Examining the Linking Number

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Math 191 Final Paper

1 Introduction

For any given link, we can observe between any two single knots what is called the *linking number*. Between two non-intersecting closed curves in $\mathbb{R}^3$, the linking number is, practically speaking, the number of times either of the curves encircles the other curve. In this paper, we highlight a combinatorial definition and an integral definition of the linking number.

The combinatorial definition is the one which is easiest and most natural to compute for a human. As long as she is given a reasonable link diagram, she can simply count the crossings, in a specific way. The second definition, often attributed to Gauss, is a double integral over two non-intersecting closed curves. Though not the easiest to use computationally, its derivation has a fascinating origin in magnetostatics. We will briefly describe a third definition, taken from topics in differential geometry and homology. It defines a continuous linear map between two manifolds, and the resulting *degree* of the map will end up being equivalent to the linking number.

Our focus here will be on the Gauss integral. Because the result appears serendipitous and because it has a particularly accessible derivation, we delve deeply into this formulation. We proceed by giving each of the mentioned definitions, providing examples, and offering arguments toward the equivalency of the definitions of linking number.

2 Combinatorial Definition

The combinatorial definition of linking number requires a link diagram of the link. This definition can be found in many texts, for example in [RN11]. Linking number is a property of two knots. Given any particular link $\mathcal{L}$ (with at least two knots) in $\mathbb{R}^3$ and any two knots γ_1 and γ_2 in $\mathcal{L}$, the linking number $L(\gamma_1, \gamma_2)$ is independent of all of the other knots in $\mathcal{L}$. So we can make a link diagram for $\gamma_1 \sqcup \gamma_2$, one which ignores all of the other knots.

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Given our link diagram for $\gamma_1 \sqcup \gamma_2$, we look at each crossing involving both curves and then assign a value of ± 1 according to the following table:

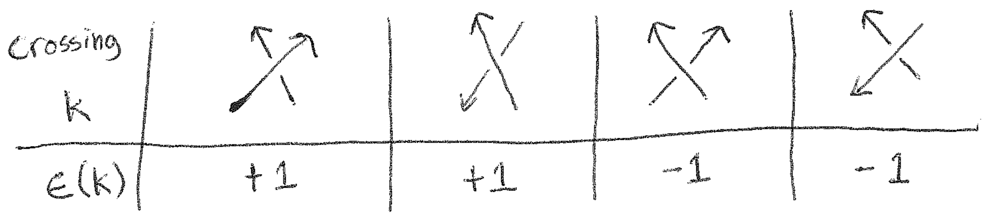


Figure 1

We can remember how to assign $\epsilon(k)$ with a right-hand rule: with your right hand, push the arrow of the curve on top toward the arrow of the curve on the bottom. If your thumb is pointing out of the page, $\epsilon(k)$ is $+1$.

With this ϵ function, the linking number is given as a summation over each crossing between the two knots in the link diagram:

$$L(\gamma_1, \gamma_2) = \frac{1}{2} \sum_{\text{crossings } k} \epsilon(k)$$

This first definition of linking number is reasonable and intuitive. It provides a simple method to calculate linking number just by looking at a carefully drawn link diagram. As an example, $\epsilon(k)$ is calculated for each crossing in the link in Figure 2. The linking number is 0.

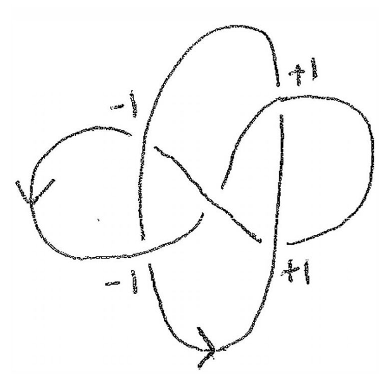


Figure 2: The Whitehead link, an example of a connected link with linking number zero.

In a sense, every pair of oriented curves γ_1 and γ_2 is classified by linking number. We state this "classification" theorem here, but we do not provide a proof:

Theorem 1. *If two non-intersecting, closed, oriented knots are embedded in $\mathbb{R}^3$, and each is allowed to pass through itself but not each other, then they can be so deformed into exactly one of the link diagrams illustrated in Figure 3.*

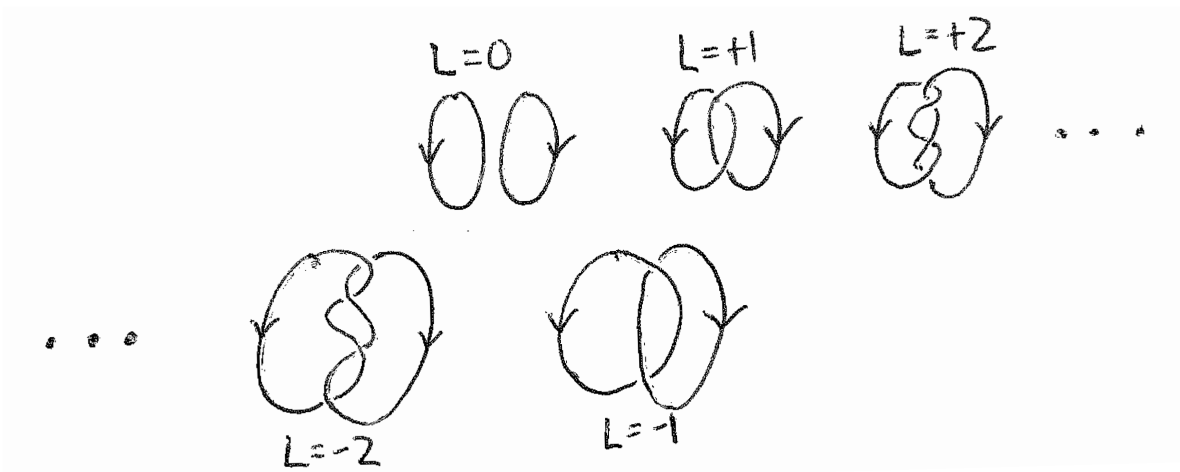


Figure 3: Every pair of oriented knots, if allowed to pass through themselves but not one another, can be deformed into exactly one of these diagrams.

Our first theorem above is a remarkable topological statement. It follows that linking number is an *invariant* of link diagram. Figure 4 shows how the Whitehead link can be deformed into the un-link.

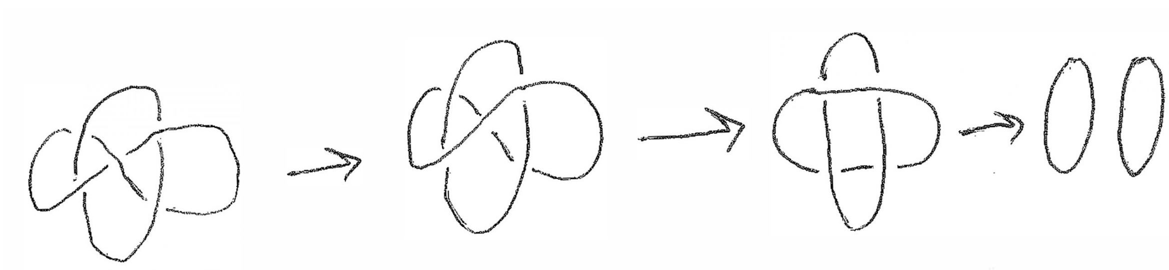


Figure 4: Unlinking the Whitehead link.

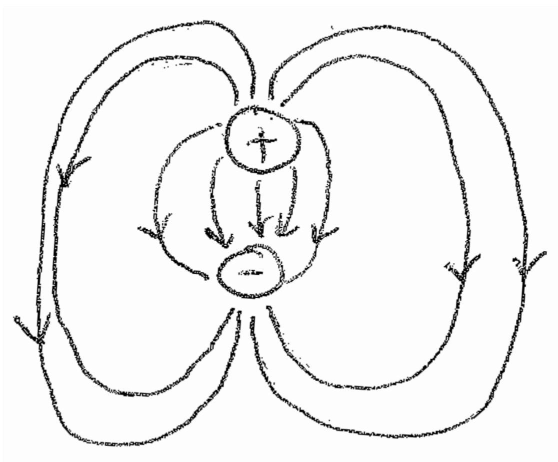
3 Gauss Integral Formulation

As early as 1832, Gauss noted in his personal diary the following expression for linking number between two curves γ_1 and γ_2 (see [RN11]):

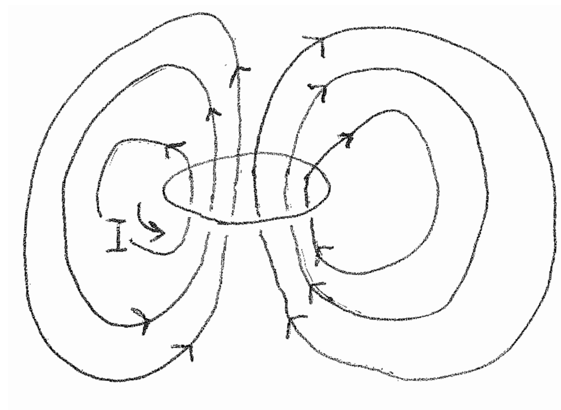
$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot \left(d\mathbf{r}_1 \times d\mathbf{r}_2\right)$$

Though he made no mention of its topological properties or physical application, it is likely he derived this expression from motivation in magnetostatics ([RN11], sec. 2).

Why magnetostatics? Recall that magnetostatics is the study of magnetic fields in the absence of changing current. In such regions, we can look at magnetic fields which are constant in time. Furthermore, in regions with non-zero and constant current, the magnetic field is no longer curl-free, by Ampère's law: $\nabla \times \mathbf{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}$. Electric fields, on the other hand, are always curl-free (except in cases of changing magnetic fields, but it will not be useful here to deal with moving magnets or changing currents). Take for instance these nearly identical fields created by dipole moments:



(a) Electric field from electric dipole.



(b) Magnetic field from magnetic dipole, a loop of current.

Figure 5

Observe the electric field created by the electrostatic dipole in Figure 5a. Since there is no changing magnetic field, the electric field is curl-free by Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = \mathbf{0}$. Moving a charged object along any closed path will require zero net work. As a result, no electrostatic configuration can be used to infer any topological properties of a closed curve taken by an electric charge.

Compare with the magnetic dipole created by the circular loop of constant current in Figure 5b. Where the current is, the magnetic field has a non-zero curl. Therefore, not all closed paths taken by a hypothetical magnetic monopole will require zero net work. Indeed, one can imagine moving the hypothetical monopole through and around the circular loop again and again, each time requiring some amount of energy to fight the magnetic vector field. This work done by a hypothetical magnetic monopole will become a measure of the linking number between two curves.

We begin by citing the upcoming theorem. First, some definitions:

Definition 2. A *magnetic shell* Σ is an infinitely thin, non-intersecting surface embedded in

$\mathbb{R}^3$ which is made of permanent magnet and has normal everywhere a magnetic north field on one side and a magnetic south field on the other side.

Think of Σ as a magnetized metallic sheet.

Definition 3. Σ will be called *uniform* if it has the same dipole strength everywhere. The dipole strength will be denoted Φ.

And now the theorem, due to Ampère, 1823 ([Gra98], sec. 367):

Theorem 4. *Every uniform magnetic shell Σ gives exactly the same magnetic field everywhere as would a constant current along the boundary of the shell: $C = \partial\Sigma$. The current would have dipole moment equal to Φ, the dipole strength of the shell Σ.*

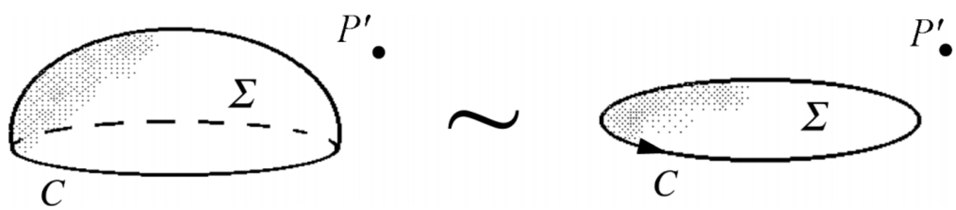


Figure 6: A uniform magnetic shell has the same magnetic field as a current along its boundary. Taken from [RN11].

Proof. Cut up Σ into a mesh of near-rectangular pieces as in Figure 7.

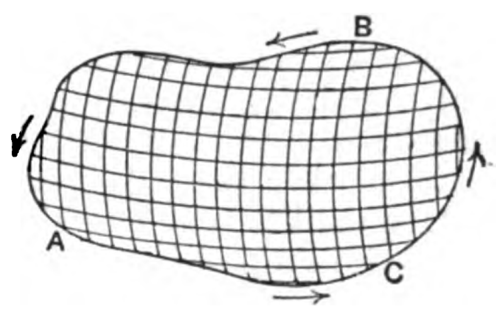


Figure 7: Loop of current divided into rectangular mesh. Taken from [Gra98], sec. 370.

If cut small enough and uniformly enough, each rectangular piece of magnetic shell gives exactly the magnetic field as would some current along the boundary of the small piece. Since Σ has equal dipole strength everywhere, the current around each small rectangle would be the same in magnitude. All of the currents would circulate in the same direction. Putting all the rectangular loops of current together, all current will cancel out except for the current at the boundary $\partial\Sigma$. $\square$

Though this theorem might appear surprising at first, there is an analogous result with electric dipoles: every path made of identical infinitesimal electric dipoles connected end-to-end gives the same electric field everywhere as a positive charge placed on one end and an equal negative charge placed on the other end. The dipole moment of the two charges $p = qd$ would be equal to the dipole moment of each infinitesimal dipole in the path. See Figure 8.

Figure 8: A path of identical infinitesimal electric dipoles connected end-to-end has the same electric field as electric charges located at the endpoints.

Theorem 4 gives us an easy corollary:

Corollary 5. *If two uniform magnetic shells Σ_1 and Σ_2 have the same dipole strength and the same boundary C, then they give the same magnetic field everywhere.*

Similarly, any two paths built end-to-end of identical infinitesimal electric dipoles and having the same endpoints will give the same electric field everywhere.

Remark. Before continuing, we address the concept of magnetic scalar potential. This value is what will be calculated to become the linking number between two curves. For any dipole, electric or magnetic, there are two types of "potential" about which we can ask. The first is the energy it takes to bring a unit electric or magnetic monopole, respectively, into the vicinity of the dipole from infinitely far away. The second is the energy it takes to rotate the dipole to a particular angle, given a stationary unit electric or magnetic monopole in the vicinity. For ideal dipoles, the former type of potential reduces to the latter. Even though we will not deal exclusively with ideal dipoles, we are primarily interested in *change* of potential from one point P' to the same point P' after traveling through a closed curve. In such a case, even for a non-ideal dipole, the contribution "from infinity" vanishes for a closed curve. Keeping these ideas in mind, we will adopt the latter definition. When we talk about potential V of a magnetic shell, we mean the energy it would take to rotate the shell a certain angle away from a specified hypothetical unit magnetic monopole at a point P' away from the shell.

With all that said, we are now ready to introduce a theorem due to Gauss ([RN11]) and Maxwell ([Max73]), independently:

Theorem 6. *The potential V of a uniform magnetic shell Σ at a point P' is the signed solid angle subtended at P' by Σ, multiplied by its magnetic dipole strength, Φ.*

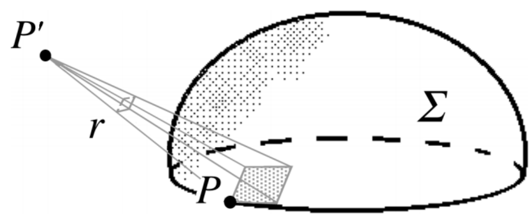


Figure 9: A uniform magnetic shell gives a potential proportional to its subtended solid angle. Taken from [RN11].

Proof. (Taken from [Max73], sec. 409) Consider a small portion of the shell with area dS located at P and whose normal vector is an angle θ from the line PP'. Let r be the distance from P to P'. Then the potential at P' due to dS is the well-known potential of a magnetic dipole:

$$dV = \frac{\Phi}{r^2} dS \cos\theta$$

Let $d\omega$ be the solid angle subtended by dS. Recall that a sphere has 4π steradians. Then

$$\frac{d\omega}{4\pi} = \frac{dS\cos\theta}{4\pi r^2}$$

so $dV = \Phi d\omega$. Integrating over all of the shell, we get that the total potential at P' is

$$V = \Phi\omega$$

□

Keep in mind that the solid angle is signed: ω is positive if P' is facing the northern magnetic field of the shell (a northern magnetic monopole should want to make the north side of the shell flip around). If the shell then wraps around behind with the opposite orientation, there will be some cancellation of solid angle. If the magnetic shell is a closed surface, like a sphere, and P' is outside the surface, then solid angle is zero. Indeed, the potential at P' would be zero. However, notice that if P' is *inside* the surface, then the potential is $V = -4\pi\Phi$, if southern field is toward the inside of the shell. This idea is important in understanding the next theorem ([Gra98], sec. 52).

Theorem 7. *Let Σ be a uniform magnetic shell of unit dipole strength. That is, $\Phi = 1$. If P' and P'' are infinitely close points on opposite sides of the shell Σ such that P' is on the north side, then the potential difference is $V(P') - V(P'') = 4\pi$.*

Proof. If Σ has no boundary, then it is a closed surface, and $V(P') - V(P'') = 4\pi(1)$. If Σ has a boundary, then close the shell by adding another magnetic shell Σ', also of unit dipole strength and having the same boundary as Σ. Since P' and P'' are infinitely close, they see the same solid angle subtended by Σ'. Then by Theorem 6, the potentials at P' and P'' change by the same amount. With Σ closed off by Σ', one of the potentials at P' or P'' is zero, and the other has potential $\pm 4\pi$, depending on which point is inside the surface. We see that the original potential difference must have been $V(P') - V(P'') = 4\pi$. □

We are finally ready to derive Gauss' integral formulation of linking number. Let γ_1 and γ_2 be two closed, non-intersecting curves embedded in $\mathbb{R}^3$. Let γ_1 be the curve with unit current and let γ_2 be the closed path to be taken by a hypothetical unit magnetic monopole. We will see later that reversal of the roles of the curves will not change the result.

Our goal is to calculate the total change in potential, ΔV, along the path γ_2 (this is also the total work required to bring the unit magnetic monopole along γ_2). Let P be a point on γ_1 and let P' be a point on the path of γ_2. Each motion along γ_2 contributes to ΔV, and so does each segment of wire on γ_1. Therefore, the total ΔV will be the result of a double integral over both curves. Notice that movement along γ_2 looks as if the wire γ_1 moves in the opposite direction. See Figure 10. The observed solid angle $d\omega$ is the change in magnetic potential from that small movement.

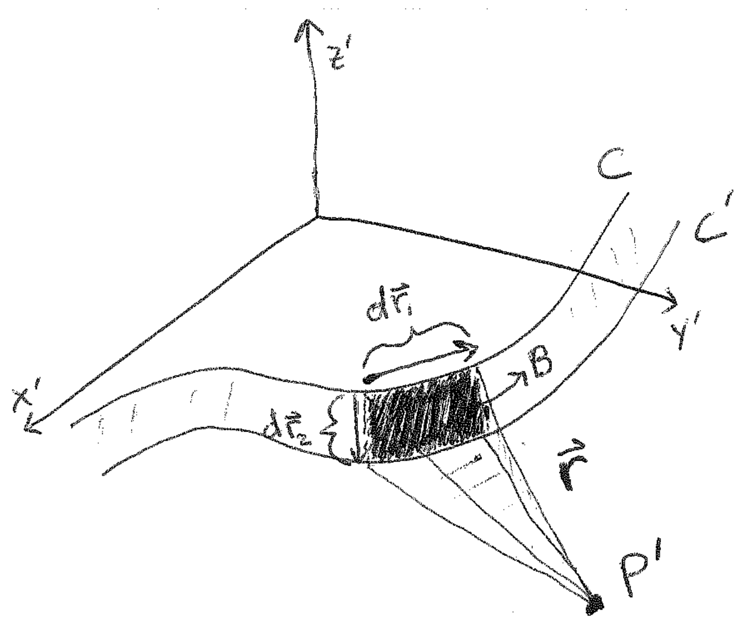
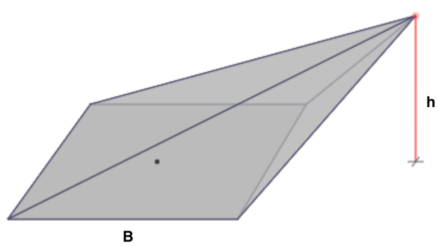


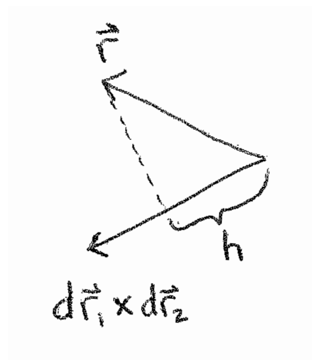
Figure 10: A monopole moving toward a loop of current can be treated as the loop of current traveling toward the monopole, creating an effective solid angle.

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the vectors which describe γ_1 and γ_2, respectively, our task is now to calculate the solid angle subtended by $d\mathbf{r}_1$ and $d\mathbf{r}_2$ from point P'. We do this by calculating the volume of the oblique pyramid segment in two different ways. Let B be the area of the patch. The volume of the pyramid is $W = \frac{1}{3}Bh$, where h is the perpendicular height from P' (see Figure 11a). Notice also that $Bh = A_\perp r$, where $A_\perp$ is the perpendicular projection of area and $r = |\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$ is the distance from P' to B. From our understanding of solid angle and from the fact that there are 4π steradians in a sphere, we can say $\frac{A_\perp}{4\pi r^2} = \frac{d\omega}{4\pi}$. So $A_\perp = r^2\omega$. This means volume of the oblique pyramid is given by

$$W = \frac{1}{3}A_\perp r = \frac{1}{3}r^3 d\omega$$



(a) Volume of an oblique pyramid is $\frac{1}{3}$ the area of the base, B, times the perpendicular height, h. Taken from [Ref].



(b) The perpendicular height for our pyramid element is the scalar projection of $\mathbf{r}$ onto $d\mathbf{r}_1 \times d\mathbf{r}_2$.

Figure 11

We can also find B and h directly. The area B is the magnitude of the cross product: $B = |d\mathbf{r}_1 \times d\mathbf{r}_2|$. For h, remember that h is the perpendicular height from P' drawn to B. Another way to write this is as the scalar projection from $\mathbf{r}$ onto $d\mathbf{r}_1 \times d\mathbf{r}_2$ (see Figure 11b). This is given by:

$$h = \frac{\mathbf{r} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2)}{|d\mathbf{r}_1 \times d\mathbf{r}_2|}$$

We can see that $W = \frac{1}{3}Bh = \frac{1}{3}\mathbf{r} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2)$.

Equating the two expressions for volume W, we get:

$$\frac{1}{3}r^3 d\omega = \frac{1}{3}\mathbf{r} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2)$$

or

$$d\omega = \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot \left(d\mathbf{r}_1 \times d\mathbf{r}_2\right)$$

Integrating over the entirety of both curves,

$$\begin{aligned}\Delta V &= \oint_{\gamma_1 \times \gamma_2} d\omega \\ &= \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot \left(d\mathbf{r}_1 \times d\mathbf{r}_2\right)\end{aligned}$$

Finally, notice that the unit monopole going around the curve γ_1 once is equivalent to crossing some unit magnetic shell Σ whose boundary is γ_1. By Theorem 7, this contributes 4π to the work done. It follows that going around the curve q times contributes $4\pi q$ to the work. So:

$$\Delta V = 4\pi L(\gamma_1, \gamma_2)$$

which gives us Gauss' result:

$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot \left(d\mathbf{r}_1 \times d\mathbf{r}_2\right)$$

Notice that switching the roles of γ_1 and γ_2 does not change the value, as there is a resulting negative from both the subtraction and the cross product.

4 Concluding Remarks

We have so far provided two equivalent definitions of linking number between two curves γ_1 and γ_2 in $\mathbb{R}^3$. There are in fact several equivalent definitions. Rolfsen cites eight different definitions, and proves their equivalence ([Rol76], ch. 5, pt. D). To conclude our discussion, we will go over one more definition of linking number, as presented in [RN11]. The background requires some differential topology; one resource often cited for review is Spivak ([Spi65]).

Let M and N be two compact, unbounded, oriented manifolds with the same dimension. Let $\mathbf{f} : M \to N$ be a continuous function.

Definition 8. Define the *degree* of $\mathbf{f}$ to be

$$\deg(\mathbf{f}) = \sum_{x \in Y} \operatorname{sign}\left[\det\left(\frac{\partial \mathbf{f}}{\partial x}\right)_x\right]$$

where Y denotes the set of *regular* points where $\det(\partial \mathbf{f}/\partial x)$ is nonzero.

Now give parametrizations to the curves γ_1 and γ_2: $\mathbf{r}_1(u)$ and $\mathbf{r}_2(v)$ where u and v range from 0 to 2π. Notice that u and v can draw out a torus. We can then define the following mapping between two-dimensional manifolds.

Definition 9. The *Gauss map* $\mathbf{n} : T^2 \to S^2$ is defined as

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_1(u) - \mathbf{r}_2(v)}{|\mathbf{r}_1(u) - \mathbf{r}_2(v)|}$$

As it turns out, we have another definition of linking number between γ_1 and γ_2.

$$L(\gamma_1, \gamma_2) = \deg(\mathbf{n})$$

Essentially, the regular points mentioned above are exactly where $\mathbf{r}_1 = \pm\mathbf{r}_2$ under a projection. These are exactly where the two curves will have a crossing with each other. $\left(\dfrac{\partial \mathbf{f}}{\partial x}\right)$ is a matrix, which ultimately explains how a small movement ∂x on a tangent plane at M makes a small movement $\partial \mathbf{f}$ on a tangent plane at N under mapping of $\mathbf{f}$ (or in this case $\mathbf{n}$). See Figure 12.

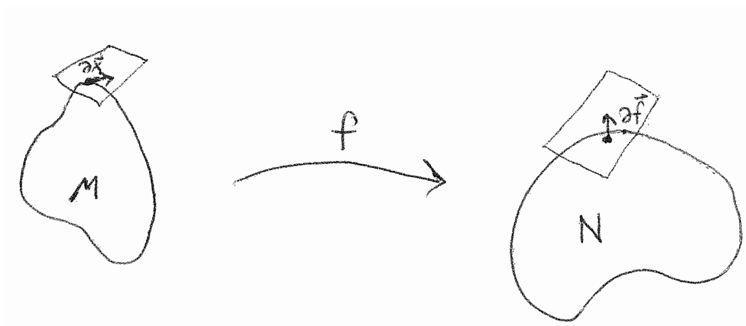


Figure 12

The determinant of that matrix gives a measure of how much the movement is scaled larger or smaller. We don't care about that; we care about the sign of the determinant. The sign indicates how permuted the axes are, after mapping. In maps between two-dimensional manifolds, such as $\mathbf{n}$, this is a question of whether a pair of axes changes handedness. See Figure 13. This contribution to the summation ultimately decides what kind of crossing is in the link diagram at that point. The total summation ends up becoming the linking number between the two curves.

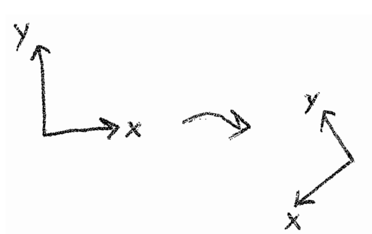


Figure 13: Axes which switch handedness.

References

[Gra98] Andrew Gray. *A Treatise on Magnetism and Electricity*, volume I. MACMILLAN & Co., Limited, New York, 1898.

[Max73] James Clerk Maxwell. *A Treatise on Electricity and Magnetism*. Clarendon Press, Oxford, 1873.

[Ref] Math Open Ref. Volume of a pyramid. http://www.mathopenref.com/pyramidvolume.html.

[RN11] Renzo L. Ricca and Bernardo Nipoti. Gauss' linking number revisited. *Journal of Knot Theory and its Ramifications*, 20(10):1325–1343, 2011.

[Rol76] Dale Rolfsen. *Knots and Links*. AMS Chelsea Publishing, Providence, RI, 1976.

[Spi65] Michael Spivak. *Calculus on Manifolds*. Addison-Wesley Publishing Company, USA, 1965.