Assignment 2

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1 Problem 2.1

Suppose $K_{3,3}$ were a planar graph. It has 6 vertices and 9 edges, and therefore it would have $2 - |V| + |E| = 5$ faces. Further, since $K_{3,3}$ is bipartite, all of its cycles have even length. Since there can be no cycles of length 2, all cycles have length greater than or equal to 4. This means each face must be adjacent to at least 4 edges. Further, each edge is adjacent to exactly two faces. Since there are 5 faces, there must be at least $(5)(4)/2 = 10$ edges. This is a contradiction because there are only 9 edges. This shows that $K_{3,3}$ cannot be planar.

Hopefully one of Alice, Bob or Charlie has a shovel. $\square$

2 Problem 2.8

Short answer: $\sim \dfrac{n^2(k-1)}{2k}$ for large n and small k, or (exactly) if $k|n$.

Long answer: If $k \geq n$, then we can put the most number of edges possible between the vertices: $\binom{n}{2} = \dfrac{n(n-1)}{2}$ (notice this becomes the formula above when $k = n$). For $k \leq n$, we realize that a graph G being k-colorable is equivalent to being able to partition the vertices $V(G) = V_1 \sqcup ... \sqcup V_k$ such that no two vertices in one coset share an edge.

Given a particular partition $\mathcal{P}$ of $V(G)$, define $\phi(\mathcal{P})$ to be the maximum number of edges between the n vertices such that no two vertices in the same coset share an edge. In order for G to be k-colorable, let $|\mathcal{P}| = k$. So then $\mathcal{P} = \{V_1, ..., V_k\}$. For each i, define $W_i = |V_i|$. We can calculate $\phi(\mathcal{P})$ by double-counting each possible edge:

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$$
\begin{aligned}
2\phi(\mathcal{P}) &= W_1(n - W_1) + W_2(n - W_2) + ... + W_k(n - W_k) \\
&= (W_1 + ... + W_k)n - (W_1^2 + ... + W_k^2) \\
&= n^2 - (W_1^2 + ... + W_k^2)
\end{aligned}
$$

We want to find a partition $\mathcal{P}$ that maximizes $\phi(\mathcal{P})$. To do this, we need to minimize the expression $W_1^2 + ... + W_k^2$ subject to the constraint $W_1 + ... + W_k = n$. We realize that $W_1 + ... + W_k = n$ is the equation for a hyperplane in k-dimensional space. Also, the different values for $W_1^2 + ... + W_k^2$ produce level surfaces which are hyperspheres in k-dimensional space. The minimum allowed value of $W_1^2 + ... + W_k^2$ corresponds to the hypersphere with least radius; this is the hypersphere which intersects the hyperplane where all the W_i are equal. We conclude that $W_1^2 + ... + W_k^2$ is minimized when each of the W_i are as close to each other as integers can be.

To conclude, we want to partition the vertices of G so that $W_1 = ... = W_{n \,(\mathrm{mod}\, k)} = \lceil \frac{n}{k} \rceil$ and $W_{n \,(\mathrm{mod}\, k)+1} = ... = W_k = \lfloor \frac{n}{k} \rfloor$. This gives our final expression for the most edges G can have while remaining k-colorable:

$$
\boxed{\frac{1}{2}\left[n^2 - \left(\big(n \,(\mathrm{mod}\, k)\big)\lceil n/k \rceil^2 + \big(n - n \,(\mathrm{mod}\, k)\big)\lfloor n/k \rfloor^2\right)\right]}
$$

When n is large and k is small, or when $k|n$, the expression reduces to

$$
\begin{aligned}
\frac{1}{2}\left(n^2 - k(n/k)^2\right) &= \frac{1}{2}\left(n^2 - \frac{n^2}{k}\right) \\
&= \frac{n^2}{2}(1 - \frac{1}{k}) \\
&= \frac{n^2(k-1)}{2k}
\end{aligned}
$$

3 Problem 2.12

Let G be a simple and connected planar graph. Let ∂_F be the boundary map on the faces of G. Applying the Rank-Nullity Theorem,

$$
|F| = \dim(\mathcal{F}) = \dim(\mathrm{Ker}(\partial_F)) + \dim(\mathrm{Im}(\partial_F))
$$

There are only two elements in ∂_F which map to zero: the zero element and the collection of all faces. Any other combination of faces maps to a non-zero linear combination of cycles. This makes $\dim(\mathrm{Ker}(\partial_F)) = 1$. Further, since G is planar, ∂_F is surjective onto the cycle space $\mathcal{C}$. To see this, let c be any element of $\mathcal{C}$. c is a linear combination of cycles. Each of these cycles corresponds to a face. The ∂_F mapping of the sum of all of these faces will map to c. Since every element in the image of ∂_F is also a combination of cycles, we see that $\mathrm{Im}(\partial_F) = \mathcal{C}$.

We have then that

$$|F| = 1 + \dim(\mathcal{C})$$

From Exercise 1.11, $\dim(\mathcal{C}) = |E| - |V| + 1$. So

$$|F| = 1 + (|E| - |V| + 1)$$

which means

$$|V| - |E| + |F| = 2 \qquad \square$$

Alternate proof: We have

$$|F| = 1 + \dim(\mathcal{C})$$

We can look at the dual graph of G and also conclude:

$$|V| = |F^*| = 1 + \dim(\mathcal{C}^*)$$

Adding these two equations, we get:

$$\begin{aligned} |V| + |F| &= 2 + \dim(\mathcal{C}) + \dim(\mathcal{C}^*) \\ &= 2 + \dim(\mathcal{C} \oplus \mathcal{C}^*) \end{aligned}$$

Recall that $\mathcal{C}^* = \mathcal{C}^\perp$. So $\mathcal{C} \oplus \mathcal{C}^* = \mathcal{C} \oplus \mathcal{C}^\perp = \mathcal{E}$. We see that:

$$\begin{aligned} |V| + |F| &= 2 + \dim(\mathcal{C} \oplus \mathcal{C}^*) \\ &= 2 + \dim(\mathcal{E}) \\ &= 2 + |E| \end{aligned}$$

and we are done. $\square$