Optimal Steering of Nonredundant Single-Gimbal CMGs using Gauss Pseudospectral Method

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Abstract—**In this paper, we consider the problem of gimbal angle trajectory planning in nonredundant control moment gyroscope (CMG) configurations—that is, configurations with n = 3 single-gimbal CMG rotors. Most single-gimbal CMG configurations would use a redundant n = 4, 5, or 6 rotor design because the resulting Jacobian would always have a nontrivial nullspace. Most of the literature focuses on analyzing the four-CMG pyramid configuration. Clearly, an n = 4 configuration is advantageous because it is the most mass-efficient while still being redundant. However, if one CMG rotor fails, then the configuration becomes nonredundant, and singularity avoidance is much more difficult. To the best of the authors' knowledge, not much literature is devoted to gimbal steering in the case of three-CMG configurations. This paper deals with the specific case of a four-CMG pyramid configuration where one CMG has failed and only three CMGs are operational. We use the Gauss pseudospectral method to find the optimal control, i.e. gimbal rates, and states of an agile spacecraft subject to mission-specific constraints while we maximally avoid the angular-momentum singularities. The proposed steering law will be tested in scenarios where the gimbal angles initially start in both nonsingular and singular positions. Results are compared with the (non-failed) four-CMG maneuvering via the generalized singularity robust steering law in the literature.**

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1. INTRODUCTION

For the past half-century, control moment gyroscopes (CMGs) have been studied extensively and used for the attitude control of various spacecraft and space missions. CMGs are often chosen as the primary means of attitude control in agile spacecraft because of their high torque output. A CMG contains a spinning rotor which is kept rotating at a constant angular speed but whose angular momentum vector changes direction relative to the spacecraft by gimballing the spinning rotor. Reaction wheels differ from CMGs by instead keeping their angular momentum vector directions constant and varying their angular speeds to apply torque. Reaction wheels have the advantage of mathematical simplicity, though their maximum torque is typically less than 2 N·m [7]. CMGs, on the other hand, can have a range of 100-5000 N·m of maximum torque [7]. The CMG has a high torque capability because it is a torque amplification device—a small gimbal torque input produces a large control torque output on the spacecraft [2], [7]. Thus CMGs are usually preferred in missions where spacecraft agility is desired.

CMGs can be single-gimbal (SGCMG) or double-gimbal (DGCMG). In DGCMGs, the angular momentum vector can point along any direction in the sphere, while SGCMGs allow pivoting of the angular momentum vector only about a gimbal axis, orthogonal to the angular momentum vector. DGCMGs offer the advantage of simpler singularity avoidance, but they have the disadvantage of more difficult hardware assembly. In recent decades, several types of SGCMG steering logic have been proposed which aptly avoid singularities [3], and so faithful SGCMG implementations are becoming more common.

In this paper, we consider the problem of gimbal angle trajectory planning in nonredundant CMG configurations—that is, configurations with $n = 3$ SGCMG rotors. Most SGCMG configurations would use a redundant $n = 4$, 5, or 6 rotor design because the resulting Jacobian would always have a nontrivial nullspace. With a nontrivial nullspace, the gimbal angles can be varied without producing an output torque on the spacecraft body, and this so-called "null motion" helps to steer the gimbal angles out of or away from singularities without excessive deviation from the commanded torque. The gimbal angles are in a singularity if the possible output torque, which should ideally span three dimensions, is reduced in dimensionality to two (or one) dimensions. Avoiding singularities is the main mathematical difficulty addressed in any SGCMG steering method. Since SGCMGs are the predominant focus of study, we will hereafter take "CMG" to mean "SGCMG."

Most of the literature [1], [3], [5], [6], [7], [8], [11], [12], [13], [14], [17], [18], [19] focuses on analyzing the four-CMG pyramid configuration, depicted in Figure 1. Clearly, an $n = 4$ configuration is advantageous because it is the most mass-efficient while still being redundant, as long as the steering algorithm can effectively deal with singular states. However, if one CMG rotor fails, then the configuration becomes nonredundant, and singularity avoidance is much more difficult. It is still possible to provide three-axis attitude control on the spacecraft as long as there are at least three functioning CMG rotors with independent gimbal axes. Not much literature is devoted to gimbal steering in the case of three-CMG configurations. Ref. [19] proposes a method by constraining the momentum envelope, but the method requires the skew angle β to be varied, which isn't usually possible in SGCMG designs. Ref. [6] provides "preferred" gimbal angles for three-CMG configurations, a useful con-

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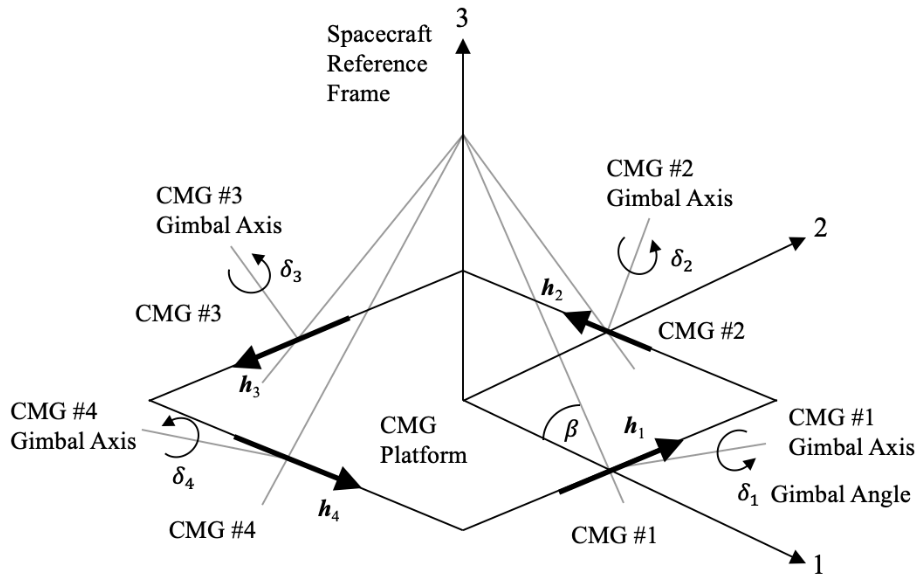


Figure 1: Four-CMG Pyramid Configuration

struct in developing a steering law. This paper deals with the specific case of a four-CMG pyramid configuration where one CMG has failed and only three CMGs are operational. (We'll call this the 3/4 pyramid configuration.)

In this paper, we will compute gimbal angle trajectories using the Gauss pseudospectral method, as outlined in [15], [16]. Pseudospectral methods are direct methods which have been useful for computational optimal control. The states' and controls' evolution over time are approximated with Lagrange interpolating polynomials collocated at certain node points. Computation of the optimal state and control values at the node points becomes a nonlinear programming problem (NLP). In the Gauss pseudospectral method, the node points are at the Legendre-Gauss (LG) points.

Optimal gimbal angle trajectory planning has been used before on CMGs [11], [12], [13], [14], [18]. Ref. [12] used a directed search method. Refs. [11], [13], [14] have used pseudospectral methods for optimization of some cost function. Ref. [11], however, provides no constraint on the singularity measure of the Jacobian, thus the method cannot guarantee that singularity will be maximally avoided during the maneuver. All of Refs. [11], [12], [13], [14] focus on redundant ($n \geq 4$) CMG configurations. Ref. [18] computes optimal gimbal angle trajectories for nonredundant CMG configurations and maximizes the integrated distance from singularity, though one drawback of the method used in [18] is that the final time is not free.

The organization of this paper is as follows. We begin by presenting the kinematic and dynamic equations of motion for a spacecraft equipped with CMGs. In doing so, we make sure to clearly state all reasonable assumptions required for the equations to be valid. Next, we outline the Gauss pseudospectral method and show how it was applied to the 3/4 pyramid configuration. With the Gauss pseudospectral method as a tool, we then describe our strategy for finding a balance between minimum time and maximum singularity avoidance in computing the optimal state trajectories. Finally, we implement our strategy on a numerical example, and plots of the resulting quaternion, angular velocity and gimbal angle trajectories are displayed. Comparisons are made to the trajectories computed using the generalized singularity robust (SR) steering [7] applied to the (non-failed) 4-CMG pyramid configuration.

2. CMG Equations of Motion

Consider a spacecraft rotating within an inertial reference frame. Fix a non-rotating reference frame onto the spacecraft so that the origin always coincides with the spacecraft's center of mass; we'll call this the "space frame." (Note that this reference frame is generally noninertial.) Attach a co-rotating reference frame to the body of the spacecraft with the origin at the spacecraft's center of mass; we'll call this the "body frame." The body frame rotates exactly with the spacecraft's rotation. Since we are only concerned with the spacecraft's orientation and not its position within the inertial frame, we only need to consider the space frame and body frame.

Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ be the angular velocity vector of the spacecraft, given in body frame coordinates. Let $\mathbf{q} = (q_1, q_2, q_3, q_4)^T$ be the quaternion that describes the orientation of the spacecraft (or body frame) relative to the space frame. Here, q_4 is the scalar component and $\mathbf{q}_v \doteq (q_1, q_2, q_3)^T$ is the vector component. The kinematic equations of motion are given by [9]:

$$\dot{\mathbf{q}}_v = -\tfrac{1}{2}\boldsymbol{\omega} \times \mathbf{q}_v + \tfrac{1}{2}q_4\boldsymbol{\omega} \tag{1a}$$

$$\dot{q}_4 = -\tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{q}_v \tag{1b}$$

Given the external torque $\mathbf{T}_{\text{ext}}$ in the body frame, the evolution of $\boldsymbol{\omega}$ is given by:

$$J\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{h}} + \boldsymbol{\omega} \times \big(J\boldsymbol{\omega} + \boldsymbol{h}\big) = \mathbf{T}_{\text{ext}} \tag{2}$$

Here, J is the spacecraft's tensor of inertia, with coordinates given in the body frame, which includes the CMG rotors.

$\boldsymbol{h}$ is the total angular momentum of all CMG rotors *from the perspective of a bystander fixed on the spacecraft and rotating with the spacecraft*. Note that this is *not* the same as the total angular momentum of the CMG rotors with coordinates taken in the body frame. See the Appendix for a clarification on the meaning of the spacecraft tensor J.

In this paper, we will focus on the four-CMG pyramid configuration, shown in Figure 1, except with one CMG rotor in failure. Without loss of generality, let CMG #4 be in failure. Let $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)^T$ be the vector of CMG gimbal angles. Assume each CMG's gimbal rate $\dot{\delta}_i$ is negligible compared to its rotor angular velocity. (This is equivalent to ignoring $\ddot{\delta}_i$ in $\dot{\boldsymbol{h}}$ within other literature.) Also assume that each CMG rotor's center of mass is fixed within the spacecraft at all times. Also assume each CMG rotor's axial direction is a principal axis of the rotor, and let each CMG rotor have the same constant axial angular momentum magnitude h. In this case, $\boldsymbol{h}$ is given by:

$$\begin{aligned} \boldsymbol{h} &= \boldsymbol{h}_1(\delta_1) + \boldsymbol{h}_2(\delta_2) + \boldsymbol{h}_3(\delta_3) \\ &= h\left(\begin{bmatrix} -c\beta\, s_1 \\ c_1 \\ s\beta\, s_1 \end{bmatrix} + \begin{bmatrix} -c_2 \\ -c\beta\, s_2 \\ s\beta\, s_2 \end{bmatrix} + \begin{bmatrix} c\beta\, s_3 \\ -c_3 \\ s\beta\, s_3 \end{bmatrix} \right) \end{aligned} \tag{3}$$

where $c\beta \doteq \cos\beta$, $c_i \doteq \cos(\delta_i)$, and likewise for sine. Then $\dot{\boldsymbol{h}}$ is:

$$\dot{\boldsymbol{h}} = A\dot{\boldsymbol{\delta}} \tag{4}$$

Here, A is called the *Jacobian*, given by:

$$A = h\begin{bmatrix} -c\beta\, c_1 & s_2 & c\beta\, c_3 \\ -s_1 & -c\beta\, c_2 & s_3 \\ s\beta\, c_1 & s\beta\, c_2 & s\beta\, c_3 \end{bmatrix} \tag{5}$$

The gimbal angles are in a singular position exactly when the Jacobian has a rank less than 3. In our case, since A is square, the Jacobian is singular if and only if $\det(A) = 0$. In the absence of external torques, equation (2) becomes:

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \left(J\boldsymbol{\omega} + \boldsymbol{h}\right) = -A\dot{\boldsymbol{\delta}} \tag{6}$$

It is clear that when the gimbal angles are singular, the CMG assembly is only able to apply torque over at most a two-dimensional space, and this is undesirable.

3. The Gauss Pseudospectral Method

In this section, we describe the Gauss pseudospectral method, as outlined in [15], [16]. First, the continuous Bolza problem with time $t \in [t_0, t_f]$ is transformed into the normalized time $\tau \in [-1, 1]$ with the affine transformation:

$$t = \frac{t_f - t_0}{2}\tau + \frac{t_f + t_0}{2} \tag{7}$$

We then wish to minimize the cost functional, generally given by:

$$\begin{aligned} C &= \Phi[\mathbf{x}(-1), t_0, \mathbf{x}(1), t_f] \\ &\quad + \frac{t_f - t_0}{2}\int_{-1}^{1} g[\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f]\, d\tau \end{aligned} \tag{8}$$

We must determine the state $\mathbf{x}(\tau) \in \mathbb{R}^r$ and the control $\mathbf{u}(\tau) \in \mathbb{R}^m$ which minimizes C. The initial time t_0 and final time t_f may be fixed or free. The state and control are subject to the constraints:

$$\begin{aligned} \frac{d\mathbf{x}}{d\tau} &= \frac{t_f - t_0}{2}\,\mathbf{f}[\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f] \in \mathbb{R}^r \quad \text{(diff. equation)} \\ \boldsymbol{\phi}\,[\mathbf{x}(-1), t_0, \mathbf{x}(1), t_f] &= \mathbf{0} \in \mathbb{R}^q \quad \text{(boundary conditions)} \\ \mathbf{C}\,[\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f] &\le \mathbf{0} \in \mathbb{R}^c \quad \text{(path constraints)} \end{aligned}$$

If we wish to have N internal node points, we use the Legendre-Gauss (LG) points, i.e. the roots of the Nth degree Legendre polynomial:

$$-1 < \tau_1 < \tau_2 < \cdots < \tau_N < 1$$

Let $\tau_0 = -1$. Let $L_i(\tau)$ $(i = 0, 1, 2, ..., N)$ be the Lagrange interpolating polynomials:

$$L_i(\tau) = \prod_{j=0, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j} \ (i = 0, 1, 2, ..., N) \tag{9}$$

Note that $L_i(\tau_j)$ equals 1 if $i = j$ and 0 otherwise. We approximate the state as:

$$\mathbf{x}(\tau) \approx \mathbf{X}(\tau) \doteq \sum_{i=0}^{N} \mathbf{X}(\tau_i) L_i(\tau) = \sum_{i=0}^{N} \mathbf{X}_i L_i(\tau) \tag{10}$$

Now let $L_i^*(\tau)$ $(i = 1, 2, ..., N)$ be defined as:

$$L_i^*(\tau) = \prod_{j=1, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j} \ (i = 1, 2, ..., N) \tag{11}$$

(Note the index starts at 1 instead of 0.) We approximate the control as:

$$\mathbf{u}(\tau) \approx \mathbf{U}(\tau) \doteq \sum_{i=1}^{N} \mathbf{U}(\tau_i) L_i^*(\tau) = \sum_{i=1}^{N} \mathbf{U}_i L_i^*(\tau) \tag{12}$$

Differentiate $\mathbf{X}(\tau)$ to obtain an approximation for $\dot{\mathbf{x}}(\tau)$:

$$\dot{\mathbf{x}}(\tau) \approx \dot{\mathbf{X}}(\tau) = \sum_{i=0}^{N} \mathbf{X}_i \dot{L}_i(\tau) \tag{13}$$

Now we define the so-called differential approximation matrix $D \in \mathbb{R}^{N \times N+1}$ with elements:

$$D_{ki} = \dot{L}_i(\tau_k) \tag{14}$$

where $k = 1, 2, ..., N$ and $i = 0, 1, 2, ..., N$. The differential equation is then approximated as:

$$\sum_{i=0}^{N} D_{ki}\mathbf{X}_i = \frac{t_f - t_0}{2}\,\mathbf{f}(\mathbf{X}_k, \mathbf{U}_k, \tau_k; t_0, t_f) \ (k = 1, 2, ..., N) \tag{15}$$

We approximate $\mathbf{x}(\tau = 1)$ with $\mathbf{X}_f$, defined by using a Gauss quadrature:

$$\mathbf{X}_f \doteq \mathbf{X}_0 + \frac{t_f - t_0}{2}\sum_{k=1}^{N} w_k\, \mathbf{f}(\mathbf{X}_k, \mathbf{U}_k, \tau_k; t_0, t_f) \tag{16}$$

where w_k are the Gauss weights, given by:

$$w_k = \frac{2}{\left(1-\tau_k^2\right)\left[\dot{P}_N(\tau_k)\right]^2} \quad (k = 1, ..., N) \tag{17}$$

where $P_N(\tau)$ is the Nth degree Legendre polynomial. The cost C is approximated also using a Gauss quadrature:

$$C = \Phi(\mathbf{X}_0, t_0, \mathbf{X}_f, t_f) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \, g(\mathbf{X}_k, \mathbf{U}_k, \tau_k; t_0, t_f) \tag{18}$$

The boundary and path constraints are approximated as:

$$\boldsymbol{\phi}(\mathbf{X}_0, t_0, \mathbf{X}_f, t_f) = \mathbf{0} \tag{19}$$

$$\mathbf{C}(\mathbf{X}_k, \mathbf{U}_k, \tau_k; t_0, t_f) \leq \mathbf{0} \;\; (k = 1, 2, ..., N) \tag{20}$$

The cost (18) must be minimized subject to the constraints (15), (16), (19), and (20). This is a nonlinear programming (NLP) problem, which can be solved with any number of established software packages. In this paper, we used the `fmincon` function in MATLAB.

Application to 3/4 Pyramid Configuration

For our spacecraft maneuver using the nonredundant 3/4 pyramid configuration, we take the state to be:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\omega} \\ \boldsymbol{\delta} \end{bmatrix} \in \mathbb{R}^{10} \tag{21}$$

The control will be taken as $\mathbf{u} = \dot{\boldsymbol{\delta}} \in \mathbb{R}^3$. The states and controls must satisfy the coupled differential equations:

$$\dot{\mathbf{q}}_v = -\tfrac{1}{2}\boldsymbol{\omega} \times \mathbf{q}_v + \tfrac{1}{2} q_4 \boldsymbol{\omega} \tag{22a}$$

$$\dot{q}_4 = -\tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{q}_v \tag{22b}$$

$$\dot{\boldsymbol{\omega}} = -J^{-1}\left[\boldsymbol{\omega} \times \left(J\boldsymbol{\omega} + \boldsymbol{h}\right) + A\mathbf{u}\right] \tag{23}$$

$$\dot{\boldsymbol{\delta}} = \mathbf{u} \tag{24}$$

The initial time will be fixed at $t_0 = 0$. The final time t_f will be free. Further constraints such as maximum spacecraft slew rate or maximum gimbal rate may also be specified.

In the next section, we will present the computation results given specified parameters of the problem. We choose a preliminary cost function $C_1 = t_f$ so as to purely optimize maneuver time. However, it is desirable to have a maneuver that maximally avoids singularity. We then use the calculated value of t_f for a second calculation. In the second calculation, we set an upper bound on the final time (~$1.35 t_f$) and use the cost function:

$$C_2 = -\int_0^{t_f} \left|\det(A)\right| dt \tag{25}$$

By minimizing C_2, we essentially are maximizing the integrated distance from singularity. Results are presented for scenarios where the gimbal angles initially start in both nonsingular and singular positions.

4. Computation Results

It is important to review the assumptions made in our maneuver problem: 1) There are no external torques on the spacecraft. 2) Each CMG's gimbal rate $\dot{\delta}_i$ is negligible compared to its rotor angular velocity. 3) The spacecraft inertia tensor J is approximated as constant, even though it technically varies according to the orientations of the CMG rotors (see Appendix). 4) Each CMG rotor's center of mass is fixed within the spacecraft at all times. 5) Each CMG rotor's axial direction is a principal axis of the rotor. Note that we made no assumption that the gyroscopic term $\boldsymbol{\omega} \times \left(J\boldsymbol{\omega} + \boldsymbol{h}\right)$ is negligible.

We assumed a 3/4 pyramid configuration with a skew angle of $\beta = 53.13°$ (i.e. $\cos\beta = 0.6$). The axial angular momentum magnitude of each CMG rotor was taken to be $h = 1000 \text{ N} \cdot \text{m} \cdot \text{s}$. We used the spacecraft inertia tensor $J = \text{diag}(21400, 20100, 5000) \text{ kg} \cdot \text{m}^2$. The gimbal angle rates must not exceed 2 rad/s, and the spacecraft maximum slew rate is 10 deg/s.

The maneuver is rest-to-rest, so the initial and final values of $\boldsymbol{\omega}$ must be zero. We consider the spacecraft initially non-rotated, so the initial quaternion was taken to be $\mathbf{q}_0 = (0, 0, 0, 1)^T$. The final quaternion was chosen as $\mathbf{q}_f = (0.4,\ 0,\ 0,\ \sqrt{0.84})^T$. This corresponds to a $47°$ counter-clockwise roll about the x-axis.

We considered two cases. In Case 1, the initial gimbal angles were chosen as $\boldsymbol{\delta}_0 = (60,\ 180,\ -60)^T$ deg. These angles are nonsingular "preferred angles," calculated using the method demonstrated by Vadali et al. [5]. In Case 2, we chose the initial gimbal angles as $\boldsymbol{\delta}_0 = (90,\ 0,\ -90)^T$ deg. This is a singular position, where $\det(A) = 0$.

We proceeded in two stages. In the first stage, we computed the maneuver using the simple cost function $C_1 = t_f$, so as to purely minimize maneuver time. In the second stage, we used the computed value of t_f to set an upper bound on maneuver time (~$1.35 t_f$) and then re-calculated the maneuver using a new cost function:

$$C_2 = -\int_0^{t_f} \left|\det(A)\right| dt \tag{26}$$

Of course, if an estimate on the maneuver time is already known, one can skip directly to the second stage. (The computation of the second stage is generally faster.) All computations were done with $N = 20$ nodes.

Results from Case 1 are plotted in Figures 2 and 3. Results from Case 2 are plotted in Figures 4 and 5. The Jacobian determinants are plotted in Figure 6. The maneuver trajectories were plotted in comparison with the trajectories computed using the generalized singularity robust (SR) steering [7] applied to the (non-failed) 4-CMG pyramid configuration. For being in a failure state, the maneuver computed by the Gauss pseudospectral method performs reasonably well when compared to the generalized SR non-failure performance; the generalized SR maneuver takes 10 seconds, and the Gauss PS maneuver takes 12 seconds, while still maximally avoiding singularity. We see in the determinant plots in Figure 6 that the CMG configuration inevitably passes through singularities, but it is able to do so quickly and without lingering at the singularities.

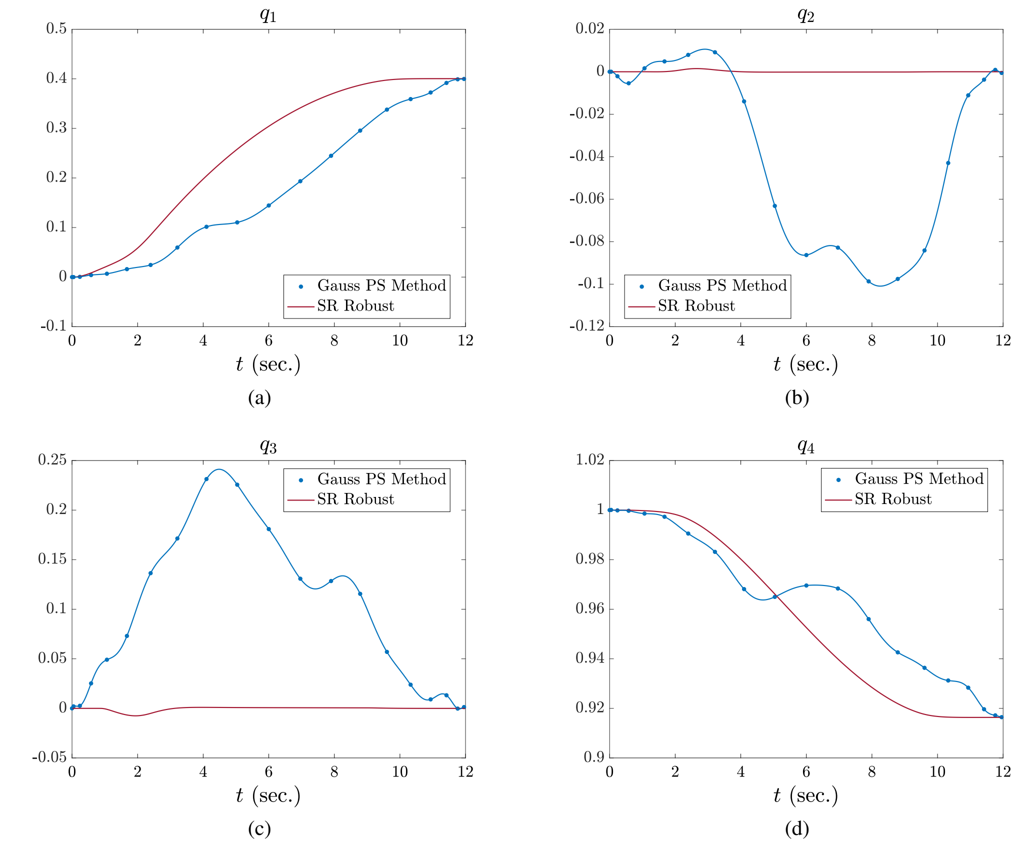


Figure 2: Quaternion components (Case 1)

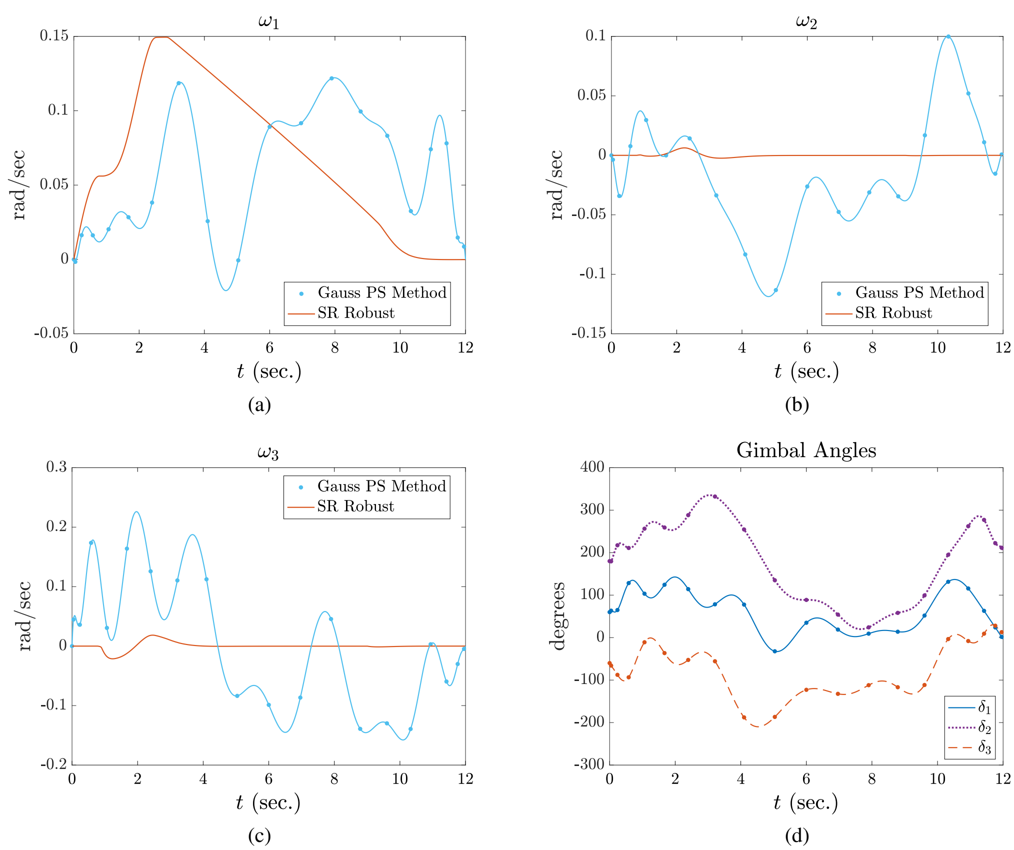


Figure 3: Angular velocity components and gimbal angles (Case 1)

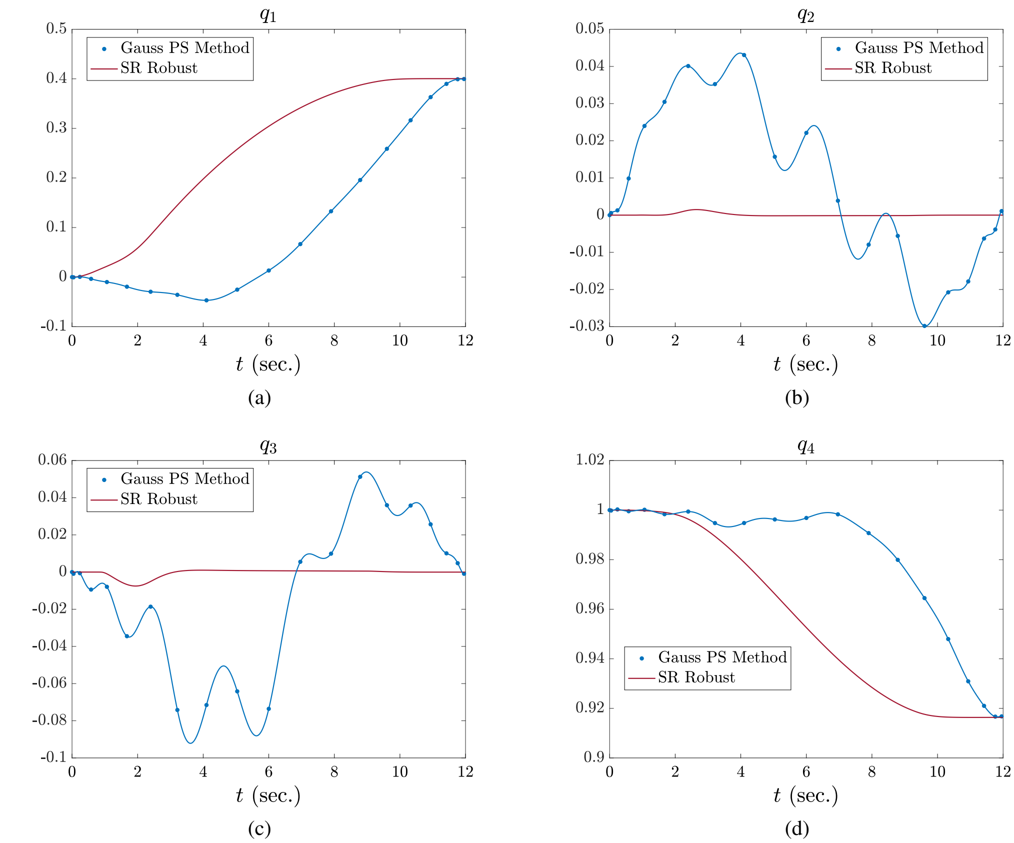


Figure 4: Quaternion components (Case 2)

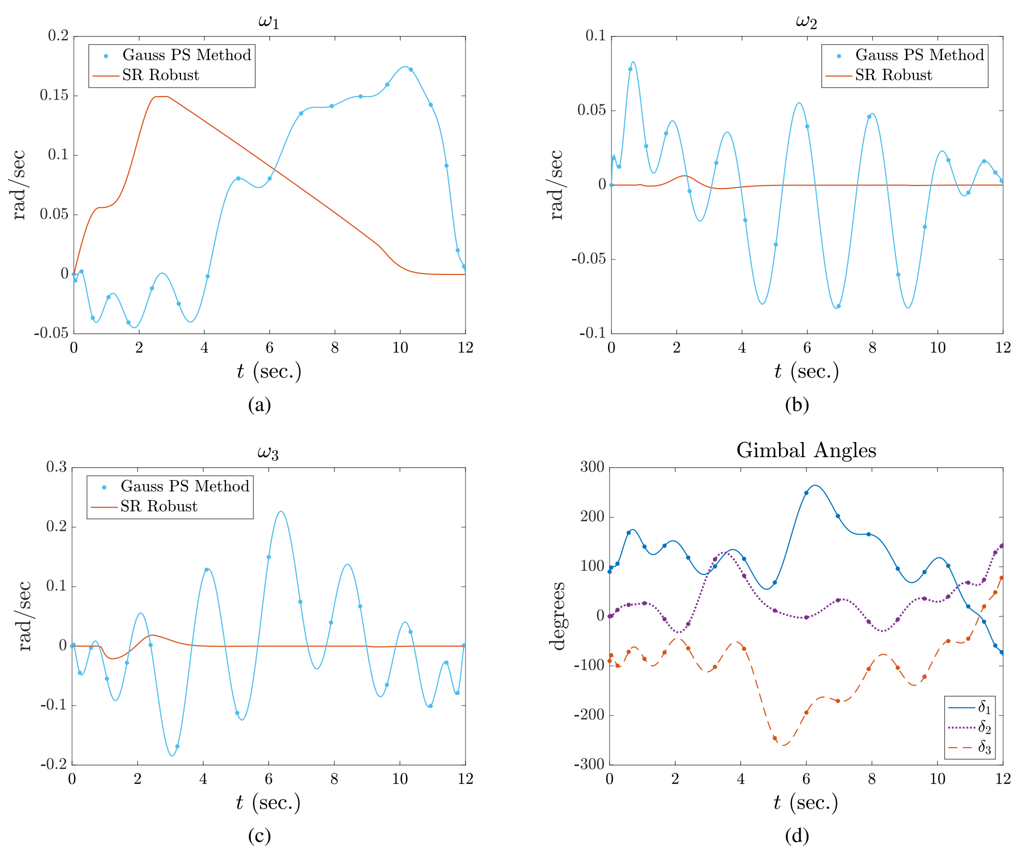
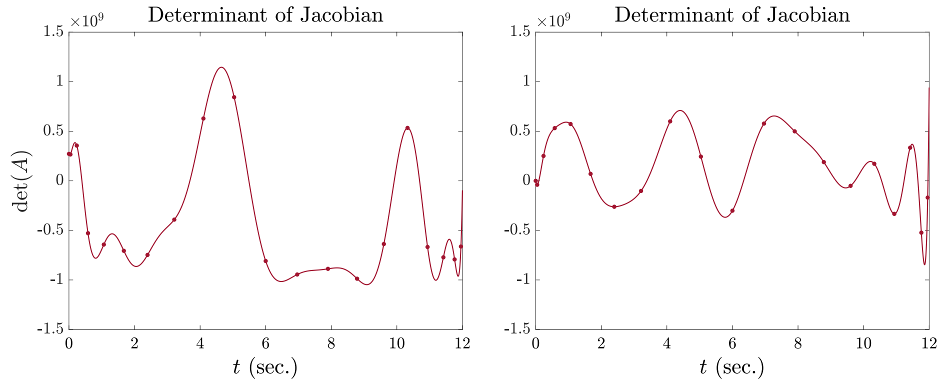


Figure 5: Angular velocity components and gimbal angles (Case 2)



(a) Case 1 (b) Case 2

Figure 6: Jacobian determinants (both cases)

5. Conclusions

In this paper, the Gauss pseudospectral method was successfully applied to computing gimbal angle trajectories of the 3/4 CMG pyramid configuration. Since the 3/4 pyramid configuration is nonredundant, care was taken to maximally avoid singularities. From the computation results, the maneuver fared reasonably well compared to the non-failure performance of the generalized singularity robust steering.

The Gauss pseudospectral method has the advantage that the cost function can be varied as desired for mission-specific preferences, whether greater emphasis should be placed on minimum time or maximal singularity avoidance. The main disadvantage of the Gauss pseudospectral method is that it is an open-loop approach; the method is trajectory planning for before the maneuver, not during. Also, it is sometimes difficult for the NLP to converge if the initial guess is not proper. Further study should hope to find closed-loop steering for the 3/4 pyramid configuration and/or methods to compute optimal trajectories more robustly to the initial guess. With stronger failure-accommodating steering, the 4-CMG configuration will become more and more faithful to implement.

Appendix: Clarification on Spacecraft Tensor J

In much of the literature on CMGs, details on the precise definition of the spacecraft tensor J are often omitted. Here, we will present a short derivation of equation (2) and a precise definition of J.

It is a well-known result of Euler that the total angular momentum $\mathbf{H}$ of a system, with coordinates taken in a reference frame rotating with angular velocity $\boldsymbol{\omega}$, satisfies the equation:

$$\dot{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{T}_{\text{ext}} \tag{27}$$

Thus equation (2) relies on the fact that the total angular momentum of the spacecraft-CMG system about the spacecraft's center of mass is $\mathbf{H} = J\boldsymbol{\omega} + \boldsymbol{h}$. Let there be n CMG rotors. Let J_s be the tensor of inertia for only the spacecraft body, not including the CMG rotors. Let $\boldsymbol{h}^{\text{total}}$ be the total angular momentum of the CMG rotors. Then the total angular momentum of the spacecraft-CMG system is:

$$\mathbf{H} = J_s\boldsymbol{\omega} + \boldsymbol{h}^{\text{total}} \tag{28}$$

Of course, $\boldsymbol{h}^{\text{total}}$ is the sum of each CMG's total angular momentum:

$$\boldsymbol{h}^{\text{total}} = \sum_{i=1}^{n} \boldsymbol{h}_i^{\text{total}} \tag{29}$$

Each CMG's total angular momentum is the angular momentum of its center of mass plus the angular momentum of itself about its center of mass:

$$\boldsymbol{h}_i^{\text{total}} = \boldsymbol{r}_i \times m_i\boldsymbol{v}_i + J_i\boldsymbol{\omega}_i^{\text{total}} \tag{30}$$

Here, m_i is the mass of the ith CMG rotor, $\boldsymbol{r}_i$ is the position vector of the ith CMG rotor's center of mass relative to the spacecraft's center of mass, and $\boldsymbol{v}_i$ is the velocity of the ith CMG rotor's center of mass relative to the spacecraft's center of mass. J_i is the tensor of inertia of the ith CMG rotor about its center of mass (not about the spacecraft center of mass), and $\boldsymbol{\omega}_i^{\text{total}}$ is the total angular velocity of the ith CMG rotor. Notice:

$$\begin{aligned} \boldsymbol{r}_i \times m_i\boldsymbol{v}_i &= m_i\,\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) \\ &= -m_i\,\boldsymbol{r}_i \times (\boldsymbol{r}_i \times \boldsymbol{\omega}) \\ &= -m_i\,\boldsymbol{r}_{i\times}^2\,\boldsymbol{\omega} \end{aligned} \tag{31}$$

where $\boldsymbol{r}_{i\times}$ is a 3-by-3 matrix written in terms of its components:

$$\boldsymbol{r}_{i\times} = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix} \tag{32}$$

Next, note that $\boldsymbol{\omega}_i^{\text{total}}$ is the spacecraft's angular velocity $\boldsymbol{\omega}$ plus the ith CMG rotor's axial angular velocity $\boldsymbol{\omega}_i^{(r)}$ plus the ith CMG's gimbal velocity $\boldsymbol{\omega}_i^{(g)}$:

$$\boldsymbol{\omega}_i^{\text{total}} = \boldsymbol{\omega} + \boldsymbol{\omega}_i^{(r)} + \boldsymbol{\omega}_i^{(g)} \tag{33}$$

We make the very common assumption that each CMG's gimbal velocity is negligible compared to its axial velocity. (This is equivalent to ignoring $\ddot{\delta}_i$ in $\dot{\boldsymbol{h}}$ within other literature.) Thus we can ignore the $\boldsymbol{\omega}_i^{(g)}$ term. This is especially reasonable considering the fact that a rotor's moment of inertia is usually largest in the axial direction. Plugging equations (29), (30), (31), and (33) back into equation (28), we get:

$$\begin{aligned} \mathbf{H} &= J_s\boldsymbol{\omega} + \sum_{i=1}^{n}\Big(-m_i\,\boldsymbol{r}_{i\times}^2\,\boldsymbol{\omega} + J_i\big(\boldsymbol{\omega} + \boldsymbol{\omega}_i^{(r)}\big)\Big) \\ &= \left[J_s + \sum_{i=1}^{n}\Big(J_i - m_i\,\boldsymbol{r}_{i\times}^2\Big)\right]\boldsymbol{\omega} + \sum_{i=1}^{n} J_i\boldsymbol{\omega}_i^{(r)} \end{aligned} \tag{34}$$

Note that the summation $\sum_{i=1}^{n} J_i \boldsymbol{\omega}_i^{(r)}$ is exactly $\boldsymbol{h}$ as defined in Section 2, as long as we have the assumption that each CMG rotor's center of mass is fixed within the spacecraft at all times. We arrive at our final desired result:

$$\mathbf{H} = J\boldsymbol{\omega} + \boldsymbol{h} \tag{35}$$

with

$$J \doteq J_s + \sum_{i=1}^{n} \left(J_i - m_i \boldsymbol{r}_{i\times}^2 \right) \tag{36}$$

Take notice that the J_i terms are *not* constant—they depend on the CMG rotors' orientations relative to the spacecraft. Thus J is technically not constant. However, it is common to make the approximation that J is constant.

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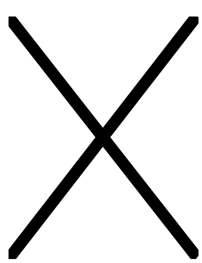

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