MATH205 Final EXAM AND Solutions August 2018

Differential & Integral Calculus II (Concordia University)

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section**
Mathematics	205	CA
Examination	**Date**	**Pages**
Final	August 2018	2
Instructors:	**T. Hughes**	**Course Examiners** A. Atoyan & H. Proppe
Special Instructions:	Only calculators approved by the Department are allowed. **For full marks show all your work.**	

1. a) *[5 marks]* Sketch the graph of the function

$$f(x) = \begin{cases} x + 6 & -7 \le x < -3 \\ 3 - \sqrt{9 - x^2} & -3 \le x \le 3 \end{cases}$$

on the interval $-6 \le x \le 3$ and calculate the definite integral $\int_{-6}^{3} f(x)dx$ in terms of *signed* area (do not antidifferentiate).

b) *[5 marks]* Use the Fundamental Theorem of Calculus, Part 1, to find $F'(x)$, given that

$$F(x) = \int_{x^2}^{x^3} \frac{\sin t}{t}\, dt$$

2. *[10 marks]* Evaluate the following indefinite integrals.

a) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$ b) $\int \frac{3x^2 + x + 1}{x\,(x^2+1)}\, dx$

3. *[10 marks]* Evaluate the following definite integrals (do not approximate):

a) $\int_0^{\frac{\pi}{4}} \sec^4 x\, dx$ b) $\int_0^2 x^2\sqrt{4 - x^2}\, dx$

4. *[10 marks]* Evaluate the given improper integral, or show that it diverges.

a) $\int_2^{\infty} \frac{1}{x\ln^2 x}\, dx$ b) $\int_2^3 \frac{dx}{\sqrt{3-x}}$

5. a) *[6 marks]* Find the area bounded by the curves $x = y$ and $4x + y^2 = 12$

b) *[6 marks]* Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$, about the line $y = 1$.

c) *[6 marks]* Find the average value of the function $f(x) = \frac{\ln x}{x^2}$ from $x = 1$ to $x = 2$.

6. *[9 marks]* Find the limit of the sequence $\{a_n\}$ as $n \to \infty$ or prove that it does not exist:

a) $a_n = \cos\frac{2n\pi}{5+6n}$ b) $a_n = \frac{(\ln n)^2}{n}$ c) $a_n = (1 + \frac{3}{n})^n$

7. *[8 marks]* Determine whether the series is divergent or convergent. If it is convergent, then determine whether it is conditionally or absolutely convergent:

a) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$ b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

8. *[6 marks]* Find (a) the radius and (b) the interval of convergence of the series

$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$

9. a) *[4 marks]* Find the Maclaurin series for $f(x) = x^2 e^{x^3}$

b) *[3 marks]* i) Express the series as a function of t. $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$

[2 marks] ii) Find the exact numerical sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!}$

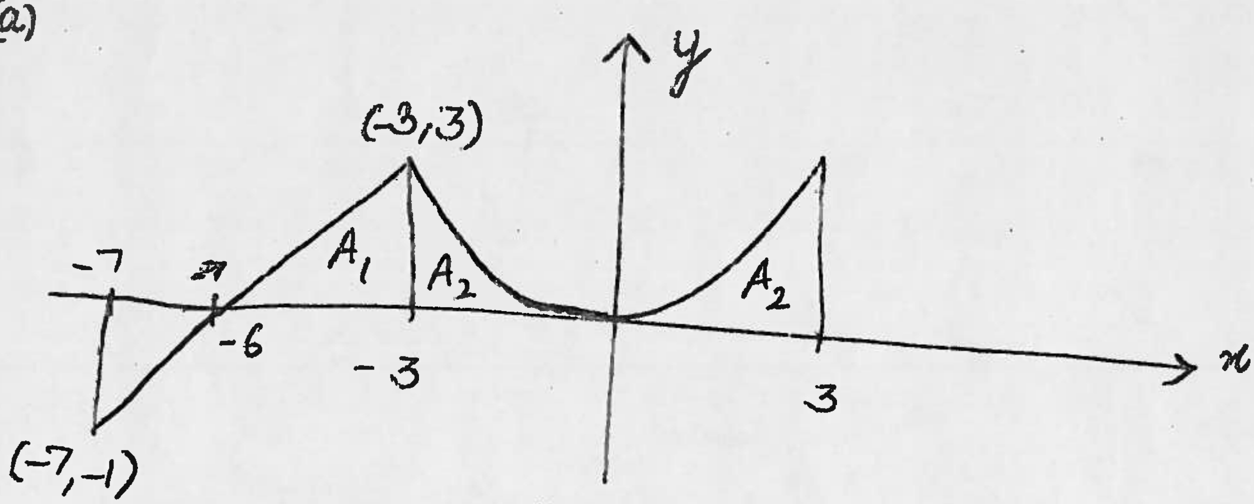
10. a) *[5 marks]* Use differentiation to find a power series representation for the function $f(x) = \frac{1}{(1+x)^2}$

b) *[5 marks]* Find the (numerical) sum of the series $\sum_{n=1}^{\infty} \frac{2^{n+1} + 3^{n-1}}{5^n}$

MATH 205

August 2018

1. (a)



$$A_1 = \frac{1}{2}\cdot b\cdot h = \frac{1}{2}(3)(3) = \frac{9}{2}$$

$$A_2 = (6)(3) - \frac{1}{2}\pi(3)^2 = 18 - \frac{9\pi}{2}$$

$$\therefore \int_{-6}^{3} f(x)\,dx = A_1 + A_2$$

$$= \left(9/2\right) + \left(18 - \frac{9\pi}{2}\right)$$

$$= 22.5 - \frac{9\pi}{2}$$

$$= \frac{45}{2} - \frac{9\pi}{2}$$

1\. (b) $F(x) = \int_{x^2}^{x^3} \frac{\sin t}{t}\, dt$

$$= \int_{x^2}^{0} \frac{\sin t}{t}\, dt + \int_{0}^{x^3} \frac{\sin t}{t}\, dt$$

$$F(x) = \int_{0}^{x^3} \frac{\sin t}{t}\, dt - \int_{0}^{x^2} \frac{\sin t}{t}\, dt$$

$$\therefore \quad F'(x) = \frac{\sin(x^3)}{x^3} \cdot 3x^2 - \frac{\sin(x^2)}{x^2} \cdot 2x$$

$$= \frac{3\sin(x^3)}{x} - \frac{2\sin(x^2)}{x}$$

2 (a) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Let $u = \sin^{-1}x$

$\therefore \quad du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

$\therefore \quad I = \displaystyle\int u\,du \quad = \frac{1}{2}u^2 + C$

$= \dfrac{1}{2}(\sin^{-1}x)^2 + C$

2 (b) $\displaystyle\int \frac{3x^2+x+1}{x(x^2+1)}\,dx$

$$\frac{3x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3x^2+x+1 = A(x^2+1) + (Bx+C)(x)$$

$$= Ax^2 + A + Bx^2 + Cx$$

$$3x^2+x+1 = x^2(A+B) + Cx + A$$

$\therefore$ $A = 1$

$C = 1$

$A+B = 3$

$\therefore B = 2$

$$\therefore I = \int \frac{1}{x}\,dx + \int \frac{2x+1}{x^2+1}\,dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1}$$

$$= \ln|x| + \ln|x^2+1| + \tan^{-1}x + C$$

3(a) $\int_0^{\pi/4} \sec^4 x \, dx$

$$\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$$

$$= \int (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int (\tan^2 x \sec^2 x) \, dx + \int \sec^2 x \, dx$$

$$= \frac{1}{3} \tan^3 x + \tan x + C$$

$$\therefore \quad I = \left(\frac{1}{3} \tan^3 x + \tan x \right) \Big|_0^{\pi/4}$$

$$= \left(\frac{1}{3} \tan^3 (\pi/4) + \tan \pi/4 \right) - \left(\frac{1}{3} \tan^3 0 + \tan 0 \right)$$

$$= \frac{1}{3}(1)^3 + 1 - 0 - 0$$

$$= \frac{4}{3}$$

3(b) $$\int_0^2 x^2\sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

When $x=0,\quad \theta=0$

$x=2,\quad \theta=\frac{\pi}{2}$

$$\therefore\quad I = \int_0^{\pi/2} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= 16\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

since,
$\sin 2\theta = 2\sin\cos\theta$

$$= 4\int_0^{\pi/2}\frac{\sin^2 2\theta}{1}\,d\theta$$

$$= 4\int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= 2\left[0 - \frac{1}{4}\sin 4\theta\right]\Big|_0^{\pi/2}$$

$$= 2\left[\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \frac{1}{4}(0-0)\right]$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

$$= \pi$$

4(a) $\displaystyle\int_2^{\infty} \frac{1}{x \ln^2 x}\, dx$

let $u = \ln x$
$du = \frac{1}{x}\, dx$ when $x = 2$, $u = \ln 2$
$x = t$, $u = \ln t$

$$\therefore I = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{du}{u^2}$$

$$= \lim_{t \to \infty} \left(-\frac{1}{u} \right) \Big|_{\ln 2}^{\ln t}$$

$$= \lim_{t \to \infty} \left[\left(\frac{-1}{\ln t} \right) - \left(\frac{-1}{\ln 2} \right) \right]$$

$$= 0 + \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}$$

4 (b) $$\int_2^3 \frac{dx}{\sqrt{3-x}} \cancel{dx}$$

$$= \lim_{t \to 3^-} \int_2^t \frac{dx}{\sqrt{3-x}}$$

$$= \lim_{t \to 3^-} -2(3-x)^{1/2} \Big|_2^t$$

$$= \lim_{t \to 3^-} \left[-2(3-t)^{1/2} + 2(3-2)^{1/2}\right]$$

$$= 0 + 2(1)$$

$$= 2$$

5 (a) $x = y$ $\quad$ $4x + y^2 = 12$

$$x = \frac{12 - y^2}{4}$$

At Intersection,

$$x = 3 - \frac{1}{4}y^2$$

$$y = \frac{12 - y^2}{4}$$

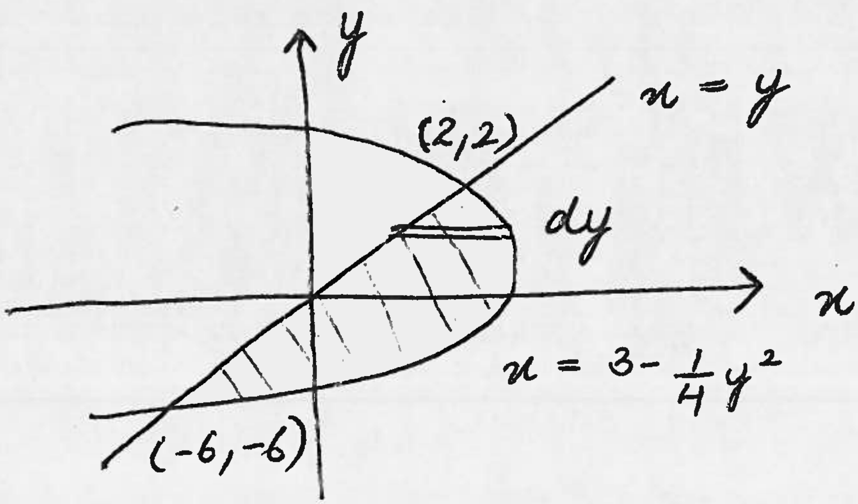
$$4y = 12 - y^2$$

$$y^2 + 4y - 12 = 0$$

$$(y+6)(y-2) = 0$$

$y = 2$ or $y = -6$

$(2, 2)$ $\quad$ $(-6, -6)$



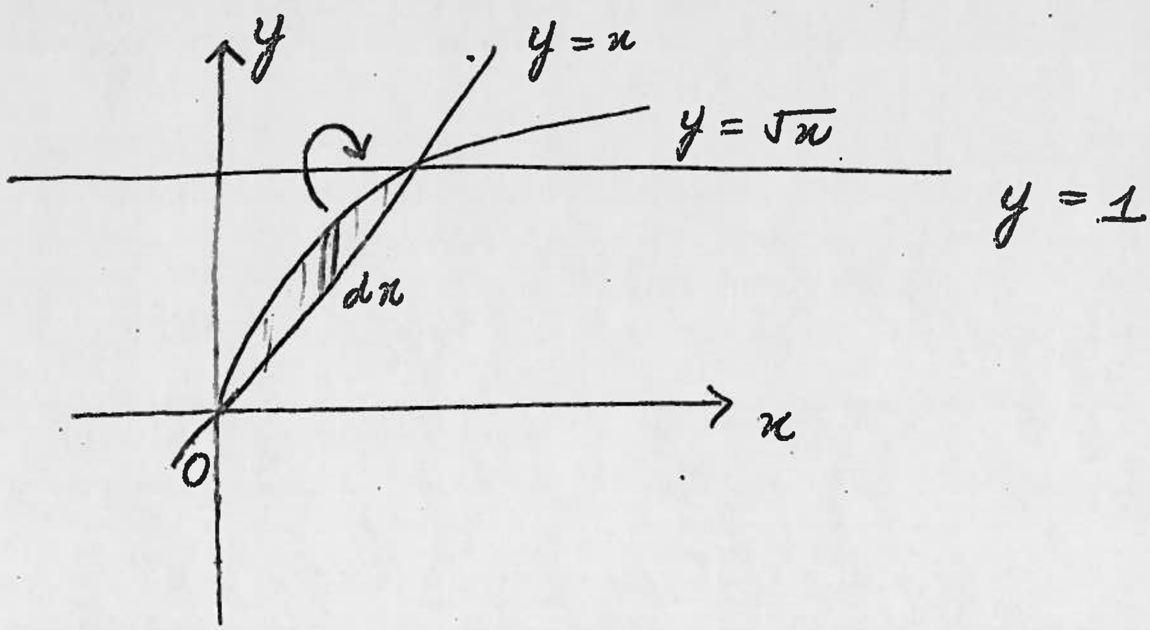
$$\text{Area} = \int_{-6}^{2} \left[\left(3 - \frac{1}{4}y^2\right) - y\right] dy$$

$$= \left(3y - \frac{y^3}{12} - \frac{y^2}{2}\right)\Bigg|_{-6}^{2}$$

$$\doteq \left(6 - \frac{2}{3} - 2\right) - \left(-18 + 18 - 18\right)$$

$$= 4 - \frac{2}{3} + 18 = 21\frac{1}{3} = \frac{64}{3}$$

5: (b)



$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left[R^2 - r^2\right] dx$$

$$= \pi \int_0^1 \left[(1-x)^2 - (1-\sqrt{x})^2\right] dx$$

$$= \pi \int_0^1 \left[(1-2x+x^2) - (1-2\sqrt{x}+x)\right] dx$$

$$= \pi \int_0^1 \left[-3x + x^2 + 2\sqrt{x}\right] dx$$

$$= \pi\left[-\frac{3x^2}{2} + \frac{x^3}{3} + 2\cdot\frac{2x^{3/2}}{3}\right]_0^1$$

$$= \pi\left[\left(-\frac{3}{2} + \frac{1}{3} + \frac{4}{3}\right) - (0)\right]$$

$$= \pi\left(\frac{5}{3} - \frac{3}{2}\right) = \frac{\pi}{6}$$

5 (c) $f_{avg} = \frac{1}{2-1} \int_1^2 \frac{\ln x}{x^2} \, dx$

$$= \int_1^2 \frac{\ln x}{x^2} \, dx$$

let $u = \ln(x)$ $\qquad dv = \frac{1}{x^2} dx$

$du = \frac{1}{x} dx \qquad v = -\frac{1}{x}$

$$\therefore \quad I = -\frac{\ln x}{x} \Big|_1^2 + \int_1^2 \frac{1}{x^2} \, dx$$

$$= \left(-\frac{\ln 2}{2} - \left(\frac{-\ln 1}{1}\right)\right) + \left(-\frac{1}{x}\right)\Big|_1^2$$

$$= -\frac{\ln 2}{2} + 0 + \left[-\frac{1}{2} - \left(-\frac{1}{1}\right)\right]$$

$$= -\frac{\ln 2}{2} - \frac{1}{2} + 1$$

$$= \frac{1 - \ln 2}{2}$$

6 (a) $\lim_{n \to \infty} \cos\left(\frac{2n\pi}{5+6n}\right)$

$= \cos\left[\lim_{n \to \infty} \frac{2n\pi}{5+6n}\right]$

$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(b) $\lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{n \to \infty} \frac{2 \ln n \cdot 1/n}{1}$

$= \lim_{n \to \infty} 2 \cdot \frac{\ln n}{n}$

$= 2 \cdot \lim_{n \to \infty} \frac{1/n}{1} = 0$

---

c) $\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n$

$= \lim_{n \to \infty} \left(\left(1 + \frac{3}{n}\right)^{n/3}\right)^3$

$= e^3$

7 (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$

$b_n = \dfrac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sin 1/n}{\sqrt{n}}\right) \cdot \left(\frac{n^{3/2}}{1}\right)$$

$$= \lim_{n\to\infty} n \cdot \sin(1/n)$$

$$= \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)} = \underline{1}$$

$\therefore$ By the Limit Comparison Test, Σa_n and Σb_n both converge or both diverge.

But $\Sigma \dfrac{1}{n^{3/2}}$ converges,

(p-series, $p = 3/2 > 1$)

$\therefore \displaystyle\sum_{n=1}^{\infty} a_n$ Converges (and converges Absolutely)

7(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Let $b_n = \dfrac{1}{\ln n}$

$\displaystyle\lim_{n \to \infty} b_n = 0$

and $\dfrac{1}{\ln(n+1)} < \dfrac{1}{\ln n}$

$\therefore$ By the Alternating Series Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ is Convergent.

Next, consider $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

$$\ln n < n$$

$$\therefore \frac{1}{\ln n} > \frac{1}{n}$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent
(p-series, $p = 1$)

Hence, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test

$\therefore$ The series is conditionally convergent.

8. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$

<u>Ratio Test</u> :

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)^{n+1}}{(n+1)} \cdot \frac{n}{(x-2)^n}\right|$$

$$= \left(\frac{n}{n+1}\right)|x-2|$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2|$$

Converges if $|x-2| < \underline{1}$

$$-\underline{1} < x-2 < \underline{1}$$

$$\underline{1} < x < 3$$

Radius of Convergence $= \underline{1}$

Check endpoints :

(a) If $x = \underline{1}$: we have

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, convergent by the A.S.T

(b) If $x = 3$: we have $\sum_{n=0}^{\infty} \frac{1}{n}$, divergent (Harmonic Series)

$\therefore$ Interval of Convergence $= [1, 3)$

9 (a) $$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$\therefore \quad e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

$$\therefore \quad x^2 e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n!}$$

9 (b) $$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n+1}}{(2n+1)!} = \sin t$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (\pi/6)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

10. (a) $f(x) = \dfrac{1}{1+x}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= 1 - x + x^2 - x^3 + x^4 - - - - -$$

$$\therefore \frac{-1}{(1+x)} = -1 + x - x^2 + x^3 - x^4 - - - -$$

Differentiate both sides,

$$\therefore \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 - - - -$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

10 (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1} + 3^{n-1}}{5^n}$

$$= 2 \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \frac{1}{5} \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{n-1}$$

$$= 2 \left[\frac{2/5}{1 - 2/5}\right] + 1/5 \left[\frac{1}{1-3/5}\right]$$

$$= 2 \left[\frac{2/5}{3/5}\right] + 1/5 \left[\frac{1}{2/5}\right]$$

$$= 2 \cdot \frac{2}{\cancel{5}} \cdot \frac{\cancel{5}}{3} + \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{2}$$

$$= \frac{4}{3} + \frac{1}{2}$$

$$= \frac{8+3}{6}$$

$$= \frac{11}{6}$$