original specification. In the case where the symmetries are independent of any particular representation of the problem, it is called *semantic*.

The presence of symmetry in a problem forces the search algorithm to fruitlessly explore symmetric search space and greatly hinders the performance of it. *Symmetry breaking* is an approach that avoid the solver to visit symmetric search space. The first step to exploit symmetry is to find them. In context of SAT, the detection of symmetry is done by a transforming the specification in a colored graph and then apply a graph automorphism tool.

When symmetries are found, the most common approach to exploit it is *static symmetry breaking*. It takes the symmetric problem as input and produces a satisfiability equivalent formula without symmetries. It is done by augmenting the problem with constraints that force solver to not explore the symmetric search space. It is a easy to integrate static symmetry breaking in an existing workflow using SAT solver, no modification of it is necessary. In addition, this approach works well on many symmetric applications. However, the number and the size of added constraints increase the solving time and the solver stucks on some highly symmetrical problems.

Another approach to handle symmetry is *dynamic symmetry breaking*. In this one, the management of symmetries is done during the search algorithm. It observes the behavior of the solver and help it to not visit symmetric search space.

The challenge of this thesis is to understand the state of the art approach in symmetry breaking and improve them.

My PhD thesis addresses the challenge of optimizing the solving of a SAT problem in presence of symmetry. In detail, my research exploits the presence of symmetry during the solving Understanding state of the art techniques in symmetry breaking allow us to improve it. Two majors contributions are detailed in this thesis. The first one use the strengths of static symmetry breaking and apply it dynamically to avoid its drawback. It add an opportunistic symmetry controller that avoid visiting symmetric search space. This idea allow us to solve very hard symmetric problems. The second contribution use the previous one and combines it with state of the art dynamic symmetry breaking approach and take the best of 2 worlds. This combination leads to important theoretical step for the usage of *partial symmetry breaking* with the usage of emphlocal symmetries.

The remaining of this document is organized in 6 chapters. Chapter 2 describes the state of the art for the Boolean satisfiability problem, Chapter 3 focuses on the symmetry present in

SAT. Chapter 4 focuses on the first contribution that use the symmetry dynamically. Chapter 5 describes our second solution to combine our first with another state-of-the-art dynamic symmetry breaking approach and Chapter 6 concludes the thesis. More precisely:

The Boolean Satisfiability Problem

The goal of chapter 2 is to better understand what is SAT. It describes in detail the basics about propositional logic that will used in the rest of the manuscript. It describes the original DPLL algorithm, and the nowadays used CDCL algorithm. Different heuristics that are embedded in modern efficient SAT sequential solvers are also explained.

Symmetry and SAT

The goal of chapter 3 is to better understand what is a symmetry. As it belong to the group theory, we first explain its basics and notation that will be used in the rest of the manuscript. This chapter also present, the computation of the search of symmetry from a SAT problem and the different existing approaches that use them to reduce the search space.

Between Static and Dynamic

Chapter 4 describes our efficient dynamic symmetry breaking approach. The first part explains our algorithm, a new way to handle symmetries, that avoid the main problem of the current static approaches. Our proposal has been implemented in state-of-the-art SAT solver. The second part presents the extensive experiments on the benchmarks of last six SAT competitions, which shows that our approach is competitive with the best state-of-the-art static symmetry breaking solutions. The last part presents different heuristics that can improve the performance of our algorithm.

Compose dynamic symmetry handling

Chapter 5 describes the theoretical and practical aspects of combining two existing symmetry breaking approach with the introduction of *local symmetries*. A extensive experiments show that the hybrid approach is better than each approach individually. The local symmetries allows to combine another symmetry breaking approach.

Finally, Chapter 6 concludes this manuscript and discusses different directions we have identified for future works.

Part I

State-of-the-art

CHAPTER 2

THE BOOLEAN SATISFIABILITY PROBLEM

Contents

In this thesis, our goal is to exploit the symmetrical properties of SAT problems. Before, we get to the heart of the matter, we first introduce the Boolean satisfiability (SAT) problem.

2.1 SAT basics

The satisfiability problem is constituted of *Boolean* or *propositional variable*. It has two possible values: true or false (noted respectively $\top$ or $\bot$). We call *literal*, a propositional variable or its negation. For a given variable x, the positive literal is represented by x and the negative one by $\neg x$. Given a formula φ, we denote $\mathcal{V}_\varphi$ ($\mathcal{L}_\varphi$) the set of variables (literals) used in the formula (the index in $\mathcal{V}_\varphi$ and $\mathcal{L}_\varphi$ is usually omitted when clear from context). To build more complex formula, it exists different operators, $\neg$, $\vee$ and $\wedge$ that are respectively

negation, disjunction and conjunction. Others operators like $\Rightarrow, \Leftrightarrow$ and $\oplus, \cdots$ respectively called implication, equivalence and exclusive disjunction (xor), $\cdots$ can be expressed with the basic ones. For example $a \Rightarrow b$ can be expressed as $\neg a \vee b$. To ensure the priority of these operators, constraints are expressed between parenthesis. In the absence of its, the following priority order applies (from the highest to the lowest priority): negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$).

The value given to each variable of a formula is called an *assignment* noted α and defined as follows:

$$\alpha : \mathcal{V} \mapsto \{\top, \bot\}$$

As usual α is said *total*, or *complete*, when all elements of $\mathcal{V}$ have an image by α, otherwise it is *partial*. By abuse of notation, an assignment is often represented by the set of its true literals. For example, $\alpha = \{\neg x_1, x_3\}$ means that x_1 have the false value and x_3 have the true value. The set of all (possibly partial) assignments of $\mathcal{V}$ is noted $Ass(\mathcal{V})$. A *truth table* gives an evaluation of all possible assignments for a given formula. Table 2.1 shows the evaluation of negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$) operators. For convenience, true value ($\top$) is also represented by 1, and false value ($\bot$) is represented by 0. When a formula is always true independently from the assignment, it is called a *tautology*: $x \vee \neg x$ is an example of tautologous formula.

x	y	$\neg x$	$x \vee y$	$x \wedge y$
0	0	1	0	0
0	1	1	1	0
1	0	0	1	0
1	1	0	1	1

Table 2.1: Truth table of basic operators

Normal forms

In Boolean logic, it exists some structural properties, called *normal form*. To introduce them, we first need to present the concepts of *cube* and *clause*.

A *cube* γ is a finite conjunction of literals represented equivalently by:

$$\gamma = \bigwedge_{i=1}^{k} l_i, \text{ or by the set of its literals } \gamma = \{l_i\}_{i \in [\![1,k]\!]}$$

A *clause* ω is a finite disjunction of literals represented equivalently by:

$$\omega = \bigvee_{i=1}^{k} l_i, \text{ or by the set of its literals } \omega = \{l_i\}_{i \in [\![1,k]\!]}$$

With respect to its size, a clause is said to be *unary, binary, ternary, n-ary* if it contains respectively one, two, three, or n literals.

The clause form has a property called *subsumption*. When a clause ω_1 is a subset of another clause ω_2 noted $\omega_1 \subset \omega_2$. And any assignment that satisfies ω_1 also satisfies ω_2, so ω_2 is *redundant* towards ω_1 and so can be removed of the formula.

Conjunctive Normal Form (CNF) of a formula is a finite conjunction of clauses represented equivalently by:

$$\varphi = \bigwedge_{i=1}^{k} \omega_i \text{ or by the set of its clauses } \omega = \{\omega_i\}_{i \in [\![1,k]\!]}$$

Disjunctive normal form (DNF) of a formula is finite disjunction of cubes represented equivalently by:

$$\varphi = \bigvee_{i=1}^{k} \gamma_i \text{ or by the set of its cubes } \omega = \{\gamma_i\}_{i \in [\![1,k]\!]}$$

The following table is a summary of laws about Boolean formulas that allow to transform any formula to a normal form.

Associativity laws	$(x \vee y) \vee z \equiv x \vee (y \vee z)$ $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$
Commutativity laws	$x \vee y \equiv y \vee x$ $x \wedge y \equiv y \wedge x$
Identity laws	$x \vee \bot \equiv x$ $x \wedge \top \equiv x$
Domination laws	$x \vee \top \equiv \top$ $x \wedge \bot \equiv \bot$
Idempotent laws	$x \vee x \equiv x$ $x \wedge x \equiv x$
Distributive laws	$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$
Negation laws	$x \vee \neg x \equiv \bot$ $x \wedge \neg x \equiv \top$
double negation law	$\neg(\neg x) \equiv x$
De Morgan's laws	$\neg x \vee \neg y \equiv \neg(x \wedge y)$ $\neg x \wedge \neg y \equiv \neg(x \vee y)$

Table 2.2: Set of laws of operators

Every formula can be transformed into a normal form with different complexity and the resulting formula is *equivalent*. In other words, every assignment α that satisfies formula φ also models the resulting formula ψ and vice-versa, denoted by $\varphi \equiv \psi$. Also, we say a formula ψ is a *logical consequence* of a formula φ is every model of φ is also a model of ψ and is denoted by $\varphi \models \psi$.

Conjunctive normal form is the input form of state-of-the-art solvers. Any propositional formula can be transformed in CNF form with polynomial time. Conversely, DNF form have an exponential memory complexity during the transformation. Note that each cube in the problem in DNF form is a solution in the equivalent CNF formulas.

An NP-complete problem

The SAT problem is the first NP-complete problem proven by Stephen Cook in 1971 [10]. NP-completeness means that a SAT problem can be solved with a non-deterministic Turing machine in polynomial time (NP) and is also NP-hard. A problem is said NP-hard if everything in NP can be transformed into it in polynomial time. One of the most important unsolved problems in theoretical computer science is the P versus NP problem. This question is one of the seven millennium prize problems.

Particular forms easy to solve

In some particular forms, SAT problems can be computed with polynomial algorithm.

2-SAT [4]. In this particular form, the given CNF formula contains only binary clauses. In this case, it suffices to create a graph in which each clause is transformed into implication. For example, the clause $x \vee y$ will be transformed into $\neg x \Rightarrow y, \neg y \Rightarrow x$. After computing *strong connected component* on this graph, to be looking for the positive and negative forms of the same variable, in the same strong connected component suffices to determine the satisfiability of the formula. If it is the case, the formula is UNSAT. Otherwise a solution can be deduce and the problem is SAT. This algorithm can be computed in linear time complexity.

Horn SAT. In this particular form, the given CNF formula contains only Horn clauses. It exists three form of Horn clause: *strict Horn clause* that contains only one positive literal and at least one negative literal, *positive Horn clause* that contains only one positive literal and no negative literals *negative Horn clause* that contains only negative literals. To solve this particular form of formula, it suffices to apply *Boolean constraint propagation* (BCP) or *unit propagation* explained thereafter in section 2.1 until fix point. Roughly speaking, It satisfies all unit clauses in cascade. Either an empty clause was deduced and the problem is UNSATor the fix point is reached and the formula is SAT. Like 2-SAT, this algorithm can also be computed in linear time complexity.

XOR SAT. In this particular form, each clause contains xor ($\oplus$) operator rather than or ($\vee$). This problem can be seen as a system of linear equations. Gaussian elimination allows to solve this kind of problem is polynomial time.

The membership of polynomial class (P) of the particular form of satisfiability problems describes above is a special case of Schaefer's dichotomy theorem [39].

Some related problems

Different kind of problems related to SAT is presented is this section. One of them is sharp-SAT (#SAT), its purpose is to count the number of solutions in a CNF.

Another related problem is maximum satisfiability problem (MAX-SAT). In this case, the problem is to find the maximum subset of clauses that can be satisfied for a formula. Different variants of this problem exist. For example, some constraints must be satisfied (hard clauses) and MAX-SAT is applied on the remaining clauses called *soft* clauses.

The last related problem is quantified Boolean formula (QBF) where the quantifiers $\exists$ and $\forall$ are present in the formula. For example, $\forall x \exists y \exists z (x \vee y) \wedge z$. This particular form is a generalization of the SAT problem with PSPACE complexity.

Solving a SAT problem

Two kinds of algorithms exist to solve satisfiability problems. First, the *incomplete* algorithm [25] which does not provide any guarantee that will eventually report either any satisfiable assignment or declare the formula unsatisfiable. This kind of algorithm is out of scope of this thesis. Second, the *complete* algorithm, which provides a guarantee that if an assignment exists it will be reached or it will declare that formula is unsatisfiable. This section describes different *complete* algorithm to solve a propositional formula.

A naive algorithm

A naive approach to solve a SAT problem is to try all possible assignments. For a propositional formula with n variables, it leads to 2^n assignments in the worse case. Figure 2.1 illustrates the search tree for a given problem with six variables. The presented formula in the figure have 6 clauses, with 2 ternary clauses and 4 binary clauses. It will be used as example in different algorithm presented below. This formula is SAT, $\alpha_{11} = \{\neg x_1, \neg x_2, x_3, \neg x_4, x_5, \neg x_6\}$ is a solution of the problem. This naive algorithm will check 10 assignments before finding the solution. In the general case, due to the number of variable in problems, this algorithm is intractable.

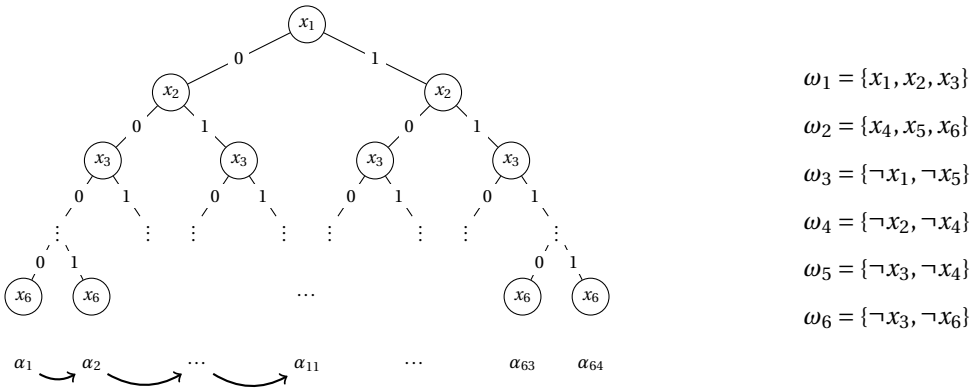


Figure 2.1: All possible assignments for a problem with 6 variables

Davis Putnam Logemann Loveland algorithm (DPLL)

One of the first non-memory-intensive algorithm developed to solve SAT problems is the Davis Putnam Logemann Loveland algorithm (DPLL) [12]. It explores a binary tree using depth first search as shown in Algorithm 1. The construction of the tree relies on the *decision* made (on line 8). Both values are checked, true value on line 9 and false value on line 11. When a leaf find a conflict (line 5) which means that the formula cannot be satisfied with the current assignment, other branches are explored. By recursive construction of the algorithm, when each value of a literal reach to a conflict, solver *backtracks* at most one level, this fact is called *chronological backtracking*. When the conflict occurs at the top of the tree, it means that the formula cannot be satisfied and the solver reports UNSAT (line 13). However, if the formula is empty in any branch, it means that current assignment satisfy the whole formula and the solver reports it on line 10 or 12.

```
1  function DPLL(φ: CNF formula, α assignment)
2      returns an assignment if φ is SAT and UNSAT otherwise
3      φ, α ← unitPropagation(φ, α);
4      if {} ∈ φ then
5          return ⊥;                                   // Conflict
6      if φ = {} then
7          return α ;                                  // φ is SAT
8      x ← assignDecisionLiteral();
9      if α ← DPLL(φ ∪ {x}, α) then
10         return α
11     if α ← DPLL(φ ∪ {¬x}, α) then
12         return α
13     return UNSAT;                                   // φ is UNSAT
```

Algorithm 1: The DPLL algorithm.

An important function in the DPLL algorithm is `unitPropagation` (line 3). It is presented in Algorithm 2. This function set value to unit clause in order to satisfy them until fix point is reached. Either, there are no more unit clause in the formula or an inconsistency is found which means that current assignment cannot satisfy the formula. In the later case, the solver will backtrack and another branch will be explored.

When DPLL algorithm is executed on the formula of Figure 2.1, after making decisions on literals $\neg x_1$ and $\neg x_2$, unit propagation detects that x_3 must be assigned to true. This propagation prevents to explore multiple assignments. Effectively, when x_3 is set to false value, the clause ω_1 is not satisfied and it remains 3 variables and so 2^3 possible assignments (from α_1 to α_8) are discarded. Moreover, in the DPLL algorithm, application of unit propagation until the fix point is reached leads to a solution.

`assignDecisionLiteral` is an important procedure that responsible of choosing the

```
1 function unitPropagation(φ: CNF formula, α assignment)
2     returns CNF formula and assignment α
3   while {l} ∈ φ and {} ∉ φ do
        // Remove all clauses containing l, all literals ¬l
4     φ ← φ|_l
5     α ← α ∪ {l}
6   return φ, α
```

Algorithm 2: Unit propagation

variable that divides the search space. Its objective is to find a literal that will generate a maximum of unit propagations. Intuitively, decision literals can be viewed as 'guesses' and propagated literals can be viewed as 'deductions'. Finding a optimal variable is NP-Hard. Different heuristics exists to choose the decision variable, some of them will be presented in the section 16.

Conflict Driven Clause Learning (CDCL) algorithm

The principal weakness of DPLL algorithm is to make the same inconsistencies several times (principally due to chronological backtracking), leading to unnecessary CPU usage. Conflict Driven Clause Learning (CDCL) algorithm 3 is a sound and complete algorithm that overcomes the weakness of DPLL.

Algorithm 3 gives an overview of CDCL, Like DPLL, it walks on a binary search tree. Initially, the assignment is empty and the decision level that indicates the depth of the search tree, noted by dl, is set to zero. The algorithm first applies unit propagation to the formula φ for the assignment α (line 6). An inconsistency or a *conflict* at level zero indicates that the formula is unsatisfiable, and the algorithm reports it (from line 7 to line 9). When the conflict is occurring at a higher level, its reason is analyzed and a clause called *conflict clause* is deduced (line 10). The work done in this procedure will be explained thereafter. This clause is *learnt* (line 12) (added to the formula). This clause is redundant w.r.t the current formula and so it does not change the satisfiability of φ. It also avoids encountering a conflict with the same causes in the future. The analysis is completed by the computation of a *backjumps level* , the assignment and decision level is updated (line 11). As the level can be much lower than the level of the current assignment, this is called *non chronological backtracking*. Finally, if no conflict appears, the algorithm chooses a new decision literal (lines 14 and 15). The above steps are repeated until the satisfiability status of the formula is determined.

Conflict Analysis

A conflict is an inconsistency discovered by the solver, a situation that requires for a variable to be set simultaneously to the $\top$and $\bot$values. Figure 2.2 shows an assignments that leads to a conflict. First the solver chooses $\neg x_1$ as a decision (marked with a D in the figure) then,

```
1  function CDCL(φ: CNF formula)
2      returns ⊤ if φ is SAT and ⊥ otherwise
3      dl ← 0;                                   // Current decision level
4      α ← ∅;
5      while not all variables are assigned do
6          (φ, α) ← unitPropagation(φ|α, α);
7          if {} ∈ φ then                        // A conflict occurs
8              if dl = 0 then
9                  return ⊥;                     // φ is UNSAT
10             ω ← analyzeConflict();
11             (dl, α) ← backjumpAndRestartPolicies();
12             φ ← φ ∪ {ω};
13         else
14             α ← α ∪ assignDecisionLiteral();
15             dl ← dl + 1;
16     return ⊤;                                 // φ is SAT
```

Algorithm 3: The CDCL algorithm.

$\neg x_6$ and, then $\neg x_5$. This last one propagates x_4 (marked with aa P in the figure), which in turn propagates x_2 and x_3. To satisfy ω_1, x_3 needs to be set to $\top$ and to satisfy ω_5, needs to be $\bot$. As a variable cannot have both values, a conflict appears (marked as a C in the figure).

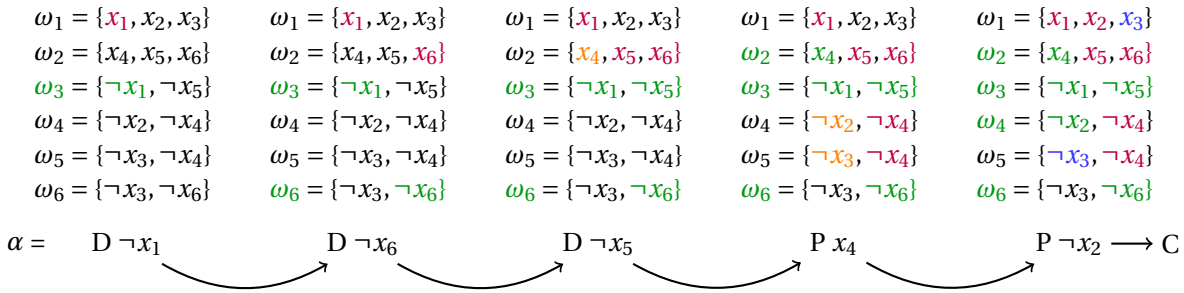


Figure 2.2: Decisions/Propagations that leads to a conflict

This series of decisions would provoke same propagation and leads to the same conflict. To escape this situation, the solver needs to analyze the reason of the conflict with so-called *implication graph*. It represents the current state of the solver and so records all dependencies between variables. It is updated either when a variable is assigned on decision/propagation, or when a variable is unassigned (on backjump's operation). The implication graph is a directed acyclic graph (DAG) in which a vertex represents an assigned variable labeled by $l@dl(l)$ where l represents assigned literal and $dl(l)$ represents the decision level of the literal l. Root vertices , that have no incoming edges, represent decision literal. The remain-

ing vertices represents propagations. Each incoming arc, labeled with a clause represents the *reason* of this propagation. This clause must be assertive, (i.e. all literals are false except one that is not yet assigned). Figure 2.3 shows the implication graph of the previous example (Figure 2.2).

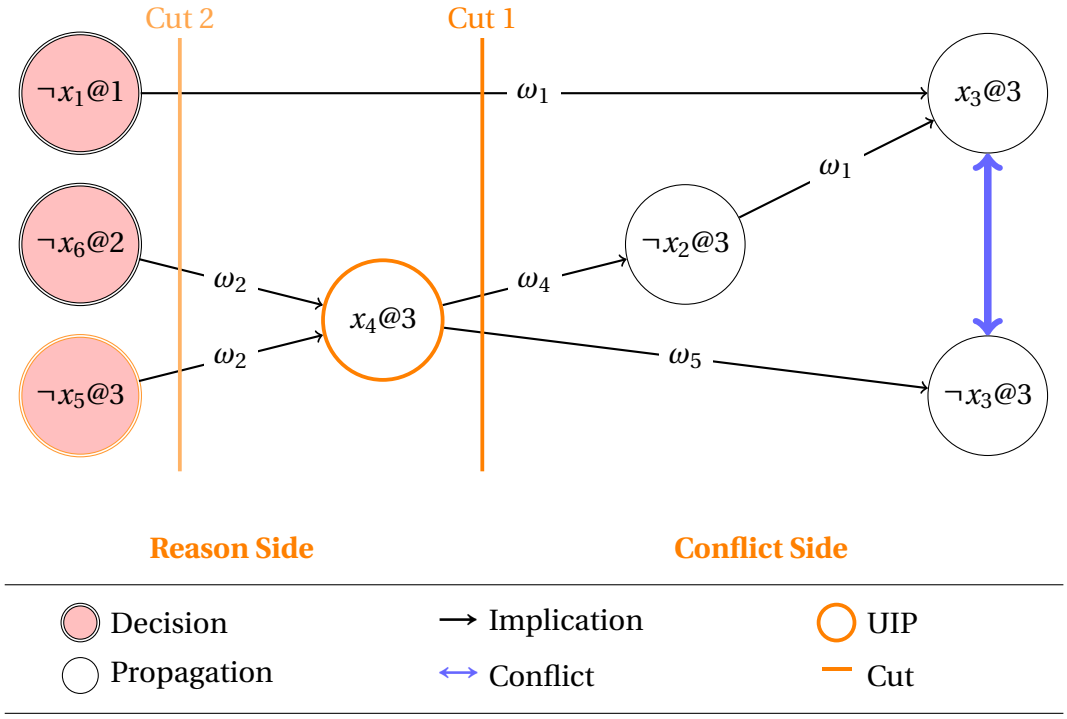


Figure 2.3: Implication graph

`analyzeConflict` procedure analyzes this graph to find the reason of the conflict. To do that, a search of *unique implication point* (UIP) is performed. UIP of the last decision level of the implication graph is a variable which lies on every path from the decision to the conflict. Note that, there are many UIP for a given decision level. In such case, UIPs are ordered according to the distance with the contradiction. The First UIP (FUIP) is the closest to the conflict. It is well known that the first UIP provides the smallest set of assignment that is responsible for the contradiction [44]. A UIP divides the implication graph in two sides with a *cut*, the *reason side* contains decision variables that is responsible of the contradiction and the *conflict side* that contains the conflict. UIP is always is the reason side. Figure 2.3 depicts two cuts in the implication graph. Once the reason side of a conflict is established, a conflict-driven clause (or simply conflict clause) is produced. To build this clause, it suffices to negate the literals that have an ongoing arc to the cut that contains the UIP. In Figure 2.3, the produced clause will be $\omega_l = \{x_1, \neg x_4\}$. Since the information of this clause is redundant regarding the original formula. It can be added without any restriction. The conflict clause can be simplified using the implication graph to reduce its size, by detecting redundancies [41]. All leaned clause are store in clause database. `backjumpAndRestartPolicies` procedure is executed after producing the conflict

clause. All responsible variables of the inconsistency are removed from the current assignment. Adding the conflict clause prunes search space that contains no solution. This is the key point of the CDCL algorithm. In our example Figure 2.3, the target decision level is 1. After backtrack, the conflict clause will be assertive.The first UIP is the only variable that have not a value and will be propagated in the next step of the algorithm. If a conflict implies only one level, the decision variable must be assigned to the opposite value at level 0. This means that this literal must be true without any decision.

Heuristics

This section gives an overview of different heuristics present in modern SAT solvers.

Decision heuristics. Decision variable has a huge impact on the overall solving time. It impacts the number of propagations and so the depth of the search tree. The Variable State Independent Decaying Sum (VSIDS) [36] measure is one of the most famous decision heuristics and is used nowadays in almost all solvers; each variable has an activity and is increased by a multiplicative factor when it participates to the resolution of a conflict. Decision heuristics choose unassigned variable with the highest activity. Learning rate based branching (LRB [30]) is a latest decision heuristic. It is a generalization of VSIDS and its goal is to optimize the *learning rate* (LR), defined as the ability to generate learned clauses. The LRB of a variable is the weighted average (computed with *exponential recency weighted average* (ERWA)) value taken by its LR over the time. Unassigned variable with the highest LRB is chose as a decision.

Restarts. Another important mechanism is *restart*. Basically, the solver abandons it current assignment and start from the top of the tree, while maintaining some information, like learned clauses, scores of variables, etc The restart prevents the solver to get stuck in the same part of the search space (heavy tailing [19]). Detecting this phenomenon has been widely treated in the literature [5, 7]. These strategies are based on counting the number of conflict or on the monitoring the current search's depth. Empirically a solver with restart has a better result [21] and is today used in almost all state-of-the-art solvers.

Cleaning clause database. Storing all learned clauses will end up by a memory issue. So, we need to develop a policy to eliminate some of them. In the literature, different criterion exists, the size of the clause is one of them and is very often used by solvers. A small clause have better chance to participate to the unit propagation and so be useful in the solving. As a consequence, large clauses are removed. *Clause activity* is another criteria, when a clause augment its activity when it participate to conflict analysis. Lower activity clauses are less used and so removed. The last often used criteria is based on Literal Block Distance (LBD). It is a measure that computes the *quality* of a clause it is based on the number of decision levels present in the clause. Clauses with high value of LBD will be deleted from the clause database. In current state-of-the-art solvers, multiple criterion are used and half of the learned clauses are removed during the clause database cleaning process.

Preprocessing / Inprocessing

In order to optimize solving time, some transformation can be applied to simplify the original formula. This is done by a *preprocessing* engine before the start of solving. When it is used during the solving, (usually after a restart), it is called *inprocessing*. Simplification of the formula is made by removing clauses and/or variables.

Variable elimination simplification is based on *resolution inference rule* [37]. Consider two clauses $\omega_1 = \{x_1, x_i, ..., x_j\}$ and $\omega_2 = \{\neg x_1, y_i, ..., y_j\}$. The resolution inference rule allows to derive a clause $\omega_3 = \{x_i, ..., x_j, y_i, ..., y_j\}$ which is called the *resolvent* as it results from solving two clauses on the literal x_1 and $\neg x_1$.

The subsumption aims at removing clauses. Consider two clauses ω_1 and ω_2, such that $\omega_1 \subset \omega_2$, then ω_2 can be safely removed from the original formula. *Self subsuming resolution* is a principle that uses resolution rules and subsumption. The resolvent clause subsumes the original one. For example, $\omega_1 = \{x_1, \neg x_2, x_3\}$ and $\omega_2 = \{x_1, \neg x_2, x_3, x_4\}$, then the resolvent clause will be $\omega_3 = \{x_1, x_3\}$ which subsumes ω_2. This principle is implemented in `SatElite` [16] preprocessor engine and is used in almost all modern SAT solvers. Other simplification techniques exist such that *Gaussian elimination* which detects sub formula in a XOR-SAT form and solve it in a polynomial time. Moreover, this technique can also be used as inprocessing [40]. Some techniques exploit the structure of the original formula and add relevant clauses to speed up the resolution time of the SAT solver. One of them use *community structure* of the formula to find good clauses to add into. A preprocessor engine doing that is `modprep` [3]. The usage of symmetries also adds relevant clauses in the formula and will be detailed in the next chapter.

Parallel SAT solving

With the emergence of multi-core architectures and increasing power of computers, one way to optimize the solving of a SAT problem is the exploitation of these cores. Actually, SAT problems are a good candidate for parallelism. *Portfolio* is a technique that launches several SAT solvers in parallel with different heuristics (decisions, restarts, ...) that communicates or not between them. When one of them finds a solution or finds that none exists, the overall computation is finished. Another technique to develop a parallel SAT solver is called *divide and conquer*. In this technique, the search space is divided dynamically and submitted to different solvers that cooperate to find a solution. Some specific techniques like load balancing and work stealing is applied to avoid a solver to be idle. A recent framework *PaInleSS* (a Framework for Parallel SAT Solving) can be used to easily create a new parallel SAT solver with different heuristics [28] [29]. Authors of this framework win the parallel tracks of SAT competition [1] in 2018.

[1] `http://www.satcompetition.org/`

CHAPTER 3

SYMMETRY AND SAT

Contents

Despite the NP-Completeness character of the SAT problem, state-of-the-art solvers are able to treat many industrial problems.This is mainly due to capacity to prune search space with, for example, the learnt clauses. Another way to accelerate the solving time is the exploitation of symmetry. Symmetries are common in real life, consider the problem of searching a pattern in butterfly wings. Most butterflies has exactly the same wings halves. After checking that both wings are symmetric (process called symmetry detection), the pattern can be search in only one wing, searching this pattern in the other wing is useless (process called symmetry exploitation).

In this chapter, we show how to detect that a given formula presents symmetries and how to exploit them to accelerate the solving.

3.1 Group basics

As symmetries belong to a branch of mathematics called group theory. This section gives us an overview of this last.

Groups

A *group* is a structure $\langle G, * \rangle$, where G is a non-empty set and $*$ a binary operation such as the following axioms are satisfied:

- *associativity*: $\forall a, b, c \in G, (a * b) * c = a * (b * c)$
- *closure*: $\forall a, b \in G, a * b \in G$.
- *identity*: $\forall a \in G, \exists e$ such that $a * e = e * a = a$
- *inverse*: $\forall a \in G, \exists b \in G$, commonly denoted a^{-1} such that $a * a^{-1} = a^{-1} * a = e$

Note that *commutativity* is not required, i.e, $a * b = b * a$, for $a, b \in G$. A group is said *abelian* if it is commutative. Moreover, the last definition leads to important properties which are: i) uniqueness of the identity element. To prove this property, assume $\langle G, * \rangle$ a group with two identity elements e and f then $e = e * f = f$. ii) uniqueness of the inverse element. To prove this property, suppose that an element x_1 has two inverses, denoted b and c in groups $\langle G, * \rangle$, then

$$
\begin{array}{lcll}
b & = & b * e & \\
 & = & b * (a * c) & c \text{ is an inverse of } a, \text{so } e = a * c \\
 & = & (b * a) * c & \textit{associativity} \text{ rule} \\
 & = & e * c & b \text{ is an inverse of } a, \text{so } e = a * b \\
 & = & c & \textit{identity} \text{ rule}
\end{array}
$$

The structure $\langle G, * \rangle$ is denoted simply G when clear from the context that G is a group with a binary operation. In this thesis, we are interested only in *finite* groups, i.e, with a finite number of elements. Given a group G, a *subgroup* is a non-empty subset of G which is also a group with the same binary operation. If H is a subgroup of G, we denote it $H \leq G$. A group has at least two subgroups: i) the subgroup composed of the identity element $\{e\}$, called *trivial* subgroup; ii) the subgroup composed of itself, called *improper* subgroup. All other subgroups are *proper*.

Generators of a group

If every element in a group G can be expressed as a linear combination of a set of elements S = $\{g_1, g_2, ..., g_n\}$ then we say that G is *generated by the S*. we denote this by G = $\langle S \rangle$ = $\langle \{g_1, g_2, ..., g_n\} \rangle$

Permutation groups

A *permutation* is a bijection from a set X to itself.
Example: given a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$,

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_1 & x_4 & x_6 & x_5 \end{pmatrix}$$

g is a permutation that maps x_1 to x_2, x_2 to x_3, x_3 to x_1, x_4 to x_4, x_5 to x_6 and x_6 to x_5. Permutations are generally written in *cycle notation*, the self-mapped elements are omitted. So the permutation in cycle notation will be

$$g = (x_1 \ x_2 \ x_3)\ (x_5 \ x_6)$$

We call *support* of the permutation g, noted $supp_g$, the elements that not mapped to themselves:

$$supp_g = \{x \in X \mid g.x \neq x\}$$

A variable x is *stable* by a permutation g iff $x \notin supp_g$.

Hakan: mettre ailleurs A clause ω is *stabilized* by a permutation g if $\omega \cap supp_g = \emptyset$.

A set of permutations of a given set X form a group G_X with the composition operation ($\circ$) and is called *permutation group*. The *symmetric group* is the set of all possible permutations of a set X and noted $\mathfrak{S}(X)$. So, $G_X \leq \mathfrak{S}(X)$.

A permutation group G induces an *equivalence relation* on the set of elements X being permuted. Two elements $x_1, x_2 \in X$ are equivalent if there exists a permutation $g \in G$ such that $g.x_1 = x_2$. The equivalence relation partitions X into *equivalence classes* referred to as the *orbits* of X under G. The orbit of an element x under group G (or simply orbit of x when clear from the context) is the set. $[x]_G = \{g.x \mid g \in G\}$

3.2 Symmetries in SAT

The previous mathematical definition of group theory can be applied to a CNF formula. The symmetric group of permutations of $\mathcal{V}$ (i.e., bijections from $\mathcal{V}$ to $\mathcal{V}$) is noted $\mathfrak{S}(\mathcal{V})$. The group $\mathfrak{S}(\mathcal{V})$ naturally acts on the set of literals: for $g \in \mathfrak{S}(\mathcal{V})$ and a literal $\ell \in \mathcal{L}$, $g.\ell = g(\ell)$ if ℓ it is a positive literal, $g.\ell = \neg g(\neg \ell)$ if ℓ is a negative literal. The group $\mathfrak{S}(\mathcal{V})$ also acts on assignments (possibly partial) of $\mathcal{V}$ as follows:

$$\forall g \in \mathfrak{S}(\mathcal{V}), \forall \alpha \in Ass(\mathcal{V}), g.\alpha = \{g.\ell \mid \ell \in \alpha\}.$$

We say that $g \in \mathfrak{S}(\mathcal{V})$ is a *symmetry of* φ if the following conditions holds:

- permutation fixes the formula, $g.\varphi = \varphi$:
- g commutes with the negation: $g.\neg l = \neg(g.l)$

The set of symmetries of φ is noted G_φ and is a subgroup of $\mathfrak{S}(\mathcal{V})$. Symmetry of a formula φ preserves the satisfaction, for every *complete* assignment α:

$$\alpha \models \varphi \Leftrightarrow g.\alpha \models \varphi$$

These symmetries can be obtained either *syntactic* or *semantic*. Semantic symmetries are independent of any particular representation of the problem. Conversely, syntactic symmetries depend of the encoding of the problem and leads to different symmetries. The next section presents the detection of syntactic symmetries.

3.3 Symmetry detection in SAT

For the detection of symmetries in SAT, we fist introduce the graph automorphism notion. Given a colored graph $Gr = (V, E, \gamma)$, with a set of vertices set $V \in [1, n]$, a set of edges E and γ a mapping : $V \rightarrow C$, where C is a set of *colors*. An automorphism of Gr is a permutation on its vertices, $g : V \rightarrow V$, such that:

- $\forall (u, v) \in E \implies (g.u, g.v) \in E$
- $\forall v \in V, \gamma(v) = \gamma(g.v)$

The graph automorphism problem is to find if a given graph has a non-trivial permutation group. The computational complexity of this algorithm is conjectured to be strictly between P and NP [27, 42]. Several tools exist to tackle this problem like `saucy3` [24], `bliss` [23], `nauty` [34], etc.

To find symmetries in a SAT problem, the formula is encoded in colored graph and an automorphism tool is applied onto. In particular, given a formula φ with m clauses and n variables, the graph is constructed as follows [9]:

- *clause nodes*: represent each of the m clauses by a node with color 0;
- *literal nodes*: represent each of the l literals by a node with color 1;
- *clause edges*: connect a clause to its literals by linking the corresponding clause node and literal nodes;
- *boolean consistency edges*: connect each pair of literals that correspond to the same variable.

Figure 3.1 shows the graph representation of a CNF. This problem has 6 variables and 11 clauses. So, the graph will have 12 + 11 = 33 vertices where 12 represent literal vertices (circles in the figure) and 11 represents the number of clause vertices (squares in the figure). The graph will also have 6 + 24 = 30 edges, 6 for Boolean consistency (red edges in the figure) and 24 edges that rely clause vertices to the literals.

$\omega_1 = \{x_1, x_2, x_3\}$
$\omega_2 = \{x_4, x_5, x_6\}$
$\omega_3 = \{x_1, x_4\}$
$\omega_4 = \{x_2, x_5\}$
$\omega_5 = \{x_3, x_6\}$
$\omega_6 = \{\neg x_1, \neg x_2\}$
$\omega_7 = \{\neg x_1, \neg x_3\}$
$\omega_8 = \{\neg x_2, \neg x_3\}$
$\omega_9 = \{\neg x_4, \neg x_5\}$
$\omega_{10} = \{\neg x_4, \neg x_6\}$
$\omega_{11} = \{\neg x_5, \neg x_6\}$

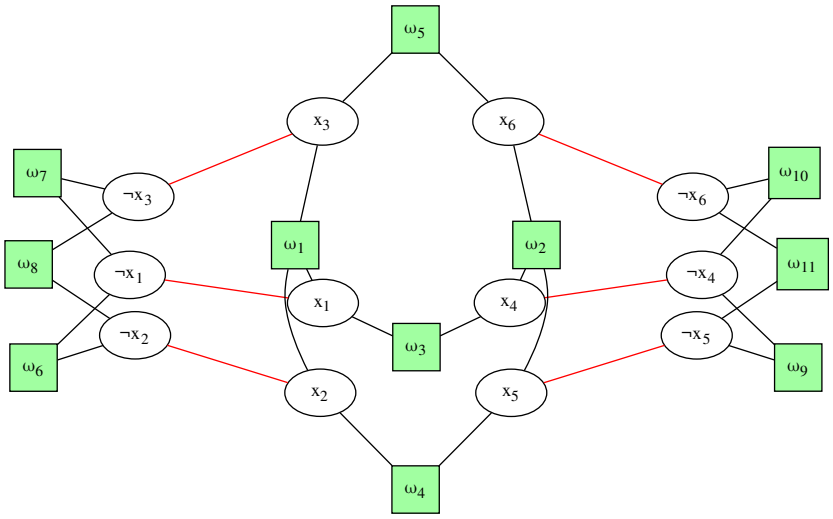


Figure 3.1: Example of constructed symmetry graph for a given CNF

An optimization to reduce the number of graph vertices is possible. It is achieved by modeling binary clause using graph edges instead of graph vertices. However, in some particular cases, it can produce spurious permutations, where Boolean consistency is not respected [2]. To ensure that the permutation is valid, it must commutes with the negation:

$$\forall x \in supp_g, g.\neg x == \neg g.x$$

Roughly speaking, we check if the image of the negation of x is equals to the negation of the image of x, for each element x in the support of the permutation. This optimization reduce considerably the size of the graph and, so accelerate the symmetry detection. In the previous example, we can remove 12 nodes and 12 edges. More generally, we can remove from the graph as many nodes and edges as binary clauses on the formula. Figure 3.2 represents the optimized version the graph.

$\omega_1 = \{x_1, x_2, x_3\}$
$\omega_2 = \{x_4, x_5, x_6\}$
$\omega_3 = \{x_1, x_4\}$
$\omega_4 = \{x_2, x_5\}$
$\omega_5 = \{x_3, x_6\}$
$\omega_6 = \{\neg x_1, \neg x_2\}$
$\omega_7 = \{\neg x_1, \neg x_3\}$
$\omega_8 = \{\neg x_2, \neg x_3\}$
$\omega_9 = \{\neg x_4, \neg x_5\}$
$\omega_{10} = \{\neg x_4, \neg x_6\}$
$\omega_{11} = \{\neg x_5, \neg x_6\}$

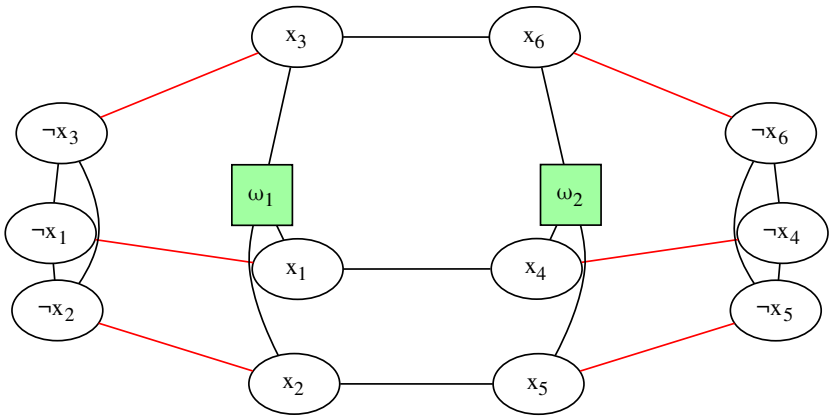


Figure 3.2: Example of constructed symmetry graph for a given CNF

After the construction of such a graph, it is given to an automorphism tool. This last will produce its set of generators. With the previous graph, the following generators are obtained with `bliss`:

$$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(\neg x_2\ \ \neg x_3)(\neg x_5\ \ \neg x_6)$$
$$g_2 = (x_1\ \ x_2)(x_4\ \ x_5)(\neg x_1\ \ \neg x_2)(\neg x_4\ \ \neg x_5)$$
$$g_3 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)(\neg x_1\ \ \neg x_4)(\neg x_2\ \ \neg x_5)(\neg x_3\ \ \neg x_6)$$

These permutations generate the group of symmetries of the formula, and so induce an equivalence relation. Figure 3.3 shows a graphical representation of an orbit, where each node represents a literal. Two literals are linked with an arc if it exists a permutation that maps one to the other. An orbit must be a *strongly connected component* (SCC) and have a particular form as in this example (matrix). Section 3.4 shows how to exploit this form.

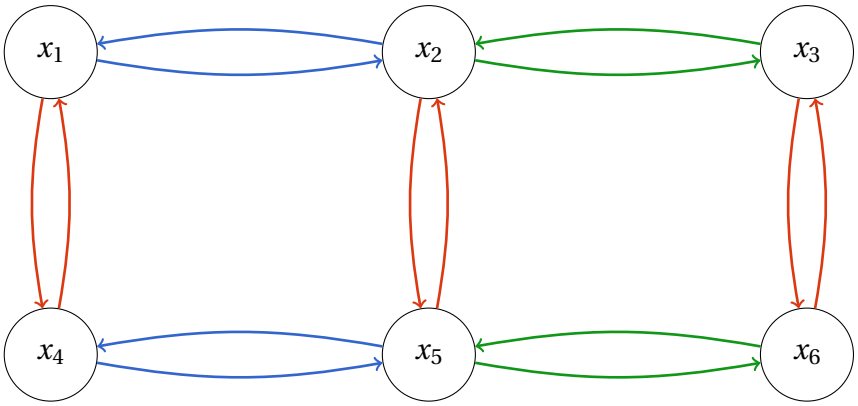


Figure 3.3: Graphical representation of an orbit

3.4 Usage of symmetries

To illustrate the usage of symmetries, consider the *pigeonhole problems* (see Figure 3.4), where n pigeons are put into $n-1$ holes, with the constraint that each pigeon must be in a different hole. This is a highly symmetric problem. Indeed, all the pigeons (resp. holes) are exchangeable without changing the initial problem.

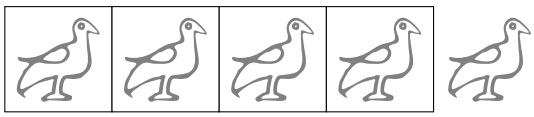


Figure 3.4: Graphical representation of an instance of the pigeonhole problem (5 pigeons, 4 holes)

The search algorithm explores fruitlessly the symmetric search space, i.e, try all possible combination of couple (pigeon, hole). Solving this problem with a standard SAT solver,

like MiniSAT [17], turns out to be very time consuming (and even impossible, in reasonable time, for high values of n). To avoid this combinatorial explosion, a technique called *symmetry breaking* allows SAT solver to avoid the visit of symmetric search space. To do so, it will exists two principles that called *static symmetry breaking* and *dynamic symmetry breaking*. In the first one, visiting one assignment for each orbit is sufficient to determine the satisfiability of the whole formula. So symmetry breaking constraints that invalidate symmetric assignment are added to the initial problem before the start of solving (statically). The second one alter the search space during the solving (dynamically) to avoid the exploration of symmetric search space. In the following sections, we present in detail the two principles.

Static symmetry breaking

This section explains how to exploit (statically) symmetrical properties of a SAT problem. In this approach, only one assignment (branch) for each orbit must be visited and others can be omitted to deduce the satisfiability of the formula by adding constraints. This leads us to the following questions:

1. How to choose branch that are equivalent to all symmetric ones?
2. How to generate constraints that forbid symmetrical assignments?

To answer question 1, we need to introduce an ordering relation between assignments

Definition 1 (Assignments ordering)**.** *We assume a total order, $\prec$, on $\mathcal{V}$. Given two assignments $(\alpha, \beta) \in Ass(\mathcal{V})^2$, we say that α is strictly smaller than β, noted $\alpha < \beta$, if there exists a variable $v \in \mathcal{V}$ such that:*

- *for all $v' \prec v$, either $v' \in \alpha \cap \beta$ or $\neg v' \in \alpha \cap \beta$.*
- *$\neg v \in \alpha$ and $v \in \beta$* [1].

In other words, the prefix of both assignment is equal according to the ordering relation $\prec$ and the next variable v has a different value, $\alpha(v) = \bot, \beta(v) = \top$, then $\alpha < \beta$. Note that $<$ coincides with the lexicographical order on *complete* assignments. Furthermore, the $<$ relation is monotonic as expressed in the following proposition:

Proposition 1 (Monotonicity of assignments ordering)**.** *Let $(\alpha, \alpha', \beta, \beta') \in Ass(\mathcal{V})^4$ be four assignments.*

$$\text{If } \alpha \subseteq \alpha' \text{ and } \beta \subseteq \beta', \text{ then } \alpha < \beta \implies \alpha' < \beta'$$

Proof. The proposition follows on directly from definition 1. □

[1] We could have chosen as well $v \in \alpha$ and $\neg v \in \beta$ without loss of generality.

Given a formula φ and its group of symmetries G, the *orbit of α under G* (or simply the *orbit of α* when G is clear from the context) is the set $[\alpha]_G = \{g.\alpha \mid g \in G\}$. The lexicographic leader (*lex-leader* for short) of an orbit $[\alpha]_G$ is defined by $min_<([\alpha]_G)$. This *lex-leader* is unique because the lexicographic order is a total order. The optimal approach to solve a symmetric SAT problem would be to explore only one assignment per orbit (for instance each lex-leader).

The lex-leader predicates for a permutation $g \in G_\varphi$ is defined as :

$$LL_g = \forall i : (\forall j < i : x_j = g.x_j) \Rightarrow x_i \preceq g.x_i$$

In other words, each assignment that have a variable such that its image under g is smaller according to the ordering relation $\prec$, is pruned by LL_g. Conjunction of LL_g, for all permutations $g \in G_\varphi$ results a sound and complete set of symmetry breaking predicates also called *full symmetry breaking*. Only lex-leader assignment will be visited for each orbit. But, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be totally computed.So, Finding the lex-leader of an orbit is computationally hard [32]. Conjunction of LL_g for some $H \subset G_\varphi$ results a set of symmetry breaking predicates that aims at visiting at least one assignment per orbit and is called *partial symmetry breaking*. In this situation, several assignments per orbit can be visited but bring often considerable reduction of the search space. Partial symmetry breaking gives a good trade-off between the number of generated constraints and the reduction of the search space. In the partial and full symmetry breaking, the set of symmetry breaking predicates generated is denoted by ψ.

Theorem 1 (Satisfiability preservation SBPs)**.** *Let φ be a formula and ψ the computed SBPs for the set of symmetries in G_φ:*

$$\varphi \text{ and } \varphi \wedge \psi \text{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under G_φ. Thus, g does not contradict β, which implies that $\beta \models \psi$. □

The generation of lex-leader constraints proposed by Crawford et al. [11] is defined as follows:

$$LL_g = \forall i : (\forall j < i : x_j = g.x_j) \Rightarrow \neg x_i \vee g.x_i$$

Figure 3.5 shows an example of the generated clauses for the permutation g_3 of the previous example and a lexicographic order. The last constraint of the figure produces tautological clauses. Actually variables x_1, x_4 are present with both polarities. The constraints of other variables produce also tautological clauses.

Here, the number of clauses generated per constraint increase exponentially by the cardinality of the support of the permutation. Hence, Aloul et al [1] proposed a more compact

Order : $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$; $(\bot < \top)$
Permutation : $g_3 = (x_1 \;\; x_4)(x_2 \;\; x_5)(x_3 \;\; x_6)(\neg x_1 \;\; \neg x_4)(\neg x_2 \;\; \neg x_5)(\neg x_3 \;\; \neg x_6)$

Constraints	Generated SBP
$x_1 \preceq x_4$	$\neg x_1 \vee x_4$
$x_1 = x_4 \Rightarrow x_2 \preceq x_5$	$x_1 \vee x_4 \vee \neg x_2 \vee x_5$ $\neg x_1 \vee \neg x_4 \vee \neg x_2 \vee x_5$
$x_1 = x_4 \wedge x_2 = x_5 \Rightarrow x_3 \preceq x_6$	$x_1 \vee x_4 \vee x_2 \vee x_5 \vee \neg x_3 \vee x_6$ $\neg x_1 \vee \neg x_4 \vee x_2 \vee x_5 \vee \neg x_3 \vee x_6$ $x_1 \vee x_4 \vee \neg x_2 \vee \neg x_5 \vee \neg x_3 \vee x_6$ $\neg x_1 \vee \neg x_4 \vee \neg x_2 \vee \neg x_5 \vee \neg x_3 \vee x_6$
$x_1 = x_4 \wedge x_2 = x_5 \wedge x_3 = x_6 \Rightarrow x_4 \preceq x_1$	$x_1 \vee x_4 \vee x_2 \vee x_5 \vee x_3 \vee x_6 \vee \neg x_4 \vee x_1$ $\cdots$ $\neg x_1 \vee \neg x_4 \vee x_2 \vee x_5 \vee x_3 \vee x_6 \neg x_4 \vee x_1$ $\cdots$

Figure 3.5: Example of generated SBPs for one permutation

representation of sbps based on the creation of auxiliary variables. These variables encodes equality of literals and are disjoint from the support of the permutation. Following clauses encode a compact lex-leader for a permutation:

$$\begin{array}{llll} \neg y_i \vee \neg x_{i-1} \vee \neg x_i \vee g.x_i & 1 \leq i \leq n & \neg y_i \vee \neg x_{i-1} \vee \neg y_{i+1} & 1 \leq i \leq n \\ \neg y_i \vee g.x_{i-1} \vee \neg x_i \vee g.x_i & 1 \leq i \leq n & \neg y_i \vee g.x_{i-1} \vee \neg y_{i+1} & 1 \leq i \leq n \end{array}$$

where $\{y_0, \cdots, y_n\}$ be the set of auxiliary variables, y_0 is a unit clause that encodes the first equality and $\{x_1, ..., x_n\}$ be the set of variables sorted with the lexicographic order.

Figure 3.6 shows the compact encoding of generated clauses. This form grows linearly with the number of variables. The auxiliary variables that encode the equality of two literals provide this reduction. Three auxiliary variables are introduced in this example x_7, x_8, x_9 such that x_7 encodes the equality of x_1 and x_4, x_8 encodes the equality of x_2 and x_5, and x_9 encodes the equality of x_3 and x_6.

`Shatter` is a tool [1] for partial symmetry breaking. It computes a compact lex-leader sbps with the symmetries produced by `saucy3` [24]. The following table shows the number of symmetry breaking predicates and the number of auxiliary variables added to the original formula.

Instances	#vars	#clause	#sbp	#auxiliary variables
battleship-12-12-unsat	936	144	1498	378
battleship-12-23-sat	1662	276	5464	1375
battleship-14-26-sat	2562	364	3688	929
battleship-14-27-sat	2653	378	7222	1814
battleship-16-16-unsat	2176	256	4388	1102
battleship-16-31-sat	3976	496	12094	3035
battleship-24-57-sat	16308	1368	40372	10113
chnl10_11	1122	220	2416	615
chnl10_12	1344	240	2736	696
chnl10_13	1586	260	3252	826
chnl11_12	1476	264	3204	813
chnl11_13	1742	286	3636	922
chnl11_20	4220	440	6760	1710
fpga10_15_uns_rcr	2130	300	4580	1160
fpga10_20_uns_rcr	3840	400	6768	1712
fpga11_12_uns_rcr	1476	264	3704	938
fpga11_13_uns_rcr	1742	286	4076	1032
fpga11_14_uns_rcr	2030	308	4740	1199
fpga11_15_uns_rcr	2340	330	5196	1314
fpga11_20_uns_rcr	4220	440	7864	1986
hole010	561	110	1054	269
hole015	1816	240	3280	828
hole020	4221	420	6478	1630
hole030	13981	930	21322	5346
hole040	32841	1640	44934	11254
hole050	63801	2550	81682	20446
Urq6_5	1756	180	109	0
Urq7_5	2194	240	143	0
Urq8_5	3252	327	200	0
x1_40	314	118	42	1
x1_80	634	238	80	0

Table 3.1: Number of sbps generated on different problem categories

Order : $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$; $(\bot < \top)$
Permutation : $g_3 = (x_1 \;\; x_4)(x_2 \;\; x_5)(x_3 \;\; x_6)(\neg x_1 \;\; \neg x_4)(\neg x_2 \;\; \neg x_5)(\neg x_3 \;\; \neg x_6)$

Constraints	Generated SBP
$x_1 \preceq x_4$	$\neg x_1 \vee x_4$
	x_7
$x_1 = x_4 \Rightarrow x_2 \preceq x_5$	$\neg x_7 \vee \neg x_1 \vee \neg x_2 \vee x_5$
	$\neg x_7 \vee \neg x_1 \vee x_8$
	$\neg x_7 \vee x_4 \vee \neg x_2 \vee x_5$
	$\neg x_7 \vee x_4 \vee x_8$
$x_1 = x_4 \wedge x_2 = x_5 \Rightarrow x_3 \preceq x_6$	$\neg x_8 \vee \neg x_2 \vee \neg x_3 \vee x_6$
	$\neg x_8 \vee \neg x_2 \vee x_9$
	$\neg x_8 \vee x_5 \vee \neg x_3 \vee x_6$
	$\neg x_8 \vee x_5 \vee \neg x_9$

Figure 3.6: Example of compact generated SBPs for one permutation

Table 3.1 presents the number of variables and clauses present in the formula and also the number of sbps generated and the number of auxiliary variables added on different problem categories that are: battleship (the battleship puzzle), chnl (channel routing instances), fpga (routing of global wires in integrated circuits) hole (the pigeonhole problem), urq (randomized instance based on expander graphs), xor (exlusive or chain) We can observe that the number of produced sbps or added auxiliary variables can be much larger than respectively the number of initial clauses, variables.

An improvement of static symmetry breaking was made by Devriendt et al [14] with a tool called `BreakID`. It exploits some properties from the structure of the generators. On some cases, a linear number of constraints can break all group. It also tries to add a maximum of binary clauses. These can accelerate the unit propagation and participate to the conflict analysis.

Special form of the group

Some formulas exhibit a specific type of symmetry, called *row (column) interchangeability*. These are a subsets of variables structured as a two-dimensional matrix. Each row (column) is interchangeable. This form of symmetry is common in different kind of problems like the pigeon hole problem in which pigeons and holes are interchangeable or in the delivery system in which trucks of a fleet are interchangeables. The usage of row (column) interchange-

ability can significantly improve SAT performance. Actually symmetries can be eliminated by the addition of only a linear number of symmetry-breaking constraints [18]. One condition must be satisfied to ensure this linear number of constraints: the lexicographic order of variables needs to respect the structure of the matrix. In practice, automorphism tools give only the set generators which contains no information on the structure of the group. Authors of `BreakID` [14] develop an algorithm to detect this specific structure and exploit it.

Binary lex-leader constraints

`BreakID` tries to generate a maximum number of binary lex-leader constraints. The first lex-leader constraint generated by each permutation is a binary clause by definition. Enumerating the whole symmetry group will generates many binary clauses but will be time consuming. To avoid this enumeration, the property of the orbit is exploited. As the orbit can be seen as a strong connected component, it must exist a permutation that permutes a variable (for example the smallest variable according to the lexicographic order) of an orbit with all other variables in the same orbit. This allow to generate as many binary lex-leader constraint as the size of the orbit. In addition, constructing a sequence of subgroups that stabilize the smallest variable results with the generation of new binary sbps. This sequence ends when trivial subgroup is reached and is called a *stabilizer chain*.

Figure 3.7 shows application of the generation of binary clauses. In the example, the considered group has three permutations and its graphical representation is show. Given the lexicographic order, the smallest variable is x_1 and all other variables are in its orbit. According to the ordering relation, five sbps are generated with the formula. Then, the subgroup that stabilizes x_1 is computed. It contains only one permutation (g_2). As it smallest variable according to the lexicographic order is x_2, the constraint $\neg x_2 \vee x_3$ is generated. The stabilizer chain leads to a trivial group and no more binary clauses are generated. In total, six binary clauses is generated without adding any auxiliary variables. Moreover, a property can be observed, when the smallest variable have the greatest value ($\top$ in this case), all variables in the orbits must have the same value.

The size of the stabilizer chain is heavily dependent of the chosen lexicographic order. To avoid to reach trivial subgroup quickly an incremental order is proposed to optimize the number of generates binary clauses. It uses the size of the orbit and occurrences of variable in the set of generators. Biggest orbit produces more binary clauses and few appearing variable keep a large set of generators.

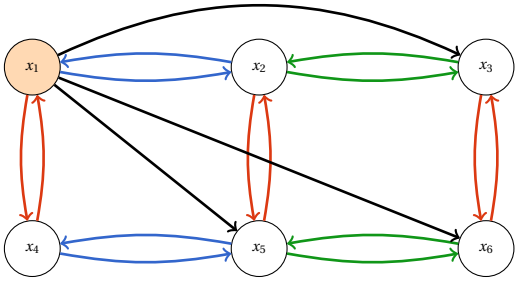
BreakID

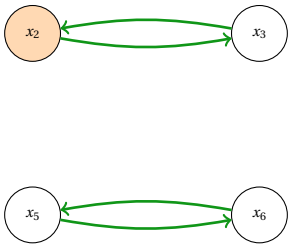
As summary, `BreakID` combines three idea. First it searches if produced generators by the automorphism tool has the special form row interchangeability and if that is the case exploit it. Secondly, it generates a maximum number of binary lex-leader constraints. Thirdly, the classical sbps are generated.

$$\text{Order} \quad : \quad x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \;\; ; \;\; (\bot < \top)$$

$g_1 = (x_2 \;\; x_3)(x_5 \;\; x_6)(\neg x_2 \;\; \neg x_3)(\neg x_5 \;\; \neg x_6)$
$g_2 = (x_1 \;\; x_2)(x_4 \;\; x_5)(\neg x_1 \;\; \neg x_2)(\neg x_4 \;\; \neg x_5)$
$g_3 = (x_1 \;\; x_4)(x_2 \;\; x_5)(x_3 \;\; x_6)(\neg x_1 \;\; \neg x_4)(\neg x_2 \;\; \neg x_5)(\neg x_3 \;\; \neg x_6)$

$g_1 = (x_2 \;\; x_3)(x_5 \;\; x_6)(\neg x_2 \;\; \neg x_3)(\neg x_5 \;\; \neg x_6)$





$\omega_1 = \{\neg x_1, x_2\}$ $\omega_4 = \{\neg x_1, x_5\}$

$\omega_2 = \{\neg x_1, x_3\}$ $\omega_5 = \{\neg x_1, x_6\}$ $\omega_6 = \{\neg x_2, x_3\}$

$\omega_3 = \{\neg x_1, x_4\}$

Figure 3.7: Generation of binary symmetry breaking predicates

Conclusion

Static symmetry breaking acts as a preprocessor that augment the initial formula with sbps. These constraints avoid exploration of isomorphic search spaces. In the general case, the number of these clauses is often too large to be effectively handled by a SAT solver [32]. On the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be perfect and its effectiveness depends heavily on the heuristically chosen symmetries [9]. An important point in static symmetry breaking is the chosen lexicographic order. Variable ordering may impact the number of generated constraints and hence the performance of the underlying solver. Different orders are studied in the literature. One of the simplest order is the lexicographical order. Some other orders exists and the exploit structural properties of the problem [14]. Combining the generation of binary sbps with the exploitation of these properties allows state-of-the-art solver to solver more symmetric instances.

Despite these optimizations and the good reduction of the search space with symmetries, some formula that exhibit symmetries still intractable for a state-of-the-art SAT solver. In the general case, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be totally computed. Even in more favorable situations, the size of the generated *sbp* is often too large to be effectively handled by a SAT solver [31]. On the other hand, if only a subset of the symmetries is considered then the resulting search

pruning will not be that interesting and its effectiveness depends heavily on the heuristically chosen symmetries [9].

Moreover, a disadvantage of static symmetry breaking is that the solver is influenced by SBPs. Internal heuristics consider these clauses as the original clauses, so the solver explores the search space with a different manner which might not be the most efficient part for the problem. This can leads to affect performance negatively.

Dynamic symmetry breaking

Dynamic symmetry breaking approaches aims to exploits the symmetries during the solving by altering search space dynamically. This is possible with the integration of symmetry breaking approach in the solving algorithm. It uses symmetries presents in the formula to propagate symmetrical facts. A consequence, the solver reduces the number of decisions, that are chosen heuristically and increase the number of propagation that are consequence of the formula. In other words, symmetries allow to transform some "guesses" to "deductions". So, it improves performance of the underlying solver. In the literature, different approaches of dynamic symmetry breaking exists, this section presents some of them.

SymChaff

One of the first tool for dynamic symmetry breaking is *SymChaff* [38] and is applicable only on particular groups, where all couple of variables are symmetric (section 3.4). To take part of this property, Instead of using the classical decision heuristic that choose exactly one variable, all symmetric variable are considered at the same time (k-branching). Roughly speaking, all variable in the same orbit are affected/unaffected in the same time. So, all possible valuation of the orbit must be checked. Computing the number of possible valuation is trivial in this particular form of group. The order in which the valuations are checked has a tremendous impact on solver performance. This approach has good results when the group of symmetry presents particular form. In the general case, when we consider any group, computing the number of possible valuation will be very difficult and this approach is not applicable.

Symmetry Propagation

A different approach can be used to reduce the search space using symmetries is *symmetry propagation*(SP) [15]. The general idea of this approach is to propagate symmetrical literals of those already propagated. In other words, it accelerates the tree traversal by “transforming some guessing (decisions) to deductions (propagation)”. Indeed, problem that presents symmetries makes possible to deduce some value for the variables that would be guessed if symmetry properties were ignored. These deductions will reduce the overall tree traversal depth and hence, eventually will accelerate the solving process. To explain this approach, some definitions are required.

Definition 2 (Logical consequence)**.** *A formula ϕ is a logical consequence of a formula φ denoted by $\varphi \models \phi$ if for any assignment α satisfying φ, it also satisfies ϕ. Two formulas are* logically equivalent *if each is a logical consequence of the other.*

Proposition 2 (Symmetry propagation)**.** *Let φ be a formula, α an assignment and l a literal. If g is a symmetry (permutation) of $\varphi \cup \alpha$ and $\varphi \models \{l\}$, then $\varphi \cup \alpha \models g.\{l\}$.*

In other words, if a literal l is propagated by the solver and g is a *valid* symmetry for the sub problem $\varphi \cup \alpha$ (in which all satisfied clauses and false literals are removed) so, the solver can also propagate the symmetric of l. The problem here is to determinate which symmetries are valid for the formula $\varphi \cup \alpha$.

Definition 3 (Active symmetry)**.** *A symmetry g is called active under a partial assignment α if $g.\alpha = \alpha$*

Definition 3 leads to the following proposition:

Proposition 3. *Let φ a formula and α a partial assignment. Let g be a symmetry of φ, if g is active under the assignment α, then g is also a symmetry of $\varphi \cup \alpha$.*

The previous proposition states that an active symmetry g for a partial assignment α is still valid for the formula $\varphi \cup \alpha$. So when a literal l is propagated, and a symmetry g is active for a partial assignment α, the solver can also propagate $g.l$. Moreover, the group theory allows to compose permutations, and the composition of two active symmetries is also an active symmetry, so the solver can also propagate. $g^2.l, g^3.l, ...$

Active symmetries needs strong requirement and can be applied rarely. Devriendt et al [15] improves the active symmetries in the SAT context, introducing *weakly active* symmetries that relax some constraints.

Definition 4 (Weakly active symmetry)**.** *Let φ be a formula and (δ, α, γ) a state of a CDCL solver in which δ is the set of decisions α is the current assignment and γ the reasons of the learned clauses. Then a symmetry g is weakly active if $g.\delta \subseteq \alpha$*

This definition leads to the following proposition:

Proposition 4. *Let φ be a formula, α an assignment. If there exists a subset $\delta \subseteq \alpha$ and a symmetry g of φ such that $g.\delta \subseteq \alpha$ and $\varphi \cup \delta \models \varphi \cup \alpha$, then g also is a symmetry of $\varphi \cup \alpha$.*

In other words, we can detect with a minimal effort, the symmetries of $\varphi \cup \alpha$ by keeping track of the set of variables δ, which are in a state-of-the-art complete SAT solving algorithms, the set of decision variables. Obviously, a weakly active symmetry can also propagate the symmetrical literals of a propagated one. Moreover, weakly active symmetries allow more propagations and so are more efficient.

Symmetry propagation gives good performances on many symmetric instances. The overall performance of the symmetry propagation is intrinsically related to the decision heuristics of the underlying SAT solver.

Note that, this approach don't discard any assignments like in the static approaches where non lex-leader assignments are eliminated.

Symmetry Explanation Learning

Another approach to exploit symmetry without removing any satisfiable assignment of the problem is *Symmetry Explanation Learning* [13] (SEL). Symmetries of a formula leaves this last invariant. Moreover, all learned clauses are logical consequences of the problem, symmetric of these clauses are also valid. Unlike Symmetry explanation scheme [6] (SLS) where all symmetrical learned clauses are added to the clause database. The idea of this approach is to learn useful symmetrical variant of learned clauses. A clause is said to be useful if it participates to the unit propagation or conflict analysis. Computing all symmetrical learnt clauses will create a huge overhead and will be intractable on real problems.

SEL uses the following fact: on the unit propagation, propagated literals has a reason clause which is assertive. Generally, symmetries permute only few literals in a clause and so symmetrical clauses may also be assertive and participate to unit propagation. These clauses are stored in different learning database and treated separately. The solver promotes these clauses when they are effectively useful at the end of unit propagation. As unit propagation is done until fix point, it ensures no duplicate clause is added to the problem. To limit memory impact, symmetrical clauses are removed when the propagated literal responsible of the computation is removed from the assignment.

SEL provides some interesting properties: First, the authors proves that its propagation are a super-set of the one provided by SP. It also does not need to track any status of symmetries (as opposed to SP). Like SP no satisfying assignment are discarded. As disadvantage, SEL may flood the solver if the used set of symmetries is big and take time to compute symmetrical clauses.

Conclusion

Dynamic symmetry breaking exploits the symmetry property of the formula during the solving. Different approaches exist, as in the static approach, some tools forbid non lex-leader assignment and so prune the search space. Others use symmetry to propagate a symmetrical fact. Essentially it transforms decisions (guesses) to propagations (deductions). Moreover, as it is integrated directly to the search engine, solvers can adapt their heuristics dynamically. This approach prevents the creation, as in static symmetry breaking, of potentially useless clause that increase the size of the original formula. Moreover, the computation of *local symmetries* becomes possible. Effectively, some symmetry can appear during the solving algorithm, exploiting these symmetries is not possible on the static approach. However, the integration of dynamic approach must be done carefully,

CDCL algorithm is highly optimized and fine tuned search engines. Integration of symmetry breaking can slow down its core engine

Part II

Contributions

CHAPTER 4

BETWEEN STATIC AND DYNAMIC SYMMETRY BREAKING

Contents

This chapter presents our first contribution published in TACAS 2018 conference 4.

4.1 General idea

In the general case, the size of the *sbp* can be exponential in the number of variables of the problem so that they cannot be entirely computed. Even in more favorable situations, the size of the generated *sbp* is often too large to be effectively handled by a SAT solver [32]. On

the other hand, if only a subset of the symmetries is considered then the resulting search pruning will not be that interesting and its effectiveness depends heavily on the heuristically chosen symmetries [9]. Besides, these approaches are preprocessors, so their combination with other techniques, such as *symmetry propagation* [15], can be very hard. Also, tuning their parameters during the solving turns out to be tough. For all these reasons, some classes of SAT problems cannot be solved yet despite the presence of symmetries. To handle these issues, we propose a new approach that reuses the principles of the static approaches, but operates dynamically: the symmetries are broken during the search process without any pre-generation of the *sbp*. It is a best effort approach that tries to eliminate, *dynamically*, the *non lex-leading* assignments with a minimal computation effort. To do so, we first introduce the notions of *reducer*, *inactive* and *active* permutations with respect to an assignment α and *effective symmetric breaking predicates* (*esbp*).

Definition 5 (Reducer, inactive and active permutation)**.** *A permutation g is a* reducer *of an assignment α if $g.\alpha < \alpha$ (hence α cannot be the lex-leader of its orbit. g reduces it and all its extensions). g is* inactive *on α when $\alpha < g.\alpha$ (so g cannot reduce α and all the extensions). A symmetry is said to be* active *with respect to α when it is neither inactive nor a reducer of α.*

Proposition 5 restates this definition in terms of variables and is the basis of an efficient algorithm to track the status of a permutation during the solving. Let us, first, recall that the *support* of a permutation g, $supp_g$, the set $\{v \in \mathcal{V} \mid g.v \neq v\}$.

Proposition 5. *Let $\alpha \in Ass(\mathcal{V})$ be an assignment, $g \in \mathfrak{S}$ a permutation and $supp_g \subseteq \mathcal{V}$ the support of g. We say that g is:*

1. a reducer of α *if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{v, \neg g^{-1}(v)\} \subseteq \alpha$;
2. inactive *on α if there exists a variable $v \in \mathcal{V}_g$ such that:*
   - $\forall\ v' \in \mathcal{V}_g$, *s. t.* $v' \prec v$, *either* $\{v', g^{-1}(v')\} \subseteq \alpha$ *or* $\{\neg v', \neg g^{-1}(v')\} \subseteq \alpha$,
   - $\{\neg v, g^{-1}(v)\} \subseteq \alpha$;
3. active *on α, otherwise.*

When g is a *reducer* of α we can define a predicate that contradicts α yet preserves the satisfiability of the formula. Such a predicate will be used to discard α, and all its extensions, from a further visit and hence pruning the search tree.

Definition 6 (Effective Symmetry Breaking Predicate)**.** *Let $\alpha \in Ass(\mathcal{V})$, and $g \in \mathfrak{S}\mathcal{V}$. We say that the formula ψ is an effective symmetry breaking predicate (esbp for short) for α under g if:*

$$\alpha \not\models \psi \text{ and for all } \beta \in Ass(\mathcal{V}), \beta \not\models \psi \Rightarrow g.\beta < \beta$$

The next definition gives a way to obtain such an effective symmetry-breaking predicate from an assignment and a reducer.

Definition 7 (A construction of an *esbp*)**.** *Let φ be a formula. Let g be a symmetry of φ that reduces an assignment α. Let v be the variable whose existence is given by item 1. in Proposition 5. Let $U = \{v', \neg v' \mid v' \in \mathcal{V}_g \text{ and } v' \preceq v\}$. We define $\eta(\alpha, g)$ as $(U \cup g^{-1}.U) \setminus \alpha$.*

Example. Let us consider $\mathcal{V} = \{x_1, x_2, x_3, x_4, x_5\}$, $g = (x_1\, x_3)(x_2\, x_4)$, and a partial assignment $\alpha = \{x_1, x_2, x_3, \neg x_4\}$. Then, $g.\alpha = \{x_1, \neg x_2, x_3, x_4\}$ and $v = x_2$. So, $U = \{x_1, \neg x_1, x_2, \neg x_2\}$ and $g^{-1}.U = \{x_3, \neg x_3, x_4, \neg x_4\}$ and we can deduce than $\eta(\alpha, g) = (U \cup g^{-1}.U) \setminus \alpha = \{\neg x_1, \neg x_2, \neg x_3, x_4\}$.

Proposition 6. *$\eta(\alpha, g)$ is an effective symmetry-breaking predicate.*

Proof. It is immediate that $\alpha \not\models \eta(\alpha, g)$.

Let $\beta \in Ass(\mathcal{V})$ such that $\beta \wedge \eta(\alpha, g)$ is UNSAT. We denote a α' and β' as the restrictions of α and β to the variables in $\{v' \in \mathcal{V}_g \mid v' \preceq v\}$. Since $\beta \wedge \eta(\alpha, g)$ is UNSAT, $\alpha' = \beta'$. But $g.\alpha' < \alpha'$, and $g.\beta' < \beta'$. By monotonicity of $<$, we thus also have $g.\beta < \beta$. □

It is important to observe that the notion of *ebsp* is a refinement of the classical concept of *sbp* defined in [1]. Specifically, like *sbp, esbp* preserve satisfiability.

Theorem 2 (Satisfiability preservation)**.** *Let φ be a formula and ψ an esp for some assignment α under $g \in S(\varphi)$. Then,*

$$\varphi \text{ and } \varphi \wedge \psi \text{ are equi-satisfiable.}$$

Proof. If $\varphi \wedge \psi$ is SAT then φ is trivially SAT. If φ is SAT, then there is some assignment β that satisfies φ. Without loss of generality, β can be chosen to be the lex-leader of its orbit under $S(\varphi)$. Thus, g does not reduce β, which implies that $\beta \models \psi$.

□

Algorithm

This section describes how to augment the state-of-the-art CDCL algorithm with the aforementioned concepts to develop an efficient symmetry-guided SAT solving algorithm. The approach is implemented using a couple of components: (1) a *Conflict Driven Clauses Learning (CDCL) search engine*; (2) *a symmetry controller*. Roughly speaking, the first component performs the classical search activity on the SAT problem, while the second observes the engine and maintains the status of the symmetries. When the controller detects a situation where the engine is starting to explore a redundant part[1], it orders the engine to operate a backjump. The detection is performed thanks to *symmetry status tracking* and the backjump order is given by a simple injection of an *esbp* computed on the fly. Principle

[1] Isomorphic to a part that has been/will be explored.

of CDCL is described in section 6, algorithm 4 explains how to extend it with a *symmetry controller* component which guides the behavior of CDCL algorithm depending on the status of symmetries.

```
1  function CDCLSym(φ: CNF formula, SymController: symmetry controller)
     returns ⊤ if φ is SAT and ⊥ otherwise
2    dl ← 0                                          // Current decision level
3    α ← ∅
4    while not all variables are assigned do
5      isConflict ← unitPropagation()
6      SymController.updateAssign(α)
7      isReduced ← SymController.isNotLexLeader(α)
8      if isConflict || isReduced then
9        if dl == 0 then
10         return ⊥                                  // φ is UNSAT
11       if isConflict then
12         ω ← analyzeConflict()
13       else
14         ω ← SymController.generateEsbp(α)
15       φ ← φ ∪ {ω}
16       dl ← backjumpAndRestartPolicies()
17       SymController.updateCancel(α)
18     else
19       α ← α ∪ assignDecisionLiteral()
20       dl ← dl + 1
21   return ⊤                                        // φ is SAT
```

Algorithm 4: the CDCLSym SAT Solving Algorithm.

The symmetry controller is initially given a set of symmetries G [2]. It observes the behavior of the SAT engine and updates its internal data according to the current assignment, to keep track of the status of the symmetries. This observation is *incremental*: whenever a literal is assigned or canceled, the symmetry controller updates the status of all the symmetries. This corresponds to lines 6 and 17 of Algorithm 3. When the controller detects that the current assignment cannot be a *lex-leader* (line 7), it generates the corresponding *esbp* (line 14).

In the remainder of this section, functions composing the symmetry controller are detailed.

[2] The generators of the group of symmetries.

Symmetries Status Tracking.

The `updateAssign`, `updateCancel` and `isNotLexLeader` functions (Algorithm 5) track the status of symmetries based on Proposition 5 ; there, resides th core of our algorithm.

All these functions rely on the pt structure: a map of variables indexed by permutations. Initially, $pt[g] = \min_{\prec}(supp_g)$ for all $g \in G$ according to the ordering relation and all permutations are marked *active*.

For each permutation, g, the symmetry controller keeps track of the smallest variable $pt[g]$ in the support of g such that $pt[g]$ and $g^{-1}(pt[g])$ does not have the same value in the current assignment. If one of the two variables is not assigned, they are considered to have different values.

When new literals are assigned, only active symmetries need to have their $pt[g]$ updated (line 2). This update is done thanks to a while loop (lines 4 – 5).

When literals are canceled, we need to update the status of symmetries for which some variable v before $pt[g]$, or $g^{-1}(v)$, becomes unassigned (lines 9 – 10). Symmetries that were inactive may be reactivated (line 11).

The current assignment is not a *lex-leader* if some symmetry g is a reducer. This is detected by comparing the value of $pt[g]$ with the value of $g^{-1}(pt[g])$ (line 16). The function `isNotLexLeader` also marks symmetries as *inactive* when appropriate (lines 18 – 19).

Generation of the *esbp*.

When the current assignment cannot be a *lex-leader,* some symmetry g is a reducer. The function `generateEsbp` computes the $\eta(\alpha, g)$ defined in Definition 7, which is an effective symmetry-breaking predicate by Proposition 6. This will prevent the SAT engine to explore further the current partial assignment.

Illustrative example

Let us illustrate the previous concepts and algorithms on a simple example. Let the ordering relation $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6 \mid \bot < \top$, and the two last previous generators. $G = \{g_2 = (x_1\ \ x_2)(x_4\ \ x_5), g_3 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)\}$ (written in cycle notation with opposite cycles omitted). Their respective supports sorted according to ordering relation are, $supp_{g_2} = \{x_1, x_2, x_4, x_5\}$ and $supp_{g_3} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

On the assignment $\alpha = \emptyset$, both permutations are active and $pt[g_1] = pt[g_2] = x_1$. When the solver updates the assignment to $\alpha = \{x_4\}$, both permutations remain active and $pt[g_2] = pt[g_3] = x_1$. On the assignment $\alpha = \{x_4, x_1\}$, the symmetry controller updates $pt[g_3]$ to x_2, while $pt[g_2]$ remains unchanged. On the assignment $\alpha = \{x_4, x_1, \neg x_2\}$, $g_2.\alpha = \{x_5, x_2, \neg x_1\}$, which is smaller than α (because $x_1 \in \alpha$ and $\neg x_1 \in g.\alpha$): g_2 is a reducer of α. The symmetry controller then generates the corresponding *esbp* $\omega = \{\neg x_1, x_2\}$.

```
1  function updateAssign(α: assignment)
2      foreach active g ∈ G do
3          v ← pt[g];
4          while {v, g^-1(v)} ⊆ α or {¬v, ¬g^-1(v)} ⊆ α do
5              v ← next variable in V_g;
6          pt[g] ← v
7  function updateCancel(α: assignment)
8      foreach g ∈ G do
9          u ← min{v ∈ V_g | {v, ¬v} ∩ α = ∅ or {g^-1(v), ¬g^-1(v)} ∩ α = ∅};
10         if u ⪯ pt[g] then
11             mark g as active;
12             pt[g] ← u;
13 function isNotLexLeader(α: assignment)
14     foreach active g ∈ G do
15         v ← pt[g];
16         if {v, ¬g^-1(v)} ⊆ α then
17             return ⊤;                                  // g is a reducer
18         if {¬v, g^-1(v)} ⊆ α then
19             mark g as inactive ;                  // g can't reduce α or its
                extensions
20     return ⊥
21 function generateEsbp(α: assignment) returns ω: generated esbp
22     ω ← {};
23     g ← the reducer of α detected in isNotLexLeader;
24     v ← min(V_g);
25     u ← pt[g];
26     while u ≠ v do
27         if v ∈ α then ω ← ω ∪ {¬v} else ω ← ω ∪ {v};
28         if g^-1(v) ∈ α then ω ← ω ∪ {¬g^-1(v)} else ω ← ω ∪ {g^-1(v)};
29         v ← next variable in V_g
30     ω ← ω ∪ {¬v, g^-1(v)};
31     return ω
```

Algorithm 5: the functions keeping track of the status of the symmetries and generating the *esbp*.

4.2 Implementation and Evaluation

In this section, we first highlight some details on our implementation of the symmetry controller. Then, we experimentally assess the performance of our algorithm against three other state-of-the-art tools.

cosy: an efficient implementation of the symmetry controller

We have implemented our method in a C++ library called cosy (1630 LoC). It implements a symmetry controller as described in the previous section, and can be interfaced with virtually any CDCL SAT solver. cosy is released under GPL v3 license and is available at `https://github.com/lip6/cosy`.

Heuristics and Options.

Let us recall that finding the optimal ordering of variables (with respect to the exploitation of symmetries) is NP-hard [31], so the choice for this ordering is heuristic. cosy offers several possibilities to define this ordering:

- a naive ordering, where variables are ordered by the lexicographic order of their names;
- an ordering based on occurrences, where variables are sorted according to the number of times they occur in the input formula. The lexicographic order of variable names is used for those having the same number of occurrences;
- an ordering based on symmetries, where variables belonging to the same orbit (under the given set of symmetries) are grouped together. Orbit are ordered by their numbers of occurrences.

The ordering of assignments we use in this paper orders positive literals before negative ones (thus, $\top < \bot$), but using the converse ordering does not change the overall method. However, it can impact the performance of the solver on some instances, so that it is an option of the library. All the symmetries we used for the presentation of our approach are permutations of variables. Our method straightforwardly extends to permutations of literals, also known as *value permutations* [9].

Integration in MiniSAT.

We show how to integrate cosy to an existing solver, through an example of `MiniSAT` [17]. First, we need an adapter that allows the communication between the solver and cosy (30 LoC). Then, we adapt Algorithm 3 to the different methods and functions of `MiniSAT`. In particular, the function `updateAssign` is moved into the `uncheckEnqueue` function of `MiniSAT` (2 LoC). The `updateCancel` function is moved to the `cancelUntil` function of `MiniSAT` that performs the backjumps (2 LoC). The `isNotLexLeader` and `generateEsbp` functions are integrated in the `propagate` function of `MiniSAT` (30

LoC). This is to keep track of the assignments as soon as they occur, then the *esbp* is produced as soon as an assignment is identified as not being *lex-leader*. Initialization issues are located in the main function of `MiniSAT`(15 LoC). The integration of `cosy` increases `MiniSAT` code by 3%.

Evaluation

This section presents the evaluation of our approach. All experiments have been performed with our modified `MiniSAT` called `MiniSym`. The symmetries of the SAT problem instances have been computed by two different state-of-the-art tools `saucy3` [24] and `bliss` [23]. For a given group of symmetries, the first tool generates less permutations to represent the group than the second one, but it is slower than the other one. We selected from the last six editions of the SAT contests [22], the CNF instances for which `bliss` finds at least 2% of the variables are involved in some symmetries that could be computed in at most 1000*s* of CPU time. We obtained a total of 1350 symmetric instances (discarding repetitions) out of 3700 instances in total. All experiments have been conducted using the following conditions: each solver has been run once on each instance, with a time-out of 5000 seconds (including the execution time of the symmetries generation except for `MiniSAT`) and limited to 8 GB of memory. Experiments were executed on a computer with an Intel Xeon X7460 2.66 GHz featuring 24 cores and 128 GB of memory, running a Linux 4.4.13, along with g++ compiler version 6.3. We compare `MiniSym` using the occurrence order, value symmetries, and without *lex-leader* forcing, against:

- `MiniSAT`, as the reference solver without symmetry handling [17];
- `Shatter`, a symmetry breaking preprocessor described in [1], coupled with the `MiniSAT` SAT engine;
- `BreakID`, another symmetry breaking preprocessor, described in [14], also coupled with the `MiniSAT` SAT engine.

Each SATsolution was successfully checked against the initial CNF. For UNSATsituations, there is no way to provide an UNSATcertificate in presence of symmetries. Nevertheless, we checked our results were also computed by the other measured tools. Unfortunately, out of the 1350 benchmarked formulas, we have no proof or evidence for the 15 UNSATformulas computed by `MiniSym` only. Results are presented Tables in 4.1, 4.2, and 4.3. We report the number of instances solved within the time and memory limits for each solver and category. We separate the UNSAT instances (Table 4.1) from the SAT ones (Table 4.2). Besides the reference with no symmetry (column `MiniSAT`), we have compared the performance of the three tools when using symmetries computed by `saucy3` (see Table 4.1a and Table 4.2a), and `bliss` (see Table 4.1b and Table 4.2b). Rows correspond to groups of instances: from each edition of the SAT contest, and when possible, we separated applicative instances (app$\langle x \rangle$ where $\langle x \rangle$ indicates the year) from hard combinatorial ones (hard$\langle x \rangle$). This separation was not possible for the editions 2015 and 2017 (all2015 and all2017). The

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	18	19	**20**	17
app2014 (161)	23	23	22	**24**
app2013 (145)	6	8	8	**10**
app2012 (367)	115	115	**120**	**120**
hard2016 (128)	8	17	**50**	42
hard2014 (107)	9	24	**30**	29
hard2013 (121)	12	24	**48**	29
hard2012 (289)	86	84	88	**93**
all2017 (124)	8	14	**15**	14
all2015 (65)	9	8	8	**10**
TOTAL (no dup)	261	302	**371**	345

(a) With saucy3

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	18	**21**	18	19
app2014 (161)	23	21	20	**24**
app2013 (145)	6	7	10	**11**
app2012 (367)	115	106	114	**123**
hard2016 (128)	8	11	**79**	77
hard2014 (107)	9	45	40	**53**
hard2013 (121)	12	51	**56**	54
hard2012 (289)	86	69	90	**93**
all2017 (124)	8	14	**15**	**15**
all2015 (65)	**9**	7	8	8
TOTAL (no dup)	261	324	415	**439**

(b) With bliss

Table 4.1: Comparison of different approaches on the UNSATinstances of the benchmarks of the six last editions of the SAT competition.

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	20	**22**	21	20
app2014 (161)	**24**	**24**	**24**	22
app2013 (145)	34	35	35	**43**
app2012 (367)	121	112	119	**126**
hard2016 (128)	**0**	**0**	**0**	**0**
hard2014 (107)	14	**17**	**17**	14
hard2013 (121)	23	23	**24**	22
hard2012 (289)	135	141	**143**	138
all2017 (124)	23	20	26	**27**
all2015 (65)	**7**	5	**7**	6
TOTAL (no dup)	325	323	**337**	335

(a) With saucy3

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
app2016 (134)	20	20	**22**	20
app2014 (161)	**24**	**24**	23	22
app2013 (145)	**34**	32	30	33
app2012 (367)	**121**	112	120	118
hard2016 (128)	**0**	**0**	**0**	**0**
hard2014 (107)	14	14	17	**18**
hard2013 (121)	23	24	**26**	25
hard2012 (289)	135	134	141	**142**
all2017 (124)	23	25	26	**29**
all2015 (65)	**7**	5	6	6
TOTAL (no dup)	325	316	334	**336**

(b) With bliss

Table 4.2: Comparison of different approaches on the SATinstances of the benchmarks of the six last editions of the SAT competition.

total number of instances for each bench is indicated between parentheses. For each row, the cells corresponding to the tools solving the most instances (within time and memory limits) are typeset in bold and grayed out. Table 4.3 shows the cumulative and average PAR-2 times of the evaluated tools. We observe that MiniSym with saucy3 solves the most

Solver	PAR-2 sum	PAR-2 avg
MiniSAT	8 074 348	5 981
Shatter	7 770 434	5 756
BreakID	6 909 999	5 119
MiniSym	7 229 700	5 355

(a) With saucy3

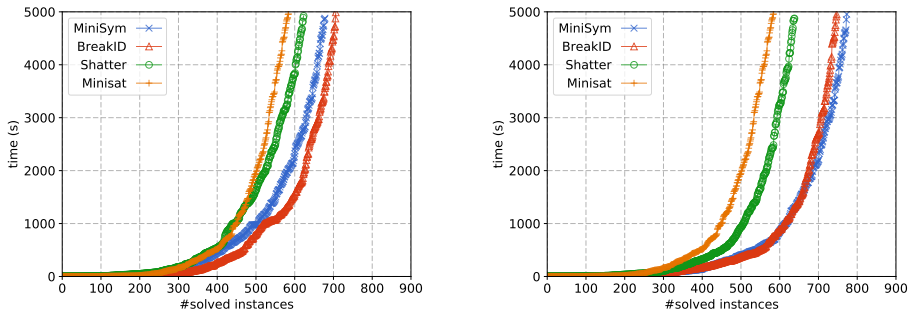
Solver	PAR-2 sum	PAR-2 avg
MiniSAT	8 074 348	5 981
Shatter	7 517 556	5 569
BreakID	6 444 954	4 774
MiniSym	6 245 448	4 626

(b) With bliss

Table 4.3: Comparison of PAR-2 times (in seconds) of the benchmarks on the six last editions of the SAT competition.

instances in only half of the UNSATcategories. However, with bliss, MiniSym solves the

most instances in all but four of the UNSATcategories ; it then also solves the highest number of instances among its competitors. This shows the interest of our approach for UNSATinstances. Since symmetries are used to reduce the search space, we were expecting that it will bring the most performance gain for UNSATinstances. The situation for SATinstances is more mitigated (Table 4.2), especially when using `saucy3`. Again, this is not very surprising: our method may cut the exploration of a satisfying assignment because it is not a *lex-leader*. This delays the discovery of a satisfying assignment. The other tools suffer less from such a delay, because they rely on symmetry breaking predicates generated in a pre-processing step. Also, when seeing the global results of `MiniSAT`, we can globally state that the use of symmetries in the case of satisfiable instances only offers a marginal improvement. We observe that performances our tool are better with `bliss` than with



(a) with `saucy3` (b) with `bliss`

Figure 4.1: cactus plot total number of instances

`saucy3` (see fig 4.1). We explain it as follows: `saucy3` is known to compute fewer generators for the group of symmetries than `bliss`. Since, the larger the symmetries set is, the earlier the detection of an *evidence* that an assignment is not a *lex-leader* will be, we generate less symmetry-breaking predicates (only the effective ones). This is shown in Table 4.4; `MiniSym` generates an order of magnitude fewer predicates than `BreakID`.

Number of SBPs	`BreakID`	`MiniSym`
UNSAT(316)	12 088 433	1 579 623
SAT(312)	13 839 689	359 352

(a) With `saucy3`

Number of SBPs	`BreakID`	`MiniSym`
UNSAT(399)	2 576 349	913 339
SAT(320)	12 179 513	457 452

(b) With `bliss`

Table 4.4: Comparison of the number of generated SBPs each time `BreakID` and `MiniSym` both compute a verdict (number of verdicts between parentheses).

We also conducted experiments on highly symmetrical instances (all variables are involved in symmetries), whose results are presented in Table 4.5. The performance of `BreakID` on this benchmark is explained by a specific optimization for the *total symmetry groups* that are found in these examples, that is neither implemented in `Shatter` nor in `MiniSym`.

However, the difference between BreakID and MiniSym is rather thin when using bliss. [redacted] Our tool still outperforms Shatter on this benchmark.

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
battleship(6)	**5**	**5**	**5**	**5**
chnl(6)	4	**6**	**6**	**6**
clqcolor(10)	3	4	5	**6**
fpga(10)	6	**10**	**10**	**10**
hole(24)	10	12	**23**	11
hole shuffle(12)	1	2	**12**	3
urq(6)	1	2	**6**	2
xorchain(2)	1	1	**2**	**2**
TOTAL	31	42	**69**	45

(a) With saucy3

Benchmark	MiniSAT	Shatter	BreakID	MiniSym
battleship(6)	5	5	5	**6**
chnl(6)	4	**6**	**6**	**6**
clqcolor(10)	3	5	8	**10**
fpga(10)	6	**10**	**10**	**10**
hole(24)	10	**24**	**24**	23
hole shuffle(12)	1	3	**7**	4
urq(6)	1	2	**6**	5
xorchain(2)	1	1	**2**	**2**
TOTAL	31	56	**68**	66

(b) With bliss

Table 4.5: Comparison of the tools on 76 highly symmetric UNSAT problems.

4.3 Related Works

Usage of symmetry property dynamically allows the solver to adapt classical heuristics and symmetry based one on the fly. For example, some restart heuristics are based on the number of conflicts, taking into account injection of ESBP may impact the performance of the overall SAT solver.

Adapt heuristics dynamically

Other heuristics on the symmetry handling may increase the performance. We present here some of them. In some cases multiple permutations are reducer at the same time, and each one generate different symmetry breaking constraints. The backtrack level of the solver and the pruning capacity depends heavily on the chosen one. In the provided library, first reducer permutation generates the constraints. Inject symmetry breaking predicates when it was detected at the beginning of the unit propagation or at the end will change the solver behavior. In the first case, ESBP is more important of a classical conflict if bot occurs and the inverse on the second case. We can even ignore the conflict and just add the constraints to clause database. This prevents the solver to get multiple time on non-minimal part of search. We can allow to go one time for satisfiable for example. Effectively, if the solution is easy to find in this symmetrical branch and much harder on the lex leader, it may have a positive impact.

Change the Order Dynamically

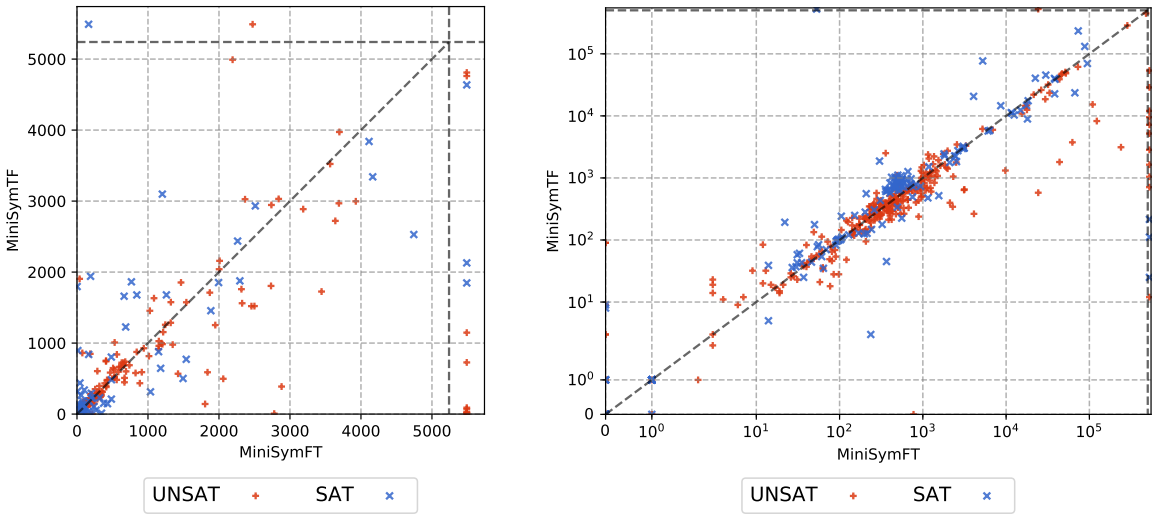
As seen before, ordering relation between variable influence lex-leader and the generated constraints. This order is chosen heuristically, changing this order dynamically is possible but with some requirement. All generated constraints and all deduced clause from a symmetry breaking predicates needs to be completely removed. If these constraints are not removed, inconsistencies may appear and a satisfiable problem become unsatisfiable.

Impact of the sign in variable ordering

With the same variable ordering, swapping the value thus, $\top < \bot$ or $\bot < \top$ may impact drastically the performance of the solver.

To illustrate it, we execute hole100, the pigeonhole problem with 100 holes and 101 pigeons with the increase order and change only the sign. With $\bot < \top$, the solver generates 20 619 ESBP and takes 13.8 seconds to solve it. With the inverse order, $\top < \bot$, it generates 33 263 ESBP ans solve it in 93.4 seconds.

Following figures 4.6 show this difference on 500 symmetric instances with a scatter plot that compare orders, `MiniSymFT` is $\top < \bot$ and `MiniSymTF` is $\bot < \top$. On the left, we compare the duration of the solver, `MiniSymTF` is more efficient on some UNSATinstances. The right figure shows the number of generated ESBP by solvers in log scale. On some instance, it will generate approximately the same number of ESBP. But the difference can an order of magnitude higher. A high number of ESBP is not necessarily better. If few number of ESBP cut-off a huge search space it can be sufficient for the solver to prove the existence or not of a solution, variable ordering impact also.



(a) Time

(b) Number of ESBP

Table 4.6: Comparison of the order with different sign on 500 symmetric instances.

`MiniSymTF` is generally better and is the default choice of the library. But on some cases, it can be worth than the opposite order like in the hole problem. With a specific application, both sign orders can be applied to evaluate which one is better.

4.4 Conclusion

SymmSAT uses same principles as static symmetry breaking but operates dynamically by injecting effective symmetry breaking during the search. This overcomes, the main prob-

lem of the static approaches , that they generate many *sbp* that is not effective in the solving (size of the generated formulas, overburden of the unit propagation procedure, etc.). The idea we bring is to break symmetries *on the fly*: when the current partial assignment cannot be a prefix of a *lex-leader* (of an orbit), an *esbp* that prunes this forbidden assignment and all its extensions are generated. This approach is implemented in the C++ library called `cosy`. It is an off-the-shelf component that can be interfaced with virtually any CDCL SAT solver. `cosy` is released under GPL license and is available at `https://github.com/lip6/cosy`.

The extensive evaluation of our approach on the symmetric formulas of the last six SAT contests shows that it outperforms the state-of-the-art techniques, in particular on unsatisfiable instances, which are the hardest class of the problem.

CHAPTER 5

COMPOSE DYNAMIC SYMMETRY HANDLING

Contents

5.1 Composition of SP and SymmSAT

Recently, we developed an approach that reuses the principles of the static approaches, but operates dynamically (namely, the effective symmetry breaking approach [35]): the symmetries are broken during the search process without any pre-generation of the *sbp*. The main advantage of this technique is to cope with the heavy (and potentially blocking) pre-generation phase of the static-based approaches. It also gives more flexibility for adjusting some parameters on the fly. Nevertheless, we also observed that many formulas easily solved by the pure dynamic approaches remained unsolvable by our approach and vice versa. This is particularly true with the *symmetry propagation* technique developed by Devriendt et al. [15]. Hence, our goal is to explore the composition of our algorithm with the *symmetry propagation* technique in a new approach that would mix the advantages of

the two classes of techniques while alleviating their drawbacks. At first sight, the two approaches appear to be orthogonal, and hence could be mixed easily. However, as we show in the rest of this section, this is not completely true: both theoretical and practical issues have to be analyzed and solved to get a running complementary. Since the approach based on symmetry propagation (later called SPA) focuses on accelerating the tree traversal and the approach based on effective symmetry breaking (later called ESBA) targets to prune the tree traversal, the question of combining these approaches, to solve a formula φ, can be reformulated as:

is it possible to accelerate the traversal while pruning the tree?

Theoretical foundations

To answer the previous questions, we analyze the evolution of φ during its solving. In ESBA, φ evolves, incrementally, to an equi-satisfiable formula of the form $\varphi \equiv \varphi \cup \varphi_e \cup \varphi_d$, where φ_e is a set of injected esbps and φ_d is a set of deduced clauses (logical consequences). Both sets are modified continuously during the solving. Hence, to be able to compose ESBA with SPA, we have to consider the symmetries of $\varphi' = \varphi \cup \varphi_e \cup \varphi_d$ as allowed permutations in place of those of φ. A first naive solution could be to recompute, dynamically, the set of symmetries of $\varphi \cup \varphi_e \cup \varphi_d$ for each new $\varphi_e \cup \varphi_d$, but this would be an intractable solution generating a huge complexity. A computationally less expensive solution would be to keep track of all globally unbroken symmetries as the clauses of φ_e are injected during the solving process: considering formula φ and a set of esbps φ_e then the set of global unbroken symmetries is:

$$GUS = \bigcap_{\omega_e \in \varphi_e} Stab(\omega_e) \cap S(\varphi)$$

Where $Stab(\omega_e) = \{g \in \mathfrak{S}\mathcal{V} \mid \omega_e = g.\omega_e\}$ is the stabilizer set of ω_e and $S(\varphi)$ is the set of symmetries of φ. Since $\varphi \cup \varphi_e \models \varphi_d$, then GUS is a valid set of symmetries for $\varphi \cup \varphi_e \cup \varphi_d$. Then, (1) each time a new set of esbp clauses is added, its stabilizer will be used to reduce GUS; (2) conversely, when a set of esbp clauses is reduced[1], GUS cannot be enlarged by the recovered broken symmetries because of the retrieved set: *at that point, we do not know which symmetries become valid*! As a consequence, the set of globally unbroken symmetries will converge very quickly to the empty set. At this point, SPA will be blocked for the rest of the solving process without any chance to recover. Therefore, this solution is of limited interest in practice. We propose here to improve the aforementioned solution by alleviating the issue cited in point (2). We first present the intuition, then we will detail and formalize it. Consider formula φ' as before. It can be rewritten as:

$$\varphi' = \varphi \bigcup_i (\varphi_e^i \cup \varphi_d^i) \text{ , such that } \varphi_e \cup \varphi_d = \bigcup_i (\varphi_e^i \cup \varphi_d^i) \text{ and } \varphi \cup \varphi_e^i \models \varphi_d^i \text{ for all } i$$

So, $GUS_i = \bigcap_{\omega_e \in \varphi_e^i} Stab(\omega_e) \cap S(\varphi)$ is a valid set of symmetries for the sub-formula $\varphi \cup \varphi_e^i \cup \varphi_d^i$, and GUS can be obtained by $GUS = \bigcap_i GUS_i$. If some esbp clauses are added to φ', then the

[1] In classical CDCL algorithm, this can be due to a back-jump or a restart.

new GUS is computed as described in (1). The novelty here comes with the retrieval of some set of clauses: by keeping track of the symmetries associated to each sub-formula (GUS_i), it is now easy to recompute a valid set of symmetries for φ' when some set $\varphi_e^k \cup \varphi_d^k$ is retrieved. It suffices to operate the intersection on the valid symmetries of the rest of the sub-formulas: $GUS = \bigcap_{i \neq k} GUS_i$. Just say your approach keeps track of a set of particular symmetries for each clause. For a deduced clause, this set of symmetry captures which esbp's were involved in a deduced clause's derivation. The intersection of these sets is a superset of the globally unbroken symmetries, and a strict superset after clause deletion.

Local Symmetries

The general and formal framework that embodies the above idea is given by the following. It first relies on the notion of *local symmetries* that we introduce in definition 8.

Definition 8. *Let φ be a formula. We define $L_{\omega,\varphi}$, the set of local symmetries for a clause ω, and with respect to a formula φ, as follows:*

$$L_{\omega,\varphi} = \{g \in \mathfrak{S}_\mathcal{V} \mid \varphi \models g.\omega\}$$

$L_{\omega,\varphi}$ is local since the set of permutations applies locally to ω. It is then straightforward to deduce the next proposition that gives us a practical framework to compute, incrementally, a set of symmetries for a formula (by using the intersection of all local symmetries).

Proposition 7. *Let φ be a formula. Then, $\bigcap_{\omega \in \varphi} L_{\omega,\varphi} \subseteq S(\varphi)$.*

Proof. Let φ be a formula. Then, $\forall \omega \in \varphi, \forall g \in L_{\omega,\varphi}, \varphi \models g.\varphi$. So, $\forall g \in \bigcap_{\omega \in \varphi} L_{\omega,\varphi}, \varphi \models g.\varphi$. This is combined with the fact that the number of satisfying assignments for a formula is not changed by permuting the variables of the formula, we have $g.\varphi \models \varphi$. Hence $\varphi \equiv g.\varphi$, and $g \in S(\varphi)$ (by definition). □

Using this proposition, it becomes easy to reconsider the symmetries on-the-fly: each time a new clause ω is added to the formula φ, we can just operate an intersection between $L_{\omega,\varphi}$ and $\bigcap_{\omega' \in \varphi} L_{\omega',\varphi}$ to get a new set of valid symmetries for $\varphi \cup \{\omega\}$. Proposition 8 establishes the relationship between the local symmetries of a deduced clause and those of the set of clauses that allow its derivation.

Proposition 8. *Let φ_1 and φ_2 be two formulas, with $\varphi_2 \subseteq \varphi_1$. Let ω be a clause such that $\varphi_2 \models \omega$. Then, $(\bigcap_{\omega' \in \varphi_2} L_{\omega',\varphi_1}) \cup Stab(\omega) \subseteq L_{\omega,\varphi_1}$;*

Proof. Let us consider a clause ω and a permutation $g \in (\bigcap_{\omega' \in \varphi_2} L_{\omega',\varphi_1}) \cup Stab(\omega)$. Since, $\varphi_2 \models \omega$, then $g.\varphi_2 \models g.\omega$. Since $\varphi_1 \models \varphi_2 (\varphi_2 \subseteq \varphi_1)$, and $g \in (\bigcap_{\omega' \in \varphi_2} L_{\omega',\varphi_1}) \cup Stab(\omega)$, then we have $\varphi_1 \models g.\varphi_2$ (from Def. 8). Hence, $\varphi_1 \models g.\varphi_2 \models g.\omega$, and then, $g \in L_{\omega,\varphi_1}$ (by definition). □

Algorithm

This section shows how to integrate the propositions developed in the previous section as the basis of our combo approach in a concrete Conflict-Driven Clause Learning (CDCL)-like solver. First recall the algorithm of symmetry propagation used for the combination of two approaches.

```
1  function CDCLSymSp(φ: CNF formula, spController: symmetry propagation controller)
2      returns ⊤ if φ is SAT and ⊥ otherwise
3      dl ← 0;                                          // Current decision level
4      α ← ∅;
5      while not all variables are assigned do
6          isConflict ← unitPropagation() ∧ spController.symPropagation();
7          if isConflict then
8              if dl = 0 then
9                  return ⊥;                                          // φ is UNSAT
10             ω ← analyzeConflict();
11             dl ← backjumpAndRestartPolicies();
12             φ ← φ ∪ {ω};
13             spController.cancelActiveSymmetries();
14         else
15             α ← α ∪ assignDecisionLiteral();
16             dl ← dl + 1;
17             spController.updateActiveSymmetries();
18     return ⊤;                                              // φ is SAT
```

Algorithm 6: The CDCLSp algorithm. Blue (or grey) parts denote additions to CDCL.

CDCLSp (see Algorithm 6) implements SPA, and also has a structure similar to the one of CDCL. In this algorithm, the symmetry propagation actions are executed by the controller component (spController) through a call to the function symPropagation (line 6). This propagation is allowed only if the conditions are met. Such conditions are evaluated by tracking on-the-fly the status of the symmetries. This is implemented by functions updateSymmetries (line 17) and cancelSymmetries (line 13).

The algorithm we propose for the composed approach is presented in algorithm 7. Let us detail the critical points.

- Lines 13 and 16: when a conflict is detected, then the analyzing procedure is triggered. According to Proposition 8, the generated conflicting clause ω, should be as-

```
1  function CDCLSymSp(φ: CNF formula, symController: symmetry controller,
2                     spController: symmetry propagation controller)
3      returns ⊤ if φ is SAT and ⊥ otherwise
4      dl ← 0;                                              // Current decision level
5      α ← ∅;
6      while not all variables are assigned do
7          isConflict ← unitPropagation() ∧ spController.symPropagation();
8          symController.updateAssign(α);
9          isReduced ← symController.isNotLexLeader(α);
10         if isConflict ∨ isReduced then
11             if dl = 0 then
12                 return ⊥;                                          // φ is UNSAT
13             if isConflict then
14                 ⟨ω, L = ⋂_{ω'∈φ1} L_{ω',φ1} ∪ Stab(ω)⟩ ← analyzeConflictSymSp();
15             else
16                 ⟨ω, L = Stab(ω)⟩ ← symController.generateEsbpSp(α);
17             dl ← backjumpAndRestartPolicies();
18             φ ← φ ∪ {ω};
19             symController.updateCancel(α);
20             spController.cancelActiveSymmetriesSym();
21             spController.updateLocalSymmetries(L);
22         else
23             α ← α ∪ assignDecisionLiteral();
24             dl ← dl + 1;
25             spController.updateActiveSymmetriesSym();
26     return ⊤;                                                  // φ is SAT
```

Algorithm 7: The CDCLSymSp algorithm. Additions derived from MiniSym and CDCLSp are reported in red and blue (or grey). Additions due to the composition of the two algorithms are reported with a gray background.

sociated with the computation of its set of local symmetries. Thus, we update the classical `analyzeConflict` procedure to `analyzeConflictSymSp` that produces such a set: φ_1 contains all the clauses that are used to derive ω[2]. So, at the end of the conflict analysis we operate the intersection of a local symmetry of these clauses to get the set of local symmetries of ω. We can thus complete this set with the stabilizer set (see as Proposition 8).

In the classical algorithm of symmSAT, when a non lex-leader assignment is detected , then the esbp generation function, `generateEsbp`, is called. In the new algorithm this function is replaced by a new one called `generateEsbpSp`. In addition to compute the esbp clause ω, it produces the stabilizer set of ω[3].

- Line 20: `cancelActiveSymmetriesSym` extends function `cancelActiveSymmetrie` of Algorithm 6 with the additional reactivation of the symmetries that have been broken (deactivated) by ESBPA. Technically speaking, each time a deduced literal is unassigned, all symmetries that became inactive because of its assignment (see `updateLocalSymmetries` and `updateActiveSymmetriesSym` functions below) are *reactivated*.

- Line 21: `updateLocalSymmetries` is a new function of `spController`. It is responsible of updating the status of the manipulated symmetries so that only those respecting Proposition 7 are active each time the `symPropagation` function is called. Technically speaking, each symmetry of the complement set (to $S(\varphi)$) of the set L is marked *inactive* (it is a broken symmetry), if it is not already marked so. Here, the asserting literal of clause ω becomes responsible of this deactivation.

- Line 25: `updateActiveSymmetriesSym` extends function `updateActiveSymmetrie` of algorithm 6. The reason clause, ω_l, of each propagated literal, l, by the `unitPropagatior` function is analyzed. Each symmetry of the complement set (to $S(\varphi)$) of the set local symmetries of ω_r is marked *inactive*, if it is not already marked so. l becomes responsible of this deactivation.

Implementation

We have implemented our combo on top of the `minisat-SPFS`[4] solver, developed by the authors of SPA. This choice has been influenced by two points: (1) take advantage of the expertise used to implement the original SPA method; (2) the easiness of integrating our implementation of ESBA to any CDCL-like solver because it is an off-the-shelf library[5]. However, this choice has the drawback of doubling the representation of symmetries. This

[2]These are clauses of the *conflict side* of the implication graph when applying the classical conflict analysis algorithm.

[3]The only allowed local symmetries in case of an esbp, according to point 2 of section **??**.

[4]https://github.com/JoD/minisat-SPFS

[5]This library is released under GPL v3 license, see `https://github.com/lip6/cosy`.

Benchmark	minisat-Sp	minisat-Sym	minisat-SymSP
Generators 0–20 (704)	194	197	**198**
Generators 20–40 (136)	33	**34**	**34**
Generators 40–60 (141)	28	28	**29**
Generators 60–80 (168)	**65**	64	**65**
Generators 80–100 (51)	28	**34**	**34**
Generators >100 (200)	58	59	**60**
TOTAL no dup (1400)	406	416	**420**

Table 5.1: Comparison of the number of SAT problems solved by each approach.

Benchmark	minisat-Sp	minisat-Sym	minisat-SymSP
Generators 0–20 (704)	**233**	220	226
Generators 20–40 (136)	50	**54**	**54**
Generators 40–60 (141)	75	**83**	**83**
Generators 60–80 (168)	**11**	**11**	10
Generators 80–100 (51)	**11**	**11**	**11**
Generators >100 (200)	90	**109**	107
TOTAL no dup (1400)	470	488	**491**

Table 5.2: Comparison of the number of UNSAT problems solved by each approach.

can be a hard limit to treat certain big problems from the memory point of view. The implemented combo solver can be found at:
`https://github.com/lip6/minisat-SymSp`

Evaluation

This section compares our combo approach against ESBA and SPA. All experiments have been performed with a modified version of the well-known `MiniSAT` solver [17]: `minisat-Sp`, for SPA; `minisat-Sym`, for ESBA; and `minisat-SymSP`, for the combo. Symmetries of the SAT problems have been computed by `bliss` [23]. We selected from the last seven editions of the SAT contest [22], the CNF problems for which `bliss` finds some symmetries that could be computed in at most 1000*s* of CPU time. We obtained a total of 1400 SAT problems (discarding repetitions) out of the 4000 proposed by the seven editions of the contest. All experiments have been conducted using the following settings: each solver has been run once on each problem, with a time-out of 7200 seconds (including the execution time of symmetry generation) and limited to 64 GB of memory. Experiments were executed on a computer with an Intel® Xeon® Gold 6148 CPU @ 2.40 GHz featuring 80 cores and 1500 GB of memory, running a Linux 4.17.18, along with g++ compiler version 8.2. Tables 5.1 and 5.2 present the obtained results for SAT and UNSAT problems respectively. The first column of each table lists the classes of problems on which we operated our experiments: we classify the problems according to the number of symmetries they admit. A line noted “generators X-Y (Z)” groups the Z problems having between X and Y generators (i.e., symmetries). Other columns show the number of solved problems for each approach. Globally, we observe that the combo approach can be effective in many

classes of symmetric problems. For SAT problems, the combo has better results than the two other approaches (4 more SAT problems when compared to the best of the two others) and this is despite the significant cost paid for the tracking of the symmetries' status. When looking at the UNSAT problems, things are more mitigated. Although, the total number of solver problems is greater than the best of the two others, we believe that the cost for tracking the symmetries' status has an impact on the performances. This can be observed on the first and last lines of Tables 5.2: when the number of generators is small (first line), the ESBA benefits greatly from the SPA. When the number of generators is high (last line), we see a small loss of the combo with respect to ESBA. It is also worth noting that the combo approach solved **8** problems that could not be handled by ESBA nor SPA. Table 5.3 compares the different techniques with respect to the PAR-2 and the CTI time measures. PAR-2 is the official ranking measure used in the yearly SAT contests [22]. CTI measures the Cumulative solving Time of the problem Intersection (i.e., 825 problems solved by all solvers). While PAR-2 value gives a global indication on the effectiveness of an approach, CTI is a good mean to evaluate its speed compared to other approaches. Hence, we observe that the combo has a better PAR-2 score, and this shows its effectiveness. However, it is the least fast when coming to solved intersection. This is clearly due to the double cost paid for tracking the symmetries' status (one for ESBA and the other for SPA). Having a unified management of symmetries tracking would probably reduce this cost.

Solvers	PAR2 (1400)	CTI (825)
`minisat-SymSp`	5,653,089	614,856
`minisat-Sym`	5,682,892	584,868
`minisat-Sp`	6,026,840	612,638

Table 5.3: Comparison of PAR-2 and CTI times (in seconds) of the global solving.

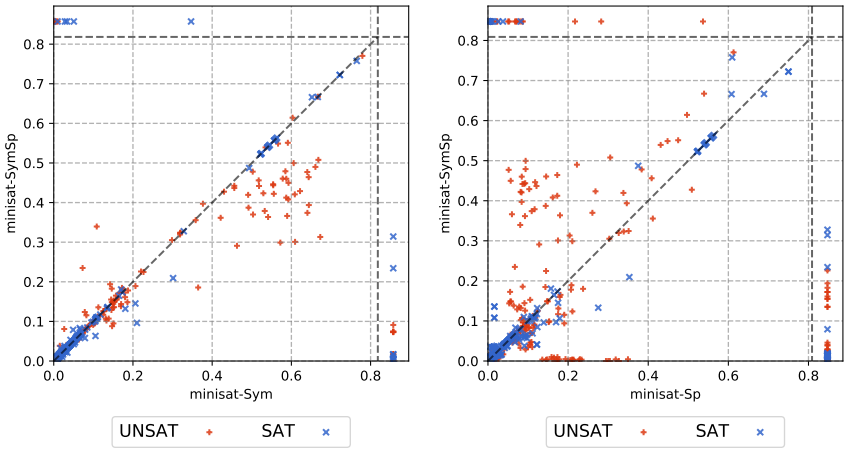


Figure 5.1: Comparison of the ratio between the number of decisions and the number of propagation for the combo w.r.t. ESBA and SPA.

To go further in our analyze, we also compare the ratio between the number of decisions and the number of propagation. This is a fair measure to assess the quality of a SAT solving approach: if the ratio is small, then this means that the developed algorithm is producing more deduced facts than making guesses, which is the best way to conclude quickly on a problem! The scatter plots of Fig.5.1 show a comparison between the aforementioned ratios. When comparing `minisat-Sp` to `minisat-SymSp` (right hand side scatter plot), we observe that the ratio goes in favor of `minisat-Sp` for the problems solved by both approaches. This is an expected result since the main objective of SPA is to minimize the number of decisions while augmenting the number of propagation. What is important to underline here is highlighted on the left hand side scatter plot: on a large majority of UNSAT problems, the ratio goes in favor of `minisat-SymSp` w.r.t. `minisat-Sym`. This confirms the positive impact of SPA when applied in conjunction with ESBA.

5.2 Another combo approach

Introducing local symmetries allow us to use it with another dynamic approach : *Symmetry Explanation Learning* (SEL). It computes the symmetrical learning clause and include it in the solver clauses when it can be used in the unit propagation or leads to a conflict. With local symmetries, each clause has a set of allowed symmetries, when a symmetry wants to add a symmetrical clause, it suffices to check if this permutation belongs to local symmetries. Hakan: Naive approach no stab .

5.3 Exploitation of local symmetries

Local symmetries seen previously allows to exploit symmetries in different ways. Effectively, we can consider symmetries on a part of the problem. Only clauses that are really used by the solver i.e in the implication graph can be processed. To obtain the symmetries of the current sub problem is the intersection of local symmetries.

CHAPTER 6

CONCLUSION AND FUTURE WORKS

In this thesis, we have presented a way to increase performance of solving Boolean satisfiability problem (SAT) on presence of symmetries.

Nowadays, SAT solvers can handled huge problems with thousands of variables and clauses. It is primary due to efficiently cut off search space.

studies of detection and exploitation of symmetry breaking techniques.

Symmetries are

6.1 Perspectives

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