Fall 2022 Midterm Solution

Vectors and Matrices (Concordia University)

Midterm – Oct 30^{th} 2022

1.

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 8 & 12 & -9 & 8 & 3 \\ 4 & 6 & 3 & -2 & 3 \\ 2 & 3 & 9 & -7 & 3 \end{array}\right] \xrightarrow[R_4' = R_4 - R_1]{R_2' = R_2 - 4R_1,\ R_3' = R_3 - 2R_1} \left[\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 0 & 0 & -5 & 4 & -1 \\ 0 & 0 & 5 & -4 & 1 \\ 0 & 0 & 10 & -8 & 2 \end{array}\right] \xrightarrow[R_4' = R_4 + 2R_2]{R_3' = R_3 + R_2}$$

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 0 & 0 & -5 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{aligned} 2x_1 + 3x_2 - x_3 + x_4 &= 1 \quad \dots (1) \\ -5x_3 + 4x_4 &= -1 \quad \dots (2) \end{aligned}$$

$\rightarrow$ Infinitely many solutions !

Let $x_4 = s$

$(2)\quad -5x_3 + 4s = -1$

$x_3 = \frac{1}{5} + \frac{4}{5}s$

Let $x_2 = t$

$(1)\quad 2x_1 + 3t - \left(\frac{1}{5} + \frac{4}{5}s\right) + s = 1$

$2x_1 + 3t - \frac{1}{5} - \frac{4}{5}s + s = 1$

$2x_1 + 3t - \frac{1}{5} + \frac{1}{5}s = 1$

$2x_1 = 1 + \frac{1}{5} - \frac{1}{5}s - 3t$

$x_1 = \frac{3}{5} - \frac{1}{10}s - \frac{3}{2}t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 0 \\ 1/5 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1/10 \\ 0 \\ 4/5 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} //$$

2. a) $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$ or use the inversion algorithm.

Since we have lots of 0's, use the latter.

$$\left[\begin{array}{cccc|cccc} 2 & 6 & 8 & 5 & 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3' = R_3 - R_4]{R_1' = R_1 - R_4,\ R_2' = R_2 - R_4} \left[\begin{array}{cccc|cccc} 2 & 6 & 8 & 0 & 1 & 0 & 0 & -1 \\ 0 & 3 & 4 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4' = \frac{1}{5}R_4]{R_1' = R_1 - 2R_3,\ R_2' = R_2 - R_3}$$

$$\left[\begin{array}{cccc|cccc} 2 & 6 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 3 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right] \xrightarrow[R_3' = \frac{1}{4}R_3]{R_1' = R_1 - 2R_2} \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{5} \end{array}\right] \xrightarrow[R_2' = \frac{1}{3}R_2]{R_1' = \frac{1}{2}R_1}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & -1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right], \quad A^{-1} = \begin{bmatrix} 1/2 & -1 & 0 & 1/2 \\ 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$$

b) $\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} X = X \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

There are 2 ways of solving this.

* 1st way

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3a+2c & 3b+2d \\ a+3c & b+3d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3a+2c & 3b+2d \\ a+3c & b+3d \end{bmatrix} = \begin{bmatrix} 2a+1 & 2b+2 \\ 2c+3 & 2d+4 \end{bmatrix}$$

$$\begin{cases} 3a+2c = 2a+1 & \ldots (1) \\ 3b+2d = 2b+2 & \ldots (2) \\ a+3c = 2c+3 & \ldots (3) \\ b+3d = 2d+4 & \ldots (4) \end{cases}$$

$(1)\ a = 1-2c \longrightarrow a = 1+4 = 5$

$(1) \to (3)$ $\quad 1-2c+3c = 2c+3$

$c-2c = 3-1$

$-c = 2,\ c = -2$

$(2)\ b = 2-2d \longrightarrow b = 2+4 = 6$

$(2) \to (4)$ $\quad 2-2d+3d = 2d+4$

$-d = 2,\ d = -2$

$\therefore X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$ //

* 2nd way

Notice that $AX = XB + C$ and $B = 2I$

So, $AX = X(2I) + C$

$AX = 2XI + C$

$AX = 2X + C$

$AX - 2X = C$

$(A-2I)X = C$

$X = (A-2I)^{-1}C$

$(A-2I) = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$(A-2I)^{-1} = \dfrac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1+6 & -2+8 \\ 1-3 & 2-4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$$ //

3.
$$\begin{vmatrix} 1202 & 1000 & 0.1 & 49 \\ 1203 & 1000 & 0.2 & 50 \\ 1204 & 1000 & 0.2 & 51 \\ 1205 & 1000 & 0.2 & 52 \end{vmatrix} \xrightarrow{C_2' = \frac{1}{1000} C_2} 1000 \begin{vmatrix} 1202 & 1 & 0.1 & 49 \\ 1203 & 1 & 0.2 & 50 \\ 1204 & 1 & 0.2 & 51 \\ 1205 & 1 & 0.2 & 52 \end{vmatrix} \xrightarrow[C_4' = C_4 - 49 C_2]{C_1' = C_1 - 1202 C_2,\ C_3' = C_3 - 0.1 C_2}$$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0.1 & 1 \\ 2 & 1 & 0.1 & 2 \\ 3 & 1 & 0.1 & 3 \end{vmatrix} \xrightarrow{C_1' = C_1 - C_4} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0.1 & 1 \\ 0 & 1 & 0.1 & 2 \\ 0 & 1 & 0.1 & 3 \end{vmatrix} = 0$$ since we have a column of 0's

4.
$$\begin{cases} x + 2y + 3z = B \\ x + 3y + 7z = 1 \\ x + 2y + 4z = -1 \end{cases}$$

Find B when $x = 2$ with Cramer's Rule!

$x = \dfrac{\det(A_1)}{\det(A)}$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 2 & 4 \end{bmatrix}$ and $A_1 = \begin{bmatrix} B & 2 & 3 \\ 1 & 3 & 7 \\ -1 & 2 & 4 \end{bmatrix}$

- $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 2 & 4 \end{vmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = 1$

- $\det(A_1) = \begin{vmatrix} B & 2 & 3 \\ 1 & 3 & 7 \\ -1 & 2 & 4 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} (-1) \begin{vmatrix} 1 & 3 & 7 \\ B & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} \xrightarrow[R_3' = R_3 + R_1]{R_2' = R_2 - BR_1}$

$$(-1) \begin{vmatrix} 1 & 3 & 7 \\ 0 & 2-3B & 3-7B \\ 0 & 5 & 11 \end{vmatrix} = (-1)\,1 \begin{vmatrix} 2-3B & 3-7B \\ 5 & 11 \end{vmatrix} = (-1)\left[22 - 3B - (15 - 35B)\right]$$

$$= -2B - 7$$

$x = \dfrac{\det(A_1)}{\det(A)}$

$2 = \dfrac{-2B-7}{1} \longrightarrow -2B - 7 = 2$

$-2B = 9$

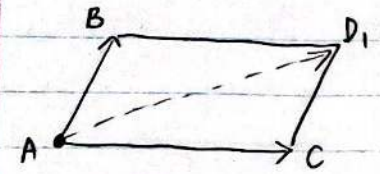
$B = -9/2$ //

(5) 3 parallelograms with common vertices
$(1,2)$, $(2,3)$ and $(1,1)$

* Find the coordinates of the 4th vertex and the longest diagonal!

$A = (1,2)$
$B = (2,3)$
$C = (1,1)$

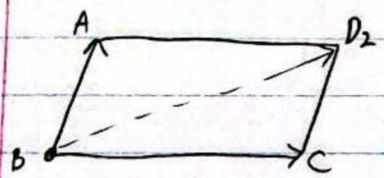
We can solve this with vector addition! Refer to the parallelogram method discussed in class.



$\vec{AB} = (1,1)$
$\vec{AC} = (0,-1)$
$\vec{AD_1} = \vec{AB} + \vec{AC}$
$= (1,0)$

1^{st} diag $= \|\vec{AD_1}\| = \sqrt{1^2+0^2} = 1$
2^{nd} diag $= \|\vec{BC}\| = \sqrt{1^2+2^2} = \sqrt{5}$

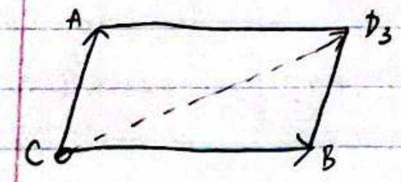
$\vec{AD_1} = D_1 - A = (1,0)$
$D_1 = (2,2)$



$\vec{BA} = (-1,-1)$
$\vec{BC} = (-1,-2)$
$\vec{BD_2} = \vec{BA} + \vec{BC} = (-2,-3)$

1^{st} diag $= \|\vec{BD_2}\| = \sqrt{2^2+3^2} = \sqrt{13}$
2^{nd} diag $= \|\vec{AC}\| = \sqrt{0^2+1^2} = 1$

$\vec{BD_2} = D_2 - B = (-2,-3)$
$D_2 = (0,0)$



$\vec{CA} = (0,1)$
$\vec{CB} = (1,2)$
$\vec{CD_3} = \vec{CA} + \vec{CB}$
$= (1,3)$

1^{st} diag $= \|\vec{CD_3}\| = \sqrt{1^2+3^2} = \sqrt{10}$
2^{nd} diag $= \|\vec{AB}\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\vec{CD_3} = D_3 - C = (1,3)$
$D_3 = (2,4)$

$\therefore$ Longest diag $= \sqrt{13}$
$D_1 = (2,2)$
$D_2 = (0,0)$
$D_3 = (2,4)$

(6) $$E_4 E_3 E_2 E_1 \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B}$$

E is an elementary matrix : I + 1 ERO

multiplying with E <=> 1 ERO on A

the goal here is to find the ERO's to perform on A, to get B

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \xrightarrow{R_2' = R_2 - 5R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1' = R_1 - 3R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1' = R_1 - \frac{1}{2} R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad //$$

* the solution is not unique; depends on your ERO's order!