$$
\begin{array}{rcll}
[\![\_]\!] & : & \mathbb{M}(A) \to ((A \to \mathbb{R}) \to \mathbb{R}) & \\
[\![\_]\!] & = & \text{LO interpretation} & \\
m & : & \mathbb{M}(A,T) & \\
m & = & \mathbf{do}\{h; a \leftarrow w; M\} & \text{input to disint} \\
\hat{m} & = & \mathbf{do}\{a \leftarrow \mu; t \leftarrow k\,a; \text{return}(a,t)\} & m \text{ to } \hat{m} \text{ by algebraic manipulation} \\
\mu & : & \mathbb{M}(A) & \\
 & \in & \{\text{lebesgue,counting,dirac}\} & \text{in concrete code, } \mu \text{ is determined by } A \text{ (type directed)} \\
k & : & A \to \mathbb{M}(T) & \\
 & = & \text{the interesting thing} & \\
[\![\hat{m}]\!] & = & \lambda f.[\![\mu]\!]\,(\lambda a.[\![k\,a]\!](\lambda t.f(a,t))) & \text{defn of } [\![\_]\!]\text{. Given } \hat{m}\text{, solve for } k \\
\hat{m} & = & \mathbf{do}\{a \leftarrow \mu; <\!\lhd\, w\,a\,\bar{M}\,h\} &
\end{array}
$$

where
$$
\begin{array}{rcl}
\bar{M} & = & \lambda h'.\mathbf{do}\{h'; M\} \\
<\!\lhd\_\_\_\_ & = & \text{constrain outcome ???}
\end{array}
$$

Target:

$$[\![\hat{m}]\!] = \lambda f. \int_{a=(-\infty,\infty)} [\![k\,a]\!](\lambda t.f(a,t)) \wedge \mathrm{d}a$$

Assuming $\mu = \text{lebesgue}$

$$[\![\hat{m}]\!] = \lambda f. \int_{a=(-\infty,\infty)} [\![<\!\lhd\, w\,a\,\bar{M}\,h]\!](f) \wedge \mathrm{d}a$$

where $[\![\bar{M}h']\!] = [\![h']\!]\big[[\![M]\!](f)\big]$ (a)

General form

$$[\![\hat{m}]\!] = \lambda f.[\![h]\!]\big[[\![w]\!](\lambda a.[\![M]\!](f))\big]$$

where $[\![h]\!]$ is an expression with holes, such that substituting a PL (patently linear) -in-f expression gives a PL-in-f expression.

assuming
$$
\begin{array}{rcl}
a & \in & \text{free-vars}([\![M]\!]) \\
\text{free-vars}(h) & \subseteq & \text{free-vars}([\![w]\!]) \cup \text{free-vars}([\![M]\!])
\end{array}
$$

Then case split on w:

$$
\begin{array}{rcll}
① \; w & = & \text{lebesgue} & \\
[\![\hat{m}]\!](f) & = & [\![h]\!]\big[\int_{a=(-\infty,\infty)} [\![M]\!](f) \wedge \mathrm{d}a\big] & \\
 & = & \int_{a=(-\infty,\infty)} [\![h]\!]\big[[\![M]\!](f)\big] \wedge \mathrm{d}a & \text{because of linearity/Tonelli/}h \text{ is nice enough}
\end{array}
$$

Thus define $<\!\lhd \text{lebesgue}\,a\,\bar{M}\,h' = \bar{M}h'$ where we used $\text{eqn}_{(a)}$

$$
\begin{array}{rcll}
② \; w & = & \text{Normal}\,\mu\,\sigma & \text{PDF is } \hat{N}_x \\
[\![\hat{m}]\!](f) & = & [\![h]\!]\big[\int_{a=(-\infty,\infty)} \hat{N}_a \cdot [\![M]\!](f) \wedge \mathrm{d}a\big] & \\
 & = & \int_{a=(-\infty,\infty)} [\![h]\!]\big[\hat{N}_a \cdot [\![M]\!](f)\big] \wedge \mathrm{d}a & \text{linearity/Tonelli/}h \text{ nice} \\
 & = & \int_{a=(-\infty,\infty)} \bar{M}\Big([\![h]\!]\big[\hat{N}_a \cdot [\,]\big]\Big) \wedge \mathrm{d}a & ???
\end{array}
$$

Thus define $<\!\lhd (\text{Normal}\,\mu\,\sigma)\,a\,\bar{M}\,h' = \bar{M}(h'; \text{factor}\,\hat{N}_{\mu,\sigma})$

$$
\begin{array}{rcll}
③ \; w & = & \text{counting} & \\
[\![\hat{m}]\!](f) & = & [\![h]\!]\big[\sum_{a=(-\infty,\infty)} [\![M]\!](f)\big] & \\
 & = & \sum_{a=(-\infty,\infty)} [\![h]\!]\big[[\![M]\!](f)\big] & \text{linearity/switch/}h \text{ nice} \\
 & = & \text{FAIL} &
\end{array}
$$

But we "knew" that the type of w is $\mathbb{M}(\mathbb{Z})$ and we are in $\mathbb{M}(\mathbb{R})$

$$
\begin{array}{rcll}
④ \; w & = & \text{Dirac}\,d & h \vdash d : \mathbb{R}
\end{array}
$$

$$
\begin{array}{lll}
[\![\hat{m}]\!](f) & = & [\![h]\!]\big[\, [\![M]\!](f)\big|_{a=d}\,\big]
\end{array}
$$

case 1: d is a constant, likely impossible?

case 2: d is a var bound in h, see POPL backup slides. case *return* x. see also outline + proofs (???)

case 3: $d = g\,x$ where x bound in h

$<\!\lhd [\![\text{Dirac } (g\,x)]\!]\, a\, c\,(h_1; x \leftarrow w_1; h_2) = <\!\lhd\, w_1\,(g^{-1}\,a)$? ?

worry: g^{-1} may involve (????)

$$
\begin{array}{rcll}
⑤ \quad w & = & \text{return}\,|e| & \\
[\![\hat{m}]\!](f) & = & [\![h]\!]\big[\, [\![M]\!](f)\big|_{a=|e|}\,\big] & \\
 & = & [\![h]\!]\big[\text{if } e \ge 0 \text{ then } \ [\![M]\!](f)\big|_{a=e} \ \text{ else } \ [\![M]\!](f)\big|_{a=-e}\big] & ? \\
 & = & [\![h]\!]\big[\, [\![\mathbf{do}\{a \leftarrow \text{return } |b|; M\}(f)]\!]\big|_{b=e}\,\big] & \\
 & = & \int_{b=(-\infty,\infty)} [\![<\!\lhd(\text{return } e)\, b\ \ M\big|_{a=|b|}\ h]\!](f) \wedge \mathrm{d}b & \text{by induction} \\
 & = & \int_{b=(-\infty,\infty)} [\![<\!\lhd(\text{return } e)\, b\, \bar{M}\, h]\!](f)\Big|_{a=|b|} \wedge \mathrm{d}b & \text{substitution commutative} \\
 & = & \int_{b=(-\infty,0)} [\![<\!\lhd(\text{return } e)\, b\, \bar{M}\, h]\!](f)\Big|_{a=-b} \wedge \mathrm{d}b & \\
 & + & \int_{b=(0,\ \ \infty)} [\![<\!\lhd(\text{return } e)\, b\, \bar{M}\, h]\!](f)\Big|_{a=\ \ b} \wedge \mathrm{d}b & \\
 & = & \int_{a=(0,\infty)} [\![<\!\lhd(\text{return } e)\, b\, \bar{M}\, h]\!](f)\Big|_{b=-a} \wedge \mathrm{d}a & \text{change of variables from } b \text{ to } -a \\
 & + & \int_{b=(0,\infty)} [\![<\!\lhd(\text{return } e)\, b\, \bar{M}\, h]\!](f)\Big|_{a=\ \ b} \wedge \mathrm{d}b & \\
 & = & \int_{a=(0,\infty)} [\![<\!\lhd(\text{return } e)\ -a\, \bar{M}\, h]\!](f) \wedge \mathrm{d}a & \\
 & + & \int_{a=(0,\infty)} [\![<\!\lhd(\text{return } e)\ \ \ \ a\, \bar{M}\, h]\!](f) \wedge \mathrm{d}a & \hat{M} \text{ is } b\text{-free} \\
 & = & \int_{a=(0,\infty)} [\![\cdots \bigoplus \cdots]\!](f) \wedge \mathrm{d}a & \\
 & = & \int_{a=(-\infty,\infty)} \mathbb{1}_{[0,\infty)}(a) \cdot [\![\cdots]\!](f) \wedge \mathrm{d}a & \\
 & = & \int_{a=(-\infty,\infty)} [\![\text{guard } [0,\infty)\, a; \cdots]\!](f) \wedge \mathrm{d}a &
\end{array}
$$

And now onto $\frac{\mathrm{d}}{\mathrm{d}a}$! Definition 2. Let:

$$
\hat{m} \quad = \quad \mathbf{do}\{x \leftarrow \mu; y \leftarrow k\,x; \text{return}(x,y)\} \quad \text{where } \mu = \text{lebesgue}
$$

Show that:

$$
\begin{array}{rcll}
[\![k\,a]\!](f) & = & \dfrac{\mathrm{d}}{\mathrm{d}a}\left([\![\hat{m}]\!]\left(\lambda(x,y).\begin{cases} f(y) & x \le a \\ 0 & x > a\end{cases}\right)\right) & \\
 & = & \dfrac{\mathrm{d}}{\mathrm{d}a}\left(\displaystyle\int_{x=(-\infty,\infty)} [\![k\,x]\!]\left(\lambda y.\begin{cases} f(y) & x \le a \\ 0 & x > a\end{cases}\right) \wedge \mathrm{d}x\right) & \text{defn. of } [\![\cdot]\!] \\
 & = & \dfrac{\mathrm{d}}{\mathrm{d}a}\left(\displaystyle\int_{x=(-\infty,\infty)} \begin{cases} [\![k\,x]\!](f) & x \le a \\ [\![k\,x]\!](\lambda\_.0) & x > a\end{cases} \wedge \mathrm{d}x\right) & \text{piecewise rules} \\
 & = & \dfrac{\mathrm{d}}{\mathrm{d}a}\left(\displaystyle\int_{x=(-\infty,a)} [\![k\,x]\!](f) \wedge \mathrm{d}x\right) & \text{linearity } T(\lambda\_.0) = 0 \\
 & = & [\![k\,a]\!](f) & \text{Leibniz integral rule } (k \text{ depends on } a?)
\end{array}
$$