MATH 205 Final Exam - December 2018

Differential & Integral Calculus II (Concordia University)

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	205	All
Examination	**Date**	**Pages**
Final	December 2018	2
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Special	Only approved calculators are allowed	
Instructions:	**Show all your work for full marks.**	

MARKS

[10] **1.** **a.** Sketch the graph of the function

$$f(x) = \begin{cases} 3x & x \le 1 \\ x\dfrac{|x-5|}{x-5} & 1 < x < 3 \\ -3 & x \ge 3 \end{cases}$$

and find the definite integral $\int_0^5 f(x)\,\mathrm{d}x$ in terms of area (**do not antidifferentiate**).

b. Use the Fundamental Theorem of Calculus to calculate the derivative of

$$F(x) = \int_0^{1-x^2} (1-t)\,e^{-t^2}\,\mathrm{d}t\ ,$$

and determine whether F is increasing or decreasing at $x = 1$.

[15] **2.** Find the following indefinite integrals:

(**a**) $\displaystyle\int \frac{\sin^3(x)}{\cos^5(x)}\,\mathrm{d}x$ (**b**) $\displaystyle\int (2x+x^2)\cos(2x)\,\mathrm{d}x$ (**c**) $\displaystyle\int \frac{x^2-8}{x^2-16}\mathrm{d}x$

[18] **3.** Evaluate the following definite integrals (give the exact answers):

(**a**) $\displaystyle\int_0^{\ln 2} \frac{e^x}{e^{2x}+4}\mathrm{d}x$ (**b**) $\displaystyle\int_0^{\pi/4} \frac{\sec^2(x)}{\sqrt{1+8\tan(x)}}\,\mathrm{d}x$ (**c**) $\displaystyle\int_1^{e^2} x\ln x\,\mathrm{d}x$

[8] **4.** Evaluate the given improper integral or show that it diverges:

(**a**) $\displaystyle\int_e^{\infty} \frac{\mathrm{d}x}{x\,[\ln(x)]^{3/2}}$ (**b**) $\displaystyle\int_0^{1} \frac{\mathrm{d}x}{(1-x)^{5/4}}$

[16] **5.** **a.** Sketch the curves $y = x^3 - x$ and $y = 3x$, and find the area enclosed.

b. Find the volume of a solid obtained by rotating the region bounded by the curve $y = \sin(x)$ and the $x-$axis on the interval $0 \leq x \leq \pi$ about the line $y = 2$.

c. Find the exact average value of $f(x) = \sqrt{9 - x^2}$ on the interval $[-3, 3]$.

[6] **6.** Find the limit of the sequence $\{a_n\}$ at $n \to \infty$ or prove that it does not exist:

(**a**) $a_n = \dfrac{3^n + (-3)^n}{4^n}$ (**b**) $a_n = \ln(1 + 3n + 4n^2) - \ln(8 + 6n + 2n^2)$

[12] **7.** Determine whether the series is divergent or convergent, and if convergent, whether absolutely or conditionally :

(**a**) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ (**b**) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2 + 1/10)^n}{(2 - 1/10)^n}$ (**c**) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n^3 + n}}{n^2}$

[6] **8.** Find **(a)** the radius of convergence, and **(b)** the interval convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{4^n\, n^2}$.

[9] **9.** **(a)** Use the integrability of the power series to express the function

$F(x) = \displaystyle\int_0^x \left(\sum_{n=1}^{\infty} n\, t^{n-1} \right) \mathrm{d}t$ as an elementary function

(i.e. sum the series for $F(x)$ within the radius of its convergence).

(b) Find the MacLaurin series for the function $f(x) = x^3 \sin(x^2)$.
(Hint: start with the series for $\sin z$ then replace z by x^2)

[5] **Bonus question.** If we know that $\sum_{n=1}^{\infty} a_n$ converges and each $a_n \neq 0$, can anything be said about the series $\sum_{n=1}^{\infty} 1/a_n$ - i.e. does it converge or diverge? Explain your answer.

Final solutions Dec. 2018

*Problem 1 :

(1a) Sketch the graph

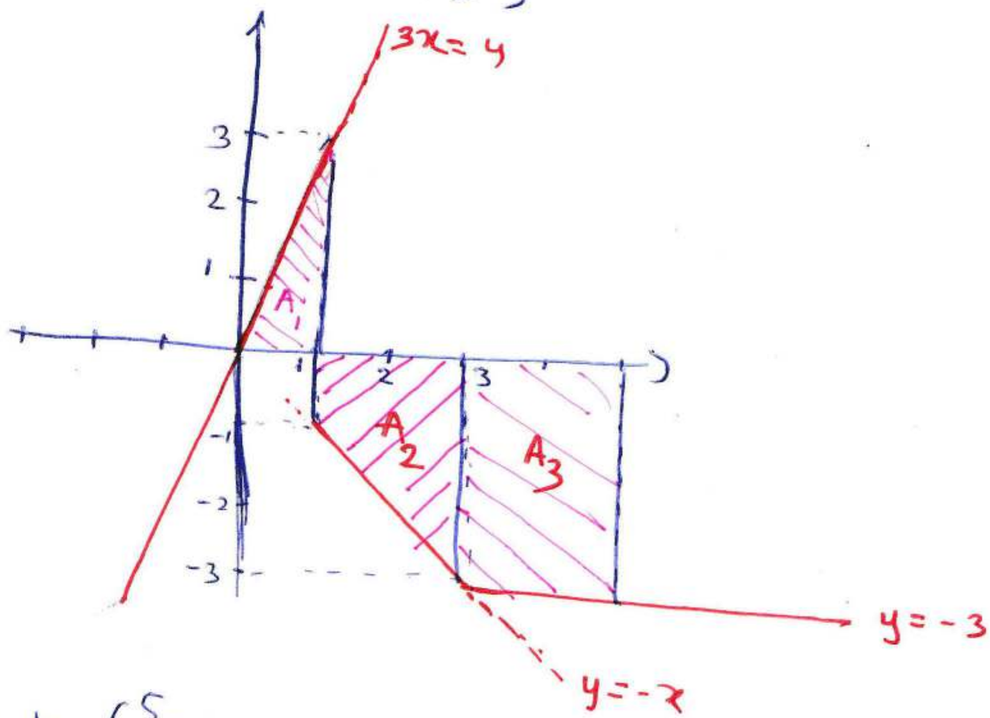
$$f(x) = \begin{cases} 3x & x \le 1 \\ x\dfrac{|x-5|}{x-5} & 1 < x < 3 \\ -3 & x \ge 3 \end{cases}$$

We have

$$|x-5| = \begin{cases} x-5 & x \ge 5 \\ -(x-5) & x \le 5 \end{cases}$$

So

$$f(x) = \begin{cases} 3x & x \le 1 \\ -x\dfrac{\cancel{(x-5)}}{\cancel{(x-5)}} & 1 < x < 3 \\ -3 & x \ge 3 \end{cases} \Leftrightarrow \begin{cases} 3x & \text{if } x \le 1 \\ -x & \text{if } 1 < x < 3 \\ -3 & \text{if } x \ge 3 \end{cases}$$



Evaluate $\int_0^5 f(x)\,dx$

$$\int_0^5 f(x)\,dx = A_1 + A_2 + A_3$$

$A_1 = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}$

$A_2 = \frac{1}{2} \cdot (1+3) \cdot 2 = 4$

$A_3 = 2 \times 3 = 6$

$\int_0^5 f(x)dx = \frac{3}{2} - 4 - 6 = -\frac{17}{2}$

$$\boxed{\int_0^5 f(x)dx = -\frac{17}{2}}$$

(1b) Use FTC to calculate the derivative of $F(x)$

$$F(x) = \int_0^{1-x^2} (1-t) e^{-t^2} dt$$

FTC: $\left(\int_0^x f(x)\right)' = f(x)$

So apply FTC on $F(x)$

$$F'(x) = \left(\int_0^{1-x^2} (1-t) e^{-t^2} dt\right)'$$

$$= \left[1 - (1-x^2)\right] e^{-(1-x^2)^2} \cdot (1-x^2)'$$

$$= \cancel{x^2 e^{(-x^2+1)^2} \cdot (-2x)} \quad x^2 e^{-(1-x^2)^2} (-2x)$$

$$\boxed{F'(x) = -2x^3 e^{-(x^2-1)^2}}$$

$F'(1) = -2 \cdot 1 e^0 = -2 < 0$

So $\boxed{F(x) \text{ is decreasing at } x = 1}$ //

* **Problem 2** : Find the indefinite integrals

(2a) $I = \int \frac{\sin^3(x)}{\cos^5(x)} dx$

$= \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^2(x)} dx$

$= \int \tan^3(x) \sec^2(x) dx$

let $u = \tan x \Rightarrow du = \sec^2(x) dx$

Using substitution u

$I = \int u^3 du = \frac{1}{4} u^4 + C$

$I = \frac{1}{4} \tan^4(x) + C$

(2b) $I = \int (2x + x^2) \cos(2x) dx$

Using Integration by parts

let $u = 2x + x^2 \Rightarrow du = (2 + 2x) dx$

$dv = \cos(2x) \Rightarrow v = \frac{\sin(2x)}{2}$

$I = uv - \int v du$

$= (2x + x^2) \frac{\sin(2x)}{2} - \int 2(1+x) \cdot \frac{\sin(2x)}{2} dx$

$= (2x + x^2) \frac{\sin(2x)}{2} - \underbrace{\int (1+x) \sin(2x) dx}_{I_1}$

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let $I_1 = \int (1+x)\sin(2x)\,dx$

using integration by parts again

let $u = 1+x \Rightarrow du = dx$

$dv = \sin(2x) \Rightarrow v = -\frac{\cos(2x)}{2}$

$$I_1 = -(1+x)\frac{\cos(2x)}{2} - \int -\frac{\cos(2x)}{2}\,dx$$

$$= -\frac{(1+x)}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$$

$$= -\frac{(1+x)}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$$

So

$$I = (2x+x^2)\frac{\sin(2x)}{2} - \left[-\frac{(x+1)}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C\right]$$

$$I = \frac{(2x+x^2)}{2}\sin(2x) + \frac{(x+1)}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C$$

(1c) $I = \int \frac{x^2-8}{x^2-16}\,dx$

$= \int \frac{x^2-16+8}{x^2-16}\,dx = \int \left[1 + \frac{8}{x^2-16}\right] dx$

$= x + 8\int \frac{1}{x^2-16}\,dx$

let $I_1 = \int \frac{1\,dx}{x^2-16} = \int \frac{1}{(x-4)(x+4)}\,dx$

$\frac{1}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4} = \frac{A(x+4)+B(x-4)}{(x-4)(x+4)}$

$\Rightarrow \quad 1 = (A+B)x + 4A - 4B$

$\Rightarrow \begin{cases} A+B=0 \\ 4A-4B=1 \end{cases} \Rightarrow A = \frac{1}{8},\ B = -\frac{1}{8}$

$I_1 = \int \frac{1}{(x-4)(x+4)}\,dx = \int \left[\frac{1}{8(x-4)} - \frac{1}{8(x+4}\right] dx$

$= \frac{1}{8}\ln|x-4) - \frac{1}{8}\ln|x+4|$

So $I = x + 8\left[\frac{1}{8}\ln|x-4| - \frac{1}{8}\ln|x+4|\right] + C$

$I = x + \ln(x-4) - \ln|x+4| + C$

$I = x + \ln\frac{|x-4|}{|x+4|} + C$

* Problem 3 : Evaluate the definite integrals

3a $\quad I = \displaystyle\int_0^{\ln 2} \frac{e^x}{e^{2x}+4}\,dx$

Using substitution

let $u = e^x \Rightarrow du = e^x dx$

$$\int \frac{e^x}{e^{2x}+4}\,dx = \int \frac{du}{u^2+4} = \int \frac{du}{4\left[\left(\frac{u}{2}\right)^2+1\right]}$$

$$= \frac{1}{4}\int \frac{du}{\left(\frac{u}{2}\right)^2+1}$$

$$= \frac{2}{4}\cdot \tan^{-1}\left(\frac{u}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + C$$

$$\Rightarrow I = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right)\Big]_0^{\ln 2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{e^{\ln 2}}{2}\right) - \tan^{-1}\left(\frac{e^0}{2}\right)\right]$$

$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$I = \frac{1}{2}\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

$\#$

(3b) $I = \int_0^{\pi/4} \frac{\sec^2(x)}{\sqrt{1+8\tan x}} dx$

let $u = 1 + 8\tan(x) \Rightarrow du = 8\sec^2(x) dx$

substitute u in to the integral

$$\int \frac{\sec^2(x) dx}{\sqrt{1+8\tan x}} = \int \frac{du}{8 \cdot \sqrt{u}} = \frac{1}{8}\int u^{-\frac{1}{2}} du$$

$$= \frac{2}{8} u^{\frac{1}{2}} = \frac{1}{4}\sqrt{1+8\tan x}$$

So $I = \frac{1}{4}\sqrt{1+8\tan x}\Big]_0^{\pi/4}$

$= \frac{1}{4}\left[\left(\sqrt{1+8\tan\frac{\pi}{4}}\right) - \left(\sqrt{1+8\tan 0}\right)\right]$

$= \frac{1}{4}(\sqrt{9} - \sqrt{1}) = \frac{1}{2}$

$I = \frac{1}{2}$

(3c) $I = \int_1^{e^2} x\ln(x) dx$

using integration by parts

let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x \Rightarrow v = \frac{x^2}{2}$

So $\int x\ln(x) = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \frac{x^2}{2} \cdot \ln x - \frac{1}{2}\int x dx$

$$\int x \ln x = \frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C$$

$$I = \left.\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2\right]_1^{e^2}$$

$$= \left(\frac{e^2}{2}\ln(e) - \frac{1}{4}(e^2)^2\right) - \left(\frac{1^2}{2}\ln(1) - \frac{1}{4}1^2\right]$$

$$I = \frac{e^2}{2} - \frac{e^4}{4} + \frac{1}{4}$$

* Problem 4 Evaluate the improper integrals

(4a) $I = \int_{e}^{\infty} \frac{dx}{x(\ln x)^{3/2}}$

since the upper bound is ∞ it is the improper integral type I

$I = \lim_{b \to \infty} \int_{e}^{b} \frac{dx}{x(\ln x)^{3/2}}$

Evaluate $\int_{e}^{b} \frac{dx}{x(\ln x)^{3/2}}$ by substitution

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$\int \frac{dx}{x(\ln(x))^{3/2}} = \int \frac{du}{u^{3/2}} = \int u^{-3/2} du$

$= -2 u^{-\frac{1}{2}} = -2(\ln x)^{-\frac{1}{2}}$

$I = \lim_{b \to \infty} -\frac{2}{\sqrt{\ln(x)}} \Big]_{e}^{b}$

$= -2 \lim_{b \to \infty} \left(\frac{1}{\sqrt{\ln b}} - \frac{1}{\sqrt{\ln e}} \right)$

$= -2 \lim_{b \to \infty} \left(\frac{1}{\sqrt{\ln b}} - 1 \right) = -2(0-1)$

$I = 2$ conv

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(46) $I = \int_0^1 \frac{dx}{(1-x)^{5/4}}$

The integrand has a discont' at $x=1$

So $I = \lim_{c \to 1} \int_0^c \frac{dx}{(1-x)^{5/4}}$

$I_1 = \int \frac{dx}{(1-x)^{5/4}}$

let $u = 1-x \Rightarrow du = -dx$

$I_1 = -\int \frac{du}{u^{5/4}} = -\int u^{-5/4} du$

$= 4u^{-\frac{1}{4}} = 4(1-x)^{-\frac{1}{4}}$

$I = \lim_{c \to 1} \frac{4}{(1-x)^{\frac{1}{4}}} \Big]_0^c$

$= 4 \lim_{c \to 1} \left(\frac{1}{(1-c)^{\frac{1}{4}}} - \frac{1}{(1-0)^{\frac{1}{4}}} \right) = 4(\infty - 1)$

$I = \infty$ so I div

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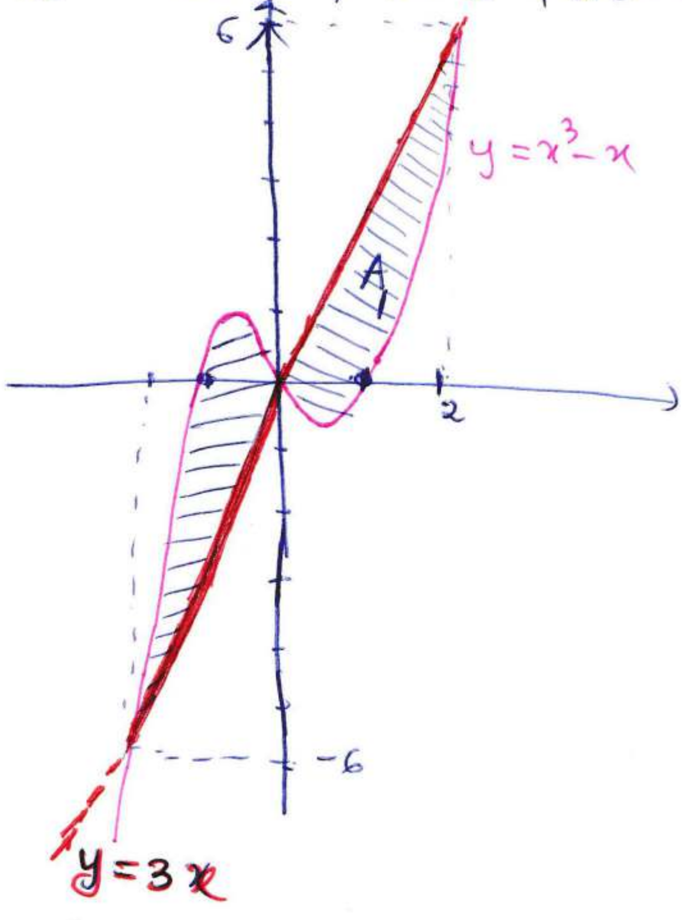
* Problem 5

(5a) Sketch & Find the area enclosed by $y = x^3 - x$ and $y = 3x$

* The intersection points

$$x^3 - x = 3x$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = x(x-2)(x+2) = 0$$

So $x = 0$, $x = 2$, $x = -2$



$$y = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$$

the zero of $y = x^3 - x$ is at $x = 0$, $x = 1$, $x = -1$

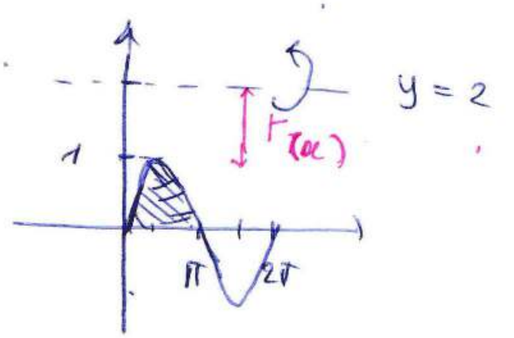
Since the solid is symmetric with respect to x-axis

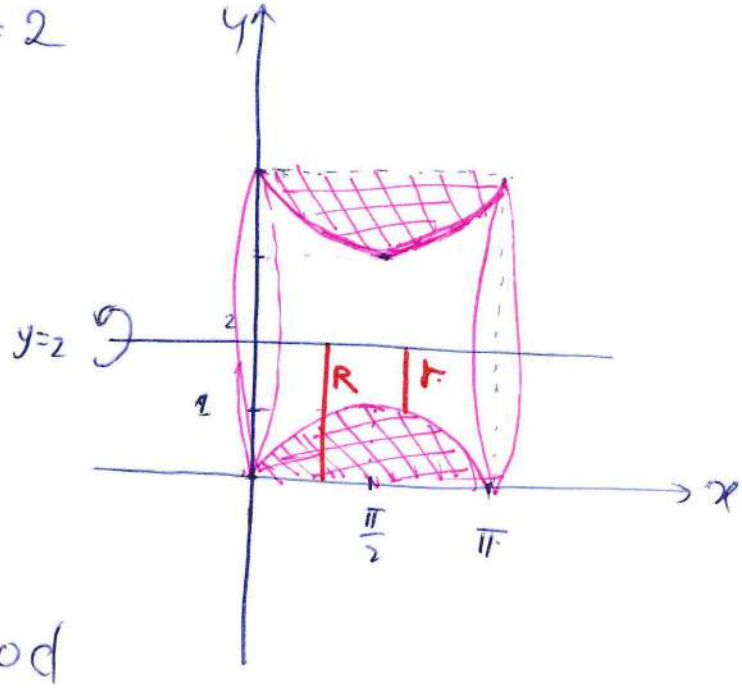
$$A = 2A_1 = 2\int_0^6 [3x - (x^3 - x)]\,dx$$

$$= 2\int_0^6 3x - x^3 + x\,dx = 2\left(\frac{3}{2}x^2 - \frac{x^4}{4} + \frac{x^2}{2}\right)\Big]_0^6$$

$$\boxed{A = 8}$$

(5b) Find the volume of the solid obtained by $y = \sin(x)$, x-axis, $0 \le x \le \pi$ about the line $y = 2$





Using washer method

$$r(x) = [2 - \sin(x)]$$

$$R(x) = 2$$

$$V = \pi\int_0^\pi [R^2_{(x)} - r^2_{(x)}]\,dx$$

$$= \pi\int_0^\pi [2^2 - (2 - \sin(x))^2]\,dx$$

$$V = \pi \int_0^{\pi} \left[4 - \left(4 - 4\sin x + \sin^2(x)\right)\right] dx$$

$$= \pi \int_0^{\pi} 4\sin(x) - \sin^2(x)\, dx$$

$$= \pi \left[\left[-4\cos(x)\right]_0^{\pi} - \int_0^{\pi} \sin^2(x)\, dx \right]$$

$$= \pi \left[-4(-1-1) - \int_0^{\pi} \frac{1-\cos 2x}{2}\, dx \right]$$

$$= 8\pi - \frac{\pi}{2} \cdot \int_0^{\pi} (1-\cos 2x)\, dx$$

$$= 8\pi - \frac{\pi}{2} \left(x - \frac{1}{2}\sin 2x\right)\Big]_0^{\pi}$$

$$= 8\pi - \frac{\pi}{2} \left[\pi - \frac{1}{2}\sin(2\pi) - 0 + 0\right]$$

$$= 8\pi - \frac{\pi}{2}(\pi)$$

$$V = 8\pi - \frac{\pi^2}{2}$$

(5c) Find the ave. of $f(x) = \sqrt{9-x^2}$ on $[-3,3]$

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx = \frac{1}{3+3} \int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{6} \int_{-3}^{3} \sqrt{9-x^2}\, dx$$

$\sqrt{9-x^2}$ is an even function

$$\Rightarrow f_{ave} = \frac{1}{6} \cdot 2 \int_0^3 \sqrt{9-x^2}\, dx = \frac{1}{3} \int_0^3 \sqrt{9-x^2}\, dx$$

let $I = \int_0^3 \sqrt{9-x^2}\, dx$

using trig. substitution

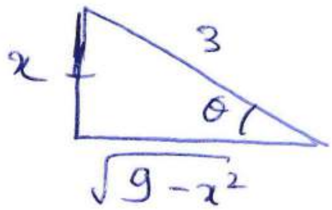
let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$$\int \sqrt{9-x^2}\, dx = \int \sqrt{9-9\sin^2\theta}\; 3\cos\theta\, d\theta$$

$$= 9\int |\cos\theta| \cdot \cos\theta\, d\theta$$

$$= 9\int \cos^2\theta\, d\theta = 9\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{9}{2}(\theta + \sin\theta\cos\theta)$$



$x = 3\sin\theta$

$\Rightarrow \sin\theta = \frac{x}{3}$

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}$$

$$I = \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)\Bigg]_0^3$$

$$= \frac{9}{2}\left(\sin^{-1}(1) + 0 - \sin^{-1}(0) + 0\right)$$

$$= \frac{9}{2}\cdot\frac{\pi}{2} = \frac{9\pi}{4}$$

$$f_{ave} = \frac{1}{3} I = \frac{1}{3}\cdot\frac{9\pi}{4} = \frac{3\pi}{4}$$

$$f_{ave} = \frac{3\pi}{4}$$

***Problem 6** Find the limit

6a $\quad a_n = \dfrac{3^n + (-3)^n}{4^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n + \left(\frac{-3}{4}\right)^n = 0+0 = \boxed{0}$$

6b $\quad a_n = \ln(1+3n+4n^2) - \ln(8+6n+2n^2)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{(1+3n^2+4n^2)}{8+6n+2n^2}$$

$$= \ln \lim_{n\to\infty} \frac{1+3n^2+4n^2}{8+6n+2n^2} = \ln \frac{4}{2} = \boxed{\ln 2}$$

*Problem 7: Determine whether the series conv/div?

7a) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Using the integral test

let $f(x) = \frac{1}{x\ln(x)}$

$$f'(x) = -\frac{(x\ln(x))'}{(x\ln(x))^2}$$

$$= -\frac{\ln(x) + \cancel{x}\cdot\frac{1}{\cancel{x}}}{(x\ln(x))^2} = -\frac{\ln(x)+1}{[x\ln(x)]^2}$$

$\ln(x) > 0 \quad \forall x \geq 2$

$\Rightarrow f'(x) < 0 \quad \forall x \geq 2$

$\Rightarrow f(x)$ is decreasing $\forall x \geq 2$

$f(x)$ also continuon $\forall x \geq 2$

$\Rightarrow$ consider

$$\int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln(x)} dx$$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$\int \frac{1}{x\ln x} dx = \int \frac{du}{u} = \ln u = \ln(\ln(x)$$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln x} dx = \lim_{b\to\infty} \ln(\ln(x]_2^b$$

$$= \lim_{b\to\infty} \left[\ln(\ln(b)) - \ln(\ln 2) \right]$$

$$= \infty$$

So $\int_2^{\infty} \frac{1}{x\ln x} dx$ div.

by the integral test

$\sum \frac{1}{n\ln(n)}$ div as well

(7b) $$\sum_{n=1}^{\infty} \frac{\left(-2+\frac{1}{10}\right)^n}{\left(2-\frac{1}{10}\right)^n}$$

$$= \sum \frac{(-1)^n \cancel{\left(2-\frac{1}{10}\right)^n}}{\cancel{\left(2-\frac{1}{10}\right)^n}} = \sum (-1)^n$$

$\sum(-1)^n$ div. by the n^{th} term test for div. series since

$\lim_{n\to\infty} (-1)^n$ do not exist

$$\Rightarrow \sum \frac{\left(-2+\frac{1}{10}\right)^n}{\left(2-\frac{1}{10}\right)^n} \text{ div}$$

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(7c) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n^3+n}}{n^2}$

Using alternating serie test

let $a_n = \frac{\sqrt{n^3+n}}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n^3+n}}{n^2}$

$= \lim_{n\to\infty} \frac{\cancel{n}\sqrt{n+\frac{1}{n}}}{n^{\cancel{2}}} = \lim_{n\to\infty} \frac{\sqrt{n+\frac{1}{n}}}{n}$

$= \lim_{n\to\infty} \sqrt{\frac{n+\frac{1}{n}}{n^2}} = \lim \sqrt{\frac{1}{n}+\frac{1}{n^3}} = 0$

$\lim_{n\to\infty} a_n = 0$

$a_n = \frac{\sqrt{n^3+n}}{n^2}$ is a decreasing sequence

let $f(x) = \frac{\sqrt{x^3+x}}{x^2} = \sqrt{\frac{1}{x}+\frac{1}{x^3}} = \left(\frac{1}{x}+\frac{1}{x^3}\right)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x^3}\right)^{-\frac{1}{2}}\left(\frac{1}{x}+\frac{1}{x^3}\right)'$

$= \frac{1}{2\left(\frac{1}{x}+\frac{1}{x^3}\right)^{\frac{1}{2}}}\left(-\frac{1}{x^2} - 3x^{-4}\right)$

$F'(x) = \frac{1}{2\left(\frac{1}{x}+\frac{1}{x^3}\right)^{\frac{1}{2}}}\left(-\frac{1}{x^2} - \frac{1}{x^4}\right) \leq 0 \quad \forall x$

So $F(x)$ is decrea. function

$\Rightarrow$ sequence a_n decreasing as well

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By the alternating series test

$$\sum (-1)^n \frac{\sqrt{n^3+n}}{n^2} \text{ conv.}$$

Is it ab. conv. or conditionally conv.

consider $\sum \left| (-1)^n \frac{\sqrt{n^3+n}}{n^2} \right| = \sum \frac{\sqrt{n^3+n}}{n^2}$

using direct comparison test

We have

$$\forall n, \quad \frac{\sqrt{n^3+n}}{n^2} \geq \frac{\sqrt{n^3}}{n^2} = \frac{n^{\frac{3}{2}}}{n^2} = \frac{1}{n^{2-\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$$

$\sum \frac{1}{n^{\frac{1}{2}}}$ div by p. series

$\Rightarrow \sum \frac{\sqrt{n^3+n}}{n^2}$ div as well

So.

$\sum (-1)^n \frac{\sqrt{n^3+n}}{n^2}$ conditionally conv

since the series of the absolute value of the term div

* Problem 8 Find R and I (interval of conv) of

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{4^n n^2}$$

using ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{(x-2)^{n+1}}{4^{n+1}(n+1)^2} \cdot \frac{4^n . n^2}{(x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-2}{4} \frac{n^2}{(n+1)^2} \right| = \frac{|x-2|}{4}$$

the serie conv if $\frac{|x-2|}{4} < 1$

$\Rightarrow |x-2| < 4$

So $R = 4$

$|x-2| < 4$

$-4 < x-2 < 4$

$-2 < x < 6$ is the interval of conv.

* Check endpoints

$x = -2 \Rightarrow \sum \frac{(x-2)^n}{4^n n^2} = \sum \frac{(-1)^n \cancel{4^n}}{\cancel{4^n} n^2} = \sum \frac{(-1)^n}{n^2}$ conv by alternating series test

$x = 4 \Rightarrow \sum \frac{(x-2)^n}{4^n n^2} = \sum \frac{4^n}{4^n n^2} = \sum \frac{1}{n^2}$ conv. by P. series

So the interval of conv. is

$-2 \leq x \leq 6$

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* **Problem 9** : Use integrability of power series

9a $F(x) = \int_0^x \left(\sum_{n=1}^{\infty} n t^{n-1}\right) dt$

By the integrability of power series

$\Rightarrow F(x) = \sum_{n=1}^{\infty} \int_0^x n t^{n-1} dt$

$= \sum_{n=1}^{\infty} \cancel{n} \cdot \frac{1}{\cancel{n}} t^n \Big]_0^x$

$= \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ where $|x| < 1$

$F(x) = \frac{x}{1-x} \quad |x| < 1$

9b Find the Maclaurin series of $f(x)$

$f(x) = x^3 \sin(x^2)$

consider. $g(t) = \sin t$

$g(t) = \sin(t) \Rightarrow g(0) = 0$

$g'(t) = \cos(t) \Rightarrow g'(0) = 1$

$g''(t) = -\sin(t) \Rightarrow g''(0) = 0$

$g^{(3)}(t) = -\cos(t) \Rightarrow g^{(3)}(0) = -1$

$g^{(4)}(t) = \sin(t) \Rightarrow g^{(4)}(0) = 0$

The patterns repeate here.

the Maclaurin series is

$$\sin(t) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}\cancel{t}^2 + \frac{f'''(0)}{3!}\cancel{t}^3 + \cdots \frac{f^n(0)}{n!}\cancel{x}^n$$

$$= 0 + \cancel{t} + 0 - \frac{\cancel{t}^3}{3!} + 0 + \frac{\cancel{t}^5}{5!} + 0 - \frac{x^7}{7!} + \cdots + \frac{(-1)^n \cancel{t}^{2n+1}}{(2n+1)!}$$

$$= \cancel{t} - \cancel{t}^3 + \cancel{t}^5 - \cancel{t}^7 + \cdots + \frac{(-1)^n \cancel{t}^{2n+1}}{(2n+1)!}$$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^2)^{2n+1}}{(2n+1)!}$$

$$\Rightarrow x^3 \sin(x^2) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum \frac{(-1)^n x^{4n+2} \cdot x^3}{(2n+1)!}$$

$$x^3 \sin(x^2) = \sum \frac{(-1)^n x^{4n+5}}{(2n+1)!}$$