Exploitation of dynamic symmetries for solving SAT problems

Doctorat de Sorbonne Université

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FABRICE KORDON	Professeur, Sorbonne Université

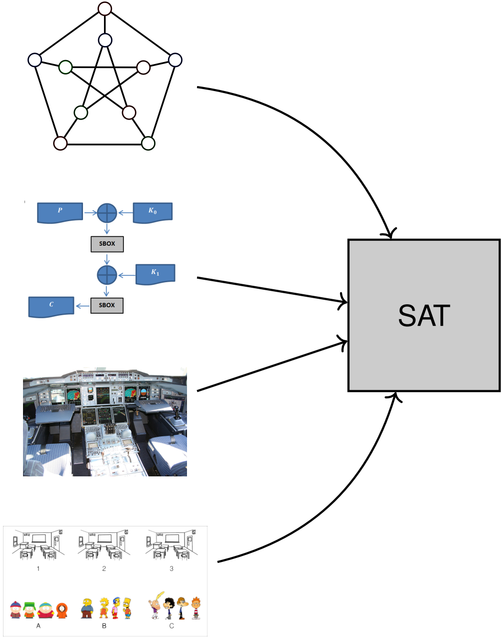
Motivation

Graph coloring

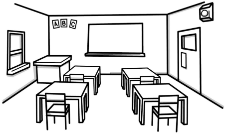
Cryptanalysis

Hardware model checking

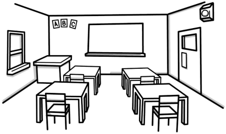
Planning



SAT: an example (1/2)



1



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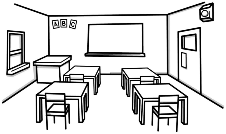
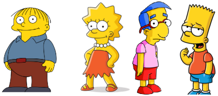


3

A



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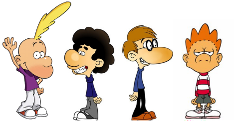
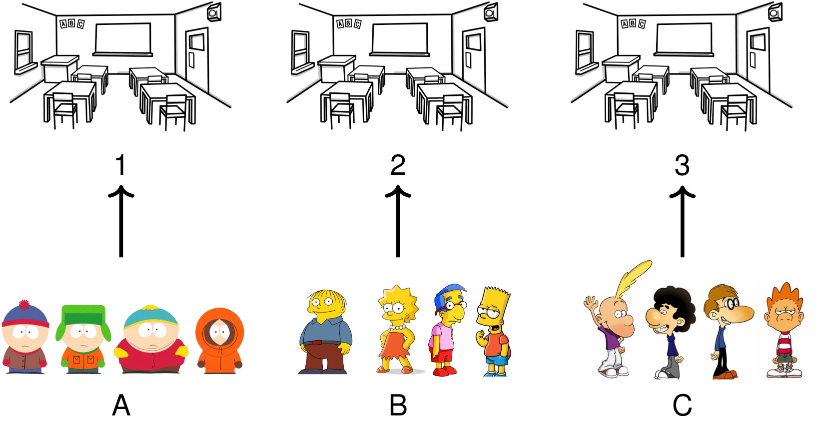


C

Is it possible to attribute each group to a unique classroom?

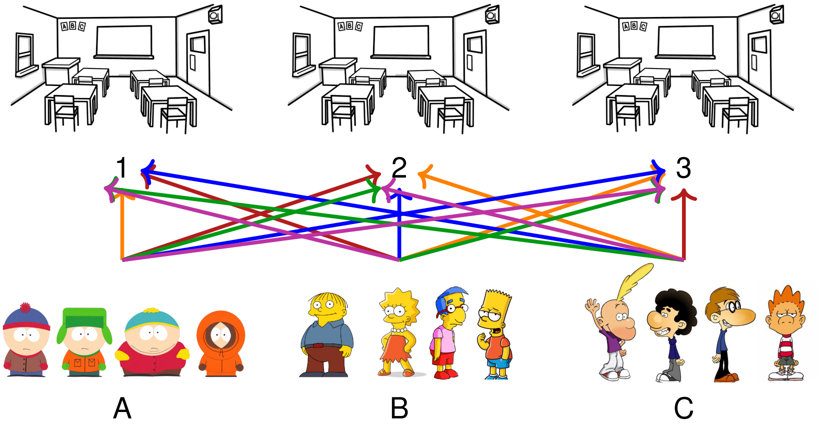
SAT: an example (1/2)



Is it possible to attribute each group to a unique classroom?

YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$

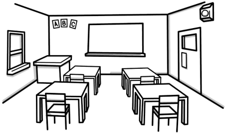
SAT: an example (1/2)



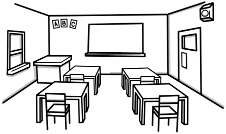
Is it possible to attribute each group to a unique classroom?

YES! SAT! $\alpha = (A, 1), (B, 2), (C, 3)$
Many solutions $\alpha = (A, 2), (B, 3), (C, 1); \cdots$

SAT: an example (2/2)

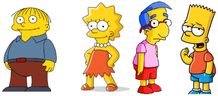


1



2

A



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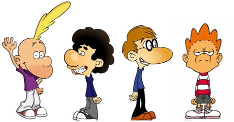
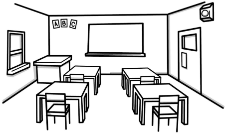


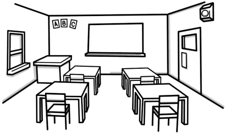
C

Is it possible to attribute each group to a unique classroom?

SAT: an example (2/2)

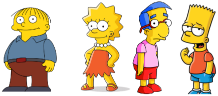


1



2

A



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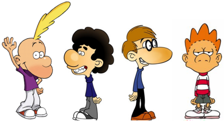


C

Is it possible to attribute each group to a unique classroom?

No! UNSAT

Encoding the problem

$$
\begin{array}{ll}
(A,1)(A,2)(A,3) & (x_1 \vee x_2 \vee x_3) \;\wedge \\
(B,1)(B,2)(B,3) & (x_4 \vee x_5 \vee x_6) \;\wedge \\
(C,1)(C,2)(C,3) & (x_7 \vee x_8 \vee x_9) \;\wedge \\
\\
\neg(A,1)\neg(B,1) & (\neg x_1 \vee \neg x_4) \;\wedge \\
\neg(A,1)\neg(C,1) & (\neg x_1 \vee \neg x_7) \;\wedge \\
\neg(B,1)\neg(C,1) & (\neg x_4 \vee \neg x_7) \;\wedge \\
\\
\neg(A,2)\neg(B,2) & (\neg x_2 \vee \neg x_5) \;\wedge \\
\neg(A,2)\neg(C,2) & (\neg x_2 \vee \neg x_8) \;\wedge \\
\neg(B,2)\neg(C,2) & (\neg x_5 \vee \neg x_8) \;\wedge \\
\\
\neg(A,3)\neg(B,3) & (\neg x_3 \vee \neg x_6) \;\wedge \\
\neg(A,3)\neg(C,3) & (\neg x_3 \vee \neg x_9) \;\wedge \\
\neg(B,3)\neg(C,3) & (\neg x_6 \vee \neg x_9) \;\wedge
\end{array}
$$

Encoding the problem

Clause

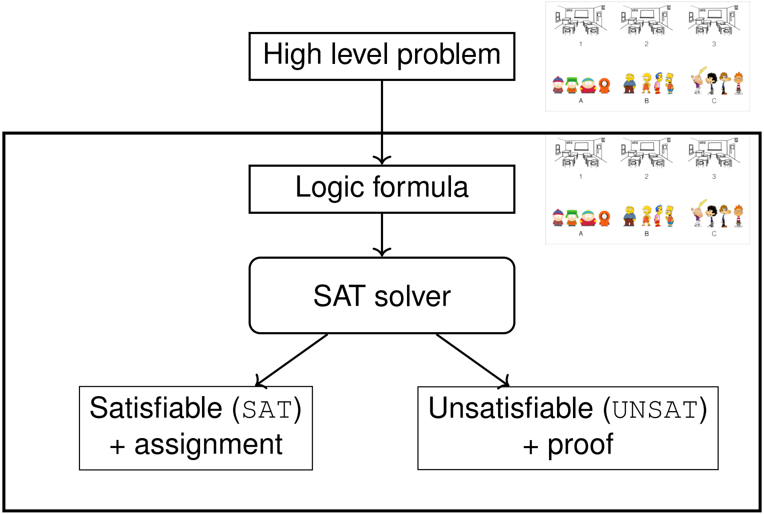
$$
\begin{array}{ll}
(A,1)(A,2)(A,3) & \boxed{(x_1 \vee x_2 \vee x_3)} \wedge \\
(B,1)(B,2)(B,3) & (x_4 \vee x_5 \vee x_6) \wedge \\
(C,1)(C,2)(C,3) & (x_7 \vee x_8 \vee x_9) \wedge \\
\\
\neg(A,1)\neg(B,1) & (\neg x_1 \vee \neg x_4) \wedge \\
\neg(A,1)\neg(C,1) & (\neg x_1 \vee \neg x_7) \wedge \\
\neg(B,1)\neg(C,1) & (\neg x_4 \vee \neg x_7) \wedge \\
\\
\neg(A,2)\neg(B,2) & (\neg x_2 \vee \neg x_5) \wedge \\
\neg(A,2)\neg(C,2) & (\neg x_2 \vee \neg x_8) \wedge \\
\neg(B,2)\neg(C,2) & (\neg x_5 \vee \neg x_8) \wedge \\
\\
\neg(A,3)\neg(B,3) & (\neg x_3 \vee \neg x_6) \wedge \\
\neg(A,3)\neg(C,3) & (\neg x_3 \vee \neg x_9) \wedge \\
\neg(B,3)\neg(C,3) & (\neg x_6 \vee \neg x_9) \wedge
\end{array}
$$

Encoding the problem

		Clause
$(A,1)(A,2)(A,3)$	$\boxed{(x_1 \vee x_2 \vee x_3)} \wedge$	
$(B,1)(B,2)(B,3)$	$(x_4 \vee x_5 \vee x_6) \wedge$	
$(C,1)(C,2)(C,3)$	$(x_7 \vee x_8 \vee x_9) \wedge$	
$\neg(A,1)\neg(B,1)$	$(\neg x_1 \vee \neg x_4) \wedge$	
$\neg(A,1)\neg(C,1)$	$(\neg x_1 \vee \neg x_7) \wedge$	
$\neg(B,1)\neg(C,1)$	$(\neg x_4 \vee \neg x_7) \wedge$	
		CNF
$\neg(A,2)\neg(B,2)$	$(\neg x_2 \vee \neg x_5) \wedge$	
$\neg(A,2)\neg(C,2)$	$(\neg x_2 \vee \neg x_8) \wedge$	
$\neg(B,2)\neg(C,2)$	$(\neg x_5 \vee \neg x_8) \wedge$	
$\neg(A,3)\neg(B,3)$	$(\neg x_3 \vee \neg x_6) \wedge$	
$\neg(A,3)\neg(C,3)$	$(\neg x_3 \vee \neg x_9) \wedge$	
$\neg(B,3)\neg(C,3)$	$(\neg x_6 \vee \neg x_9) \wedge$	

Conjunctive Normal Form (CNF)

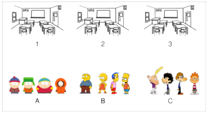
SAT



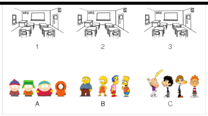
High level problem
1
2
3
A
B
C
Logic formula
1
2
3
A
B
C
SAT solver
Satisfiable (SAT)
+ assignment
Unsatisfiable (UNSAT)
+ proof

SAT

High level problem



Logic formula



SAT solver

Satisfiable (SAT)
+ assignment

Unsatisfiable (UNSAT)
+ proof

SAT Solving

Solving SAT formula is known to be **NP-complete** [Coo71]

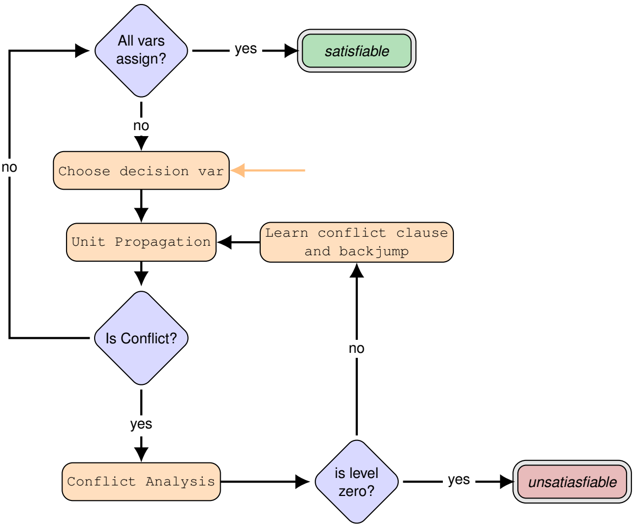
Good performance in practice:

- Handle large problem (million variables and clauses)
- International SAT competition each year on academic and industrial problems

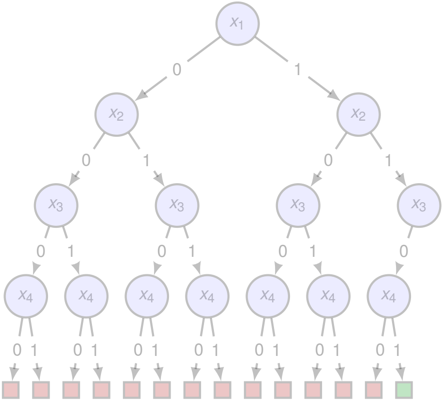
Enumerative algorithms:

- Davis, Putnam, Logemann, and Loveland (DPLL) [DLL62]
  - Boolean Constraint Propagation (BCP)
- Conflict Driven Clause Learning (CDCL) [MSS99]
  - Derived from DPLL
  - Clause learning

CDCL in detail

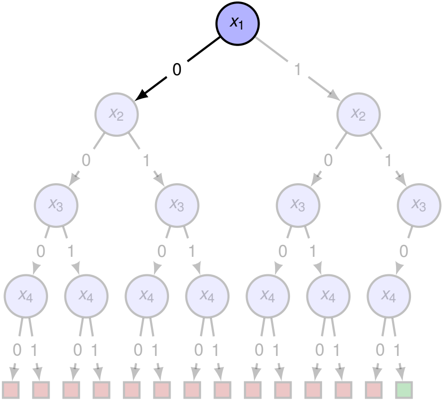


CDCL in action



$$\omega_1 = \{x_1, x_2, x_3, x_4\}$$
$$\omega_2 = \{x_1, \neg x_4\}$$
$$\omega_3 = \{x_1, x_4\}$$
$$\omega_4 = \{x_2, \neg x_4\}$$
$$\omega_5 = \{x_2, x_4\}$$
$$\omega_6 = \{x_3, x_4\}$$

CDCL in action



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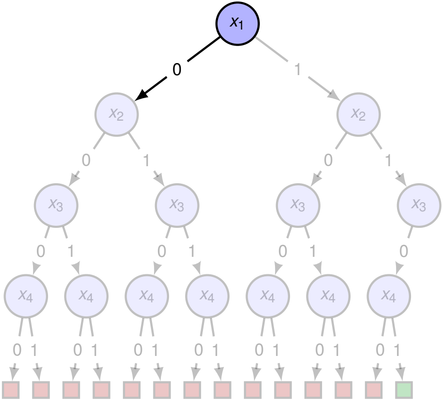

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\end{aligned}
$$

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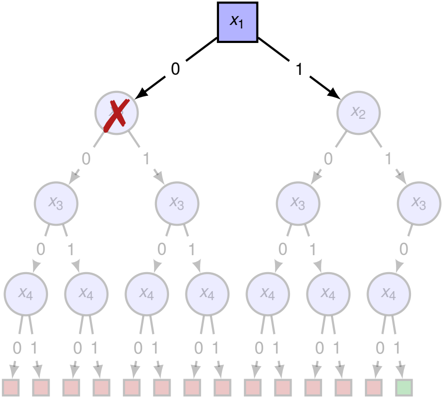

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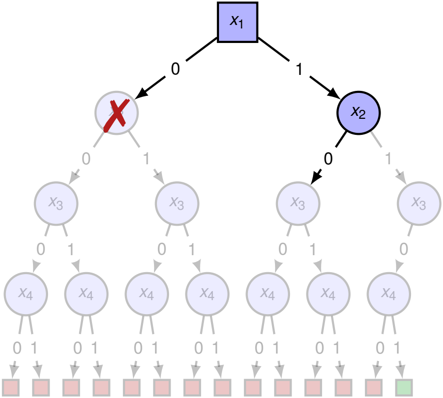

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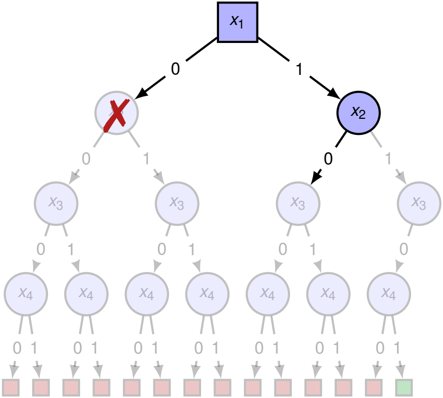

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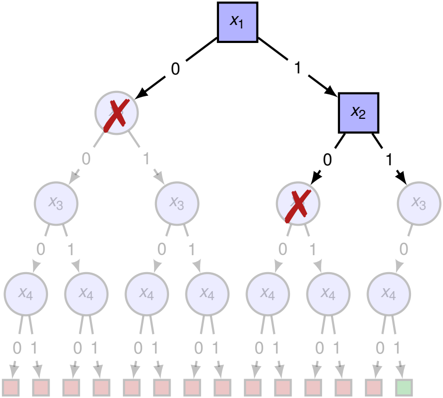

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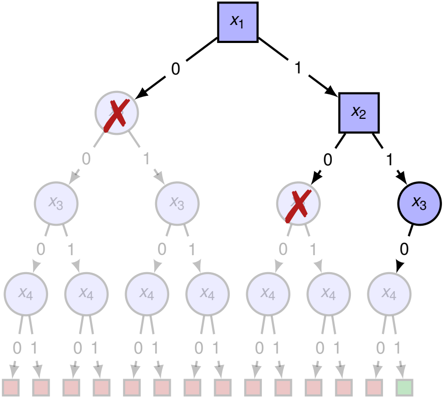
CDCL in action



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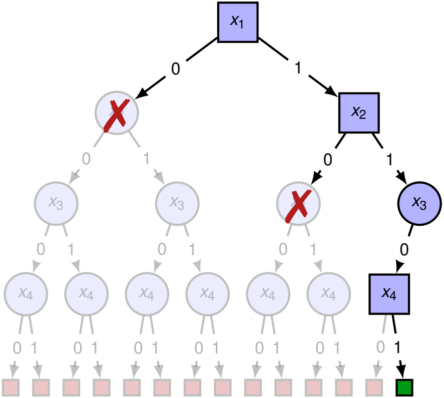

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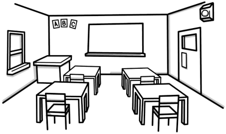
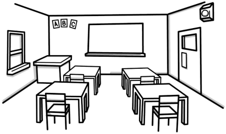
SAT and symmetries

Presence of symmetries

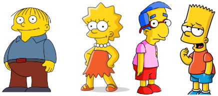


1



2

A



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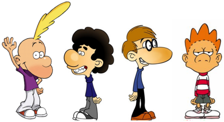
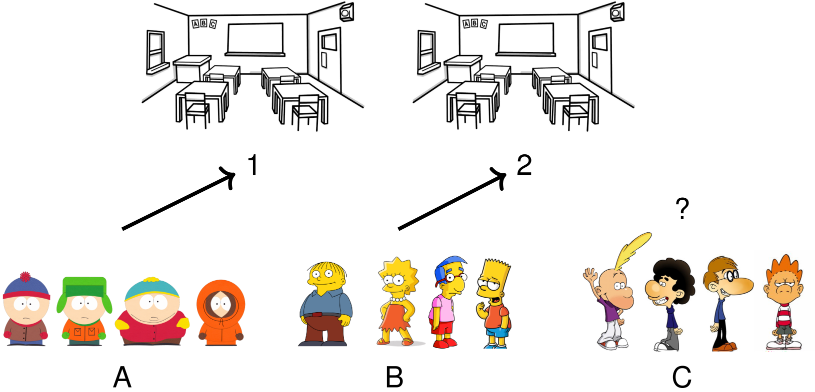
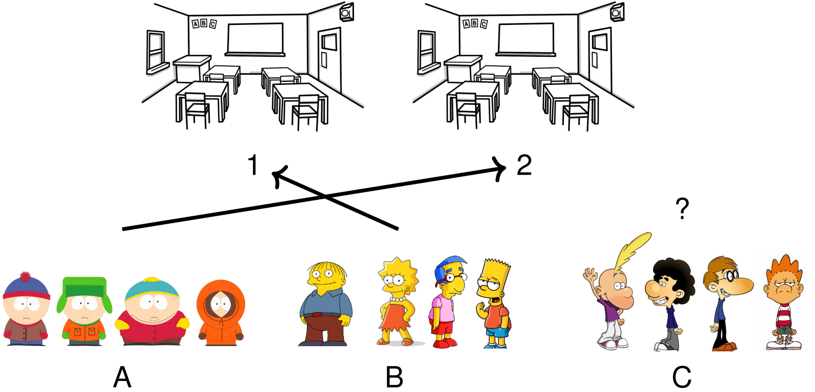


C

Presence of symmetries



Presence of symmetries



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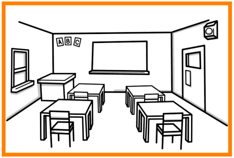

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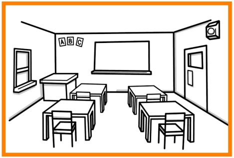

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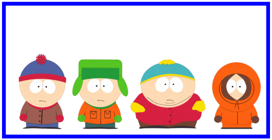

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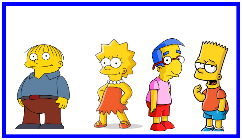
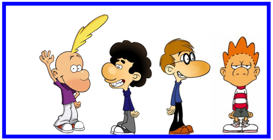
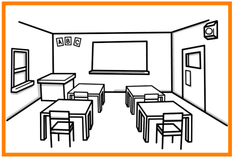


B



C

Presence of symmetries



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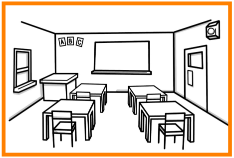

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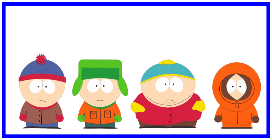

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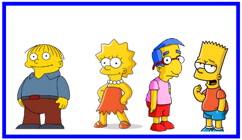

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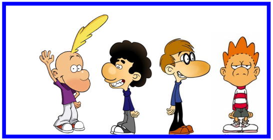


B

Symmetry (Syntactic)

A symmetry (permuation) g is a bijective function (on variables) that leaves the formula invariant

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_2 & x_1 & x_3 & x_5 & x_4 & x_6 & x_8 & x_7 & x_9 \end{pmatrix} \rightarrow (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$$

$$\begin{array}{lcl}
\omega_1 = \{x_1, x_2, x_3\} & \longleftrightarrow & \omega_1 = \{x_2, x_1, x_3\} \\
\omega_2 = \{x_4, x_5, x_6\} & \longleftrightarrow & \omega_2 = \{x_5, x_4, x_6\} \\
\omega_3 = \{x_7, x_8, x_9\} & \longleftrightarrow & \omega_3 = \{x_8, x_7, x_9\} \\
\omega_4 = \{\neg x_1, \neg x_4\} & & \omega_7 = \{\neg x_2, \neg x_5\} \\
\omega_5 = \{\neg x_1, \neg x_7\} & & \omega_8 = \{\neg x_2, \neg x_8\} \\
\omega_6 = \{\neg x_4, \neg x_7\} & & \omega_9 = \{\neg x_5, \neg x_8\} \\
\omega_7 = \{\neg x_2, \neg x_5\} & & \omega_4 = \{\neg x_1, \neg x_4\} \\
\omega_8 = \{\neg x_2, \neg x_8\} & & \omega_5 = \{\neg x_1, \neg x_7\} \\
\omega_9 = \{\neg x_5, \neg x_8\} & & \omega_6 = \{\neg x_4, \neg x_7\} \\
\omega_{10} = \{\neg x_3, \neg x_6\} & \longleftrightarrow & \omega_{10} = \{\neg x_3, \neg x_6\} \\
\omega_{11} = \{\neg x_3, \neg x_9\} & \longleftrightarrow & \omega_{11} = \{\neg x_3, \neg x_9\} \\
\omega_{12} = \{\neg x_6, \neg x_9\} & \longleftrightarrow & \omega_{12} = \{\neg x_6, \neg x_9\}
\end{array}$$

Equi-satisfiability:

$$\alpha \models \varphi \Leftrightarrow g.\alpha \models \varphi$$

Computing symmetries of a SAT problem

CNF formula

$$
\begin{aligned}
&(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)\\
&\wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)\\
&\wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)\\
&\wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)
\end{aligned}
$$

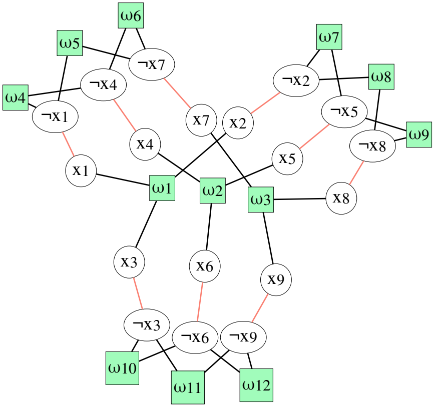
Computing symmetries of a SAT problem

CNF formula

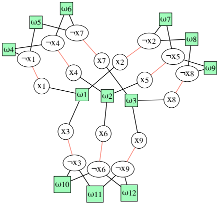
$$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)$$
$$\wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)$$
$$\wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)$$
$$\wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)$$

$\Downarrow$ $\Downarrow$

colored graph



Computing symmetries of a SAT problem

CNF formula	$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)$ $\wedge(\neg x_1 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_7) \wedge (\neg x_4 \vee \neg x_7)$ $\wedge(\neg x_2 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_8) \wedge (\neg x_5 \vee \neg x_8)$ $\wedge(\neg x_3 \vee \neg x_6) \wedge (\neg x_3 \vee \neg x_9) \wedge (\neg x_6 \vee \neg x_9)$
$\Downarrow$	$\Downarrow$
colored graph	
$\Vert$	$\Vert$
graph automorphism	(`bliss`, `saucy`, ...)

Computing symmetries of a SAT problem

CNF formula

$$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee x_8 \vee x_9)$$
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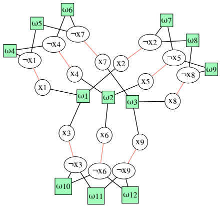
$\Downarrow$ $\Downarrow$

colored graph



$\|$ $\|$

graph automorphism — (`bliss`, `saucy`, ...)

$\Downarrow$ $\Downarrow$

set of symmetries

$$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(x_8\ \ x_9)$$
$$g_2 = (x_4\ \ x_7)(x_5\ \ x_8)(x_6\ \ x_9)$$
$$g_3 = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$$
$$g_4 = (x_1\ \ x_4)(x_2\ \ x_5)(x_3\ \ x_6)$$

The set of symmetries of a formula is a group noted G

Exploitation of symmetries

Static symmetry breaking

Orbit

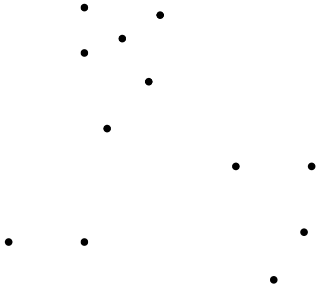
Orbit of an assignment $\alpha = G.\alpha = \{g.\alpha \mid g \in G\}$

Orbit

Orbit of an assignment $\alpha = G.\alpha = \{g.\alpha \mid g \in G\}$

Example:

- full assignment

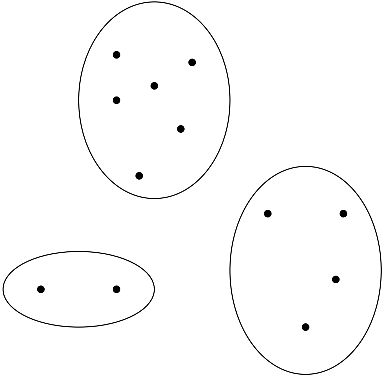


Orbit

Orbit of an assignment $\alpha = G.\alpha = \{g.\alpha \mid g \in G\}$

Example:

- full assignment
- orbit



Orbit

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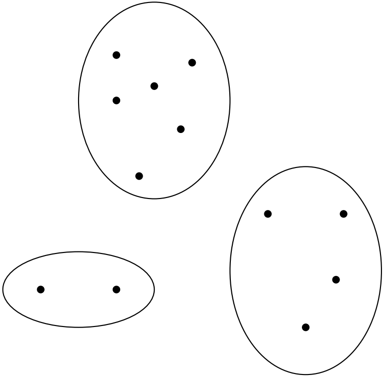
Example:

- full assignment
- orbit



Equivalence relation with respect to SAT:

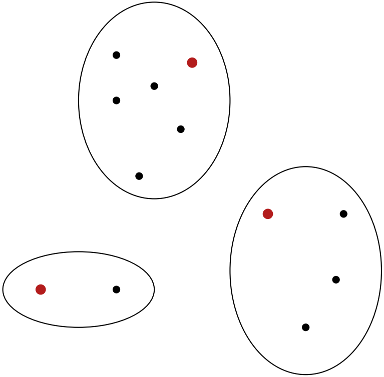
- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

Orbit

Orbit of an assignment $\alpha = G.\alpha = \{g.\alpha \mid g \in G\}$

Example:

- full assignment
- orbit
- representative



Equivalence relation with respect to SAT:

- Either $G.\alpha$ contains no solution
- Or all elements of $G.\alpha$ are solutions

Compare assignment

Define an ordering relation to compare assignment

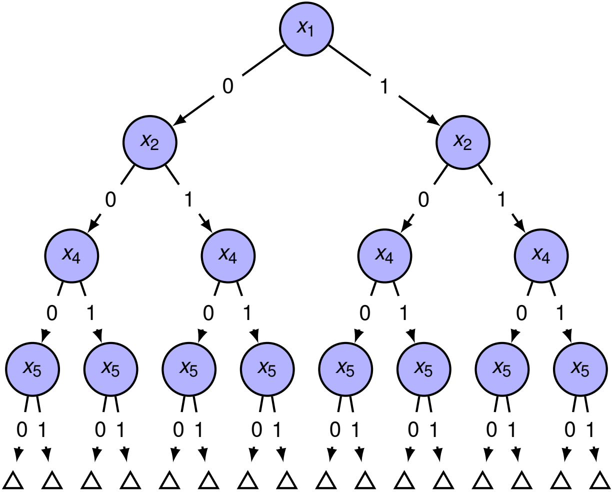
- Total ordering on variables
- Minimum value: $\mathtt{F} < \mathtt{T}$ or $\mathtt{T} < \mathtt{F}$

Allow only minimal (maximal) value

Forbids other assignment in each orbit
Adds Symmetry breaking predicates (SBP)

Using symmetries to prune the search space

$g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$



Using symmetries to prune the search space

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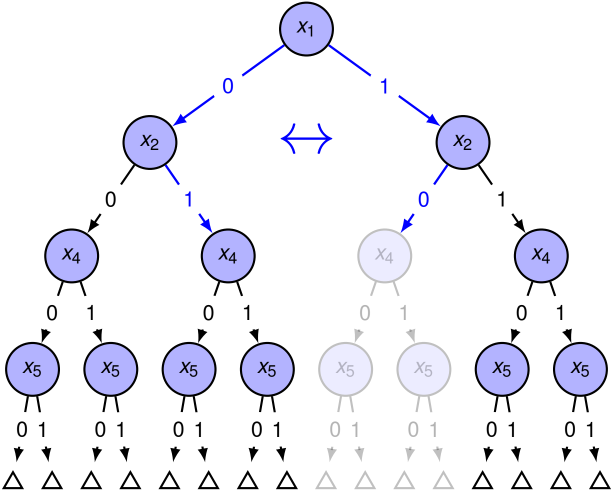
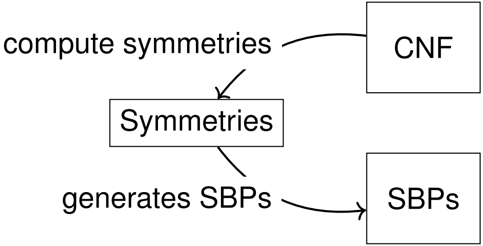
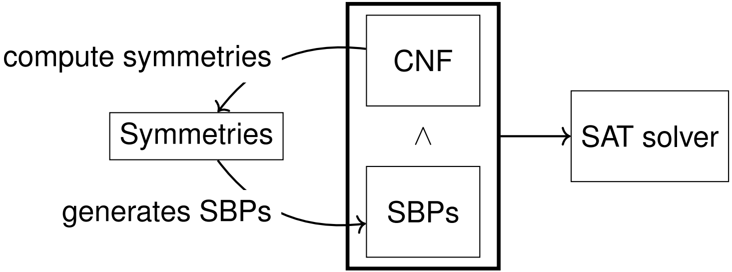


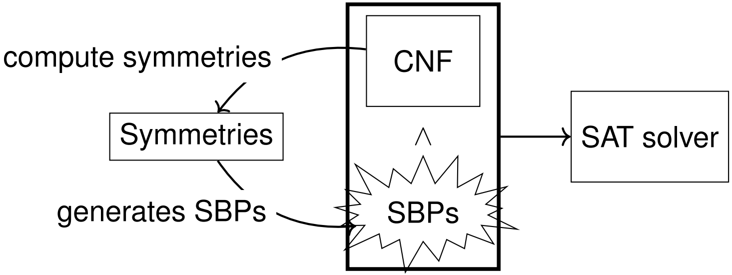
Static symmetry breaking



Static symmetry breaking



Static symmetry breaking



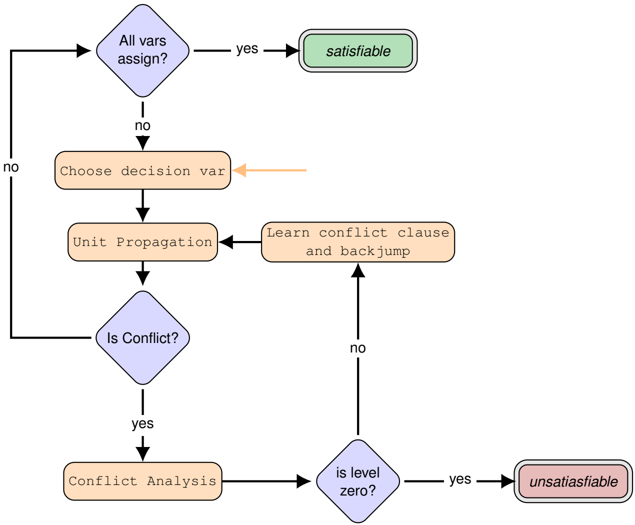
The solver can ”explode” instead of being helped

Our contribution CDCL[sym]

TACAS'18 [MBCK18]

Our contribution CDCL[Sym]

Compute and inject SBP opportunistically, during the solving



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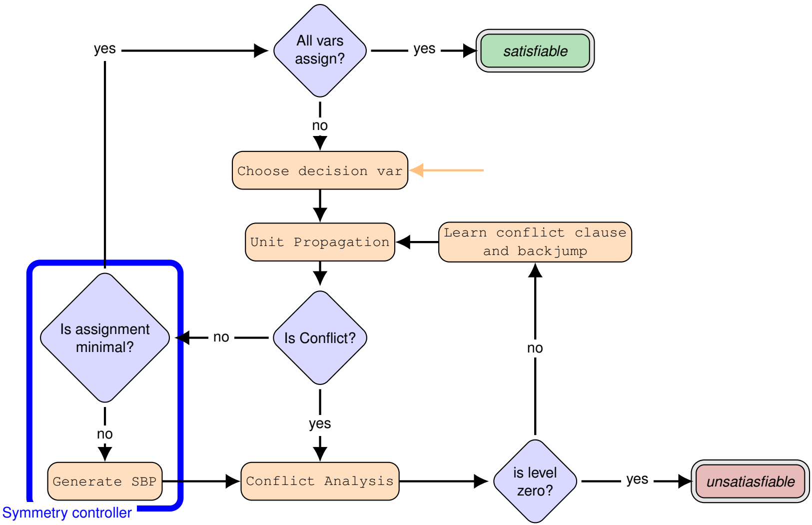


Symmetry status

- `reducer`: $g.\alpha \prec \alpha$
- `inactive` : $\alpha \prec g.\alpha$
- `active` : *not enough information*

Efficient implementation of symmetry status

Keep track the smallest unassigned variable x :

1. $\alpha(g.x) \leq \alpha(x)$, then g is `reducer` $\Rightarrow$ Effective SBP (ESBP)
2. $\alpha(x) \leq \alpha(g.x)$, then g is `inactive` $\Rightarrow$ g cannot reduce α
3. $\alpha(g.x)$ or $\alpha(x)$ is unassigned then g is `active`

Update whenever variables are assigned / unassigned

CDCL[Sym] Implementation

- Packaged as a library **cosy**[1]
- Lightweight
- Fast update and low memory
- Follows symmetry status

- Works with any enumerative SAT solver
- Can be integrated easily

$\rightarrow$ e.g. +3% LOC on `MiniSAT`.

[1] `https://github.com/lip6/cosy`

Experiments

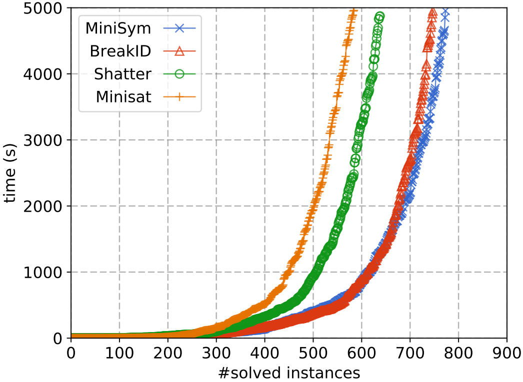
Benchmark:

- from SAT contests 2012 – 2017
- filter: `bliss` finds symmetries in 1000s
- 36 % of instances, $1\,350/3\,700$

Setup:

- four tools
  - MiniSat (no symmetry, baseline)
  - MiniSat + BreakID (SOTA SAT solver using symmetries)
  - MiniSat + Shatter (SOTA SAT solver using symmetries)
  - **MiniSym** = MiniSat + CDCL[Sym] (our approach)
- 5000s timeout, 8GB memory
- includes time to compute symmetries (except for MiniSat)

Experimental results

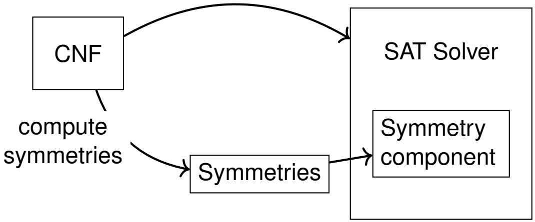


Solver	PAR-2	ALL	SAT	UNSAT
MiniSAT	2243h	586	325	261
Shatter	2088h	640	316	324
BreakID	1790h	749	334	415
MiniSym	**1735h**	**775**	**336**	**439**

Exploitation of symmetries

Dynamic symmetry breaking

Dynamic Symmetry Breaking

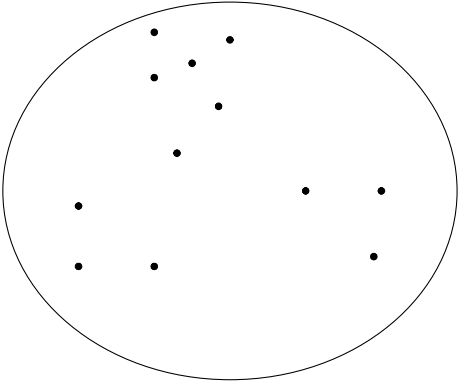


State-of-the-art:

- Symmchaff [Sab05]
- Symmetry Propagation (SP) [DBdC+12]
- Symmetry Learning Scheme (SLS) [BNOS10]
- Symmetry Explanation Leaning (SEL) [DBB17]
- ...

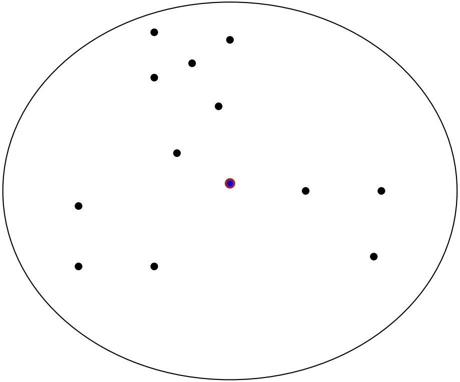
Learning symmetrical clauses

formula
clause



Learning symmetrical clauses

- formula
- clause
- learnt clause



Learning symmetrical clauses

- formula
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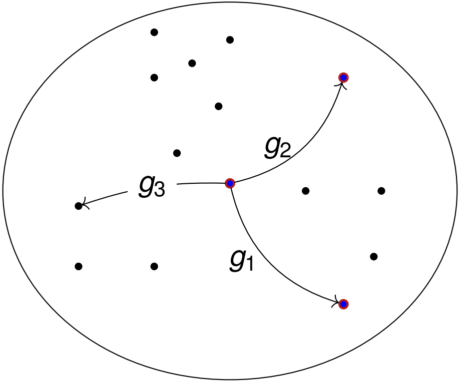
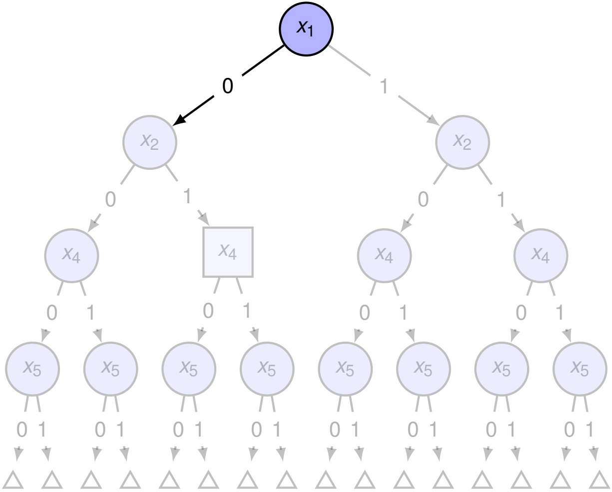


Using symmetries to accelerate the tree traversal

$g = (x_1 \;\; x_2)(x_4 \;\; x_5)(x_7 \;\; x_8)$



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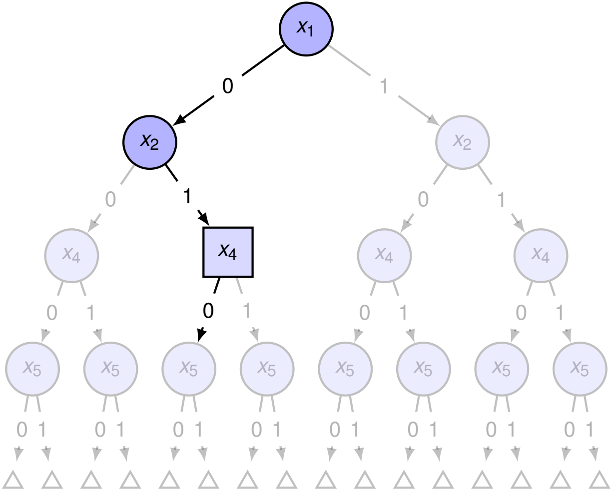

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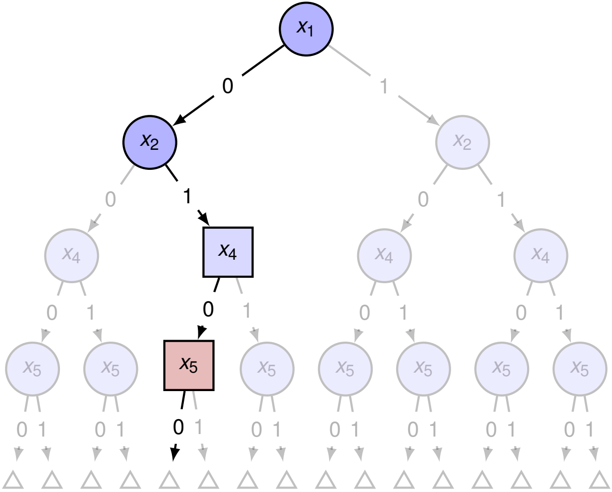
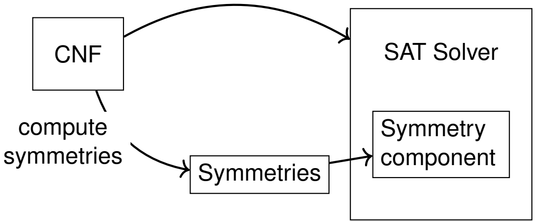


Use symmetries to deduce symmetrical facts.

Dynamic Symmetry Breaking



State-of-the-art:

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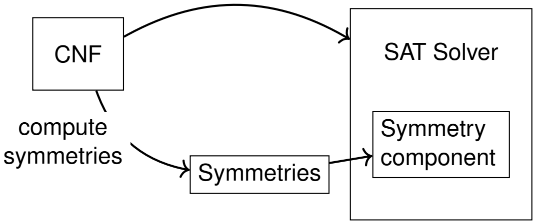
Dynamic Symmetry Breaking



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- ...

Cannot handle some instances solved by static approach

Our contribution CDCL[sym]

NFM'19 [MBK19]

ESBP + SP [MBK19]

Compose the symmetry propagation and the ESBP

prune the decision tree while accelerating its traversal

Problems:

- ESBP breaks symmetries (incrementally)
- SP considers the manipulated symmetries valid all time

In a hybrid approach, SP must be able to identify
valid symmetries

Local symmetries

$$\text{Formula} \leftarrow \text{(Symmetries)}$$

$$\begin{aligned}\omega_1 &\leftarrow \text{(Local symmetries)}\\ \omega_2 &\leftarrow \text{(Local symmetries)}\\ \omega_3 &\leftarrow \text{(Local symmetries)}\\ \omega_4 &\leftarrow \text{(Local symmetries)}\end{aligned}$$

Macro level $\rightarrow$ Micro level

Local symmetries

$\omega_1 \leftarrow$ (Local symmetries)

$\omega_2 \leftarrow$ (Local symmetries)

Formula $\leftarrow$ (Symmetries)

$\omega_3 \leftarrow$ (Local symmetries)

$\omega_4 \leftarrow$ (Local symmetries)

ω_5

Macro level $\rightarrow$ Micro level

Local symmetries

$$\omega_1 \leftarrow \text{(Local symmetries)}$$
$$\omega_2 \leftarrow \text{(Local symmetries)}$$
$$\text{Formula} \leftarrow \text{(Symmetries)} \qquad \omega_3 \leftarrow \text{(Local symmetries)}$$
$$\omega_4 \leftarrow \text{(Local symmetries)}$$
$$\omega_5 \leftarrow \text{(Local symmetries)}$$

Macro level $\rightarrow$ Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

Local symmetries

$\omega_1 \leftarrow$ (Local symmetries)

$\omega_2 \leftarrow$ (Local symmetries)

Formula $\leftarrow$ (Symmetries)

$\omega_3 \leftarrow$ (Local symmetries)

$\omega_4 \leftarrow$ (Local symmetries)

$\omega_5 \leftarrow$ (Local symmetries)

Macro level $\rightarrow$ Micro level

Compute valid local symmetries on-the-fly at a minimal cost.

- Inductive construction of the valid symmetries
- During the solving
- At a minimal cost

Experimental results

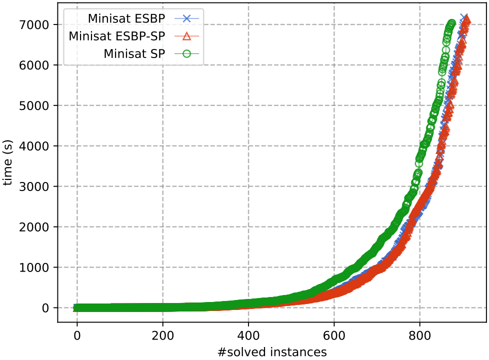
Benchmark:

- from SAT contests 2012 – 2018
- retain only instances for which `bliss` finds significant symmetries in 1000s
- 1400 symmetric instances (out of 4000)

Setup:

- three tools
  - MiniSat SP (Minisat with Symmetry Propagation)
  - MiniSat ESBP (Minisat with CDCL[Sym])
  - **Minisat ESBP-SP** (our approach)
- 7200s timeout

Experimental results



Solver	PAR-2	ALL	SAT	UNSAT
SP	1674h00	876	406	470
ESBP	1578h30	904	416	488
ESBP-SP	**1570h15**	**911**	**420**	**491**

Conclusion

- A new dynamic symmetry breaking approach
  - Generation of SBP on the fly
  - Package as a library cosy usable with any CDCL solver
  - Overcomes drawbacks of the existing approaches

- A new hybrid approach (ESBP-SP)
  - Take advantage of static and dynamic approach
  - Introduce local symmetries

Perspectives

- Combination of CDCL[Sym] with other dynamic symmetry breaking approach
- Combination with parallel SAT solver
- Exploitation of partial symmetries

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Thanks !

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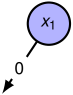
CDCL in action TODO

$$\begin{aligned}
\omega_1 &= \{x_1, x_2, x_3\} \\
\omega_2 &= \{x_4, x_5, x_6\} \\
\omega_3 &= \{\neg x_1, \neg x_5\} \\
\omega_4 &= \{\neg x_2, \neg x_4\} \\
\omega_5 &= \{\neg x_3, \neg x_4\} \\
\omega_6 &= \{\neg x_3, \neg x_6\}
\end{aligned}$$

CDCL in action TODO



$$\omega_1 = \{x_1, x_2, x_3\}$$
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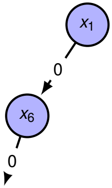

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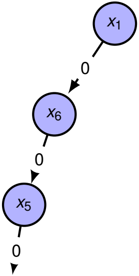
CDCL in action TODO



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$$\omega_2 = \{x_4, x_5, x_6\}$$
$$\omega_3 = \{\neg x_1, \neg x_5\}$$
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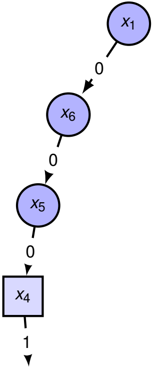
CDCL in action TODO



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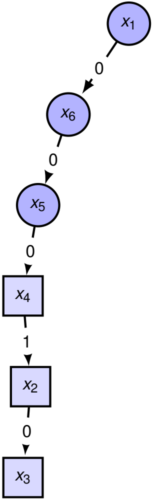

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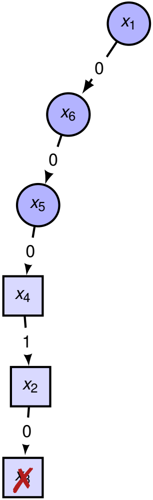

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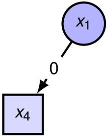

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Weakly active symmetries

Logical consequence

When ω is satisfied in all satisfying assignments of φ, we say that ω is a logical consequence of φ, and we denote this by $\varphi \vdash \omega$.

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Symmetry propagation

Let σ a weakly active symmetry, then

$$\varphi \cup \alpha \vdash \{l\} \Leftrightarrow \varphi \cup \alpha \vdash \sigma.\{l\}$$

Local symmetries

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Local Symmetries

Let φ be a formula. We define $L_{\omega,\varphi}$, the set of *local symmetries* for a clause ω, and with respect to a formula φ, as follows:

$$L_{\omega,\varphi} = \{\sigma \in \mathfrak{S} \mid \varphi \vdash \sigma.\omega\}$$

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We can state that:

$$\bigcap_{\omega \in \varphi} L_{\omega,\varphi} \subseteq G.$$

Computing local symmetries

Formula can be decomposed as : $\varphi = \varphi_o \cup \varphi_e \cup \varphi_d$ where

- φ_o is the set of the original clauses
- φ_e is the set of ESBPs
- φ_d is the set of deduced clauses.

Local symmetries

- $\omega \in \varphi_o, L_{\omega,\varphi} \supseteq G$
- $\omega \in \varphi_e, L_{\omega,\varphi} \supseteq Stab(\omega) = \{\sigma \in G \mid \omega = \sigma.\omega\}$
- $\omega \in \varphi_d, L_{\omega,\varphi} \supseteq (\bigcap_{\omega' \in \varphi_1} L_{\omega',\varphi}) \cup Stab(\omega)$

  where φ_1 is the set of clauses that derives ω.

Generates symmetry breaking predicates (SBP)

- Define lexicographic order
  - Define total order on variables
  - Define minimal value
- Forbid non minimal assignment for each orbit

Example:

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8; \mathtt{F} < \mathtt{T}$$

$$g = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$$

	x_1	x_2	x_3	x_4	x_5	$\cdots$	lex-leader	SBP

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O_1	F	T	–	–	–	$\cdots$	✓	
	T	F	–	–	–	$\cdots$	✗	$\rightarrow \neg x_1 \vee x_2$
O_2	F	F	–	F	T	$\cdots$	✓	

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				$\cdots$				

Example

$$g_1 = (x_2\ \ x_3)(x_5\ \ x_6)(x_8\ \ x_9) \qquad g_2 = (x_1\ \ x_2)(x_4\ \ x_5)(x_7\ \ x_8)$$

$$\mathtt{F} < \mathtt{T} \quad x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 \leq x_9$$

$\overline{g_1}$

$$\alpha = \underline{\mathrm{U}}\quad \overline{\mathrm{U}}\quad \mathrm{U}\quad \mathrm{U}\quad \mathrm{U}\quad \mathrm{U}\quad \mathrm{U}\quad \mathrm{U}\quad \mathrm{U}$$

$\underline{g_2}$

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$$\alpha = \quad \underline{\mathtt{U}} \quad \mathtt{F} \quad \mathtt{F} \quad \mathtt{U} \quad \overline{\mathtt{U}} \quad \mathtt{U} \quad \mathtt{U} \quad \mathtt{U} \quad \mathtt{U}$$

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$\underline{g_2}$

g_2 generates ESBP $\omega = \{\neg x_1, x_2\}$

Example

1. `reducer`: $\alpha(g.x) \leq \alpha(x)$
2. `inactive`: $\alpha(x) \leq \alpha(g.x)$
3. `active`: $\alpha(g.x)$ or $\alpha(x)$ is unassigned

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq x_6 \leq x_7 \leq x_8 \; ; \; \mathtt{F} < \mathtt{T}$$

$$g_1 = \begin{array}{cccccc} (x_2 & x_3) & (x_5 & x_6) & (x_8 & x_9) \\ \uparrow & & & & & \end{array} \left| \begin{array}{ll} x = x_2 & g.x = x_3 \\ & \mathtt{active} \end{array} \right.$$

$$g_2 = \begin{array}{cccccc} (x_1 & x_2) & (x_4 & x_5) & (x_7 & x_8) \\ \uparrow & & & & & \end{array} \left| \begin{array}{ll} x = x_1 & g.x = x_2 \\ & \mathtt{active} \end{array} \right.$$

$$\ldots$$

$$\alpha = \{ \qquad \qquad \qquad \}$$

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$$\dots$$

$$\alpha = \{\neg x_2 \qquad\qquad \}$$

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$$\ldots$$

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

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$$\ldots$$

$$\alpha = \{\neg x_2, \neg x_3, x_1\}$$

g_2 generates $\omega = \{\neg x_1, x_2\}$

Encoding the problem

$$
\begin{array}{ll}
(A,1)(A,2)(A,3) & x_1 \vee x_2 \vee x_3 \\
(B,1)(B,2)(B,3) & x_4 \vee x_5 \vee x_6 \\
(C,1)(C,2)(C,3) & x_7 \vee x_8 \vee x_9 \\
\\
\neg(A,1)\neg(B,1) & \neg x_1 \vee \neg x_4 \\
\neg(A,1)\neg(C,1) & \neg x_1 \vee \neg x_7 \\
\neg(B,1)\neg(C,1) & \neg x_4 \vee \neg x_7 \\
\\
\neg(A,2)\neg(B,2) & \neg x_2 \vee \neg x_5 \\
\neg(A,2)\neg(C,2) & \neg x_2 \vee \neg x_8 \\
\neg(B,2)\neg(C,2) & \neg x_5 \vee \neg x_8 \\
\\
\neg(A,3)\neg(B,3) & \neg x_3 \vee \neg x_6 \\
\neg(A,3)\neg(C,3) & \neg x_3 \vee \neg x_9 \\
\neg(B,3)\neg(C,3) & \neg x_6 \vee \neg x_9
\end{array}
$$