MATH204 Midterm WITH Solutions Winter 2022

Vectors and Matrices (Concordia University)

Math 204, Class test, March 13, 2022

Course Examiner: A. Kokotov

Instructors: Carlos-Luis Santana, Stephanie Saputra, Hamid Pezeshk, Thomas Hughes, Maryam Jabbari Khasraghi

Time: 1 Hour and 30 minutes

Answer all questions. **Only approved calculators are allowed**

1. **(10 points)** Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ such that $p(-1) = 0$, $p(1) = 4$, $p(2) = 3$ and $p(3) = 16$.

2. **(10 points)**

   (A) Find the inverse matrix A^{-1} if

$$A = \begin{pmatrix} 5 & 2 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   (B) Solve the following equation for matrix X:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} X = X + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

3. **(10 points)** Compute the determinant

$$\begin{vmatrix} 1001 & 100 & 0.01 & 39 \\ 1002 & 100 & 0.03 & 40 \\ 1003 & 100 & 0.03 & 40 \\ 1004 & 100 & 0.01 & 41 \end{vmatrix}$$

4. **(10 points)** Using Cramer's rule, find a number A such that the system

$$\begin{cases} x + y + z = 1 \\ x + 2y + 2z = 1 \\ x + y + 3z = A \end{cases}$$

has a solution (x, y, z) with $y = 7$. (You don't need to find x and z.)

5. **(10 points)** Find the area of the triangle with vertices $A = (1, 2, 3)$, $B = (2, 4, 7)$ and $C = (3, 3, 4)$.

6. **(10 points)** Find elementary matrices E_1, E_2 and E_3 such that

$$E_3 E_2 E_1 \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

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1.

$$-a+b-c+d=0$$
$$a+b+c+d=4$$
$$8a+4b+2c+d=3$$
$$27a+9b+3c+d=16$$

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 4 \\ 8 & 4 & 2 & 1 & 3 \\ 27 & 9 & 3 & 1 & 16 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow$$

$a=2$

$b=-5$

$c=0$

$d=7.$

2. A.

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

B. $$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X = X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X - X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} - I_2 \right) X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{-1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}.$$

$\therefore$)

$$-\frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} X = -\frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$I_2 X = -\frac{1}{2}\begin{bmatrix} -5 & -6 \\ -1 & -2 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}.$$

3.

$$\begin{vmatrix} 1001 & 100 & 0.01 & 39 \\ 1002 & 100 & 0.03 & 40 \\ 1003 & 100 & 0.03 & 40 \\ 1004 & 100 & 0.01 & 41 \end{vmatrix} \xrightarrow[-R_1+R_4 \to R_4]{\substack{-R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}}$$

$$\begin{vmatrix} 1001 & 100 & 0.01 & 39 \\ 1 & 0 & 0.02 & 1 \\ 2 & 0 & 0.02 & 1 \\ 3 & 0 & 0 & 2 \end{vmatrix} =$$

$$(-1)^{(1+2)} 100 \begin{vmatrix} 1 & 0.02 & 1 \\ 2 & 0.02 & 1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -100(-0.04) = 4.$$

4. $|A| = 2.$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & A & 3 \end{bmatrix}.$$

$|A_2| = 1 - A.$

$\therefore$

$$y = 7 = \frac{|A_2|}{|A|} = \frac{1-A}{2}$$

$$7 = \frac{1-A}{2} \Rightarrow A = -13.$$

5. $\overrightarrow{AB} = (1,2,4)$; $\overrightarrow{AC} = (2,1,1)$.

area of Δ: $\text{area}_{\Delta} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\|(-2,7,-3)\| = \frac{\sqrt{62}}{2} = \text{area}_{\Delta}.$$

6. $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_1]{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow[E_2]{-3R_2 + R_3 \to R_3}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow{-4R_1 + R_3 \to R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$