Computational Biofluids Writeup

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1 Euler-Bernoulli Beam Equation

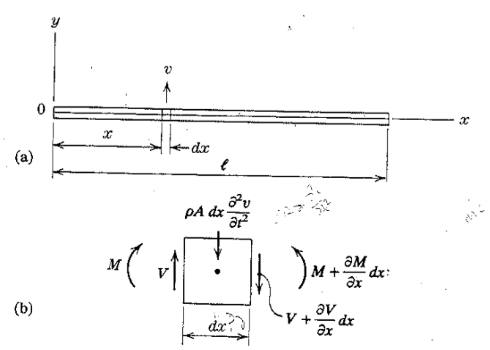

FIG. 5.13

[?]

In the graphical representation of the beam shown above, heaving movements occur in the y-direction, while the length of the beam is represented from 0 to L in the x-direction.

We begin with the equation of dynamic equilibrium condition for forces in the y-direction, represented as the sum of the forces above:

$$V - (V + \frac{\partial V}{\partial x}dx) - \rho A dx \frac{\partial^2 v}{\partial t^2} = 0 \tag{1}$$

where V is the shearing force applied to the beam, A is the area of the cross section, ρ is the density of the beam, and v is the deflection of the beam. Equation (1) describes the forces of the beam at equilibrium. We can rewrite the equation as

$$\frac{\partial V}{\partial x}dx = -\rho A dx \frac{\partial^2 v}{\partial t^2} \tag{2}$$

We introduce the bending moment equilibrium condition:

$$-Vdx + \frac{\partial M}{\partial x}dx = 0, \tag{3}$$

where M is the bending moment of the beam. If we solve for Vdx and substitute equation (3) into equation (2), we get

$$\frac{\partial^2 M}{\partial x^2}dx = -\rho A dx \frac{\partial^2 v}{\partial t^2} \tag{4}$$

We're given the relationship

$$M = EI\frac{\partial^2 v}{\partial x^2}, \tag{5}$$

where EI is flexural rigidity, which is constant in the case of a prismatic beam.

Plugging in our equation for M in equation (4), we arrive at the homogeneous version of the Euler-Bernoulli Beam Equation

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} = 0 \tag{6}$$

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To model the motion of a beam with length L, width W, and height δ in a fluid, we let $A = \delta W$, and specify ρ_p as the density of the beam, while setting the equation equal to an external force, F_{ext}. In our case, an external force would be fluid force.

$$EI\frac{\partial^4 v}{\partial x^4} + \rho_p \delta W\frac{\partial^2 v}{\partial t^2} = F_{ext} \tag{7}$$

1.1 Non-Dimensionalization of the Beam Equation

We now non-dimensionalize the Euler Bernoulli Beam Equation to separate the inertial forces from the bending forces. To begin, consider the Euler-Bernoulli Beam Equation, labeled as equation (8). Since added mass forces will dominate over inertial forces on the panel, $\rho_p \delta W$ can be rewritten as $\rho L W$. Additionally, substituting in dimensionless variables of $x^* = \frac{x}{L}$, $t^* = t\omega$, and $v^* = \frac{v}{a'}$ into the beam equation yields the following equation:

$$\frac{\partial^4 v^* a'}{\partial (x^* L)^4} + \rho L W\frac{\partial^2 v^* a'}{\partial (t^* \omega^{-1})^2} = F^*_{ext} EI a' L^{-4},$$

where a' is the trailing-edge amplitude of the beam, ω is the heaving frequency, . Simplifying the equation, we obtain

$$\frac{\partial^4 v^*}{\partial (x^*)^4} + \rho L^5 W EI^{-1} f^2 \frac{\partial^2 v^*}{\partial (t^*)^2} = F^*_{ext} \tag{8}$$

Setting

$$\Pi^2 = \rho L^5 W E I^{-1} \omega^2 \tag{9}$$

and substituting Π^2 into the Beam Equation

$$\frac{\partial^4 v^*}{\partial (x^*)^4} + \Pi^2 \frac{\partial^2 v^*}{\partial (t^*)^2} = F^*_{ext} \tag{10}$$

The Euler-Bernoulli Beam Equation now has been transformed into a non-dimensional equation. With the introduction of the effective flexibility Π, we are able to characterize the ratio of added mass forces and internal bending forces.

We would now like to solve the non-dimensionalized Euler-Bernoulli beam equation. For simplification, we assume that our variables are non-dimensionalized. We consider the homogeneous case for the Euler-Bernoulli beam equation, such that $F_{ext} = 0$:

$$\frac{\partial^4 v}{\partial x^4} + \Pi^2 \frac{\partial^2 v}{\partial t^2} = 0 \tag{11}$$

When a beam vibrates transversely in one of its natural modes, the deflection $v(x, t)$ at any time can be measured using the following equation:

$$v(x, t) = X(Acos(\omega t) + Bsin(\omega t)) \tag{12}$$

The equation for v appears to be the product of two functions, $X(x)$ and $T(t)$, where T is a temporal function of time t, and X is a spatial function of the distance x from the left-hand side of the beam. Thus we can rewrite v as

$$v = X(x)T(t)$$

Substituting v into equation (12) and simplifying yields the equation:

$$\frac{\partial^4 X}{\partial x^4} T - \Pi^2 \omega^2 T X = 0 \tag{13}$$

To help us solve the homogeneous, fourth order differential equation above, let $k^4 = \Pi^2 \omega^2$.

Therefore,

$$\frac{\partial^4 X}{\partial x^4} - k^4 X = 0.$$

If we let $X = e^{nx}$, we obtain

$$e^{nx}(n^4 - k^4) = 0$$

Solving for n, we have four solutions for $n : n_1 = k, n_2 = -k, n_3 = ik, n_4 = -ik$, where $i = \sqrt{-1}$. Thus, the general solution of our fourth order differential equation is

$$X = Ae^{kx} + Be^{-kx} + Ce^{ikx} + De^{-ikx}$$

Which, for our purposes, we can rewrite as

$$\begin{aligned} X = {} & C_1(cos(kx) + cosh(kx)) + C_2(cos(kx) - cosh(kx)) \\ & + C_3(sin(kx) + sinh(kx)) + C_4(sin(kx) - sinh(kx)) \end{aligned}$$

Since we are investigating the behavior of beams with one end fixed and one end free, we will use the boundary conditions:

$$(X)_{x=0} = 0 \quad \left(\frac{dX}{dx}\right)_{x=0} = 0 \quad \left(\frac{d^2X}{dx^2}\right)_{x=L} = 0 \quad \left(\frac{d^3X}{dx^3}\right)_{x=L} = 0$$

These conditions describe the beam's boundaries; at the leading edge of the beam ($x = 0$), the displacement X is fixed and is thus zero. The slope at the leading edge also maintains a flat shape, characteristic of a clamped leading edge. The concavity at the trailing edge of the beam is zero, and thus the change in concavity is also zero.

From the first two conditions, we notice that $C_1 = 0$ and $C_3 = 0$. We continue solving for our remaining coefficients C_2 and C_4 by taking the second derivative of X with respect to x:

$$\frac{d^2X}{dx^2} = k^2[C_2(-cos(kx) - cosh(kx)) + C_4(-sin(kx) - sinh(kx))]$$

We then evaluate $\frac{d^2X}{dx^2}$ at $x = L$ in order to solve for C_2:

$$\begin{aligned} 0 &= C_2(cos(kL) + cosh(kL)) + C_4(sin(kL) + sinh(kL)) \\ C_2 &= -C_4 \frac{sin(kL) + sinh(kL)}{cos(kL) + cosh(kL)} \end{aligned}$$

By evaluating $\left.\frac{d^3X(x)}{dx^3}\right|_{x=L}$ we are able to solve the equation for our remaining coefficient, C_4:

$$\begin{aligned} \frac{d^3X(L)}{dx^3} &= k^3[C_2(sin(kL) - sinh(kL)) + C_4(-cos(kL) - cosh(kL))] = 0 \\ 0 &= C_2(sin(kL) - sinh(kL)) + C_4(-cos(kL) - cosh(kL)) \end{aligned}$$

We can then plug C_2 into our equation and solve for C_4:

$$
\begin{aligned}
0 &= (-C_4\frac{sin(kL)+sinh(kL)}{cos(kL)+cosh(kL)})(sin(kL)-sinh(kL)) - C_4(cos(kL)+cosh(kL)) \\
0 &= C_4(\frac{sin^2(kL)-sinh^2(kL)}{cos(kL)+cosh(kL)} + cos(kL)+cosh(kL)) \\
0 &= C_4(sin^2(kL)-sinh^2(kL)+cos^2(kL)+2cosh(kL)cos(kL)+cosh^2(kL)) \\
0 &= C_4(2+2cosh(kL)cos(kL)) \\
-C_4 &= C_4cosh(kL)cos(kL) \\
-1 &= cosh(kL)cos(kL)
\end{aligned}
$$

We note that attempting to solve for C_4 using C_2 leads us to the frequency equation

$$-1 = cosh(kL)cos(kL) \tag{14}$$

Therefore, the values for each of the four weights in X are:

$$C_1 = 0 \quad C_2 = -C_4\tfrac{sin(kL)+sinh(kL)}{cos(kL)+cosh(kL)} \quad C_3 = 0 \quad C_4 = s, \forall s \in \mathbb{R}$$

1.2 Solving for k Using Newton's Method

To get closer to a numeric value for C_2, we use Newton's method to find the roots of $-1 = cosh(kL)cos(kL)$.

We begin by guessing an approximate solution k_0L of $-1 = cosh(kL)cos(kL)$, and then we plug it into the formula:

$$k_1L = k_0L - \frac{f(k_0L)}{f'(k_0L)}$$

Note that this is the formula for Newton's method, where k_0L is our initial guess, and k_1L is the next estimate for finding a root. We created an algorithm in Matlab that allows us to find a root using an initial estimate. The program iterates the function until $|k_1L - k_0L| < 1 \times 10^{-9}$.

We found 4 roots using Newton's method in Matlab:

$$k_1L = 1.875 \quad k_2L = 4.694 \quad k_3L = 7.855 \quad k_4L = 10.996$$

With these solutions, we can approximate k for our X function by dividing each root by a given length. We now have an equation for the transverse vibrations of a prismatic beam, dependent on C_4:

$$X_j = C_4\left(-\frac{sin(k_jL)+sinh(k_jL)}{cos(k_jL)+cosh(k_jL)}(cos(k_jx)-cosh(k_jx)) + (sin(k_jx)-sinh(k_jx))\right)$$

If we let $C_4 = 1$, then we have the spatial solution

$$\phi_j = -\frac{sin(k_jL) + sinh(k_jL)}{cos(k_jL) + cosh(k_jL)}(cos(k_jx) - cosh(k_jx)) + (sin(k_jx) - sinh(k_jx)))$$

We now see that the vibrations of a prismatic beam can be decomposed to the sum of infinite eigenmodes, all of which have a modal contribution dependent on t:

$$v(x,t) = \sum_{j=1}^{\infty} \phi_j(x)C_i(t).$$

In order to accurately see the effects of each modal contribution, we normalize ϕ_j to negate the scaling effect from C_4.

To normalize ϕ_j, we set $\hat{\phi}_j = \frac{\phi_j}{\sqrt{<\phi_j,\phi_j>}}$

Thus, the solution to the Euler-Bernoulli beam equation represents the deflection in terms of displacement functions $\hat{\phi}$ and time functions $C_i(t^*)$:

$$v(x,t) = \sum_{j=1}^{\infty} \hat{\phi}_j(x)C_j(t) \tag{15}$$

where ϕ_i are orthonormal eigenfunctions and C_j are the modal contributions of each eigenfunction, which is calculated using the experimental deflection, v_{exp}:

$$C_i(t) = \int_0^L v_{exp}(x,t)\hat{\phi}_j(x)dx$$

Finally, we are able to use $v(x,t)$ to analyze the modal decomposition of an experimental deflection. We did so with four values of k_j, $L = .025$, and $v_{exp} = sin(\phi x)$

2 Navier-Stokes Equations

In order to model the movement of a heaving flexible panel in fluid, we incorporate equations of fluid motion into our simulation. The equations that account for fluid motion are the Navier-Stokes equations for viscous, incompressible fluid flow. By numerically solving for the Navier-Stokes equations, we are able to compute the velocity of fluid elements surrounding our beam. This is necessary in simulating the interaction of the beam with the surrounding fluid, accomplished using the Immersed Boundary Method, which we discuss in the subsequent sections.

We begin by remarking that fluid flows will be represented in an Eulerian coordinate system, rather than in a Lagrangian coordinate system. The difference between the two is that in the Eulerian coordinate system, points are fixed in space, and fluid properties are monitored as functions of time as the flow passes those fixed spatial locations. In the Lagrangian reference frame, fluid particles are followed throughout different locations in the fluid domain. It is conventional to use the Eulerian view of fluid motion, because velocity can be measured directly at each fixed spatial coordinate, rather than calculated as the time-rate of change of a fluid particle location moving through the fluid.

The Navier-Stokes equations are derived from the equations for conservation of mass and conservation of momentum. However, rather than viewing the equation for the conservation of momentum with point masses, we consider the time rate of change of momentum of a fluid element, or $\frac{\partial u}{\partial t}$. In addition to the local acceleration of a fluid element, we also consider the convective acceleration $(u \cdot \nabla)u$, which accounts for force exerted on a particle of fluid by the other particles of fluid surrounding it. This is how the fluid accelerates in a non-uniform flow. We derive these acceleration components using the substantial derivative of fluid velocity:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u$$

The substantial derivative allows us to describe Lagrangian acceleration in an Eulerian reference frame. The forces represented in the Navier-Stokes equations are pressure forces from the fluid, viscous forces within the fluid, and additional body-forces, which in our case are the forces associated with the beam.

Thus, the form of the Navier-Stokes Equations as a fluid version of the equation of conservation of momentum is:

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\nabla p + \mu \Delta u + f(x,t) \tag{16}$$

where ρ is density, u is the fluid velocity, p is pressure, μ is the dynamic viscosity, and $f(x,t)$ are external forces acting on the fluid. The left-hand side of the equation represents the inertial terms, and the right-hand side of the equation represents the sum of pressure forces ∇p, viscous forces $\mu \Delta u$, and body-forces $f(x,t)$. We assume constant density throughout the fluid, and thus the conservation of mass implies that

$$\nabla \cdot u = 0 \tag{17}$$

This relation is the continuity equation for an incompressible flows, which sets the pressure field in an incompressible flow. Together, equations (17) and (18) constitute the Navier-Stokes Equations of incompressible flow.

At this point, we would like to non-dimensionalize the Navier-Stokes equations to identify dimensionless parameters that characterize the behavior of the

solutions of the Navier-Stokes equations. By non-dimensionalizing the Navier-Stokes equations, we determined the Reynolds Number, which gives insight into the ratio of inertial forces of the object to the forces of the fluid.
To begin we start with the dimensional Navier-Stokes Equation:

$$\rho(\frac{\partial u}{\partial t} + (u \cdot \nabla)u) = -\nabla p + \mu \triangle u \tag{18}$$

where ρ is the fluid mass density of the object, μ is the fluid dynamic viscosity, u is the velocity of the fluid, p is the pressure, and t is time.

To non-dimensionalize equation (1) we must introduce non-dimensional parameters related to the parameters listed above. Let $X' = \frac{x}{L}$ where x is the position of the fluid and L is the characteristic length. Let $t' = \frac{t}{T} = \frac{t}{\frac{L}{u}}$ where u is the characteristic velocity, Let $\nabla' = L \cdot \nabla$ Let $\triangle' = L^2 \cdot \triangle$ Let $p' = \frac{p'}{\rho U^2}$.

After performing the desired substitutions our new Navier-Stokes Equation is

$$\rho(u(\frac{u}{L}\frac{\partial u'}{\partial t'} + u^2(L^{-1})(u \cdot \nabla)u) = (L^{-1)}(\rho u^2)(-\nabla' p') + (L^{-2})(U\mu\triangle' u') \tag{19}$$

After combining and cancelling terms

$$(\frac{\partial u'}{\partial t'} + (u \cdot \nabla)u) = (-\nabla' p') + L^{-1}U^{-1}\rho\mu u^2(\triangle' u') \tag{20}$$

Recoginzing that

$$L^{-1}U^{-1}\rho\mu = \frac{1}{Re} \tag{21}$$

where Re is the Reynolds Number
Our final form of the Non-dimensional Navier Stokes Equation is

$$(\frac{\partial u'}{\delta t'} + (u \cdot \nabla)u) = (-\nabla' p') + \frac{1}{Re}u^2(\triangle' u') \tag{22}$$