On the Curve with Height-based Derivative

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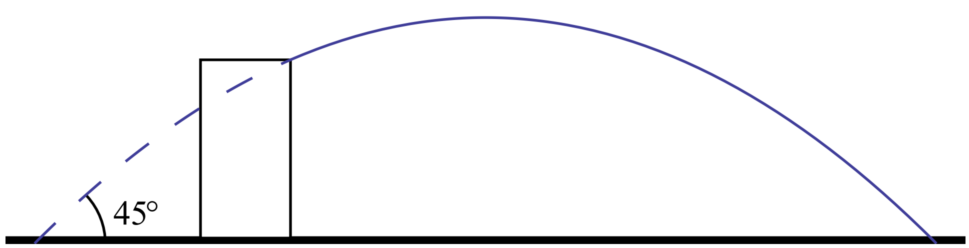
Introduction

A typical problem in physics is to find a formula for the range of a trajectory, given that some point particle is fired with an initial velocity, height and angle from the horizontal. It may be yet further asked what value of initial angle must be taken in order to maximize the range that the point particle should travel before hitting the ground, given a starting velocity and height. Common knowledge suggests the optimal angle to be 45^o, but this is only valid when the initial height is zero (starting from the ground).

One physical explanation is that if a projectile does not start from the ground, it already has some upwardness to its path, so to speak. The optimal angle should be slightly lower than 45^o and should decrease as initial height increases. A downward parabola of a trajectory with zero initial height and 45^o initial launch angle is often drawn to show why. For each initial velocity, there is a unique such parabola. One can say that as the initial launch height of a trajectory is increased, a hypothetical building of equal height can be placed on the parabola such that the height of the building matches the height of a point on the parabola. If, from that initial height, a projectile is launched in the direction of the derivative of the parabola, the range of the particle will be the maximum range attainable. However, this only is true for small heights relative the height of the parabola's peak and for positive derivates (left of the peak). When given an intial velocity and initial height equal to the height of the peak of the parabola particular to that velocity, the optimal angle is not zero, and it does not approach zero so rapidly, or finitely for that matter.



But this inspires the question of what an actual "optimal launch curve" looks like. If it exists, it needs to be able to accomodate for heights approaching infinite value and should have a positive derivative closer and closer to zero as height increases arbitrarily. I attempt to create such a curve, which I will call the curve with height-based derivative. The derivative, in the direction of the optimal angle, must be a function of initial launch height, which should be the height of the curve.

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1 The Range of a Projectile

To find the range of a projectile, the time to hit ground must first be calculated. If a point particle is launched with initial velocity v_o, initial height H, and an angle θ_i, then the time, t, can be described by the following equation.

$$0 = -\frac{1}{2}gt^2 + (v_o \sin\theta_i)t + H$$

where g is the gravitational constant. Solving the quadratic and neglecting the negative answer,

$$t = \frac{v_o \sin\theta_i + \sqrt{v_o^2 \sin^2\theta_i + 2gH}}{g}$$

The distance traveled by the projectile, R, is the time in the air multiplied by the horizontal component of the initial velocity:

$$R = (v_o \cos\theta_i)t = \frac{(v_o \sin\theta_i)v_o \cos\theta_i + v_o \cos\theta_i \sqrt{v_o^2 \sin^2\theta_i + 2gH}}{g}$$

Using the identiy $\sin(2\phi) = 2\sin\phi\cos\phi$,

$$\boxed{R = \frac{\frac{1}{2}v_o^2 \sin(2\theta_i) + \sqrt{\frac{1}{4}v_o^4 \sin^2(2\theta_i) + 2gHv_o^2\cos^2\theta_i}}{g}}$$

2 Deriving the Formula for Optimal Angle

Knowing the expression for the range of a projectile, there should be an optimal maximizing the cosine and sine elements of the equation. Intuitively, shooting too low or too high will not yield a long distance traveled by the fired particle. As the launch angle varies, there should be one specific angle at which R is maximized. In order to find this optimal, the rate of change of R with respect to θ_i must be calculated, allowing the optimal to be found when the value of this derivative is zero. Taking the derivative with respect to θ_i,

$$\frac{dR}{d\theta_i} = \frac{v_o^2\cos(2\theta_i)}{g} + \frac{1}{g}\left(\frac{1}{2\sqrt{\frac{1}{4}v_o^4\sin^2(2\theta_i) + 2gHv_o^2\cos^2\theta_i}}\right)\left(v_o^4\sin(2\theta_i)\cos(2\theta_i) - 4gHv_o^2\cos\theta_i\sin\theta_i\right)$$

$$\frac{dR}{d\theta_i} = \frac{v_o^2\cos(2\theta_i)}{g} + \frac{v_o^4\sin(2\theta_i)\cos(2\theta_i) - 2gHv_o^2\sin(2\theta_i)}{2g\sqrt{\frac{1}{4}v_o^4\sin^2(2\theta_i) + 2gHv_o^2\cos^2\theta_i}}$$

$$\frac{dR}{d\theta_i} = \frac{v_o^2\cos(2\theta_i)}{g} + \frac{v_o\sin(2\theta_i)\left(v_o^2\cos(2\theta_i) - 2gH\right)}{2g\sqrt{\frac{1}{4}v_o^2\sin^2(2\theta_i) + 2gH\cos^2\theta_i}}$$

It is appropriate to allow the derivative to be zero and replace the intial angle, θ_i, with optimal angle, θ_o. Taking the square,

$$v_o^2 \cos^2(2\theta_o) = \frac{\sin^2(2\theta_o)\left(v_o^2\cos(2\theta_o) - 2gH\right)^2}{v_o^2\sin^2(2\theta_o) + 8gH\cos^2\theta_o}$$

$$v_o^4 \sin^2(2\theta_o)\cos^2(2\theta_o) + 8gHv_o^2\cos^2(2\theta_o)\cos^2\theta_o = \sin^2(2\theta_o)\left(v_o^4\cos^2(2\theta_o) - 4gHv_o^2\cos(2\theta_o) + 4g^2H^2\right)$$

$$\begin{aligned}
8gHv_o^2\cos^2(2\theta_o)\cos^2\theta_o &= -4gHv_o^2\sin^2(2\theta_o)\cos(2\theta_o) + 4g^2H^2\sin^2(2\theta_o)\\
2v_o^2\cos^2(2\theta_o)\cos^2\theta_o &= gH\sin^2(2\theta_o) - v_o^2\sin^2(2\theta_o)\cos(2\theta_o)
\end{aligned}$$

Substituting $\cos^2(2\theta_o) = 1 - \sin^2(2\theta_o)$,

$$2v_o^2\cos^2\theta_o - 2v_o^2\cos^2\theta_o\sin^2(2\theta_o) = gH\sin^2(2\theta_o) - v_o^2\sin^2(2\theta_o)\cos(2\theta_o)$$

Using the identity $\cos(2\phi) = \cos^2\phi - \sin^2\phi$,

$$\begin{aligned}
2v_o^2\cos^2\theta_o &= gH\sin^2(2\theta_o) + v_o^2\cos^2\theta_o\sin^2(2\theta_o) + v_o^2\sin^2(2\theta_o)\sin^2\theta_o\\
2v_o^2\cos^2\theta_o &= gH\sin^2(2\theta_o) + v_o^2\sin^2(2\theta_o)(\cos^2\theta_o + \sin^2\theta_o)\\
2v_o^2\cos^2\theta_o &= gH\sin^2(2\theta_o) + v_o^2\sin^2(2\theta_o)\\
2v_o^2 &= (v_o^2 + gH)(4\sin^2\theta_o)\\
\sin^2\theta_o &= \frac{v_o^2}{2(v_o^2 + gH)}
\end{aligned}$$

$$\boxed{\theta_o = \sin^{-1}\sqrt{\frac{v_o^2}{2(v_o^2 + gH)}}}$$

The argument of the root turns out dimensionless, and notice that if $H = 0$, the optimal angle would simplify to 45^o. The positive root is adopted so that the inverse sine will be positive, i.e. we would be launching upward, not downward.

3 Finding the Optimal Launch Curve

Let us denote the formula for optimal launch angle by $\theta_o = f(H)$. Our so-called optimal launch curve we will denote as a function by $z(x)$. Although the curve need not be a function, solving for z should lead to either a function or family of functions, each having exactly one derivative for each height. Also, $z(x)$ must relate the value H to its function, $f(H)$.

The first, and most basic, condition on $z(x)$ must be:

$$z(x) = H$$

because the curve is simply defined to have a "height" of the initial launching height. Next, the derivative of $z(x)$ must be in the same direction as $\theta_o = f(H)$ above the horizontal. Since the derivative is most fundamentally a ratio between rise and run, the tangent of the angle from the horizontal should be equal to that ratio. Therefore, the second condition is:

$$z'(x) = \tan(f(H))$$

Proceedingly, the identity $\tan(\sin^{-1}\alpha) = \frac{\alpha}{\sqrt{1-\alpha^2}}$ can be easily deduced. After some algebra, one will obtain,

$$\tan\left(f(H)\right) = \sqrt{\frac{v_o^2}{v_o^2 + 2gH}}$$

leading to the two conditions on z:

$$z(x) = H$$

$$z'(x) = \sqrt{\frac{v_o^2}{v_o^2 + 2gH}}$$

It is easy here to substitute $z(x)$ for H:

$$z'(x) = \sqrt{\frac{v_o^2}{v_o^2 + 2gz(x)}}$$

This is evidently a separable first-order differential equation.

$$\frac{dz}{dx} = \sqrt{\frac{v_o^2}{v_o^2 + 2gz}}$$

$$\sqrt{\frac{v_o^2 + 2gz}{v_o^2}}\, dz = dx$$

$$\frac{2}{3}\left(\frac{v_o^2}{2g}\right)\left(\frac{2g\,z}{v_o^2} + 1\right)^{3/2} = x + C_1$$

$$\left(\frac{2g\,z}{v_o^2} + 1\right)^{3/2} = \frac{3g\,x}{v_o^2} + C_2$$

Which leaves,

$$z = \frac{v_o^2}{2g}\left[\left(\frac{3g}{v_o^2}x + C\right)^{2/3} - 1\right]$$

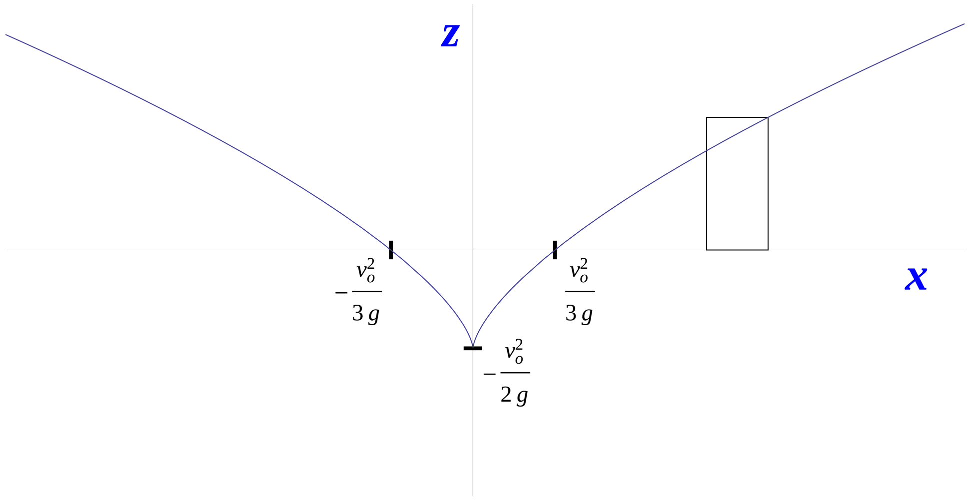
or

$$\boxed{z = \frac{v_o^2}{2g}\left[\left(\frac{3g}{v_o^2}x\right)^{2/3} - 1\right]}$$

We can set the constant of integration equal to zero because it will only move the curve left or right. The actual curve, which is of interest, will not be altered.

$z(x)$ intersects the x-axis at $x = \frac{v_o^2}{3g}$. To the right of this point, the values of the height of the curve z form a bijection with all nonnegative real numbers. As can be seen, our optimal launch curve increases by a power of two-thirds, approaching zero as height approaches infinity. It is also appropriate that the units of $z(x)$ pan out to be the units of distance, the same as the units of H.

If launching a projectile with speed v_o atop a building of height H, find the point on $z(x)$ at which $z(x) = H$, then fire the projectile in the same direction as $z'(x)$. The range of the projectile will be the maximum range.



Finally, we notice that our optimal launch curve is symmetric. To the left of $x = -\frac{v_o^2}{3g}$, launching in the direction of the derivative will yield maximum range in the *other direction*. The negative side is yet another bijection with all nonnegative real numbers.

4 Further Analysis

One can show that the derivative of the curve $z(x)$ yields the tangent of the optimal launch angle, better stated as the direction to attain maximum range. Taking as an example $H = 0$, the optimal angle from the ground should be 45^o. Therefore, the derivative where z is zero should be one, or $z'(\frac{v_o^2}{3g}) = 1$. I first compute the derivative of z:

$$z'(x) = \frac{2}{3}\left(\frac{v_o^2}{2g}\right)\left(\frac{3gx}{v_o^2}\right)^{-\frac{1}{3}}\left(\frac{3g}{v_o^2}\right)$$

$$z'(x) = \left(\frac{v_o^2}{3gx}\right)^{1/3}$$

Quite fittingly,

$$z'\left(\frac{v_o^2}{3g}\right) = 1$$

Moreover, it should be feasible to derive the curve $z(x)$ by equating its derivative with the tangent of the optimal launch angle. Since $\theta_o = f(H)$, we have

$$z'(x) = \tan\theta_0 = \tan\left(f(H)\right)$$

Or

$$\left(\frac{v_o^2}{3gx}\right)^{1/3} = \sqrt{\frac{v_o^2}{v_o^2 + 2gH}}$$

One can substitue z for H and retrieve the original curve of $z(x)$:

$$z(x) = \frac{v_o^2}{2g}\left[\left(\frac{3g}{v_o^2}x\right)^{2/3} - 1\right]$$

Finally, it was stated earlier that the parabolic trajectory drawn out by a particle fired with velocity v_o from the ground at 45^o initial angle can approximate the "launch curve" $z(x)$, but only for small intial heights relative the height of the peak of the parabola. Put another way, the derivative of this parabola where H is close to zero has a value close to the value of $z'(x)$. I will show that the ratio of the difference of the two derivatives to the value of $z'(x)$ is close enough to zero so as to be negligible if H is close to zero. I will first write the function $y(x)$ which will describe the parabola mentioned earlier. Shooting with $H = 0$ and $\theta_i = 45^o$, the curve can be described by the time parameter:

$$x = (v_o \cos 45^o)t$$
$$y = -\tfrac{1}{2}gt^2 + (v_o \sin 45^o)t$$

Evaluating the trigonometric expressions and substituting for t,

$$y = -g\left(\frac{x}{v_o}\right)^2 + x$$
$$y = -\frac{g}{v_o^2}x^2 + x$$

But to properly compare with $z(x)$, we must shift the curve to the right by $\dfrac{v_o^2}{3g}$, producing the final $y(x)$:

$$y(x) = -\frac{g}{v_o^2}\left(x - \frac{v_o^2}{3g}\right)^2 + \left(x - \frac{v_o^2}{3g}\right)$$

Taking the derivative,

$$y'(x) = -2\frac{g}{v_o^2}\left(x - \frac{v_o^2}{3g}\right) + 1$$

Now the ratio of the difference of the derivatives to $z'(x)$ is computed:

$$\frac{\frac{dz}{dx} - \frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\left(\frac{v_o^2}{3gx}\right)^{1/3} - \left(-2\frac{g}{v_o^2}\left(x - \frac{v_o^2}{3g}\right) + 1\right)}{\left(\frac{v_o^2}{3gx}\right)^{1/3}}$$

But if H is close to zero, then $z'(x) = \left(\frac{v_o^2}{3gx}\right)^{\frac{1}{3}} \approx 1$! We have:

$$\frac{\frac{dz}{dx} - \frac{dy}{dx}}{\frac{dz}{dx}} \approx 2\frac{g}{v_o^2}\left(x - \frac{v_o^2}{3g}\right)\left(\frac{3g}{v_o^2}x\right)^{1/3}$$

If H is close to zero, then the value of x for which $z(x) = H$ must be close to its x-intercept. Therefore, $x \approx \dfrac{v_o^2}{3g}$ and the above ratio must always be arbitrarily small as H is small.

We have shown that a parabolic launch curve can be used in place of the more proper $z(x)$ if H is a small value. The parabolic launch curve drawn out must be the projectile of a particle shot from the ground with angle 45^o and initial velocity v_o.

5 Application to Compton Scattering

Using the conservation of relativistic energy and conservation of relativistic momentum laws, one can show that a photon will redshift and deflect after colliding with a lone electron, described by the equation,

$$\Delta\lambda = \frac{h}{m_e c}\left(1 - \cos\theta\right)$$

where h is Planck's constant, c is the speed of light, and m_e is the accepted mass of the electron. θ is the angle from the horizontal which the photon deflects, and $\Delta\lambda$ is the photon's increase in wavelength. This phenomenon is known as *Compton Scattering*.

We can apply the same method to this formula and build a height-based derivative. Doing so will provide visual aid in how $\Delta\lambda$ changes as θ changes. We proceed to build the corresponding curve $z(x)$. The two conditions on z are:

$$z(x) = \Delta\lambda$$
and
$$z'(x) = \tan\left(f(\Delta\lambda)\right)$$

where $f(\Delta\lambda) = \cos^{-1}\left(1 - \frac{m_e c}{h}\Delta\lambda\right)$. We can easily find that

$$\begin{aligned}\tan\left(f(\Delta\lambda)\right) &= \frac{\sqrt{1 - \left(1 + \left(\frac{m_e c}{h}\right)^2 \Delta\lambda^2 - 2\left(\frac{m_e c}{h}\right)\Delta\lambda\right)}}{\left(1 - \frac{m_e c}{h}\Delta\lambda\right)} \\ &= \frac{\sqrt{\left(\frac{m_e c}{h}\right)\Delta\lambda\left(2 - \left(\frac{m_e c}{h}\right)\Delta\lambda\right)}}{\left(1 - \frac{m_e c}{h}\Delta\lambda\right)}\end{aligned}$$

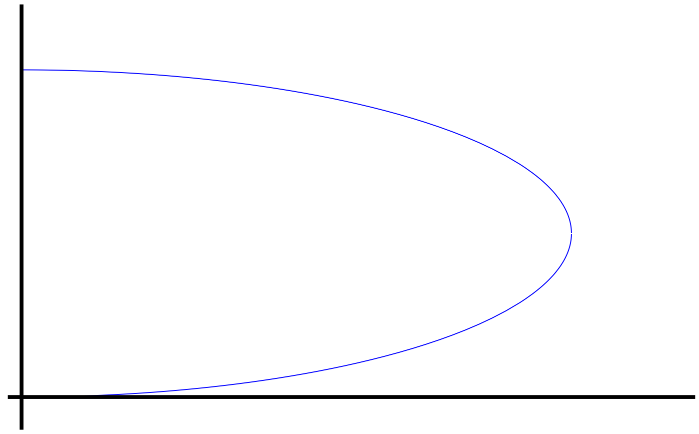
Substituting z for $\Delta\lambda$, we get the separable first order differential equation,

$$\frac{dz}{dx} = \frac{\sqrt{\left(\frac{m_e c}{h}\right)z\left(2 - \left(\frac{m_e c}{h}\right)z\right)}}{\left(1 - \frac{m_e c}{h}z\right)}$$

Which eventually yields:

$$\boxed{x = \sqrt{\frac{2h}{m_e c}z - z^2}}$$

This can be plotted implicitly, the graph of such a result given below:



We finish by observing the interesting properties of this graph. It strongly resembles the characteristics of the original formula for Compton Scattering as there seems to be a maximum redshift in the photon's wavelength. Where $z = 0$, the photon did not deflect at all (there was no electron there). The graph has a tip with an infinite-derivative spot because the photon will redshift that value of $z(x)$ if it scatters exactly 90^o. Theoretically, if the photon were to deflect more than 90^o, the redshift would be even greater, as described by the upper part of the graph. The theoretical maximum redshift is the value given where the photon bounces backward entirely (highly unlikely) at an angle of 180^o.

Conclusion

The curve created with a height based on the curve's derivative allows an analysis of mathematical equations in such a way so as to allow a highly physical and visual aspect. Where there was only a formula for optimal angle, there became a launch curve after which one could fit hypothetical building height-values to see geometrically what the optimal angle for range is. With this analysis, not only will an angle be visual, but how it changes quantitatively can be measured. The method of creating height-based curves can be applied to any set of conditions with one angle and one gradually increasing scalar value.

Sources

Moses, Clement J., Curt A. Moyer & Raymond A. Serway. *Modern Physics*. Third Edition. United States: Brooks/Cole, 2005. Print.