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Math 204 midterm 2023 winter solution

Vectors and Matrices (Concordia University)

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Midterm Math 204 - March 12, 2023

1. $\left[\begin{array}{cccc|c} 1 & 1 & -6 & -4 & 6 \\ 3 & -1 & -6 & -4 & 2 \\ 2 & 3 & 9 & 2 & 6 \\ 3 & 2 & 3 & 8 & -7 \end{array}\right] \xrightarrow[R_4' = R_4 - 3R_1]{R_2' = R_2 - 3R_1,\ R_3' = R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -6 & -4 & 6 \\ 0 & -4 & 12 & 8 & -16 \\ 0 & 1 & 21 & 10 & -6 \\ 0 & -1 & 21 & 20 & -25 \end{array}\right] \xrightarrow[R_4' = R_4 + R_3]{R_2' = -\frac{1}{4}R_2}$

$\left[\begin{array}{cccc|c} 1 & 1 & -6 & -4 & 6 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 1 & 21 & 10 & -6 \\ 0 & 0 & 42 & 30 & -31 \end{array}\right] \xrightarrow[R_1' = R_1 - R_2]{R_3' = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 2 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & 24 & 12 & -10 \\ 0 & 0 & 42 & 30 & -31 \end{array}\right] \xrightarrow{R_3' = \frac{1}{24}R_3}$

$\left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 2 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{12} \\ 0 & 0 & 42 & 30 & -31 \end{array}\right] \xrightarrow{R_4' = R_4 - 42R_3} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 2 \\ 0 & 1 & -3 & -2 & 4 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{12} \\ 0 & 0 & 0 & 9 & -\frac{27}{2} \end{array}\right]$

Back-substitution:

$$\begin{cases} x_1 - 3x_3 - 2x_4 = 2 & \text{---(1)} \\ x_2 - 3x_3 - 2x_4 = 4 & \text{---(2)} \\ x_3 + \frac{1}{2}x_4 = -\frac{5}{12} & \text{---(3)} \\ 9x_4 = -\frac{27}{2} & \text{...(4)} \end{cases}$$

From (4), $x_4 = -\frac{3}{2}$

(3) $x_3 + \frac{1}{2}\cdot\left(-\frac{3}{2}\right) = -\frac{5}{12}$

$x_3 = -\frac{5}{12} + \frac{3}{4}$

$= \frac{1}{3}$

(2) $x_2 = 4 + 3x_3 + 2x_4$

$= 4 + 3\left(\frac{1}{3}\right) + 2\left(-\frac{3}{2}\right)$

$= 4 + 1 - 3$

$= 2$

(1) $x_1 = 2 + 3x_3 + 2x_4$

$= 2 + 3\left(\frac{1}{3}\right) + 2\left(-\frac{3}{2}\right) = 2 + 1 - 3 = 0$

Soln: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ \frac{1}{3} \\ -\frac{3}{2} \end{pmatrix}$

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2. a) Find the inverse of A.

$$A = \begin{bmatrix} 3 & 8 & 9 & 6 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 \\ 3 & 8 & 9 & 8 \end{bmatrix}$$

easier to do inversion algorithm than $\frac{1}{\det} \cdot \operatorname{adj}(A)$.

$$\left[\begin{array}{cccc|cccc} 3 & 8 & 9 & 6 & 1 & 0 & 0 & 0 \\ 0 & 2 & 6 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 & 0 & 0 & 1 & 0 \\ 3 & 8 & 9 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_3' = \frac{1}{3}R_3 \\ R_2' = \frac{1}{2}R_2}]{R_4' = R_4 - R_1} \left[\begin{array}{cccc|cccc} 3 & 8 & 9 & 6 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2' = R_2 - R_4 \\ R_3' = R_3 - R_4}]{R_1' = R_1 - 3R_4}$$

$$\left[\begin{array}{cccc|cccc} 3 & 8 & 9 & 0 & 4 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2' = R_2 - 3R_3 \\ R_4' = \frac{1}{2}R_4}]{R_1' = R_1 - 9R_3} \left[\begin{array}{cccc|cccc} 3 & 8 & 0 & 0 & -5 & 0 & -3 & 6 \\ 0 & 1 & 0 & 0 & -2 & \frac{1}{2} & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1' = R_1 - 8R_2}$$

$$\left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 11 & -4 & 5 & -10 \\ 0 & 1 & 0 & 0 & -2 & \frac{1}{2} & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1' = \frac{1}{3}R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{11}{3} & -\frac{4}{3} & \frac{5}{3} & -\frac{10}{3} \\ 0 & 1 & 0 & 0 & -2 & \frac{1}{2} & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{11}{3} & -\frac{4}{3} & \frac{5}{3} & -\frac{10}{3} \\ -2 & \frac{1}{2} & -1 & 2 \\ 1 & 0 & \frac{1}{3} & -1 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

b) $$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X = X \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$3IX = XA + B$$
$$X \cdot 3I = XA + B$$
$$X(3I - A) = B$$
$$X = B(3I - A)^{-1} \longrightarrow X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$3I - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$(3I - A)^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-2 & -2-2 \\ -3-4 & -6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ -7 & -10 \end{bmatrix} //$$

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3. $$\begin{vmatrix} 1303 & 100 & 0.01 & 39 \\ 1304 & 100 & 0.02 & 40 \\ 1305 & 100 & 0.01 & 41 \\ 1306 & 100 & 0.02 & 42 \end{vmatrix} \xrightarrow{C_2' = \frac{1}{100} C_2} 100 \begin{vmatrix} 1303 & 1 & 0.01 & 39 \\ 1304 & 1 & 0.02 & 40 \\ 1305 & 1 & 0.01 & 41 \\ 1306 & 1 & 0.02 & 42 \end{vmatrix} \xrightarrow[\substack{C_3' = C_3 - 0.01C_2 \\ C_4' = C_4 - 39C_2}]{C_1' = C_1 - 1303C_2}$$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0.01 & 1 \\ 2 & 1 & 0 & 2 \\ 3 & 1 & 0.01 & 3 \end{vmatrix}$$

since $C_1 = C_4$, the determinant will be 0 because we can create a column of 0's like so:

$$\xrightarrow{C_4' = C_4 - C_1} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0.1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0.1 & 0 \end{vmatrix}$$

$\therefore$ det $= 0$.

4. $$y = 2 = \frac{\det(A_2)}{\det(A)} \quad \text{where}$$

$$\det(A_2) = \begin{vmatrix} 1 & 1 & 6 \\ 1 & A & 7 \\ 1 & -1 & 4 \end{vmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{vmatrix} 1 & 1 & 6 \\ 0 & A-1 & 1 \\ 0 & -2 & -2 \end{vmatrix} = 1 \cdot \begin{vmatrix} A-1 & 1 \\ -2 & -2 \end{vmatrix}$$

$$= -2(A-1) - (-2)$$

$$= -2A + 2 + 2$$

$$= -2A + 4$$

$$\det(A) = \begin{vmatrix} 1 & 5 & 6 \\ 1 & 4 & 7 \\ 1 & 5 & 4 \end{vmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{vmatrix} 1 & 5 & 6 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{vmatrix} = 1 \times (-1)(-2) = 2.$$

$$2 = \frac{-2A + 4}{2}$$

$$4 = -2A + 4$$

$$A = 0$$

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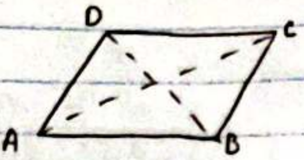
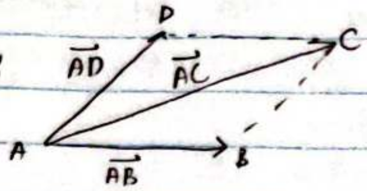
5.



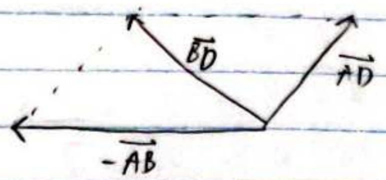
we know that $\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AD}\|^2 = 12$

we need to find $\|\overrightarrow{AC}\|^2 + \|\overrightarrow{BD}\|^2$.

Notice that:



$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ ... (1)



$(-\overrightarrow{AB}) + \overrightarrow{AD} = \overrightarrow{BD}$ ... (2)

Using dot product, $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$. Thus:

(1) $\|\overrightarrow{AC}\|^2 = \overrightarrow{AC} \cdot \overrightarrow{AC}$

$= (\overrightarrow{AB} + \overrightarrow{AD}) \cdot (\overrightarrow{AB} + \overrightarrow{AD})$

$= \overrightarrow{AB} \cdot \overrightarrow{AB} + \overrightarrow{AB} \cdot \overrightarrow{AD} + \overrightarrow{AD} \cdot \overrightarrow{AB} + \overrightarrow{AD} \cdot \overrightarrow{AD}$

$= \|\overrightarrow{AB}\|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \|\overrightarrow{AD}\|^2$

(2) $\|\overrightarrow{BD}\|^2 = \overrightarrow{BD} \cdot \overrightarrow{BD}$

$= (-\overrightarrow{AB} + \overrightarrow{AD}) \cdot (-\overrightarrow{AB} + \overrightarrow{AD})$

$= -\overrightarrow{AB} \cdot -\overrightarrow{AB} - \overrightarrow{AB} \cdot \overrightarrow{AD} - \overrightarrow{AD} \cdot \overrightarrow{AB} + \overrightarrow{AD} \cdot \overrightarrow{AD}$

$= \|\overrightarrow{AB}\|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \|\overrightarrow{AD}\|^2$

So, $\|\overrightarrow{AC}\|^2 + \|\overrightarrow{BD}\|^2 = \|\overrightarrow{AB}\|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \|\overrightarrow{AD}\|^2 +$

$\|\overrightarrow{AB}\|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{AD} + \|\overrightarrow{AD}\|^2$

$= 2(\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AD}\|^2)$

$= 2(12)$

$= 24$ //

important note: The diagonals are not equal to each other,

$\|\overrightarrow{AC}\| \neq \|\overrightarrow{BD}\|$

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(6.) Find E_1, E_2, E_3, E_4 so that

$$E_4 E_3 E_2 E_1 \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 7 & 2 & 0 \\ 5 & 8 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{B}$$

This is equal to reducing A to B. See chapter 1.5!

$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 2 & 0 \\ 5 & 8 & 3 \end{bmatrix} \xrightarrow[ERO_1]{R_1 \leftrightarrow R_2} \begin{bmatrix} 7 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & 8 & 3 \end{bmatrix} \xrightarrow[ERO_2]{R_1' = R_1 - 7R_2} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 5 & 8 & 3 \end{bmatrix} \xrightarrow[ERO_3]{R_3' = R_3 - 5R_2}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow[ERO_4]{R_3' = R_3 - 4R_1} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

. Based on E's definition, then

$$\left.\begin{array}{l} I_3 \xrightarrow{ERO_1} E_1 \\ I_3 \xrightarrow{ERO_2} E_2 \\ I_3 \xrightarrow{ERO_3} E_3 \\ I_3 \xrightarrow{ERO_4} E_4 \end{array}\right\}$$

So, $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

* solutions are not unique.

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So, $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & -7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

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