Final - Fall 2022

1. $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 3 & 7 \\ 3 & 2 & 4 \end{bmatrix} \longrightarrow$ Find A^{-1}

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 3 & 1 & 0 & 0 \\ 4 & 3 & 7 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 5 \\ 0 & 1 & 0 & 5 & 3 & -9 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$AXA^{-1} = C + A$

$X = A^{-1}(C+A)A$

$$= \begin{bmatrix} -2 & -2 & 5 \\ 5 & 3 & -9 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 4 \\ 4 & 4 & 7 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 4 & 3 & 7 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -4 \\ 12 & 11 & 35 \\ -1 & -1 & -3 \end{bmatrix}$$

2. $AXB = C$

$X = A^{-1} C B^{-1}$

$\det(X) = \det(A^{-1} C B^{-1})$

$= \det(A^{-1}) \ \det(C) \ \det(B^{-1})$

$= \dfrac{1}{\det(A)} \ \det(C) \ \dfrac{1}{\det(B)}$

$= \dfrac{1}{\det(A)} \cdot 1 \cdot \dfrac{1}{1}$

$= \dfrac{1}{160}$ //

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \xrightarrow[R_4' = R_4 - 4R_1]{R_2' = R_2 - 2R_1,\ R_3' = R_3 - 3R_1}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{vmatrix} \xrightarrow[R_4' = R_4 - 7R_2]{R_3' = R_3 - 2R_2}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{vmatrix} \xrightarrow{R_4' = R_4 + R_3}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 40 \end{vmatrix} = 160$$

3. $$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 3 & 0 \\ 11 & 4 & a & 5 & 0 \\ 4 & 3 & 0 & 7 & 0 \\ 3 & 2 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_4' = R_4 - R_1} \left[\begin{array}{cccc|c} 3 & 2 & 0 & 3 & 0 \\ 11 & 4 & a & 5 & 0 \\ 4 & 3 & 0 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1' = R_1 - R_3} \left[\begin{array}{cccc|c} -1 & -1 & 0 & -4 & 0 \\ 11 & 4 & a & 5 & 0 \\ 4 & 3 & 0 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3' = R_3 + 4R_1]{R_2' = R_2 + 11R_1} \left[\begin{array}{cccc|c} -1 & -1 & 0 & -4 & 0 \\ 0 & -7 & a & -39 & 0 \\ 0 & -1 & 0 & -9 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} -1 & -1 & 0 & -4 & 0 \\ 0 & -1 & 0 & -9 & 0 \\ 0 & -7 & a & -39 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_1' = -R_1]{R_3' = R_3 - 7R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & -1 & 0 & -9 & 0 \\ 0 & 0 & a & 24 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2' = -R_2]{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & a & 24 & 0 \end{array}\right] \xrightarrow{R_4' = R_4 - aR_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 9 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 24-a & 0 \end{array}\right]$$

To have ∞ many soln : $a - 24 = 0$

$a = 24$ //

4. $l_1 : t_1(0,4,2)$

$l_2 : t_2(-1,-3,-2)$

$l_3 : t_3(10,2,4)$

The lines intersects with $x+2y-3z=1$

Find the area of Δ !

get 3 points : 5/10

get 2 vectors : 7/10

wrong calc : 8/10 or 7/10 ↳ depends

For l_1 : $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_1 \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4t_1 \\ 2t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ ← point A

$x+2y-3z=1$

$0+2(4t_1)-3(2t_1)=1$

$8t_1-6t_1=1$

$2t_1=1, \; t_1=\frac{1}{2}$

For l_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_2 \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -t_2 \\ -3t_2 \\ -2t_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ← point B

$x+2y-3z=1$

$-t_2+2(-3t_2)-3(-2t_2)=1$

$-t_2-6t_2+6t_2=1$

$t_2=-1$

For l_3 : $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_3 \begin{pmatrix} 10 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 10t_3 \\ 2t_3 \\ 4t_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ ← Point C

$x+2y-3z=1$

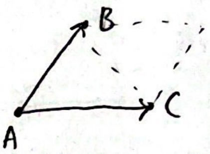
$10t_3+2(2t_3)-3(4t_3)=1$

$10t_3+4t_3-12t_3=1$

$2t_3=1$

$t_3=\frac{1}{2}$

Construct the $\triangle$



$\vec{AB} = B-A = (1,1,1)$

$\vec{AC} = C-A = (5,-1,1)$

$\vec{AB}\times\vec{AC} = 2\vec{i}+4\vec{j}-6\vec{k}$

Area of $\Delta = \dfrac{\sqrt{2^2+4^2+(-6)^2}}{2}$

$= \dfrac{\sqrt{56}}{2} = 3.742$ //

or $\dfrac{\sqrt{7\cdot 8}}{2} = \dfrac{\cancel{2}\sqrt{14}}{\cancel{2}} = \sqrt{14}$ //

5. $\alpha : ax+by+12z=2$

$\beta : x+2y+4z=\sqrt{2}$

$\alpha \parallel \beta$ then $n_\alpha \parallel n_\beta$

$\frac{a}{1}=\frac{b}{2}=\frac{12}{4}=3$, so $a=3$ and $b=6$

α: $3x+6y+12z=2$ intersects $x+y+z=1$

$\begin{cases} x+y+z=1 \\ 3x+6y+12z=2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 6 & 12 & 2 \end{array}\right] \xrightarrow{R_2' = R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 9 & -1 \end{array}\right] \xrightarrow{R_3' = \frac{1}{3}R_3}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -\frac{1}{3} \end{array}\right]$

$\begin{cases} x+y+z=1 \\ y+3z=-\frac{1}{3} \end{cases}$

let $z=t$

$y=-\frac{1}{3}-3t$

$x=1-y-z = 1-\left(-\frac{1}{3}-3t\right)-t$

$= 1+\frac{1}{3}+3t-t$

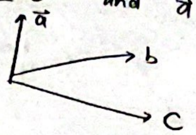
$= \frac{4}{3}+2t$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4/3 \\ -1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = l$ //

(6.) $\|\vec{a}\| = 2$ $\quad \vec{a} \perp \vec{b}$ and $\vec{a} \perp c$

$\vec{b} = (7, 3, 4)$

$\vec{c} = (-1, 1, 1)$



$\vec{b} \times \vec{c} = -i - 11j + 10k = \vec{w}$

since $\|\vec{a}\| = 2$, let $\vec{a} = \dfrac{\vec{w}}{\|\vec{w}\|} \times 2 = \dfrac{(-1, -11, 10)}{\sqrt{1+121+100}} \cdot 2$

$\vec{a} = (-1, -11, 10) \cdot \dfrac{2}{\sqrt{222}} = \left(-\dfrac{2}{\sqrt{222}}, \dfrac{-22}{\sqrt{222}}, \dfrac{20}{\sqrt{222}} \right)$

$\star$ $2\vec{a} - \frac{1}{2}\vec{b} + \vec{c} = \left(\dfrac{-4}{\sqrt{222}}, \dfrac{-44}{\sqrt{222}}, \dfrac{40}{\sqrt{222}} \right) - \left(\dfrac{7}{2}, \dfrac{3}{2}, 2 \right) + (-1, 1, 1)$

$= (-4.77, -3.45, 1.684)$

Norm ≈ 6.123 // (this is an approximation)

(7.) $(A+B)^2 = A^2 + AB + BA + B^2$

$\begin{pmatrix} 10 & 12 \\ 18 & 22 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} + BA + \begin{pmatrix} 2 & 4 \\ 4 & 10 \end{pmatrix}$

$BA = \begin{pmatrix} 10 & 12 \\ 18 & 22 \end{pmatrix} - \begin{pmatrix} 7 & 10 \\ 11 & 18 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$

(8.) Basis for solution space

$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & -1 & -1 & 0 & 1 & 0 \\ 4 & 1 & -1 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_3' = R_3 - 4R_1]{R_2' = R_2 - 2R_1} \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & -2 & 1 & 0 \\ 0 & -3 & -5 & -3 & -6 & 1 & 0 \end{array}\right] \xrightarrow{R_3' = R_3 - R_2}$

$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & -4 & 0 & 0 \end{array}\right] \xrightarrow{R_3' = -\frac{1}{2}R_3} \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 & \ldots (1) \\ -3x_2 - 3x_3 - 3x_4 - 2x_5 + x_6 = 0 & \ldots (2) \\ x_3 + 2x_5 = 0 & \ldots (3) \end{cases}$

let $x_5 = s$, $x_3 = -2s$

let $x_6 = t$, $x_4 = u$

(2) $-3x_2 - 3(-2s) - 3u - 2s + t = 0$

$-3x_2 = -6s + 3u + 2s - t$

$x_2 = -\frac{1}{3}(-4s + 3u - t)$

$= \frac{4}{3}s - u + \frac{1}{3}t$

(1) $x_1 = -x_2 - x_3 - x_4 - x_5$

$= -\frac{4}{3}s + \cancel{u} - \frac{1}{3}t + 2s - \cancel{u} - s$

$= -\frac{1}{3}s - \frac{1}{3}t$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = s \overset{V_1}{\begin{pmatrix} -1/3 \\ 4/3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + t \overset{V_2}{\begin{pmatrix} -1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} + u \overset{V_3}{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$

Basis $= \{V_1, V_2, V_3\}$

9. Eigenvalue for A corresponding to eigenvector $\perp$ to $x-y+2z=\sqrt{3}$

the eigenvector $\propto$ the normal = $(1,-1,2)$.

$Ax=\lambda x$.

$$\begin{bmatrix} 7 & 7 & 1 \\ 13 & 13 & -1 \\ 1 & -3 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 7-7+2 \\ 13-13-2 \\ 1+3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

10. $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda-17 & 6 \\ -35 & \lambda+12 \end{vmatrix} = 0$$

$(\lambda-17)(\lambda+12)+210=0$

$\lambda^2-5\lambda-204+210=0$

$\lambda^2-5\lambda+6=0$

$(\lambda-3)(\lambda-2)=0$

$\lambda_1=2, \quad \lambda_2=3$

$\lambda_1=2$

$$\left[\begin{array}{cc|c} -15 & 6 & 0 \\ -35 & 14 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ -35 & 14 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x-\frac{2}{5}y=0$

$x=\frac{2}{5}y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 2/5 \\ 1 \end{pmatrix}$$

$\lambda_2=3$

$$\left[\begin{array}{cc|c} -14 & 6 & 0 \\ -35 & 15 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3/7 & 0 \\ -35 & 15 & \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3/7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x-3/7\,y=0$

$x=\frac{3}{7}y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 3/7 \\ 1 \end{pmatrix}$$

$$P=\begin{bmatrix} 2/5 & 3/7 \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 3/7 & 2/5 \\ 1 & 1 \end{bmatrix}$$