Exploitation of dynamic symmetries for solving SAT problems

Doctorat de Sorbonne Université

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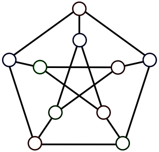
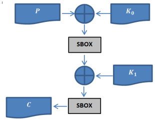
Motivation

Graph coloring

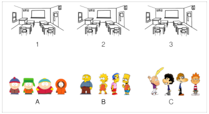


Cryptanalysis



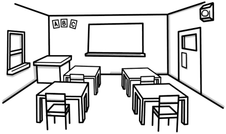
Hardware and sofware verification

Planning

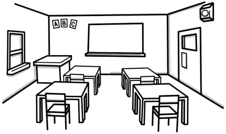


Boolean SATisfiability

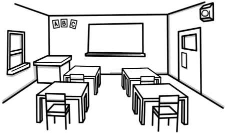
SAT: an example (1/2)



1

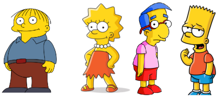


2

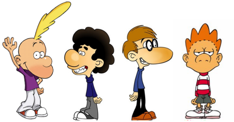


3

A



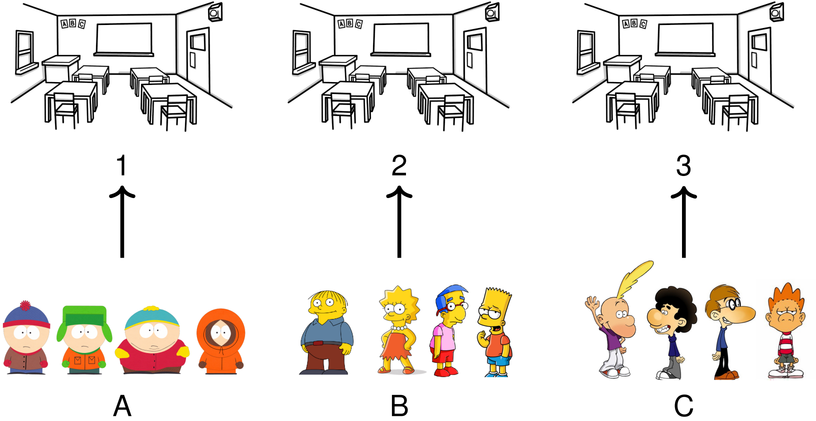
B



C

Is it possible to attribute each group to a unique classroom?

SAT: an example (1/2)



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YES! SATisfiable $\alpha = \{(A, 1), (B, 2), (C, 3)\}$

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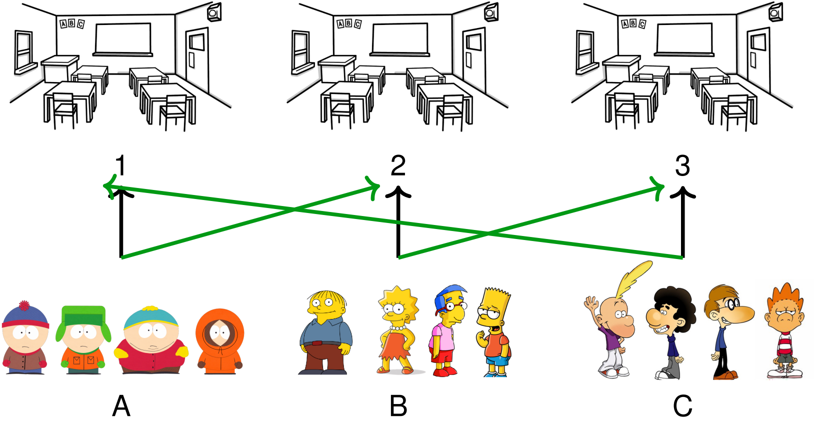

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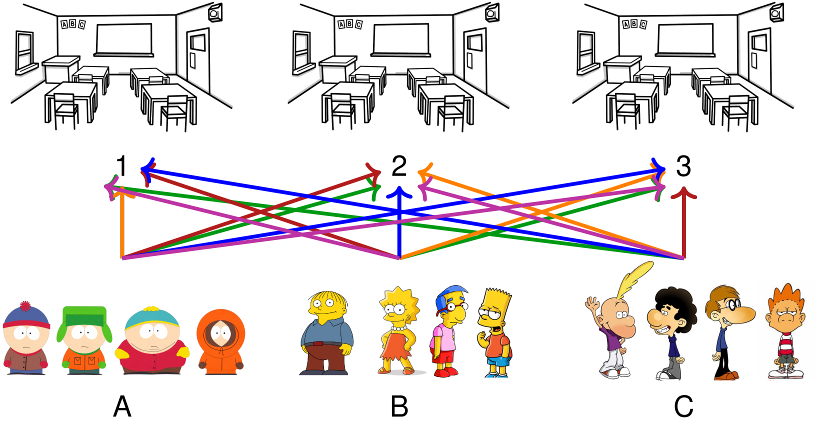

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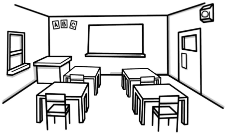
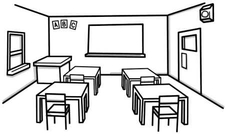
YES! $\texttt{SAT}$isfiable $\alpha = \{(A,1),(B,2),(C,3)\}$
Many solutions $\alpha' = \{(A,2),(B,3),(C,1)\}$

$$\vdots$$

SAT: an example (2/2)

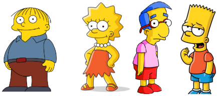


1

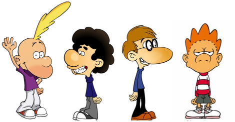


2

A



B



C

Is it possible to attribute each group to a unique classroom?

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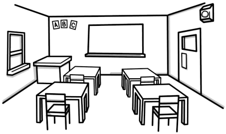
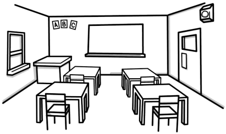
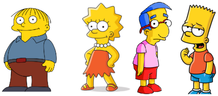


1



2

A



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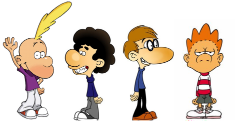
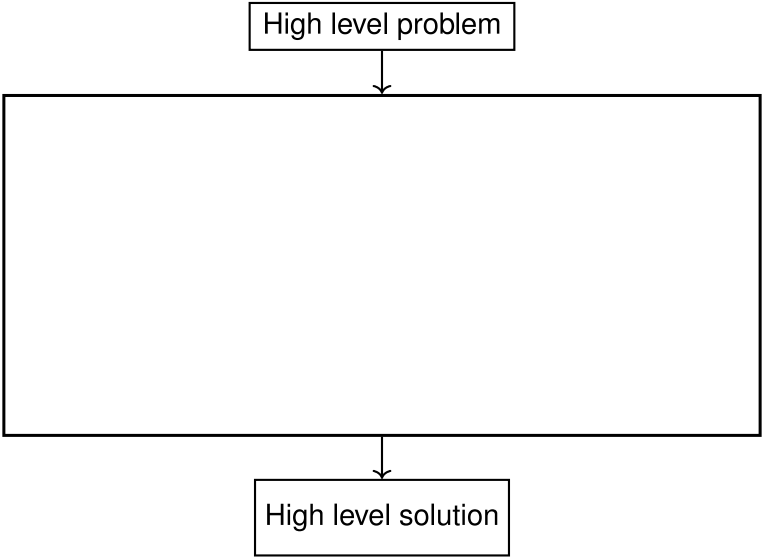


C

Is it possible to attribute each group to a unique classroom?

No! UNSATisfiable

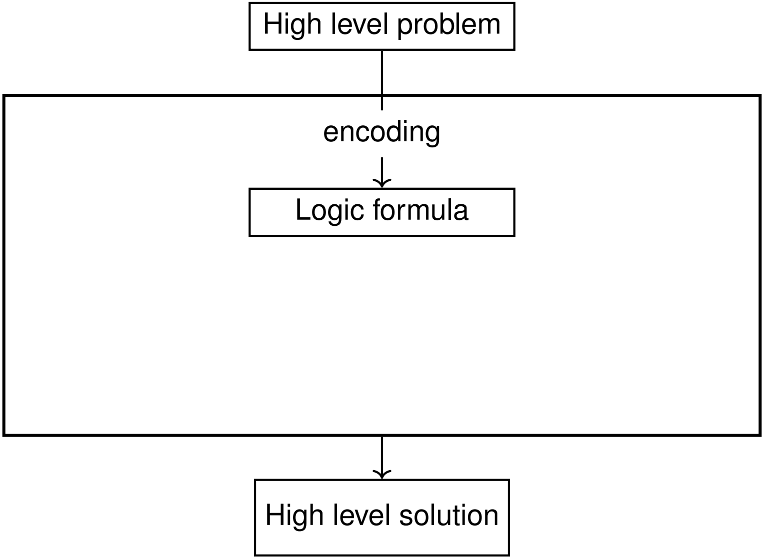
From high level problem to the solution through SAT solving



High level problem

High level solution

From high level problem to the solution through SAT solving



High level problem

encoding

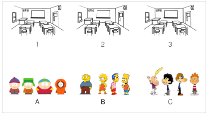
Logic formula

High level solution

Encoding the problem

$$\overbrace{(A,1)}^{x_1}\overbrace{(A,2)}^{x_2}\overbrace{(A,3)}^{x_3}$$
$$(B,1)(B,2)(B,3)$$
$$(C,1)(C,2)(C,3)$$

$$(x_1 \vee x_2 \vee x_3) \quad \wedge$$
$$(x_4 \vee x_5 \vee x_6) \quad \wedge$$
$$(x_7 \vee x_8 \vee x_9) \quad \wedge$$



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$$\neg(A,1)\neg(B,1)$$
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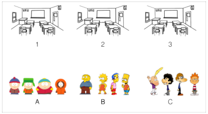

$$(x_1 \vee x_2 \vee x_3) \ \wedge$$
$$(x_4 \vee x_5 \vee x_6) \ \wedge$$
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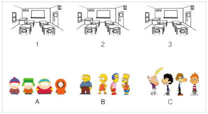
$$(\neg x_1 \vee \neg x_4) \ \wedge$$
$$(\neg x_1 \vee \neg x_7) \ \wedge$$
$$(\neg x_4 \vee \neg x_7) \ \wedge$$

Encoding the problem



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$$\neg(A,2)\neg(B,2)$$
$$\neg(A,2)\neg(C,2)$$
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Encoding the problem

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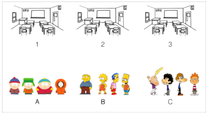

Clause

$$\boxed{(x_1 \vee x_2 \vee x_3)} \wedge$$
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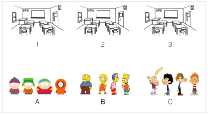

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$$(\neg x_3 \vee \neg x_9) \wedge$$
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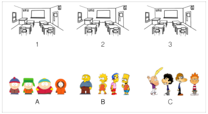
← Conjunctive Normal Form (CNF)

Encoding the problem



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$$\begin{array}{l}
\boxed{(x_1 \vee x_2 \vee x_3)} \wedge \\
(x_4 \vee x_5 \vee x_6) \wedge \\
(x_7 \vee x_8 \vee x_9) \wedge \\
\\
(\neg x_1 \vee \neg x_4) \wedge \\
(\neg x_1 \vee \neg x_7) \wedge \\
(\neg x_4 \vee \neg x_7) \wedge \\
\\
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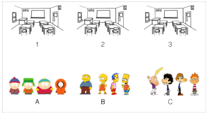
(\neg x_2 \vee \neg x_8) \wedge \\
(\neg x_5 \vee \neg x_8) \wedge \\
\\
(\neg x_3 \vee \neg x_6) \wedge \\
(\neg x_3 \vee \neg x_9) \wedge \\
(\neg x_6 \vee \neg x_9)
\end{array}$$

Clause

Conjunctive Normal Form (CNF)

Any Boolean formula can be transformed into CNF in polynomial time

Encoding the problem



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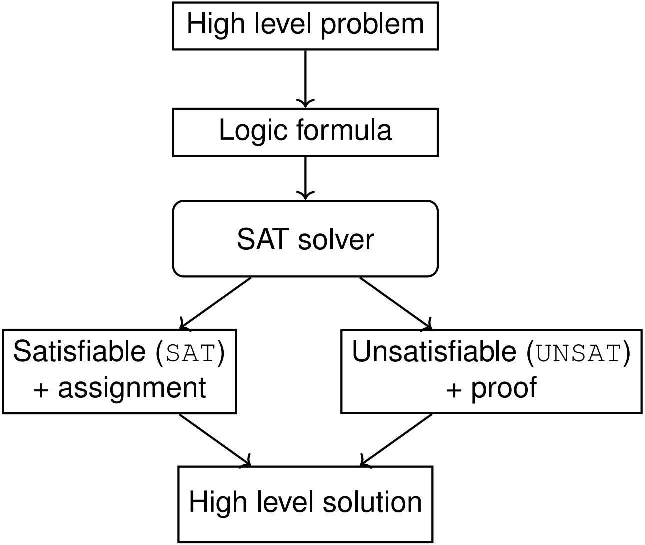
Conjunctive Normal Form (CNF)

Clause represented as a set:

$$(x_1 \vee x_2 \vee x_3) \rightarrow \{x_1, x_2, x_3\}$$

From high level problem to the solution through SAT solving



High level problem

Logic formula

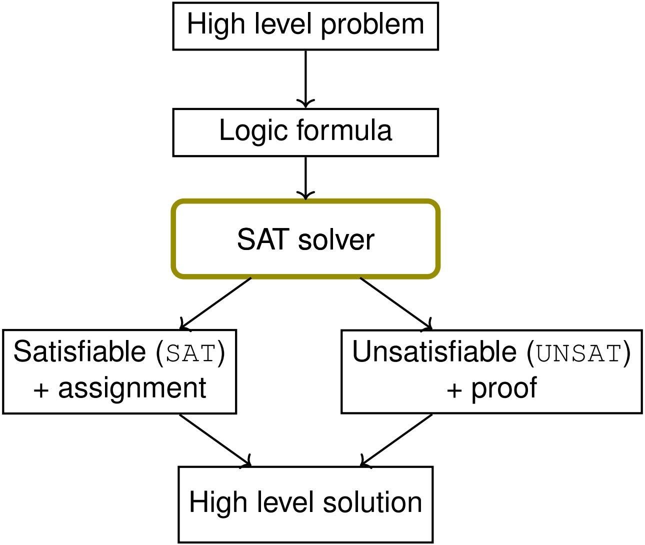
SAT solver

Satisfiable (`SAT`)
+ assignment

Unsatisfiable (`UNSAT`)
+ proof

High level solution

From high level problem to the solution through SAT solving



High level problem

Logic formula

SAT solver

Satisfiable (`SAT`) + assignment

Unsatisfiable (`UNSAT`) + proof

High level solution

SAT Solving

Solving SAT formula is known to be **NP-complete** [Coo71]

SAT Solving

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Good performance in practice:

- Handle large problem (million variables and clauses)
- International SAT competition each year on academic and industrial problems

SAT Solving

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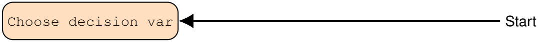
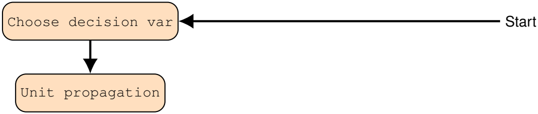
Enumerative algorithms:

- Davis, Putnam, Logemann, and Loveland (DPLL) [DLL62]
  - Boolean Constraint Propagation (BCP)
- **Conflict Driven Clause Learning** (CDCL) [MSS99]
  - Derived from DPLL
  - Clause learning

CDCL in detail



CDCL in detail

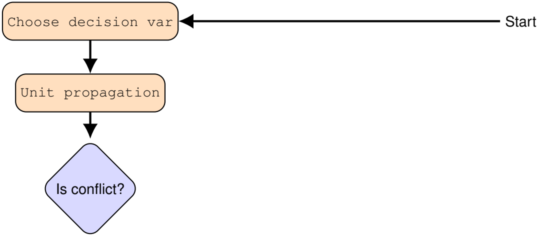


Choose decision var

Unit propagation

Start

CDCL in detail



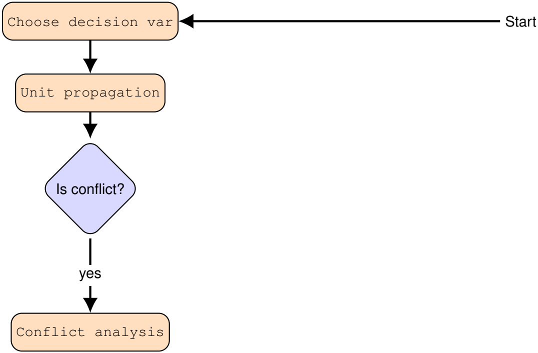
`Choose decision var`

`Unit propagation`

Is conflict?

Start

CDCL in detail



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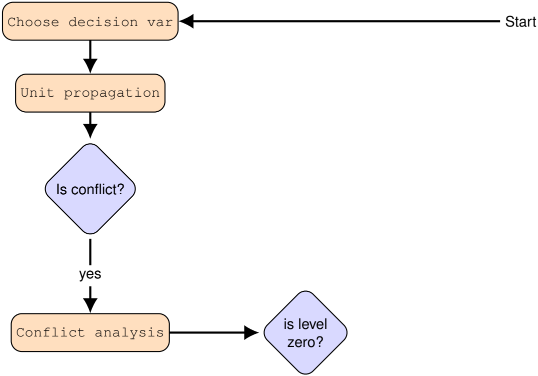

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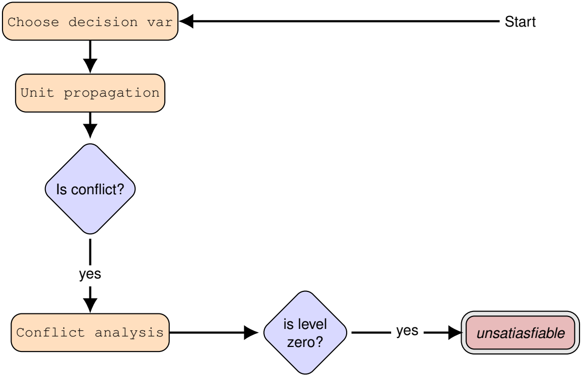

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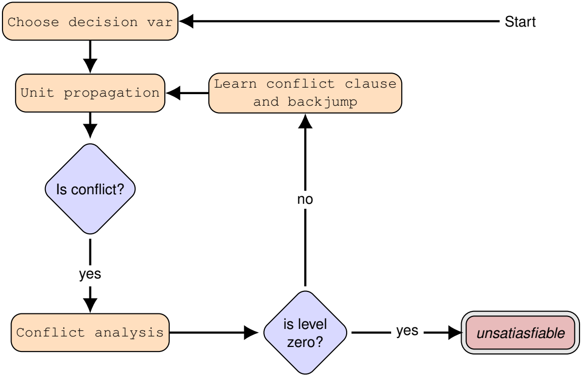

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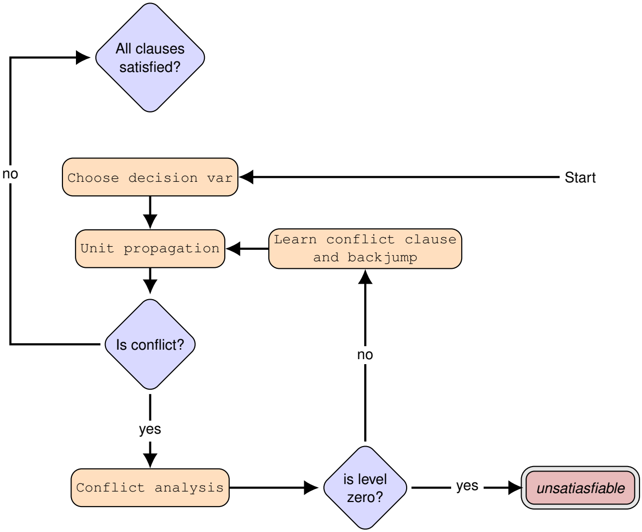

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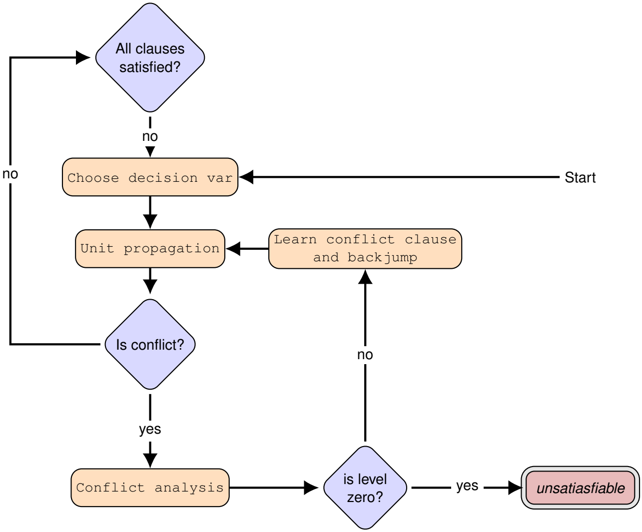


All clauses satisfied?

no

Choose decision var

Start

Unit propagation

Learn conflict clause and backjump

Is conflict?

no

yes

Conflict analysis

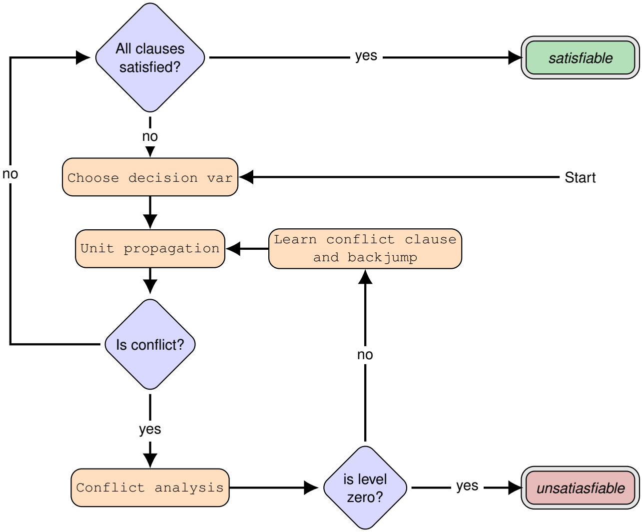
is level zero?

no

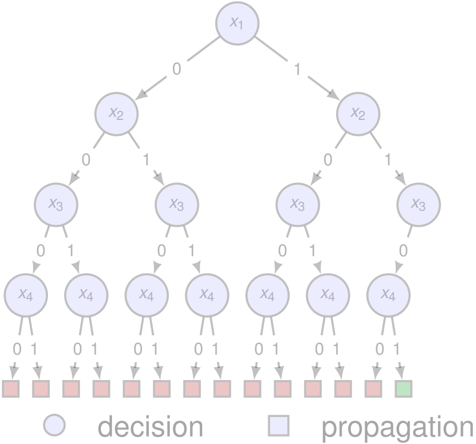
yes

unsatiasfiable

CDCL in detail



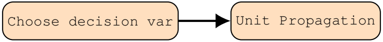
CDCL in action



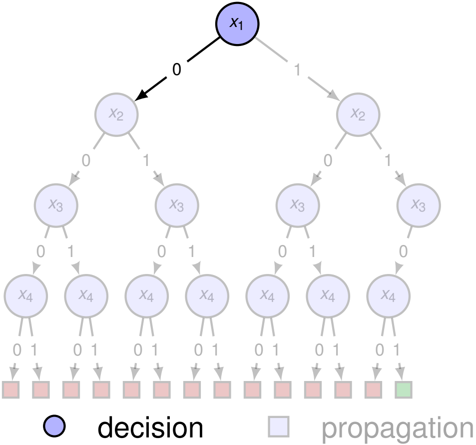
$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$
$$\omega_2 = \{x_1, \neg x_4\}$$
$$\omega_3 = \{x_1, x_4\}$$
$$\omega_4 = \{x_2, \neg x_4\}$$
$$\omega_5 = \{x_2, x_4\}$$
$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\}$$

CDCL in action



Choose decision var → Unit Propagation



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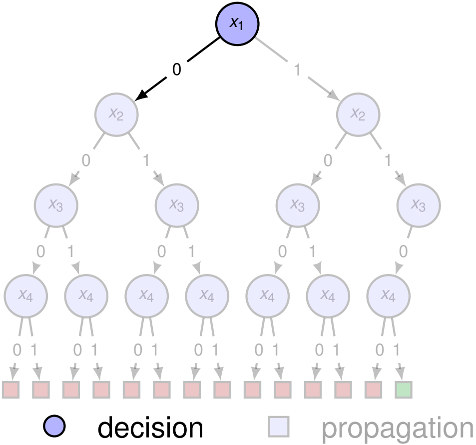
$$\omega_5 = \{x_2, x_4\}$$

$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg x_1\}$$

CDCL in action

Conflict Analysis



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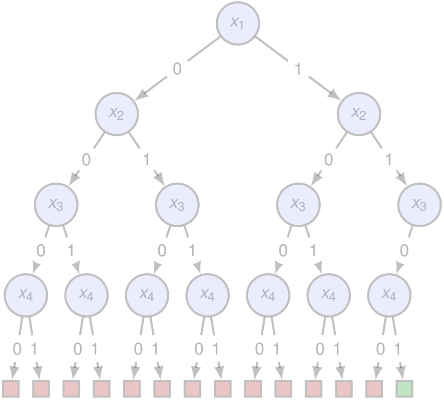
$$\omega_6 = \{x_3, x_4\}$$

$$\alpha = \{\neg x_1\}$$

CDCL in action

Learn conflict clause and backjump



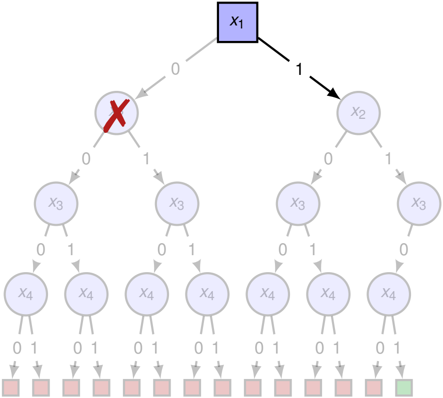
● decision ☐ propagation

$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$
$$\omega_2 = \{x_1, \neg x_4\}$$
$$\omega_3 = \{x_1, x_4\}$$
$$\omega_4 = \{x_2, \neg x_4\}$$
$$\omega_5 = \{x_2, x_4\}$$
$$\omega_6 = \{x_3, x_4\}$$
$$\omega_7 = \{x_1\}$$

$$\alpha = \{\}$$

CDCL in action

Unit Propagation



○ decision ▢ propagation

$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$
$$\omega_2 = \{x_1, \neg x_4\}$$
$$\omega_3 = \{x_1, x_4\}$$
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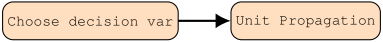
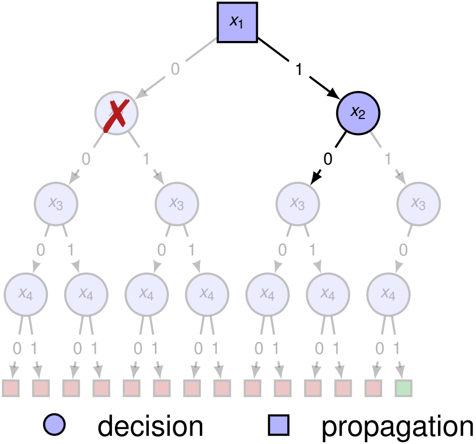
$$\alpha = \{x_1\}$$

CDCL in action


Choose decision var → Unit Propagation



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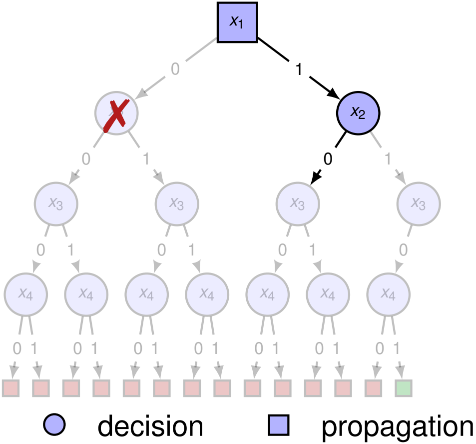
$$\omega_4 = \{x_2, \neg x_4\}$$
$$\omega_5 = \{x_2, x_4\}$$
$$\omega_6 = \{x_3, x_4\}$$
$$\omega_7 = \{x_1\}$$

$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

Conflict Analysis



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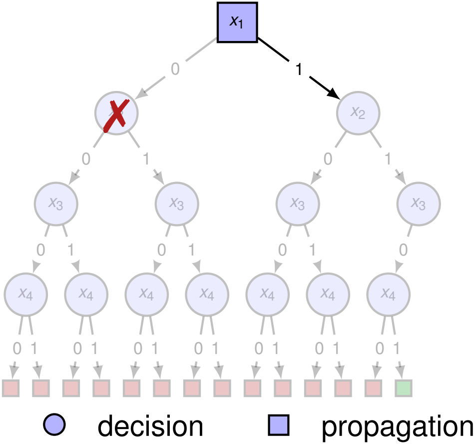
$$\alpha = \{x_1, \neg x_2\}$$

CDCL in action

`Learn conflict clause and backjump`



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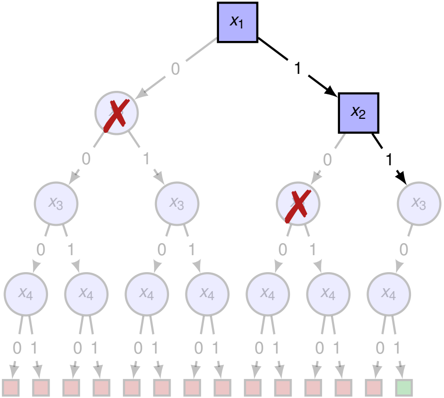
$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1\}$$

CDCL in action

Unit Propagation



○ decision ☐ propagation

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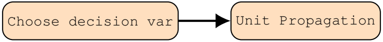
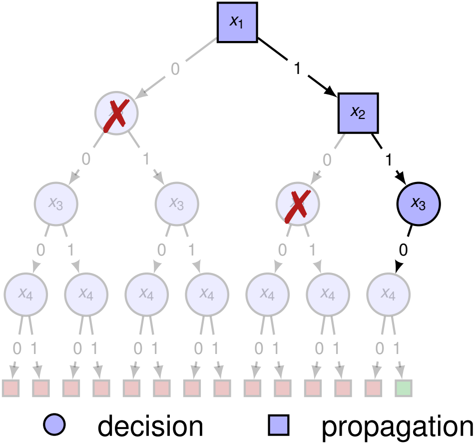
CDCL in action



`Choose decision var` → `Unit Propagation`



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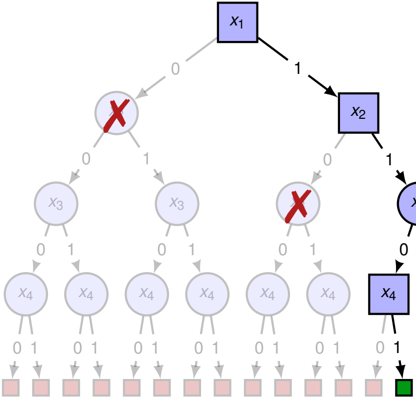
$$\omega_7 = \{x_1\}$$

$$\omega_8 = \{x_2\}$$

$$\alpha = \{x_1, x_2, \neg x_3\}$$

CDCL in action

satisfiable



○ decision □ propagation

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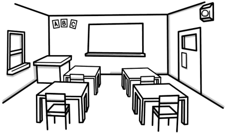
$$\omega_7 = \{x_1\}$$
$$\omega_8 = \{x_2\}$$

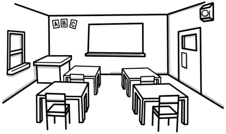
$$\alpha = \{x_1, x_2, \neg x_3, x_4\}$$

SAT and symmetries

Presence of symmetries

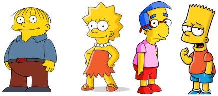


1

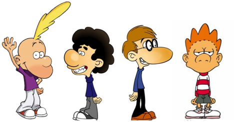


2

A

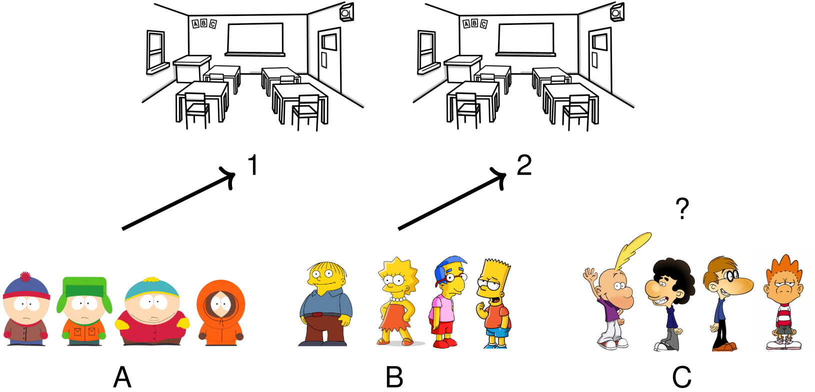


B

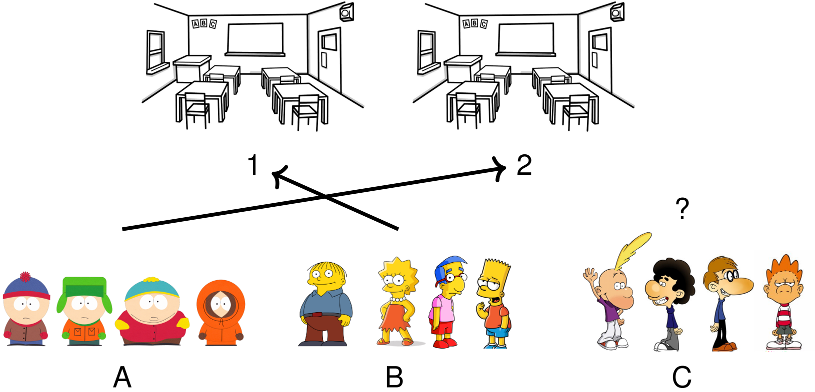


C

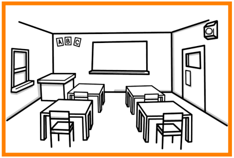
Presence of symmetries



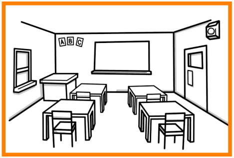
Presence of symmetries



Presence of symmetries



1



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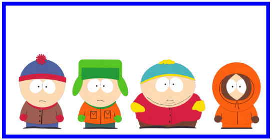
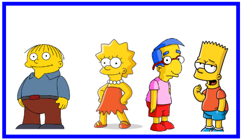
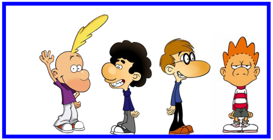


A

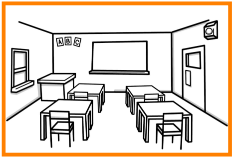


B

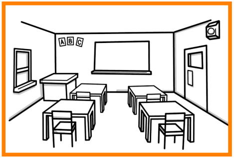


C

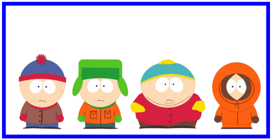
Presence of symmetries



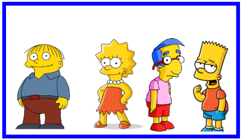
~~1~~ 2



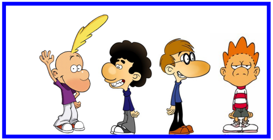
~~2~~ 1



A



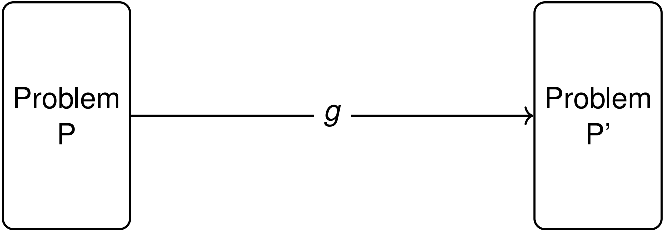
~~B~~ C



~~C~~ B

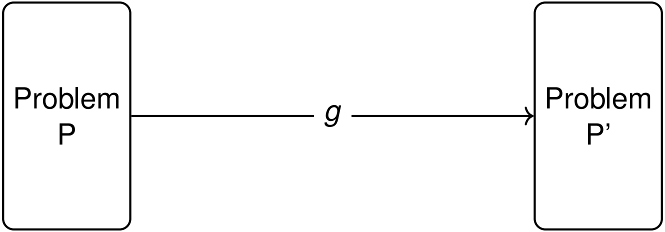
Symmetry in high level

g a symmetry



Symmetry in high level

g a symmetry



Problem P

g

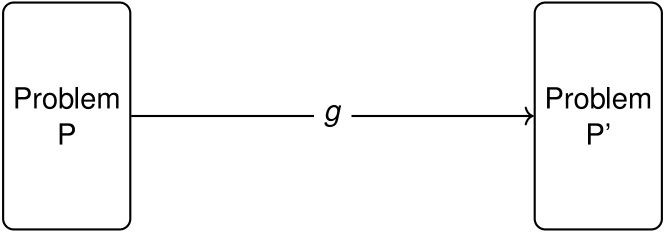
Problem P’

Equi-satisfiability

$$solution \models P \Leftrightarrow g.solution \models P'$$

Symmetry in high level

g a symmetry



Equi-satisfiability

$$solution \models P \Leftrightarrow g.solution \models P'$$

- Semantic symmetries
- **Syntactic symmetries**

Syntactic symmetry

A symmetry (permuation) g is a bijective function (on variables) that leaves the formula φ invariant

Syntactic symmetry

A symmetry (permuation) *g* is a bijective function (on variables) that leaves the formula φ invariant

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_2 & x_1 & x_3 & x_5 & x_4 & x_6 & x_8 & x_7 & x_9 \end{pmatrix} \rightarrow (x_1\ x_2)(x_4\ x_5)(x_7\ x_8)$$

Syntactic symmetry

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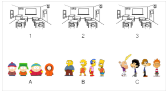
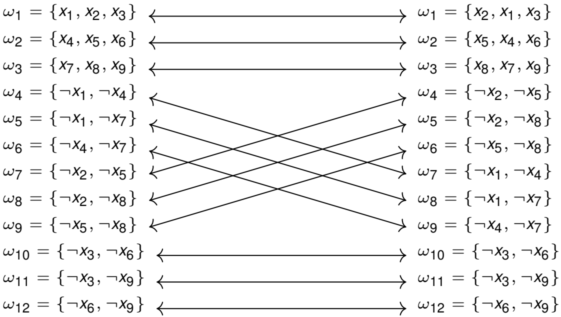
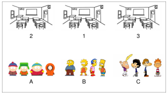




$$\begin{array}{ll}
\omega_1 = \{x_1, x_2, x_3\} & \omega_1 = \{x_2, x_1, x_3\} \\
\omega_2 = \{x_4, x_5, x_6\} & \omega_2 = \{x_5, x_4, x_6\} \\
\omega_3 = \{x_7, x_8, x_9\} & \omega_3 = \{x_8, x_7, x_9\} \\
\omega_4 = \{\neg x_1, \neg x_4\} & \omega_4 = \{\neg x_2, \neg x_5\} \\
\omega_5 = \{\neg x_1, \neg x_7\} & \omega_5 = \{\neg x_2, \neg x_8\} \\
\omega_6 = \{\neg x_4, \neg x_7\} & \omega_6 = \{\neg x_5, \neg x_8\} \\
\omega_7 = \{\neg x_2, \neg x_5\} & \omega_7 = \{\neg x_1, \neg x_4\} \\
\omega_8 = \{\neg x_2, \neg x_8\} & \omega_8 = \{\neg x_1, \neg x_7\} \\
\omega_9 = \{\neg x_5, \neg x_8\} & \omega_9 = \{\neg x_4, \neg x_7\} \\
\omega_{10} = \{\neg x_3, \neg x_6\} & \omega_{10} = \{\neg x_3, \neg x_6\} \\
\omega_{11} = \{\neg x_3, \neg x_9\} & \omega_{11} = \{\neg x_3, \neg x_9\} \\
\omega_{12} = \{\neg x_6, \neg x_9\} & \omega_{12} = \{\neg x_6, \neg x_9\}
\end{array}$$



P

g.P = P’ = P

Computing symmetries of a SAT problem

CNF formula

$$\begin{gathered}(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)\\ \wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)\\ \wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)\\ \wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)\end{gathered}$$

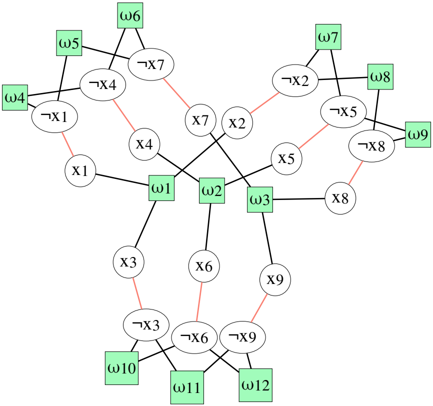
Computing symmetries of a SAT problem

CNF formula

$$\begin{aligned}(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)\\ \wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)\\ \wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)\\ \wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)\end{aligned}$$

$\Downarrow$ $\Downarrow$

colored graph



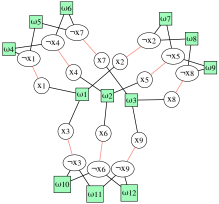
Computing symmetries of a SAT problem

CNF formula

$$
\begin{gathered}
(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9) \\
\wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7) \\
\wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8) \\
\wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)
\end{gathered}
$$

$\Downarrow$ $\Downarrow$

colored graph



$\|$ $\|$

graph automorphism

(`bliss`, `saucy`, ...)

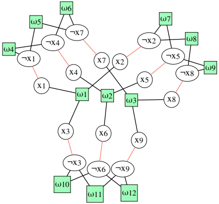
Computing symmetries of a SAT problem

CNF formula

$$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)$$
$$\wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)$$
$$\wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)$$
$$\wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)$$

$\Downarrow$ $\Downarrow$

colored graph



$\|$ $\|$

graph automorphism

(`bliss`, `saucy`, ...)

$\Downarrow$ $\Downarrow$

set of symmetries

$$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(x_8\ \ x_9)$$
$$g_2 = (x_4\ \ x_7)(x_5\ \ x_8)(x_6\ \ x_9)$$
$$g_3 = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$$
$$g_4 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)$$

The set of symmetries of a formula is a group noted $< G, \circ >$

Exploitation of symmetries:

Static symmetry breaking

Orbit

Orbit of an assignment α for a group G:

$$G.\alpha = \{g.\alpha \mid g \in G\}$$

Orbit

Orbit of an assignment α for a group G:

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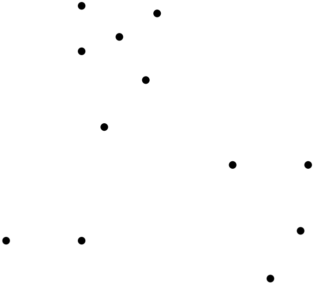
Example:

- full assignment



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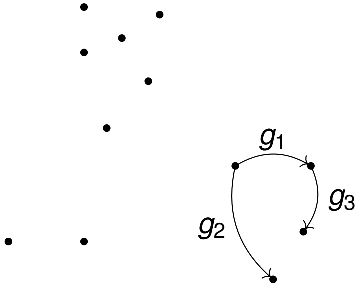

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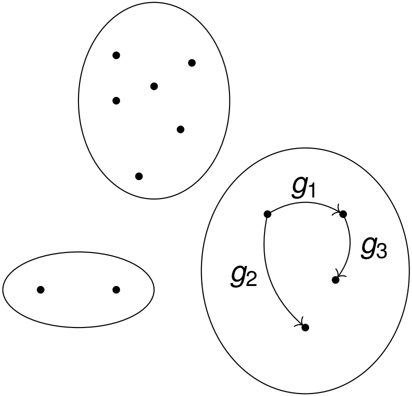
Example:

- • full assignment
- ⬭ orbit



Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

Orbit

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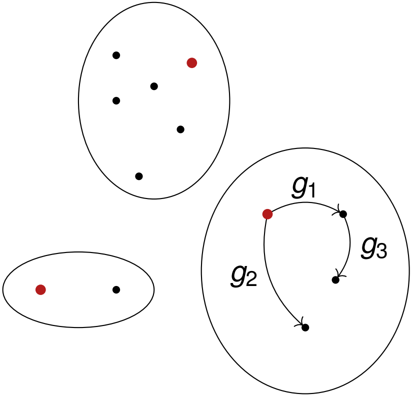
Example:

- • full assignment
- ⬭ orbit
- ● representative



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Comparing assignments: Assessments

Define an ordering relation to compare assignments ($\prec$)

- Total ordering on variables
- Minimum value: $\texttt{F} < \texttt{T}$ or $\texttt{T} < \texttt{F}$

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Allow only minimal value (lex-leader)

Comparing assignments: Assessments

Define an ordering relation to compare assignments ($\prec$)

- Total ordering on variables
- Minimum value: $\mathtt{F} < \mathtt{T}$ or $\mathtt{T} < \mathtt{F}$

Allow only minimal value (lex-leader)

Forbid other assignments in each orbit
→ Add Symmetry breaking predicates (SBP)

Comparing assignments: Example

Ordering relation: $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8$; $\mathtt{F} < \mathtt{T}$

Symmetry: $g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$

Assignments:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	T	F	F	F	F	F	F	F

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Comparing: $g.\alpha \prec \alpha$

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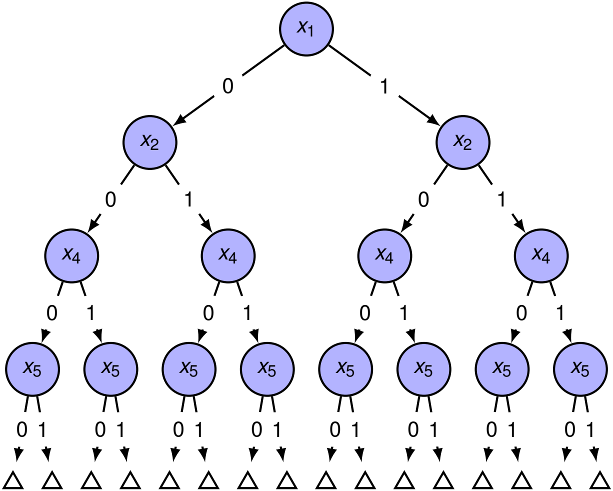
Comparing: $g.\alpha \prec \alpha \Rightarrow$ SBP: $\omega = \{\neg x_1, x_2\}$

Using symmetries to prune the search space

$g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$

$\omega = \{\neg x_1, x_2\}$



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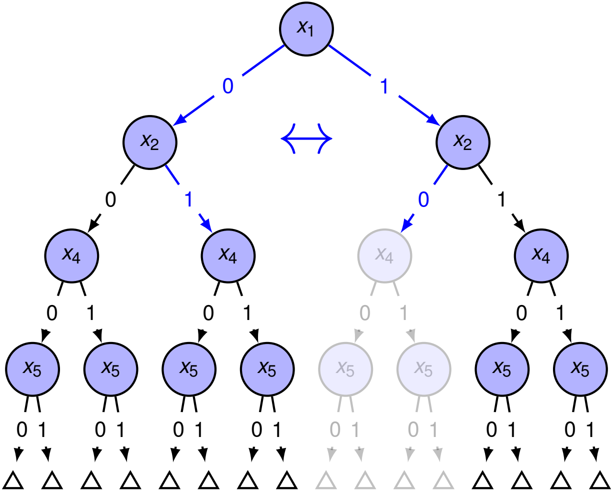
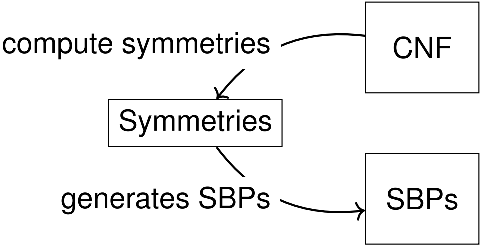


State-of-the-art static symmetry breaking

State-of-the-art:

- Shatter [ASM06]
- BreakID [DBBD16]



compute symmetries

CNF

Symmetries

generates SBPs

SBPs

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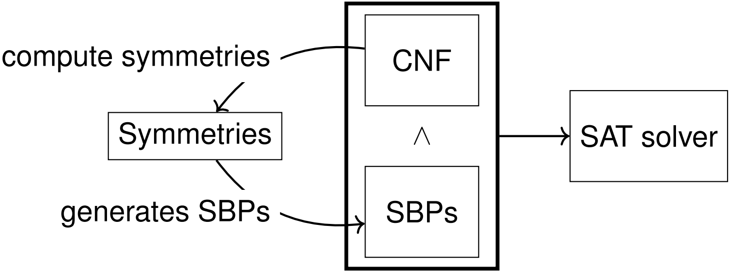

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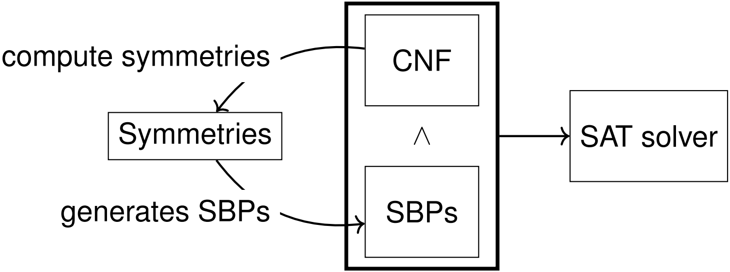

compute symmetries

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CNF

$\wedge$

SBPs

SAT solver

Works well on many symmetrical problems

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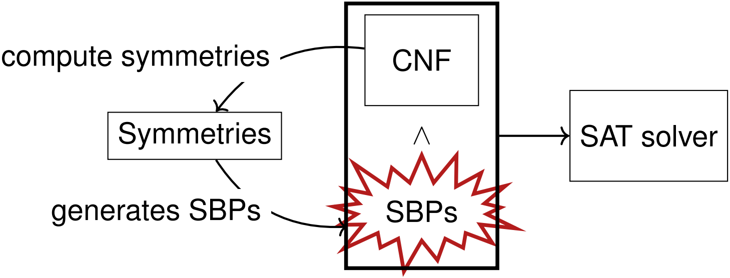


Works well on many symmetrical problems

The solver can ”explode” instead of being helped

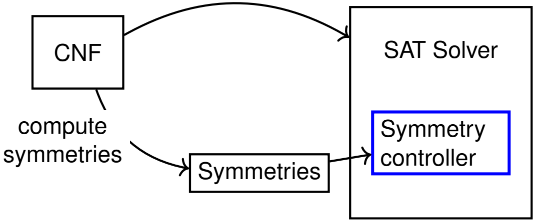
- generate not needed clause
- flooding the solver

First contribution:

CDCL[sym] Introducing Effective Symmetry Breaking in SAT Solving

TACAS'18 [MBCK18]

General idea

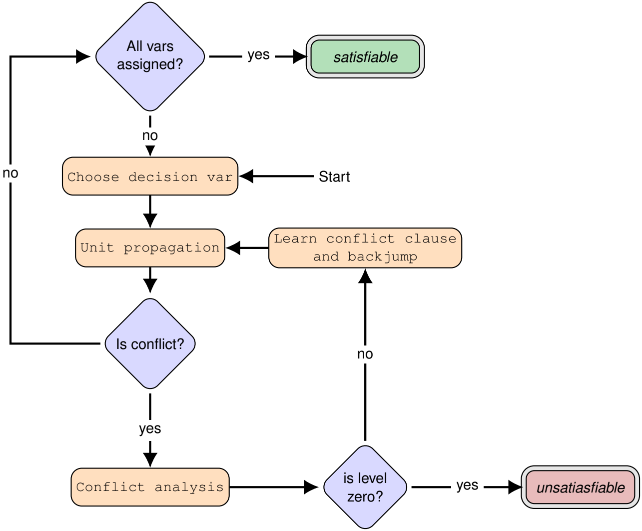


Symmetry controller:

- Generates SBP on-the-fly
- Only when needed
- Intrusive on solver

CDCL[Sym]

Compute and inject SBP opportunistically, during the solving



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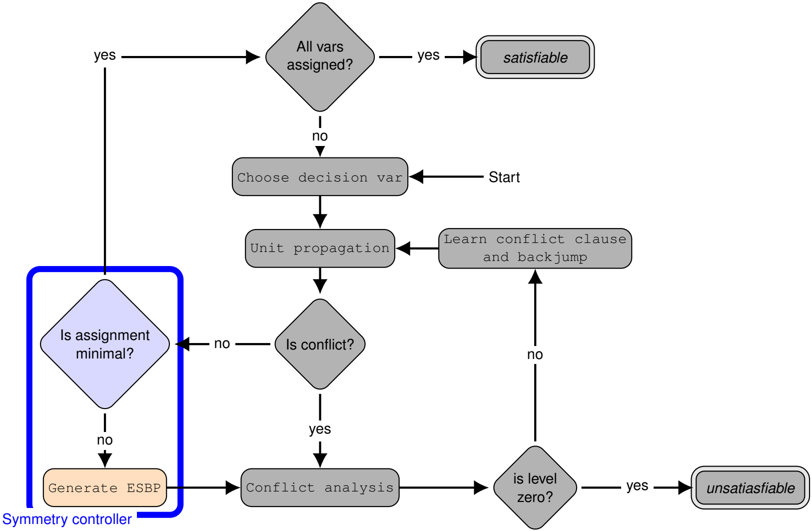

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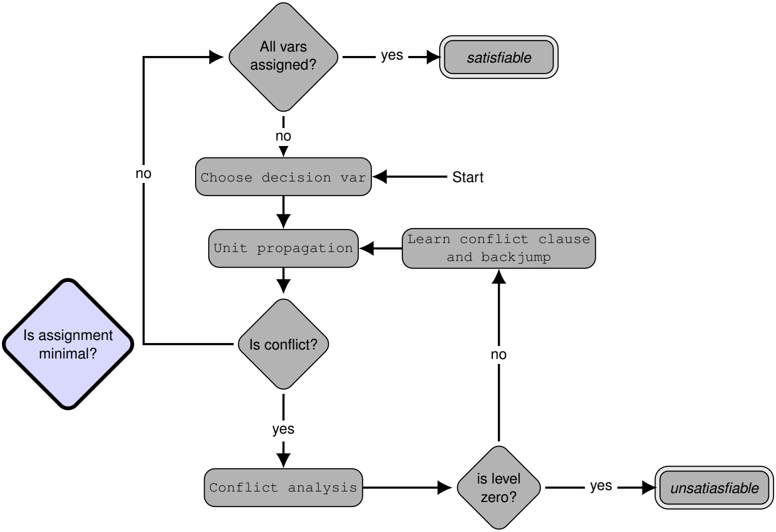

Is assignment minimal?

Our proposal: Symmetry status

- `reducer`: $\boldsymbol{g}.\alpha \prec \alpha$
- `inactive`: $\alpha \prec \boldsymbol{g}.\alpha$
- `active`: *not enough information*

Is assignment minimal?

Our proposal: Symmetry status

- `reducer`: $g.\alpha \prec \alpha$
- `inactive`: $\alpha \prec g.\alpha$
- `active`: *not enough information*

Efficient implementation of symmetry status tracking

Keep track the smallest unassigned variable x:

1. $\alpha(g.x) \leq \alpha(x)$, then g is `reducer` ⇒ Effective SBP (ESBP)
2. $\alpha(x) \leq \alpha(g.x)$, then g is `inactive` ⇒ g cannot reduce α
3. $\alpha(g.x)$ or $\alpha(x)$ is unassigned then g is `active`

Example

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; \mathtt{F} < \mathtt{T}$$

$$g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$$

	$\downarrow$							
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
α	U	U	U	U	U	U	U	U
$g.\alpha$	U	U	U	U	U	U	U	U

$g.\alpha \qquad \alpha$

status of permutation g: `active`

Example

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α	T	F	U	F	U	U	U	U
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$$g.\alpha \prec \alpha$$

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$$g.\alpha \;\prec\; \alpha$$

status of permutation g: `reducer`

On-the-fly generation of ESBP: $\omega = \{\neg x_1, x_2\}$

CDCL[Sym] implementation

- C++ Implementation: 1780 Loc
- Packaged as a library **cosy** (Controller of Symmetry)
  `https://github.com/lip6/cosy`
- Low memory consumption

- Virtually works with any enumerative CDCL SAT solver
- Can be integrated easily

→ e.g. +3% LOC on `MiniSAT`
90 lines out of 3090

Experiments

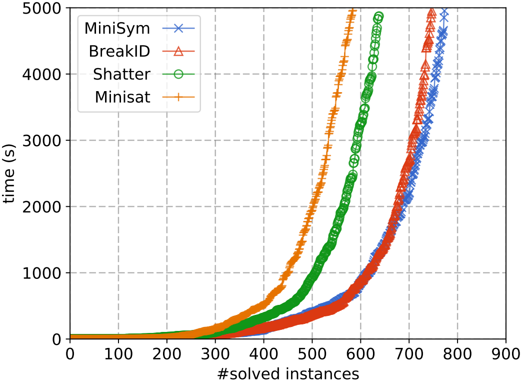
Benchmark:

- from SAT contests 2012 – 2017
- filter: `bliss` finds symmetries in 1000 seconds
- 36 % of instances, 1 350/3 700

Setup:

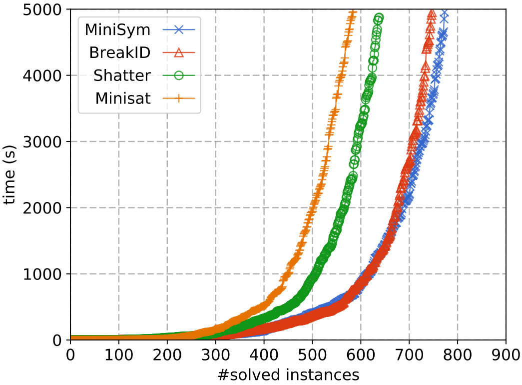
- four tools
    - MiniSat (no symmetry, baseline)
    - MiniSat + BreakID (SOTA SAT solver using symmetries)
    - MiniSat + Shatter (SOTA SAT solver using symmetries)
    - **MiniSym** = MiniSat + CDCL[Sym] (our approach)
- 5000 seconds timeout, 8GB memory
- includes time to compute symmetries (except for MiniSat)

Experimental results



Solver	PAR-2	SAT	UNSAT
MiniSAT	2243h	325	261
Shatter	2088h	316	324
BreakID	1790h	334	415
MiniSym	**1735h**	**336**	**439**

Experimental results



Number of SBPs	`BreakID`	`MiniSym`
UNSAT (399)	2 576 349	913 339
SAT (320)	12 179 513	457 452

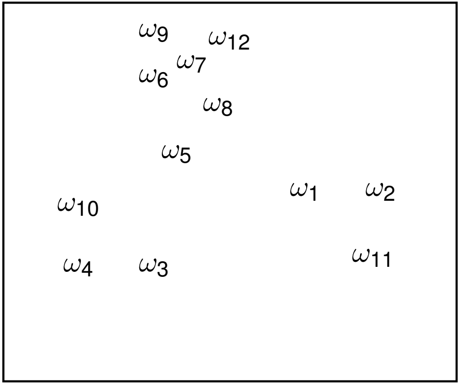
recul

Exploitation of symmetries:

Dynamic symmetry breaking

Learn symmetrical clauses

□	formula
ω	clause



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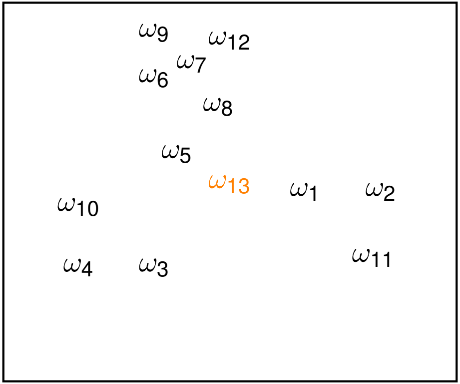

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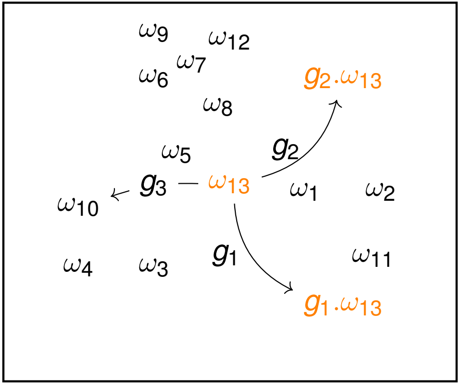
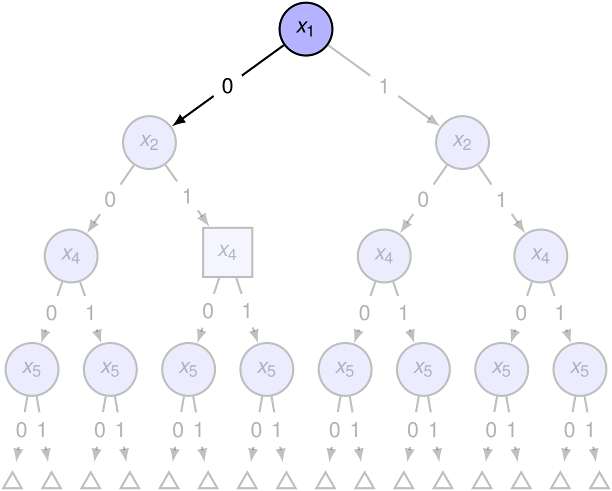


Using symmetries to accelerate the tree traversal

$g = (x_1 \ \ x_2)(x_4 \ \ x_5)(x_7 \ \ x_8)$



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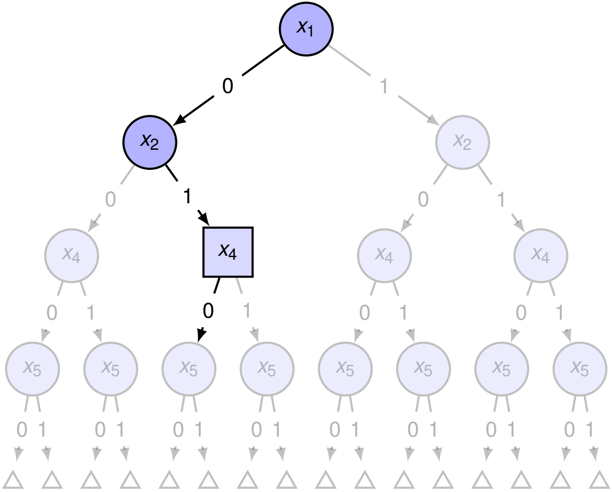

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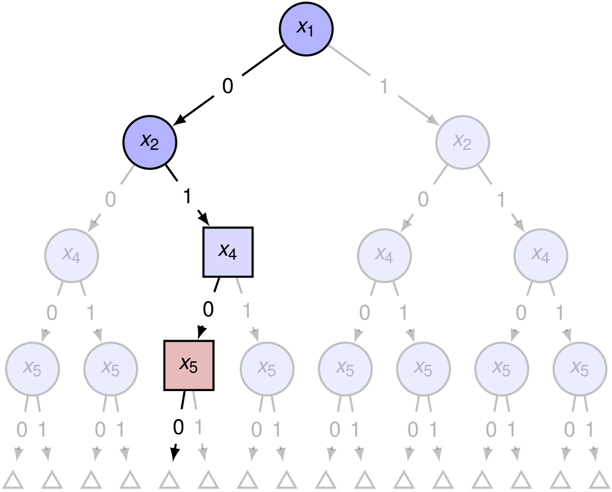
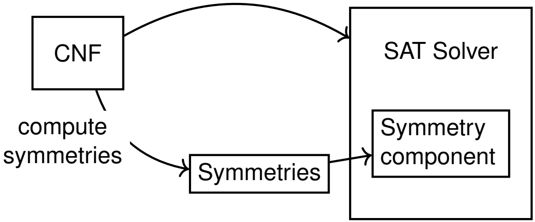


Use symmetries to deduce symmetrical facts.

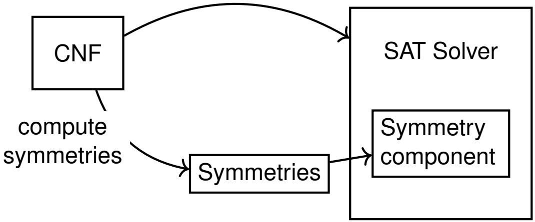
Dynamic Symmetry Breaking



State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC$^+$12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Learning (SEL) [DBB17]

Dynamic Symmetry Breaking



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Observation:

Cannot handle some instances solved by static approach

Second contribution

Composing Symmetry Propagation and Effective Symmetry Breaking for SAT Solving

NFM’19 [MBK19]

ESBP + SP

Compose the symmetry propagation and the ESBP

prune the decision tree while accelerating its traversal

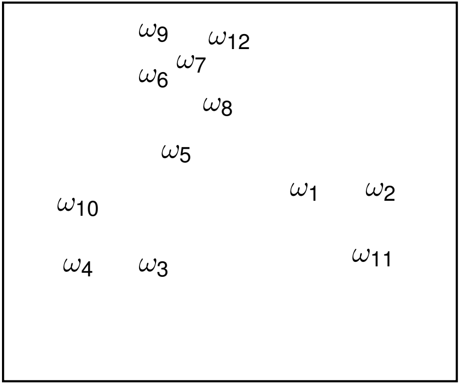
Problems:

- ESBP breaks symmetries (incrementally)
- SP considers the manipulated symmetries valid all time

In a hybrid approach, SP must be able to identify
valid symmetries

Local symmetry

- □ formula
- ω clause
- ω learnt clause



Local symmetries:

$\omega \leftarrow \{g_1, g_2, g_3\}$

Local symmetry

☐	formula
ω	clause
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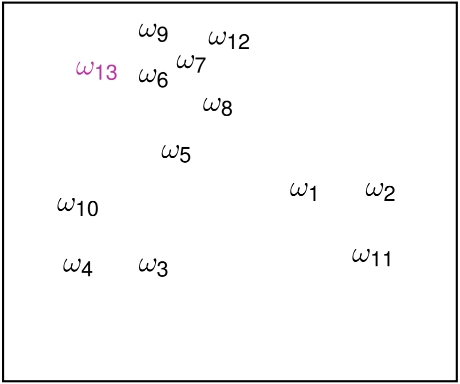
ω	ESBP



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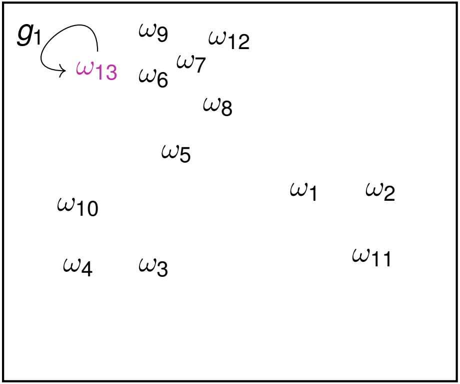

Local symmetries:

$\omega \leftarrow \{g_1, g_2, g_3\}$

$\omega \leftarrow \{g_1\}$

- Compute valid local symmetries
- On the fly
- At minimal cost

Local symmetry

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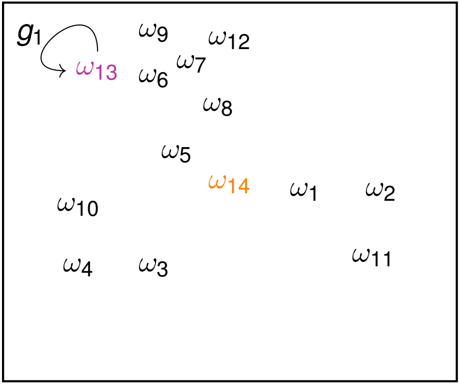

Local symmetries:

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Inductive construction

Local symmetry

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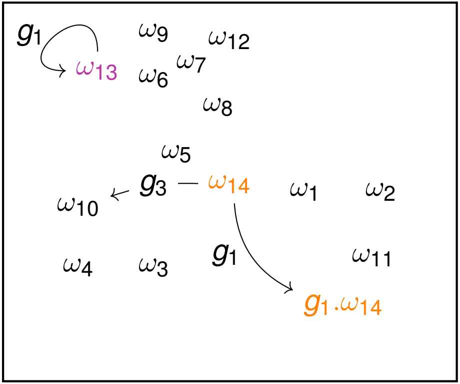
ω	learnt clause
ω	ESBP



Local symmetries:

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- Compute valid local symmetries
- On the fly
- At minimal cost

Inductive construction

Experimental results

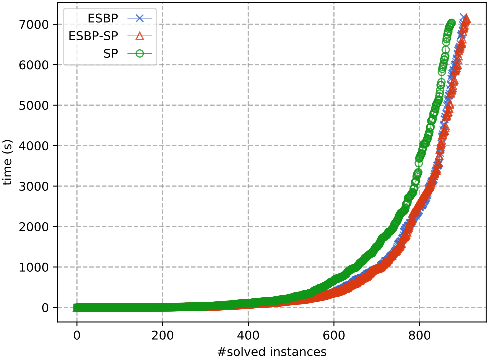
Benchmark:

- from SAT contests 2012 – 2018
- filter: `bliss` finds symmetries in 1000 seconds
- 1400 symmetric instances (out of 4000)

Setup:

- three tools
    - MiniSat SP (Minisat with Symmetry Propagation)
    - MiniSat ESBP (Minisat with CDCL[Sym])
    - **Minisat ESBP-SP** (our approach)
- 7200 seconds timeout

Experimental results



Solver	PAR-2	SAT	UNSAT
SP	1674h00	406	470
ESBP	1578h30	416	488
ESBP-SP	**1570h15**	**420**	**491**

Conclusion & Perspectives

Conclusion

- A new dynamic symmetry breaking approach
  - Generation of SBP on the fly
  - Package as a library cosy usable with any CDCL solver
- A new hybrid approach (ESBP-SP)
  - Take advantage of static and dynamic approach

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Perspectives

- Exploitation of partial symmetries
- Symmetries and parallel SAT solver

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Thanks !

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CDCL in action TODO

$$\omega_1 = \{x_1, x_2, x_3\}$$
$$\omega_2 = \{x_4, x_5, x_6\}$$
$$\omega_3 = \{\neg x_1, \neg x_5\}$$
$$\omega_4 = \{\neg x_2, \neg x_4\}$$
$$\omega_5 = \{\neg x_3, \neg x_4\}$$
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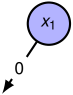

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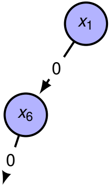

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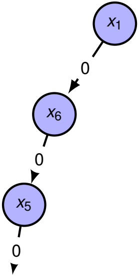

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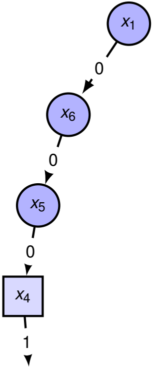

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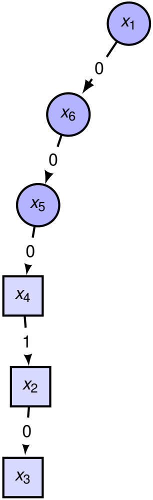

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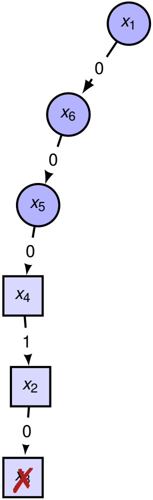

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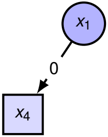

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$$\omega_7 = \{x_1, \neg x_4\}$$

Weakly active symmetries

Logical consequence

When ω is satisfied in all satisfying assignments of φ, we say that ω is a logical consequence of φ, and we denote this by $\varphi \vdash \omega$.

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Symmetry propagation

Let σ a weakly active symmetry, then

$$\varphi \cup \alpha \vdash \{l\} \Leftrightarrow \varphi \cup \alpha \vdash \sigma.\{l\}$$

Local symmetries

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Local Symmetries

Let φ be a formula. We define $L_{\omega,\varphi}$, the set of *local symmetries* for a clause ω, and with respect to a formula φ, as follows:

$$L_{\omega,\varphi} = \{\sigma \in \mathfrak{S} \mid \varphi \vdash \sigma.\omega\}$$

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We can state that:

$$\bigcap_{\omega \in \varphi} L_{\omega,\varphi} \subseteq G.$$

Computing local symmetries

Formula can be decomposed as : $\varphi = \varphi_o \cup \varphi_e \cup \varphi_d$ where

- φ_o is the set of the original clauses
- φ_e is the set of ESBPs
- φ_d is the set of deduced clauses.

Local symmetries

- $\omega \in \varphi_o, L_{\omega,\varphi} \supseteq G$
- $\omega \in \varphi_e, L_{\omega,\varphi} \supseteq Stab(\omega) = \{\sigma \in G \mid \omega = \sigma.\omega\}$
- $\omega \in \varphi_d, L_{\omega,\varphi} \supseteq (\bigcap_{\omega' \in \varphi_1} L_{\omega',\varphi}) \cup Stab(\omega)$

  where φ_1 is the set of clauses that derives ω.

Generates symmetry breaking predicates (SBP)

- Define lexicographic order
  - Define total order on variables
  - Define minimal value
- Forbid non minimal assignment for each orbit

Example:

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; \texttt{F} < \texttt{T}$$

$$g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$$

	x_1	x_2	x_3	x_4	x_5	$\cdots$	lex-leader	SBP

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	T	F	–	–	–	$\cdots$	✗	$\rightarrow \neg x_1 \vee x_2$
O_2	F	F	–	F	T	$\cdots$	✓	

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O_1	F	T	–	–	–	$\cdots$	✓	
	T	F	–	–	–	$\cdots$	✗	$\rightarrow \neg x_1 \vee x_2$
O_2	F	F	–	F	T	$\cdots$	✓	
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O_1	F	T	–	–	–	⋯	✓	
	T	F	–	–	–	⋯	✗	$\rightarrow \neg x_1 \vee x_2$
O_2	F	F	–	F	T	⋯	✓	
	F	F	–	T	F	⋯	✗	$\rightarrow x_1 \vee x_2 \vee \neg x_4 \vee x_5$
					⋯			

Example

$$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(x_8\ \ x_9) \qquad g_2 = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$$

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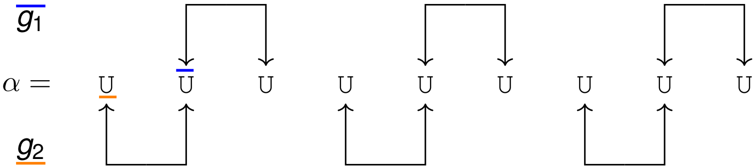

$\overline{g_1}$

$\alpha = \underline{\mathrm{U}}\ \ \overline{\mathrm{U}}\ \ \mathrm{U}\ \ \mathrm{U}\ \ \mathrm{U}\ \ \mathrm{U}\ \ \mathrm{U}\ \ \mathrm{U}\ \ \mathrm{U}$

$\underline{g_2}$

Example

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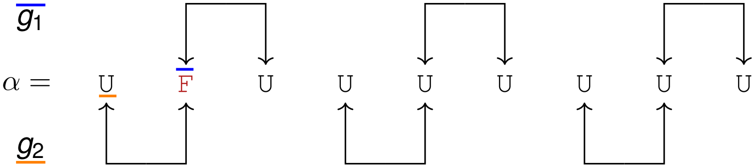

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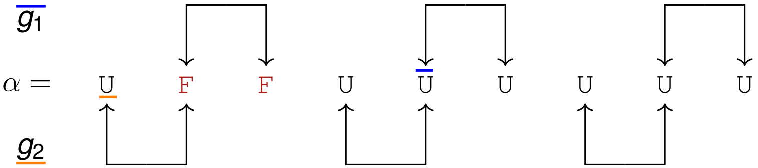

$\overline{g_1}$

$$\alpha = \underline{\mathtt{U}} \quad \mathtt{F} \quad \mathtt{F} \quad \mathtt{U} \quad \overline{\mathtt{U}} \quad \mathtt{U} \quad \mathtt{U} \quad \mathtt{U} \quad \mathtt{U}$$

$\underline{g_2}$

Example

$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(x_8\ \ x_9)$ $g_2 = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$

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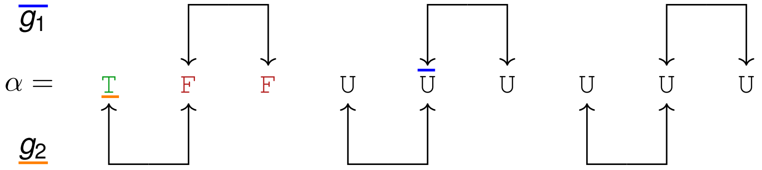

$\overline{g_1}$

$\alpha = \underline{\mathtt{T}}\ \ \mathtt{F}\ \ \mathtt{F}\ \ \mathtt{U}\ \ \overline{\mathtt{U}}\ \ \mathtt{U}\ \ \mathtt{U}\ \ \mathtt{U}\ \ \mathtt{U}$

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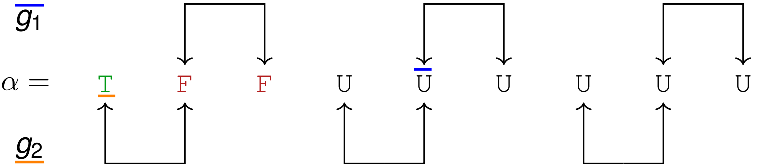

g_2 generates ESBP $\omega = \{\neg x_1, x_2\}$

Example

1. `reducer`: $\alpha(g.x) \leq \alpha(x)$
2. `inactive`: $\alpha(x) \leq \alpha(g.x)$
3. `active`: $\alpha(g.x)$ or $\alpha(x)$ is unassigned

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 \;\; ; \;\; \texttt{F} < \texttt{T}$$

$$\begin{array}{rcccccc|cc} g_1 = & (x_2 & x_3) & (x_5 & x_6) & (x_8 & x_9) & x = x_2 & g.x = x_3 \\ & \uparrow & & & & & & & \texttt{active} \end{array}$$

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$$\ldots$$

$$\alpha = \{ \qquad\qquad\qquad \}$$

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$$\alpha = \{\neg x_2 \qquad \qquad \}$$

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$$\dots$$

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

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$$\ldots$$

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g_2 generates $\omega = \{\neg x_1, x_2\}$

Encoding the problem

$(A,1)(A,2)(A,3)$	$x_1 \vee x_2 \vee x_3$
$(B,1)(B,2)(B,3)$	$x_4 \vee x_5 \vee x_6$
$(C,1)(C,2)(C,3)$	$x_7 \vee x_8 \vee x_9$
$\neg(A,1)\neg(B,1)$	$\neg x_1 \vee \neg x_4$
$\neg(A,1)\neg(C,1)$	$\neg x_1 \vee \neg x_7$
$\neg(B,1)\neg(C,1)$	$\neg x_4 \vee \neg x_7$
$\neg(A,2)\neg(B,2)$	$\neg x_2 \vee \neg x_5$
$\neg(A,2)\neg(C,2)$	$\neg x_2 \vee \neg x_8$
$\neg(B,2)\neg(C,2)$	$\neg x_5 \vee \neg x_8$
$\neg(A,3)\neg(B,3)$	$\neg x_3 \vee \neg x_6$
$\neg(A,3)\neg(C,3)$	$\neg x_3 \vee \neg x_9$
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