Mathematical Backing for Womanium Hackathon

Bora Basyildiz, Joshua Moler, Caleb Rotello, Hakan Ayaz

August 2022

1 Introduction

For our Womanium Hackathon project, we are using a Quantum Annealing to program our AI. To do this, we will take our game and reduce it to a Min-Cut-Max-Flow problem where Quantum Annealing has already been thoroughly studied and implemented. The process is the following

$$\text{Breedy} \propto \text{NAE SAT} \propto \text{NAE 3-SAT} \propto \text{Max-Cut} \tag{1}$$

Thus when we reduce our problem to the well studied Max-cut problem, we can use the max-cut quantum annealing optimization to find the optimal move for the AI in our Breedy game. For those who are unfamiliar to NAE SAT, it is similar to the well studied SAT problem but with one difference. For SAT, a clause like $T \cup T \cup T$ would be a True clause. However, in NAE SAT, the clause must be true and not all be equal. NAE stands for (Not All Equal). Since all terms in this clause are equal, the $T \cup T \cup T$ clause would resolve to false, while clauses like $T \cup F \cup T$ and $F \cup F \cup T$ would resolve to True.

Now to illustrate the chain of reductions, the reduction between our game to NAE SAT has been already studied [1]. The reduction is within polynomial time. The reduction from NAE-3 SAT to Max Cut has also been well studied and is also on the order of polynomial time [2]. The reduction that hasn't been formally proven to be in polynomial time is from NAE SAT to NAE-3 SAT. We will demonstrate this reduction.

2 NAE SAT Reduction

For this reduction, there are four respective cases, $k = 1, k = 2, k = 3,$ and $k > 3$ where k is equal to the number of terms within a clause.

2.1 Single Term Clause

If there is a single term within a clause, such as T or F, then every term within the clause is the same since there is only one term. Thus for all single clauses they reduce to logically False, and will have the following mapping

$$NAE(T) = NAE(T \cup T \cup T) = F \tag{2}$$
$$NAE(F) = NAE(F \cup F \cup F) = F. \tag{3}$$

2.2 Two Term Clause

If there are two terms within a clause, like $T \cup F$ or $T \cup T$, we will only have logically True NAE statements for $T \cup F$ and $F \cup T$, as these are the only clauses where the terms are not all equal. Modeling the reduction after the $SAT \propto 3 - SAT$ reduction, for a clause $Z_1 \cup Z_2$ we will reduce this to

$$NAE(Z_1 \cup Z_2) = NAE(Z_1 \cup Z_2 \cup V) \cap NAE(Z_1 \cup Z_2 \cup \neg V). \tag{4}$$

The logical table for this reduction is rather trivial and will be left as an exercise for the reader.

2.3 Three term Clause

IF there are three terms within a clause, then we already have a 3-SAT clause. Thus, this is trivial case.

2.4 Three Plus term Clause

If we have a clause that is greater than three terms, we will use a reduction similar to the SAT to 3-SAT reduction. For k terms $\{Z_1, Z_2, \ldots, Z_k\}$ within the clause, we will introduce $k-3$ new variables $\{V_1, V_2, \ldots, V_{k-3}\}$ and $k-2$ new clauses in the chain. Then we will create a system of clauses like the following

$$NAE(Z_1 \cup Z_2 \cup \ldots \cup Z_k) = \tag{5}$$
$$= NAE(Z_1 \cup Z_2 \cup \neg V_1) \cap NAE(V_1 \cup Z_3 \cup \neg V_2) \cap \ldots \cap NAE(V_{k-3} \cup Z_{k-1} \cup Z_k). \tag{6}$$

It can be shown that there are values of V that satisfy the NAE 3-SAT similar to the SAT to 3-SAT reduction. This will also be left as an exercise to the reader.

3 Quantum Optimization

Since we have shown that for all cases of k clauses, we have shown that we can reduce NAE SAT to NAE 3-SAT. Now we have fully shown that the reduction in equation 1. Now we our game is proportional to the Max-Cut problem. Since this problem is well studied, there are quantum optimizations in Qiskit to optimize our problem [3]. Hence we can create an AI that uses the quantum optimizations for Max-Cut problem to generate optimal moves for the Breedy game. This is done through a cost function where we compare the current game state to the optimal game state, and then the AI will chose the move that approaches the optimal game-state the fastest. If there are degenerate optimal moves, the AI will pick one at random.

References

[1] Laurent Gourvès and Jérôme Monnot. On strong equilibria in the max cut game. 12 2009.

[2] David Steurer. Reduction from 3 sat to max cut, 3 2014.

[3] Joao Basso, Edward Farhi, Kunal Marwaha, Benjamin Villalonga, and Leo Zhou. The quantum approximate optimization algorithm at high depth for maxcut on large-girth regular graphs and the sherrington-kirkpatrick model. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022.