MATH204 Final EXAM AND Solutions DEC 2019

Vectors and Matrices (Concordia University)

MATH 204.
FINAL EXAMINATION.
DECEMBER 2019.

Each question 10 marks.
Justify all your answers.

1. (a) Find the cosine of the angle between vectors (2, 1, 5) and (1, 2, 3).
   (b) Find the distance between point (1, −4, −3) and the plane $3x - y + 4z = 5$.

2. (a) Verify the triangle inequality for $\bar{u} = (2, 4, 5)$ and $\bar{v} = (2, 0, 1)$.
   (b) Use part (a) to find a vector orthogonal to the plane of the triangle.
   (c) Find $(3\bar{u} + 2\bar{v}) \cdot (-5\bar{u} + \bar{v})$ if $\bar{u} \cdot \bar{u} = 3$, $\bar{u} \cdot \bar{v} = -2$, $\bar{v} \cdot \bar{v} = 8$.

3. (a) Use Cramer's rule to solve: (no marks if Cramer's Rule not used!)

$$\begin{aligned} x + y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$$

   (b) Compute the determinant of:

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 7 & 3 & 2 & 1 \\ 4 & 1 & 1 & 1 \\ 5 & 2 & 6 & -3 \end{bmatrix}$$

4. If $M = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \\ 0 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.

   Find the matrix C such that MC = B.

5. Solve by Gauss-Jordan Elimination.

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

6. (a) Let P(1, 1, 1), Q(−2, 3, 5), R(1, 7, 2) be 3 points in space. Find the area of the triangle having P, Q, R as vertices.
   (b) Find the volume of the parallelepiped determined by the vectors $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}$ where S is the point (2, 2, 3).

7. Find the basis of the solution space of the following homogeneous system of linear equations:

$$\begin{aligned} x_1 + x_2 + \quad 3x_4 + 2x_5 &= 0 \\ x_3 - x_4 + 7x_5 &= 0 \\ x_4 - 8x_5 &= 0 \end{aligned}$$

8. (a) Find the linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^3$ if: T(1, 1, 1) = (1 ,2, 3), T(0, 1, 1) = (2, 3, 4), T(0, 0, 1) = (3, 4, 5).
   (b) Let T_1: $\mathbb{R}^2 \to \mathbb{R}^2$ be the rotation operator through $\frac{\pi}{2}$. Find the standard matrix of T_1.

9. (a) Let A = $\begin{bmatrix} 2 & 7 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$. Find an invertible matrix P, and a diagonal matrix D, such that $P^{-1}AP$ = D.

   (b) Let A = $\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$. Compute A^{100}.

10. (a) Show $\bar{u}$ = (1, 1, 1), $\bar{v}$ = (1, 1, 0), $\bar{w}$ = (1, 0, 0) are linearly independent.
    (b) Write vector (2, −4, 5) as a linear combination of $\bar{u}, \bar{v}, and\ \bar{w}$.

MATH. 204.

Final Exam Solutions.

Fall 2019.

1. A.

$$\cos\theta = \frac{(2,1,5)\cdot(1,2,3)}{\|(2,1,5)\|\,\|(1,2,3)\|} = \frac{19}{\sqrt{30}\sqrt{14}} = \frac{19}{\sqrt{420}}.$$

B.

$$D = \frac{|3(1)-(-4)+4(-3)-5|}{\sqrt{26}} = \frac{10}{\sqrt{26}} = \frac{5\sqrt{26}}{13}.$$

2. A. $\|\bar{u}+\bar{v}\| \le \|\bar{u}\| + \|\bar{v}\|$

$\|(4,4,6)\| \le \|(2,4,5)\| + \|(2,0,1)\|$

$\sqrt{68} \le \sqrt{45} + \sqrt{5}$

$8.25 \le 8.94$. True!

B. $\bar{u}\times\bar{v} = \begin{vmatrix} 2 & 4 & 5 \\ 2 & 0 & 1 \end{vmatrix} = (4,8,-8).$

C. $(3\bar{u}+2\bar{v})\cdot(-5\bar{u}+\bar{v})$

$= (3\bar{u}+2\bar{v})\cdot(-5\bar{u}) + (3\bar{u}+2\bar{v})\cdot(\bar{v})$

$= (3\bar{u})\cdot(-5\bar{u}) + (2\bar{v})\cdot(-5\bar{u}) + (3\bar{u})\cdot(\bar{v}) + (2\bar{v})\cdot(\bar{v})$

$= -15(\bar{u}\cdot\bar{u}) - 10(\bar{v}\cdot\bar{u}) + 3(\bar{u}\cdot\bar{v}) + 2(\bar{v}\cdot\bar{v})$

$= -15(3) - 10(-2) + 3(-2) + 2(8)$

$= -15.$

3\. A. $|A_1| = 156$

$|A_2| = 52$

$|A_3| = 104$

$|A| = 52.$

$x = \dfrac{|A_1|}{|A|} = \dfrac{156}{52} = 3$

$y = \dfrac{|A_2|}{|A|} = \dfrac{52}{52} = 1$

$z = \dfrac{|A_3|}{|A|} = \dfrac{104}{52} = 2.$

B. $\det(A) = 0.$

4\. $MC = B$

$C = M^{-1}B$

$C = \dfrac{1}{18}\begin{bmatrix} -8 & -12 \\ -2 & -12 \\ 13 & 24 \end{bmatrix};$

$M^{-1} = \begin{bmatrix} 0.2222 & -0.7778 & 0.3333 \\ -0.4444 & -0.4444 & 0.3333 \\ 0.3889 & 0.3889 & -0.1667 \end{bmatrix}.$

5. $$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ infinite solutions!

$x = -1 + w$

$y = 2z$

Parameterizing. Let $z = s$; $w = t$.

$x = -1 + t$

$y = 2s$

$z = s$

$w = t$.

$\therefore$

$$S.S.: X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

6. A. $\vec{PQ} = (-3, 2, 4)$; $\vec{PR} = (0, 6, 1)$.

$$A = \frac{1}{2} \| \vec{PQ} \times \vec{PR} \| = \frac{1}{2} \begin{vmatrix} -3 & 2 & 4 \\ 0 & 6 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \| (-22, 3, -18) \|$$

$$A \doteq 14.29 \text{ units}^2.$$

B. $\vec{PS} = (1, 1, 2)$.

$$V = | \vec{PS} \cdot (\vec{PQ} \times \vec{PR}) |$$

$$= | (1, 1, 2) \cdot (-22, 3, -18) |$$

$$V = 55 \text{ units}^3.$$

7.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & -1 & 7 & 0 \\ 0 & 0 & 0 & 1 & -8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 26 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -8 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} x_1 &= -x_2 - 26x_5 \\ x_3 &= x_5 \\ x_4 &= 8x_5 \end{aligned}$$

Parameterizing:

$$\begin{aligned} x_1 &= -s - 26t \\ x_3 &= t \\ x_4 &= 8t \\ x_2 &= s \\ x_5 &= t \end{aligned}$$

$\therefore$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -26 \\ 0 \\ 1 \\ 8 \\ 1 \end{bmatrix}.$$

Basis are:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} -26 \\ 0 \\ 1 \\ 8 \\ 1 \end{bmatrix}$$

8. A.

$$T = \begin{bmatrix} -1 & -1 & 3 \\ -1 & -1 & 4 \\ -1 & -1 & 5 \end{bmatrix}.$$

Continued below.

B. T_1: $$\begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A. Continued.

$$T(x, y, z) = (-x - y + 3z, -x - y + 4z, -x - y + 5z).$$

9. A. $\lambda = 2, 3, 4.$

$$P = \begin{bmatrix} 1 & 7 & 41/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

B. $\lambda = -1; 2.$

$$P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}; \quad P^{-1} = -\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\therefore A^{100} = PD^{100}P^{-1} = 10^{30}\begin{bmatrix} -1.2677 & -2.5353 \\ 1.2677 & 2.5353 \end{bmatrix}.$$

10.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -4 \\ 1 & 0 & 0 & 5 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{array}\right]}_{R}.$$

A and B.

The first 3 columns of matrix R prove vectors $\bar{u}, \bar{v}, \bar{w}$ are L.I.

$$(2,-4,5) = 5\bar{u} - 9\bar{v} + 6\bar{w}.$$