Course	Number	Section(s)
Mathematics	204	All

Instructors **Course Examiner**

S.Jessup, C. Santana, A. Panait, K. Benchekroun, E.Wright, A. Johnson; A. Kokotov

Problem N1

(a). A system of six linear equations with five unknowns x_1, x_2, x_3, x_4, x_5 has two solutions

$$(x_1, x_2, x_3, x_4, x_5) = (1, 2, 3, 4, 5)$$

and

$$(x_1, x_2, x_3, x_4, x_5) = (7, 8, 9, 10, 11)\,.$$

Find a solution $(x_1, x_2, x_3, x_4, x_5)$ of this system such that $x_1 = 19$.

[**Marks= 5**]

(b). Three points A, B, C in $\mathbb{R}^3$ are given:

$$A = (1, 2, 3), \quad B = (2, 2, -1), \quad C = (3, 1, 0)\,.$$

Find the coordinates of the point D lying on the line passing through A and B such that the vector $\overrightarrow{CD}$ is orthogonal to the vector $\overrightarrow{AB}$.

[**Marks= 5**]

Problem N2

(a). Compute the determinant

$$\begin{vmatrix} 1001 & 100 & 1 & 0.02 \\ 1002 & 100 & 2 & 0.01 \\ 1003 & 100 & 1 & 0.02 \\ 1004 & 100 & 2 & 0.01 \end{vmatrix}$$

[**Marks= 5**]

(b). The system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 2 \\ ax_1 + 2x_2 + 4x_3 = b \end{cases}$$

has infinitely many solutions. Find b.

[**Marks= 5**]

Problem N3

(a). Find the coordinates of the point $A = \alpha \cap l$ of intersection of the plane α defined by the equation

$$x + 2y - z = 1$$

and the line l that is orthogonal to α and passes through the point $(1, 1, 1)$.

[Marks= 5]

(b). Find a basis of the subspace of $\mathbb{R}^4$ spanned by five vectors

$$(1, 2, 1, 1),$$

$$(1, 1, 1, 1),$$

$$(2, 3, 2, 2),$$

$$(4, 7, 4, 4),$$

and

$$(6, 10, 6, 6)\,.$$

[Marks= 5]

Problem N4

(a). The vector $\vec{U} = (a, 1, 2)$ is parallel to the plane

$$x + 2y + 3z = 1$$

Find the norm $||\vec{U}||$ of the vector $\vec{U}$.

[**Marks= 5**]

(b). Show that the normal vector to the plane

$$3x + 6y + 9z = 11$$

is an eigenvector of the matrix

$$\begin{pmatrix} 2 & 3 & -2 \\ -3 & 2 & 1 \\ 4 & -2 & 2 \end{pmatrix}$$

and find the corresponding eigenvalue.

[**Marks= 5**]

Problem N5 Find the absolute value of the cosine of the angle between the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

[Marks= 10]

Problem N6

(a). The matrices A and B

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & x \\ 3 & y \end{pmatrix}$$

are commuting (i. e. $AB = BA$). Find x and y.

[Marks= 5]

(b). Let $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. The linear transformation T in $\mathbb{R}^2$ transforms $\vec{i} - \vec{j}$ to $3\vec{i} + 2\vec{j}$ and $\vec{i} + 2\vec{j}$ to $2\vec{i} + \vec{j}$. Find the matrix of the linear transformation T.

[Marks= 5]

MATH. 204.

Final Exam Solutions. Fall 2021.

1.A. Let $X = (x_1, x_2, x_3, x_4, x_5)$.

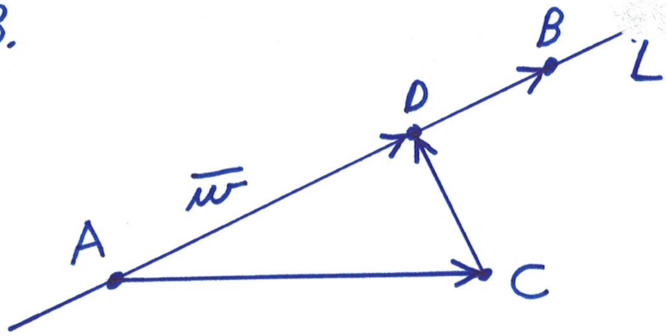
$$X = 12(1,2,3,4,5) + (7,8,9,10,11)$$

$$X = (19, 32, 45, 58, 71);$$

OR

$$X = 19(1,2,3,4,5) = (19, 38, 57, 76, 95).$$

B.



$$\overrightarrow{AC} = (2,-1,-3); \ \overrightarrow{AB} = (1,0,-4).$$

$$\overline{w} = Proj_{\overrightarrow{AB}}(\overrightarrow{AC}) = \left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|^2}\right)\overrightarrow{AB} = \left(\frac{14}{17}, 0, \frac{-56}{17}\right).$$

$$D = A + \overline{w} = \left(\frac{31}{17}, 2, \frac{-5}{17}\right).$$

$$\overrightarrow{CD} = \left(-\frac{20}{17}, 1, \frac{-5}{17}\right). \quad \overrightarrow{CD}\cdot\overrightarrow{AB} = 0.$$

2.A.

$$\begin{vmatrix} 1001 & 100 & 1 & 0.02 \\ 1002 & 100 & 2 & 0.01 \\ 1003 & 100 & 1 & 0.02 \\ 1004 & 100 & 2 & 0.01 \end{vmatrix} \xrightarrow[R_4 - R_1 \to R_4]{\substack{R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3}}$$

$$\begin{vmatrix} 1001 & 100 & 1 & 0.02 \\ 1 & 0 & 1 & -0.01 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 1 & -0.01 \end{vmatrix} = 2 \begin{vmatrix} 100 & 1 & 0.02 \\ 0 & 1 & -0.01 \\ 0 & 1 & -0.01 \end{vmatrix} = 0.$$

B.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ a & 2 & 4 & b \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & (-2a+6) & b-a \end{array}\right]$$

For infinite solutions $\Rightarrow a = 3,\ b = 3.$

3. A.

Since line is $\perp$ plane, direction vector of line is same as direction vector of plane, $(1,2,-1)$.

$$L: (x, y, z) = (1,1,1) + t(1,2,-1)$$

$x = 1+t$

$y = 1+2t$

$z = 1-t$.

Substituting into equation of plane:

$$(1+t) + 2(1+2t) - (1-t) = 1 \Rightarrow t = -\frac{1}{6}$$

$\Rightarrow$ point of intersection, $A = \left(\frac{5}{6}, \frac{4}{6}, \frac{7}{6}\right)$.

B.

$$\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \end{array} \qquad\qquad \begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 & 6 \\ 2 & 1 & 3 & 7 & 10 \\ 1 & 1 & 2 & 4 & 6 \\ 1 & 1 & 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis vectors are $\alpha_1 = (1,2,1,1)$ and

$\alpha_2 = (1,1,1,1)$.

4. A. $x+2y+3z=1$; $\bar{u}=(a,1,2)$.

Direction vector of plane: $(1,2,3)$.

Direction vector of plane is $\perp$ direction vector $\bar{u} \Rightarrow (1,2,3)\cdot(a,1,2)=0$

$$a+2+6=0$$

$$a=-8.$$

$$\|\bar{u}\| = \|(-8,1,2)\| = \sqrt{64+1+4} = \sqrt{69}.$$

B. Normal vector of plane is: $(3,6,9)$ or equivalently $(1,2,3)$.

If normal vector is an eigenvector of the given matrix, A, then it must satisfy the following equation for some λ:

$$A\bar{x} = \lambda\bar{x}$$

where λ: scalar, $\lambda \in \mathbb{R}$, eigenvector.

$$\bar{x} = (1,2,3).$$

$$A\bar{x} = \lambda\bar{x}$$

$$\begin{bmatrix} 2 & 3 & -2 \\ -3 & 2 & 1 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \lambda = 2 \Rightarrow \bar{v} \text{ is an eigenvector}$$

and λ is the corresponding eigenvalue of A.

5. Eigenvalues and Eigenvectors.

$$\lambda_1 = -1 \qquad \lambda_2 = 5$$

$$\bar{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \bar{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$|\cos\theta| = \left| \frac{\bar{x}_1 \cdot \bar{x}_2}{\|\bar{x}_1\| \, \|\bar{x}_2\|} \right| = \frac{1}{\sqrt{10}}.$$

6. A. $AB = BA$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & x\\3 & y\end{bmatrix} = \begin{bmatrix}2 & x\\3 & y\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$$

$$\begin{bmatrix}8 & (x+2y)\\18 & (3x+4y)\end{bmatrix} = \begin{bmatrix}(2+3x) & (4+4x)\\(3+3y) & (6+4y)\end{bmatrix} \Rightarrow \begin{array}{l}x=2\\y=5.\end{array}$$

B. $\bar{i}-\bar{j} = (1,-1)$; $3\bar{i}+2\bar{j} = (3,2)$.

$\bar{i}+2\bar{j} = (1,2)$; $2\bar{i}+\bar{j} = (2,1)$.

Let T be the transformation matrix.

$$T = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

$$T\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix};\ T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

$$\left.\begin{array}{l}\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix} \Rightarrow \begin{array}{l}a-b=3\\c-d=2\end{array}\\ \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \Rightarrow \begin{array}{l}a+2b=2\\c+2d=1\end{array}\end{array}\right\} \begin{array}{l}a=8/3\\b=-1/3\\c=5/3\\d=-1/3\end{array}$$

$\therefore$, $$T = \frac{1}{3}\begin{bmatrix}8 & -1\\5 & -1\end{bmatrix}.$$