Concordia University
ENGR 233: Applied Advanced Calculus
Midterm #1 – Section T (Winter 2023)

2023-02-23

Name: ______________________________

Student ID: ______________________________

This exam consists of 1 page and 5 questions, worth a total of 100 points. Books and notes are not allowed. A formula sheet is provided. Only ENCS-approved calculators are allowed. Submit all sheets and papers at the end of the exam.
You have 60 minutes to complete the exam. Good luck!

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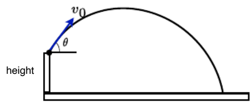
1. (20 points) Consider the two planes in 3-space given by $x - y - 2z = 1$ and $x - 2y - z = 5$.
   (a) Find the intersection of the planes and write its equation.
   (b) Write the equation for the plane passing through the point (2, 2, 1) that is perpendicular to the intersection of the two planes. Sketch the plane in the first octant.

2. (20 points) A shell is fired from the top of a building with a height of 40 m, at an angle of 45°, with a speed of 40 m/s (only consider the force of gravity (g=10m/s^2) and an ideal situation; assume shell doesn't move/roll when it hits the ground, no air friction, flat ground, etc.).
   (a) Find the range of the shell and maximum altitude of the shell from the ground.
   (b) Determine if the shell hits wall that is located 50 m horizontally from its fired point and has a height of 30 m (wall has negligible thickness). Justify your answer.



3. (20 points) Find the equation of the tangent plane at point $(0, 0, 4)$ and the equation of the normal line at point $(0, 2\pi/3, 2)$ to the equation $z = 4e^{2\pi x}\cos(y/2)$.

4. (20 points) For a moving particle given by $x = \cos t$, $y = \sin t$, and $z = t$, find the vectors **T**, **N**, and **B**. What is the curvature κ?

5. (20 points) Answer **only** two parts in this question.
   (a) For the scalar function $f(x, y, z) = x - \cos x + 2y^2 - 4y + z^3 - z$, find all points at which $\|\nabla f\| = 0$.
   (b) A vector filed **F** is said to be incompressible if $\nabla \cdot \mathbf{F} = 0$. Given $\mathbf{F} = (axy + xy + bx^2z^2 + 3x)\mathbf{i} + (by^2 + aczy^2 - 2y)\mathbf{j} + (3az^2y - z + z^3x)\mathbf{k}$, determine the non-zero possible values of the constants a, b and c such that **F** is incompressible.
   (c) Calculate the curl for the vector field $\mathbf{F} = (yz\ln x)\mathbf{i} + (yxe^{-z})\,\mathbf{j} + (x\cos(yz))\,\mathbf{k}$

Solutions of Midterm I   Winter 2023   ENGR 233:

Q1. a) $n_1 = k n_2 \Rightarrow$ they intersect, the intersection is a line!

Let's consider $z = t$

$$\begin{cases} x - y - 2t = 1 \\ x - 2y - t = 5 \end{cases} \Rightarrow \begin{array}{l} \circledast \; x - y = 1 + 2t \xrightarrow{\times(-1)} \\ x - 2y = t + 5 \end{array} \quad \oplus \begin{array}{l} -x + y = -(1 + 2t) \\ x - 2y = t + 5 \\ \hline -y = -t + 4 \Rightarrow y = t - 4 \end{array}$$

$\circledast$
$$x = y + (1 + 2t)$$
$$x = 3t - 3$$

line: $(3, 1, 1)t + (-3, -4, 0)$

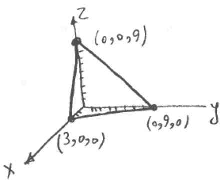
or

$$\frac{x+3}{3} = \frac{y+4}{1} = \frac{z}{1}$$

b) $\begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = -3i - j - k \Rightarrow$ plane Eq. $\Rightarrow n \cdot (r - r_0) = 0$, $r_0 \to (2, 2, 1)$

$\Rightarrow -3(x - 2) - 1(y - 2) - 1(z - 1) = 0$

$-3x + 6 - y + 2 - z + 1 = 0 \Rightarrow 3x + y + z = 9$



Q2.

$\vec{v}(t) = v_0 \cos\theta\, i + (v_0 \sin\theta - gt)\, j$

$\vec{r}(t) = v_0 \cos\theta\, t\, i + \left(-\frac{g t^2}{2} + v_0 \sin\theta t + \underbrace{h_0}_{\text{initial height} = 40m}\right) j$

$\vec{a} = -gt\, j$

$v_0 = 40\,m/s$
$h_0 = 40\,m$
$\theta = 45°$

a) max. altitude $\Rightarrow v_y = 0 \Rightarrow v_0 \sin\theta = gt \Rightarrow t_{\text{highest alt.}} = \frac{v_0 \sin\theta}{g}$ (t_h)

$\Rightarrow r_y(t_h) = \left(-\frac{g t_h^2}{2} + v_0 \sin\theta\, t_h + h_0\right) = -\frac{g}{2}\left(\frac{v_0 \sin\theta}{g}\right)^2 + \frac{(v_0 \sin\theta)(v_0 \sin\theta)}{g} + h_0$

$\Rightarrow r_y(t_h) = h_0 + \frac{v_0^2 \sin^2\theta}{2g}$ $\Rightarrow$ max altitude $= 40 + \frac{40 \sin^2 45}{2g}$ $\xrightarrow{g=9.81\,m/s^2}$ $\boxed{80.77\,m}$ or $\xrightarrow{g=10\,m/s^2}$ $\boxed{80\,m}$

$r_x(t_r) = ? \Leftarrow$ range: $r_y(t_r) = 0 \Rightarrow -\frac{g t_r^2}{2} + t_r v_0 \sin\theta + h_0 = 0 \Rightarrow t_r = \frac{-v_0 \sin\theta \pm \sqrt{(v_0 \sin\theta)^2 - 4(-\frac{g}{2})}}{2(-\frac{g}{2})}$

$\Rightarrow t_r = \frac{v_0 \sin\theta + \sqrt{(v_0 \sin\theta)^2 + 2gh_0}}{g}$ $\xrightarrow{g=9.81}$ $t_r = 6.94\,s$; $\xrightarrow{g=10}$ $t_r = 6.8284\,s$

$r_x(t_r) = v_0 \cos\theta\, t = 193.1371\,m$ $(g = 10)$
196.3286 $(g = 981)$

b) if $r_x = 50 \Rightarrow r_y = ?$ $\Rightarrow$ then compare it with the height of the wall.

$r_x = 50\,m \Rightarrow v_0 \cos\theta\, t = 50\,m \Rightarrow t_b = \frac{50}{v_0 \cos\theta} = 1.7678\,s$

$\Rightarrow r_y(t_b) = -\frac{g t_b^2}{2} + v_0 \sin\theta\, t_b + h_0 =$ $74.6719\,m$ $(g = 9.81\,m/s^2)$
74.3750 $(g = 10\,m/s^2)$

It doesn't hit the wall as $r_y(t_b) = 74.3750\,m > \underbrace{30\,m}_{\text{height of the wall}}$

Q3

tangent plane: ∇f = vector $f = 4e^{2\pi x}\cos y/2 - z = 0$

f scalare

$\Rightarrow \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$= 8\pi e^{2\pi x}\cos y/2\, \hat{i} + \frac{4}{2}(-1)\, e^{2\pi x}\sin y/2\, \hat{j} - 1k$

$\Rightarrow \nabla f(0,0,4) = \langle 8\pi, 0, -1\rangle \Rightarrow 8\pi(x-0) + 0(y-0) + (-1)(z-4) = 0$

$\Rightarrow 8\pi x - z + 4 = 0 \Rightarrow \underline{8\pi x - z = -4}$

normal line $\nabla f(0, 2\pi/3, 2) = 8\pi\cos\pi/3\, \hat{i} - 2\sin\pi/3\, \hat{j} - k$

$= 4\pi\hat{i} - \sqrt{3}\,\hat{j} - k$

$\frac{x-0}{4\pi} = \frac{y - 2\pi/3}{-\sqrt{3}} = \frac{z-2}{-1}$ or $\vec{r} = (4\pi, -\sqrt{3}, -1)t + (0, +2\pi/3, 2)$

Q.4 $\vec{r} = \cos t\, i + \sin t\, \hat{j} + t\, k \Rightarrow r'(t) = -\sin t\, \hat{i} + \cos t\, \hat{j} + k \Rightarrow \|r'(t)\| = \sqrt{2}$

$T = -\frac{\sin t}{\sqrt{2}}\hat{i} + \frac{\cos t}{\sqrt{2}} j + \frac{1}{\sqrt{2}} k \Rightarrow T'(t) = -\frac{\cos t}{\sqrt{2}} i - \frac{\sin t}{\sqrt{2}}\hat{j} + 0k$

$\Rightarrow \|dT/dt = T'(t)\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{1}{2}} = \sqrt{2}/2$

$N = \frac{dT/dt}{\|dT/dt\|} = -\cos t\,\hat{i} - \sin t\,\hat{j}$

$B = T \times N = \begin{vmatrix} i & j & k \\ -\sin t/\sqrt{2} & -\cos t/\sqrt{2} & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\sin t\, i - \frac{1}{\sqrt{2}}\cos t\,\hat{j} + \frac{1}{\sqrt{2}} k$

$K = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\sqrt{2}/2}{\sqrt{2}} = 1/2$

Q.5 $\|\nabla f\| = 0 \Rightarrow \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = 0$

a) $\|\nabla f\| = 0 \Rightarrow \frac{\partial f}{\partial x} = 0 \quad 1 + \sin x = 0 \Rightarrow x = 2k\pi + \frac{3\pi}{2}\ k \in \mathbb{Z}$ ($\sin x = -1$)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y - 4 = 0 \Rightarrow y = 1$

$\frac{\partial f}{\partial z} = 0 \Rightarrow 3z^2 - 1 = 0 \Rightarrow z = \pm \frac{1}{\sqrt{3}}$

all points with the coordinates of

$\left(\underset{k \in \mathbb{Z}}{2k\pi + 3\pi/2},\ 1,\ \pm \frac{1}{\sqrt{3}}\right)$

b) $\operatorname{div} F = 0 \Rightarrow \nabla \cdot F = 0 \Rightarrow \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$

$P i + Q j + R k$

$\Rightarrow \nabla \cdot F = 0 \Rightarrow ay + y + 2bxz^2 + \cancel{3} + 2by + 2aczy - \cancel{2}$

$+ 6azy - \cancel{1} + 3z^2 x = 0$

$a + 1 + 2b = 0 \Rightarrow a = 2$

$3 + 2b = 0 \Rightarrow b = -3/2$

$6a = -2ac \overset{a \neq 0}{\Rightarrow} c = -3$

given in the question!

c) $\operatorname{Curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\ln x & yxe^{-z} & x\cos yz \end{vmatrix} = i\left(\frac{\partial}{\partial y}R - \frac{\partial Q}{\partial z}\right) - j\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) =$

$F = Pi + Qj + Rk$ ($P = yz\ln x$, $Q = yxe^{-z}$, $R = x\cos yz$)

$\Rightarrow \left[-xz\sin yz - (-yxe^{-z})\right] i - j\left[\cos yz - y\ln x\right] + k\left(ye^{-z} - z\ln x\right)$

$\Rightarrow \left[-xz\sin yz + yxe^{-z}\right] i + \left[y\ln x - \cos yz\right] j + k\left(ye^{-z} - z\ln x\right)$