Assignment 1

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Math 191

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1 Problem 1.3

1. Let G be a tree. Suppose G is *not* minimally connected. Then there is an edge e such that $G \setminus e$ is connected. Consider the two vertices which were adjacent to e; call them v_1 and v_2. Since $G \setminus e$ is connected, there exists a path from v_1 to v_2 in $G \setminus e$. Inserting the edge e back in to get G, we see that there must have been a cycle in G. This is a contradiction, so G must indeed be minimally connected.

2. Let G be minimally connected. Suppose G has a cycle. Then let e be any edge on the cycle and let v_1 and v_2 be the two vertices adjacent to e. Let a and b be any two arbitrary vertices in G. Since G is connected, there exists a path P from a to b. If the edge e is not contained in P, then P is a path from a to b in $G \setminus e$. If the edge e is contained in P, then P is not a path from a to b in $G \setminus e$. However, since e was an edge on a cycle, there exists another path from v_1 to v_2 in $G \setminus e$. Therefore, there exists a different path P' from a to b in $G \setminus e$. In either case, there exists a path from a to b in $G \setminus e$. Since a and b were arbitrary vertices, $G \setminus e$ is connected. This contradicts the fact that G is minimally connected, so G must have no cycles.

   Now let c and d be any two vertices in G that do not already share an edge. Since G is connected, there exists a path from c to d. Adding an edge between c and d would create a cycle. This shows that adding an edge between two vertices which do not have an edge already induces a cycle.

3. Let G have no cycles and let adding an edge between two vertices in G which do not have an edge already induce a cycle in G. Let c and d be any two vertices in G. If they share an edge, then that edge is a path from c to d. If not, then adding an edge between c and d induces a cycle. This means there must have been another path originally from c to d. Either way, there is a path from c to d. Since c and d were arbitrary, there always exists a path between any two vertices in G.

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Now we show that the path is unique. Suppose there are two vertices a and b such that there are two distinct paths from a to b. Then there exist two distinct sets of edges $\{e_1, ..., e_k\}$ and $\{f_1, ..., f_l\}$ such that

$$\partial(\{e_1\} + ... + \{e_k\}) = \{a, b\}$$

and

$$\partial(\{f_1\} + ... + \{f_l\}) = \{a, b\}$$

We can add the two equations. On the right-hand side, the sum is zero (or the empty set of vertices). On the left-hand side, we use linearity to write as $\partial(\cdot)$, and the argument is non-zero because the two paths are distinct. With this construction, we have found a non-trivial element in the nullspace of ∂. This element must be a linear combination of one or more cycles since the nullspace of ∂ is also the cycle space of G. This is a contradiction because G cannot have any cycles. We have shown there is always a unique path between any two vertices in G.

4. Let there always exist a unique path between any two vertices of G. G is trivially connected. We use induction on the number of vertices in G to show that $|V| = |E| + 1$. For the base case, let $|V| = 1$. Then there are no edges, and $|V| = 1 = 0 + 1 = |E| + 1$.

   For the inductive step, let the formula $|V| = |E| + 1$ hold for all such graphs with n vertices. We want to show that for any way we adjoin an $(n + 1)$-st vertex, the formula still holds. It suffices to show that after adjoining the $(n + 1)$-st vertex in any way, the number of edges increases by exactly 1.

   Suppose not. Call the $(n + 1)$-st vertex v. After adjoining v, there are k new edges attached to v for some integer $k \geq 2$. Then there must be at least two distinct vertices a and b that are now each adjacent to v. By adjoining v in this way, we have created a new path of length 2 from a to b through v. Before adjoining v, there was a path from a to b already, since G was connected. This contradicts the requirement that there must exist a unique path between any two points. Therefore, adjoining the $(n + 1)$-st vertex must increase the number of edges by exactly 1. Since the number of vertices also increases by exactly 1, the formula holds for all numbers of vertices, by induction.

5. Let G be connected and let its vertices and edges satisfy $|V| = |E| + 1$. We need to show that G has no cycles. We use induction on the number of vertices of G. For the base case, let $|V| = 1$. It is trivial that there are no cycles.

   Now let all such graphs have no cycles if they have n vertices. We need to show that no matter how an $(n + 1)$-st vertex is adjoined, as long as the graph stays connected and still follows the formula $|V| = |E| + 1$, then there will still be no cycles. Call the $(n + 1)$-st vertex v. In order to follow the formula after adjoining v, the number of edges must increase by exactly 1. In order for the graph to remain connected, that extra edge must connect v with another vertex. However, in order for a cycle to be induced, we require at least two new edges, enough to pass through v and connect back to another vertex in the original graph. This is not possible if the number of edges increases by exactly 1 after adjoining the $(n+1)$-st vertex. By induction, all such graphs G must have no cycles. Since G is connected, it must be a tree.

We are done. $\square$

2 Problem 1.6

First, we show there exists a graph H' such that G is a minor of H' and all vertices in H' have degree less than or equal to 3. We will construct H' from G appropriately by inserting edges into vertices. If G has only vertices of degree less than or equal to 3, then let $H' = G$. Otherwise, for each vertex of degree more than 3, do the following procedure.

Let $v \in V(G)$ such that $\deg(v) = k > 3$ for some integer k. Call the edges adjacent to v $\{e_1, e_2, ..., e_k\}$. Replace v with a new edge e adjacent to two new vertices v_1 and v_2 so that v_1 is adjacent to exactly the edges e_1, e_2, and e and so that v_2 is adjacent to exactly the edges $e, e_3, ..., e_k$. Now the vertex v_1 has degree 3 and v_2 has degree $k-1$ (see Figure 1). It is clear that contraction along the edge e will lead back to the original graph with vertex v. Repeating this process enough times for vertex v_2 will ultimately give a new graph which replaces v with only vertices of degree exactly 3. Repeating this entire process for all vertices in G of degree more than 3 will result in a graph H' such that G is a minor of H' and all vertices of H' have degree less than or equal to 3.

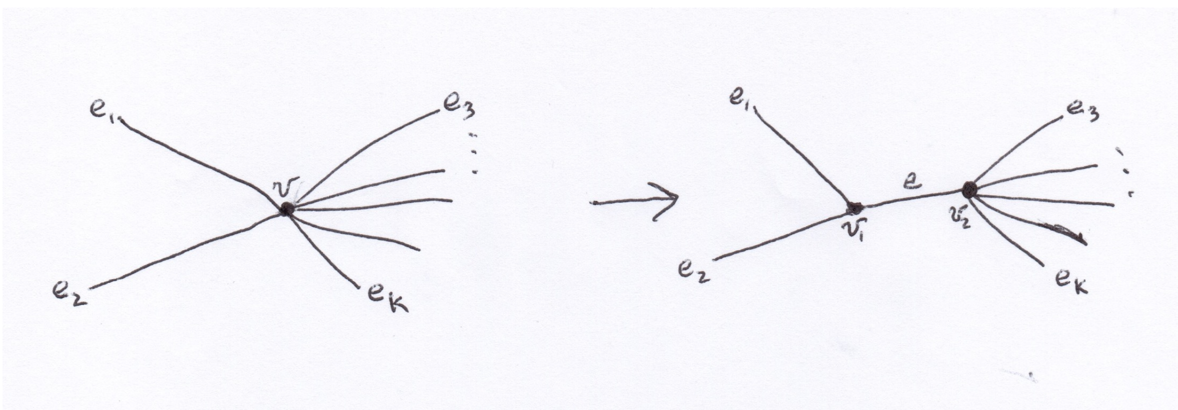

Figure 1

Next, we show there exists a graph H such that $H' \subseteq H$ and all vertices in H have degree exactly 3. We will construct H from H' by adjoining new edges and vertices. H' has vertices each of degree less than or equal to 3. This means H' has vertices only of degree 0, 1 or 2 if not 3. For every vertex v of degree 2, adjoin a single new edge, adjacent to one new vertex v' of degree 1. This leaves v with degree 3 instead of 2 as before. If this is done to all vertices of degree 2 in H', then we will have constructed a new graph $\widetilde{H}$ such that $H' \subseteq \widetilde{H}$ and all vertices in $\widetilde{H}$ have degree 3, 1 or 0.

For all vertices in $\widetilde{H}$ with degree 1, adjoin 4 new vertices and 7 new edges as shown in Figure 2.

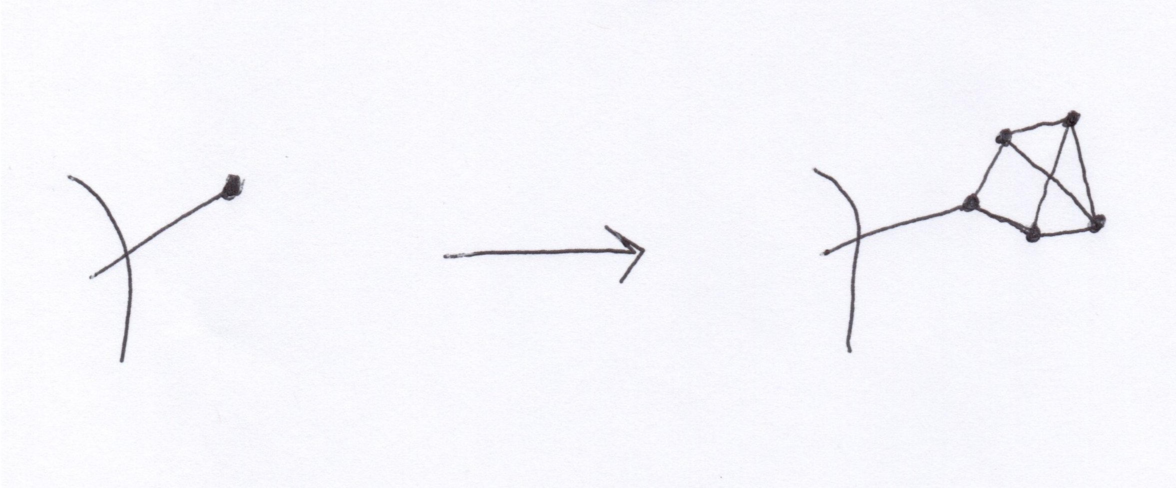

Figure 2

For all vertices in $\widetilde{H}$ with degree 0, adjoin 3 new vertices and 6 new edges as shown in Figure 3.

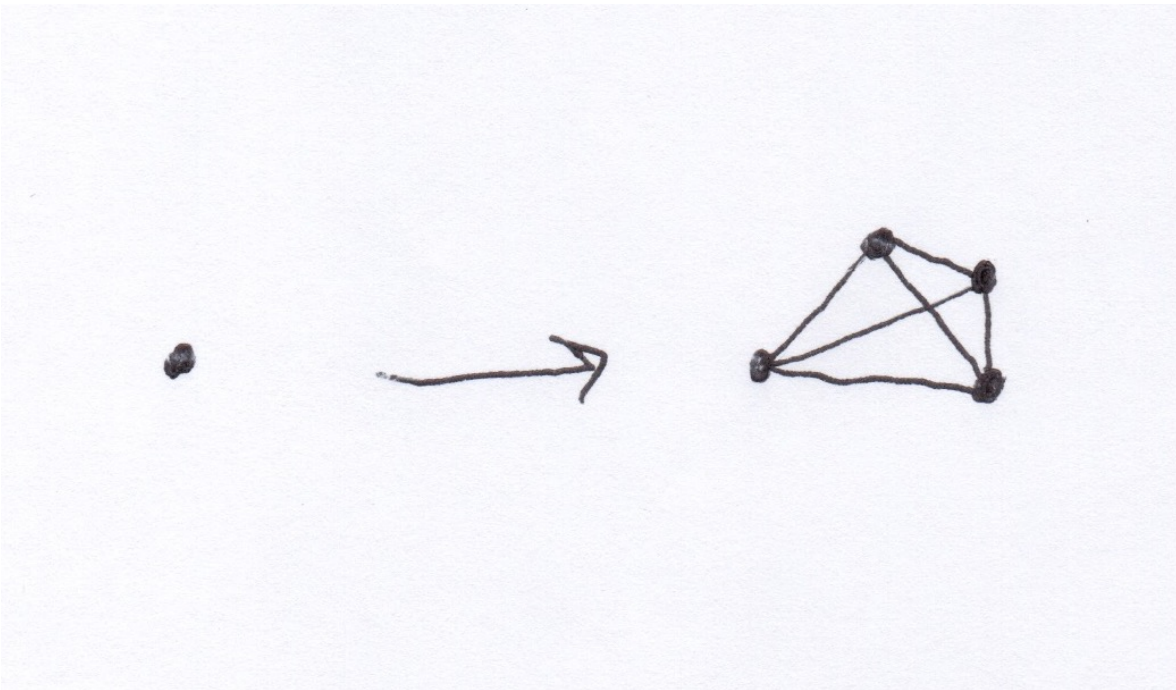

Figure 3

The result is a graph $H \supseteq \widetilde{H}$ where all vertices in H have degree 3. Since $H' \subseteq \widetilde{H} \subseteq H$, it is clear that $H' \subseteq H$. Therefore, we have constructed a graph H such that G is a minor of H and all vertices in H have degree 3. $\square$

3 Problem 1.10

1. There exists a cycle c such that e is in c. To see this, let a and b be the two vertices adjacent to e. Since $G \setminus e$ is connected, there exists a path from a to b in $G \setminus e$. Adding back the edge e, we see e must be part of a cycle.

   Since c is a cycle, it is an element of the cycle space $\mathcal{C}(G)$. Consider the cycle space $\mathcal{C}(G \setminus e)$. Let $\{c_1, ..., c_k\}$ be a basis for $\mathcal{C}(G \setminus e)$. Notice that c is not a cycle in $\mathcal{C}(G \setminus e)$ since it would be missing one of its edges, e. Therefore, c is linearly independent of the basis $\{c_1, ..., c_k\}$. So the set $B = \{c, c_1, ..., c_k\}$ is a linearly independent set in $\mathcal{C}(G)$.

   Now we show that B also spans $\mathcal{C}(G)$. Because $\mathcal{C}(G)$ is generated by all the cycles of G, it suffices to show that any cycle d is in the span of B. So let d be any cycle in G. If e is not an edge of d, then d was an element of $\mathcal{C}(G \setminus e)$, so some linear combination of $\{c_1, ..., c_k\}$ equals d. Now consider a cycle d where e is an edge of d. Notice that $(d+c)+c = d$. Also, $(d+c)$ is an element of the cycle space $\mathcal{C}(G)$ because the sum of any two cycles is an element of the cycle space, by definition of cycle space. Further, since d and c are both cycles which contain e as an edge, the sum $(d+c)$ does not contain e, so $(d+c)$ is an element of $\mathcal{C}(G \setminus e)$. Therefore, $(d+c)$ is equal to a linear combination from the set $\{c_1, ..., c_k\}$. We can see that $(d+c)+c = d$ is a linear combination from the set B. We have shown that B spans $\mathcal{C}(G)$.

   Since B is a linearly independent spanning set for $\mathcal{C}(G)$, it is a basis for $\mathcal{C}(G)$. We see that $\dim(\mathcal{C}(G)) = k+1 = \dim(\mathcal{C}(G \setminus e)) + 1$. So $b_1(G) = b_1(G \setminus e) + 1$, as was needed to be shown.

2. Define a *nonessential edge* to be an edge e such that $G \setminus e$ is connected.

   We will calculate the formula for $b_1(G)$ by removing nonessential edges one by one. (Here, removal consists of removing just the edge, not also the vertices adjacent to the edge.) By what we just proved above, each time we remove a nonessential edge to get a new graph G', $b_1(G')$ decreases by 1. This can be done until there are no more nonessential edges to be removed; the resulting graph T is by definition minimally connected. We can conclude from Exercise 1.3 above that T is a tree. Trees have no cycles, and so $b_1(T) = 0$. Also from Exercise 1.3, $|E(T)| + 1 = |V(T)| = |V(G)|$. So $b_1(T) = 0 = |E(T)| - |V(G)| + 1$. Since, when we started with G, removing each nonessential edge would decrease both $b_1(\cdot)$ and the number of edges by 1, the difference $b_1(G) - b_1(T)$ must be equal to $|E(G)| - |E(T)|$. Substituting, we see that

   $$\begin{aligned} b_1(T) = |E(T)| - |E(G)| + b_1(G) &= |E(T)| - |V(G)| + 1 \\ -|E(G)| + b_1(G) &= -|V(G)| + 1 \\ b_1(G) &= |E(G)| - |V(G)| + 1 \end{aligned}$$

   and we are done.

3. Break down G into its disjoint union of connected components:

$$G = G_1 \sqcup G_2 \sqcup ... \sqcup G_k$$

where $k = b_0(G) \geq 2$. Let V_i and E_i be the number of vertices and edges, respectively, of the ith component. In the vector space $\mathcal{E}(G)$, the cycles from different components should be linearly independent. So we can say:

$$b_1(G) = b_1(G_1) + b_1(G_2) + ... + b_1(G_k)$$

We can use the equation from (2) to see that

$$\begin{aligned} b_1(G) &= (E_1 - V_1 + 1) + ... + (E_k - V_k + 1) \\ &= (E_1 + ... + E_k) - (V_1 + ... + V_k) + (1 + ... + 1) \\ &= |E| - |V| + k \end{aligned}$$

And so we get our final formula:

$$\boxed{b_1(G) = |E| - |V| + b_0(G)}$$