CONCORDIA UNIVERSITY
Department of Mathematics and Statistics

Course	Number	Section(s)
Mathematics	204	All

Examination	Date	Pages
Final	April 2022	3

Instructors	Course Examiner
C. L. Santana, S. Saputra, H. Pezeshk, T. Hughes, M. Khasraghi;	A. Kokotov

Special Instructions: Only approved calculators are allowed !

1. **[10]** Solve the system of equations

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 10 \\ 2x_1 - x_2 + x_3 - x_4 = -1 \\ 3x_1 - x_3 - 2x_4 = -8 \\ -x_1 + 3x_2 - x_4 = 1 \\ 5x_1 - x_2 - 3x_4 = -9 \end{cases}$$

2. **[10]** Solve the following equation for (4×4) matrix X:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} X \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3. [10] For which a the system

$$\begin{cases} x+y+z=2 \\ x+2y+2z=3 \\ x+a(a-1)z=a+1 \end{cases}$$

- has infinitely many solutions?
- has no solutions?
- has the unique solution?

4. [10] The plane α contains the line l with parametric equation

$$\begin{cases} x=1+t \\ y=2-3t \\ z=3-t \end{cases}$$

and passes through the point $P=(1,2,1)$. Find the distance $\operatorname{dist}(Q,\alpha)$ from the point $Q=(1,1,1)$ to the plane α.

5. [10] The plane α passes through the points $A=(1,2,3)$, $B=(2,-1,4)$ and $C=(1,4,1)$.

The plane β is defined via the equation

$$x+2y+3z=1\,.$$

Write down the parametric equation of the line of intersection of α and β.

6. [10] The vectors

$$\vec{u}=(a/2,\ 1,\ 3/2)$$
$$\vec{v}=(a/3,\ a/3,\ 4/3)$$

and

$$\vec{w}=(a,\ 2,\ a+6)$$

have the same initial point and lie in the same plane. Find all the possible values of a.

7. [10] Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 0 & 3 & 4 \\ 2 & 3 & 0 & 4 & 1 \\ 7 & 8 & 5 & 9 & 5 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 1 & 0 & 2 & 3 \end{vmatrix}$$

8. [10] Find a basis for the solution space of the homogeneous system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 - x_5 = 0 \\ 2x_1 - x_2 + x_3 - x_4 + x_6 = 0 \\ 4x_1 + x_2 + 3x_3 + x_4 - 2x_5 + x_6 = 0 \end{cases}$$

9. [10] The matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & -1 & 0 & 3 \end{pmatrix}$$

has an eigenvector $(1, 2, 3, 4)$. Find an eigenvalue of the matrix A.

10. [10] Find an invertible 2×2 matrix P such that the matrix

$$P^{-1} \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix} P$$

is diagonal.

MATH. 204.

Final Exam Solutions. Winter. 2022.

1. $x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$x_4 = 4.$

2. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B \quad \times \quad A \quad = \quad C$$

$BXA = C.$

Must compute A^{-1}, B^{-1}, if they exist.

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}; B^{-1} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

$$BXA = C$$

$$BXAA^{-1} = CA^{-1} = C$$

$$B^{-1}BX = B^{-1}C$$

$$X = B^{-1}C = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 0 & a(a-1) & a+1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & (a^2-a) & a \end{array}\right].$$

no. Sol.: $a=1$.

one Sol.: $a \neq 0, 1$.

Infinite Sols.: $a=0$.

4. $P_0 = (1,2,3)$; $\bar{n} = (1,-3,-1)$

$P = (1,2,1)$; $Q = (1,1,1)$.

$\overrightarrow{P_0P} = (0,0,-2)$.

$\bar{n} \times \overrightarrow{P_0P} = (6,2,0)$.

Equation of plane.

$6(x-1) + 2(y-2) = 0$

$3x + y = 5$.

Distance from Q to plane.

$$D = \frac{|3(1)+(1)-5|}{\sqrt{(3)^2+(1)^2}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}.$$

5. $A(1,2,3)$, $B(2,-1,4)$, $C(1,4,1)$.

$\vec{AB} = (1,-3,1)$.

$\vec{AC} = (0,2,-2)$.

$\vec{AB} \times \vec{AC} = (4,2,2)$.

Plane α: $4(x-1)+2(y-2)+2(z-3)=0$

$$2x+y+z=7.$$

Computing line of intersection of the two planes.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 13/3 \\ 0 & 1 & 5/3 & -5/3 \end{array}\right] \Rightarrow$$

$$\text{S.S.: } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{13}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \end{bmatrix}.$$

$x = \frac{13}{3} + \frac{t}{3}$

$y = -\frac{5}{3} - \frac{5}{3}t$

$z = t$.

6. $\bar{u} = \left(\frac{a}{2}, 1, \frac{3}{2}\right)$

$\bar{v} = \left(\frac{a}{3}, \frac{a}{3}, \frac{4}{3}\right)$

$\bar{w} = (a, 2, a+6)$.

For the three vectors to lie on the same plane, $\bar{w} \cdot (\bar{u} \times \bar{v})$ must equal 0.

$\bar{u} \times \bar{v} = (8-3a, -a, a^2-2a)$.

$(a, 2, a+6) \cdot (8-3a, -a, a^2-2a) = 0$

$a^3 + a^2 - 6a = 0$

$a(a^2 + a - 6) = 0$

$a(a+3)(a-2) = 0$

$a = 0, -3, 2$.

7. $\det(A) = 800$.

8.

$$\begin{array}{cccccc} x & y & z & u & v & w \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & -1 & 0 & 1 \\ 4 & 1 & 3 & 1 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2/3 & 0 & -1/3 & 1/3 \\ 0 & 1 & 1/3 & 1 & -2/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Infinite Solutions.

$$S.S.: X = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = q \underbrace{\begin{bmatrix} -2/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{\alpha}} + r \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{\beta}} + s \underbrace{\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{\gamma}} + t \underbrace{\begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{\delta}}.$$

The basis vectors are: $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$.

9. $A\bar{x} = \lambda \bar{x}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \lambda = 3.$$

10. $\det(\lambda I - A) = \det\begin{pmatrix} (\lambda-2) & -\sqrt{2} \\ -\sqrt{2} & (\lambda-3) \end{pmatrix} = 0$

$(\lambda-2)(\lambda-3)-2=0$

$\lambda^2-5\lambda+4=0$

$(\lambda-4)(\lambda-1)=0$

$\lambda=1; \lambda=4.$

$\lambda=1$:

$\left[\begin{array}{cc|c} -1 & -\sqrt{2} & 0 \\ -\sqrt{2} & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x=-\sqrt{2}y$

Let $y=t$

S.S: $X=\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

OR

$x=-\sqrt{2}t$

$y=t$.

$\lambda=4$:

$\left[\begin{array}{cc|c} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$x=\dfrac{\sqrt{2}}{2}y$ OR $x=\dfrac{\sqrt{2}}{2}t$

Let $y=t$ $\quad y=t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} = t \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}.$$

$$P = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 2 \end{bmatrix}.$$

$$P^{-1} = \frac{-1}{3\sqrt{2}} \begin{bmatrix} 2 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}.$$

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$