Solved Final Exam April 2017

Vectors and Matrices (Concordia University)

CONCORDIA UNIVERSITY
Department of Mathematics and Statistics

Course	Number	Section(s)
Mathematics	204	All - Except EC

Examination	Date	Pages
Final	April 2017	3

Instructors	Course Examiner
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Special Instructions: Only approved calculators are allowed !

1. **[10]** Solve the system of equations

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 1 \\ 3x_1 - x_2 - x_3 + 2x_4 = 3 \\ 2x_1 + 6x_2 + 4x_3 - x_4 = 11 \\ -5x_1 + x_2 - x_3 + 7x_4 = 2 \\ 2x_1 - 4x_2 + x_3 + 5x_4 = 4 \end{cases}$$

2. **[10]** Solve the following equation for (3×3) matrix X:

$$X - \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

3. **[10]** For which a the following matrix

$$\begin{pmatrix} a+1 & 1 & 1 & 1 \\ a+2 & a & 2 & 2 \\ a+2 & 1 & a+2 & 2 \\ 2a+3 & 1 & 1 & a+3 \end{pmatrix}$$

is not invertible?

4. **[10]**

Let the planes α and β be given by equations

$$(\alpha) \quad x + y - z = 1$$

and

$$(\beta) \quad x - y + 2z = -1$$

and let P be the point $(2, 1, 1)$.

(A) Write down the equation of the plane passing through the point P and the line $\mathcal{L}$ of intersection of α and β.

(B) Find the coordinates of the intersection of the line $\mathcal{L}$ and the plane

$$2x - 5y + z = 4$$

5. **[10]**

Find a for which all three vectors $(1, 2, 3)$, $(1, 2 + a, 6)$ and $(1, 10, a - 1)$ are parallel to the same plane.

6. **[10]** Find the value of the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 1 & 4 \\ 2 & 3 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 2 & 0 & 4 & 5 \\ 7 & 7 & 5 & 6 & 9 \end{vmatrix}$$

7. [10] Find a basis of the subspace of $\mathbb{R}^4$ generated by the vectors $(1,1,1,1)$, $(1,2,3,4)$, $(2,3,4,5)$, $(3,5,7,9)$ and $(4,6,8,10)$.

8. [10] Find a basis for the solution space of the homogeneous system

$$\begin{cases} x_1 + x_2 + 3x_3 + x_4 - x_5 = 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 - x_6 = 0 \\ 4x_1 + 5x_2 + 5x_3 + 4x_4 - 2x_5 - x_6 = 0 \end{cases}$$

9. [10] Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

Is A diagonalizable?

10. [10] Find an invertible 2×2 matrix P such that the matrix

$$P^{-1} \begin{pmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix} P$$

is diagonal.

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1. x = y = z = w = 1.
2.

$$X = \begin{bmatrix} -1 & -0.6 & -1 \\ 0 & -0.6 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}$$

5. The scalar triple product of the three vectors must be 0.

Let: $\bar{u}$, $\bar{v}$, $\bar{w}$ be the three vectors.

$(\bar{u} \times \bar{v}) \cdot \bar{w} = 0.$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & (2+a) & 6 \end{vmatrix} = (6 - 3a, -3, a).$$

$(1, 10, a - 1) \cdot (6 - 3a, -3, a) = a^2 - 4a - 24.$

$a^2 - 4a - 24 = 0$

$a = 2 \pm 2\sqrt{7}\,.$

3. $$\begin{vmatrix} (a+1) & 1 & 1 & 1 \\ (a+2) & a & 2 & 2 \\ (a+2) & 1 & (a+2) & 2 \\ (2a+3) & 1 & 1 & (a+3) \end{vmatrix} \xrightarrow[\substack{-R_1+R_3 \to R_3 \\ -R_1+R_4 \to R_4}]{-aR_1+R_2 \to R_2}$$

$$\begin{vmatrix} (a+1) & 1 & 1 & 1 \\ (2-a^2) & 0 & (2-a) & (2-a) \\ 1 & 0 & (a+1) & 1 \\ (a+2) & 0 & 0 & (a+2) \end{vmatrix} =$$

$$(1)\begin{vmatrix} (2-a^2) & (2-a) & (2-a) \\ 1 & (a+1) & 1 \\ (a+2) & 0 & (a+2) \end{vmatrix} =$$

$-a^4-2a^3+a^2+2a=0$

$-a(a^3+2a^2 \mp a \mp 2)=0$

$-a(a+1)(a-1)(a+2)=0$

$a=0; -1; 1; -2.$

Matrix is not invertible when determinant equals 0.

4. $\alpha : x + y - z = 1 \qquad P(2,1,1).$

$\beta : x - y + 2z = -1$

A.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right] \Rightarrow \begin{array}{l} x = -\frac{1}{2} z \\ y = \frac{3}{2} z + 1 \end{array}$$

Let $z = t$

$x = -\frac{1}{2} t$

$y = 1 + \frac{3}{2} t$

$$\overline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix}$$

$\therefore\ \mathcal{L}: \bar{X} = (0, 1, 0) + \left(-\frac{1}{2}, \frac{3}{2}, 1\right)t = Q + \bar{d}t.$

Plane:

$\overrightarrow{QP} = (2-0, 1-1, 1-0) = (2, 0, 1).$

$\bar{d} = \left(-\frac{1}{2}, \frac{3}{2}, 1\right) = (-1\ 3\ 2).$

$\bar{d} \times \overrightarrow{QP} = \bar{n} = \begin{vmatrix} -1 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = (3, +5, -6).$

$X = (x, y, z).$

Plane: $\bar{n} \cdot (X - P) = 0$

$$(3, 5, -6) \cdot (x-2, y-1, z-1) = 0$$

$$3(x-2) + 5(y-1) + (-6)(z-1) = 0$$

$$3x + 5y - 6z = 5.$$

B. $2x - 5y + z = 4.$

$\mathcal{L}$: $x = -\frac{1}{2}t$

$y = 1 + \frac{3}{2}t$

$z = t.$

$$2\left(-\frac{1}{2}t\right)-5\left(1+\frac{3}{2}t\right)+(t)=4$$

$$-t-5-\frac{15}{2}t+t=4$$

$$-\frac{15}{2}t=9$$

$$-15t=18$$

$$t=-\frac{18}{15}=-\frac{6}{5}.$$

$\therefore$, Point of intersection: $\left(\frac{3}{5},-\frac{4}{5},-\frac{6}{5}\right).$

6. $\begin{vmatrix} 1 & 1 & 2 & 1 & 4 \\ 2 & 3 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 2 & 0 & 4 & 5 \\ 7 & 7 & 5 & 6 & 9 \end{vmatrix} \xrightarrow{-C_3+C_4 \to C_4}$

$$\begin{vmatrix} 1 & 1 & 2 & -1 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 4 & 5 \\ 7 & 7 & 5 & 1 & 9 \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & -1 & 3 \\ 2 & 2 & 4 & 5 \\ 7 & 7 & 1 & 9 \end{vmatrix} \xrightarrow[-7R_1+R_4 \to R_4]{-2R_1+R_2 \to R_2,\ -2R_1+R_3 \to R_3}$$

$$\begin{vmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 8 & -19 \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 & -5 \\ 0 & 6 & -3 \\ 0 & 8 & -19 \end{vmatrix} =$$

$$(1)\begin{vmatrix} 6 & -3 \\ 8 & -19 \end{vmatrix} = -90.$$

7.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 & 6 \\ 1 & 3 & 4 & 7 & 8 \\ 1 & 4 & 5 & 9 & 10 \end{bmatrix} \xrightarrow[-R_1+R_4 \to R_4]{\substack{-R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 4 & 4 \\ 0 & 3 & 3 & 6 & 6 \end{bmatrix} \xrightarrow[-3R_2+R_4 \to R_4]{\substack{-R_2+R_1 \to R_1 \\ -2R_2+R_3 \to R_3}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis $\bar{v}_1 ; \bar{v}_2$.

8. $$\begin{array}{cccccc} x & y & z & u & v & w \end{array}$$
$$\left[\begin{array}{cccccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 2 & 3 & -1 & 2 & 0 & -1 \\ 4 & 5 & 5 & 4 & -2 & -1 \end{array}\right] \xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -7 & 0 & 2 & -1 \\ 0 & 1 & -7 & 0 & 2 & -1 \end{array}\right] \xrightarrow[-R_2+R_1 \to R_1]{-R_2+R_3 \to R_3}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 10 & 1 & -3 & 1 \\ 0 & 1 & -7 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$$

$x = -10z - u + 3v - w$

$y = +7z \quad -2v + w$

Let $z = s$

$u = t$

$v = a$

$w = b$

$$X = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = s\begin{bmatrix} -10 \\ 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\therefore$) Basis: $\begin{bmatrix} -10 \\ 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

9. $\det(\lambda I - A) = 0$

$$\det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 4 & (\lambda-4) & 0 \\ 2 & -1 & (\lambda-2) \end{bmatrix}\right) = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$(\lambda-2)^3 = 0$$

$$\lambda = 2.$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ 2 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1/2 & 0 \\ 4 & -2 & 0 \\ 2 & -1 & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x = \frac{1}{2} y \\ \text{Let } y = s \\ \quad z = t \end{array}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t.$$

Eigenvectors: $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

A is not diagonalizable since the three λ's are not distinct.

10.

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{pmatrix}1 & \sqrt{6}\\ \sqrt{6} & 2\end{pmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}(\lambda-1) & -\sqrt{6}\\ -\sqrt{6} & (\lambda-2)\end{bmatrix}\right) = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda-4)(\lambda+1) = 0$$

$$\lambda = -1; 4.$$

$\lambda = -1$:

$$\begin{pmatrix}-2 & -\sqrt{6}\\ -\sqrt{6} & -3\end{pmatrix} \longrightarrow \begin{pmatrix}1 & \frac{\sqrt{6}}{2}\\ 0 & 0\end{pmatrix} \Rightarrow x = \frac{-\sqrt{6}}{2} y$$

Let $y = t$

$$X = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\frac{-\sqrt{6}}{2}\\ 1\end{bmatrix} t.$$

$\lambda = 4$:

$$\begin{pmatrix} 3 & -\sqrt{6} \\ -\sqrt{6} & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{\sqrt{6}}{3} \\ 0 & 0 \end{pmatrix} \Rightarrow x = \frac{\sqrt{6}}{3} y$$

Let $y = t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ 1 \end{bmatrix} t.$$

$$P = \begin{bmatrix} -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{3} \\ 1 & 1 \end{bmatrix}; \quad P^{-1} = -\frac{\sqrt{6}}{5} \begin{bmatrix} 1 & -\frac{\sqrt{6}}{3} \\ -1 & -\frac{\sqrt{6}}{2} \end{bmatrix}.$$

$$P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$