الإســـــــــــراء

ملزمة (٣)

رياضـــــــــة

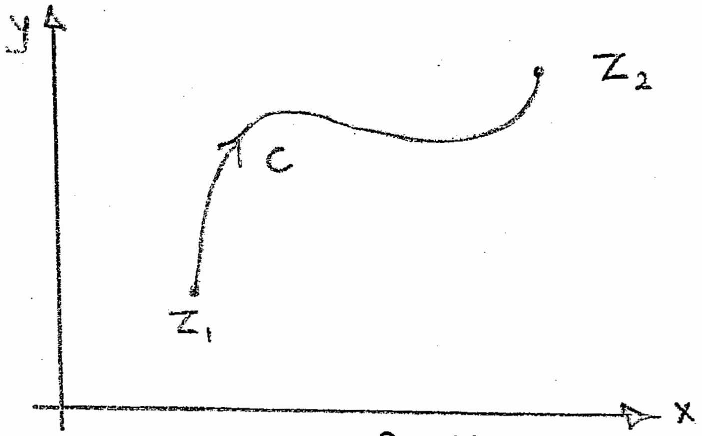
Complex Integral

الترم الثاني

ثانية كهرباء

- Complex Integral -

- To get $\int_{z_1}^{z_2} f(z)\,dz$ along the path c joining z_1 & z_2 :-



① we get the equation of the path as

$y = y(x) \rightarrow z = x + i\,y(x) \rightarrow dz = (1 + i\,y'(x))\,dx$

or $x = x(y) \rightarrow z = x(y) + i\,y \rightarrow dz = (x'(y) + i)\,dy$

or $r = r(\theta) \rightarrow z = r(\theta)\,e^{i\theta}$

or $\theta = \theta(r) \rightarrow z = r\,e^{i\theta(r)}$

or its parametric form $x = x(t)$ & $y = y(t)$

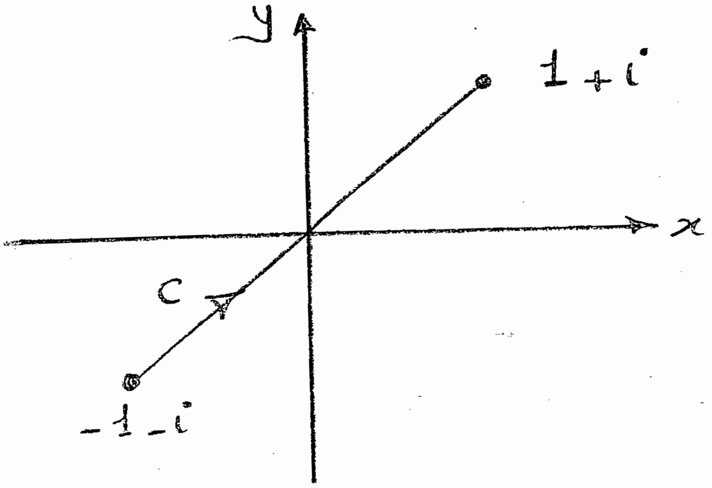
$\rightarrow z = x(t) + i\,y(t) \rightarrow dz = (\dot{x}(t) + i\,\dot{y}(t))\,dt$

② Subsitute for every z in the integral.

<u>For example</u>, to integrate $f(z) = z^2$ from

$z_1 = -1-i$ to $z_2 = 1+i$:-

* <u>Along the line segment joining z_1 & z_2</u>



we have the equation of C is $y = x$

$$\Rightarrow z = x + iy = x + ix$$

$$\Rightarrow dz = (1+i)\,dx$$

$$f(z) = z^2 = (x+ix)^2 = 2x^2 i$$

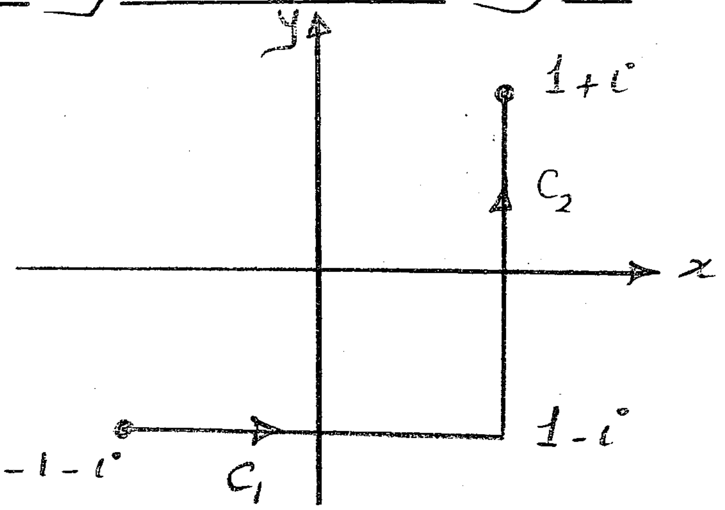
$$\Rightarrow I = \int_C f(z)\,dz = \int_{-1}^{1} (2x^2 i)(1+i)\,dx$$

$$= (1+i)2i \int_{-1}^{1} x^2\,dx = (1+i)(2i)\frac{x^3}{3}\Big|_{-1}^{1}$$

$$= (1+i)(2i)(2/3)$$

$$= -\frac{4}{3} + \frac{4}{3}i$$

* Horizontally then Vertically :



for C_1 we have $y=-1 \Rightarrow z = x - i$

$$\Rightarrow \int_{C_1} f(z)\,dz$$

$$= \int_{-1}^{1} (x-i)^2 \, dx$$

$$= \frac{(x-i)^3}{3}\Big|_{-1}^{1} = \frac{(1-i)^3}{3} - \frac{(-1-i)^3}{3}$$

$$= -4/3$$

for C_2 we have $x = 1 \Rightarrow z = 1 + iy$

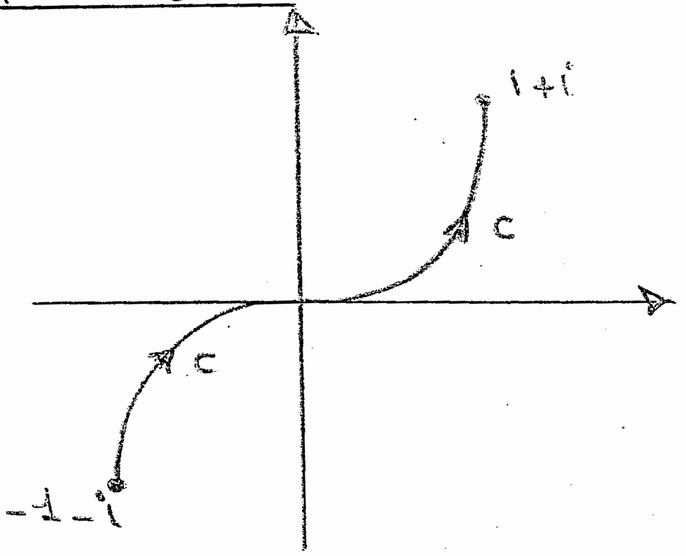
$$\int_{C_2} f(z)\,dz = \int_{-1}^{1} (1+iy)^2 \, i\,dy$$

$$= \frac{i}{i} \frac{(1+iy)^3}{3}\Big|_{-1}^{1} = \frac{1}{3}\left((1+i)^3 - (1-i)^3\right)$$

$$= \frac{4}{3} i$$

$$\Rightarrow \int_{-1-i}^{1+i} f(z)\,dz = \int_{C_1} + \int_{C_2} = -\frac{4}{3} + i\frac{4}{3}$$

* Along the path $y = x^3$



$$Z = x + iy = x + ix^3 \longrightarrow dz = (1 + 3x^2 i)\, dx$$

$$\int_c f(z)\, dz = \int_{z_1}^{z_2} z^2\, dz = \int_{-1}^{1} (x + ix^3)^2 (1 + 3x^2 i)\, dx$$

$$= \left. \frac{(x + ix^3)^3}{3} \right|_{-1}^{1} = \frac{(1+i)^3}{3} - \frac{(-1-i)^3}{3}$$

$$= \frac{2}{3}(1+i)^3 = -\frac{4}{3} + i\frac{4}{3}$$

Note :-

If the fn $f(z)$ is analytic everywhere (entire) we Can integrate it as areal fn , for example , the fn $f(z) = z^2$ is entire fn , So

$$\int_{-1-i}^{1+i} z^2\, dz = \left. \frac{z^3}{3} \right|_{-1-i}^{1+i} = \frac{1}{3}\left((1+i)^3 - (-1-i)^3 \right)$$

$$= -\frac{4}{3} + i\frac{4}{3}.$$

Example:- find a) $\int_0^i z e^{z^2} dz$

b) $\int_1^i z \cos z \, dz$

Solution:-

a) $\int_0^i z e^{z^2} dz = \frac{1}{2} \int_0^i e^{z^2} (2z) dz$

$= \frac{1}{2} e^{z^2} \Big|_0^i$

$= \frac{1}{2} (e^{-1} - e^{0}) = \frac{1}{2} (\frac{1}{e} - 1)$

b) $\int_1^i z \cos z \, dz$

$= z (\sin z) - (- \cos z) \Big|_1^i$

$= i \sin i + \cos i - \sin 1 - \cos 1$

$= \cosh 1 - \sinh 1 - \sin 1 - \cos 1$

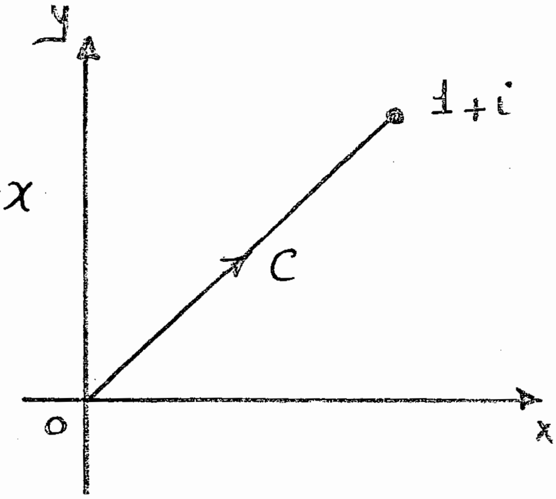
Example :- Evaluate $\int_C f(z)\,dz$ where

$f(z) = y - x - 3x^2 i$ & C is

i) the segment from 0 to $1+i$

ii) the segment from 0 to i & then horizontally to $1+i$

Solution :-



i) The equation of C is $y = x$

$\Rightarrow z = x + iy = x + ix$

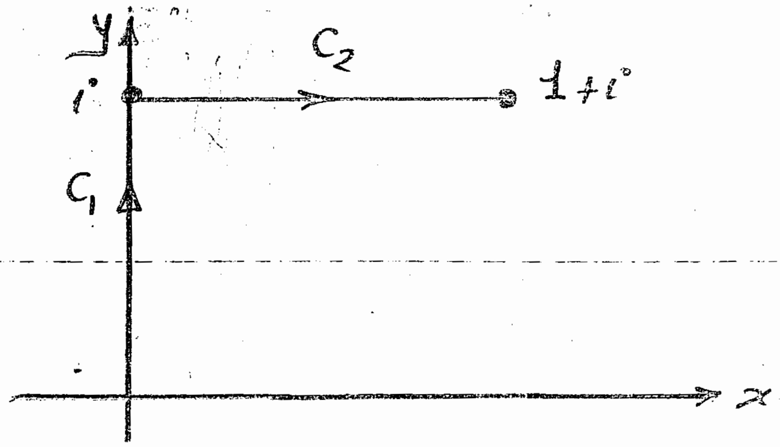
$\Rightarrow dz = (1+i)\,dx$

$$I = \int_C f(z)\,dz = \int_0^1 (y - x - 3x^2 i)(1+i)\,dx$$

$$= \int_0^1 -3x^2 i\,(1+i)\,dx = -i(1+i)\,x^3\Big|_0^1$$

$$= 1 - i$$

ii)



- Along C_1 we have $x=0$

$$\Rightarrow z = iy \Rightarrow dz = i\,dy$$

$$f(z) = y$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_0^1 y\, i\, dy = i\,\frac{y^2}{2}\Big|_0^1 = i/2$$

- Along C_2 we have $y=1$

$$\Rightarrow z = x + i \Rightarrow dz = dx$$

$$f(z) = 1 - x - 3x^2 i$$

$$\int_{C_2} f(z)\,dz = \int_0^1 1 - x - 3x^2 i \quad dx$$

$$= x - \frac{x^2}{2} - x^3 i \Big|_0^1 = 1 - \frac{1}{2} - i$$

$$= \frac{1}{2} - i$$

$$\Rightarrow \int_C f(z)\,dz = \int_{C_1} + \int_{C_2}$$

$$= \frac{1}{2} - \frac{i}{2}$$

Example :-

Evaluate these integrals :-

a) $\int_c |z|\, dz$; c is the upper half of the unit circle from -1 to 1.

b) $\int_c f(z)\, dz$; where $f(z) = \frac{z+2}{z}$ & C is

i) the Semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$

ii) the Semicircle $z = 2e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$

iii) the Semicircle $z = 2e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$

Solution :-

a) we have $z = e^{i\theta}$; $\theta : \pi \longrightarrow 0$ along the given path

$\Rightarrow$ $|z| = 1$

$dz = i e^{i\theta} d\theta$

$$\int_c |z|\, dz = \int_{\pi}^{0} 1 .\, i e^{i\theta} d\theta = i \int_{\pi}^{0} e^{i\theta} d\theta$$

$$= i \left. \frac{e^{i\theta}}{i} \right|_{\pi}^{0} = e^{0} - e^{i\pi} = 1 - (-1) = 2$$

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b) we have $z = 2e^{i\theta}$ $\Rightarrow$ $dz = 2i\, e^{i\theta} d\theta$

i) $\int_C f(z)\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{\cancel{2e^{i\theta}}} \cdot \cancel{2i e^{i\theta}}\, d\theta$

$= i\left(\frac{2}{i} e^{i\theta} + 2\theta\right)\Big|_0^{\pi}$

$= 2e^{i\theta} + 2i\theta \Big|_0^{\pi} = -2 + 2\pi i - 2 - 0$

$= -4 + 2\pi i$

ii) $\int_C f(z)\,dz = \int_{\pi}^{2\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2i e^{i\theta}\, d\theta$

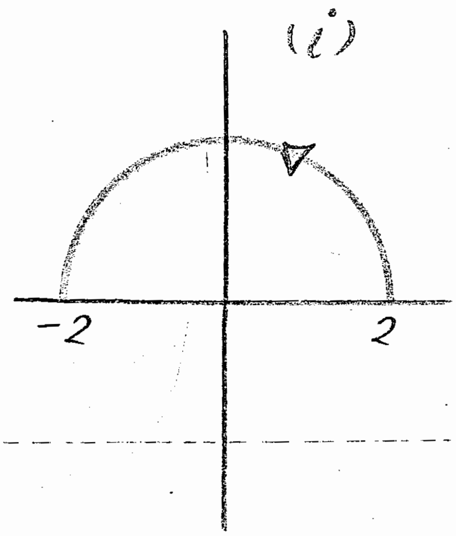
$= 2e^{i\theta} + 2i\theta \Big|_{\pi}^{2\pi} = 2 + 4\pi i + 2 - 2\pi i$

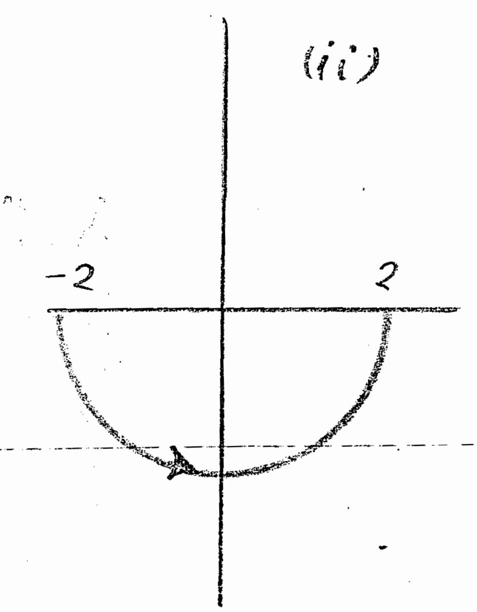
$= 4 + 2\pi i$

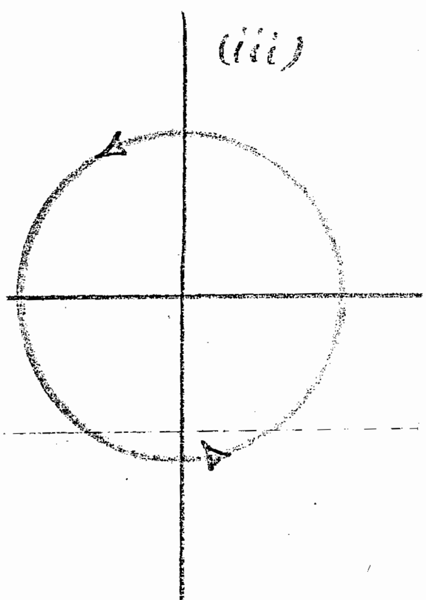
iii) $\int_C f(z)\,dz = 2e^{i\theta} + 2i\theta \Big|_{-\pi}^{\pi}$

$= -2 + 2i\pi + 2 + 2i\pi = 4\pi i$

OR by adding (i) & (ii) we get also $4\pi i$







* **Example:** Evaluate $\int_c \bar{z}\, dz$ from 0 to $1+2i$

where c is the parametric curve

$$x = \frac{2}{\pi} t \quad , \quad y = 2 \sin t$$

Solution:-

$$z = x + iy = \frac{2}{\pi} t + i\, 2 \sin t$$

$$\Rightarrow dz = \frac{2}{\pi} + i\, 2 \cos t \quad dt$$

when $z : 0 \longrightarrow 1+2i$ we have $t : 0 \longrightarrow \pi/2$

$$f(z) = \bar{z} = \frac{2}{\pi} t - i\, 2 \sin t$$

$$\int_c \bar{z}\, dz = \int_0^{\pi/2} \left(\frac{2}{\pi} t - i\, 2 \sin t\right)\left(\frac{2}{\pi} + i\, 2 \cos t\right) dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{2}{\pi}\right)^2 t - i \frac{4}{\pi} \sin t + i \frac{4}{\pi} t \cos t + 4 \sin t \cos t \, dt$$

$$= \frac{2}{\pi^2} t^2 + i \frac{4}{\pi} \cos t + i \frac{4}{\pi} \left(t \sin t + \cos t \right)$$

$$+ 2 (\sin t)^2 \Big|_0^{\pi/2}$$

$$= \frac{1}{2} + 2i + 2 - i \frac{4}{\pi} - i \frac{4}{\pi}$$

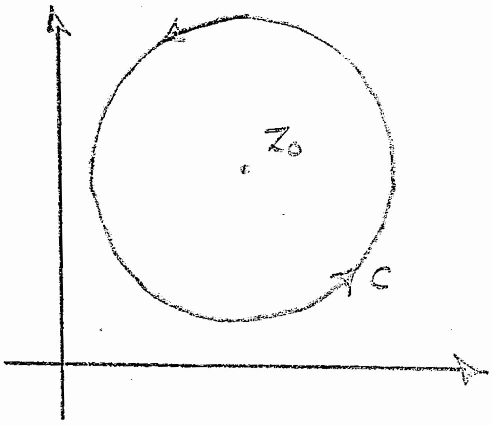
$$= \checkmark$$

Example (A) Evaluate $\int_c (z-z_0)^m dz$ along the circle $|z-z_0|=r$, where m is integer.

Solution :-

$$|z-z_0|=r \longrightarrow z-z_0 = re^{it} ; 0 \le t \le 2\pi$$

$$\longrightarrow z = z_0 + re^{it} \longrightarrow dz = ire^{it} dt$$



$$\int_c (z-z_0)^m dz =$$

$$\int_0^{2\pi} (re^{it})^m (ire^{it})\, dt$$

$$= r^{m+1} i \int_0^{2\pi} e^{i(m+1)t}\, dt$$

If $m=-1 \Rightarrow \int = i \int_0^{2\pi} dt = 2\pi i$

If $m \neq -1 \Rightarrow \int = i r^{m+1} \left(\frac{e^{i(m+1)t}}{i(m+1)} \right) \Big|_0^{2\pi}$

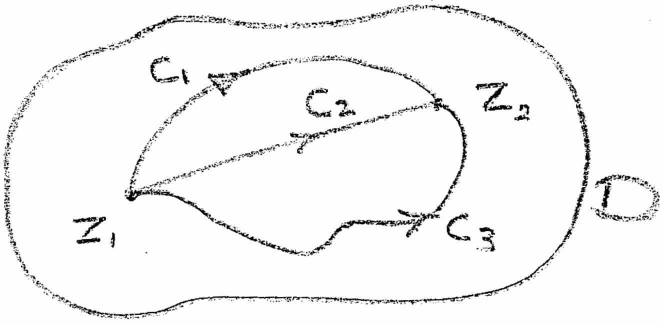
$$= \frac{r^{m+1}}{m+1} \left(e^{i(m+1)(2\pi)} - 1 \right) = \text{Zero}$$

$$\Rightarrow \int_c (z-z_0)^m dz = \begin{cases} 2\pi i & ; m=-1 \\ \text{Zero} & ; m \neq -1 \end{cases}$$

$\Rightarrow \int_c \frac{1}{z-z_0} dz = 2\pi i$, for any path c Containing z_0 " not necessary a circle "

Cauchy Integral theorm :-

(I) If $f(z)$ is analytic in a simply Connected domain D, as shown :-



then $\int_{z_1}^{z_2} f(z)\,dz$ is independant on the path , i.e.

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = \int_{C_3} f(z)\,dz \;---$$

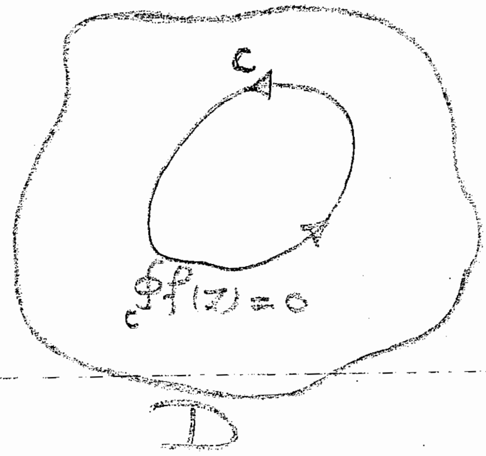
(II) If $f(z)$ is analytic in a simply Connected domain D ; then $\oint_c f(z)\,dz =$ zero, for any closed path c Contained in D.

<u>Proof:-</u>

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \oint_c u\,dx - v\,dy$$

$$+ i \oint_c v\,dx + u\,dy$$



Using Green's theorm $\oint_c P\,dx + Q\,dy$

$$= \int_{R:\text{ interior of } c} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\Rightarrow \oint_c f(z)\,dz = \int_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + i \int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

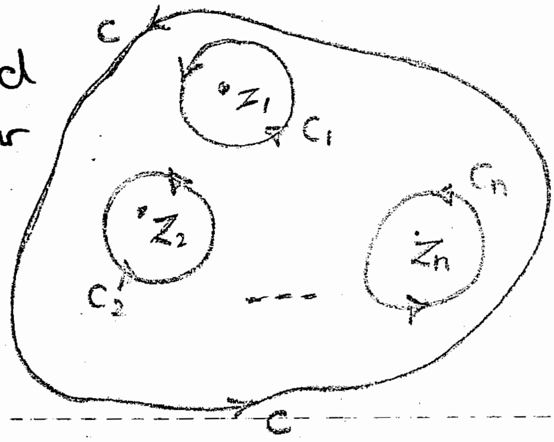
But $f(z)$ is analytic inside R $\Rightarrow$ Cauchy Reiman are Satisfied $\Rightarrow$ $u_x = v_y$, $u_y = -v_x$

$$\Rightarrow \oint_c f(z)\,dz = \text{Zero}.$$

(III) If the closed path C Contains singular Points (points making $f(z)$ not analytic), then:

$$\oint_c f(z)\,dz = \oint_{c_1} f(z)\,dz + \oint_{c_2} f(z)\,dz + \cdots + \oint_{c_n} f(z)\,dz.$$

where $C_1, C_2, \ldots, C_n$ are closed pathes Containing the Singular points inside C.



Examples :-

1) $\oint_c \sin z \, dz = 0$ for any closed path c

because $\sin z$ is every where analytic (entire)

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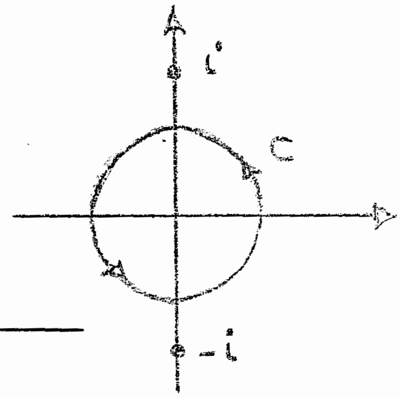
2) $$\oint_{|z|=\frac{1}{2}} \frac{z e^{z}}{z^{2}+1} \, dz$$

$f(z)$ is analytic every where except at $z = \pm i$

$f(z)$ is analytic on & inside C $\Rightarrow$

$\oint f(z) \, dz$ = Zero



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3) $\oint_{|z|=1} \tan z \, dz$

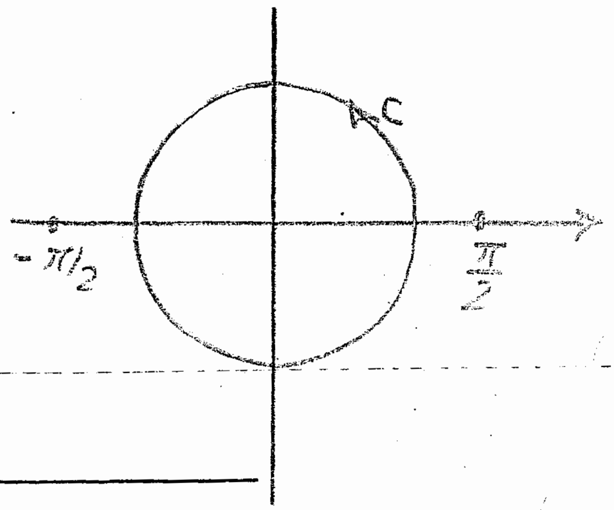
$f(z) = \tan z = \dfrac{\sin z}{\cos z}$ is analytic except for

$\cos z = 0 \longrightarrow z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

all the singular points lie outside c

$f(z)$ is analytic on & inside c

$\Rightarrow \oint_c f(z) \, dz =$ Zero



Example:-

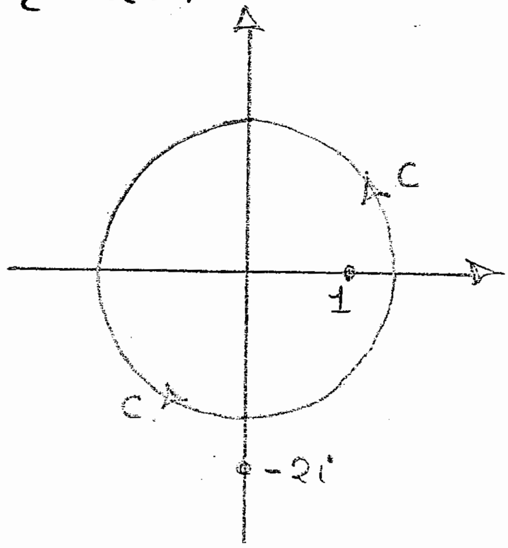
Find $\oint_c \frac{3}{z+2i} - \frac{7}{z-1} \, dz$

where c: (i) $|z| = 3/2$

(ii) $|z+i| = 3$

Solution

(i) $I = \oint_c \frac{3}{z+2i} \, dz - \oint_c \frac{7}{z-1} \, dz$



$\frac{3}{z+2i}$ is analytic inside $c \Rightarrow I_1 = 0$

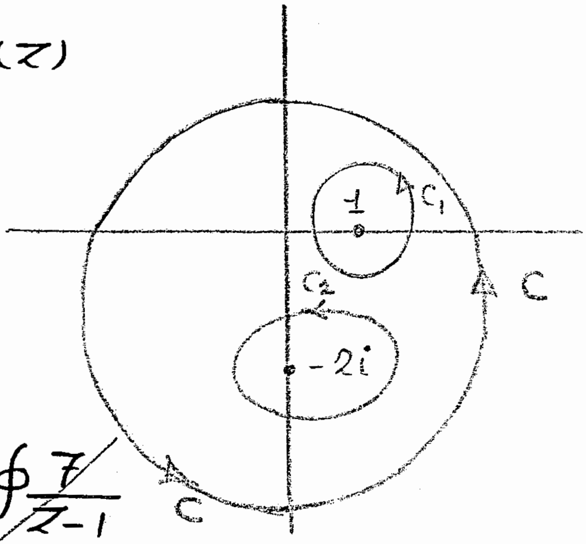
$I_2 = 7 \oint_c \frac{1}{z-1} \, dz$

Using the result of Example (A)

$\Rightarrow \quad I_2 = 7(2\pi i)$

$I = I_1 - I_2 = -14\pi i$

(ii) $\oint_c f(z) = \oint_{c_1} f(z) + \oint_{c_2} f(z)$



$= \oint_{c_1} \frac{3}{z+2i} - \frac{7}{z-1} \, dz$

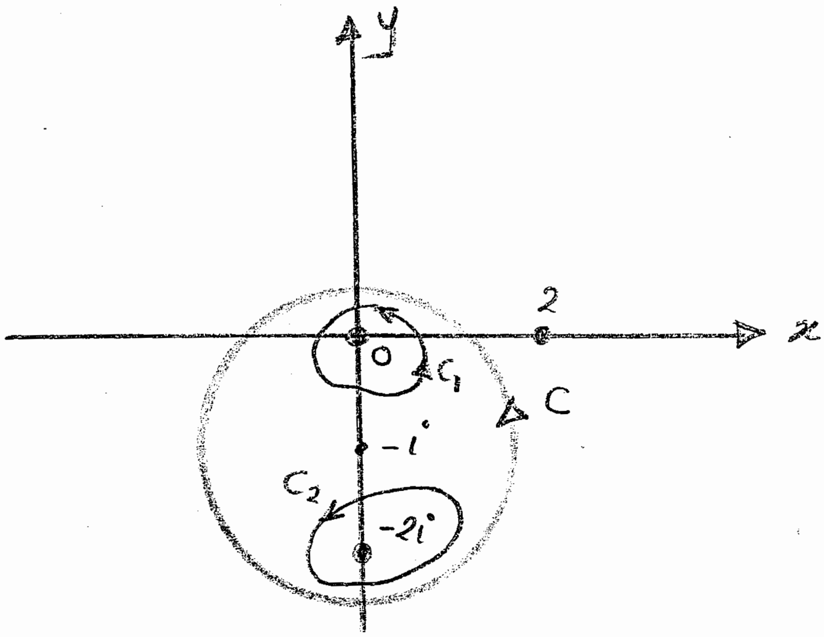
$+ \oint_{c_2} \frac{3}{z+2i} - \frac{7}{z-1} \, dz$

$= \underbrace{\cancel{\oint_{c_1} \frac{3}{z+2i}}}_{0} - \oint_{c_1} \frac{7}{z-1} + \oint_{c_2} \frac{3}{z+2i} - \underbrace{\cancel{\oint_{c_2} \frac{7}{z-1}}}_{0}$

$= -7(2\pi i) + 3(2\pi i) = -8\pi i$

Example :- $\oint_C \frac{e^z (z+1)}{z(z+2i)(z-2)^2}\, dz$ where C is the closed path $|z+i| = 3/2$

Solution :-



we have $\oint_C = \oint_{C_1} + \oint_{C_2}$

$$\Rightarrow \oint_C \frac{e^z(z+1)}{z(z+2i)(z-2)^2} = \oint_{C_1} \frac{e^z(z+1)/(z+2i)(z-2)^2}{z}$$

$$+ \oint_{C_2} \frac{e^z(z+1)/(z)(z-2)^2}{z+2i}$$

$$= 2\pi i \cdot \frac{e^0(0+1)}{(0+2i)(0-2)^2} + 2\pi i \cdot \frac{e^{-2i}(-2i+1)}{(-2i)(-2i-2)^2}$$

$$= \frac{\pi}{4} - \frac{\pi e^{-2i}(1-2i)}{(2+2i)^2}$$

Example :-

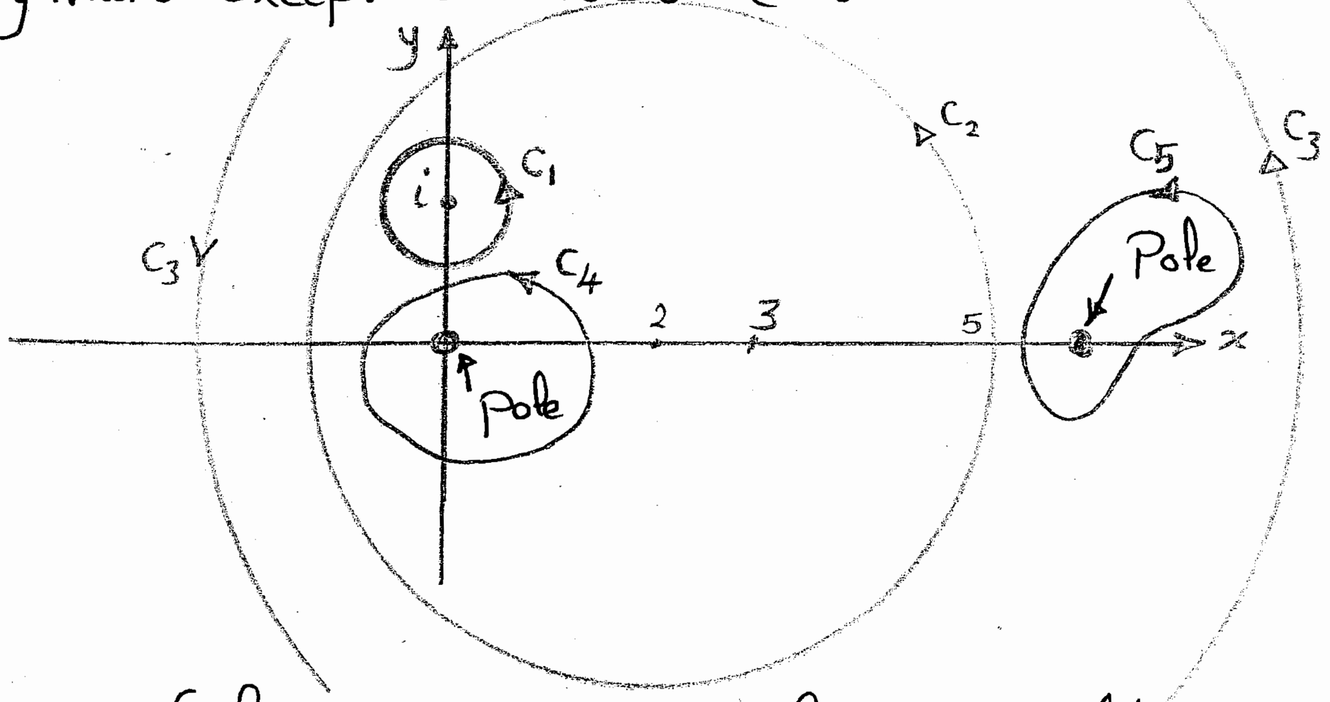
Evaluate $\oint_C \frac{e^{z^2}\,dz}{z^2-6z}$; where C is

i) $|z-i| = 1/2$ ii) $|z-2| = 3$ iii) $|z-3| = 5$

Solution :-

$$f(z) = \frac{e^{z^2}}{z^2-6z} = \frac{e^{z^2}}{z(z-6)}$$ is analytic

every where except at $z=0$ & $z=6$.



i) $I = \oint_{C_1} f(z)\,dz$ = Zero , $f(z)$ is analytic on & inside C_1

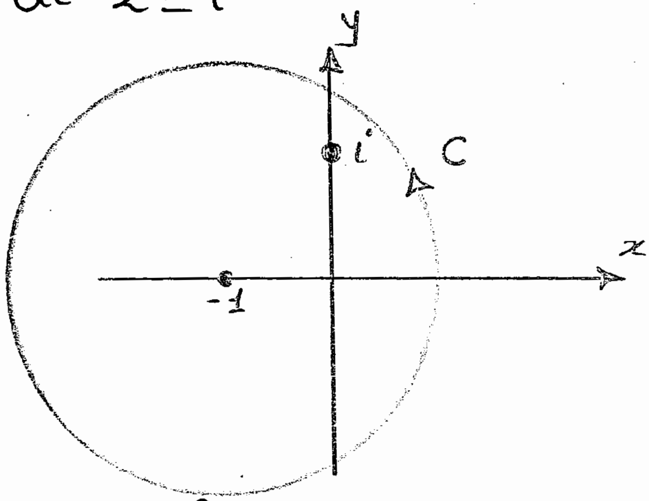
ii) $I = \oint_{C_2} \frac{e^{z^2}/(z-6)}{z}\,dz = 2\pi i * \frac{e^0}{-6} = -\frac{\pi i}{3}$

iii) $I = \oint_{C_3} = \oint_{C_4} + \oint_{C_5} = -\frac{\pi i}{3} + \oint_{C_5} \frac{e^{z^2}/z}{z-6}$

$$= -\frac{\pi i}{3} + 2\pi i * \frac{e^{36}}{6} = \frac{\pi i}{3}(e^{36} - 1)$$

Example: find $\oint_c \frac{\sin(z-i)}{z-i}\,dz$; where c is the path $|z+1| = \sqrt{3}$

Solution: we have a pole at $z = i$

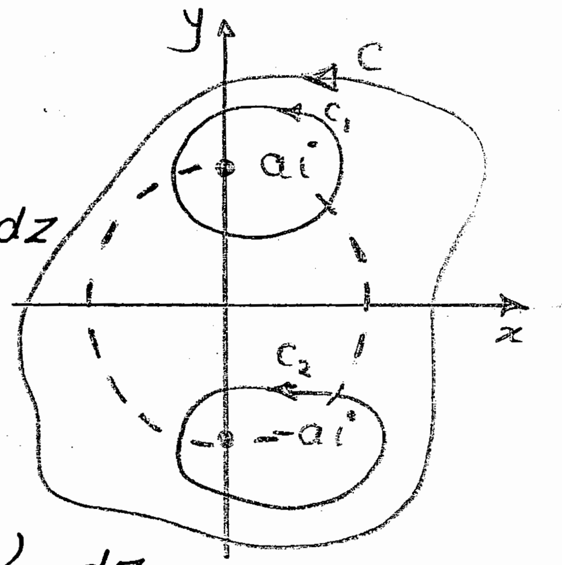


$$I = 2\pi i * \sin(z-i)\Big|_{z=i}$$

$$= \text{Zero}$$

Example:- evaluate $\oint_c \frac{e^z}{z^2+a^2}\,dz$ if the contour c contains the circle $|z| = a$ within it.

Solution



$$\oint_c \frac{e^z}{z^2+a^2}\,dz = \oint_c \frac{e^z}{(z-ai)(z+ai)}\,dz$$

$$= \oint_{c_1} \frac{e^z/(z+ai)}{z-ai}\,dz + \oint_{c_2} \frac{e^z/(z-ai)}{z+ai}\,dz$$

$$= 2\pi i \cdot \frac{e^{ai}}{2ai} + 2\pi i \cdot \frac{e^{-ai}}{-2ai} = \frac{\pi}{a}\left(e^{ia} - e^{-ia}\right)$$

$$= 2\pi i \sin a$$

Example:-

$$\oint_c \frac{e^z}{(z+2)^4}\, dz \quad ; \quad c \text{ is } |z| = 3$$

Solution:-

$z = -2$ is a pole of order 4

$$\oint_c \frac{e^z}{(z+2)^4} = \frac{2\pi i}{3!} \frac{d^3}{dz^3} (e^z) \Big|_{z=-2}$$

$$= \frac{2\pi i}{6} e^{-2} = \frac{\pi i}{3e^2}$$

Example:- $\oint_c \frac{ch z}{z^4 + 4z^2}\, dz$, c is

a) $|z| = 1$ b) $|z + i| = 3/2$

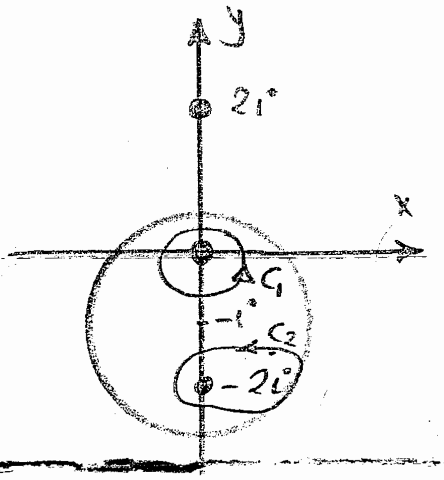
Solution:-

$$f(z) = \frac{ch z}{z^4 + 4z^2} = \frac{ch z}{z^2 (z+2i)(z-2i)}$$

has a poles (is not analytic) at $z = 0, \pm 2i$

a) $$\oint_{|z|=1} f(z) = \oint_{|z|=1} \frac{(ch z)/(z^2+4)}{z^2}$$

$$= \frac{2\pi i}{1!} \frac{d}{dz} \left(\frac{ch z}{z^2+4} \right) \Big|_{z=0} = \text{zero}$$



b) $$\oint_{|z+i| = 3/2} f(z) = \oint_{C_1} + \oint_{C_2}$$

$$= \text{zero} + \oint_{C_2} \frac{(ch z)/(z^2)(z-2i)}{z+2i}$$

$$= 2\pi i \frac{ch\,{-2i}}{(-2i)^2(-4i)} = \frac{\pi}{8} \cos 2$$

Example:- find $\oint_C \tan z \, dz$ where C is the unit Circle

Solution:-

$\tan z = \dfrac{\sin z}{\cos z}$ is analytic every where except

for $\cos z = 0$ i.e. $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

which all lies outside C

$\Rightarrow$ $\tan z$ is analytic on & inside C

$\Rightarrow$ $\oint_C \tan z =$ zero.

Example:- find $\displaystyle\oint_{|z|=1} \frac{e^{i\pi z}}{2z^2 - 5z + 2} dz$

Solution:- $f(z)$ is not analytic for $2z^2 - 5z + 2 = 0$

$$\Rightarrow z = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5}{4} \pm 3/4$$

$$= 2 , \ 1/2$$

$$\oint f(z) dz = \oint_{|z|=1} \frac{e^{i\pi z}}{(2z-1)(z-2)} dz$$

$$= \oint_{|z|=1} \frac{e^{i\pi z}/(z-2)}{2z-1} dz = \frac{1}{2} \oint \frac{e^{i\pi z}/z-2}{z - 1/2}$$

$$= \frac{1}{2} \cdot 2\pi i \cdot \frac{e^{i\frac{\pi}{2}}}{1/2 - 2} = \frac{2}{3} \pi$$

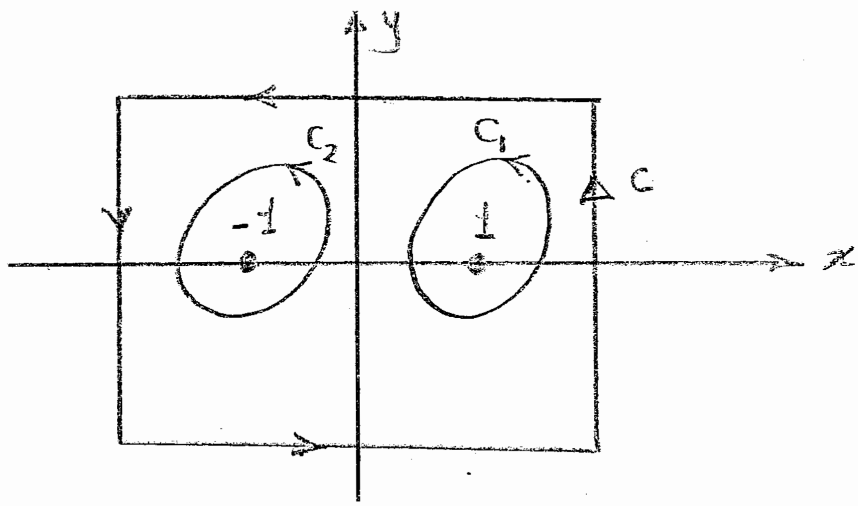
Example :-

Evaluate $\oint \frac{\sin \pi z}{(z^2-1)^2} dz$;

where C is the boundary of the square $x = \pm 2$ & $y = \pm$:

Solution :-

$f(z) = \frac{\sin \pi z}{(z^2-1)^2} = \frac{\sin \pi z}{(z-1)^2 (z+1)^2}$ is not analytic at $z = \pm 1$



$$\oint_C = \oint_{C_1} + \oint_{C_2} = \oint_{C_1} \frac{(\sin \pi z)/(z+1)^2}{(z-1)^2} dz$$

$$+ \oint_{C_2} \frac{(\sin \pi z)/(z-1)^2}{(z+1)^2} dz$$

$$= \frac{2\pi i}{1!} \frac{d}{dz} \left(\frac{\sin \pi z}{(z+1)^2} \right) \Bigg|_{z=1} + \frac{2\pi i}{1!} \frac{d}{dz} \left(\frac{\sin \pi z}{(z-1)^2} \right) \Bigg|_{z=-1}$$

$$= 2\pi i \left(\frac{\pi \cos \pi z \, (z+1)^2 - 2(z+1) \sin \pi z}{(z+1)^4} \Bigg|_{z=1} + \cdots \right)$$

$$= 2\pi i \left(\frac{-\pi}{4} + \frac{-\pi}{4} \right) = -\pi^2 i$$

Example:- If C is the circle $|z| = 2$ and if

$$g(z_0) = \oint \frac{z^2+2z}{(z-z_0)^3}\, dz \quad , \text{ find}$$

a) $g(i)$ b) $g(z_0)$ when $|z_0| > 2$

Solution:

we have 2 Cases:- (i) z_0 is inside C and $z_0 \neq 0$

$$\Rightarrow g(z_0) = \frac{2\pi i}{2!} \frac{d^2}{dz^2} (z^2+2z)\Big|_{z=z_0}$$

$$= 2\pi i$$

(ii) z_0 is outside C

$$\Rightarrow f(z) = \frac{z^2+2z}{(z-z_0)^3} \text{ is analytic on \& inside } C$$

$$\Rightarrow \oint_C f(z) = g(z_0) = \text{zero}$$

* So,

a) $g(i) = 2\pi i$

b) $g(z_0) = \text{zero}$ when $|z_0| > 2$

Example :- Evaluate $\oint_C \frac{\sin \pi z}{(z-i)^4} dz$ where

(i) $C: |z+i| + |z-i| = 4$

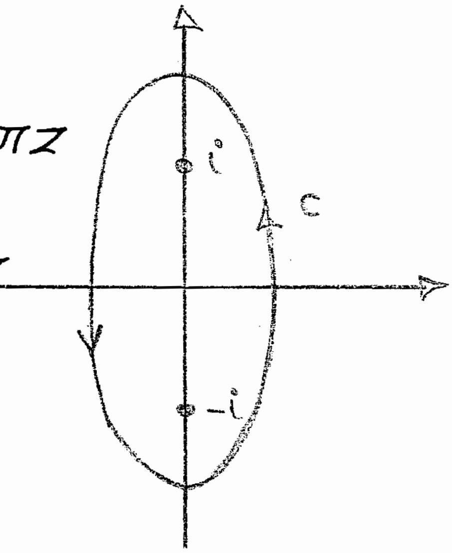
(ii) $C: |z+1| + |z-1| = 5/2$

Solution :-

The only singular point is at $z = i$

The equation $|z-a| + |z-b| = c$ is an ellipse with focus at $z = a$ & $z = b$.

(i)



$z = i$ inside C & $f(z) = \sin \pi z$

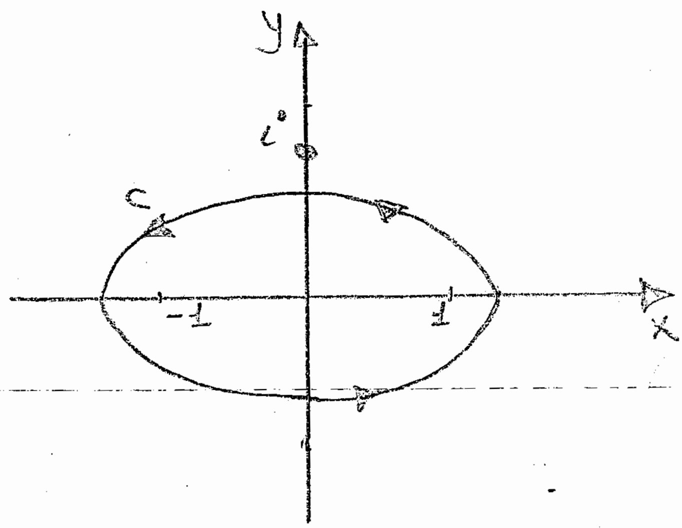
$f'(z) = \pi \cos \pi z \rightarrow f'' = -\pi^2 \sin \pi z$

$\rightarrow f''' = -\pi^3 \cos \pi z$

$f'''(i) = -\pi^3 \cos i\pi = -\pi^3 \operatorname{ch} \pi$

$I = \frac{2\pi i}{n!} f^{(n)}(z_0) = \frac{2\pi i}{3!} (-\pi^3 \operatorname{ch} \pi) = -i \frac{\pi^4}{3} \operatorname{ch} \pi$

(ii) the singular point lies outside C $\Rightarrow$



$f(z)$ is analytic on & inside $C \Rightarrow I = \text{Zero}$

Proof of Cauchy formula :-

1) $$\oint \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0).$$

Proof:- Since we have $\oint \frac{f(z)}{z-z_0}\, dz$ may be written

as $$\oint \frac{f(z_0) + (f(z) - f(z_0))}{z-z_0}\, dz$$

$$= \oint \frac{f(z_0)}{z-z_0}\, dz + \oint \frac{f(z) - f(z_0)}{z-z_0}\, dz$$

for the first integral $= f(z_0) \oint \frac{dz}{z-z_0} = f(z_0) \cdot 2\pi i$

But the 2nd integral = zero, because the fn.

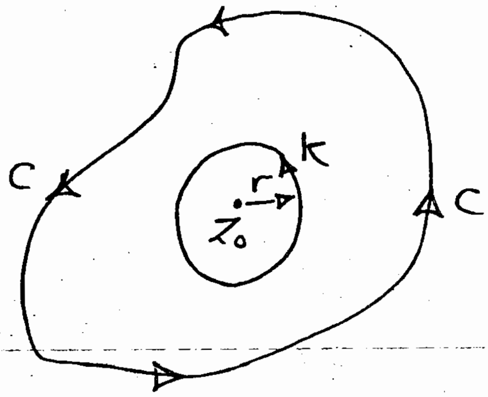
$\frac{f(z) - f(z_0)}{z-z_0}$ is analytic inside C except at z_0 &

$f(z)$ is a Continous fn. on C $\Rightarrow$

$$|f(z) - f(z_0)| < \epsilon \quad \text{for all } |z-z_0| < \delta$$

Choose a circle k with Center at z_0 & radius $r < \delta$

$$\Rightarrow \frac{|f(z) - f(z_0)|}{|z-z_0|} < \frac{\epsilon}{r} \quad \text{for points on } k$$



$$\Rightarrow \oint \frac{f(z) - f(z_0)}{z-z_0} < M\ell$$

$$< \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon$$

Since ϵ is arbitrary $\Rightarrow \oint = \text{zero}$

$$\Rightarrow \oint \frac{f(z)}{z-z_0}\, dz = 2\pi i + \text{zero} = 2\pi i$$

2) $$\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Proof: from the previous relation $\oint_c \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0)$

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-z_0}\,dz$$

where z_0 is a point inside C, let Δz be so small

so that $z_0 + \Delta z$ is inside C

$$\Rightarrow \quad f(z_0+\Delta z) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-(z_0+\Delta z)}\,dz$$

$$\Rightarrow \quad \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \oint \frac{f(z)}{z-z_0-\Delta z} - \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \oint \frac{(z-z_0)f(z) - (z-z_0-\Delta z)f(z)}{(z-z_0-\Delta z)(z-z_0)}$$

$$= \frac{1}{2\pi i} \cdot \frac{1}{\Delta z} \cdot \oint \frac{\Delta z\, f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

let $\Delta z \to 0 \Rightarrow \quad f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^2}\,dz$

$$\Rightarrow \quad \oint \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i\, f'(z_0)$$

one can also prove $\oint \frac{f(z)}{(z-z_0)^3} = \frac{2\pi i}{2!} f''(z_0)$

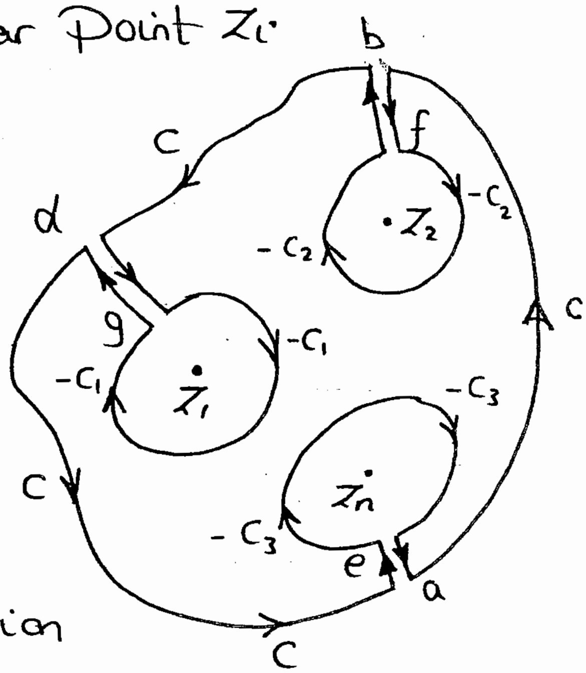
& more generally $\oint \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$.

* Proof of the Cauchy Integral Theorm

$\oint_C f(z)\, dz$; where $f(z)$ is analytic inside C except at some points $z_1, z_2, \ldots, z_n$

$$= \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \cdots + \oint_{C_n} f(z)\, dz$$

where C_i contain only the singular point z_i



we have

$$\int_a^b + \cancel{\int_b^f} + \int_{-C_2} + \cancel{\int_f^b} + \int_b^d$$

$$+ \cancel{\int_d^g} + \int_{-C_1} + \cancel{\int_g^d} + \int_d^a$$

$$+ \cancel{\int_a^e} + \int_{-C_3} + \cancel{\int_e^a} = \text{Integration}$$

of an analytic fn on closed path = zero

$$\Rightarrow \int_a^b + \int_b^d + \int_d^a = -\int_{-C_2} - \int_{-C_1} - \int_{-C_3}$$

$$\Rightarrow \oint_C = \oint_{C_2} + \oint_{C_1} + \oint_{C_3}$$